# Diff Geo II: Problem Set III

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**Abstract**

This work contains solutions for the problem set III.

**Question 1-1.**

1. Suppose that $a\colon (M,g) \to (N,h)$ is an isometry.
   - Show that $a$ takes geodesics to geodesics.
   - Suppose that $M$ is connected. Show that if $b\colon (M,g) \to (N,h)$ is an isometry and $x \in M$ is such that $a(x) = b(x)$ and $a_*(v) = b_*(v)$ for all $v \in T_xM$, then $a = b$.

**Solution.**
Let $\gamma$ be a geodesic in $M$. Then,

$$D_{t(\gamma)}\dot{\gamma} \quad = \quad 0.$$

As $a$ is an isometry, and $a_*$ is linear,

$$D_{t(a\circ\gamma)}a \dot{\circ} \gamma = a_*(D_{t(\gamma)}\dot{\gamma}) = 0,$$

because local isometries preserve metric tensor components, and Chirstoffel symbols in coordinates. Hence, $a \circ \gamma$ is a geodesic.

Let $U = \{q \in M : da = db\}$. Note that $U$ is closed, and $p \in U$. But, $A$ is open, since for any $q \in U$, any normal neighborhood of $q$, $N$, must be contained in $U$, as for any $s \in N$, there exists $v \in TqM$, such that $\gamma_V(1) = r$, so

$$a(r) = \gamma_{da(v)}(1) = \gamma_{db(v)}(1) = b(r).$$

Hence $U$ is clopen, and $U = M$. $\square$

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## Question 1-2.

2. Suppose that $M$ and $N$ are connected Riemannian manifolds and that $a\colon M \to N$ is a map that preserves the distance function (i.e., $d_M(x,y) = d_N(a(x),a(y))$ for all $x, y \in M$). Show that $a$ preserves the Riemannian metric (i.e., $\langle v,w\rangle_M = \langle a_*(v), a_*(w)\rangle_N$ for all $x \in M$ and $v, w \in T_xM$). (First show that $a$ is smooth; this can be done using the smoothness of the exponential map.)

**Solution.**
Suppose for the moment that $a$ is smooth. Let $v \in TM$, and set $\gamma(t) = exp_p(tv)$. For $t$ small enough, we know that $\gamma$ has a constant speed. Therefore,

$$|v||t-s| \quad = \quad d_M(\gamma(t), \gamma(s)) = d_N(F \circ \gamma(t), F \circ \gamma(s))$$

for any $t, s$ small enough. Hence,

$$|Da(v)| \quad = \quad |\frac{d(F \circ \gamma)}{dt}|_{t=0} = \lim_{t \downarrow 0} \frac{d_M(\gamma(t), \gamma(0))}{|t|} = |\dot{\gamma}(0)| = |v|.$$

Therefore, $a$ is a Riemannian isometry, if we show that $a$ is smooth. For now, assume that $a$ is in fact bijective, which is necessary. Let $p \in M$. and $q = a(p)$. Then, at $q$ choose a radial coordinate from a normal neighborhood $(r_{q_1}, ..., r_{q_n})$, and choose $a(p_i) = q_i$ for all $i$. Then,

$$r_{q_i} \circ a(x) \quad = \quad d(a(x), q_i) = d(x, p_i)$$

for all $x$. By distance preserving property, we can choose $p_i$ and $q_i$ such that $r_{p_i}(x) = d(x, p_i)$ is smooth at $p$ for all $i$. Hence, $r \circ a$ is smooth at $p$, so $a$ is smooth. $\square$

**Question 1-3.**

3. Suppose that $M$ is a compact Riemannian manifold.

 (a) Prove that there is an $\epsilon > 0$ such that every closed curve of length $< \epsilon$ is null-homotopic.

 (b) Suppose that $\gamma : S^1 \to M$ is a closed curve. Prove that there is a piecewise-smooth curve $\gamma'$ that is homotopic to $\gamma$ and such that $\ell(\gamma') \le \ell(\gamma)$.

 (c) If $\gamma : S^1 \to M$ is a closed curve, the free homotopy class of $\gamma$ is the set of curves that are homotopic to $\gamma$. Show that this set has a minimal-length element and that this element is a geodesic.

**Solution.**
**(a)** Consider the cover of $M$ obtained by for each point $p \in M$, choose a uniformly normal neighborhood of $p$. Then, via compactness, we can extract a finite subcover from the cover, which we denote by $\{N_i\}_{i \le n}$. Now, set $\epsilon = \dfrac{1}{2} \min_i \epsilon(N_i)$, where $\epsilon(N_i)$s are chosen such that for each i, and each point in $N_i$ is contained in a geodesic ball of radius $\epsilon(N_i)$. Then, by corollary 6.11 from John M lee, we know that the Riemmanian distance and the radial distance are the same, so any closed curve of length $\epsilon$ must lie inside a geodesic ball. Therefore, for any closed curve $\gamma$ of length less than $\epsilon$, we can find $\exp_p$ such that $\exp_p^{-1}(\gamma)$ maps into $B(0, \epsilon)$ in $T_pM$ and $\gamma(0) = p$. With the homotopy map

$$H(s,t) \quad = \quad sexp_p^{-1}\gamma(t)$$

for $s, t \in [0,1] \times [0,1]$, we see that ${\exp_p}^{-1}(\gamma)$ is nulhomotopic. Since $\exp_p$ is a diffeomorphism, so it is a homeomorphism, and we see that $\gamma$ is nulhomotopic as well.

**(b)** We cover the gamma with finitely many uniformly normal neighborhoods. On the lifted paths, we can do a straight line homotopy with the straight line from the origin to $\exp_p(\gamma(1))$. Then, the inverse image of the homotopic path pieced together will form a piecewise smooth curve, that is still homotopic to $\gamma$. Since the Riemmanian distance is same as the radial function, in normal neigbhorhods, and we have chosen to do a straight line homotopy with the striaght line from the origin to $\exp_p(\gamma(1))$, $L(\gamma') \le L(\gamma)$.

**(c)** We denote the free homotopy class of $\gamma$ as $\mathfrak{H}$, and set $L = \inf_{\gamma' \in \mathfrak{H}} L(\gamma')$. From (b), we can choose a minimizing sequence $\{\gamma_n\} \subset \mathfrak{H}$ such that $\gamma_n$ is piecewise smooth, and unit speed. Then,

$$\begin{aligned} d(\gamma_n(t_1), \gamma_n(t_2)) &\le L(\gamma_n|_{[t_1,t_2]}) && (1) \\ &\le |t_1 - t_2| && (2) \end{aligned}$$

for each $n$ and $t_1, t_2 \in S^1$, where (1) holds by definition of Riemmanian distance, and (2) holds by the choice of unit speed parametrization, Therefore, $\{\gamma_n\}$ are $1-$Lipschitz family. Furthermore, for any metric ball in $S^1$, denoted by $B$,

$$\gamma_n(B) \subset M$$

for each $n$ trivially, so by Arzela-Ascoli, we can extract a subsequence, which we still denote as $\{\gamma_n\}$, converges uniformly to some $\gamma^*$. We now claim that the $\gamma^*$ is homotopic to $\gamma$, $L(\gamma^*) = L$, and geodesic. Using the uniformly normal covering as in a, and via compactness, we can have some small $\delta > 0$, such that for each $p \in M$, there exists a geodesic ball of radius $\delta$. Now, choose $\gamma_{n_0}$ from $\{\gamma_n\}$ such that $\gamma_{n_0}$ is uniformly $\frac{\delta}{2}$ away from $\gamma^*$. Then, we see that we can form a piecewise homotopy between $\gamma_{n_0}$ and $\gamma^*$ with finite number of geodesic balls, using straight line homotopy available at the tangent spaces. This shows that the uniform convergence preserves the property of being homotopic to $\gamma$, so $\gamma^*$ is homotopic to $\gamma$. Now, to prove that $\gamma^*$ is indeed a minimal length element, since $L(\gamma_n) \to L$, it suffices to show that $L(\gamma_n) \to L(\gamma)$, which is equivalent to

$$\int_{S_1} |\dot{\gamma_n}(t)| dt \quad \to \quad \int_{S_1} |\dot{\gamma^*}(t)| dt.$$

Since each $\gamma_n$ is piecewise smooth, and $\gamma_n$ converges uniformly $\gamma$, we formally expect that

$$|\dot{\gamma_n}(\cdot)| \quad \to \quad |\dot{\gamma^*}(\cdot)| \;\; \text{almost everywhere.}$$

and $\dot{\gamma_n}(\cdot)$ are $L^1$. Then, by DCT, we would have the desired conclusion. The limit object should be locally minimizing curve, which implies that it is a geodesic. $\square$