Diff Geo II: Problem Set III

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Abstract

This work contains solutions for the problem set III.

Question 1-1.

1. Suppose that $a\colon (M,g) \to (N,h)$ is an isometry.
   - Show that a takes geodesics to geodesics.
   - Suppose that M is connected. Show that if $b\colon (M,g) \to (N,h)$ is an isometry and $x \in M$ is such that $a(x) = b(x)$ and $a_*(v) = b_*(v)$ for all $v \in T_xM$, then $a = b$.

Solution.
Let γ be a geodesic in M. Then,

$$D_{t(\gamma)}\dot{\gamma} \quad = \quad 0.$$

As a is an isometry, and a_* is linear,

$$D_{t(a\circ\gamma)} a \,\dot{\circ}\, \gamma = a_*(D_{t(\gamma)}\dot{\gamma}) = 0,$$

because local isometries preserve metric tensor components, and Chirstoffel symbols in coordinates. Hence, $a \circ \gamma$ is a geodesic.

Let $U = \{q \in M : da = db\}$. Note that U is closed, and $p \in U$. But, A is open, since for any $q \in U$, any normal neighborhood of q, N, must be contained in U, as for any $s \in N$, there exists $v \in TqM$, such that $\gamma_V(1) = r$, so

$$a(r) = \gamma_{da(v)}(1) = \gamma_{db(v)}(1) = b(r).$$

Hence U is clopen, and $U = M$. $\square$

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Question 1-2.

2. Suppose that M and N are connected Riemannian manifolds and that $a\colon M \to N$ is a map that preserves the distance function (i.e., $d_M(x,y) = d_N(a(x),a(y))$ for all $x,y \in M$). Show that a preserves the Riemannian metric (i.e., $\langle v,w\rangle_M = \langle a_*(v),a_*(w)\rangle_N$ for all $x \in M$ and $v,w \in T_xM$). (First show that a is smooth; this can be done using the smoothness of the exponential map.)

Solution.

Suppose for the moment that a is smooth. Let $v \in TM$, and set $\gamma(t) = exp_p(tv)$. For t small enough, we know that γ has a constant speed. Therefore,

$$|v||t-s| \;=\; d_M(\gamma(t),\gamma(s)) = d_N(F\circ\gamma(t),F\circ\gamma(s))$$

for any t, s small enough. Hence,

$$|Da(v)| \;=\; |\frac{d(F\circ\gamma)}{dt}|_{t=0} = \lim_{t\downarrow 0}\frac{d_M(\gamma(t),\gamma(0))}{|t|} = |\dot\gamma(0)| = |v|.$$

Therefore, a is a Riemannian isometry, if we show that a is smooth. For now, assume that a is in fact bijective, which is necessary. Let $p \in M$. and $q = a(p)$. Then, at q choose a radial coordinate from a normal neighborhood $(r_{q_1},...,r_{q_n})$, and choose $a(p_i) = q_i$ for all i. Then,

$$r_{q_i}\circ a(x) \;=\; d(a(x),q_i) = d(x,p_i)$$

for all x. By distance preserving property, we can choose p_i and q_i such that $r_{p_i}(x) = d(x,p_i)$ is smooth at p for all i. Hence, $r\circ a$ is smooth at p, so a is smooth. $\square$

Question 1-3.

3. Suppose that M is a compact Riemannian manifold.

   (a) Prove that there is an $\epsilon > 0$ such that every closed curve of length $< \epsilon$ is null-homotopic.

   (b) Suppose that $\gamma : S^1 \to M$ is a closed curve. Prove that there is a piecewise-smooth curve γ' that is homotopic to γ and such that $\ell(\gamma') \leq \ell(\gamma)$.

   (c) If $\gamma : S^1 \to M$ is a closed curve, the free homotopy class of γ is the set of curves that are homotopic to γ. Show that this set has a minimal-length element and that this element is a geodesic.

Solution.
(a) Consider the cover of M obtained by for each point $p \in M$, choose a uniformly normal neighborhood of p. Then, via compactness, we can extract a finite subcover from the cover, which we denote by $\{N_i\}_{i \leq n}$. Now, set $\epsilon = \dfrac{1}{2}\min_i \epsilon(N_i)$, where $\epsilon(N_i)$s are chosen such that for each i, and each point in N_i is contained in a geodesic ball of radius $\epsilon(N_i)$. Then, by corollary 6.11 from John M lee, we know that the Riemmanian distance and the radial distance are the same, so any closed curve of length ϵ must lie inside a geodesic ball. Therefore, for any closed curve γ of length less than ϵ, we can find $\exp_p$ such that $\exp_p^{-1}(\gamma)$ maps into $B(0, \epsilon)$ in T_pM and $\gamma(0) = p$. With the homotopy map

$$H(s,t) \quad = \quad sexp_p^{-1}\gamma(t)$$

for $s, t \in [0,1] \times [0,1]$, we see that $\exp_p{}^{-1}(\gamma)$ is nulhomotopic. Since $\exp_p$ is a diffeomorphism, so it is a homeomorphism, and we see that γ is nulhomotopic as well.

(b) We cover the gamma with finitely many uniformly normal neighborhoods. On the lifted paths, we can do a straight line homotopy with the straight line from the origin to $\exp_p(\gamma(1))$. Then, the inverse image of the homotopic path pieced together will form a piecewise smooth curve, that is still homotopic to γ. Since the Riemmanian distance is same as the radial function, in normal neigbhorhods, and we have chosen to do a straight line homotopy with the striaght line from the origin to $\exp_p(\gamma(1))$, $L(\gamma') \leq L(\gamma)$.

(c) We denote the free homotopy class of γ as $\mathfrak{H}$, and set $L = \inf_{\gamma' \in \mathfrak{H}} L(\gamma')$. From (b), we can choose a minimizing sequence $\{\gamma_n\} \subset \mathfrak{H}$ such that γ_n is piecewise smooth, and unit speed. Then,

$$\begin{aligned} d(\gamma_n(t_1), \gamma_n(t_2)) &\leq L(\gamma_n|_{[t_1,t_2]}) && (1) \\ &\leq |t_1 - t_2| && (2) \end{aligned}$$

for each n and $t_1, t_2 \in S^1$, where (1) holds by definition of Riemmanian distance, and (2) holds by the choice of unit speed parametrization, Therefore, $\{\gamma_n\}$ are $1-$Lipschitz family. Furthermore, for any metric ball in S^1, denoted by B,

$$\gamma_n(B) \subset M$$

for each n trivially, so by Arzela-Ascoli, we can extract a subsequence, which we still denote as $\{\gamma_n\}$, converges uniformly to some γ^*. We now claim that the γ^* is homotopic to γ, $L(\gamma^*) = L$, and geodesic. Using the uniformly normal covering as in a, and via compactness, we can have some small $\delta > 0$, such that for each $p \in M$, there exists a geodesic ball of radius δ. Now, choose γ_{n_0} from $\{\gamma_n\}$ such that γ_{n_0} is uniformly $\frac{\delta}{2}$ away from γ^*. Then, we see that we can form a piecewise homotopy between γ_{n_0} and γ^* with finite number of geodesic balls, using straight line homotopy available at the tangent spaces. This shows that the uniform convergence preserves the property of being homotopic to γ, so γ^* is homotopic to γ. Now, to prove that γ^* is indeed a minimal length element, since $L(\gamma_n) \to L$, it suffices to show that $L(\gamma_n) \to L(\gamma)$, which is equivalent to

$$\int_{S_1} |\dot{\gamma_n}(t)| dt \quad \to \quad \int_{S_1} |\dot{\gamma^*}(t)| dt.$$

Since each γ_n is piecewise smooth, and γ_n converges uniformly γ, we formally expect that

$$|\dot{\gamma_n}(\cdot)| \quad \to \quad |\dot{\gamma^*}(\cdot)| \quad \text{almost everywhere.}$$

and $\dot{\gamma_n}(\cdot)$ are L^1. Then, by DCT, we would have the desired conclusion. The limit object should be locally minimizing curve, which implies that it is a geodesic. $\square$