

Diff Geo II: Problem Set I

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Abstract

This work contains solutions for the problem set I.

Question 1-1.

1. Show that the covariant derivative can be expressed in terms of parallel transport in the following sense.

Let $\gamma : [0, 1] \rightarrow M$ be a smooth curve in M and let $p_t : T_{\gamma(0)}M \rightarrow T_{\gamma(t)}M$ be the parallel transport maps along γ . Show that if $X \in \mathbf{V}(\gamma)$, then

$$D_t X(0) = \frac{d}{dt} p_t^{-1}(X(t)).$$

Solution.

Let $E_1(t), \dots, E_n(t)$ be a parallel frame along γ , and $X \in \mathcal{V}(\gamma)$. Then,

$$D_t X(0) = \sum_i D_t(X^i(0)E_i(0)) = \sum_i \dot{X}^i(0)E_i(0) + \sum_i X^i(0)D_t(E_i(0)) \quad (1)$$

$$\begin{aligned} &= \sum_i \dot{X}^i(0)E_i(0) = \frac{d}{dt} \sum_i X^i(0)P_t^{-1}(E_i(0)) = \frac{d}{dt} P_t^{-1}(X(0)) \quad (2) \\ &= \left. \frac{d}{dt} \right|_{t=0} P_t^{-1}(X(t)) \end{aligned}$$

where (1) follows from the Leibniz rule with $D_t(E_i(0)) = 0$ for all i , and (2) holds by the linear isomorphism property of P_t . \square

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Question 1-2.

2. Prove that the tangential connection ∇^T on an embedded submanifold $M \subset \mathbb{R}^n$ is compatible with the metric induced by the dot product on \mathbb{R}^n .

Solution.

Let $\gamma(t)$ be a curve and $Y, Z \in \mathcal{V}(\gamma)$. Then,

$$\begin{aligned} \partial_t \langle Y, Z \rangle &= \langle \bar{\nabla}_{\gamma(t)} Y, Z \rangle + \langle Y, \bar{\nabla}_{\gamma(t)} Z \rangle \\ &= \langle \pi^T(\bar{\nabla}_{\gamma(t)} Y), Z \rangle + \langle Y, \pi^T(\bar{\nabla}_{\gamma(t)} Z) \rangle \end{aligned} \quad (3)$$

$$= \langle \nabla_{\gamma(t)}^T Y, Z \rangle + \langle Y, \nabla_{\gamma(t)}^T Z \rangle \quad (4)$$

where (3) holds as $Y, Z \in \mathcal{V}(\gamma)$, and (4) follows from definition of tangential connection. Therefore, we have precisely shown the compatibility of the tangential connection with the metric induced by the dot product on \mathbb{R}^n . \square

Question 1-3.

3. Prove that ∇^T is torsion-free. (Since τ is a tensor, it suffices to show that for any patch $(u^1, \dots, u^n): U \rightarrow M$, $\tau(\partial_i, \partial_j) = 0$.)

Solution.

Let $X, Y \in \mathcal{V}(M)$. Then,

$$\begin{aligned} \nabla_X^T Y - \nabla_Y^T X &= \pi^T(\bar{\nabla}_X Y) - \pi^T(\bar{\nabla}_Y X) = \pi^T(\bar{\nabla}_X Y - \bar{\nabla}_Y X) \\ &= \pi^T([X, Y]) \end{aligned} \tag{5}$$

$$= [X, Y] \tag{6}$$

where (5) follows from the fact that $\bar{\nabla}$ is torsion-free, and (6) follows from $[X, Y] \in \mathcal{V}(M)$. \square

Question 1-4.

Let $\phi \in (0, \pi/2)$ and let $M \subset \mathbb{R}^3$ be the cone

$$M = \{(r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi) \mid r > 0, \theta \in [0, 2\pi)\}$$

with axis the positive z -axis and angle ϕ .

4. The parametrization $u(r, \phi) = (r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi)$ has coordinate vector fields $\partial_r = \frac{\partial}{\partial r}$ and $\partial_\theta = \frac{\partial}{\partial \theta}$. Let ∇^T be the tangential connection on M and calculate $\nabla_{\partial_r}^T \partial_r$, $\nabla_{\partial_r}^T \partial_\theta$, $\nabla_{\partial_\theta}^T \partial_r$, and $\nabla_{\partial_\theta}^T \partial_\theta$.

Solution.

We compute

$$\begin{aligned}\partial_r &= (\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)) \\ \partial_\theta &= (-r \sin(\theta) \sin(\phi), r \cos(\theta) \sin(\phi), 0)\end{aligned}$$

and hence

$$\begin{aligned}\bar{\nabla}_{\partial_r} \partial_r &= \frac{\partial}{\partial r}(\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)) = (0, 0, 0) \\ \bar{\nabla}_{\partial_r} \partial_\theta &= \frac{\partial}{\partial r}(-r \sin(\theta) \sin(\phi), r \cos(\theta) \sin(\phi), 0) \\ &= (-\sin(\theta) \sin(\phi), \cos(\theta) \sin(\phi), 0) \\ \bar{\nabla}_{\partial_\theta} \partial_r &= \frac{\partial}{\partial r}(\cos(\theta) \sin(\phi), \sin(\theta) \sin(\phi), \cos(\phi)) \\ &= (-\sin(\theta) \sin(\phi), \cos(\theta) \sin(\phi), 0) \\ \bar{\nabla}_{\partial_\theta} \partial_\theta &= \frac{\partial}{\partial r}(-r \sin(\theta) \sin(\phi), r \cos(\theta) \sin(\phi), 0) \\ &= (-r \cos(\theta) \sin(\phi), -r \sin(\theta) \sin(\phi), 0)\end{aligned}$$

Now, to carry out the projection onto the tangent plane, we must obtain the normal. We compute

$$\partial_r \times \partial_\theta = (-r \cos(\theta) \sin(\phi) \cos(\phi), -r \sin(\theta) \sin(\phi) \cos(\phi), r \sin^2(\phi))$$

and hence

$$N = \frac{\partial_r \times \partial_\theta}{\|\partial_r \times \partial_\theta\|} = (-\cos(\theta) \cos(\phi), -\sin(\theta) \cos(\phi), \sin(\phi)).$$

Therefore,

$$\begin{aligned}\nabla_{\partial_r}^T \partial_r &= (0, 0, 0) \\ \nabla_{\partial_\theta}^T \partial_r &= \nabla_{\partial_r}^T \partial_\theta = \pi^T(\bar{\nabla}_{\partial_\theta} \partial_r) = \pi^T\left(\frac{1}{r} \partial_\theta\right) = \frac{1}{r} \partial_\theta \\ \nabla_{\partial_\theta}^T (\partial_\theta) &= \pi^T(\bar{\nabla}_{\partial_\theta} \partial_\theta) = \pi^T(\cos(\phi) \sin(\phi) r N - r \sin(\phi)^2 \partial_r) = -r \sin(\phi)^2 \partial_r\end{aligned}$$

as required.

Later, I realized that this mess maybe be simplified by noting that the non vanishing Christoffel symbols are

$$\begin{aligned}\Gamma_{\theta r}^{\theta} &= \Gamma_{r\theta}^{\theta} = \frac{1}{r} \\ \Gamma_{\theta\theta}^r &= -r \sin(\phi)^2\end{aligned}$$

so by the Christoffel formula, we can verify the above results as well.

□

Question 1-5.

5. Let $\gamma: [0, 2\pi] \rightarrow M$ be the circle $\gamma(t) = u(1, t)$. Calculate the parallel transport map $P_{0,2\pi}: T_{\gamma(0)}M \rightarrow T_{\gamma(2\pi)}M$.

Solution.

Let $\gamma(t) = u(1, t) = (\cos(t) \sin(\phi), \sin(t) \sin(\phi), \cos(\phi))$, and $V_0 = (V_0^r, V_0^\theta) \in T_{\gamma(0)}M$. From the transport equation is given by $D_t V = 0$, and the computations from 1 – 4,

$$D_t V = \frac{dV^r}{dt} \partial_r + \frac{dV^\theta}{dt} \partial_\theta + V^r \partial_\theta - V^\theta \sin(\phi)^2 \partial_r = 0.$$

Therefore, V must satisfy the following system of odes: $\frac{dV^r}{dt} = V^\theta \sin(\phi)^2$, $\frac{dV^\theta}{dt} = -V^r$, with initial conditions being $V(0) = V_0$. With separation of variables, we see that

$$\begin{aligned} V^r(t) &= V_0^r \cos(t \sin(\phi)) + V_0^\theta \sin(\phi) \sin(t \sin(\phi)) \\ V^\theta(t) &= V_0^\theta \cos(t \sin(\phi)) - \frac{V_0^r}{\sin(\phi)} \sin(t \sin(\phi)) \end{aligned}$$

uniquely solves the system. Therefore, we see that we can give an explicit characterization of the transport map as

$$P_{0,2\pi}(V_0) = (\cos(2\pi \sin(\phi)), \sin(\phi) \sin(2\pi \sin(\phi)); -\frac{\sin(2\pi \sin(\phi))}{\sin(\phi)}, \cos(2\pi \sin(\phi)))(V_0^r, V_0^\theta)^T$$

for each $V_0 \in T_{\gamma(0)}M$, as required. □