PDE II: Final

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Abstract

This work contains solutions for the final of the PDE II course at Courant.

Question 1-1.

Exercise 1 (The Allen-Cahn equation) We consider the equation

$$\begin{cases} \partial_t u - \Delta u + (u^2 - 1)u = 0 \\ u(t = 0) = u_0 \end{cases} \tag{1}$$

where $(t, x) \in [0, \infty) \times \mathbb{T}^d$, and $u(t, x) \in \mathbb{R}$.

1. By a fixed point argument, prove local well-posedness in $H^s(\mathbb{T}^d)$, for $s > \frac{d}{2}$.

2. By a fixed point argument, prove local well-posedness in $L^\infty(\mathbb{T}^d)$.

3. Describe the solutions of the ODE $\dot{u} + (u^2 - 1)u = 0$, with data $u(t = 0) = u_0 \in \mathbb{R}$, in particular indicating their limit as $t \to \infty$ depending on u_0. We can think of the PDE (1) as a combination of the dynamics given by this ODE, with diffusion given by the Laplacian

4. Write down the energy for which (1) is a gradient flow.

5. Three stationary (and constant in space) solutions are given by $u = 0, \pm 1$. Linearize the PDE around these solutions. Which ones of these linearized problems are stable (converge to zero as $t \to \infty$), and which ones are not?

6. From now on, $d = 1$. Show that (1) is globally well-posed in $H^1(\mathbb{R}^d)$.

7. Show that the solution $u = 1$ is asymptotically stable in L^∞, namely: there exists $\epsilon > 0$ such that, if $\|u_0(x) - 1\|_{L^\infty} < \epsilon$, then $\|u(t, x) - 1\|_{L^\infty} \to 0$ as $t \to \infty$.

8. Is $u = 0$ asymptotically stable?

9. Prove that

$$\int_0^\infty \int_{\mathbb{T}^d} |\partial_t u|^2 \, dx \, dt + \sup_{t > 0} \int_{\mathbb{T}^d} |\partial_x u|^2 \, dx < \infty.$$

10. (Harder question!) Classify the possible asymptotic behaviors of u as $t \to \infty$.

Solution.
(1) We will roughly follow the strategy shown in class with the cubic NLH problem. By definition, our solution of the equation will be functions in $C([0, T], H^s)$ such that

$$u(t) \quad = \quad e^{t\triangle} u_0 + \int_0^t e^{(t-s)\triangle}(u - u^3) ds$$

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for some $T > 0$. This in turn can be rewritten as a fixed point problem:

$$u \quad = \quad Ru \ \text{ with } \ R : C([0,T], H^s) \to C([0,T], H^s), \quad v \mapsto u$$

$$\text{and} \quad u = e^{t\triangle} u_0 + \int_0^t e^{(t-l)\triangle}(v(l) - v(l)^3)dl. \tag{1}$$

for some $T > 0$. Set $\rho = ||u_0||_{H^s}$. We claim that there exists T, K suitably chosen such that R is a contraction in $B_{(C[0,T],H^s)}(0, K\rho)$. We compute, for any $u \in B_{(C[0,T],H^s)}(0, K\rho)$,

$$
\begin{aligned}
||R(u)||_{C([0,T],H^s)} &\leq\ ||u_0||_{H^s} + \sup_{0<t<T} \int_0^t ||e^{(t-l)\triangle}(u - u^3)(l)||_{H^s} dl \\
&\leq\ \rho + CT \sup_{0<t<T} (||u||_{H^s} + ||u||_{H^s}^3) \quad &(2) \\
&\leq\ \rho + CT(K\rho + K^3\rho^3) \quad &(3)
\end{aligned}
$$

where C is an universal constant independent of u, whose existence is justified along with (2) from the fact that $e^{t\triangle}$ is bounded on H^s for each $t > 0$ and H^s is an algebra for each $s > \dfrac{d}{2}$, and (3) follows from $u \in B_{C([0,T],H^s)}(0, K\rho)$. Hence, if

$$CT(K\rho + K^3\rho^3) < (K-1)\rho \quad \Longleftrightarrow \quad T \leq \frac{K-1}{C(K + K^3\rho^2)}$$

then, R stabilizes $B_{C([0,T],H^s)}(0, K\rho)$. We now investigate the condition on K and T, which will imply that R is a contraction in $B_{C([0,T],H^s)}(0, K\rho)$ i.e. for any $u, v \in B_{C([0,T],H^s)}(0, K\rho)$,

$$||Ru - Rv||_{C([0,T],H^s)} < \frac{1}{2}||u - v||_{C([0,T],H^s)}. \tag{4}$$

We compute, for any $u, v \in B_{C[0,T],H^s)}(0, K\rho)$, with the same reasoning as above,

$$
\begin{aligned}
\sup_{0<t<T} ||R(u)(t) - R(v)(t)||_{H^s} &\leq\ \sup_{0<t<T} \int_0^t ||e^{(t-l)\triangle}(u(l) - u^3(l) - v(l) + v^3(l))||dl \\
&=\ \sup_{0<t<T} \int_0^t ||e^{(t-l)\triangle}(v(l) - u(l))(v^2(l) + v(l)u(l) + u^2(l) - 1)||dl \\
&\leq\ C \sup_{0<t<T} \int_0^t ||u(l) - v(l)||((||u(l)|| + ||v(l)||)^2 + 1)dl \\
&\leq\ C(K^2\rho^2 + 1)T||u - v||_{C([0,T],H^s)}.
\end{aligned}
$$

Hence,

$$T \leq \frac{1}{2C(K^2\rho^2 + 1)} \quad \Longrightarrow \quad R \ \text{ is a contraction on } \ B_{C([0,T],H^s)}(0, K\rho).$$

Therefore, choose any $K > 1$, and then choose

$$T \quad = \quad \min\{\frac{K-1}{C(K + K^3\rho^2)}, \frac{1}{2C(K^2\rho^2 + 1)}\}.$$

Now, with respect to these choices, by a general fixed point theorem, we have proved existence and uniqueness in $B_{C([0,T],H^s)}(0, K\rho)$. Continuous dependence on the data and uniqueness on the whole space $C([0,T], H^s)$ can be argued in the same way as the lecture note 7. So we are done.
$\square$

(2) As L^∞ is an algebra, the technique used above will work if we show that for each $t > 0$

$$e^{t\triangle} \text{ is bounded on } L^\infty.$$

(3) We solve the given ODE. Observe that

$$\frac{1}{u - u^3} = \frac{1}{u} + \frac{u}{1 - u^2}.$$

Hence,

$$du =$$

so

$$u(t) = \ln|t| - \frac{1}{2}\ln|(1-t)(1+t)| + u_0$$

(4) We derive the energy with the heuristic shown in class. Multiply both sides of the equation by u and taking the integral with respect to x,

$$\int \partial_t u u dx - \int \triangle u u dx = \int u^2 - u^4 dx$$

which simplifies to

$$\frac{1}{2}\frac{d}{dt}\int u^2 dx = -\int |\nabla u|^2 dx - \int u^2 - u^4 dx.$$

Taking the time integral with the above equation,

$$\int u^2(T)dx = \int u_0^2 dx - \int_0^T |\nabla u|^2 dx dt - \int_0^T \int |u|^2 dx dt + \int_0^T \int |u|^4 dx dt.$$

(5) Let $u = 0$. Ignoring the non-linear term in $u - u^3$, we get the linearized pde as

$$\partial_t - \triangle u = u.$$

Let $u = 1$. Set $v = u - 1$, so

$$\partial_t v - \triangle v = \partial_t u - \triangle u = (v+1)^2 - (v+1)^3$$
$$= -v^3 - 3v^2 - 2v.$$

Ignoring the nonlinear term, we get the linearized pde as

$$\partial_t v - \triangle v = -2v.$$

Similarly, for $u = -1$, we get the linearized pde as

$$\partial_t v - \triangle v = -2v.$$

For the $u = 0$ case, set $u = e^t v$ to get

$$u(x,t) \;\; = \;\; e^t \int \Phi(x - y, t) u_0(y) dy$$

where Φ is the heat kernel. Then, for any $\delta > 0$, and $||u_0||_{H^s} < \delta$ such that $u \neq 0$, $u(x,t)$ blows up by the fact that e^t term dominates. For the other two cases, we have e^{-2t} factor instead of e^t, so we see that the linearized PDE is unstable for $u = 0$ and stable for $u = \pm 1$. $\qquad \square$

(6) As $s = 1 > \dfrac{1}{2} = \dfrac{d}{2}$, by 1, we have local well-posedness of the equation in $H^1(\mathbb{T})$. Let T^* be the maximal time of existence T^*. Then, for some constant C,

$$\sup_{0<t<T^*} ||u(t)||_{H^1} \;\; \leq \;\; C \sup_{0<t<T^*} (||u(t)||_{L^2} + ||\nabla u(t)||_{L^2}))$$
$$\leq$$
$$\leq \;\; C \sup_{0<t<T^*} E(u(t)) \leq CE(u_0) \leq$$

Hence u is globally bounded in H^1, so

$$\limsup_{t \to T^*} ||u(t)||_{H^1} < \infty$$

and by the lemma in page 2 of lecture note 8, $T^* = \infty$.

Question 1-2.

Exercise 2 (A hypoelliptic operator) We want to study the differential operator L on $\mathbb{R}^2$ defined by
$$[Lu](x,y) = -\partial_x^2 u(x,y) - x\partial_y u(x,y)$$
and its parabolic flow.

We consider first the equation
$$f = Lu$$
and aim at understanding the smoothing properties of L.

1. By an energy estimate, show that $\|\partial_x u\|_{L^2}^2 \leq \|f\|_{L^2}\|u\|_{L^2}$.

2. By explicitly solving the differential equation $v'(s) + s^2 v(s) = g(s)$, for $v \in C_0^\infty$, show that $\|v\|_{L^2} \leq C\|g\|_{L^2}$.

3. Deduce by scaling that $\forall \lambda \in \mathbb{R}$, $\|v' + \lambda s^2 v\| \geq C|\lambda|^{1/3}\|v\|_{L^2}$

4. (Harder question!) By using the previous question, switching to Fourier space, and denoting ξ and η for the dual variables of x and y respectively, show that
$$\int (1 + |\xi|^2 + |\eta|^{4/3})|\hat{u}(\xi,\eta)|^2 \, d\eta \, d\xi \leq C(\|f\|_{L^2}^2 + \|u\|_{L^2}^2).$$

5. Deduce that any solution in $u \in H^2$ of $Lu = u^3$ is actually C^∞.

We turn to the associated parabolic flow
$$\begin{cases} \partial_t u + Lu = 0 \\ u(t=0) = u_0 \end{cases}$$

6. Setting $v(t,x,y) = u(t,x,y-xt)$, show that it solves $\partial_t v = (\partial_x + t\partial_y)^2 v$.

7. Arguing in Fourier space, deduce from the previous question that
$$v(t,x,y) = \Phi(t,x,y) * u_0(x,y) \quad \text{where} \quad \Phi(t,x,y) = \frac{C_0}{t^2}e^{-\frac{x^2}{t}+\frac{3xy}{t^2}-\frac{3y^2}{t^3}}.$$

8. Show that the parabolic flow preserves positivity: $u \geq 0$ if $u_0 \geq 0$.

9. Prove decay estimates of the type
$$\|u(t)\|_{L^p} \leq \frac{C}{t^\alpha}\|u_0\|_{L^q},$$
for appropriately chosen p, q, α.

10. Does this equation enjoy smoothing estimates?

Solution.

(1) We proceed by our usual formal derivation of the energy estimate. Multiplying both sides of the equation by u, integrating over x, y, and integrating by parts,

$$\int\int uf\,dxdy = -\int\int uu_{xx}\,dxdy - \int\int ux u_y\,dydx$$

$$= \int\int (u_x)^2\,dxdy - \int x(\frac{1}{2}u^2|_{-\infty}^{\infty})\,dx$$

$$= \int\int (u_x)^2\,dxdy$$

and hence, by Holder's inequality,

$$||u_x||_{L^2}^2 = |\int\int uf\,dxdy| \le \int\int |uf|\,dxdy \le ||f||_{L^2}||u||_{L^2},$$

as required. $\qquad\square$

(2) Firstly, observe that as v vanishes at ∞, if g solves the ODE then, g has compact support. Now, we solve the ODE. Multiplying both sides of the equation by $e^{\frac{1}{3}s^3}$ (integrating factor),

$$e^{\frac{1}{3}s^3}dv + e^{\frac{1}{3}s^3}s^2 v\,ds = e^{\frac{1}{3}s^3}g(s)\,ds$$

and hence

$$e^{\frac{1}{3}s^3}g(s)\,ds = d(e^{\frac{1}{3}s^3}v).$$

Therefore,

$$v(s) = e^{-\frac{1}{3}s^3}\int_0^s g(t)e^{\frac{1}{3}t^3}\,dt.$$

solves the ODE. Now, we compute

$$||v||_{L^2}^2 = \int (e^{-\frac{1}{3}s^3}\int_0^s g(t)e^{\frac{1}{3}t^3}\,dt)^2\,ds \le \int (e^{-\frac{1}{3}s^3}\int_0^s g(t)e^{\frac{1}{3}s^3}\,dt)^2\,ds$$

$$= \int (\int_0^s g(t)\,dt)^2\,ds \le \int (\int_{\mathbb{R}} |g(t)|\,dt)^2\,ds$$

$$\le \int\int_{\mathbb{R}} |g(t)|^2\,dtds \le m(\text{supp}(v))||g||_{L^2}^2$$

where the second last inequality holds by Jensen's inequality on finite measure space, as g has compact support, and $m(\text{supp}(v)) < \infty$ by assumption. $\qquad\square$

(8) Suppose $u_0 \ge 0$. Note that Φ is non-negative everywhere. Then, as for each t, x, y, by 7, for each t, x, y,

$$u(t, x, y - xt) = v(t, x, y) = \Phi(t, x, y) * u_0(x, y) \ge 0$$