Diff Geo II: Problem Set VI

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Abstract

This work contains solutions for the problem set VI.

Question 1-1.

1. Show that CAT(0) spaces have the approximate midpoint property. That is, if X is CAT(0) and $x, y \in X$ are points such that $r = d(x, y) > 0$, then for any ϵ, there is a δ such that if $d(z, x) \leq \frac{r}{2} + \delta$ and $d(z, y) \leq \frac{r}{2} + \delta$, then $d(z, m) \leq \epsilon$, where m is the midpoint of x and y.

Solution.
Consider $\angle_m(z, x)$ and $\angle_m(z, y)$ such that $d(z, x) \leq \dfrac{r}{2} + \delta$ for some $\delta > 0$. As X is non-positively curved, we can assume without loss of generality that $\angle_m(z, x) \geq \frac{\pi}{2}$. Then, the angle at $\bar{m}$ for the comparison triangle $\bar{\triangle}(m, z, x)$ is greater than equal to $\frac{\pi}{2}$. Hence, by the Cosine law,

$$
\begin{aligned}
(\frac{r}{2})^2 + d(m, z)^2 &= d(x, m)^2 + d(m, z)^2 \\
&\leq d(z, x)^2 \leq (\frac{r}{2} + \delta)^2
\end{aligned}
$$

so

$$d(m, z)^2 \leq r\delta + \delta^2.$$

Therefore, as $r > 0$ is fixed, for any given $\epsilon > 0$, we can choose $\delta > 0$ small enough, $d(m, z) \leq \epsilon$, and we are done. $\square$

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Question 1-2.

2. Suppose that X is a Riemannian manifold that is CAT(0). Show that X has nonpositive sectional curvature. It may help to go back to problem set 4.

Solution.
In this problem, we show that in the Riemannian category, if the Riemannian distance, induced by the Riemannian metric is a metric, which satisfies the curvature condition of bounded above by 0 in the sense of Alexandrov, then the sectional curvature of the Riemannian metric is everywhere bounded above by 0. Hence, we see that the theory of Alexandrov spaces generalizes the sectional curvature part of Riemannian geometry in a particular way by focusing on the structure of sectional curvature having uniform bound from above. We now proceed with the proof. Let $x \in M$, and U, $V \in T_xM$, such that U and V are orthonormal. Then, let K be the sectional curvature of the plane, spanned by U and V, in T_xM. Via normal coordinates, for all t small enough, we have two unique geodesics $t \mapsto \exp_x(tU)$ and $t \mapsto \exp_x(tV)$ such that the tangent vectors at x at $t = 0$ are U and V respectively. Set $a(t) = d(\exp_x(tU), \exp_x(tV))$. From the homework set 4, squaring the expression in the problem 2 gives in normal coordinate,

$$a(t)^2 \quad = \quad 2t^2 - \frac{K}{6}t^4 + O(t^5). \tag{1}$$

Now, consider a geodesic triangle(straight lines) in $\mathbb{R}^2$ with side lengths t, and orthogonal. Then, we claim that

$$a(t)^2 \geq s(t)^2 = 2t^2 \tag{2}$$

where the $s(t)$ is the length of the third side on the Euclidean triangle. Clearly, the last equality is from the Pythagorean theorem, and in view of (1) and considering t small, it suffices to show (2) to conclude that $K \leq 0$. From the solution to the problem 3, we can deduce that the Alexandrov angle between sides of any geodesic in X with distinct vertices is less than or equal to the angle between the corresponding sides of the comparison triangle in $\mathbb{R}^2$. Now, as $d(\exp_x(tV), x) = t$ and $d(\exp_x(tU), x) = t$ by a property of normal neighborhoods, and by the Cosine law, we see that

$$a(t)^2 \quad = \quad d(\exp_x(TU), \exp_X(tV)) \geq s(t)^2 = 2t^2.$$

Since K was arbitrary, we are done. $\square$

Question 1-3.

3. Let $\gamma, \gamma' \colon [0, \infty) \to X$ be unit-speed geodesic rays such that $\gamma(0) = \gamma'(0) = p$. In class, we defined the comparison angle between γ and γ' by

$$\angle_p(\gamma, \gamma') = \lim\sup_{t,t' \to 0} \angle_{\bar{p}}(\overline{\gamma(t)}, \overline{\gamma'(t')}),$$

where $\triangle(\bar{p}, \overline{\gamma(t)}, \overline{\gamma'(t')})$ is the comparison triangle for $\triangle(p, \gamma(t), \gamma'(t))$. That is, by the law of cosines,

$$\cos \angle_p(\gamma, \gamma') = \lim\inf_{t,t' \to 0} \frac{t^2 + t'^2 - d(\gamma(t), \gamma'(t'))^2}{2tt'}.$$

Show that this limit exists (not just the lim sup) when X is CAT(0). (Suppose that $s < t$ and $s' < t'$. How does $\angle_{\bar{p}}(\overline{\gamma(s)}, \overline{\gamma'(s')})$ compare to $\angle_{\bar{p}}(\overline{\gamma(t)}, \overline{\gamma'(t')})$?)

Solution.
The key idea in this proof is that the lines on the compared triangle have the same length as the original length, as opposed to general lines inside the compared triangle. With the given set up, it suffices to show that

$$\angle_{\bar{p}}(\overline{\gamma(s)}, \overline{\gamma(s')}) \quad \leq \quad \angle_{\bar{p}}(\overline{\gamma(t)}, \overline{\gamma(t')}),$$

since monotonicity of the limit will show that the a priori $\lim\sup$ is in fact lim. Denote compared triangles on $\mathbb{R}^2$ as $\bar{\triangle}(p, \gamma(s), \gamma(s'))$, and $\bar{\triangle}(p, \gamma(t), \gamma(t'))$. Now, observe that by the first comparison,

$$d(\overline{\gamma(s)}_1, \overline{\gamma(s')}_1) \quad = \quad d(\gamma(s), \gamma(s'))$$

and from the second comparison

$$d(\overline{\gamma(s)}_2, \overline{\gamma(s')}_2) \quad \geq \quad d(\gamma(s), \gamma(s'))$$

so

$$d(\overline{\gamma(s)}_2, \overline{\gamma(s')}_2) \quad \geq \quad d(\overline{\gamma(s)}_1, \overline{\gamma(s')}_1).$$

Then, by the Cosine law on the Euclidean space (recall that all lines on the compared triangle have the same length as the original triangle, as oppposed to lines inside the compared triangle in general, and cosine is decreasing as the angle increases to π)

$$\begin{aligned} \angle_{\bar{p},2}(\overline{\gamma(t)}, \overline{\gamma(t')}) &= \angle_{\bar{p},2}(\overline{\gamma(s)}, \overline{\gamma(s')}) \\ &\geq \angle_{\bar{p},1}(\overline{\gamma(s)}, \overline{\gamma(s')}) \end{aligned}$$

where we denoted the comparison angle explicitly with respect to the first and the second triangle. Hence, we are done. $\square$