Diff Geo II: Problem Set III

Youngduck Choi *

Abstract

This work contains solutions for the problem set III.

Question 1-1.

1. Let M be a Riemannian manifold and let d be the distance function on M. Show that M is complete with respect to d if and only if M is geodesically complete.

Solution.
Firstly, it is necessary to assert that the Riemmanian manifold must be connected in this context; otherwise, there can be two points $x, y \in M$, such that $d(x, y) = \infty$ where d is the standard Riemmanian distance, as the infimum of an emptyset by convention is ∞. Therefore, d does not even satisfy the property of a metric, hence we cannot make sense of the "completeness" of the metric space of Riemmian distance, induced by the Riemmanian metric. If we assume connected, since connected manifolds are path connected, and doing replacement of a piecewise continuous path to piecewise smooth path, if necessary, we can gurantee the well-definedness of the Riemmnian distance as a metric function on a set. Now, we proceed to prove the statement. Suppose M is complete, but not geodesically complete. Then, we can choose a unit speed geodesic $\gamma : [0, b) \to M$ such that it does not extend to $[0, b+\epsilon)$ for any $\epsilon > 0$. Let $\{a_n\}$ be an increasing sequence, limiting to b, and set $\{q_n = \gamma(a_n)\}$. By construction,

$$d(q_n, q_m) \quad \leq \quad |a_n - b_m|$$

for any n, m, so $\{q_n\}$ is Cauchy, and via completeness q_n converges to some point $q \in M$. Now, choose N be a uniformly normal neighborhood of q, and $\delta > 0$ be chosen that the geodesic δ ball at each point contains N, and for convenience we can assume without loss of generality that $\delta < b$. Now, for n large enough $a_n > b - \delta$, so every geodesic starting at q_n exists for at least time δ for n large enough, via the property of normal neighborhoods. This, in particular, implies that there is a geodesic γ^* such that $\gamma^*(0) = q_n$ and $\dot{\gamma}^*(a_n)$ for some n large enough. By uniqueness $\tilde{\gamma}(t) = \sigma(a_n + t)$ is an extension of γ upto $a_n + b - \delta > 0$, which contradicts the minimality of b. Now, for the converse, suppose M is geodesically complete. From class, we know that geodesically

*Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

complete is equvivalent to exp defined on all of the tangent bundle. Also, by Hopf-Rinow proven in class, we have that there exists a minimal geodesic for any two points on M, which we denote as property $(*)$. Fix $p \in M$, and set $M_n = \exp_p(\overline{B_n(0)})$ for all n. These are compact, as its an image of compact set of a continuous map. From (*), we see

$$\bigcup_{k=1}^{\infty} M_n = M$$

Now, let $\{x_n\}$ be cauchy in M. This implies that the Riemmanian distance is bounded on $\{x_n\}$, and since Riemmanian distance agrees with the distance in T_pM, through exp map, we can find k large enough such that $\{x_n\} \subset M_k$. Since, compactness implies completness in a general metric space, we see that x_n converges to some $x \in M_k \subset M$, and we are done. $\square$

Question 1-2.

2. Show that the curvature tensor of a Riemannian metric is determined by its sectional curvatures.

Solution.

We here assume the well-definedness of the sectional curvature, K: it does not depend on the choice of the pair of vectors of the tangent space at the point. Now, fix $p \in M$. We claim that If $K = 0$ at p, then $R = 0$ at p. Firstly, let $v, w \in T_p(M)$. Then, if v and w span a non-degenerate plane then, $< R_{vw}v, w >= 0$. Otherwise, there is a technical lemma, which asserts that any pair of vectors is a limit of non-degenerate vectors. By multilinearity, and continutiy of $< R_{vw}x, y >$ on $T_p(M)^4$, we see that $< R_{vw}v, w >= 0$. We now claim that $R_{vw}v = 0$. By polarization identity,

$$< R_{v,w+x}v, w + x > \quad = \quad < R_{vw}v, w > + < R_{vx}v, w > + < R_{vw}, x > + < R_{vx}v, x >$$

and by the previous remark, and symmetry of pairs,

$$< R_{vw}v, x >= 0$$

for any $x \in T_pM$. Now, again by polarization,

$$R_{v+x,w}(v + x) \quad = \quad R_{vw}x + R_{xw}v + R_{vw}x + R_{xw}x$$

and by the previous remark, and skew-symmetry in subscripts,

$$R_{vw}x = R_{wx}v$$

for all $x \in T_pM$. Hence, by first Bianchi identity, we see that

$$R_{vw}x = 0$$

for any $x \in T_pM$, so $R = 0$ at p. Now, suppose F is a multilinear on $T_p(M)^4$. and has symmetries properties enjoyed by $< R_{vw}x, y >$. Then, from the previous remark, we see that R is completely characterized by

$$K(v, w) \quad = \quad \frac{F(v, w, v, w)}{< v, v >< w, w, > - < v, w >^2},$$

for any $x, y, v, w \in T_p(M)$, as taking the difference and applying the previous remark shows that the difference is 0. $\square$

Question 1-3.

3. (a) Let $p \in M$ and let $V, W \in T_pM$. Let $\gamma = \gamma_V$ be a geodesic such that $\gamma(0) = p$ and $\gamma'(0) = V$. Let J be the Jacobi field on γ such that $J(0) = 0$, $D_tJ(0) = W$. Find $D_t^2J(0)$ and $D_t^3J(0)$. (It may help to let $E_1, \dots, E_n \in \mathcal{V}(\gamma)$ be a set of orthonormal parallel fields on γ and write $J = \sum_i f_i E_i$.)

   (b) Let $\epsilon > 0$ be such that $\exp_p \colon B(0, \epsilon) \to U \subset M$ is a diffeomorphism and let $(x^1, \dots, x^n)\colon U \to \mathbb{R}^n$ be the corresponding normal coordinate system. (See Prop. 5.11 of Lee for some properties of normal coordinates.) Use the calculation above to show that the second-order Taylor series of g with respect to normal coordinates is given by

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{kl}\langle R(\partial_i, \partial_k)\partial_l, \partial_j\rangle x^k x^l + O(|x|^3),$$

   where $x = (x^1, \dots, x^m)$.

Solution.

(a) We compute

$$\begin{aligned}
< J, J >' &= 2 < J', J > \\
< J, J >'' &= 2 < V'', V > +2 < V', V' > \\
&= 2 < R(\gamma', J)\gamma', J > +2 < J', J' > \\
< J, J >''' &= 2 < R(\gamma', J)'\gamma', J > +6 < R(\gamma, J)\gamma', J' > \\
< J, J >''' &= 8 < R(\gamma', J)', \gamma', J' > +2 < R(\gamma', J)'', \gamma', J > +6 < R(\gamma', J)\gamma', J'' > .
\end{aligned}$$

Plugging in $t = 0$, first and third terms disappear and

$$\begin{aligned}
< J, J >'' (0) &= < w, w > \\
< J, J >'''' (0) &= 8 < R(v, w)v, w > .
\end{aligned}$$

We will use that

$$< V(t), V(t) >=< w, w > t^2 + \frac{1}{3} < R(v, w), v, w > t^4 + O(t^5).$$

(b) Let $w \in T_pM$, and consider the geodesic $\gamma(t) = \exp_p(tw)$, and let c be a curve in T_pM such that $c(0) = w$ and $\dot{c}(0) = v$, which can be chosen by setting $c(s) = w + sv$. Then, set

$$\Gamma(t, s) = \exp_p(tc(s)).$$

Observe that γ is a geodesic variation of γ, with the variation field over γ, is given by,

$$V(t) \quad = \quad \Gamma_s(t, 0) = d(\exp_p)_{tw}(tv) = td\exp_p(tw)(v).$$

Then, $V(0) = 0$, and $V'(0) = \Gamma_{st}(0, 0) = \Gamma_{ts}(0, 0)$. Furthermore, $\Gamma_s(t, 0) = \exp_p(tv)$ and differntiating with respect to t and taking value at $t = 0$ yields v. Hence,

$$d(\exp_p)(tw)(v) = t^{-1}V(t) \tag{1}$$

where V is the Jacobi field along $\gamma(t) = \exp_p(tw)$ with $V(0) = 0$ and $V'(0) = v$. Now, apply the above result with each $e_1, ..., e_n$ orthonormal basis of T_pM, to obtain the corresponding Jacobi fields $J_1, ..., J_n$. Then, by a property of normal coordinates, and (1),

$$\partial_j(\exp_p(tw)) = d(\exp_p)_{tw}e_j = t^{-1}J_j(t) \tag{2}$$

for all j. Hence,

$$\begin{aligned} g_{jk}(\exp_p(tw)) &= <\partial_j(\exp_p(tw)), \partial_k(\exp_p(tw)> \\ &= t^{-2} < J_i, J_k > (t) = t^{-2}(t^2 < e_j, e_k > + \frac{t^4}{3} < R(w, e_j)w, e_k > + O(t^5)) \quad (3) \\ &= \delta_{jk} + \frac{t^2}{3} < R(w, e_j)w, e_k > + O(t^3) = \delta_{jk} + \frac{1}{3} R_{ijkl} x^i x^l + O(t^3) \quad (4) \\ &= \delta_{jk} - \frac{1}{3} R_{iklj} x^k x^l + O(t^3); \quad (5) \end{aligned}$$

where (3) holds by (2) and (4) holds via $w = t^{-1}x$ substitution.