PDE II: Final

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Abstract

This work contains solutions for the final of the PDE II course at Courant.

Question 1-1.

Exercise 1 (The Allen-Cahn equation) We consider the equation

$$\begin{cases} \partial_t u - \Delta u + (u^2 - 1)u = 0 \\ u(t=0) = u_0 \end{cases} \tag{1}$$

where $(t, x) \in [0, \infty) \times \mathbb{T}^d$, and $u(t, x) \in \mathbb{R}$.

1. By a fixed point argument, prove local well-posedness in $H^s(\mathbb{T}^d)$, for $s > \frac{d}{2}$.
2. By a fixed point argument, prove local well-posedness in $L^\infty(\mathbb{T}^d)$.
3. Describe the solutions of the ODE $\dot{u} + (u^2 - 1)u = 0$, with data $u(t = 0) = u_0 \in \mathbb{R}$, in particular indicating their limit as $t \to \infty$ depending on u_0. We can think of the PDE (1) as a combination of the dynamics given by this ODE, with diffusion given by the Laplacian
4. Write down the energy for which (1) is a gradient flow.
5. Three stationary (and constant in space) solutions are given by $u = 0, \pm 1$. Linearize the PDE around these solutions. Which ones of these linearized problems are stable (converge to zero as $t \to \infty$), and which ones are not?
6. From now on, $d = 1$. Show that (1) is globally well-posed in $H^1(\mathbb{R}^d)$.
7. Show that the solution $u = 1$ is asymptotically stable in L^∞, namely: there exists $\epsilon > 0$ such that, if $\|u_0(x) - 1\|_{L^\infty} < \epsilon$, then $\|u(t, x) - 1\|_{L^\infty} \to 0$ as $t \to \infty$.
8. Is $u = 0$ asymptotically stable?
9. Prove that
$$\int_0^\infty \int_{\mathbb{T}^d} |\partial_t u|^2 \, dx \, dt + \sup_{t>0} \int_{\mathbb{T}^d} |\partial_x u|^2 \, dx < \infty.$$
10. (Harder question!) Classify the possible asymptotic behaviors of u as $t \to \infty$.

Solution.
(1) We will roughly follow the strategy shown in class with the cubic NLH problem. By definition, our solution of the equation will be functions in $C([0, T], H^s)$ such that

$$u(t) \quad = \quad e^{t\triangle}u_0 + \int_0^t e^{(t-s)\triangle}(u - u^3)ds$$

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for some $T > 0$. This in turn can be rewritten as a fixed point problem:

$$\begin{aligned} u &= Ru \ \text{ with } \ R : C([0,T], H^s) \to C([0,T], H^s), \quad v \mapsto u \\ \text{and} \qquad & u = e^{t\triangle} u_0 + \int_0^t e^{(t-l)\triangle}(v(l) - v(l)^3)dl. \end{aligned} \tag{1}$$

for some $T > 0$. Set $\rho = ||u_0||_{H^s}$. We claim that there exists T, K suitably chosen such that R is a contraction in $B_{(C[0,T],H^s)}(0, K\rho)$. We compute, for any $u \in B_{(C[0,T],H^s)}(0, K\rho)$,

$$\begin{aligned} ||R(u)||_{C([0,T],H^s)} &\leq ||u_0||_{H^s} + \sup_{0<t<T} \int_0^t ||e^{(t-l)\triangle}(u - u^3)(l)||_{H^s} dl \\ &\leq \rho + CT \sup_{0<t<T} (||u||_{H^s} + ||u||^3_{H^s}) && (2) \\ &\leq \rho + CT(K\rho + K^3\rho^3) && (3) \end{aligned}$$

where C is an universal constant independent of u, whose existence is justified along with (2) from the fact that $e^{t\triangle}$ is bounded on H^s for each $t > 0$ and H^s is an algebra for each $s > \dfrac{d}{2}$, and (3) follows from $u \in B_{C([0,T],H^s)}(0, K\rho)$. Hence, if

$$CT(K\rho + K^3\rho^3) < (K-1)\rho \quad \Longleftrightarrow \quad T \leq \frac{K-1}{C(K + K^3\rho^2)}$$

then, R stabilizes $B_{C([0,T],H^s)}(0, K\rho)$. We now investigate the condition on K and T, which will imply that R is a contraction in $B_{C([0,T],H^s)}(0, K\rho)$ i.e. for any $u, v \in B_{C([0,T],H^s)}(0, K\rho)$,

$$||Ru - Rv||_{C([0,T],H^s)} < \frac{1}{2}||u - v||_{C([0,T],H^s)}.$$

We compute, for any $u, v \in B_{C[0,T],H^s)}(0, K\rho)$, with the same reasoning as above,

$$\begin{aligned} \sup_{0<t<T} ||R(u)(t) - R(v)(t)||_{H^s} &\leq \sup_{0<t<T} \int_0^t ||e^{(t-l)\triangle}(u(l) - u^3(l) - v(l) + v^3(l))||dl \\ &= \sup_{0<t<T} \int_0^t ||e^{(t-l)\triangle}(v(l) - u(l))(v^2(l) + v(l)u(l) + u^2(l) - 1)||dl \\ &\leq C \sup_{0<t<T} \int_0^t ||u(l) - v(l)||((||u(l)|| + ||v(l)||)^2 + 1)dl \\ &\leq C(K^2\rho^2 + 1)T||u - v||_{C([0,T],H^s)}. \end{aligned}$$

Hence,

$$T \leq \frac{1}{2C(K^2\rho^2 + 1)} \quad \Longrightarrow \quad R \ \text{ is a contraction on } \ B_{C([0,T],H^s)}(0, K\rho).$$

Therefore, choose any $K > 1$, and then choose

$$T = \min\{\frac{K-1}{C(K + K^3\rho^2}, \frac{1}{2C(K^2\rho^2 + 1)}\}.$$

Now, with respect to these choices, by a general fixed point theorem, we have proved existence and uniqueness in $B_{C([0,T],H^s)}(0, K\rho)$. Continuous dependence on the data and uniqueness on the whole space $C([0,T], H^s)$ can be argued in the same way as the lecture note 7. So we are done. $\square$

(2) As L^∞ is an algebra, the fixed point argument, used above will work if we show that for each $t > 0$

$$e^{t\triangle} \text{ is bounded on } L^\infty.$$

This, however, is clear, as the eigenfunction expansion picks up $e^{-4\pi^2 n^2 t}$ term, which is bounded by 1 for all n, for each coefficients. $\square$

(3) We solve the given ODE. By the separation of variables,

$$dt = (\frac{1}{u} + \frac{u}{1-u^2})du$$

so

$$t = \ln|u| - \frac{1}{2}\ln|1-u^2| + C.$$

and

$$e^{t-C} = |u||1-u|^{-\frac{1}{2}}.$$

For $1 - u^2 > 0$ and $1 - u^2 < 0$ respectively,

$$\begin{aligned} u &= (1 + e^{c-t})^{-\frac{1}{2}} \\ u &= (1 - e^{c-t})^{-\frac{1}{2}}. \end{aligned}$$

For the $1 - u^2 < 0$ case, we see that at $t = \ln(|1-u_0|^{\frac{1}{2}}|u_0|^{-1})$ the solution blows up. $\square$

(4) We derive the energy formally, as shown in class. Multiplying by u_t on both sides of the equation, and integrating by parts,

$$\int u_t \partial_t u - u_t \triangle u + u_t u^3 - u_t u = 0$$

so

$$\int |\partial_t u|^2 dx = -\frac{d}{dt}(\frac{1}{4}\int u^4 dx - \frac{1}{2}\int u^2 dx + \frac{1}{2}|\triangle u|^2 dx).$$

Therefore,

$$E(u) = \frac{1}{4}\int u^4 dx - \frac{1}{2}\int u^2 dx + \frac{1}{2}|\triangle u|^2 dx,$$

as required. $\square$

(5) Let $u = 0$. Ignoring the non-linear term in $u - u^3$, we get the linearized pde as

$$\partial_t - \triangle u = u.$$

Let $u = 1$. Set $v = u - 1$, so

$$\begin{aligned} \partial_t v - \triangle v &= \partial_t u - \triangle u = (v+1)^2 - (v+1)^3 \\ &= -v^3 - 3v^2 - 2v. \end{aligned}$$

Ignoring the nonlinear term, we get the linearized pde as

$$\partial_t v - \triangle v = -2v.$$

Similarly, for $u = -1$, we get the linearized pde as

$$\partial_t v - \triangle v = -2v.$$

For the $u = 0$ case, set $u = e^t v$ to get

$$u(x,t) = e^t \int \Phi(x-y,t) u_0(y) dy$$

where Φ is the heat kernel. Then, for any $\delta > 0$, and $||u_0||_{H^s} < \delta$ such that $u \neq 0$, $u(x,t)$ blows up by the fact that e^t term dominates. For the other two cases, we have e^{-2t} factor instead of e^t, so we see that the linearized PDE is unstable for $u = 0$ and stable for $u = \pm 1$. $\square$

(6) As $s = 1 > \frac{1}{2} = \frac{d}{2}$, by 1, we have local well-posedness of the equation in $H^1(\mathbb{T})$. Let T^* be the maximal time of existence T^*. Now, observe that by Soblev embedding, we have

$$u_0 \in H^1(\mathbb{T}) \subset C^{\frac{1}{2}}(\mathbb{T}) \subset C(\mathbb{T}) \subset L^\infty(\mathbb{T}) \subset \bigcap_{1 \le p} L^p(\mathbb{T}).$$

Therefore, $E(u_0) < \infty$, and since the energy is decreasing $E(u) \le E(u_0)$ for all time up to T^*. Observe that by Holder, for all time up to T^*,

$$\frac{1}{2}|\triangle u|^2 + \frac{1}{4}\left(\int u^2 dx\right)^2 - \frac{1}{2}\int u^2 dx \le E(u).$$

Hence, for u with sufficiently large L^2 norm, we have a bound on the L^2 norm from above by $E(u_0)$. Therefore, there exists a constant $C > 0$ such that for all time up to T^*,

$$||u||_{H^1} \le C E(u_0).$$

Hence u is globally bounded in H^1, so

$$\limsup_{t \to T^*} ||u(t)||_{H^1} < \infty$$

and by the lemma in page 2 of lecture note 8, $T^* = \infty$. $\square$

(8)

Question 1-2.

Exercise 2(A hypoelliptic operator) We want to study the differential operator L on $\mathbb{R}^2$ defined by

$$[Lu](x,y) = -\partial_x^2 u(x,y) - x\partial_y u(x,y)$$

and its parabolic flow.

We consider first the equation

$$f = Lu$$

and aim at understanding the smoothing properties of L.

1. By an energy estimate, show that $\|\partial_x u\|_{L^2}^2 \leq \|f\|_{L^2}\|u\|_{L^2}$.

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2. By explicitly solving the differential equation $v'(s) + s^2 v(s) = g(s)$, for $v \in \mathcal{C}_0^\infty$, show that $\|v\|_{L^2} \leq C\|g\|_{L^2}$.
3. Deduce by scaling that $\forall \lambda \in \mathbb{R}$, $\|v' + \lambda s^2 v\| \geq C|\lambda|^{1/3}\|v\|_{L^2}$
4. (Harder question!) By using the previous question, switching to Fourier space, and denoting ξ and η for the dual variables of x and y respectively, show that
$$\int (1 + |\xi|^2 + |\eta|^{4/3})|\widehat{u}(\xi,\eta)|^2 \, d\eta \, d\xi \leq C(\|f\|_{L^2}^2 + \|u\|_{L^2}^2).$$
5. Deduce that any solution in $u \in H^2$ of $Lu = u^3$ is actually $\mathcal{C}^\infty$.

We turn to the associated parabolic flow

$$\begin{cases} \partial_t u + Lu = 0 \\ u(t=0) = u_0 \end{cases}$$

6. Setting $v(t,x,y) = u(t,x,y-xt)$, show that it solves $\partial_t v = (\partial_x + t\partial_y)^2 v$.
7. Arguing in Fourier space, deduce from the previous question that
$$v(t,x,y) = \Phi(t,x,y) * u_0(x,y) \qquad \text{where} \qquad \Phi(t,x,y) = \frac{C_0}{t^2}e^{-\frac{x^2}{t}+\frac{3xy}{t^2}-\frac{3y^2}{t^3}}.$$
8. Show that the parabolic flow preserves positivity: $u \geq 0$ if $u_0 \geq 0$.
9. Prove decay estimates of the type
$$\|u(t)\|_{L^p} \leq \frac{C}{t^\alpha}\|u_0\|_{L^q},$$
for appropriately chosen p, q, α.
10. Does this equation enjoy smoothing estimates?

Solution.

(1) We proceed by our usual formal derivation of the energy estimate. Multiplying both sides of the equation by u, integrating over x, y, and integrating by parts,

$$\begin{aligned}\int\int uf dxdy &= -\int\int uu_{xx}dxdy - \int\int uxu_y dydx \\ &= \int\int (u_x)^2 dxdy - \int x(\frac{1}{2}u^2|_{-\infty}^{\infty})dx \\ &= \int\int (u_x)^2 dxdy\end{aligned}$$

and hence, by Holder's inequality,

$$||u_x||_{L^2}^2 = |\int\int uf dxdy| \leq \int\int |uf|dxdy \leq ||f||_{L^2}||u||_{L^2},$$

as required. □

(2) Firstly, observe that as v vanishes at ∞, if g solves the ODE then, g has compact support. Now, we solve the ODE. Multiplying both sides of the equation by $e^{\frac{1}{3}s^3}$(integrating factor),

$$e^{\frac{1}{3}s^3}dv + e^{\frac{1}{3}s^3}s^2 v ds = e^{\frac{1}{3}s^3}g(s)ds$$

and hence

$$e^{\frac{1}{3}s^3}g(s)ds = d(e^{\frac{1}{3}s^3}v).$$

Therefore,

$$v(s) = e^{-\frac{1}{3}s^3}\int_0^s g(t)e^{\frac{1}{3}t^3}dt.$$

solves the ODE. Now, we compute

$$\begin{aligned}||v||_{L^2}^2 &= \int (e^{-\frac{1}{3}s^3}\int_0^s g(t)e^{\frac{1}{3}t^3}dt)^2 ds \leq \int (e^{-\frac{1}{3}s^3}\int_0^s g(t)e^{\frac{1}{3}s^3}dt)^2 ds \\ &= \int (\int_0^s g(t)dt)^2 ds \leq \int (\int_{\mathbb{R}} |g(t)|dt)^2 ds \\ &\leq \int\int_{\mathbb{R}} |g(t)|^2 dtds \leq m(\mathrm{supp}(v))||g||_{L^2}^2\end{aligned}$$

where the second last inequality holds by Jensen's inequality on finite measure space, as g has compact support, and $m(\mathrm{supp}(v)) < \infty$ by assumption. □

(3) Fix $\lambda, C_1, C_2 \in \mathbb{R}$. Then, set

$$z(s) = \lambda^{C_1}v(\lambda^{C_2}s)$$

so

$$\begin{aligned} z'(s) &= \lambda^{C_1+C_2} v'(\lambda^{C_2} s) \\ s^2 z(s) &= s^2 \lambda^{C_1} v(\lambda^{C_2} s). \end{aligned}$$

We compute, via a change of variable $l = \lambda^{C_2} s$,

$$\int |z'(s) + s^2 z(s)|^2 ds = \lambda^{-C_2} \int |\lambda^{C_1+C_2} v'(l) + l^2 \lambda^{C_1-2C_2} v(l)|^2 dl$$

Setting $C_1 = \frac{4}{3}$, and $C_2 = -\frac{1}{3}$,

$$\begin{aligned} ||z||_{L^2}^2 &= \lambda^{\frac{11}{3}} ||v||_{L_2}^2 \\ ||z' + l^2 z||_{L^2}^2 &= \lambda^3 ||v' + l^2 \lambda v||_{L^2}^2. \end{aligned} \tag{4}$$

Now, by (2), for some constant $C > 0$,

$$||z||_{L^2}^2 \leq C||z' + s^2 z||_{L^2}^2$$

and substituting (4) to above,

$$C^{-1} \lambda^{\frac{2}{3}} ||v||_{L^2}^2 \leq ||v' + l\lambda v||_{L^2}^2.$$

Taking the square root on both sides gives the desired inequality. □

(4) From the energy estimate (1), and taking the Fourier transform of $\partial_x u$,

$$\int\int |\xi|^2 |\hat{u}(\xi,\eta)|^2 d\xi d\eta = ||\partial_x||_{L^2}^2 \leq ||\hat{f}||_{L^2} ||\hat{u}||_{L^2} \leq \frac{1}{2}(||\hat{f}||_{L^2}^2 + ||\hat{u}||_{L^2}^2)$$

and hence, for some constant C,

$$\int\int (1 + |\xi|^2)|\hat{u}|^2 \leq C(||\hat{f}||_{L^2} + ||\hat{u}||_{L^2}).$$

Therefore, it suffices to show that, for some constant $C > 0$,

$$\int\int |\eta|^{\frac{4}{3}} |\hat{u}(\xi,\eta)|^2 d\xi d\eta \leq C||\hat{f}||_{L^2}^2 \tag{5}$$

Taking the Fourier transform of the equation,

$$|\xi|^2 \hat{u} + \eta \partial_\xi \hat{u} = \hat{f}$$

and dividing both sides of the above equation by η,

$$\frac{1}{\eta}|\xi|^2 \hat{u} + \partial_\xi \hat{u} = \frac{1}{\eta}\hat{f}.$$

Now, by the scaling estimate from (3), for some constant $C_1 > 0$,

$$|\frac{1}{\eta}|||\hat{f}(\xi,\eta)||_{L^2_\xi} \quad \geq \quad C_1|\eta|^{-\frac{1}{3}}||\hat{u}(\xi,\eta)||_{L^2_\xi}$$

and hence, rearranging and squaring both sides, for some constant $C_2 > 0$.

$$C_2 \int |\hat{f}(\xi,\eta)|^2 d\xi \quad \geq \quad |\eta|^{\frac{4}{3}} \int |\hat{u}(\xi,\eta)|^2 d\xi.$$

Integrating the above equation with respect to η gives (5), so we are done. $\square$

(5) Since $s = 2 > 1 = \frac{d}{2}$, H^2 is an algebra, and $u^3 \in H^2$. From (4), $u \in H^{\frac{4}{3}}$, so $u \in H^{\frac{10}{3}}$. Repeating the argument inductively, $u \in H^s$ for all $2 \leq s$, and by Morrey's inequality, $u \in C^\alpha$ for all $1 \leq \alpha$. Therefore, $u \in C^\infty$ as required. $\square$

(6) We wish to show that

$$v_t \quad = \quad v_{xx} + 2tv_{xy} + t^2 v_{yy} \tag{6}$$

To that end, we compute

$$\begin{aligned} v_x &= u_x - tu_y \\ v_{xx} &= u_{xx} - 2tu_{yx} + t^2 u_{yy} \\ v_t &= u_t - xu_y \\ 2tv_{xy} &= 2tu_{xy} - 2t^2 u_{yy} \\ t^2 v_{yy} &= t^2 u_{yy}. \end{aligned}$$

Substituting the above equations to (6),

$$u_t - xu_y \quad = \quad u_{xx} = u_{xx} - 2tu_{xy} + t^2 u_{yy} + 2tu_{xy} - 2t^2 u_{yy} + t^2 u_{yy}$$

which simplifies to

$$u_t - u_x x - xu_y \quad = \quad 0.$$

Therefore, we see that v solves (6), as required. $\square$

(7) Taking the Fourier transform of the equation in (6),

$$\partial_t \hat{v} \quad = \quad -\xi^2 \hat{v} - 2t\xi\eta\hat{v} - t^2\eta^2\hat{v}$$

and hence

$$\hat{v} \quad = \quad \hat{u_0}(\xi,\eta)e^{-\xi^2 t - \xi\eta t^2 - \frac{1}{3}\eta^2 t^3}.$$

Therefore,it suffices to show that

$$\mathscr{F}(\Phi) \quad = \quad e^{-\xi^2 t - \xi\eta t^2 - \frac{1}{3}\eta^2 t^3}.$$

This can be checked (don't have time to type it).

(8) Suppose $u_0 \geq 0$. Note that Φ is non-negative everywhere. Hence, by 6 and 7, for each t, x, y,

$$u(t,x,y) \quad = \quad v(t,x,y+xt) = \Phi(t,x,y+xt) * u_0(x,y+xt) \geq 0,$$

as required. $\square$

(9) By (7),

$$u(t,x,z) = \int \frac{C_0}{t^2} e^{-\frac{(x-z_1)^2}{t} + \frac{3(x-z_1)(z+xt-z_2)}{t^2} - \frac{3(z+xt-z_2)^2}{t^3}} u_0(z_1,z_2)dz_1dz_2.$$

Then, if Young's inequality was to be applied, then

$$||u(t)||_p \quad \leq \quad ||\Phi(t,x,xt+z)||_r ||u_0||_q$$

for $1 \leq p,q,r \leq \infty$, with $\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{q}$,

$\square$

(10) Yes, as in general parabolic equations, like heat, enjoys smoothing estimates. $\square$