

Limit Theorems II: Final

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Abstract

This work contains solutions for the final of the Limit Theorems II by Professor McKean.

Question 1-1.

1. *Rw(3): CAPACITY.* Compute the capacity of a pair of lattice points $a \neq b$ and also of a singleton c in terms of the Green's function $G(a, b) = \sum_0^\infty P_a[\mathbf{X}(n) = b]$. This already shows that capacity is *not* additive. Yes?

Solution.

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Question 1-2.

2. *Bm(1) CONDITIONAL ARCSINE.* Recall the usual arcsine law

$$P_0[\text{measure}(t \leq 1 : \mathbf{b}(t) \geq 0) \leq x] = \frac{1}{\pi} \int_0^x \frac{dy}{\sqrt{y(1-y)}} \quad (0 \leq x \leq 1)$$

showing that the bulk of the density lies (surprisingly) near $y = 0$ and $y = 1$. It comes as a further surprise that for the tied Brownian motion, conditioned to end at $\mathbf{b}(1) = 0$, that the density is *perfectly flat* as in

$$P_0[\text{measure}(t \leq 1 : \mathbf{b}(t) \geq 0) \leq x] = x \quad (0 \leq x \leq 1).$$

I will lead you through it via Feynman-Kac.

Write $\mathbf{m} = \text{measure}(t' \leq t : \mathbf{b}(t') \geq 0)$ and recall that

$$\int_0^\infty e^{-\alpha t} \cdot E_0[e^{-\beta \mathbf{m}} f \circ \mathbf{b}(t)] dt$$

may be expressed as

$$h_+(0) \int_{-\infty}^0 h_- f dy + h_-(0) \int_0^\infty h_+ f dy \quad \text{over} \quad \frac{1}{2} (h'_- h_+ - h_- h'_+) \quad \text{at } x = 0$$

with $h_-(x) = e^{\sqrt{2}x}$ for $x \leq 0$ and $h_+(x) = e^{-\sqrt{2}(\alpha+\beta)x}$ for $x \geq 0$ for reasonable functions f . That is now used for the unreasonable function $f = \text{Dirac's delta function } \delta(x)$ to produce

$$\int_0^\infty e^{-\alpha t} E_0[e^{-\beta \mathbf{m}}, \mathbf{b}(t) = 0] dt = \frac{2}{\sqrt{2\alpha} + \sqrt{2(\alpha+\beta)}}$$

as you will check. This may also be written

$$\int_0^\infty e^{-\alpha t} E_0[e^{-\beta \mathbf{m}} | \mathbf{b}(t) = 0] \frac{dt}{\sqrt{2\pi t}}$$

and if you knew, what is the fact, that \mathbf{m} is now uniformly distributed over the interval $0 \leq m \leq t$, then you could put this into the last display to see if it checks. Do it. It

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works. Is that a proof? Could the conditional law of \mathbf{m} be anything else? Of course this is only a consistency check, based so to speak on proprietary information. Insider trading if you will. The honest way would be to invert $\frac{2}{\sqrt{2\alpha} + \sqrt{2(\alpha+\beta)}}$. Not a standard exercise. Try it if you like.

Solution.

Question 1-3.

3. $BM(1)$: LOCAL TIME. Begin with the function $f''(x) = \frac{1}{\epsilon}$ for $0 \leq x \leq \epsilon$, vanishing elsewhere, and compute f' and f subject to $f'(0) = f(0) = 0$. Apply Itô's lemma for small ϵ to obtain

$$\begin{aligned} \max[0, \mathbf{b}(t)] &\simeq f \circ \mathbf{b}(t) \\ &= \int_{(0,t) \cap \{0 \leq \mathbf{b}(t') < \epsilon\}} \frac{b}{\epsilon} db \\ &\quad + \int_{(0,t) \cap \{\mathbf{b}(t') > \epsilon\}} db \\ &\quad + \frac{1}{2\epsilon} \text{measure } (t' \leq t : 0 \leq \mathbf{b}(t') < \epsilon) \\ &= \textcircled{1} + \textcircled{2} + \textcircled{3}. \end{aligned}$$

Here $\textcircled{1}$ is a fair game. Show that it vanishes, uniformly for $t \leq 1$ say, if $\epsilon \downarrow 0$ fast enough, and likewise that $\textcircled{2}$ tends to $\int_0^t e d\mathbf{b}$ with $e(t, \mathbf{b}) =$ the indicator of $\mathbf{b}(t) > 0$, and the same uniformity. Then $\textcircled{3}$ converges in the same style and you have

$$\max[0, \mathbf{b}(t)] = \int_0^t e d\mathbf{b} \text{ plus } \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \text{measure } (t' \leq t : 0 \leq \mathbf{b}(t') < \epsilon) = \textcircled{4}$$

with the same uniformity in $\textcircled{4}$, this being P. Levy's *mesure de voisinage*, more aptly called in English *local time*, written $\mathbf{t}(t)$. Think a little how extraordinary it is that this limit should exist, due solely to the wildness of the Brownian path. Evidently $\mathbf{t}(t)$ is continuous and increasing but only on $Z = \{t \geq 0 : \mathbf{b}(t) = 0\}$ which is of Lebesgue measure zero(!) and cannot vanish for any $t > 0$. Give me a little argument for *that*.

FYI. The fact is that it does not matter how fast $\epsilon \downarrow 0$. This lies deeper. See Itô-McKean if you're interested.

Solution.

Recall the Ito's formula for single martingales: If f is C^2 and for all $0 \leq t$,

$$E \int_0^t [f'(M(s))]^2 dA(s) < \infty \quad \text{and} \quad E \int_0^t |f''(M(s))| dA(s) < \infty$$

then, for all $0 \leq t$,

$$f(M(t)) - f(M(0)) = \int_0^t f'(M(s)) dM(s) + \frac{1}{2} \int_0^t f''(M(s)) dA(s) \quad (1)$$

where $A(t)$ is the unique increasing, continuous process, corresponding to $M(t)$. Of course, for the problem at hand, we have $M(t) = B(t)$ and $A(t) = t$, so $db(s)$ and ds will be used to denote

the integrators. Now, to compute local time, we cannot naively apply this formula, as $\max(0, \cdot)$ is not $C^2(\mathbb{R})$. Hence, we approximate with the suggested strategy. Let f_ϵ be defined as given for each $\epsilon > 0$. Then, for any $0 \leq t$, and $\epsilon > 0$,

$$f_\epsilon \circ b(t) = f_\epsilon \circ b(t) - f_\epsilon \circ b(0) = \int_0^t f'_\epsilon(B(s))db(s) + \frac{1}{2} \int_0^t f''_\epsilon(B(s))ds \quad (2)$$

$$\begin{aligned} &= \int_{\{s \in [0, t] : 0 \leq b(s) < \epsilon\}} \frac{b(s)}{\epsilon} db(s) + \int_{\{s \in [0, t] : b(s) > \epsilon\}} 1 db(s) + \frac{1}{2} \int_{\{s \in [0, t] : 0 \leq b(s) < \epsilon\}} \frac{1}{\epsilon} ds \quad (3) \\ &= \int_{\{s \in [0, t] : 0 \leq b(s) < \epsilon\}} \frac{b(s)}{\epsilon} db(s) + \int_{\{s \in [0, t] : b(s) > \epsilon\}} 1 db(s) + \frac{1}{2\epsilon} \lambda_1(\{s \in [0, t] : 0 \leq b(s) < \epsilon\}) \\ &=: (I) + (II) + (III) \end{aligned}$$

where (??) follows from (??), (??) follows from definition of f_ϵ for each $\epsilon > 0$, and λ_1 denotes 1-dimensional Lebesgue measure as hinted. With the given fact that (I) is a martingale, we show that

$$\int_{\{s \in [0, 1] : 0 \leq b(s) < \epsilon\}} \frac{b(s)}{\epsilon} db(s) \text{ converges to } 0 \text{ as } \epsilon \rightarrow 0 \text{ almost surely}$$

dd

Question 1-4.

4. SOME PICTURES. Make me some diagrams or so to say *caricatures* like

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showing in a single picture $\mathbf{b}(t)$, $\max(\mathbf{b}(t') : t' \leq t)$, and the passage times $T_x = \min(t : \mathbf{b}(t) = x)$ together, and in a second picture, $\mathbf{b}(t)$, $\mathbf{t}(t)$, and the inverse local time $\mathbf{t}^{-1}(x) = \min(t : \mathbf{t}(t) = x)$. You will see a similarity between the passage times and the inverse local time and likewise between $\mathbf{t}(t)$ and $\max(\mathbf{b}(t') : t' \leq t)$. Tell me what you see. More coming at once.

Solution.

Question 1-5.

5. A SURPRISE. You have seen how the distribution of the passage times is determined by stopping the fair game

$$e^{\alpha \mathbf{b}(t) - \alpha^2 t/2} \quad \text{as in}$$

$$1 = E_0 \left[e^{\alpha \mathbf{b}(T_x) - \alpha^2 T_x/2} \right] = e^{\alpha x} E_0 \left(e^{-\alpha^2 T_x/2} \right).$$

The same idea can be applied to the inverse local time. You are to show that $\mathbf{t}^{-1}(x)$ is a stopping time, that $\mathbf{b} \circ \mathbf{t}^{-1}(x) = x$ and to conclude that the two processes $(T_x : x \geq 0)$ and $(\mathbf{t}^{-1}(x) : x \geq 0)$ are not the *same* but *identical in law*.

HINT: Both processes are *additive* in the sense that $\mathbf{x}(a+b) = \mathbf{x}(a)$ plus an independent copy of $\mathbf{x}(b)$ as you will need to check for the inverse local time. It follows that the local time itself is a copy of the inverse function of the passage times, namely $\max(\mathbf{b}(t') : t' \leq t)$. Indeed, wonders never cease! BM(1) has lots of such complicated internal imitations of one thing with another, the copies $c\mathbf{b}(t/c^2)$ and $t\mathbf{b}(1/t)$ of $\mathbf{b}(t)$ being the simplest of these. I tell you another. Take $\mathbf{b}(0) = x \geq 0$, run it to the passage time T_0 , then start it over from $x = 0$ in the form $\max(\mathbf{b}(t') : t' \leq t) - \mathbf{b}(t)$.

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What is that? Draw a picture and you may guess, what P. Levy discovered, that it's a copy of the reflecting Brownian motion! Useless but remarkable.

Solution.

Question 1-6.

6. AN OLD FRIEND EXPLAINED. Write $h(t, x)$ for $e^{\alpha x - \alpha^2 t/2}$ and define

$$\begin{aligned} P^h[x(t_2) = y | x(t_1) = x] \\ = \frac{1}{h(t_1, x)} \frac{e^{-(x-y)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} h(t_2, y). \end{aligned}$$

Check that this is a perfectly good Markovian transition density, compute its associated infinitesimal operator, and tell me what the underlying process is. Note that h solves the backward heat equation $\partial h / \partial t + \frac{1}{2} \partial^2 h / \partial x^2$, so it's something very special since heat does not like to flow backwards. More can be said but let's leave it here.

Solution.