Diff Geo II: Problem Set II

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Abstract

This work contains solutions for the problem set II.

Question 1-1.

1. In class, we used the fact that the curvature tensor can be expressed in terms of parallel transport. That is, we claimed that if $q \in M$ and if $\alpha \colon \mathbb{R}^2 \to M$ is a smooth map such that $\frac{\partial \alpha}{\partial x} = X$, $\frac{\partial \alpha}{\partial y} = Y$, then for all $Z \in T_q M$, we have

$$R(X,Y)Z = \lim_{s \to 0} \frac{Z - p_{\gamma_s}(Z)}{s^2},$$

   where $\gamma_s \colon [0,4] \to M$ is the image under α of the boundary of an $s \times s$ square, i.e.,

$$\gamma_s(t) = \begin{cases} \alpha(st,0) & t \in [0,1] \\ \alpha(s,s(t-1)) & t \in [1,2] \\ \alpha(s(3-t),s) & t \in [2,3] \\ \alpha(0,s(4-t)) & t \in [3,4]. \end{cases}$$

   Prove this fact.

   (Hint: Construct a frame of vector fields $V_1, \ldots, V_m \in \mathbf{V}(\alpha)$ such that $\nabla_X V_i = 0$ and $\nabla_Y V_i(u_1, 0) = 0$. Any vector field W along γ_s can be expressed as a linear combination of the V_i — when is W parallel?)

Solution.

Let $s > 0$. Set

$$V_i(\alpha(s_1, s_2)) = P_{\alpha(0,s_2),\alpha(s_1,s_2)} \circ P_{q,\alpha(0,s_2)}(V_i(q))$$

for all $1 \le i \le m$ and $(s_1, s_2) \in [0,s] \times [0,s]$. As $\dfrac{\partial \alpha}{\partial x} = X$, $\dfrac{\partial \alpha}{\partial y} = Y$, we have

$$\nabla_X V_i = 0 \quad \text{and} \quad \nabla_Y V_i(u_1, 0) = 0$$

for all i. Then, we see that any $W \in V(\gamma)$ can only change along 2nd part of the path. We compute

$$
\begin{aligned}
V_i(q)(P_\gamma W(q) - W(q)) &= <V_i, W>\gamma_s(2) + <V_i, W>\gamma_s(1) \\
&= \int_0^s \partial_t <V_i, W> dt = \int_0^s V_i \cdot \nabla_Y W(1,t) dt \\
&= \int_0^s V_i \cdot \nabla_Y W(1,t) dt - \int_0^s \partial_r (V_i \cdot \nabla_Y W(r,t)) dr dt \\
&= -\int_0^s \int_0^s V_i \cdot \nabla_X \nabla_Y W(r,t) dr dt
\end{aligned}
$$

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for all i. Now, with $W(0,0) = Z$, and $[X, Y] = 0$, letting $s \downarrow 0$, gives

$$\lim_{s \downarrow 0} \frac{< Z - P_\gamma(Z), V_i >}{s^2} = < R(X, Y)Z, V_i >$$

for all i, and hence,

$$R(X, Y)Z = \lim_{s \downarrow 0} \frac{Z - P_\gamma(Z)}{s^2}.$$

$\square$

Question 1-2.

2. Suppose that M is a two-dimensional Riemannian manifold with curvature tensor R.

    (a) Use the symmetries of the curvature tensor to show that if $p \in M$, $V, W \in T_pM$, then

$$K = \frac{\langle R(V,W)W, V \rangle}{\|V\|^2\|W\|^2 - 2\langle V,W \rangle^2}$$

    is independent of V and W.

    (b) Prove that if K is as above, then

$$R(X,Y)Z = K(\langle Y,Z \rangle X - \langle X,Z \rangle Y)$$

    for all $X, Y, Z \in T_pM$.

Solution.

I believe there is a typo in the problem: 2 in the denominator should be gone.

(a) Let E_1, E_2 be orthonormal frame, so $V = V^1E_1 + V^2E_2$ and $W = W^1E_1 + W^2E_2$. Then,

$$
\begin{aligned}
< R(V,W)W, V > \; &= \; < R(V^1E_1 + V^2E_2, W^1E_1 + W^2E_2)W, V > \\
&= \; < V^1W^2R(E_1, E_2)W, V > + < V^2W^1R(E_2, E_1)W, V > \quad\quad (1)\\
&= \; (V^1W^2 - V^2W^1)(W^1V^2 < R(E_1, E_2)E_1, E_2 > + W^2V^1 < R(E_1, E_2)E_2, E_1 >)\\
&= \; -(V^1W^2 - V^2W^1)^2 < R(E_1, E_2)E_1, E_2 >
\end{aligned}
$$

where **(??)**, in particular, holds by $R(E_1, E_1)W = R(E_2, E_2)W = 0$. Furthermore,

$$\|V\|^2\|W\|^2 - < V, W >^2 \; = \; (V^1W^2 - V^2W^1)^2$$

and hence

$$
\begin{aligned}
K \; &= \; \frac{-(V^1W^2 - V^2W^1)^2 < R(E_1, E_2)E_1, E_2 >}{(V^1W^2 - V^2W^1)^2} \\
&= \; < R(E_1, E_2)E_2, E_1 >
\end{aligned}
$$

Therefore, we see that K only depend on the chosen frame, so K is independent of V and W.

(b) We compute

$$
\begin{aligned}
R(X,Y)Z \; &= \; \sum_{i,j,k=1}^{2} X^iY^jZ^kR(E_i, E_j)E_k \\
&= \; (X^1Y^2Z^1 - X^2Y^1Z^1)R(E_1, E_2)E_1 + (X^1Y^2Z^2 - X^2Y^1Z^2)R(E_1, E_2)E_2
\end{aligned}
$$

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and

$$\begin{aligned}
K(<Y,Z>X- <X,Z>Y) &= <R(E_1,E_2)E_2,E_1>X(Y^1Z^1+Y^2Z^2) \\
&\quad - <R(E_1,E_2)E_2,E_1>Y(X^1Z^1+X^2Z^2) \\
&= <R(E_1,E_2)E_2,E_1>(X^1Y^2Z^2-X^2Z^2Y^1)E_1 \\
&\quad + <R(E_1,E_2)E_2,E_1>(Y^1Z^1X^2-X^1Y^2Z^1)E_2
\end{aligned}$$

Therefore,

$$\begin{aligned}
<R(X,Y)Z,E_1> &= <R(E_1,E_2)E_2,E_1>(X^1Y^2Z^2-X^2Y^1Z^2) \\
&= <K(<Y,Z>X- <X,Z>Y),E_1>
\end{aligned}$$

Similarly,

$$<R(X,Y)Z,E_2> = <K(<Y,Z>X- <X,Z>Y),E_2> .$$

As the equality holds for the basis elements, we have

$$R(X,Y)Z = K(<Y,Z>X- <X,Z>Y)$$

as required. $\square$

Question 1-3.

3. Find a sequence $\gamma_i \colon [0, 1] \to S^2$ of piecewise-smooth curves with the same endpoints such that the γ_i converge to a piecewise-smooth curve γ, but the parallel transport maps $P_{\gamma_i} \colon T_p M \to T_q M$ do not converge to the map P_γ.

Solution.

All triangles in the discussion are great circle triangles. From class, we saw that the parellel transport along a path of triangle, starting at a vertex and coming back to the chosen vertex, is given by a rotation of $\theta - \pi$ where θ is the sum of three angles of the triangle, formed by the path. To discuss pointwise convergence, we choose the geodesic distance on S^2. Let p be the north pole of S^2. Then, choose a triangle path, starting at p and coming back to p such that $\theta - \pi > 0$. Now, for each $i \in \mathbb{N}$, choose a triangle path, γ_i^*, whose angle sum is θ', yet $\theta - \pi = k\theta'$ for some $k \in \mathbb{N}$, and the path of the triangle is contained in $B(p, \dfrac{1}{i})$, and set $\gamma_i = (\gamma_i^*)^k$, which is the standard path composition of γ_i k times. Then, set γ be the trivial path from p to p itself. Then, we see that γ_i converges to γ pointwise, as for any $t \in [0, 1]$, for any $\epsilon > 0$, there exists i such that $d(\gamma_i(t), \gamma(t)) < \epsilon$, whenever $n \geq i$. Furthermore, P_{γ_i} is given by a rotation of $\theta - \pi > 0$, yet $P_\gamma = I$, so we see that P_{γ_i} does not converge to P_γ as a linear map. Hence, we have the desired construction. $\qquad \square$

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Question 1-4.

Solution.

Cutting the cone and unfolding it, and using elementary geometry tell us that, in order to have an isometry, we must satisfy $r \sin(\phi)\theta = r\theta'$, because the radius of the circle changes from $r \sin(\phi)$ to r, by unfolding. As a parellel transport along any curve in $\mathbb{R}^2$ is an identity, we have that a parellel transport on the image is given by $(\cos(\theta'), -r \sin(\theta); \sin(\theta'), -r \cos(\theta'))$.