

PDE II: Final

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Abstract

This work contains solutions for the final of the PDE II course at Courant.

Question 1-1.

Exercise 1 (The Allen-Cahn equation) We consider the equation

$$\begin{cases} \partial_t u - \Delta u + (u^2 - 1)u = 0 \\ u(t=0) = u_0 \end{cases} \quad (1)$$

where $(t, x) \in [0, \infty) \times \mathbb{T}^d$, and $u(t, x) \in \mathbb{R}$.

1. By a fixed point argument, prove local well-posedness in $H^s(\mathbb{T}^d)$, for $s > \frac{d}{2}$.
2. By a fixed point argument, prove local well-posedness in $L^\infty(\mathbb{T}^d)$.
3. Describe the solutions of the ODE $\dot{u} + (u^2 - 1)u = 0$, with data $u(t=0) = u_0 \in \mathbb{R}$, in particular indicating their limit as $t \rightarrow \infty$ depending on u_0 . We can think of the PDE (1) as a combination of the dynamics given by this ODE, with diffusion given by the Laplacian
4. Write down the energy for which (1) is a gradient flow.
5. Three stationary (and constant in space) solutions are given by $u = 0, \pm 1$. Linearize the PDE around these solutions. Which ones of these linearized problems are stable (converge to zero as $t \rightarrow \infty$), and which ones are not?
6. From now on, $d = 1$. Show that (1) is globally well-posed in $H^1(\mathbb{R}^d)$.
7. Show that the solution $u = 1$ is asymptotically stable in L^∞ , namely: there exists $\epsilon > 0$ such that, if $\|u_0(x) - 1\|_{L^\infty} < \epsilon$, then $\|u(t, x) - 1\|_{L^\infty} \rightarrow 0$ as $t \rightarrow \infty$.
8. Is $u = 0$ asymptotically stable?

9. Prove that

$$\int_0^\infty \int_{\mathbb{T}^d} |\partial_t u|^2 dx dt + \sup_{t>0} \int_{\mathbb{T}^d} |\partial_x u|^2 dx < \infty.$$

10. (Harder question!) Classify the possible asymptotic behaviors of u as $t \rightarrow \infty$.

Solution.

(1) We will roughly follow the strategy shown in class with the cubic NLH problem. By definition, our solution of the equation will be functions in $C([0, T], H^s)$ such that

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (u - u^3) ds$$

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for some $T > 0$. This in turn can be rewritten as a fixed point problem:

$$\begin{aligned} u &= Ru \text{ with } R : C([0, T], H^s) \rightarrow C([0, T], H^s), \quad v \mapsto u \\ \text{and} \quad u &= e^{t\Delta} u_0 + \int_0^t e^{(t-l)\Delta} (v(l) - v(l)^3) dl. \end{aligned} \quad (1)$$

for some $T > 0$. Set $\rho = \|u_0\|_{H^s}$. We claim that there exists T, K suitably chosen such that R is a contraction in $B_{C([0, T], H^s)}(0, K\rho)$. We compute, for any $u \in B_{C([0, T], H^s)}(0, K\rho)$,

$$\begin{aligned} \|R(u)\|_{C([0, T], H^s)} &\leq \|u_0\|_{H^s} + \sup_{0 < t < T} \int_0^t \|e^{(t-l)\Delta} (u - u^3)(l)\|_{H^s} dl \\ &\leq \rho + CT \sup_{0 < t < T} (\|u\|_{H^s} + \|u\|_{H^s}^3) \end{aligned} \quad (2)$$

$$\leq \rho + CT(K\rho + K^3\rho^3) \quad (3)$$

where C is an universal constant independent of u , whose existence is justified along with (2) from the fact that $e^{t\Delta}$ is bounded on H^s for each $t > 0$ and H^s is an algebra for each $s > \frac{d}{2}$, and (3) follows from $u \in B_{C([0, T], H^s)}(0, K\rho)$. Hence, if

$$CT(K\rho + K^3\rho^3) < (K - 1)\rho \iff T \leq \frac{K - 1}{C(K + K^3\rho^2)}$$

then, R stabilizes $B_{C([0, T], H^s)}(0, K\rho)$. We now investigate the condition on K and T , which will imply that R is a contraction in $B_{C([0, T], H^s)}(0, K\rho)$ i.e. for any $u, v \in B_{C([0, T], H^s)}(0, K\rho)$,

$$\|Ru - Rv\|_{C([0, T], H^s)} < \frac{1}{2} \|u - v\|_{C([0, T], H^s)}.$$

We compute, for any $u, v \in B_{C([0, T], H^s)}(0, K\rho)$, with the same reasoning as above,

$$\begin{aligned} \sup_{0 < t < T} \|R(u)(t) - R(v)(t)\|_{H^s} &\leq \sup_{0 < t < T} \int_0^t \|e^{(t-l)\Delta} (u(l) - u^3(l) - v(l) + v^3(l))\|_{H^s} dl \\ &= \sup_{0 < t < T} \int_0^t \|e^{(t-l)\Delta} (v(l) - u(l))(v^2(l) + v(l)u(l) + u^2(l) - 1)\|_{H^s} dl \\ &\leq C \sup_{0 < t < T} \int_0^t \|u(l) - v(l)\| (\|u(l)\| + \|v(l)\|)^2 + 1) dl \\ &\leq C(K^2\rho^2 + 1)T \|u - v\|_{C([0, T], H^s)}. \end{aligned}$$

Hence,

$$T \leq \frac{1}{2C(K^2\rho^2 + 1)} \implies R \text{ is a contraction on } B_{C([0, T], H^s)}(0, K\rho).$$

Therefore, choose any $K > 1$, and then choose

$$T = \min\left\{\frac{K - 1}{C(K + K^3\rho^2)}, \frac{1}{2C(K^2\rho^2 + 1)}\right\}.$$

Now, with respect to these choices, by a general fixed point theorem, we have proved existence and uniqueness in $B_{C([0,T],H^s)}(0, K\rho)$. Continuous dependence on the data and uniqueness on the whole space $C([0,T],H^s)$ can be argued in the same way as the lecture note 7. So we are done. \square

(2) As L^∞ is an algebra, the technique used above will work if we show that for each $t > 0$

$$e^{t\Delta} \text{ is bounded on } L^\infty.$$

(3) We solve the given ODE. Observe that

$$\frac{1}{u - u^3} = \frac{1}{u} + \frac{u}{1 - u^2}.$$

Hence,

$$du =$$

so

$$u(t) = \ln |t| - \frac{1}{2} \ln |(1-t)(1+t)| + u_0$$

(4) We derive the energy formally, as shown in class. Multiplying by u_t on both sides of the equation, and integrating by parts,

$$\int u_t \partial_t u - u_t \Delta u + u_t u^3 - u_t u = 0$$

so

$$\int |\partial_t u|^2 dx = -\frac{d}{dt} \left(\frac{1}{4} \int u^4 dx - \frac{1}{2} \int u^2 dx + \frac{1}{2} |\Delta u|^2 dx \right).$$

Therefore,

$$E(u) = \frac{1}{4} \int u^4 dx - \frac{1}{2} \int u^2 dx + \frac{1}{2} |\Delta u|^2 dx,$$

as required. \square

(5) Let $u = 0$. Ignoring the non-linear term in $u - u^3$, we get the linearized pde as

$$\partial_t - \Delta u = u.$$

Let $u = 1$. Set $v = u - 1$, so

$$\begin{aligned}\partial_t v - \Delta v &= \partial_t u - \Delta u = (v+1)^2 - (v+1)^3 \\ &= -v^3 - 3v^2 - 2v.\end{aligned}$$

Ignoring the nonlinear term, we get the linearized pde as

$$\partial_t v - \Delta v = -2v.$$

Similarly, for $u = -1$, we get the linearized pde as

$$\partial_t v - \Delta v = -2v.$$

For the $u = 0$ case, set $u = e^t v$ to get

$$u(x, t) = e^t \int \Phi(x - y, t) u_0(y) dy$$

where Φ is the heat kernel. Then, for any $\delta > 0$, and $\|u_0\|_{H^s} < \delta$ such that $u \neq 0$, $u(x, t)$ blows up by the fact that e^t term dominates. For the other two cases, we have e^{-2t} factor instead of e^t , so we see that the linearized PDE is unstable for $u = 0$ and stable for $u = \pm 1$. \square

(6) As $s = 1 > \frac{1}{2} = \frac{d}{2}$, by 1, we have local well-posedness of the equation in $H^1(\mathbb{T})$. Let T^* be the maximal time of existence T^* . Now, observe that by Sobolev embedding, we have

$$u_0 \in H^1(\mathbb{T}) \subset C^{\frac{1}{2}}(\mathbb{T}) \subset C(\mathbb{T}) \subset L^\infty(\mathbb{T}) \subset \bigcap_{1 \leq p} L^p(\mathbb{T}).$$

Therefore, $E(u_0) < \infty$, and since the energy is decreasing

$$\begin{aligned}\sup_{0 < t < T^*} \|u(t)\|_{H^1} &\leq C \sup_{0 < t < T^*} (\|u(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}) \\ &\leq \\ &\leq C \sup_{0 < t < T^*} E(u(t)) \leq CE(u_0) \leq\end{aligned}$$

Hence u is globally bounded in H^1 , so

$$\limsup_{t \rightarrow T^*} \|u(t)\|_{H^1} < \infty$$

and by the lemma in page 2 of lecture note 8, $T^* = \infty$.

Question 1-2.

Exercise 2(A hypoelliptic operator) We want to study the differential operator L on \mathbb{R}^2 defined by

$$[Lu](x, y) = -\partial_x^2 u(x, y) - x\partial_y u(x, y)$$

and its parabolic flow.

We consider first the equation

$$f = Lu$$

and aim at understanding the smoothing properties of L .

1. By an energy estimate, show that $\|\partial_x u\|_{L^2}^2 \leq \|f\|_{L^2} \|u\|_{L^2}$.

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2. By explicitly solving the differential equation $v'(s) + s^2 v(s) = g(s)$, for $v \in C_0^\infty$, show that $\|v\|_{L^2} \leq C\|g\|_{L^2}$.
3. Deduce by scaling that $\forall \lambda \in \mathbb{R}$, $\|v' + \lambda s^2 v\| \geq C|\lambda|^{1/3} \|v\|_{L^2}$
4. (Harder question!) By using the previous question, switching to Fourier space, and denoting ξ and η for the dual variables of x and y respectively, show that

$$\int (1 + |\xi|^2 + |\eta|^{4/3}) |\hat{u}(\xi, \eta)|^2 d\eta d\xi \leq C(\|f\|_{L^2}^2 + \|u\|_{L^2}^2).$$

5. Deduce that any solution in $u \in H^2$ of $Lu = u^3$ is actually C^∞ .

We turn to the associated parabolic flow

$$\begin{cases} \partial_t u + Lu = 0 \\ u(t=0) = u_0 \end{cases}$$

6. Setting $v(t, x, y) = u(t, x, y - xt)$, show that it solves $\partial_t v = (\partial_x + t\partial_y)^2 v$.
7. Arguing in Fourier space, deduce from the previous question that

$$v(t, x, y) = \Phi(t, x, y) * u_0(x, y) \quad \text{where} \quad \Phi(t, x, y) = \frac{C_0}{t^2} e^{-\frac{x^2}{t} + \frac{3xy}{t^2} - \frac{3y^2}{t^3}}.$$

8. Show that the parabolic flow preserves positivity: $u \geq 0$ if $u_0 \geq 0$.
9. Prove decay estimates of the type

$$\|u(t)\|_{L^p} \leq \frac{C}{t^\alpha} \|u_0\|_{L^q},$$

for appropriately chosen p, q, α .

10. Does this equation enjoy smoothing estimates?

Solution.

(1) We proceed by our usual formal derivation of the energy estimate. Multiplying both sides of the equation by u , integrating over x, y , and integrating by parts,

$$\begin{aligned}\int \int u f dx dy &= - \int \int u u_{xx} dx dy - \int \int u x u_y dy dx \\ &= \int \int (u_x)^2 dx dy - \int x \left(\frac{1}{2} u^2 \right) \Big|_{-\infty}^{\infty} dx \\ &= \int \int (u_x)^2 dx dy\end{aligned}$$

and hence, by Holder's inequality,

$$\|u_x\|_{L^2}^2 = \left| \int \int u f dx dy \right| \leq \int \int |u f| dx dy \leq \|f\|_{L^2} \|u\|_{L^2},$$

as required. \square

(2) Firstly, observe that as v vanishes at ∞ , if g solves the ODE then, g has compact support. Now, we solve the ODE. Multiplying both sides of the equation by $e^{\frac{1}{3}s^3}$ (integrating factor),

$$e^{\frac{1}{3}s^3} dv + e^{\frac{1}{3}s^3} s^2 v ds = e^{\frac{1}{3}s^3} g(s) ds$$

and hence

$$e^{\frac{1}{3}s^3} g(s) ds = d(e^{\frac{1}{3}s^3} v).$$

Therefore,

$$v(s) = e^{-\frac{1}{3}s^3} \int_0^s g(t) e^{\frac{1}{3}t^3} dt.$$

solves the ODE. Now, we compute

$$\begin{aligned}\|v\|_{L^2}^2 &= \int (e^{-\frac{1}{3}s^3} \int_0^s g(t) e^{\frac{1}{3}t^3} dt)^2 ds \leq \int (e^{-\frac{1}{3}s^3} \int_0^s g(t) e^{\frac{1}{3}t^3} dt)^2 ds \\ &= \int \left(\int_0^s g(t) dt \right)^2 ds \leq \int \left(\int_{\mathbb{R}} |g(t)| dt \right)^2 ds \\ &\leq \int \int_{\mathbb{R}} |g(t)|^2 dt ds \leq m(\text{supp}(v)) \|g\|_{L^2}^2\end{aligned}$$

where the second last inequality holds by Jensen's inequality on finite measure space, as g has compact support, and $m(\text{supp}(v)) < \infty$ by assumption. \square

(3)

(4) From the energy estimate (1), and taking the Fourier transform of $\partial_x u$,

$$\int \int |\xi|^2 |\hat{u}(\xi, \eta)|^2 d\xi d\eta = \|\partial_x\|_{L^2}^2 \leq \|\hat{f}\|_{L^2} \|\hat{u}\|_{L^2} \leq \frac{1}{2} (\|\hat{f}\|_{L^2}^2 + \|\hat{u}\|_{L^2}^2)$$

and hence, for some constant C ,

$$\int \int (1 + |\xi|^2) |\hat{u}|^2 \leq C (\|\hat{f}\|_{L^2} + \|\hat{u}\|_{L^2}).$$

Therefore, it suffices to show that, for some constant $C > 0$,

$$\int \int |\eta|^{\frac{4}{3}} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \leq C \|\hat{f}\|_{L^2}^2 \quad (4)$$

Taking the Fourier transform of the equation,

$$|\xi|^2 \hat{u} + \eta \partial_\xi \hat{u} = \hat{f}$$

and dividing both sides of the above equation by η ,

$$\frac{1}{\eta} |\xi|^2 \hat{u} + \partial_\xi \hat{u} = \frac{1}{\eta} \hat{f}.$$

Now, by the scaling estimate from (3), for some constant $C_1 > 0$,

$$\frac{1}{\eta} \|\hat{f}(\xi, \eta)\|_{L_\xi^2} \geq C_1 |\eta|^{-\frac{1}{3}} \|\hat{u}(\xi, \eta)\|_{L_\xi^2}$$

and hence, rearranging and squaring both sides, for some constant $C_2 > 0$.

$$C_2 \int |\hat{f}(\xi, \eta)|^2 d\xi \geq |\eta|^{\frac{4}{3}} \int |\hat{u}(\xi, \eta)|^2 d\xi.$$

Integrating the above equation with respect to η gives (4), so we are done. \square

(5)

(6) We compute

$$\begin{aligned} v_t &= u_t - x u_y \\ v_x &= u_x - t u_y \\ v_{xx} &= u_{xx} - 2t u_{yx} + t^2 u_{yy} \end{aligned}$$

(8) Suppose $u_0 \geq 0$. Note that Φ is non-negative everywhere. Hence, by 6 and 7, for each t, x, y ,

$$u(t, x, y) = v(t, x, y + xt) = \Phi(t, x, y + xt) * u_0(x, y + xt) \geq 0,$$

as required. \square