Limit Theorems II: Final

Youngduck Choi *

Abstract

This work contains solutions for the final of the Limit Theorems II by Professor McKean.

Question 1-1.

1. $Rw(3)$: CAPACITY. Compute the capacity of a pair of lattice points $a \neq b$ and also of a singleton c in terms of the Green's function $G(a,b) = \sum_0^\infty P_a[\mathbf{X}(n) = b]$. This already shows that capacity is *not* additive. Yes?

Solution.

*Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

Question 1-2.

2. $Bm(1)$ CONDITIONAL ARCSINE. Recall the usual arcsine law

$$P_0\left[\text{measure}\,(t \le 1 : \mathrm{b}(t) \ge 0) \le x\right] = \frac{1}{\pi}\int_0^x \frac{dy}{\sqrt{y(1-y)}} \quad (0 \le x \le 1)$$

showing that the bulk of the density lies (surprisingly) near $y = 0$ and $y = 1$. It comes as a further surprise that for the tied Brownian motion, conditioned to end at $\mathrm{b}(1) = 0$, that the density is *perfectly flat* as in

$$P_0\left[\text{measure}\,(t \le 1 : \mathrm{b}(t) \ge 0) \le x\right] = x \; (0 \le x \le 1).$$

I will lead you through it via Feynman-Kac.

Write $\mathbf{m} =$ measure $(t' \le t : \mathrm{b}(t') \ge 0)$ and recall that

$$\int_0^\infty e^{-\alpha t} \cdot E_0\left[e^{-\beta \mathbf{m}}\, f \circ \mathrm{b}(t)\right] dt$$

may be expressed as

$$h_+(0)\int_{-\infty}^0 h_- f\,dy + h_-(0)\int_0^\infty h_+ f\,dy \;\text{ over }\; \frac{1}{2}\left(h_-' h_+ - h_- h_+'\right) \;\text{ at } x = 0$$

with $h_-(x) = e^{\sqrt{2\alpha}\,x}$ for $x \le 0$ and $h_+(x) = e^{-\sqrt{2(\alpha+\beta)}}$ for $x \ge 0$ for reasonable functions f. That is now used for the unreasonable function $f =$ Dirac's delta function $\delta(x)$ to produce

$$\int_0^\infty e^{-\alpha t}\, E_0\left[e^{-\beta \mathbf{m}},\, \mathrm{b}(t) = 0\right] dt = \frac{2}{\sqrt{2\alpha} + \sqrt{2(\alpha+\beta)}}$$

as you will check. This may also be written

$$\int_0^\infty e^{-\alpha t}\, E_0\left[e^{-\beta \mathbf{m}}\,|\mathrm{b}(t) = 0\right] \frac{dt}{\sqrt{2\pi t}}$$

and if you knew, what is the fact, that $\mathbf{m}$ is now uniformly distributed over the interval $0 \le m \le t$, then you could put this into the last display to see if it checks. Do it. It

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works. Is that a proof? Could the conditional law of $\mathbf{m}$ be anything else? Of course this is only a consistency check, based so to speak on proprietary information. Insider trading if you will. The honest way would be to invert $\frac{2}{\sqrt{2\alpha}+\sqrt{2(\alpha+\beta)}}$. Not a standard exercise. Try it if you like.

Solution.

Question 1-3.

3. $BM(1)$: LOCAL TIME. Begin with the function $f''(x) = \frac{1}{\epsilon}$ for $0 \le x \le \epsilon$, vanishing elsewhere, and compute f' and f subject to $f'(0) = f(0) = 0$. Apply Itô's lemma for small ϵ to obtain

$$
\begin{aligned}
\max[0, \mathbf{b}(t)] &\simeq f \circ \mathbf{b}(t) \\
&= \int_{(0,t)\cap(0\le \mathbf{b}(t')<\epsilon)} \frac{b}{\epsilon}\, db \\
&\quad + \int_{(0,t)\cap(\mathbf{b}(t')>\epsilon)} db \\
&\quad + \frac{1}{2\epsilon} \text{ measure } (t' \le t : 0 \le \mathbf{b}(t') < \epsilon) \\
&= ① + ② + ③.
\end{aligned}
$$

Here ① is a fair game. Show that it vanishes, uniformly for $t \le 1$ say, if $\epsilon \downarrow 0$ fast enough, and likewise that ② tends to $\int_0^t e\, d\mathbf{b}$ with $e(t, \mathbf{b}) =$ the indicator of $\mathbf{b}(t) > 0$, and the same uniformity. Then ③ converges in the same style and you have

$$\max[0, \mathbf{b}(t)] = \int_0^t e\, d\mathbf{b} \text{ plus } \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \text{ measure } (t' \le t : 0 \le \mathbf{b}(t') < \epsilon) = ④$$

with the same uniformity in ④, this being P. Levy's *mesure de voisinage*, more aptly called in English *local time*, written $\mathfrak{t}(t)$. Think a little how extraordinary it is that this limit should exist, due solely to the wildness of the Brownian path. Evidently $\mathfrak{t}(t)$ is continuous and increasing but only on $Z = (t \ge 0 : \mathbf{b}(t) = 0)$ which is of Lebesgue measure zero(!) and cannot vanish for any $t > 0$. Give me a little argument for *that*.

FYI. The fact is that it does not matter how fast $\epsilon \downarrow 0$. This lies deeper. See Itô-McKean if you're interested.

Solution.
Recall the Ito's formula for single martingales: If f is C^2 and for all $0 \le t$,

$$E\int_0^t [f'(M(s))]^2 dA(s) < \infty \qquad \text{and} \qquad E\int_0^t |f''(M(s))| dA(s) < \infty$$

then, for all $0 \le t$,

$$f(M(t)) - f(M(0)) \;=\; \int_0^t f'(M(s))dM(s) + \frac{1}{2}\int_0^t f'(M(s))dA(s) \tag{1}$$

where $A(t)$ is the unique increasing, continuous process, corresponding to $M(t)$. Of course, for the problem at hand, we have $M(t) = B(t)$ and $A(t) = t$, so $db(s)$ and ds will be used to denote the integrators. Now, to compute local time, we cannot naively apply this formula, as $\max(0,\cdot)$ is not $C^2(\mathbb{R})$. Hence, we approximate with the suggested strategy. Let f_ϵ be defined as given for each $\epsilon > 0$. Then, for any $0 \leq t$, and $\epsilon > 0$,

$$\begin{aligned}
f_\epsilon \circ b(t) &= f_\epsilon \circ b(t) - f_\epsilon \circ b(0) = \int_0^t f_\epsilon^{'}(B(s))db(s) + \frac{1}{2}\int_0^t f_\epsilon^{''}(B(s))ds && (2)\\
&= \int_{\{s\in[0,t]:0\leq b(s)<\epsilon\}} \frac{b(s)}{\epsilon}db(s) + \int_{\{s\in[0,t]:b(s)>\epsilon\}} 1db(s) + \frac{1}{2}\int_{\{s\in[0,t]:0\leq b(s)<\epsilon\}} \frac{1}{\epsilon}ds && (3)\\
&= \int_{\{s\in[0,t]:0\leq b(s)<\epsilon\}} \frac{b(s)}{\epsilon}db(s) + \int_{\{s\in[0,t]:b(s)>\epsilon\}} 1db(s) + \frac{1}{2\epsilon}\lambda_1(\{s\in[0,t]:0\leq b(s)<\epsilon\})\\
&=: (I) + (II) + (III)
\end{aligned}$$

where (**??**) follows from (**??**), (**??**) follows from definition of f_ϵ for each $\epsilon > 0$, and λ_1 denotes 1-dimensional Lebesgue measure as hinted. With the given fact that (I) is a martingale, we claim that, for any $0 \leq t \leq 1$ uniformly,

$$\int_{\{s\in[0,t]:0\leq b(s)<\epsilon\}} \frac{b(s)}{\epsilon}db(s) \text{ converges to } 0 \tag{4}$$

and

$$\int_{\{s\in[0,t]:b(s)>\epsilon\}} 1db(s) \text{ converges to } \int_0^t e(s)db(s) \tag{5}$$

as $\epsilon \downarrow 0$ $\mathbb{P}$ almost surely, so that

$$\max(0, b(t)) = \int_0^t e(s)db(s) + \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\lambda_1(\{s\in[0,t]:0\leq b(s)<\epsilon\}) \;\; \mathbf{P} \text{ almost surely}$$

where the last limit exists, since for any $\mathbb{P}$-a.s. $\omega \in \Omega$, $0 \leq t \leq 1$,

$$\lambda_1(\{s\in[0,t]:0\leq b(s)(w)<\epsilon\}).$$

Then, for each $0 \leq t \leq 1$, for some universal constant C, independent of t,

$$\begin{aligned}
E\left[\int_0^t \frac{b(s)}{\epsilon}1_{\{0\leq b(s)<\epsilon\}} - 0db(s)\right]^2 &= E\int_0^t \left[\frac{b(s)}{\epsilon}1_{\{0\leq b(s)<\epsilon\}}\right]^2 ds && (6)\\
&\leq E\int_0^t 1_{\{0\leq b(s)<\epsilon\}}ds\\
&= \int_0^t P(0\leq b(s)<\epsilon)ds\\
&= \int_0^t P(0\leq b(1)<\frac{\epsilon}{\sqrt{s}})ds && (7)\\
&\leq \int_0^t P(|b(1)|<\frac{\epsilon}{\sqrt{s}})ds \leq C\sqrt{t}\epsilon, && (8)
\end{aligned}$$

where (**??**) holds by the Ito L^2 isometry, (**??**) follows by the scaling of BW(1), and (**??**) follows from the fact that

$$\frac{P(|b(1) \leq x)}{x} \text{ is uniformly bounded } \quad \text{for all } \; x > 0.$$

Hence, if $\epsilon = \Omega(n^2)$, then, by Doob's inequality,

$$(I) \to 0 \;\text{ uniformly for } \quad 0 \leq t \leq 1 \;\; P \;\text{ almost surely}$$

showing (**??**).

Now, from uniformity of convergence and continuity of Lebesgue measure, we see that $\mathbf{t}(t)$ is continuous, and clearly increasing as (II) and (III) are increasing in t almost surely.

Question 1-4.

4. Some Pictures. Make me some diagrams or so to say *caricatures* like

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showing in a single picture $\mathbf{b}(t)$, $\max(\mathbf{b}(t') : t' \leq t)$, and the passage times $T_x = \min(t : \mathbf{b}(t) = x)$ together, and in a second picture, $\mathbf{b}(t)$, $\mathbf{t}(t)$, and the inverse local time $\mathbf{t}^{-1}(x) = \min(t : \mathbf{t}(t) = x)$. You will see a similarity between the passage times and the inverse local time and likewise between $\mathbf{t}(t)$ and $\max(\mathbf{b}(t') : t' \leq t)$. Tell me what you see. More coming at once.

Solution.

Question 1-5.

5. A SURPRISE. You have seen how the distribution of the passage times is determined by stopping the fair game

$$e^{\alpha\, \mathrm{b}(t)-\alpha^2 t/2} \quad \text{as in}$$

$$1 = E_0\left[e^{\alpha \mathrm{b}(T_x)-\alpha^2 T_x/2}\right] = e^{\alpha x}\, E_0\left(e^{-\alpha^2 T_x/2}\right).$$

The same idea can be applied to the inverse local time. You are to show that $\mathbf{t}^{-1}(x)$ is a stopping time, that $\mathbf{b}\circ\mathbf{t}^{-1}(x) = x$ and to conclude that the two processes $(T_x : x \geq 0)$ and $(\mathbf{t}^{-1}(x) : x \geq 0)$ are not the *same* but *identical in law.*

HINT: Both processes are *additive* in the sense that $\mathbf{x}(a+b) = \mathbf{x}(a)$ plus an independent copy of $\mathbf{x}(b)$ as you will need to check for the inverse local time. It follows that the local time itself is a copy of the inverse function of the passage times, namely $\max(\mathrm{b}(t') : t' \leq t)$. Indeed, wonders never cease! BM(1) has lots of such complicated internal imitations of one thing with another, the copies $c\,\mathrm{b}(t/c^2)$ and $t\mathrm{b}(1/t)$ of $\mathrm{b}(t)$ being the simplest of these. I tell you another. Take $\mathrm{b}(0) = x \geq 0$, run it to the passage time T_0, then start it over from $x = 0$ in the form $\max(\mathrm{b}(t') : t' \leq t) - \mathrm{b}(t)$.

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What is that? Draw a picture and you may guess, what P. Levy discovered, that it's a copy of the reflecting Brownian motion! Useless but remarkable.

Solution.

Question 1-6.

6. AN OLD FRIEND EXPLAINED. Write $h(t,x)$ for $e^{\alpha x - \alpha^2 t/2}$ and define

$$P^h\left[\mathbf{x}(t_2) = y | \mathbf{x}(t_1) = x\right]$$
$$= \frac{1}{h(t_1,x)} \frac{e^{-(x-y)^2/2(t_2-t_1)}}{\sqrt{2\pi\,(t_2-t_1)}} h(t_2,y).$$

Check that this is a perfectly good Markovian transition density, compute its associated infinitessimal operator, and tell me what the underlying process is. Note that h solves the backward heat equation $\partial h/\partial t + \frac{1}{2}\,\partial^2 h/\partial x^2$, so it's something very special since heat does not like to flow backwards. More can be said but let's leave it here.

Solution.
We check that P^h is a Markovian transition density by checking that it integrates to 1 with respect to y. We compute, via completing the square,

$$\begin{aligned}
\int P^h(\mathbf{x}(t_2) = y|\mathbf{x}(t_1) = x)dy &= \int \frac{h(t_2,y)}{h(t_1,x)} \frac{e^{-(x-y)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}} dy \\
&= \frac{1}{h(t_1,x)} \frac{1}{\sqrt{2\pi(t_2-t_1)}} \int e^{-(y^2-2xy+x^2)/2(t_2-t_1)+\alpha y - \alpha^2 t_2/2} dy \\
&= \frac{1}{h(t_1,x)} e^{\alpha x - \frac{1}{2}\alpha^2 t_1} \int e^{\frac{-(y-(x+\alpha(t_2-t_1))^2}{2(t_2-t_1)}} dy \\
&= \frac{1}{h(t_1,x)} e^{\alpha x - \frac{1}{2}\alpha^2 t_1} = \frac{h(t_1,x)}{h(t_1,x)} = 1.
\end{aligned}$$

Now, we compute the associated infinitessimal operator. From the general theory, we know that the generator must be defined by

$$\mathscr{G}(t_1)f(x) = \lim_{t_2 \downarrow t_1} \frac{\int f(y) P^h(\mathbf{x}(t_2) = y|\mathbf{x}(t_1) = x)dy - f(x)}{t_2 - t_1} \tag{9}$$
$$\tag{10}$$

whenever the limit is well-defined for $f \in C_0(\mathbb{R})$.

By observing that h satisfies the backward heat equation, we further compute for each $0 \leq t$,

$$\begin{aligned}
\mathscr{G}(t)f &= \frac{1}{h}\partial_t h f + \frac{1}{2}\partial_x(f_x h + h_X f) \\
&= \frac{1}{h}\partial_t h f + \frac{1}{2h}(f_{xx}h + 2f_x h_x + h_{xx} f) \\
&= \frac{1}{2}
\end{aligned}$$