Diff Geo II: Problem Set II

Youngduck Choi *

Abstract

This work contains solutions for the problem set II.

Question 1-1.

1. In class, we used the fact that the curvature tensor can be expressed in terms of parallel transport. That is, we claimed that if $q \in M$ and if $\alpha \colon \mathbb{R}^2 \to M$ is a smooth map such that $\frac{\partial \alpha}{\partial x} = X$, $\frac{\partial \alpha}{\partial y} = Y$, then for all $Z \in T_qM$, we have

$$R(X,Y)Z = \lim_{s \to 0} \frac{Z - p_{\gamma_s}(Z)}{s^2},$$

where $\gamma_s : [0,4] \to M$ is the image under α of the boundary of an $s \times s$ square, i.e.,

$$\gamma_s(t) = \begin{cases} \alpha(st, 0) & t \in [0,1] \\ \alpha(s, s(t-1)) & t \in [1,2] \\ \alpha(s(3-t), s) & t \in [2,3] \\ \alpha(0, s(4-t)) & t \in [3,4]. \end{cases}$$

Prove this fact.

(Hint: Construct a frame of vector fields $V_1, \dots, V_m \in \mathbf{V}(\alpha)$ such that $\nabla_X V_i = 0$ and $\nabla_Y V_i(u_1, 0) = 0$. Any vector field W along γ_s can be expressed as a linear combination of the V_i — when is W parallel?)

Solution.
Let $s > 0$. Set

$$V_i(\alpha(s_1, s_2)) = P_{\alpha(0,s_2),\alpha(s_1,s_2)} \circ P_{q,\alpha(0,s_2)}(V_i(q))$$

for all $1 \le i \le m$ and $(s_1, s_2) \in [0,s] \times [0,s]$. As $\dfrac{\partial \alpha}{\partial x} = X$, $\dfrac{\partial \alpha}{\partial y} = Y$, we have

$$\nabla_X V_i = 0 \qquad \text{and} \qquad \nabla_Y V_i(u_1, 0) = 0$$

for all i. Then, we see that any $W \in V(\gamma)$ can only change along 2nd part of the path. We compute

$$\begin{aligned} V_i(q)(P_\gamma W(q) - W(q)) &= < V_i, W > \gamma_s(2) + < V_i, W > \gamma_s(1) \\ &= \int_0^s \partial_t < V_i, W > dt = \int_0^s V_i \cdot \nabla_Y W(1,t) dt \\ &= \int_0^s V_i \cdot \nabla_Y W(1,t) dt - \int_0^s \partial_r (V_i \cdot \nabla_Y W(r,t) dr dt \\ &= -\int_0^s \int_0^s V_i \cdot \nabla_X \nabla_Y W(r,t) dr dt \end{aligned}$$

*Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

for all i. Now, with $W(0,0)=Z$, and $[X,Y]=0$, letting $s \downarrow 0$, gives

$$\lim_{s\downarrow 0} \frac{< Z - P_\gamma(Z), V_i >}{s^2} =< R(X,Y)Z, V_i >$$

for all i, and hence,

$$R(X,Y)Z = \lim_{s\downarrow 0} \frac{Z - P_\gamma(Z)}{s^2}.$$

$\square$

Question 1-2.

2. Suppose that M is a two-dimensional Riemannian manifold with curvature tensor R.
   (a) Use the symmetries of the curvature tensor to show that if $p \in M$, $V, W \in T_pM$, then
   $$K = \frac{\langle R(V,W)W, V\rangle}{\|V\|^2\|W\|^2 - 2\langle V, W\rangle^2}$$
   is independent of V and W.
   (b) Prove that if K is as above, then
   $$R(X,Y)Z = K(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$$
   for all $X, Y, Z \in T_pM$.

Solution.
I believe there is a typo in the problem: 2 in the denominator should be gone.

(a) Let E_1, E_2 be orthonormal frame, so $V = V^1E_1 + V^2E_2$ and $W = W^1E_1 + W^2E_2$. Then,

$$\begin{aligned}
< R(V,W)W, V > &= < R(V^1E_1 + V^2E_2, W^1E_1 + W^2E_2)W, V > \\
&= < V^1W^2R(E_1,E_2)W, V > + < V^2W^1R(E_2,E_1)W, V > \qquad (1) \\
&= (V^1W^2 - V^2W^1)(W^1V^2 < R(E_1,E_2)E_1, E_2 > + W^2V^1 < R(E_1,E_2)E_2, E_1 >) \\
&= -(V^1W^2 - V^2W^1)^2 < R(E_1,E_2)E_1, E_2 >
\end{aligned}$$

where (1), in particular, holds by $R(E_1,E_1)W = R(E_2,E_2)W = 0$. Furthermore,

$$||V||^2||W||^2 - < V, W >^2 = (V^1W^2 - V^2W^1)^2$$

and hence

$$\begin{aligned}
K &= \frac{-(V^1W^2 - V^2W^1)^2 < R(E_1,E_2)E_1, E_2 >}{(V^1W^2 - V^2W^1)^2} \\
&= < R(E_1,E_2)E_2, E_1 >
\end{aligned}$$

Therefore, we see that K only depend on the chosen frame, so K is independent of V and W.

(b) We compute

$$\begin{aligned}
R(X,Y)Z &= \sum_{i,j,k=1}^{2} X^iY^jZ^kR(E_i,E_j)E_k \\
&= (X^1Y^2Z^1 - X^2Y^1Z^1)R(E_1,E_2)E_1 + (X^1Y^2Z^2 - X^2Y^1Z^2)R(E_1,E_2)E_2
\end{aligned}$$

and

$$
\begin{aligned}
K(<Y,Z>X-<X,Z>Y) &= <R(E_1,E_2)E_2,E_1>X(Y^1Z^1+Y^2Z^2) \\
&- <R(E_1,E_2)E_2,E_1>Y(X^1Z^1+X^2Z^2) \\
&= <R(E_1,E_2)E_2,E_1>(X^1Y^2Z^2-X^2Z^2Y^1)E_1 \\
&+ <R(E_1,E_2)E_2,E_1>(Y^1Z^1X^2-X^1Y^2Z^1)E_2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
<R(X,Y)Z,E_1> &= <R(E_1,E_2)E_2,E_1>(X^1Y^2Z^2-X^2Y^1Z^2) \\
&= <K(<Y,Z>X-<X,Z>Y),E_1>
\end{aligned}
$$

Similarly,

$$
<R(X,Y)Z,E_2> = <K(<Y,Z>X-<X,Z>Y),E_2>.
$$

As the equality holds for the basis elements, we have

$$
R(X,Y)Z = K(<Y,Z>X-<X,Z>Y)
$$

as required. $\square$

Question 1-3.

3. Find a sequence $\gamma_i \colon [0,1] \to S^2$ of piecewise-smooth curves with the same endpoints such that the γ_i converge to a piecewise-smooth curve γ, but the parallel transport maps $P_{\gamma_i} \colon T_pM \to T_qM$ do not converge to the map P_γ.

Solution.
All triangles in the discussion are great circle triangles. From class, we saw that the parellel transport along a path of triangle, starting at a vertex and coming back to the chosen vertex, is given by a rotation of $\theta - \pi$ where θ is the sum of three angles of the triangle, formed by the path. To discuss pointwise convergence, we choose the geodesic distance on S^2. Let p be the north pole of S^2. Then, choose a triangle path, starting at p and coming back to p such that $\theta - \pi > 0$. Now, for each $i \in \mathbb{N}$, choose a triangle path, γ_i^*, whose angle sum is θ', yet $\theta - \pi = k\theta'$ for some $k \in \mathbb{N}$, and the path of the triangle is contained in $B(p, \dfrac{1}{i})$, and set $\gamma_i = (\gamma_i^*)^k$, which is the standard path composition of γ_i k times. Then, set γ be the trivial path from p to p itself. Then, we see that γ_i converges to γ pointwise, as for any $t \in [0,1]$, for any $\epsilon > 0$, there exists i such that $d(\gamma_i(t), \gamma(t)) < \epsilon$, whenever $n \geq i$. Furthermore, P_{γ_i} is given by a rotation of $\theta - \pi > 0$, yet $P_\gamma = I$, so we see that P_{γ_i} does not converge to P_γ as a linear map. Hence, we have the desired construction. $\square$

Question 1-4.

Let $M \subset \mathbb{R}^3$ be the cone

$$M = \{(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) \mid r > 0, \theta \in [0, 2\pi)\}$$

as in the first problem set.

4. Construct an isometry from the complement of the line $\theta = 0$ to a subset of $\mathbb{R}^2$. Use this isometry to calculate the parallel transport map $P_{0,2\pi}$ from problem 5 of the first problem set.

Solution.
Cutting the cone and unfolding it, and using elementary geometry tell us that, in order to have an isometry, we must satisfy $r\sin(\phi)\theta = r\theta'$, because the radius of the circle changes from $r\sin(\phi)$ to r, by unfolding. As a parellel transport along any curve in $\mathbb{R}^2$ is an identity, we have that a parellel transport on the image is given by $(\cos(\theta'), -r\sin(\theta); \sin(\theta'), -r\cos(\theta'))$. I don't quite recover the parellel transport computed in the last homework, by naively composing the isometry map, with the tranport on the image yet. I will try more later!