

PDE II: Final

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Abstract

This work contains solutions for the final of the PDE II course at Courant.

Question 1-1.

Exercise 1 (The Allen-Cahn equation) We consider the equation

$$\begin{cases} \partial_t u - \Delta u + (u^2 - 1)u = 0 \\ u(t=0) = u_0 \end{cases} \quad (1)$$

where $(t, x) \in [0, \infty) \times \mathbb{T}^d$, and $u(t, x) \in \mathbb{R}$.

1. By a fixed point argument, prove local well-posedness in $H^s(\mathbb{T}^d)$, for $s > \frac{d}{2}$.
2. By a fixed point argument, prove local well-posedness in $L^\infty(\mathbb{T}^d)$.
3. Describe the solutions of the ODE $\dot{u} + (u^2 - 1)u = 0$, with data $u(t=0) = u_0 \in \mathbb{R}$, in particular indicating their limit as $t \rightarrow \infty$ depending on u_0 . We can think of the PDE (1) as a combination of the dynamics given by this ODE, with diffusion given by the Laplacian
4. Write down the energy for which (1) is a gradient flow.
5. Three stationary (and constant in space) solutions are given by $u = 0, \pm 1$. Linearize the PDE around these solutions. Which ones of these linearized problems are stable (converge to zero as $t \rightarrow \infty$), and which ones are not?
6. From now on, $d = 1$. Show that (1) is globally well-posed in $H^1(\mathbb{R}^d)$.
7. Show that the solution $u = 1$ is asymptotically stable in L^∞ , namely: there exists $\epsilon > 0$ such that, if $\|u_0(x) - 1\|_{L^\infty} < \epsilon$, then $\|u(t, x) - 1\|_{L^\infty} \rightarrow 0$ as $t \rightarrow \infty$.
8. Is $u = 0$ asymptotically stable?

9. Prove that

$$\int_0^\infty \int_{\mathbb{T}^d} |\partial_t u|^2 dx dt + \sup_{t>0} \int_{\mathbb{T}^d} |\partial_x u|^2 dx < \infty.$$

10. (Harder question!) Classify the possible asymptotic behaviors of u as $t \rightarrow \infty$.

Solution.

(1) We will roughly follow the strategy shown in class with the cubic NLH problem. By definition, our solution of the equation will be functions in $C([0, T], H^s)$ such that

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (u - u^3) ds$$

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for some $T > 0$. This in turn can be rewritten as a fixed point problem:

$$\begin{aligned} u &= Ru \text{ with } R : C([0, T], H^s) \rightarrow C([0, T], H^s), \quad v \mapsto u \\ \text{and} \quad u &= e^{t\Delta} u_0 + \int_0^t e^{(t-l)\Delta} (v(l) - v(l)^3) dl. \end{aligned} \quad (1)$$

for some $T > 0$. Set $\rho = \|u_0\|_{H^s}$. We claim that there exists T, K suitably chosen such that R is a contraction in $B_{C([0, T], H^s)}(0, K\rho)$. We compute, for any $u \in B_{C([0, T], H^s)}(0, K\rho)$,

$$\begin{aligned} \|R(u)\|_{C([0, T], H^s)} &\leq \|u_0\|_{H^s} + \sup_{0 < t < T} \int_0^t \|e^{(t-l)\Delta} (u - u^3)(l)\|_{H^s} dl \\ &\leq \rho + CT \sup_{0 < t < T} (\|u\|_{H^s} + \|u\|_{H^s}^3) \end{aligned} \quad (2)$$

$$\leq \rho + CT(K\rho + K^3\rho^3) \quad (3)$$

where C is an universal constant independent of u , whose existence is justified along with (2) from the fact that $e^{t\Delta}$ is bounded on H^s for each $t > 0$ and H^s is an algebra for each $s > \frac{d}{2}$, and (3) follows from $u \in B_{C([0, T], H^s)}(0, K\rho)$. Hence, if

$$CT(K\rho + K^3\rho^3) < (K - 1)\rho \iff T \leq \frac{K - 1}{C(K + K^3\rho^2)}$$

then, R stabilizes $B_{C([0, T], H^s)}(0, K\rho)$. We now investigate the condition on K and T , which will imply that R is a contraction in $B_{C([0, T], H^s)}(0, K\rho)$ i.e. for any $u, v \in B_{C([0, T], H^s)}(0, K\rho)$,

$$\|Ru - Rv\|_{C([0, T], H^s)} < \frac{1}{2} \|u - v\|_{C([0, T], H^s)}.$$

We compute, for any $u, v \in B_{C([0, T], H^s)}(0, K\rho)$, with the same reasoning as above,

$$\begin{aligned} \sup_{0 < t < T} \|R(u)(t) - R(v)(t)\|_{H^s} &\leq \sup_{0 < t < T} \int_0^t \|e^{(t-l)\Delta} (u(l) - u^3(l) - v(l) + v^3(l))\|_{H^s} dl \\ &= \sup_{0 < t < T} \int_0^t \|e^{(t-l)\Delta} (v(l) - u(l))(v^2(l) + v(l)u(l) + u^2(l) - 1)\|_{H^s} dl \\ &\leq C \sup_{0 < t < T} \int_0^t \|u(l) - v(l)\| (\|u(l)\| + \|v(l)\|)^2 + 1) dl \\ &\leq C(K^2\rho^2 + 1)T \|u - v\|_{C([0, T], H^s)}. \end{aligned}$$

Hence,

$$T \leq \frac{1}{2C(K^2\rho^2 + 1)} \implies R \text{ is a contraction on } B_{C([0, T], H^s)}(0, K\rho).$$

Therefore, choose any $K > 1$, and then choose

$$T = \min\left\{\frac{K - 1}{C(K + K^3\rho^2)}, \frac{1}{2C(K^2\rho^2 + 1)}\right\}.$$

Now, with respect to these choices, by a general fixed point theorem, we have proved existence and uniqueness in $B_C([0, T], H^s)(0, K\rho)$. Continuous dependence on the data and uniqueness on the whole space $C([0, T], H^s)$ can be argued in the same way as the lecture note 7. So we are done. \square

(2) As L^∞ is an algebra, the fixed point argument, used above will work if we show that for each $t > 0$

$$e^{t\Delta} \text{ is bounded on } L^\infty.$$

This, however, is clear, as the eigenfunction expansion picks up $e^{-4\pi^2 n^2 t}$ term, which is bounded by 1 for all n , for each coefficients. \square

(3) We solve the given ODE. By the separation of variables,

$$dt = \left(\frac{1}{u} + \frac{u}{1-u^2}\right) du$$

so

$$t = \ln|u| - \frac{1}{2} \ln|1-u^2| + C.$$

and

$$e^{t-C} = |u||1-u|^{-\frac{1}{2}}.$$

For $1-u^2 > 0$ and $1-u^2 < 0$ respectively,

$$\begin{aligned} u &= (1+e^{c-t})^{-\frac{1}{2}} \\ u &= (1-e^{c-t})^{-\frac{1}{2}}. \end{aligned}$$

For the $1-u^2 < 0$ case, we see that at $t = \ln(|1-u_0|^{\frac{1}{2}}|u_0|^{-1})$ the solution blows up. \square

(4) We derive the energy formally, as shown in class. Multiplying by u_t on both sides of the equation, and integrating by parts,

$$\int u_t \partial_t u - u_t \Delta u + u_t u^3 - u_t u = 0$$

so

$$\int |\partial_t u|^2 dx = -\frac{d}{dt} \left(\frac{1}{4} \int u^4 dx - \frac{1}{2} \int u^2 dx + \frac{1}{2} |\Delta u|^2 dx \right).$$

Therefore,

$$E(u) = \frac{1}{4} \int u^4 dx - \frac{1}{2} \int u^2 dx + \frac{1}{2} |\Delta u|^2 dx,$$

as required. □

(5) Let $u = 0$. Ignoring the non-linear term in $u - u^3$, we get the linearized pde as

$$\partial_t v - \Delta u = u.$$

Let $u = 1$. Set $v = u - 1$, so

$$\begin{aligned} \partial_t v - \Delta v &= \partial_t u - \Delta u = (v+1)^2 - (v+1)^3 \\ &= -v^3 - 3v^2 - 2v. \end{aligned}$$

Ignoring the nonlinear term, we get the linearized pde as

$$\partial_t v - \Delta v = -2v.$$

Similarly, for $u = -1$, we get the linearized pde as

$$\partial_t v - \Delta v = -2v.$$

For the $u = 0$ case, set $u = e^t v$ to get

$$u(x, t) = e^t \int \Phi(x - y, t) u_0(y) dy$$

where Φ is the heat kernel. Then, for any $\delta > 0$, and $\|u_0\|_{H^s} < \delta$ such that $u \neq 0$, $u(x, t)$ blows up by the fact that e^t term dominates. For the other two cases, we have e^{-2t} factor instead of e^t , so we see that the linearized PDE is unstable for $u = 0$ and stable for $u = \pm 1$. □

(6) As $s = 1 > \frac{1}{2} = \frac{d}{2}$, by 1, we have local well-posedness of the equation in $H^1(\mathbb{T})$. Let T^* be the maximal time of existence T^* . Now, observe that by Soblev embedding, we have

$$u_0 \in H^1(\mathbb{T}) \subset C^{\frac{1}{2}}(\mathbb{T}) \subset C(\mathbb{T}) \subset L^\infty(\mathbb{T}) \subset \bigcap_{1 \leq p} L^p(\mathbb{T}).$$

Therefore, $E(u_0) < \infty$, and since the energy is decreasing $E(u) \leq E(u_0)$ for all time up to T^* . Observe that by Holder, for all time up to T^* ,

$$\frac{1}{2} |\Delta u|^2 + \frac{1}{4} \left(\int u^2 dx \right)^2 - \frac{1}{2} \int u^2 dx \leq E(u).$$

Hence, for u with sufficiently large L^2 norm, we have a bound on the L^2 norm from above by $E(u_0)$. Therefore, there exists a constant $C > 0$ such that for all time up to T^* ,

$$\|u\|_{H^1} \leq CE(u_0).$$

Hence u is globally bounded in H^1 , so

$$\limsup_{t \rightarrow T^*} \|u(t)\|_{H^1} < \infty$$

and by the lemma in page 2 of lecture note 8, $T^* = \infty$. \square

(7) We follow the strategy given in the page 6 of the lecture note 8. Denote $u - 1$ simply by u . Then, there exists C constant such that, for $t - 1 < s$,

$$\begin{aligned} \left\| \int_{t-1}^t e^{(t-s)\Delta} (u - u^3)(s) ds \right\|_{\infty} &\leq C \int_{t-1}^t \|u(s)\|_{\infty}^3 + \|u(s)\|_{\infty} ds \\ &\leq \frac{C(\epsilon + \epsilon^3)}{t} \end{aligned}$$

and for $0 < s < t - 1$,

$$\begin{aligned} \left\| \int_0^{t-1} e^{(t-s)\Delta} (u - u^3) ds \right\|_{\infty} &\leq \int_0^{t-1} \|e^{(t-s)\Delta} (u - u^3)\|_{\infty} ds \\ &\leq \int_0^{t-1} \left\| \frac{C}{(t-s)^{\frac{1}{2}}} (u - u^3) \right\|_1 ds \\ &\leq \int_0^{t-1} \frac{C}{t^{\frac{1}{2}}} (\|u(s)\|_2^2 \|u(s)\|_{\infty} + \|u(s)\|_{\infty}) ds \\ &\leq \frac{C(\epsilon + \epsilon^3)}{t^{\frac{1}{2}}}. \end{aligned}$$

Hence, for ϵ small enough, the solution $u = 1$ is asymptotically stable. \square

(8) No. From (5), we saw that the linearized PDE is unstable for the $u = 0$ case. Hence, we do not have a linear decay estimate that overcomes the nonlinear effects even for small data. \square

(9)

(10)

Question 1-2.

Exercise 2(A hypoelliptic operator) We want to study the differential operator L on \mathbb{R}^2 defined by

$$[Lu](x, y) = -\partial_x^2 u(x, y) - x\partial_y u(x, y)$$

and its parabolic flow.

We consider first the equation

$$f = Lu$$

and aim at understanding the smoothing properties of L .

1. By an energy estimate, show that $\|\partial_x u\|_{L^2}^2 \leq \|f\|_{L^2} \|u\|_{L^2}$.

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2. By explicitly solving the differential equation $v'(s) + s^2 v(s) = g(s)$, for $v \in C_0^\infty$, show that $\|v\|_{L^2} \leq C\|g\|_{L^2}$.
3. Deduce by scaling that $\forall \lambda \in \mathbb{R}$, $\|v' + \lambda s^2 v\| \geq C|\lambda|^{1/3}\|v\|_{L^2}$
4. (Harder question!) By using the previous question, switching to Fourier space, and denoting ξ and η for the dual variables of x and y respectively, show that

$$\int (1 + |\xi|^2 + |\eta|^{4/3}) |\hat{u}(\xi, \eta)|^2 d\eta d\xi \leq C(\|f\|_{L^2}^2 + \|u\|_{L^2}^2).$$

5. Deduce that any solution in $u \in H^2$ of $Lu = u^3$ is actually C^∞ .

We turn to the associated parabolic flow

$$\begin{cases} \partial_t u + Lu = 0 \\ u(t=0) = u_0 \end{cases}$$

6. Setting $v(t, x, y) = u(t, x, y - xt)$, show that it solves $\partial_t v = (\partial_x + t\partial_y)^2 v$.
7. Arguing in Fourier space, deduce from the previous question that

$$v(t, x, y) = \Phi(t, x, y) * u_0(x, y) \quad \text{where} \quad \Phi(t, x, y) = \frac{C_0}{t^2} e^{-\frac{x^2}{t} + \frac{3xy}{t^2} - \frac{3y^2}{t^3}}.$$

8. Show that the parabolic flow preserves positivity: $u \geq 0$ if $u_0 \geq 0$.
9. Prove decay estimates of the type

$$\|u(t)\|_{L^p} \leq \frac{C}{t^\alpha} \|u_0\|_{L^q},$$

for appropriately chosen p, q, α .

10. Does this equation enjoy smoothing estimates?

Solution.

(1) We proceed by our usual formal derivation of the energy estimate. Multiplying both sides of the equation by u , integrating over x, y , and integrating by parts,

$$\begin{aligned}\iint u f dx dy &= - \iint u u_{xx} dx dy - \iint u x u_y dy dx \\ &= \iint (u_x)^2 dx dy - \int x \left(\frac{1}{2} u^2 \Big|_{-\infty}^{\infty} \right) dx \\ &= \iint (u_x)^2 dx dy\end{aligned}$$

and hence, by Holder's inequality,

$$\|u_x\|_{L^2}^2 = \left| \iint u f dx dy \right| \leq \iint |u f| dx dy \leq \|f\|_{L^2} \|u\|_{L^2},$$

as required. \square

(2) Firstly, observe that as v vanishes at ∞ , if g solves the ODE then, g has compact support. Now, we solve the ODE. Multiplying both sides of the equation by $e^{\frac{1}{3}s^3}$ (integrating factor),

$$e^{\frac{1}{3}s^3} dv + e^{\frac{1}{3}s^3} s^2 v ds = e^{\frac{1}{3}s^3} g(s) ds$$

and hence

$$e^{\frac{1}{3}s^3} g(s) ds = d(e^{\frac{1}{3}s^3} v).$$

Therefore,

$$v(s) = e^{-\frac{1}{3}s^3} \int_0^s g(t) e^{\frac{1}{3}t^3} dt.$$

solves the ODE. Now, we compute

$$\begin{aligned}\|v\|_{L^2}^2 &= \int (e^{-\frac{1}{3}s^3} \int_0^s g(t) e^{\frac{1}{3}t^3} dt)^2 ds \leq \int (e^{-\frac{1}{3}s^3} \int_0^s g(t) e^{\frac{1}{3}t^3} dt)^2 ds \\ &= \int \left(\int_0^s g(t) dt \right)^2 ds \leq \int \left(\int_{\mathbb{R}} |g(t)| dt \right)^2 ds \\ &\leq \int \int_{\mathbb{R}} |g(t)|^2 dt ds \leq m(\text{supp}(v)) \|g\|_{L^2}^2\end{aligned}$$

where the second last inequality holds by Jensen's inequality on finite measure space, as g has compact support, and $m(\text{supp}(v)) < \infty$ by assumption. \square

(3) Fix $\lambda, C_1, C_2 \in \mathbb{R}$. Then, set

$$z(s) = \lambda^{C_1} v(\lambda^{C_2} s)$$

so

$$\begin{aligned} z'(s) &= \lambda^{C_1+C_2} v'(\lambda^{C_2} s) \\ s^2 z(s) &= s^2 \lambda^{C_1} v(\lambda^{C_2} s). \end{aligned}$$

We compute, via a change of variable $l = \lambda^{C_2} s$,

$$\int |z'(s) + s^2 z(s)|^2 ds = \lambda^{-C_2} \int |\lambda^{C_1+C_2} v'(l) + l^2 \lambda^{C_1-2C_2} v(l)|^2 dl$$

Setting $C_1 = \frac{4}{3}$, and $C_2 = -\frac{1}{3}$,

$$\begin{aligned} \|z\|_{L^2}^2 &= \lambda^3 \|v\|_{L^2}^2 \\ \|z' + l^2 z\|_{L^2}^2 &= \lambda^{\frac{7}{3}} \|v' + l^2 \lambda v\|_{L^2}^2. \end{aligned} \tag{4}$$

Now, by (2), for some constant $C > 0$,

$$\|z\|_{L^2}^2 \leq C \|z' + s^2 z\|_{L^2}^2$$

and substituting (4) to above,

$$C^{-1} \lambda^{\frac{2}{3}} \|v\|_{L^2}^2 \leq \|v' + l \lambda v\|_{L^2}^2.$$

Taking the square root on both sides gives the desired inequality. \square

(4) From the energy estimate (1), and taking the Fourier transform of $\partial_x u$,

$$\int \int |\xi|^2 |\hat{u}(\xi, \eta)|^2 d\xi d\eta = \|\partial_x\|_{L^2}^2 \leq \|\hat{f}\|_{L^2} \|\hat{u}\|_{L^2} \leq \frac{1}{2} (\|\hat{f}\|_{L^2}^2 + \|\hat{u}\|_{L^2}^2)$$

and hence, for some constant C ,

$$\int \int (1 + |\xi|^2) |\hat{u}|^2 \leq C (\|\hat{f}\|_{L^2} + \|\hat{u}\|_{L^2}).$$

Therefore, it suffices to show that, for some constant $C > 0$,

$$\int \int |\eta|^{\frac{4}{3}} |\hat{u}(\xi, \eta)|^2 d\xi d\eta \leq C \|\hat{f}\|_{L^2}^2 \tag{5}$$

Taking the Fourier transform of the equation,

$$|\xi|^2 \hat{u} + \eta \partial_\xi \hat{u} = \hat{f}$$

and dividing both sides of the above equation by η ,

$$\frac{1}{\eta} |\xi|^2 \hat{u} + \partial_\xi \hat{u} = \frac{1}{\eta} \hat{f}.$$

Now, by the scaling estimate from (3), for some constant $C_1 > 0$,

$$\left| \frac{1}{\eta} \right| \|\hat{f}(\xi, \eta)\|_{L_\xi^2} \geq C_1 |\eta|^{-\frac{1}{3}} \|\hat{u}(\xi, \eta)\|_{L_\xi^2}$$

and hence, rearranging and squaring both sides, for some constant $C_2 > 0$.

$$C_2 \int |\hat{f}(\xi, \eta)|^2 d\xi \geq |\eta|^{\frac{4}{3}} \int |\hat{u}(\xi, \eta)|^2 d\xi.$$

Integrating the above equation with respect to η gives (5), so we are done. \square

(5) Since $s = 2 > 1 = \frac{d}{2}$, H^2 is an algebra, and $u^3 \in H^2$. From (4), $u \in H^{\frac{4}{3}}$, so $u \in H^{\frac{10}{3}}$. Repeating the argument inductively, $u \in H^s$ for all $2 \leq s$, and by Morrey's inequality, $u \in C^\alpha$ for all $1 \leq \alpha$. Therefore, $u \in C^\infty$ as required. \square

(6) We wish to show that

$$v_t = v_{xx} + 2tv_{xy} + t^2v_{yy} \tag{6}$$

To that end, we compute

$$\begin{aligned} v_x &= u_x - tu_y \\ v_{xx} &= u_{xx} - 2tu_{yx} + t^2u_{yy} \\ v_t &= u_t - xu_y \\ 2tv_{xy} &= 2tu_{xy} - 2t^2u_{yy} \\ t^2v_{yy} &= t^2u_{yy}. \end{aligned}$$

Substituting the above equations to (6),

$$u_t - xu_y = u_{xx} = u_{xx} - 2tu_{xy} + t^2u_{yy} + 2tu_{xy} - 2t^2u_{yy} + t^2u_{yy}$$

which simplifies to

$$u_t - u_{xx} - xu_y = 0.$$

Therefore, we see that v solves (6), as required. \square

(7) Taking the Fourier transform of the equation in (6),

$$\partial_t \hat{v} = -\xi^2 \hat{v} - 2t\xi\eta \hat{v} - t^2\eta^2 \hat{v}$$

and hence

$$\hat{v} = \hat{u}_0(\xi, \eta) e^{-\xi^2 t - \xi\eta t^2 - \frac{1}{3}\eta^2 t^3}.$$

Therefore, it suffices to show that

$$\mathcal{F}(\Phi) = e^{-\xi^2 t - \xi \eta t^2 - \frac{1}{3} \eta^2 t^3}.$$

This can be checked by definition and completing the square in the integral.

(8) Suppose $u_0 \geq 0$. Note that Φ is non-negative everywhere. Hence, by 6 and 7, for each t, x, y ,

$$u(t, x, y) = v(t, x, y + xt) = \Phi(t, x, y + xt) * u_0(x, y + xt) \geq 0,$$

as required. □

(9) By (7),

$$u(t, x, z) = \int \frac{C_0}{t^2} e^{-\frac{(x-z_1)^2}{t} + \frac{3(x-z_1)(z+xt-z_2)}{t^2} - \frac{3(z+xt-z_2)^2}{t^3}} u_0(z_1, z_2) dz_1 dz_2.$$

Then, if Young's inequality were to be applied, then

$$\|u(t)\|_p \leq \|\Phi(t, x, xt + z)\|_r \|u_0\|_q$$

for $1 \leq p, q, r \leq \infty$, with $\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{q}$. With this in mind, we compute

$$\begin{aligned} \|\Phi(t, x, xt + z)\|_r^r &= \int \frac{C_0^r}{t^{2r}} e^{r(-\frac{x^2}{t} + \frac{3x(xt+z)}{t^2} - \frac{3(xt+z)^2}{t^3})} dx dz \\ &= C t^{-2r} \int e^{r\Theta(t^{-1})} dx dz \end{aligned}$$

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(10) Yes, as in general parabolic equations, like heat, enjoys smoothing estimates. □