Limit Theorems II: Final

Youngduck Choi *

Abstract

This work contains solutions for the final of the Limit Theorems II by Professor McKean.

Question 1-1.

1. $Rw(3)$: CAPACITY. Compute the capacity of a pair of lattice points $a \neq b$ and also of a singleton c in terms of the Green's function $G(a,b) = \sum_0^\infty P_a[\mathbf{X}(n) = b]$. This already shows that capacity is *not* additive. Yes?

Solution.
For $0 \leq t$ and $x \in \mathbb{R}$, set

$$w(t,x) \quad = \quad E_x(e^{-\beta m(t) f \circ b(t)})$$

with $m(t) = \int_0^t K(x)ds$. Then, we see from Mckean's boo

$$w \;\; \text{solves} \;\; \partial_t w = \frac{1}{2}\partial_{xx} w - \beta$$

dd

*Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

Question 1-2.

2. $Bm(1)$ Conditional Arcsine. Recall the usual arcsine law

$$P_0\left[\text{measure}\,(t \le 1 : \mathrm{b}(t) \ge 0) \le x\right] = \frac{1}{\pi}\int_0^x \frac{dy}{\sqrt{y(1-y)}} \quad (0 \le x \le 1)$$

showing that the bulk of the density lies (surprisingly) near $y=0$ and $y=1$. It comes as a further surprise that for the tied Brownian motion, conditioned to end at $\mathrm{b}(1)=0$, that the density is *perfectly flat* as in

$$P_0\left[\text{measure}\,(t \le 1 : \mathrm{b}(t) \ge 0) \le x\right] = x \,(0 \le x \le 1).$$

I will lead you through it via Feynman-Kac.

Write $\mathbf{m} =$ measure $(t' \le t : \mathrm{b}(t') \ge 0)$ and recall that

$$\int_0^\infty e^{-\alpha t} \cdot E_0\left[e^{-\beta \mathbf{m}} f \circ \mathrm{b}(t)\right] dt$$

may be expressed as

$$h_+(0)\int_{-\infty}^0 h_- f\,dy + h_-(0)\int_0^\infty h_+ f\,dy \ \text{ over } \ \frac{1}{2}\left(h'_- h_+ - h_- h'_+\right) \ \text{ at } x=0$$

with $h_-(x) = e^{\sqrt{2\alpha}\,x}$ for $x \le 0$ and $h_+(x) = e^{-\sqrt{2(\alpha+\beta)}}$ for $x \ge 0$ for reasonable functions f. That is now used for the *unreasonable* function $f =$ Dirac's delta function $\delta(x)$ to produce

$$\int_0^\infty e^{-\alpha t}\, E_0\left[e^{-\beta \mathbf{m}},\, \mathrm{b}(t)=0\right] dt = \frac{2}{\sqrt{2\alpha} + \sqrt{2\,(\alpha+\beta)}}$$

as you will check. This may also be written

$$\int_0^\infty e^{-\alpha t}\, E_0\left[e^{-\beta \mathbf{m}}\,|\mathrm{b}(t)=0\right] \frac{dt}{\sqrt{2\pi\, t}}$$

and if you knew, what is the fact, that $\mathbf{m}$ is now uniformly distributed over the interval $0 \le m \le t$, then you could put this into the last display to see if it checks. Do it. It

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works. Is that a proof? Could the conditional law of $\mathbf{m}$ be anything else? Of course this is only a consistency check, based so to speak on proprietary information. Insider trading if you will. The honest way would be to invert $\frac{2}{\sqrt{2\alpha}+\sqrt{2(\alpha+\beta)}}$. Not a standard exercise. Try it if you like.

Solution.

Question 1-3.

3. $BM(1)$: LOCAL TIME. Begin with the function $f''(x) = \frac{1}{\epsilon}$ for $0 \le x \le \epsilon$, vanishing elsewhere, and compute f' and f subject to $f'(0) = f(0) = 0$. Apply Itô's lemma for small ϵ to obtain

$$\begin{aligned} \max[0, \mathbf{b}(t)] &\simeq f \circ \mathbf{b}(t) \\ &= \int_{(0,t)\cap(0\le \mathbf{b}(t')<\epsilon)} \frac{b}{\epsilon}\, db \\ &\quad + \int_{(0,t)\cap(\mathbf{b}(t')>\epsilon)} db \\ &\quad + \frac{1}{2\epsilon} \text{ measure } (t' \le t : 0 \le \mathbf{b}(t') < \epsilon) \\ &= ① + ② + ③. \end{aligned}$$

Here ① is a fair game. Show that it vanishes, uniformly for $t \le 1$ say, if $\epsilon \downarrow 0$ fast enough, and likewise that ② tends to $\int_0^t e\, d\mathbf{b}$ with $e(t, \mathbf{b}) =$ the indicator of $\mathbf{b}(t) > 0$, and the same uniformity. Then ③ converges in the same style and you have

$$\max[0, \mathbf{b}(t)] = \int_0^t e\, d\mathbf{b} \text{ plus } \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon} \text{ measure } (t' \le t : 0 \le \mathbf{b}(t') < \epsilon) = ④$$

with the same uniformity in ④, this being P. Levy's *mesure de voisinage*, more aptly called in English *local time*, written $\mathfrak{t}(t)$. Think a little how extraordinary it is that this limit should exist, due solely to the wildness of the Brownian path. Evidently $\mathfrak{t}(t)$ is continuous and increasing but only on $Z = (t \ge 0 : \mathbf{b}(t) = 0)$ which is of Lebesgue measure zero(!) and cannot vanish for any $t > 0$. Give me a little argument for *that*.

FYI. The fact is that it does not matter how fast $\epsilon \downarrow 0$. This lies deeper. See Itô-McKean if you're interested.

Solution.
Recall the Ito's formula for single martingales: If f is C^2 and for all $0 \le t$,

$$E\int_0^t [f'(M(s))]^2 dA(s) < \infty \qquad \text{and} \qquad E\int_0^t |f''(M(s))| dA(s) < \infty$$

then, for all $0 \le t$,

$$f(M(t)) - f(M(0)) = \int_0^t f'(M(s))dM(s) + \frac{1}{2}\int_0^t f'(M(s))dA(s) \tag{1}$$

where $A(t)$ is the unique increasing, continuous process, corresponding to $M(t)$. Of course, for the problem at hand, we have $M(t) = B(t)$ and $A(t) = t$, so $db(s)$ and ds will be used to denote the integrators. Now, to compute local time, we cannot naively apply this formula, as $\max(0,\cdot)$ is not $C^2(\mathbb{R})$. Hence, we approximate with the suggested strategy. Let f_ϵ be defined as given for each $\epsilon > 0$. Then, for any $0 \leq t$, and $\epsilon > 0$,

$$\begin{aligned} f_\epsilon \circ b(t) &= f_\epsilon \circ b(t) - f_\epsilon \circ b(0) = \int_0^t f_\epsilon^{'}(B(s))db(s) + \frac{1}{2}\int_0^t f_\epsilon^{''}(B(s))ds && (2) \\ &= \int_{\{s\in[0,t]:0\leq b(s)<\epsilon\}} \frac{b(s)}{\epsilon}db(s) + \int_{\{s\in[0,t]:b(s)>\epsilon\}} 1db(s) + \frac{1}{2}\int_{\{s\in[0,t]:0\leq b(s)<\epsilon\}} \frac{1}{\epsilon}ds && (3) \\ &= \int_{\{s\in[0,t]:0\leq b(s)<\epsilon\}} \frac{b(s)}{\epsilon}db(s) + \int_{\{s\in[0,t]:b(s)>\epsilon\}} 1db(s) + \frac{1}{2\epsilon}\lambda_1(\{s\in[0,t]:0\leq b(s)<\epsilon\}) \\ &=: (I) + (II) + (III) \end{aligned}$$

where (2) follows from (1), (3) follows from definition of f_ϵ for each $\epsilon > 0$, and λ_1 denotes 1-dimensional Lebesgue measure as hinted. With the given fact that (I) is a martingale, we claim that, for any $0 \leq t \leq 1$ uniformly,

$$\int_{\{s\in[0,t]:0\leq b(s)<\epsilon\}} \frac{b(s)}{\epsilon}db(s) \text{ converges to } 0 \tag{4}$$

and

$$\int_{\{s\in[0,t]:b(s)>\epsilon\}} 1db(s) \text{ converges to } \int_0^t e(s)db(s) \tag{5}$$

as $\epsilon \downarrow 0$ $\mathbb{P}$ almost surely. To show (4), there exists a constant C, each $0 \leq t \leq 1$, for some universal constant C, independent of t,

$$\begin{aligned} E\left[\int_0^t \frac{b(s)}{\epsilon}1_{\{0\leq b(s)<\epsilon\}} - 0db(s)\right]^2 &= E\int_0^t \left[\frac{b(s)}{\epsilon}1_{\{0\leq b(s)<\epsilon\}}\right]^2 ds && (6) \\ &\leq E\int_0^t 1_{\{0\leq b(s)<\epsilon\}}ds \\ &= \int_0^t P(0\leq b(s)<\epsilon)ds \\ &= \int_0^t P(0\leq b(1)<\frac{\epsilon}{\sqrt{s}})ds && (7) \\ &\leq \int_0^t P(|b(1)|<\frac{\epsilon}{\sqrt{s}})ds \leq C\sqrt{t}\epsilon, && (8) \end{aligned}$$

where (10) holds by the Ito L^2 isometry, (11) follows by the scaling of BW(1), and (8) follows from the fact that

$$\frac{P(|b(1)|\leq x)}{x} \text{ is uniformly bounded for all } x > 0.$$

Hence, if $\epsilon = O(n^{-2})$, then, by Doob's inequality,

$$(I) \to 0 \ \text{ uniformly for } \ 0 \le t \le 1 \ \ P \ \text{ almost surely}$$

showing (4). Now, denote the speed chosen above as $\gamma(n)$, which is required to be $\lim_{n\to\infty} \gamma(n) = 0$ and $O(n^{-2})$. Now, by the Ito L^2 Isometry, for each n and $0 \le t \le 1$,

$$\begin{aligned} E\left[\int_0^t (-1_{\{b(s)>\gamma(n)\}} + 1_{\{b(s)>0\}}) db(s)\right]^2 &= E\left[\int_0^t (-1_{\{b(s)>\gamma(n)\}} + 1_{\{b(s)>0\}})^2 ds\right] \\ &= E\left[\int_0^t 1_{\{\gamma(n)>b(s)>0\}} ds\right]. \end{aligned} \tag{9}$$

Now, as $\gamma(n) \to 0$, for almost surely $w \in \Omega$, and n large enough

$$1_{\{\gamma(n)>b(s)(w)>0\}} = 0$$

and hence

$$\int_0^t 1_{\{\gamma(n)>b(s)(w)>0\}} ds = 0 \tag{10}$$

Since, for each n,

$$\int_0^t 1_{\{\gamma(n)>b(s)>0\}} ds \le 1_{\{[0,t]\times\{\gamma(n)>b(s)>0\}\}} \in L^1(\Omega \times [0,\infty))$$

by DCT on the product space, we see that the right hand side of (5) converges to 0, and

$$\int_0^t 1_{\{b(s)>\gamma(n)\}} db(s) \to \int_0^t 1_{\{b(s)>0\}} db(s) \ \text{ in } \ L^2.$$

Hence, we can choose a subsequence of $\gamma(n)$, denote it as $\rho(n)$, such that

$$\int_0^t 1_{\{b(s)>\rho(n)\}} db(s) \to \int_0^t 1_{\{b(s)>0\}} db(s) \ \text{ in } \ \text{almost surely}.$$

We see that uniformity of convergence persists by monotonicity (10) with t. Hence, we see that (4) and (5) holds by choosing the speed $\rho(n)$ uniformly in $t \le 1$. Now, for almost surely $w \in \Omega$, and $m < n$,

$$\left| \frac{1}{2\rho(n)} \lambda_1(\{s \in [0,t] : 0 \le b(s)(w) < \rho(n)\}) - \frac{1}{2\rho(m)} \lambda_1(\{s \in [0,t] : 0 \le b(s)(w) < \rho(m)\}) \right| \tag{11}$$

is bounded above by

$$\frac{1}{2\rho(m)} \lambda_1(\{s \in [0,t] : \rho(n) < b(s)(w) < \rho(m)\})$$

which in turn is bounded above by

$$\frac{1}{2\rho(m)}\frac{2}{\pi}(\rho(m)-\rho(n)) = \frac{\rho(m)-\rho(n)}{\pi\rho(m)}$$

By choosing a subsequence of $\rho(n)$ of speed $O(2^{-n})$, $\psi(n)$, we see that for almost surely $w \in \Omega$ and $m < n$, (11) is bounded above by

$$\frac{\psi(m)-\psi(n)}{\pi\psi(m)} = O(2^{-m})$$

and hence, for almost surely $w \in \Omega$,

$$\frac{1}{2\psi(n)}\lambda_1(\{s \in [0,t] : 0 \leq b(s)(w) < \psi(n)\} \quad \text{is cauchy, thus convergent.}$$

Hence, we have established that with the sequence $\psi(n)$, the claimed convergence of (I), (II), and (III) all hold. We are told that the convergence will hold for any speed, i.e. independent of the choice of the countable sequence converging to 0, but we will forgo the work for now. From uniformity of convergence and continuity of Lebesgue measure, we see that $\mathbf{t}(t)$ is continuous, and clearly increasing as (II) and (III) are increasing in t almost surely.

Question 1-4.

4. Some Pictures. Make me some diagrams or so to say *caricatures* like

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showing in a single picture $\mathbf{b}(t)$, $\max(\mathbf{b}(t') : t' \le t)$, and the passage times $T_x = \min(t : \mathbf{b}(t) = x)$ together, and in a second picture, $\mathbf{b}(t)$, $\mathbf{t}(t)$, and the inverse local time $\mathbf{t}^{-1}(x) = \min(t : \mathbf{t}(t) = x)$. You will see a similarity between the passage times and the inverse local time and likewise between $\mathbf{t}(t)$ and $\max(\mathbf{b}(t') : t' \le t)$. Tell me what you see. More coming at once.

Solution.
Fix $x \in \mathbb{R}$. We first show that $\mathbf{t}^{-1}$ is a stopping time. Observe that

$$\{\mathbf{t}^{-1}(x) \le t\} \quad =$$

Question 1-5.

5. A SURPRISE. You have seen how the distribution of the passage times is determined by stopping the fair game

$$e^{\alpha\,\mathbf{b}(t)-\alpha^2 t/2} \quad \text{as in}$$

$$1 = E_0\left[e^{\alpha\mathbf{b}(T_x)-\alpha^2 T_x/2}\right] = e^{\alpha x}\, E_0\left(e^{-\alpha^2 T_x/2}\right).$$

The same idea can be applied to the inverse local time. You are to show that $\mathbf{t}^{-1}(x)$ is a stopping time, that $\mathbf{b}\circ\mathbf{t}^{-1}(x) = x$ and to conclude that the two processes $(T_x : x \geq 0)$ and $(\mathbf{t}^{-1}(x) : x \geq 0)$ are not the *same* but *identical in law.*

HINT: Both processes are *additive* in the sense that $\mathbf{x}\,(a+b) = \mathbf{x}(a)$ plus an independent copy of $\mathbf{x}(b)$ as you will need to check for the inverse local time. It follows that the local time itself is a copy of the inverse function of the passage times, namely $\max(\mathbf{b}(t') : t' \leq t)$. Indeed, wonders never cease! BM(1) has lots of such complicated internal imitations of one thing with another, the copies $c\,\mathbf{b}\,(t/c^2)$ and $t\mathbf{b}(1/t)$ of $\mathbf{b}(t)$ being the simplest of these. I tell you another. Take $\mathbf{b}(0) = x \geq 0$, run it to the passage time T_0, then start it over from $x = 0$ in the form $\max(\mathbf{b}(t') : t' \leq t) - \mathbf{b}(t)$.

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What is that? Draw a picture and you may guess, what P. Levy discovered, that it's a copy of the reflecting Brownian motion! Useless but remarkable.

Solution.

Question 1-6.

6. AN OLD FRIEND EXPLAINED. Write $h(t,x)$ for $e^{\alpha x-\alpha^2 t/2}$ and define

$$P^h\left[\mathbf{x}(t_2)=y|\mathbf{x}(t_1)=x\right] = \frac{1}{h(t_1,x)}\,\frac{e^{-(x-y)^2/2(t_2-t_1)}}{\sqrt{2\pi\,(t_2-t_1)}}\,h(t_2,y).$$

Check that this is a perfectly good Markovian transition density, compute its associated infinitessimal operator, and tell me what the underlying process is. Note that h solves the backward heat equation $\partial h/\partial t+\frac{1}{2}\,\partial^2 h/\partial x^2$, so it's something very special since heat does not like to flow backwards. More can be said but let's leave it here.

Solution.
We check that P^h is a Markovian transition density by checking that it integrates to 1 with respect to y. We compute, via completing the square,

$$\begin{aligned}
\int P^h(\mathbf{x}(t_2)=y|\mathbf{x}(t_1)=x)dy &= \int \frac{h(t_2,y)}{h(t_1,x)}\frac{e^{-(x-y)^2/2(t_2-t_1)}}{\sqrt{2\pi(t_2-t_1)}}dy \\
&= \frac{1}{h(t_1,x)}\frac{1}{\sqrt{2\pi(t_2-t_1)}}\int e^{-(y^2-2xy+x^2)/2(t_2-t_1)+\alpha y-\alpha^2 t_2/2}dy \\
&= \frac{1}{h(t_1,x)}e^{\alpha x-\frac{1}{2}\alpha^2 t_1}\int e^{\frac{-(y-(x+\alpha(t_2-t_1))^2}{2(t_2-t_1)}}dy \\
&= \frac{1}{h(t_1,x)}e^{\alpha x-\frac{1}{2}\alpha^2 t_1} = \frac{h(t_1,x)}{h(t_1,x)} = 1.
\end{aligned}$$

Now, we compute the associated infinitessimal operator. Setting $s=t_2-t_1$ and $t=t_1$, from the general theory, we know that the generator must be defined by

$$\begin{aligned}
\mathscr{G}f(x) &= \lim_{s\downarrow 0}\frac{\int f(y)P^h(\mathbf{x}(t+s)=y|\mathbf{x}(t)=x)dy-f(x)}{s} \\
&= \frac{1}{h(t,x)}\lim_{s\downarrow 0}\frac{1}{s}\int f(y)\frac{e^{-\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}}h(t+s,y)dy-f(x)h(t,x) \\
&= \frac{1}{h(t,x)}\lim_{s\downarrow 0}\frac{1}{s}\int f(y)\frac{e^{-\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}}e^{\alpha y}e^{-\alpha^2\frac{(t+s)}{2}}dy-f(x)e^{\alpha x-\alpha^2\frac{t}{2}}
\end{aligned} \tag{12}$$

whenever the limit is well-defined for $f\in C_0(\mathbb{R})$. Recall that we saw from class that

$$u(s,x) = \int f(y)\frac{e^{\frac{-(x-y)^2}{\sqrt{2s}}}}{\sqrt{2\pi s}}e^{\alpha y}dy$$

solves

$$u_t - \frac{1}{2}u_{xx} = 0 \quad \text{and} \quad u(0,x) = f(x)e^{\alpha x}.$$

Therefore, we can continue (12) as

$$\begin{aligned}
&= \frac{1}{h(t,x)} \lim_{s\downarrow 0} \frac{1}{s}(u(s,x)e^{\frac{-\alpha^2(t+s)}{2}} - u(0,x)e^{\frac{-\alpha^2 t}{2}}) \\
&= \frac{1}{h(t,x)} \lim_{s\downarrow 0} \frac{1}{s}((u(s,x) - u(0,x))e^{-\frac{\alpha^2(t+s)}{2}} + u(0,x)(e^{-\frac{\alpha^2(t+s)}{2}} - e^{-\frac{\alpha^2 t}{2}})) \\
&= \frac{1}{h(t,x)}(\frac{1}{2}\partial_{xx}(f(x)h(t,x)) + f(x)\partial_t h(t,x))
\end{aligned}$$

Hence, the generator of the process is

$$\mathscr{G}f = \frac{1}{h}(\frac{1}{2}\partial_{xx} + \partial_t)(fh),$$

defined for each $f \in C(\mathbb{R})$ with 1-time partial and 2-space partials. Furthermore, by noting that h solves the backward heat equation,

$$\begin{aligned}
\mathscr{G}f &= \frac{1}{h}\partial_t hf + \frac{1}{2}\partial_x(f_x h + h_X f) \\
&= \frac{1}{h}\partial_t hf + \frac{1}{2h}(f_{xx}h + 2f_x h_x + h_{xx}f) \\
&= \frac{1}{2}f_{xx} + \frac{1}{h}h_x f_x,
\end{aligned}$$

and hence the underlying process $\mathbf{x}_t$ solves the stochastic differential equation

$$d\mathbf{x}_t \quad = \quad \frac{\partial_x h(t,\mathbf{x}_t)}{h(t,\mathbf{x}_t)}dt + dB_t.$$