Diff Geo II: Problem Set V

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Abstract

This work contains solutions for the problem set V.

Question 1-1.

1. Suppose that $\gamma\colon [0,1] \to M$ is a geodesic. The second variation formula states that if $V = \frac{d\gamma}{dt}$ and $W_1, W_2 \in \mathcal{V}(\gamma)$ are piecewise-smooth vector fields with $W_i(0) = W_i(1) = 0$, then

$$\frac{1}{2}H(E)(W_1, W_2) = -\sum_t \langle W_2, \delta_t D_t W_1 \rangle - \int_0^1 \langle W_2, D_t^2 W_1 - R(V, W_1)V \rangle \, dt.$$

Show that this can be rewritten in the more symmetric form

$$\frac{1}{2}H(E)(W_1, W_2) = \int_0^1 \langle D_t W_1, D_t W_2 \rangle + \langle R(V, W_1)V, W_2 \rangle \, dt.$$

This expression is known as the *index form*.

Solution.
From

$$\frac{d}{dt} < W_2, D_t W_1 > \quad = \quad - < D_t W_1, D_t W_2 > + < W_2, {D_t}^2 W_1 >$$

it follows that

$$\begin{aligned}
\frac{1}{2}H(E)(W_1, W_2) &= = -\sum_i < W_2, \triangle_t D_t W_1 > - \int_0^1 < W_2, D_t^2 W_1 - R(V, W_1)V > dt \\
&= -\sum_i \int_{t_{i-1}}^{t_i} (\frac{d}{dt} < W_2, D_t W_1 >) dt + \int_0^1 < W_2, D_t^2 W_1 > + < R(V, W_1)V, W_2 > dt \\
&= \int_0^1 < D_t W_1, D_t W_2 > + < R(V, W_1)V, W_2 > dt.
\end{aligned}$$

□

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Question 1-2.

2. Let M be a compact n–manifold and suppose that $\text{Ric}(U,U) \geq (n-1)k$ for all $U \in TM$ such that $\|U\| = 1$. Prove that if γ is a geodesic in M of length greater than $\frac{m\pi}{\sqrt{k}}$, then index $\gamma \geq m$.

Solution.
From the given Ricci lower bound, we see that the hypothesis of Myer's theorem is satisfied with the positive constant $\frac{1}{\sqrt{k}}$, and hence any geodesic γ with length greater than $\frac{\pi}{\sqrt{k}}$ contains conjugate points, which implies that the index of γ is at least 1. We now proceed by induction. Suppose the statement is true for some $m > 1$. Then, split a geodesic γ with length greater than $\frac{(m+1)\pi}{\sqrt{k}}$ into two parts where γ_1 has the length greater length $m\frac{\pi}{\sqrt{k}}$, and γ_2 has the length greater than $\frac{\pi}{\sqrt{k}}$. Then, by Morse Index theorem, and the inductive hypothesis

$$\begin{aligned}
\text{Index}(\gamma) &= \sum_{q \text{ conjugate to } p;\, q\in\gamma((0,1])} Ord(q) \\
&\geq \sum_{q \text{ conjugate to } p;\, q\in\gamma_1((0,1])} Ord(q) \geq m
\end{aligned}$$

where $p = \gamma(0) = \gamma_1(0)$. Hence, it suffices to show that there is a conjugate point to p along γ_2. Now, as γ_2 has length greater than $\frac{\pi}{\sqrt{k}}$, we know that there is a point on the image of γ_2 such that it is conjugate to $\gamma_2(0)$. Hence, it now suffices to show the following: let p, q, r be $p = \gamma(0), q = \gamma(t_1)$, and $r = \gamma(t_2)$, such that $0 < t_1 < t_2$, and r is conjugate to q. Then, there exists $t^* \in [t_1, t_2]$ such that $\gamma(t^*)$ is conjugate to p. The statement is true, since if r is conjugate to q, then the index along the geodesic from p to r must increase from q to r, by monotonicity, so using Morse theorem one more time, we are done. $\square$

Question 1-3.

3. Recall that the Poincaré disc model of the hyperbolic plane $\mathbb{H}^2$ consists of the open unit disc $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ with the metric $dg^2 = \frac{dx^2+dy^2}{(1-x^2-y^2)^2}$. If we identify $\mathbb{R}^2$ with the complex plane, we can write the metric as

$$dg^2 = \frac{|dz|^2}{(1-|z|^2)^2}.$$

Prove that for any $a \in D^2$, $\theta \in \mathbb{R}$,

$$f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$$

is an isometry of $\mathbb{H}^2$.

Solution.
Using the isometry between the upper half plane model and the ball model, $\dfrac{zi+1}{z+i}$, we can instead argue that $z \mapsto \dfrac{az+b}{cz+d}$ for any $a,b,c,d \in \mathbb{R}$ with $ad-bc=1$, is an isometry on $\mathbb{H}^2$. Now, it suffices to show that for any piecewise differentiable path in $\mathbb{H}^2$, $\gamma : I \to \mathbb{H}^2$, $L(\gamma) = L(f(\gamma))$, as this will determine the hyperbolic distance. Set $z(t) = (x(t), y(t))$ and $w(t) = f(z(t)) = u(t) + iv(t)$. Then,

$$\frac{dw}{dz} = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

and

$$v = Im(w) = \frac{w-\bar{w}}{2i} = \frac{Im(z)}{|cz+d|^2} = \frac{y}{|cz+d|^2}.$$

Hence,

$$\left|\frac{dw}{dz}\right| = \frac{v}{y}$$

and

$$L(f(\gamma)) = \int_0^1 \frac{|\frac{dw}{dt}|}{v(t)}dt = \int_0^1 \frac{|\frac{dw}{dz}\frac{dz}{dt}|}{v(t)}dt = \int_0^1 \frac{|\frac{dz}{dt}|}{y(t)}dt = L(\gamma),$$

as required. $\square$