Diff Geo II: Problem Set I

Youngduck Choi *

Abstract

This work contains solutions for the problem set I.

Question 1-1.

1. Show that the covariant derivative can be expressed in terms of parallel transport in the following sense.

   Let $\gamma : [0,1] \to M$ be a smooth curve in M and let $p_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$ be the parallel transport maps along γ. Show that if $X \in \mathbf{V}(\gamma)$, then

$$D_t X(0) = \frac{d}{dt} p_t^{-1}(X(t)).$$

Solution.
Let $E_1(t), ..., E_n(t)$ be a parallel frame along γ, and $X \in \mathscr{V}(\gamma)$. Then,

$$\begin{aligned} D_t X(0) &= \sum_i D_t(X^i(0)E_i(0)) = \sum_i \dot{X}^i(0)E_i(0) + \sum_i X^i(0)D_t(E_i(0)) && (1) \\ &= \sum_i \dot{X}^i(0)E_i(0) = \frac{d}{dt}\sum_i X^i(0)P_t^{-1}(E_i(0)) = \frac{d}{dt}P_t^{-1}(X(0)) && (2) \\ &= \frac{d}{dt}|_{t=0} P_t^{-1}(X(t)) \end{aligned}$$

where (1) follows from the Leibiniz rule with $D_t(E_i(0)) = 0$ for all i, and (2) holds by the linear isomorphism property of P_t. $\square$

*Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

Question 1-2.

2. Prove that the tangential connection ∇^T on an embedded submanifold $M \subset \mathbb{R}^n$ is compatible with the metric induced by the dot product on $\mathbb{R}^n$.

Solution.

Let $\gamma(t)$ be a curve and $Y, Z \in \mathscr{V}(\gamma)$. Then,

$$\begin{aligned}\partial_t < Y, Z > &= < \overline{\nabla}_{\gamma(t)} Y, Z > + < Y, \overline{\nabla}_{\gamma(t)} Z > \\ &= < \pi^T(\overline{\nabla}_{\gamma(t)} Y), Z > + < Y, \pi^T(\overline{\nabla}_{\gamma(t)} Z) > \qquad (3) \\ &= < \nabla^T_{\gamma(t)} Y, Z > + < Y, \nabla^T_{\gamma(t)} Z > \qquad (4)\end{aligned}$$

where (3) holds as $Y, Z \in \mathscr{V}(\gamma)$, and (4) follows from definition of tangential connection. Therefore, we have precisely shown the compatibility of the tangential connection with the metric induced by the dot product on $\mathbb{R}^n$. $\square$

Question 1-3.

3. Prove that ∇^T is torsion-free. (Since τ is a tensor, it suffices to show that for any patch $(u^1, \ldots, u^n)\colon U \to M$, $\tau(\partial_i, \partial_j) = 0$.)

Solution.

Let $X, Y \in \mathscr{V}(M)$. Then,

$$\begin{aligned} \nabla^T_X Y - \nabla^T_Y X &= \pi^T(\overline{\nabla}_X Y) - \pi^T(\overline{\nabla}_Y X) = \pi^T(\overline{\nabla}_X Y - \overline{\nabla}_Y X) \\ &= \pi^T([X, Y]) \qquad (5) \\ &= [X, Y] \qquad (6) \end{aligned}$$

where (5) follows from the fact that $\overline{\nabla}$ is torsion-free, and (6) follows from $[X, Y] \in \mathscr{V}(M)$. $\square$

Question 1-4.

Let $\phi \in (0, \pi/2)$ and let $M \subset \mathbb{R}^3$ be the cone

$$M = \{(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi) \mid r > 0, \theta \in [0, 2\pi)\}$$

with axis the positive z–axis and angle ϕ.

4. The parametrization $u(r, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$ has coordinate vector fields $\partial_r = \frac{\partial}{\partial r}$ and $\partial_\theta = \frac{\partial}{\partial\theta}$. Let ∇^T be the tangential connection on M and calculate $\nabla^T_{\partial_r}\partial_r$, $\nabla^T_{\partial_r}\partial_\theta$, $\nabla^T_{\partial_\theta}\partial_r$, and $\nabla^T_{\partial_\theta}\partial_\theta$.

Solution.
We compute

$$\begin{aligned}
\partial_r &= (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) \\
\partial_\theta &= (-r\sin(\theta)\sin(\phi), r\cos(\theta)\sin(\phi), 0)
\end{aligned}$$

and hence

$$\begin{aligned}
\overline{\nabla}_{\partial r}\partial r &= \frac{\partial}{\partial_r}(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) = (0,0,0) \\
\overline{\nabla}_{\partial r}\partial\theta &= \frac{\partial}{\partial_r}(-r\sin(\theta)\sin(\phi), r\cos(\theta)\sin(\phi), 0) \\
&= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\
\overline{\nabla}_{\partial\theta}\partial r &= \frac{\partial}{\partial_r}(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) \\
&= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\
\overline{\nabla}_{\partial\theta}\partial\theta &= \frac{\partial}{\partial_r}(-r\sin(\theta)\sin(\phi), r\cos(\theta)\sin(\phi), 0) \\
&= (-r\cos(\theta)\sin(\phi), -r\sin(\theta)\sin(\phi), 0)
\end{aligned}$$

Now, to carry out the projection onto the tangent plane, we must obtain the normal. We compute

$$\partial_r \times \partial_\theta = (-r\cos(\theta)\sin(\phi)\cos(\phi), -r\sin(\theta)\sin(\phi)\cos(\phi), r\sin^2(\phi))$$

and hence

$$N = \frac{\partial_r \times \partial_\theta}{||\partial_r \times \partial_\theta||} = (-\cos(\theta)\cos(\phi), -\sin(\theta)\cos(\phi), \sin(\phi)).$$

Therefore,

$$\begin{aligned}
\nabla^T_{\partial_r}\partial_r &= (0,0,0) \\
\nabla^T_{\partial_\theta}\partial_r &= \nabla^T_{\partial_r}\partial_\theta = \pi^T(\overline{\nabla}_{\partial\theta}\partial_r) = \pi^T(\frac{1}{r}\partial_\theta) = \frac{1}{r}\partial_\theta \\
\nabla^T_{\partial_\theta}(\partial_\theta) &= \pi^T(\overline{\nabla}_{\partial\theta}\partial_\theta) = \pi^T(\cos(\phi)\sin(\phi)rN - r\sin(\phi)^2\partial_r) = -r\sin(\phi)^2\partial_r
\end{aligned}$$

as required.

One can note that the non-vanishing Chirstoffel symbols are

$$\begin{aligned}\Gamma^{\theta}_{\theta r} &= \Gamma^{\theta}_{r\theta} = \frac{1}{r} \\ \Gamma^{r}_{\theta\theta} &= -r\sin(\phi)^2\end{aligned}$$

so by the Christofeel formula, we can verify the above results as well.

$\square$

Question 1-5.

5. Let $\gamma\colon [0,2\pi] \to M$ be the circle $\gamma(t) = u(1,t)$. Calculate the parallel transport map $P_{0,2\pi}\colon T_{\gamma(0)}M \to T_{\gamma(2\pi)}M$.

Solution.
Let $\gamma(t) = u(1,t) = (\cos(t)\sin(\phi), \sin(t)\sin(\phi), \cos(\phi))$, and $V_0 = (V_0^r, V_0^\theta) \in T_{\gamma(0)}M$. From the transport equation is given by $D_tV = 0$, and the computations from $1-4$,

$$D_tV = \frac{dV^r}{dt}\partial_r + \frac{dV^\theta}{dt}\partial_\theta + V^r\partial_\theta - V^\theta\sin(\phi)^2\partial_r = 0.$$

Therefore, V must satisfy the following system of odes: $\dfrac{dV^r}{dt} = v^\theta\sin(\phi)^2$, $\dfrac{dV^\theta}{dt} = -V^r$, with initival conditions being $V(0) = V_0$. With separation of variables, we see that

$$\begin{aligned} V^r(t) &= V_0^r\cos(t\sin(\phi)) + V_0^\theta\sin(\phi)\sin(t\sin(\phi)) \\ V^\theta(t) &= V_0^\theta\cos(t\sin(\phi)) - \frac{V_0^r}{\sin(\phi)}\sin(t\sin(\phi)) \end{aligned}$$

uniquely solves the system. Therefore, we see that we can give an explicit charaterization of the transport map as

$$P_{0,2\pi}(V_0) \quad = \quad (\cos(2\pi\sin(\phi)), \sin(\phi)\sin(2\pi\sin(\phi)); -\frac{\sin(2\pi\sin(\phi))}{\sin(\phi)}, \cos(2\pi\sin(\phi)))(V_0^r, V_0^\theta)^T$$

for each $V_0 \in T_{\gamma(0)}M$, as required. $\square$