Random Graphs:
Final Exam

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

Every human activity, good or bad, except mathematics, must come to an end.
- Paul Erdos

Question 1.

pm-f-1.png

Solution.

Question 2.

pm-f-2.png

Solution. (a) Let $\{X_i\}_{i \in I}$ be the collection of all subsets of Ω, with size 3, such that the sum of the subset is precisely n, indexed by I. Let B_i be the event $X_i \subseteq R$. In particular, $A = \bigwedge_{i \in I} \overline{B_i}$. We note that

$$\epsilon \quad = \quad P(B_i) = p^3 = (cn^{-\frac{2}{3}})^3 = c^3 n^{-2} = o(1),$$

and

$$\mu \quad = \quad E[X] = \sum_{i \in I} P(B_i) = \binom{n-1}{2} c^3 n^{-2} \sim c^3,$$

as a simple case counting with $x = i$ for $i = 1, .., n-1$, shows that there are $\binom{n-1}{2}$ solutions to $x + y + z = n$, such that $x, y, z \in \Omega$, and x, y, z are distinct. To apply Janson, it remains to show that $\triangle = o(1)$. From the given combinatorial structure, we have the following equivalence:

$$\begin{aligned} i \sim j \quad &\iff \quad i \neq j \text{ and } X_i \cap X_j \neq \emptyset \\ &\iff \quad |X_i \cap X_j| = 1, \end{aligned}$$

as $|X_i \cap X_j| = 2$ implies that the triple is identical. Now, fix a solution (x^*, y^*, z^*). The number of solutions that share exactly one coordinate is clearly bounded above by $3n$. Therefore, the number of ordered pair (i, j), with $|X_i \cap X_j| = 1$ is $O(n^3)$, as there are $\Theta(n^2)$ of them in total. Therefore, it follows that

$$\begin{aligned} \triangle \quad &= \quad \sum_{i \sim j} P(B_i \wedge B_j) = \sum_{|X_i \cap X_j| = 1} P(B_i \wedge B_j) \\ &= \quad O(n^3 p^5) = O(n^3 c^5 n^{-\frac{10}{3}}) = O(c^5 n^{-\frac{1}{3}}) = o(1). \end{aligned}$$

As $\epsilon = o(1)$ and $\triangle = o(1)$, Janson gives

$$P(A) \to e^{-c^3},$$

as required.

(b) Choose $p = n^{-\frac{2}{3}} \ln(n)^{\frac{1}{3}}$ with foresight. Substituting the new parametrization to the above calculation gives

$$\epsilon = P(B_i) = p^3 = (n^{-\frac{2}{3}} \ln(n)^{\frac{1}{3}})^3 = n^{-2} \ln(n) = o(1),$$

and

$$\mu = E[X] = \sum_{i \in I} P(B_i) = \binom{n-1}{2} n^{-2} \ln(n) \sim \ln(n),$$

with

$$\triangle = O(n^{-\frac{1}{3}} \ln(n)) = o(1).$$

Hence, Janson gives

$$P(A) = n^{-1+o(1)},$$

as required.

$\square$

Question 3.

pm-f-3.png

Solution. □

Question 4.

pm-f-4.png

Solution.

Question 5.

pm-f-5.png

Solution. □

Question 6.

pm-f-6.png

Solution. □

Question 7.

pm-f-7.png

Solution. Let $\{A_i\}_{i\in I}$ be the collection of 4-cycles of G, and let $\{X_i\}_{i\in I}$ be the corresponding indicator random variables. In particular, $X = \sum_{i\in I} X_i$. Now, by linearity of expectation, it follows that

$$E[X] \quad = \quad \sum_{i\in I} P(A_i) \sim n^2p^4.$$

Therefore, with $p = \dfrac{1}{\sqrt{n}}$, it follows that

$\square$

Question 8.

pm-f-8.png

Solution. (a) Let X_i be the indicator of triangles, so $X = \sum_{i \in I} X_i$. By linearity of expectation, we obtain

$$E[X] = E[\sum_{i \in I} X_i] = \sum_{i \in I} E[X_i] = \binom{n}{3} p^3 = \binom{n}{3} \left(\frac{c}{n}\right)^3,$$

and asymptotically

$$E[X] \sim \frac{c^3}{6},$$

as required.

(b) From the definition of variance, it follows that

$$Var[X] = \sum_S Var[X_S] + \sum_{S \neq T} Cov[X_S, X_T].$$

In particular, the variance formula for discrete random variable yields

$$\sum_S Var[X_S] = \binom{n}{3} p^3 (1 - p^3).$$

Since $p = o(1)$, we have that $(1 - p^3) = o(1)$. Combining the result gives

$$Var[X_S] = p^3(1 - p^3) \sim p^3 \text{ and } Var[X_S] \sim E[X_S] \sim \frac{1}{6} c^3.$$

Now, observe that covariance is 0 for S, T pair, where $|S \cap T| \neq 2$. Now, for S, T pair, where $|S \cap T| = 2$, we have, by definition of covariance,

$$Cov(X_S, X_T) = E[X_S X_T] - E[X_S]E[X_T] = p^5 - p^6.$$

Since there are $\binom{n}{3}3(n-3)$ choices (fix the first triangle, pick the one that will not be shared, and choose the remaining one from the rest of the graph), we finally have

$$\sum_{S \neq T} Cov(X_S, X_T) \quad = \quad \binom{n}{3}3(n-3)(p^5 - p^6) = o(1).$$

Therefore, we can conclude that $Var[X] \sim \dfrac{c^3}{6}$ as well.

(c) Using Janson's inequality (to be precise a corolloary of it (pg.129), we have that as $n \to \infty$

$$P[X = 0] \to e^{\frac{c^3}{6}},$$

as required.

$\square$

Question 9.

pm-f-9.png

Solution. For notational convenience, we replace X_k with X. Let $\{X_i\}_{i\in I}$ be the set of indicators defined on each k-clique of a complete graph with n vertices, indexed by I. In particular, we have $\sum_{i\in I} X_i = X$. By linearity of expectation, it follows that

$$E[X] \quad = \quad E[\sum_{i\in I} X_i] = \sum_{i\in I} E[X_i] = \binom{n}{k}\left(\frac{1}{3}\right)^{\binom{k}{2}}.$$

Since k is fixed, as $n \to \infty$, we have

$$E[X] \quad \sim \quad \frac{n^k}{k!}(\frac{1}{3})^{\binom{k}{2}}$$

$\square$

Question 10.

pm-f-10.png

Solution. **(a)** We denote the event that X_i indicates as A_i. With the given setup, by linearity of expectation, we obtain

$$\begin{aligned} E[X] &= \sum_{0 \le i < m} E[X_i] = \sum_{0 \le i < m} P(A_i) \\ &= \sum_{0 \le i < m} 2^{-1-n} = 2^{-1}. \end{aligned}$$

(b)

(c)

(d) By Brun's Sieve (theorem 8.3.1 from PM), it follows that

□

Question 11.

pm-f-11.png

Solution. **(a)** For simplicity, consider the complete graph case in $G(n,p)$, where we have pre-exposed a square in the gradation without its diagonals being exposed just yet. Now, exposing a diagonal edge will increase the number of the triangles at least by 2, so the Lipschitz condition is not satisfied.

(b) Let Y be the max size family of edge disjoint triangles. Trivially, $X = 0 \iff Y = 0$. Suppose that an edge exposure introduced k new triangles to the current state of the graph. Consider the collection of families of all edge disjoint triangles at the given exposure. Each family can only increase by one, as they can pick at most 1 triangle out of the k new triangles. Therefore, each family of edge disjoint triangles increase at most by 1 and the maximum number thus increase at most by 1. We have shown that this graph theoretic property is Lipschitz under the edge exposure gradation.

(c) From the proof of theorem 7.3.2 from PM, which is a direct consequence of the current setup and the Azuma's inequality, we obtain

$$\begin{aligned} P[X=0] &= P[Y=0] \leq P[Y - E[Y] \leq -E[Y]] \\ &\leq e^{-\frac{E[Y]^2}{2\binom{n}{2}}} \leq e^{-\Theta(\frac{n^2}{k^8})} = e^{-\Theta(n^2)}, \end{aligned}$$

as required.

□

Question 12.

pm-f-12.png

Solution. (a)

(b) From LLL, we can show that

$$R(k,k) > \frac{\sqrt{2}}{e} k 2^{\frac{k}{2}} (1 + o(1)),$$

which is a factor of 2 improvement from the union bound argument.

(c)

$\square$

Question 13.

pm-f-13.png

Solution. □

Question 14.

pm-f-14.png

Solution. □