

PDE II: Problem Set II

Youngduck Choi *

Abstract

This work contains solutions for the problem set II.

Question 1-1.

① Consider a function $f \in L^1$ which admits a weak derivative in $L^1(\mathbb{R})$. Show that there exists a continuous function ψ which agrees with f almost everywhere.

Solution.

Let $g \in L^1(\mathbb{R})$ be the weak-derivative of f , and set $\psi(x) = \int_{-\infty}^x g(y)dy$. Then, $\psi \in C(\mathbb{R})$, and

$$\int (\psi - f)\phi = 0$$

for any $\phi \in C_0^\infty(\mathbb{R})$. Therefore, $\psi - f = c$ a.e. for some constant c . Since $\psi(x) - c$ is continuous, we are done. \square

*Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

Question 1-2.

- ② @ If $0 < s < 1$, find a function in $H^s(\mathbb{R}^2)$ which is not bounded.
- ⓑ If $s = 1$, find a function in $H^1(\mathbb{R}^2)$ which is not bounded (Think of a logarithmic singularity $\sim (\ln|x|)^\alpha$).
- ⓒ If $s \leq 1$, find a function in $H^s(\mathbb{R}^2)$ which is not bounded on any open set.

Solution.

(a) Set $f(x) = X(x)|x|^\alpha$ for some $\alpha \in (0, 1)$ and $X(x)$ defined as HW1-2. From HW1-2-b,

$$\hat{f}(\xi) \sim C_\alpha |\xi|^{-1-\alpha} + O(|\xi|^{-N})$$

and

$$|\langle \xi \rangle^s \hat{f}|^2 \sim |\xi|^{2s-2-2\alpha}.$$

Let $s \in (0, \frac{1}{2})$. Choose $\alpha \in (s - \frac{1}{2}, 0)$. Then, $\alpha > s - \frac{1}{2}$, so $f \in H^s(\mathbb{R})$ and f is unbounded, as $\alpha < 0$.

(b) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x) = \ln(\ln(1 + \frac{1}{|x|}))$ for $|x| < \frac{1}{e-1}$ and 0 otherwise, so f is unbounded. Observe that

$$\partial_i f = \frac{-x_i}{\ln(1 + \frac{1}{|x|})(|x|^3 + |x|^2)} \text{ in } B(0, \frac{1}{e-1}) \setminus \{0\}.$$

for $i = 1, 2$. Then, we can see that g_i for $i = 1, 2$ defined by $\frac{-x_i}{\ln(1 + \frac{1}{|x|})(|x|^3 + |x|^2)}$ for $0 < |x| < \frac{1}{e-1}$ and 0 otherwise are weak derivatives of f by testing against Schwartz class functions. Furthermore, as $r(\ln(r))^2 \sim r(\ln(\ln(1 + \frac{1}{r})))^2$ as $r \rightarrow 0^+$,

$$\int_{\mathbb{R}^2} |f|^2 = 2\pi \int_0^{\frac{1}{e-1}} \ln(\ln(1 + \frac{1}{r}))^2 r dr < \infty$$

and as $\frac{r^3}{(r^3 + r^2)^2 (\ln(1 + \frac{1}{r}))^2} \sim r$ as $r \rightarrow 0^+$

$$\int_{\mathbb{R}^2} |g_i|^2 = \pi \int_0^{\frac{1}{e-1}} \frac{r^3}{(r^3 + r^2)^2 (\ln(1 + \frac{1}{r}))^2} dr < \infty$$

for $i = 1, 2$. Therefore, $f \in H^1(\mathbb{R}^2)$ and is unbounded. By the trace estimate,

$$\|f|_{\mathbb{R}}\|_{H^{\frac{1}{2}}(\mathbb{R})} \leq C\|f\|_{H^1(\mathbb{R}^2)}.$$

Therefore, $f|_{\mathbb{R}} \in H^{\frac{1}{2}}(\mathbb{R})$ and unbounded.

(c) Set $g(x)$ be $f(x)$ from (a) for $0 < s < \frac{1}{2}$, and from (b) for $s = \frac{1}{2}$. Then, set

$$h(x) = \sum_{n=1}^{\infty} 2^{-n} g(x - a_n)$$

where $\{a_n\}$ is an enumeration of \mathbb{Q} . Then, $h(x)$ is unbounded on any open set, by density of \mathbb{Q} , and

$$\|h(x)\|_{H^s} \leq \sum_{n=1}^{\infty} 2^{-n} \|g(x - a_n)\|_{H^s} = \sum_{n=1}^{\infty} 2^{-n} \|g(x)\|_{H^s} = \|g(x)\|_{H^s} < \infty,$$

which shows $h(x) \in H^s(\mathbb{R})$. □

Question 1-3.

③ @ If $d \geq 2$, show that the trace inequality

$$\|f\|_{\mathbb{R}^{d-1}} \|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \leq C \|f\|_{H^1(\mathbb{R}^d)}$$

 cannot be improved by replacing $\frac{1}{2}$ by $\sigma > \frac{1}{2}$ on the ~~left~~ ^{right} hand side
 (give an example)
 @ We want to find a lifting operator which, for any $g \in H^\sigma(\mathbb{R}^{d-1})$, associates $Lg \in H^{\sigma+\frac{1}{2}}(\mathbb{R}^d)$ such that $(Lg)|_{\mathbb{R}^{d-1}} = g$
 (this shows that the trace operator is onto). Show that

$$L(v)(x', x_d) = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{ix' \xi'} X(x_d < \xi' >) \hat{v}(\xi') d\xi'$$

 has this property

Solution.

(a) Set $\hat{g}(\xi) = \frac{1}{(\sqrt{1+|\xi|^2})^\alpha}$, so

$$g \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}) \iff 2\alpha - 2\sigma \leq d-1 \iff \alpha > \frac{d}{2}.$$

Furthermore,

$$g \notin H^\sigma(\mathbb{R}^{d-1}) \iff 2\alpha - 2\sigma \leq d-1 \iff \alpha \leq \frac{d}{2} + \sigma - \frac{1}{2}$$

for any $\sigma > \frac{1}{2}$. Fix $\sigma > \frac{1}{2}$. Choose $\alpha \in (\frac{d}{2}, \frac{d}{2} + \sigma - \frac{1}{2}]$. From 3-(b),

$$Lg \in H^1(\mathbb{R}^d) \quad \text{and} \quad Tr Lg = g \quad \text{such that} \quad \|Tr Lg\|_{H^\sigma(\mathbb{R}^{d-1})} = \infty$$

so we are done.

(b) Set $\hat{X}(\xi) \in C_0^\infty(\mathbb{R})$, $\int \hat{X}(\xi) d\xi = \sqrt{2\pi}$, and $\text{supp } \hat{X} \subset [-1, 1]$, so $Tr L(v) = v$. Then,

$$\begin{aligned} L(v)(x', x_d) &= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{ix' \xi'} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix_d \xi_d} \hat{L}(v)(\xi', \xi_d) d\xi_d d\xi' \\ &= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{ix' \xi'} X(x_d < \xi' >) \hat{v}(\xi') d\xi'. \end{aligned}$$

Therefore,

$$X(x_d < \xi' >) \hat{V}(\xi') = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix_d \xi_d} \hat{L}(\xi', \xi_d) d\xi_d$$

so

$$\hat{V}(\xi') \frac{1}{< \xi' >} \hat{X}\left(\frac{\xi_d}{< \xi' >}\right) = \hat{V}(\xi') \widehat{X(x_d < \xi' >)}(\xi_d) = \hat{L}(v)(\xi', \xi_d).$$

Hence,

$$\begin{aligned} ||L(v)||_{H^{s+\frac{1}{2}}(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} < \xi >^{2s+1} |\widehat{L(v)}(\xi', \xi_d)|^2 d\xi \\ &\leq C \int_{\mathbb{R}^{d-1}} < \xi' >^{2s-1} |\hat{v}(\xi')|^2 \int_{< \xi' > \geq |\xi_d|} |\hat{X}\left(\frac{\xi_d}{< \xi' >}\right)|^2 d\xi_d d\xi' \\ &\leq C \int_{\mathbb{R}^{d-1}} < \xi' >^{2s} |\hat{v}(\xi')|^2 d\xi' < \infty \end{aligned}$$

so $L(v) \in H^{s+\frac{1}{2}}$ as required. □