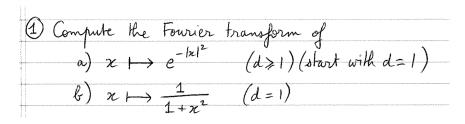
PDE II: Problem Set I

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Abstract

This work contains solutions for the problem set I.

Question 1-1.



Solution.

(a) Set $u = x^2$ and du = 2xdx. Then,

$$\int_{-\infty}^{\infty} e^{-x^2} = \int_{0}^{\infty} u^{-\frac{1}{2}} e^{-u} du = \Gamma(\frac{1}{2}).$$
 (1)

As

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$$

for any $s\in\mathbb{C},$ setting $s=\frac{1}{2}$ in the above and substituting to (1) gives

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}.$$
 (2)

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We now compute

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-|x|^2}
= \frac{1}{(2\pi)^{\frac{d}{2}}} \prod_{k=1}^d \int_{\mathbb{R}} e^{-(x_k + \frac{i\xi_k}{2})^2 - \frac{\xi_k^2}{4}} dx_k
= \frac{1}{(2\pi)^{\frac{d}{2}}} (\sqrt{\pi})^d e^{-\frac{|\xi|^2}{4}}
= 2^{-\frac{d}{2}} e^{-\frac{|x_i|^2}{4}}$$
(3)

for any $\xi \in \mathbb{R}^d$, where (3) follows from (2).

(b) Firstly,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \frac{1}{1+x^2} dx \tag{4}$$

for all $\xi \in \mathbb{R}$. Let $\xi < 0$. By Residue theroem,

$$\int_{H+C_R} \frac{e^{-iz\xi}}{1+z^2} dz = 2\pi i \text{Res}(\frac{e^{-iz\xi}}{1+z^2}; i) = \pi e^{\xi}$$

for all R > 0 sufficiently large, where C_R is the standard arc and H is the horizontal part of the upper half circle, oriented counter-clockwise. As

$$\left| \int_{C_R} \frac{e^{-iz\xi}}{1+z^2} dz \right| \leq \frac{R}{R^2 - 1} \int_0^{\pi} e^{R\xi \sin(\theta)} d\theta \leq \frac{\pi R}{R^2 - 1}$$

for all R > 0, taking $R \to \infty$ gives

$$\int_{\mathbb{D}} e^{-ix\xi} \frac{1}{1+x^2} dx = \pi e^{\xi}.$$

Similarly, for $\xi \geq 0$, considering the lower half circle containing -i gives

$$\int_{\mathbb{R}} e^{-ix\xi} \frac{1}{1+x^2} dx = \pi e^{-\xi}$$

and hence, by (4),

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{2}}e^{-|\xi|}$$

for all $\xi \in \mathbb{R}$.

Question 1-2.

(2) For $\chi \in \mathcal{E}^{\infty}(\mathbb{R})$, $\chi(x) = 1$ for $ x \leq 1$, define, for $x > -1$, $f(x) = \chi(x) x ^{\infty}$ for $x > -1$
a) For which $k \in IN$ & does $2^k f \in L'$? $\in L'$?
(ie: there exists a weak der., and it belongs to L'or L')
b) Show that $\hat{f}(\xi) \sim C_{\alpha} \xi ^{-\alpha-1} + O(\xi ^{-\alpha})$ for all N, as $ \xi \rightarrow \infty$, if $\alpha \in (-1, 0)$
c) What if $\alpha > 0$? Compare with the results of a): for which
TE WES & P. E. L.
d) State an extension of a), b), c) to higher dimension, and prove it.

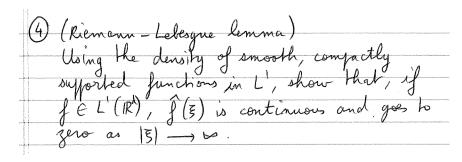
Solution. ddd

Question 1-3.

3 For y as	above,	let f	(x) = x	(n) e 4	wind	x e IR
For which	k does	and gray	€ L2	?		

Solution. ddd

Question 1-4.



Solution.

The key property in this problem is that smooth and compactly supported functions on \mathbb{R}^d are dense in $L^1(\mathbb{R}^d)$, and L^1 convergence gives uniform control on the Fourier domain.

Let $f \in L^1(\mathbb{R}^d), \xi \in \mathbb{R}^d$. Then,

$$|\hat{f}(\xi + \delta) - \hat{f}(\xi)| = \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot (\xi + \delta)} f(x) dx - \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx \right|$$

$$\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)| |e^{-ix \cdot (\xi + \delta)} - e^{-ix \cdot \xi}| dx$$

$$\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} 2f(x) dx \in L^1(\mathbb{R}^d)$$

for all $\delta \in \mathbb{R}^d$. As

$$|e^{-ix\cdot(\xi+\delta)} - e^{-ix\cdot\xi}| \to 0 \text{ as } \delta \to 0$$

for all $x \in \mathbb{R}^d$, by DCT,

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)| |e^{-ix \cdot (\xi + \delta)} - e^{-ix \cdot \xi}| dx \to 0 \text{ as } \delta \to 0$$

and hence,

$$\lim_{\delta \to 0} \hat{f}(\xi + \delta) - \hat{f}(\xi) = 0,$$

which shows that \hat{f} is continuous.

Let $f \in C_0^{\infty}(\mathbb{R}^d)$. Then,

$$|\hat{f}(\xi)| = \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx \right|$$

Therefore, the Riemann-Lebesgue lemma is true for $f \in C_0^{\infty}(\mathbb{R}^d)$. Now, suppose $f \in L^1(\mathbb{R}^d)$. Then, by density of $C_0^{\infty}(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$, we can choose $\{f_n\}_{n\in\mathbb{N}} \subset C_0^{\infty}(\mathbb{R}^d)$ such that $f_n \to_{L^1} f$ as $n \to \infty$. Then,

$$|\hat{f}(\xi) - \hat{f}_n(\xi)| = dd$$

Question 1-5.

(5) (Generalized Leibniz formula)
Show that the formula $(fg)' = fg + fg'$
remains true in the sense of weak derivatives
of $f,g \in L^{\infty}(\mathbb{R})$, and $f,g' \in L'(\mathbb{R})$.
(Use an approximation argument)

Solution.

We have

$$\int_{\mathbb{R}} f \phi' dx = -\int_{\mathbb{R}} f' \phi dx$$

and

$$\int_{\mathbb{R}} g\phi' dx = -\int_{\mathbb{R}} g'\phi dx$$

for any $\phi \in \mathscr{S}$. We wish to show that

$$\int_{\mathbb{R}} (fg)\phi'dx = \int_{\mathbb{R}} (f'g + fg')\phi dx$$

for any $\phi in \mathscr{S}$.