PDE II: Problem Set I

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Abstract

This work contains solutions for the problem set I.

Question 1-1.

① Consider a function $f \in L^1$ which admits a weak derivative in $L^1(\mathbb{R})$. Show that there exists a continuous function φ which agrees with f almost everywhere.

Solution.
Let $g \in L^1(\mathbb{R})$ be the weak-derivative of f, and set $\psi(x) \int_{-\infty}^{x} g(y)dy$. Then, $\psi \in C(\mathbb{R})$, and

$$\int (\psi - f)\phi = 0$$

for any $\phi \in C_0^\infty(\mathbb{R})$. Therefore, $\psi - f = c$ a.e. for some constant c. Since $\phi(x) - c$ is continuous, we are done. □

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Question 1-2.

② ⓐ If $0<s<1$, find a function in $H^s(\mathbb{R}^2)$ which is not bounded.
ⓑ If $s=1$, find a function in $H^1(\mathbb{R}^2)$ which is not bounded (Think of a logarithmic singularity $\sim(\ln|x|)^\alpha$).
ⓒ If $s\leq 1$, find a function in $H^s(\mathbb{R}^2)$ which is not bounded on <u>any</u> open set

Solution.
(a) Set $f(x) = X(x)|x|^\alpha$ for some $\alpha \in (0,1)$ and $X(x)$ defined as $HW1-2$. From $HW1-2-b$,

$$\hat{f}(\xi) \quad \sim \quad C_\alpha|\xi|^{-1-\alpha} + O(|\xi|^{-N})$$

and

$$| < \xi >^s \hat{f}|^2 \quad \sim \quad |\xi|^{2s-2-2\alpha}.$$

Let $s \in (0, \frac{1}{2})$. Choose $\alpha \in (s - \frac{1}{2}, 0)$. Then, $\alpha > s - \frac{1}{2}$, so $f \in H^s(\mathbb{R})$ and f is unbounded, as $\alpha < 0$.

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x) = \ln(\ln(1 + \frac{1}{|x|}))$ for $|x| < \frac{1}{e-1}$ and 0 otherwise, so f is unbounded. Observe that

$$\partial_i f \quad = \quad \frac{-x_i}{\ln(1+\frac{1}{|x|})(|x|^3+|x|^2)} \quad \text{in} \quad B(0, \frac{1}{e-1}) \setminus \{0\}.$$

for $i = 1, 2$. Then, we can see that g_i for $i = 1, 2$ defined by $\frac{-x_i}{\ln(1+\frac{1}{|x|})(|x|^3+|x|^2)}$ for $0 < |x| < \frac{1}{e-1}$ and 0 otherwise are weak derivatives of f by testing against Schwartz class functions. Furthermore, as $r(\ln(r))^2 \sim r(\ln(\ln(1+\frac{1}{r}))^2$ as $r \to 0^+$,

$$\int_{\mathbb{R}^2} |f|^2 = 2\pi \int_0^{\frac{1}{e-1}} \ln(\ln(1+\frac{1}{r}))^2 r dr < \infty$$

and as $\frac{r^3}{(r^3+r^2)^2(\ln(1+\frac{1}{r}))^2} \sim r$ as $r \to 0^+$

$$\int_{\mathbb{R}^2} |g_i|^2 = \pi \int_0^{\frac{1}{e-1}} \frac{r^3}{(r^3+r^2)^2(\ln(1+\frac{1}{r}))^2} dr < \infty$$

for $i = 1, 2$. Therefore, $f \in H^1(\mathbb{R}^2)$ and is unbounded. By the trace estimate,

$$||f|_{\mathbb{R}}||_{H^{\frac{1}{2}}(\mathbb{R})} \leq C||f||_{H^1(\mathbb{R}^2)}.$$

Therefore, $f|_{\mathbb{R}} \in H^{\frac{1}{2}}(\mathbb{R})$ and unbounded.

(c) Set $g(x)$ be $f(x)$ from (a) for $0 < s < \dfrac{1}{2}$, and from (b) for $s = \dfrac{1}{2}$. Then, set

$$h(x) = \sum_{n=1}^{\infty} 2^{-n} g(x - a_n)$$

where $\{a_n\}$ is an enumeration of $\mathbb{Q}$. Then, $h(x)$ is unbounded on any open set, by density of $\mathbb{Q}$, and

$$||h(x)||_{H^s} \leq \sum_{n=1}^{\infty} 2^{-n} ||g(x - a_n)||_{H^s} = \sum_{n=1}^{\infty} 2^{-n} ||g(x)||_{H^s} = ||g(x)||_{H^s} < \infty,$$

which shows $h(x) \in H^s(\mathbb{R})$. $\square$

Question 1-3.

③ ⓐ If $d \geqslant 2$, show that the trace inequality

$$\|f|_{\mathbb{R}^{d-1}}\|_{H^{\frac{1}{2}}(\mathbb{R}^{d-1})} \leqslant C\|f\|_{H^1(\mathbb{R}^d)}$$

cannot be improved by replacing $\frac{1}{2}$ by $\sigma > \frac{1}{2}$ on the ~~right~~ left-hand side.
(give an example)

ⓑ We want to find a lifting operator which, to any $g \in H^s(\mathbb{R}^{d-1})$, associates $Lg \in H^{s+\frac{1}{2}}(\mathbb{R}^d)$ such that $(Lg)|_{\mathbb{R}^{d-1}} = g$
(this shows that the trace operator is onto). Show that

$$L(v)(x', x_d) = \frac{1}{(2\pi)^{(d-1)/2}} \int_{\mathbb{R}^{d-1}} e^{ix'\xi'} \chi(x_d \langle \xi' \rangle) \hat{v}(\xi') d\xi'$$

has this property.

Solution.
(a) Set $\hat{g}(\xi) = \dfrac{1}{(\sqrt{1+|\xi|^2})^\alpha}$, so

$$g \in H^{\frac{1}{2}}(\mathbb{R}^{d-1}) \iff 2\alpha - 2\sigma \leq d-1 \iff \alpha > \frac{d}{2}.$$

Furthermore,

$$g \notin H^\sigma(\mathbb{R}^{d-1}) \iff 2\alpha - 2\sigma \leq d-1 \iff \alpha \leq \frac{d}{2} + \sigma - \frac{1}{2}$$

for any $\sigma > \frac{1}{2}$. Fix $\sigma > \frac{1}{2}$. Choose $\alpha \in (\frac{d}{2}, \frac{d}{2} + \sigma - \frac{1}{2}]$. From 3-(b),

$$Lg \in H^1(\mathbb{R}^d) \quad \text{and} \quad TrLg = g \text{ such that } ||TrLg||_{H^\sigma}(\mathbb{R}^{d-1}) = \infty$$

so we are done.

(b) Set $\hat{X}(\xi) \in C_0^\infty(\mathbb{R})$, $\int \hat{X}(\xi)d\xi = \sqrt{2\pi}$, and $\text{supp}\hat{X} \subset [-1,1]$, so $TrL(v) = v$. Then,

$$\begin{aligned} L(v)(x', x_d) &= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{ix'\xi'} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix_d\xi_d} \hat{L}(v)(\xi', \xi_d) d\xi_d d\xi' \\ &= \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{ix'\xi'} X(x_d < \xi' >) \hat{V}(\xi') d\xi'. \end{aligned}$$

Therefore,

$$X(x_d < \xi' >)\hat{V}(\xi') = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix_d\xi_d} \hat{L}(\xi', \xi_d) d\xi_d$$

so

$$\hat{V}(\xi') \frac{1}{< \xi' >} \hat{X}(\frac{\xi_d}{< \xi' >}) = \hat{V}(\xi') \widehat{X(x_d} < \xi' >)(\xi_d) = \hat{L}(v)(\xi', \xi_d).$$

Hence,

$$\begin{aligned} ||L(v)||_{H^{s+\frac{1}{2}}}(\mathbb{R}^d) &= \int_{\mathbb{R}^d} < \xi >^{2s+1} |\widehat{L(v)}(\xi', \xi_d)|^2 d\xi \\ &\leq C \int_{\mathbb{R}^{d-1}} < \xi' >^{2s-1} |\hat{v}(\xi')|^2 \int_{<\xi'> \geq |\xi_d|} |\hat{X}(\frac{\xi_d}{< \xi' >})|^2 d\xi_d d\xi' \\ &\leq C \int_{\mathbb{R}^{d-1}} < \xi' >^{2s} |\hat{v}(\xi')|^2 d\xi' < \infty \end{aligned}$$

so $L(v) \in H^{s+\frac{1}{2}}$ as required. $\square$