# PDE II: Problem Set I

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**Abstract**

This work contains solutions for the problem set I.

**Question 1-1.**

(1) Compute the Fourier transform of

a) $x \longmapsto e^{-|x|^2}$ $(d \geq 1)$ (start with $d = 1$)

b) $x \longmapsto \frac{1}{1+x^2}$ $(d = 1)$

**Solution.**

**(a)** Set $u = x^2$ and $du = 2xdx$. Then,

$$\int_{-\infty}^{\infty} e^{-x^2} = \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du = \Gamma(\frac{1}{2}). \tag{1}$$

As

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$$

for any $s \in \mathbb{C}$, setting $s = \frac{1}{2}$ in the above and substituting to (1) gives

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}. \tag{2}$$

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We now compute

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}e^{-|x|^2} \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}\prod_{k=1}^{d}\int_{\mathbb{R}} e^{-(x_k+\frac{i\xi_k}{2})^2-\frac{\xi_k^2}{4}}dx_k \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}}(\sqrt{\pi})^d e^{-\frac{|\xi|^2}{4}} \qquad (3) \\
&= 2^{-\frac{d}{2}}e^{-\frac{|xi|^2}{4}}
\end{aligned}$$

for any $\xi \in \mathbb{R}^d$, where (3) follows from (2).

**(b)** Firstly,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\frac{1}{1+x^2}dx \qquad (4)$$

for all $\xi \in \mathbb{R}$. Let $\xi < 0$. By Residue theroem,

$$\int_{H+C_R}\frac{e^{-iz\xi}}{1+z^2}dz = 2\pi i\mathrm{Res}(\frac{e^{-iz\xi}}{1+z^2};i) = \pi e^{\xi}$$

for all $R > 0$ sufficiently large, where $C_R$ is the standard arc and $H$ is the horizontal part of the upper half circle, oriented counter-clockwise. As

$$\left|\int_{C_R}\frac{e^{-iz\xi}}{1+z^2}dz\right| \le \frac{R}{R^2-1}\int_0^{\pi} e^{R\xi\sin(\theta)}d\theta \le \frac{\pi R}{R^2-1}$$

for all $R > 0$, taking $R \to \infty$ gives

$$\int_{\mathbb{R}} e^{-ix\xi}\frac{1}{1+x^2}dx = \pi e^{\xi}.$$

Similarly, for $\xi \ge 0$, considering the lower half circle containing $-i$ gives

$$\int_{\mathbb{R}} e^{-ix\xi}\frac{1}{1+x^2}dx = \pi e^{-\xi}$$

and hence, by (6),

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{2}}e^{-|\xi|}$$

for all $\xi \in \mathbb{R}$. $\square$

**Question 1-2.**

(2) For $\chi \in \mathscr{C}_0^\infty(\mathbb{R})$, $\chi(x) = 1$ for $|x| \leq 1$, define, for $\alpha > -1$, $f(x) = \chi(x)|x|^\alpha$ for $\alpha > -1$

a) For which $k \in \mathbb{N}$ ~~f~~ does $\partial_x^k f \in L^1$? $\in L^2$? (ie: there exists a weak der., and it belongs to $L^1$ or $L^2$)

b) Show that $\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-1} + O(|\xi|^{-N})$ for all $N$, as $|\xi| \to \infty$, if $\alpha \in (-1, 0)$

c) What if $\alpha \geqslant 0$? Compare with the results of a): for which $k$ does $\xi^k \hat{f} \in L^2$?

d) State an extension of a), b), c) to higher dimension, and prove it.

**Solution.**

Now, consider $\alpha \in \mathbb{Z}_+$. For $\alpha$ even, we see

$$f(x) \quad = \quad X(x)|x|^\alpha = X(x)x^\alpha \in C_0^\infty(\mathbb{R}).$$

Thus, $\partial_x^k f \in L^1 \cap L^2$ for all $k \in \mathbb{Z}_+$. For $\alpha$ odd, as $f \in C^{\alpha-1}$, and $f^{(\alpha-1)}$ has $\text{sgn}(x)(\alpha-1)!$ on $[-1,1]$ as weak-derivative, which we know does not have a weak derivative, $\partial_x^k f \in L^1 \cap L^2$ for any $k \leq \alpha$.

Now, let $-1 < \alpha < 0$. Then, if $f$ has a weak-derivative, then, for any $\phi \in \mathscr{S}$, and $\epsilon > 0$,

$$\begin{aligned} \int_{\mathbb{R}} X(x)|x|^\alpha \phi'(x)dx &= \int_\epsilon^\infty + \int_0^\epsilon + \int_{-\epsilon}^0 + \int_{-\infty}^{-\epsilon} X(x)x^\alpha \phi'(x)dx \\ &= \epsilon^\alpha(\phi(-\epsilon) - \phi(\epsilon)) + \int_0^\epsilon + \int_{-\epsilon}^0 - \int_{|x|>\epsilon} g(x)\phi(x)dx \qquad (5) \end{aligned}$$

where (6) holds by integration by parts, and

$$g(x) = X'(x)|X|^\alpha + \text{sgn}(x)\alpha X(x)|x|^{\alpha-1}.$$

Therefore, as $g \notin L^1_{\text{loc}}$, and $P.V_{|x|>\epsilon}g(x)$ is not a function, $\partial_x^k f \in L^1 \iff k = 0$, and $\partial_x^k f \in L^2 \iff k = 0$ and $\alpha > -\dfrac{1}{2}$.

Now, consider $\alpha \geq 0$ and $\alpha \notin \mathbb{Z}$. Observe that $X(x)|x|^\alpha \in C^{\lceil \alpha \rceil}$ and

$$(X(x)|x|^\alpha)^{\lceil \alpha \rceil} = \binom{\alpha}{\lfloor \alpha \rfloor} \lfloor \alpha \rfloor! (\mathrm{sgn}(x))^{\lfloor x \rfloor} |x|^{\alpha - \lfloor \alpha \rfloor} \text{ on } [0,1].$$

Now, for any $\phi \in \mathscr{S}$ and $\epsilon > 0$,

$$\begin{aligned}\int_{\mathbb{R}} f^{(\lfloor \alpha \rfloor)}(x)\phi'(x)dx &= -f^{(\lfloor \alpha \rfloor)}(x)\phi(x)|_{\epsilon}^{-\epsilon} + \binom{\alpha}{\lfloor \alpha \rfloor} \lfloor \alpha \rfloor! (\mathrm{sgn}(x))^{\lfloor \alpha \rfloor} \\ &\quad \int_{-\epsilon}^{\epsilon} |x|^{\alpha - \lfloor \alpha \rfloor} \phi^{'}](x)dx - \int_{|x|>\epsilon} \phi(x)g(x) \to \int_{\mathbb{R}} \phi(x)g(x)dx \text{ as } \epsilon \to 0^+\end{aligned}$$

where $g(x) = h(x)|x|^{\alpha - \lfloor \alpha \rfloor - 1}$ on $[-1,1]$ for some $h$ smooth and compactly supported. Therefore, by the above discussion, $f$ only has weak derivative upto $\lfloor \alpha \rfloor$, and $\partial_x^k f \in L^1$ for $k = 0, 1..., \lfloor \alpha \rfloor$ and $\partial_x^k f \in L^2$ for $k = 0, ..., \lceil \alpha \rceil$ for $(\alpha) \leq \frac{1}{2}$ and $k = 0, ..., \lfloor \alpha \rfloor$ for $(\alpha) > \frac{1}{2}$.

**(b)** We compute

$$\begin{aligned}\widehat{|x|^\alpha}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x|^\alpha e^{-ix\xi} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |y|^\alpha |\xi|^{-\alpha-1} e^{-iy} dy \qquad (6) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |y|^\alpha e^{-iy} dy |\xi|^{-\alpha-1} =: C_\alpha |\xi|^{-\alpha-1}\end{aligned}$$

where (6) holds by a change of variable of $y = x\xi$. Set $g(x) = f - |x|^\alpha = (X(x) - 1)|x|^\alpha$, so

$$\hat{g}(\xi) = \widehat{f - |x|^\alpha}(\xi) = \hat{f}(\xi) - C_\alpha |\xi|^{-\alpha-1}.$$

Since

$$\partial_x^N g = \sum_{k=0}^{N} \binom{N}{k} (X(x) - 1)^{(N-k)} \binom{\alpha}{k} k! \mathrm{sgn}(x)^k |x|^{\alpha-k}$$

for any $N$, we have

$$|||\xi|^N \hat{g}||_{L^\infty} \leq C||\partial_x^N g||_{L^1} < \infty$$

and hence

$$\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-1} + O(|\xi|^{-N})$$

for any $N$.

**(c)** Let $\alpha \notin \mathbb{Z}$. Then, $f$ has weak derivative up to $\lceil \alpha \rceil$.

$$\partial_x^{\lceil \alpha \rceil} f(x) = \sum_{k=0}^{\lceil \alpha \rceil - 1} X(x)^{\lceil \alpha \rceil - k}(x) \binom{\alpha}{k} k! |x|^{\alpha-k} (\mathrm{sgn} x)^k \binom{\lceil \alpha \rceil}{k} + CX(x)|x|^{\alpha - \lceil x \rceil}.$$

Now, by (b),

$$\widehat{\partial_x^{\lceil\alpha\rceil} f}(\xi) \quad \sim \quad C_\alpha |\xi|^{-1\lceil\alpha\rceil-\alpha} + O(|\xi|^{-N})$$

so

$$\hat{f}(\xi) \quad \sim \quad C_\alpha |\xi|^{-1-\alpha} + O(|\xi|^{-N})$$

and hence $|\xi|^k \hat{f} \in L^2$ iff $k - 1 - \alpha < -\dfrac{1}{2}$. Now, for $\alpha$ even, if $f \in C_0^\infty(\mathbb{R})$, then $|||\xi|^N \hat{f}||_\infty \leq C|||\partial_x^N f||_{L^1} < \infty$, so $\hat{f} \sim O(|\xi|^{-N})$ for all $N$, and $|\xi|^k \hat{f} \in L^2$ for all $k$.

For $\alpha$ even, if $f \in C_0^\infty(\mathbb{R})$, then $||\xi|^N \hat{f}||_\infty \leq C||\partial_x^N f||_{L^1} < \infty$, so $\hat{f} \sim O(|\xi|^{-N})$ for all $N$, and $|\xi|^k \hat{f} \in L^2$ for all $k$.

For $\alpha$ odd, we have that $f$ has weak derivative only up to order $\alpha$. So, $|||\xi|^N \hat{f}||_{L^1} \leq C||\partial_x^N f||_{L^1} < \infty$, and $\hat{f}(\xi) \sim O(|\xi|^{-N})$ for $N \leq \alpha + 1$. Therefore, $|\xi|^k \hat{f} \in L^2$ for $k \leq \alpha$.

**(d)** The proof does not change for higher dimension from the first parts.

a) For $\alpha$ odd, $\partial_x^k f \in L^1 \cap L^2$ for all $k \leq \alpha$. For $\alpha$ even, $\partial_x^k f \in L^1 \cap L^2$ for all $k$. For $\alpha \notin \mathbb{Z}$, if $k < \alpha + \dfrac{d}{2}$, $|\xi|^k \hat{f} \in L^2 \implies \partial_x^k f \in L^2 \implies \partial_x^k f \in L^1$.

b) $\alpha \in (-d, 0)$, $\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-d} + O(|\xi|^{-N})$ for all $N$.

c) $\alpha \notin \mathbb{Z}$. If $\alpha + \dfrac{d}{2} > k$, then, $\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-d} + O(|\xi|^{-N})$ and $|\xi|^k \hat{f} \in L^2$. $\alpha$ even is the same. For $\alpha$ odd, $\hat{f} \sim O(|\xi|^{-\alpha-d})$ and $|\xi|^k \hat{f} \in L^2$ for $k < \alpha + \dfrac{d}{2}$.

**Question 1-3.**

(3) For $\chi$ as above, let $f(x) = \chi(x) e^{ix^\alpha}$ $\alpha \in \mathbb{R}$
For which $k$ does $\partial^x_k f \in L^2$ ?

**Solution.**
By the same line of reasoning as above, for $\alpha \in \mathbb{Z}$, $\partial_x^k f \in L^2$ for all $k \in \mathbb{Z}_+$. Let $\alpha \notin \mathbb{Z}$. Integrating by parts, for any $\phi \in \mathscr{S}$, and $\epsilon > 0$,

$$\int_{\mathbb{R}} X(x) e^{ix^\alpha} \phi'(x) dx \quad = \quad -X(x)e^{ix^\alpha}\phi(x)|_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{\epsilon} X(x)e^{ix^\alpha}\phi(x)dx - \int_{|x|>\epsilon} \phi(x) g(x) dx$$

where

$$g(x) \quad = \quad X'(x)e^{ix^\alpha} + \operatorname{sgn}(x) i\alpha X(x) x^{\alpha-1} e^{ix^\alpha}$$

for $x \in \mathbb{R}$. Taking $\epsilon \to 0^+$ $f' \sim x^{\alpha-1}$, and similarly $f^{(k)} \sim x^{\alpha-k}$. Therefore, $\partial_x^k f \in L^2 \iff 2\alpha - 2k > -1$ and $\alpha + \dfrac{1}{2} > k$. $\square$

**Question 1-4.**

(4) (Riemann – Lebesgue lemma)
Using the density of smooth, compactly supported functions in $L^1$, show that, if $f \in L^1(\mathbb{R}^d)$, $\hat{f}(\xi)$ is continuous and goes to zero as $|\xi| \longrightarrow \infty$.

**Solution.**
The key property in this problem is that smooth and compactly supported functions on $\mathbb{R}^d$ are dense in $L^1(\mathbb{R}^d)$, and $L^1$ convergence gives uniform control on the Fourier domain.

Let $f \in L^1(\mathbb{R}^d), \xi \in \mathbb{R}^d$. Then,

$$\begin{aligned}
|\hat{f}(\xi+\delta) - \hat{f}(\xi)| &= \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot(\xi+\delta)} f(x)dx - \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx \right| \\
&\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)||e^{-ix\cdot(\xi+\delta)} - e^{-ix\cdot\xi}|dx \\
&\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} 2|f(x)|dx \in L^1(\mathbb{R}^d)
\end{aligned}$$

for all $\delta \in \mathbb{R}^d$. As

$$|e^{-ix\cdot(\xi+\delta)} - e^{-ix\cdot\xi}| \to 0 \quad \text{as} \quad \delta \to 0$$

for all $x \in \mathbb{R}^d$, by DCT,

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)||e^{-ix\cdot(\xi+\delta)} - e^{-ix\cdot\xi}|dx \to 0 \quad \text{as} \quad \delta \to 0$$

and hence,

$$\lim_{\delta\to 0} \hat{f}(\xi+\delta) - \hat{f}(\xi) \quad = \quad 0,$$

which shows that $\hat{f}$ is continuous.

Let $f \in C_0^\infty(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$. Then,

$$\begin{aligned}
|\hat{f}(\xi)| &= \left|\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{-ix\cdot\xi}f(x)dx\right| = \left|-\frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla(\frac{e^{ix\cdot\xi}}{-|\xi|^2})\right| &(7)\\
&\leq \frac{1}{(2\pi)^{\frac{d}{2}}|\xi|^2}\int_{\mathbb{R}^d}\left|\nabla f(x)\cdot(-i\xi e^{-ix\cdot\xi})\right|dx \leq \frac{1}{(2\pi)^{\frac{d}{2}}|\xi|^2}\int_{\mathbb{R}^d}|\nabla f(x)\cdot\xi|\,dx\\
&\leq \frac{1}{(2\pi)^{\frac{d}{2}}|\xi|^2}\sum_{i=1}^{d}|\xi_i|\int_{\mathbb{R}}|f_i(x)|dx \leq \frac{1}{(2\pi)^{\frac{d}{2}}|\xi|^2}\max_{i\leq d}|\xi_i|\sum_{i\leq d}\int_{\mathbb{R}^d}|f_i(x)|dx\\
&\lesssim \frac{1}{|\xi|}\max_{i\leq d}||f_i||_{L_1} &(8)
\end{aligned}$$

where (7) holds by integration by parts, and (8) holds by the topological equivalence of the norms in $\mathbb{R}^d$. By the compactness assumption, taking $|\xi| \to \infty$ sends RHS to 0. Therefore, the Riemann-Lebesgue lemma holds for $f \in C_0^\infty(\mathbb{R}^d)$. Now, suppose $f \in L^1(\mathbb{R}^d)$. Then, by density of $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$, we can choose $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that $f_n \to_{L^1} f$ as $n \to \infty$. Observe that there is some $C > 0$ such that for all $\xi \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$|\hat{f}(\xi) - \hat{f}_n(\xi)| \leq C||f_n - f||$$

and hence $\hat{f}_n$ converges uniformly to $\hat{f}$. Fix $\epsilon > 0$. Then, we can choose $f_n$ such that $||\hat{f}_n - \hat{f}||_\infty < \frac{\epsilon}{2}$ and choose $M$ large enough that $|\hat{f}_n(\xi)| < \frac{\epsilon}{2}$ for all $\xi$ such that $|\xi| > M$. Therefore, for any $\xi$ with $|\xi| > M$, $|\hat{f}(\xi)| < \epsilon$, so

$$\hat{f}(\xi) \to 0 \quad \text{as} |\xi| \to \infty,$$

as required. □

**Question 1-5.**

(5) (Generalized Leibniz formula)
Show that the formula $(fg)' = f'g + fg'$
remains true in the sense of weak derivatives
if $f, g \in L^\infty(\mathbb{R})$, and $f', g' \in L^1(\mathbb{R})$.
(Use an approximation argument)

**Solution.**
Consider an approximate identity $\{\phi_\epsilon\}_{\epsilon>0} \subset C_0^\infty$. Then,

$$(\phi_\epsilon * f)^{'}(x) \;\; = \;\; \int_{\mathbb{R}} \phi_\epsilon^{'}(x-y) = -(-1)\int_{\mathbb{R}} \phi_\epsilon(x-y)f^{'}(y)dy = \phi_\epsilon * f'$$

for any $x \in \mathbb{R}, \epsilon > 0$, so $(\phi_\epsilon * f)^{'} = \phi_\epsilon * f'$. Similarly, $(\phi_\epsilon * g)^{'} = \phi_\epsilon * g'$. Furthermore, by a property of approximate identity

$$\phi_\epsilon * g' \to_{L^1} g' \qquad \text{and} \qquad \phi_\epsilon * f' \to_{L^1} f'.$$

Now, for any $\phi \in \mathscr{S}$, and $\epsilon > 0$,

$$\begin{aligned}
\int_{\mathbb{R}} (\phi_\epsilon * f \phi_\epsilon * g)\phi^{'} dx &= \int_{\mathbb{R}} (\phi_\epsilon * f \phi_\epsilon * g)' \phi dx \\
&= \int_{\mathbb{R}} (\phi_\epsilon * f' \phi_\epsilon * g + \phi_\epsilon * g' \phi_\epsilon * f)\phi dx
\end{aligned}$$

where (9) holds by $\phi_\epsilon * f, \phi_\epsilon * g \in C_0^\infty$. Taking $\epsilon \to 0$, by DCT,

$$\int_{\mathbb{R}} fg\phi^{'} \;\; = \;\; \int_{\mathbb{R}} (f'g + g'f)\phi$$

for all $\phi \in \mathscr{S}$, so $(fg)' = f'g + fg'$ as required. $\square$