PDE II: Problem Set II

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Abstract

This work contains solutions for the problem set II.

Question 1-1.

① We want to prove the Hardy inequality.

$$\left(\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2}dx\right)^{1/2} \leq \frac{2}{d-2}\left(\int |\nabla f|^2 dx\right)^{1/2} \quad \text{if } d \geq 3$$

a) Does it follow from the Sobolev embedding theorem

b) Check that the scaling is satisfied (consider $f_\lambda \cdots$)

c) Can it hold for $d = 1, 2$, $f \in \mathcal{C}_0^\infty$?

d) Prove this identity for $f \in \mathcal{C}_0^\infty$ by using the identity $(x \cdot \nabla)|x|^{-2} = -2|x|^{-2}$ $(x \cdot \nabla = \Sigma x^i \partial_i)$ to integrate by parts in $\int \frac{|f(x)|^2}{|x|^2}$, followed by Cauchy-Schwarz.

e) Extend this result to any $f \in H^1(\mathbb{R}^d)$, $d \geq 3$

Solution.

(a) It does not follow from sobolev embedding, as $||\cdot||_{H^1(\mathbb{R}^d)}$ is not equivalent to $||\nabla \cdot||_{L^2(\mathbb{R}_d)}$.

(b) Let f satisfy the inequality. Then, by a change of variables,

$$\begin{aligned}
\left(\int_{\mathbb{R}^d} \frac{|f(\lambda x)|^2}{|x|^2}dx\right)^{\frac{1}{2}} &= |\lambda|^{\frac{2-d}{2}}\left(\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2}dx\right)^{\frac{1}{2}} \\
&\leq \frac{2}{d-2}|\lambda|^{\frac{2-d}{2}}\left(\int_{\mathbb{R}} |\nabla f(x)|^2 dx\right)^{\frac{1}{2}} \\
&= \frac{2}{d-2}\left(\int_{\mathbb{R}} |\nabla f(\lambda x)|^2 dx\right)^{\frac{1}{2}}
\end{aligned}$$

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(c) For $d = 1$, the proof in part (d) would work. For $d = 2$, let $f = 1$ on $B(0, 1)$ and $f \in C_0^\infty$. Then, RHS is finite, but the LHS is infinite, so the inequality does not for for $d = 2$.

(d) By integrating by parts,

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx &= -\frac{1}{2} \sum_{i=1}^{d} \int_{\mathbb{R}^d} |f(x)|^2 x^i \partial_i |x|^{-2} dx \\
&= \frac{1}{2} \sum_{i=1}^{d} \int_{\mathbb{R}^d} \partial_i (|f(x)|^2 x_i) |x|^{-2} dx \\
&= \frac{1}{2} \int_{\mathbb{R}^d} \frac{2|f(x)|}{|x|^2} x \cdot \nabla f(x) dx + \frac{d}{2} \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx
\end{aligned}$$

and hence, by Cauchy-Schwartz,

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx &\leq \frac{2}{d-2} \int_{\mathbb{R}^d} \frac{|f(x)|}{|x|^2} x \cdot \nabla f dx \\
&\leq \frac{2}{d-2} (\int_{\mathbb{R}^d} |\nabla f(x)|^2 dx)^{\frac{1}{2}} (\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, dividing by the second term on the RHS, we see that the inequality holds for the case considered.

(e) Fix $f \in H^1(\mathbb{R}^d)$. By density of $C_0^\infty(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)$, we can choose a sequence in C_0^∞, $\{f_n\}$ that converges to f in $H^1(\mathbb{R}^d)$. Then, we can choose a subsequence of $\{f_n\}$ denoted by $\{g_n\}$ such that g_n converges to f a.e. and ∇g_n converges to f a.e. Now, by DCT, we see that the inequality can be extended. $\square$

Question 1-2.

(2) Assume that $\varphi \in \mathcal{C}_0^\infty$, with $|\mathrm{Jac}\,\varphi| = \left|\det \frac{\partial \varphi}{\partial x}\right| < 1$,
and let $\psi : \mathbb{R}^d \to \mathbb{R}^d$
$x \longmapsto x + \varphi(x)$.
Prove then that $f \circ \psi \in H^1(\mathbb{R}^d)$ if $f \in H^1(\mathbb{R}^d)$.
What are optimal conditions on ψ for such a result to hold?

Solution.
As $|\det \frac{\partial \phi}{\partial x}| < 1$,

$$||\partial_i (f \circ \psi)||_{L^2}^2 \;=\; \int_{\mathbb{R}^d} |\triangle f|^2 + |\triangle f \cdot \frac{\partial \phi}{\partial x_i}|^2 dx \le C||\triangle f||_{L^2}^2.$$

Let $y = \Psi(x)$, and $\inf |\det \frac{\partial \Psi}{\partial x}| > 0$. Then, by inverse function theorem, we have h such that $\sup |\det h(y)| < \infty$, and $h(y) = (D\Psi)^{-1}(y)$. Then,

$$\int_{\mathbb{R}^d} |f \circ \Psi(x)|^2 dx \le \sup |\det h(y)| ||f||_{L^2}^2 < \infty.$$

Now, consider $d \ge 3$, $f(x) = |x|^\alpha X(x)$, $X(x) \in C_0^\infty(\mathbb{R}^d)$, $X = 1$ on $B(0,1)$, $\alpha > 1 - \frac{d}{2}$. We from hw1 that $f \in H^1(\mathbb{R}^d)$. Now, set $\Psi(x) = |x|^2 x$ if $|x| < \frac{1}{2}$ and x if $|x| > 1$. Then, we see $f \circ \Psi \notin H^1(\mathbb{R}^d)$ is equivalent $\alpha \ge \frac{1}{3} - \frac{d}{6}$. As α can be chosen to be in $(1 - \frac{d}{2}, \frac{1}{3} - \frac{d}{6}]$, we see that the condition is optimal. $\square$

Question 1-3.

③ Prove that the $H^s(\mathbb{R}^d)$ norm is equivalent to the following norm:

$$\left(\int_{\mathbb{R}^d} |u(x)|^2 dx\right)^{1/2} + \left(\iint \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}} dx\, dy\right)^{1/2}$$

if $s \in (0,1)$ and $d \geqslant 2$. [Argue in Fourier space]

Solution.
We compute

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|e^{iy\xi}-1|^2}{|y|^{d+2s}} dy &\leq C\left(\int_{|y|<\frac{1}{|\xi|}} \frac{|y|^2|\xi|^2}{|y|^{d+2s}} dy + \int_{|y|>\frac{1}{|\xi|}} \frac{1}{|y|^{d+2s}} dy\right) \\
&= C\left(|\xi|^2 \int_0^{\frac{1}{|\xi|}} r^{-d-2s+2} r^{d-1} dr + \int_{\frac{1}{|\xi|}}^{\infty} r^{-d-2s+d-1} dr\right) \\
&= C\left(|\xi|^2 r^{-2s+2}\Big|_0^{\frac{1}{|\xi|}} + r^{2s}\Big|_{\frac{1}{|\xi|}}^{\infty}\right) \sim < \xi >^{2s} .
\end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^d} \frac{|e^{iy\xi}-1|^2}{|y|^{d+2s}} dy \geq C|\xi|^{2s} \sim < \xi >^{2s} .$$

Since, by Fubini,

$$\begin{aligned}
\int\int \frac{|u(x+y)-u(x)|^2}{|y|^{d+2s}} dxdy &= \int\int |e^{iy\xi}-1|^2 |\hat{u}(\xi)|^2 d\xi \frac{1}{|y|^{d+2s}} dy \\
&= \int\int |e^{iy\xi}-1|^2 \frac{1}{|y|^{d+2s}} dy |\hat{u}(\xi)|^2 d\xi,
\end{aligned}$$

and the established asymtotics, we see the norm equivalence. □

Question 1-4.

(4) We want to prove the Rellich Kondrakov theorem.
Let K be a compact set of $\mathbb{R}^d$.

a) For $s>0$, $\varepsilon>0$, $s-\varepsilon>0$, show that the embedding $H^s(\mathbb{R}^d) \subset H^{s-\varepsilon}(\mathbb{R}^d)$ is not compact (ie: there are bounded sequences in $H^s(\mathbb{R}^d)$ which do not admit any convergent subsequence in $H^{s-\varepsilon}$).

b) If $0<s<d/2$, $p=\frac{2d}{d-2s}$, show that the embedding $H^s(K) \subset L^p(\mathbb{R}^d)$ is not compact.

c) Prove that $H^s(K) \subset\subset H^{s-\varepsilon}(\mathbb{R}^d)$ (compact embedding)
[Hint: show that it suffices to find a subsequence f_n s.t. $\hat{f_n} \to \hat{f}$ uniformly on $B(0,N)$ for all N, and then use Arzela-Ascoli].

d) Prove that $H^s(\mathbb{T}^d) \subset\subset H^{s-\varepsilon}(\mathbb{T}^d)$
[~~So~~ Argue in Fourier space; the definition of $H^s(\mathbb{T}^d)$ is identical to that of $H^s(\mathbb{R}^d)$ replacing the continuous frequencies ξ by discrete frequencies k, and $\int_{\mathbb{R}^d}$ by $\sum_{\mathbb{Z}^d}$]

Solution.
(a) Let $B_n = B(0, \dfrac{1}{n})$, for $n \in \mathbb{N}$, $\Omega_1 = B_1$, and $\Omega_n = B_n \setminus B_{n-1}$ for $n \geq 2$. Furthermore, for each $n \in \mathbb{N}$, let

$$c_n = \left(\int_{\Omega_n} <\xi>^{2s-2\epsilon} \, d\xi \right)^{-\frac{1}{2}} \text{ and } \hat{f_n} = c_n 1_{\Omega_n}.$$

Then,

$$||f_n||_{H^{s-\epsilon}(\mathbb{R})} = \left(\int_{\mathbb{R}^d} (<\xi>^{s-\epsilon} \hat{f}(\xi))^2 d\xi \right)^{\frac{1}{2}} = \left(\int_{\Omega_n} c_n^2 <\xi>^{2s-2\epsilon} \, d\xi \right)^{\frac{1}{2}} = 1$$

for any $n \in \mathbb{N}$, and

$$<f_n, f_m>_{H^{s-\epsilon}(\mathbb{R}^d)} = \int_{\mathbb{R}_d} <\xi>^{2s-2\epsilon} \hat{f_n}(\xi)\hat{f_m}(\xi) d\xi = 0 \tag{1}$$

for any $n, m \in \mathbb{N}$ with $n \neq m$, where (1) follows from the fact that $\hat{f_n}$ and $\hat{f_m}$ have disjoint supports, if $n \neq m$. Therefore,

$$\begin{aligned} ||f_n - f_m||^2_{H^{s-\epsilon}(\mathbb{R}^d)} &= < f_n - f_m, f_n - f_m > \\ &= ||f_n||^2_{H^{s-\epsilon}(\mathbb{R}^d)} + ||f_m||^2_{H^{s-\epsilon}(\mathbb{R}^d)} - 2 < f_n, f_m >= 2 \end{aligned}$$

for any $n, m \in \mathbb{N}$, with $n \neq m$, so $\{f_n\}$ does not have a convergent subsequence in $H^{s-\epsilon}(\mathbb{R}^d)$. Furthermore,

$$||f_n||_{H^s(\mathbb{R}^d)} = \int_{\Omega_n} {c_n}^2 < \xi >^{2s} d\xi = c_n^2 \int_{\Omega_n} {c_n}^2 < \xi >^{2s} d\xi \leq 2^\epsilon c_n^2 \int_{\Omega_N} < \xi >^{2s-2\epsilon} d\xi = 2^\epsilon \qquad (2)$$

for any $n \in \mathbb{N}$, where (2) holds by $< \xi > \leq 2$ for any $\xi \in B(0,1)$. Therefore, the embedding is not compact.

(b) Let $f \in H^s(K)$ such that $||f||_{H^s} > 0$, and set $f_n = n^{\frac{d}{p}} f(n\cdot)$ for any $n \in \mathbb{N}$. Then, $||f_n||_{L^p} = ||f||_{L^p}$ for all n, and $f_n \to 0$ almost everywhere. We see that $\{f_n\}$ does not have any convergence subsequence, as if some $\{f_{n_k}\}$ converges to some g in L^p, then there is a further subsequence that convergence to g, so $g = 0$, which is a contradiction, as $||g||_{L^p} = ||f||_{L^p} \geq ||f||_{H^s(K)} > 0$, by continuity of the norm. Now,

$$\begin{aligned} ||f_n||^2_{H^s(K)} &= \int < \xi >^{2s} n^{2d(\frac{1}{p}-1)} |f(\frac{\xi}{n})|^2 d\xi \\ &\leq n^{2d(\frac{1}{p}-\frac{1}{2})+2s} \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \\ &= ||f||^2_{H_s(K)} \end{aligned}$$

for all n, so we see that the inclusion is not compact.

(c) Let $\{f_n\}$ be a sequence in $H^s(K)$ such that there exists a constant C such that $||f_n||_{H^s(K)} \leq C$ for all $n \in \mathbb{N}$. Then,

$$||\hat{f}_n||_L^\infty \leq (2\pi)^{\frac{d}{2}} \sqrt{m(K)} \sup_n ||f_n||_{H^s(K)} < \infty$$

for all n. Furthermore, for any $\xi_1, \xi_2 \in \mathbb{R}^d$, and $n \in \mathbb{N}$,

$$\begin{aligned} |\hat{f}_n(\xi_1) - \hat{f}_n(\xi_2)| &= C| \int_K (e^{-i\xi_1 \cdot x} - e^{-i\xi_2 \cdot x}) f_n(x) dx| \\ &\leq C \int_K |f_n||\xi_1 - \xi_2||x| dx \leq C|\xi_1 - \xi_2|. \end{aligned}$$

Therefore, $\{\hat{f}_n\}$ is uniformly bounded on a compact set, and equicontinuous. Now, upto diagonal extraction, we have a subsequence denoted as $\{\hat{f}_n\}$, which converges uniformly on $B(0,k)$ for all k. Let g be defined by a.e. convergence of $\{\hat{f}_n\}$. Fix $\delta > 0$. Choose N large enough such that

$$\frac{1}{(1+N^2)^\epsilon} \leq \frac{\delta}{\delta \sup_n ||f_n||^2_{H_s}} \quad \text{and} \quad \int <\xi>^{2s-2\epsilon} |g|^2 X_{B(0,N)^c} < \delta.$$

and hence

$$||f_n||^2_{H^{s-\epsilon}} \quad \leq \quad \int <\xi>^{2s-2\epsilon} X_{B(0,N)}|\hat{f}_n - \hat{f}|^2 + 2\int <\xi>^{2s-2\epsilon} X_{B(0,N)^c}|\hat{f}_n|^2|g|^2) \leq C\delta.$$

So the embedding is compact.

(d) Observe that

$$\sum <\xi>^{2s-2\epsilon} |\hat{f}(\xi) - \hat{f}_n(\xi)|^2 \quad \leq \quad \sum_{\xi| \leq N} +2\sum_{|\xi|>N} <\xi>^{2s-2\epsilon} |\hat{f}(\xi)|^2 + 2\sum_{|\xi|>N} <\xi>^{2s} |\hat{f}_n(\xi)|^2 \frac{1}{(1+N^2)^\epsilon}$$

for all n. Therefore, it's again enough to choose for fixed $\delta > 0$, large enough N such that

$$\frac{\sup_n ||f_n||^2_{H_s}}{(1+N^2)^\epsilon} < \frac{\delta}{8} \quad \text{and} \sum_{|\xi|>N} <\xi>^{2s-2\epsilon} |\hat{f}(\xi)|^2 < \frac{\delta}{8}.$$

This can be done in the analogous way with part c, by employing Weistrauss theorem and diagonal extraction. $\square$