PDE II: Problem Set I

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Abstract

This work contains solutions for the problem set I.

Question 1-1.

① Compute the Fourier transform of

a) $x \longmapsto e^{-|x|^2}$ $(d \geqslant 1)$ (start with $d=1$)

b) $x \longmapsto \dfrac{1}{1+x^2}$ $(d=1)$

Solution.

(a) Set $u = x^2$ and $du = 2xdx$. Then,

$$\int_{-\infty}^{\infty} e^{-x^2} = \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du = \Gamma(\frac{1}{2}). \quad (1)$$

As

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$$

for any $s \in \mathbb{C}$, setting $s = \dfrac{1}{2}$ in the above and substituting to (1) gives

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}. \quad (2)$$

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We now compute

$$
\begin{aligned}
\hat{f}(\xi) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-|x|^2} \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \prod_{k=1}^{d} \int_{\mathbb{R}} e^{-(x_k+\frac{i\xi_k}{2})^2-\frac{\xi_k^2}{4}} dx_k \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} (\sqrt{\pi})^d e^{-\frac{|\xi|^2}{4}} \qquad (3) \\
&= 2^{-\frac{d}{2}} e^{-\frac{|xi|^2}{4}}
\end{aligned}
$$

for any $\xi \in \mathbb{R}^d$, where (3) follows from (2).

(b) Firstly,

$$
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \frac{1}{1+x^2} dx \qquad (4)
$$

for all $\xi \in \mathbb{R}$. Let $\xi < 0$. By Residue theroem,

$$
\int_{H+C_R} \frac{e^{-iz\xi}}{1+z^2} dz = 2\pi i \mathrm{Res}(\frac{e^{-iz\xi}}{1+z^2}; i) = \pi e^{\xi}
$$

for all $R > 0$ sufficiently large, where C_R is the standard arc and H is the horizontal part of the upper half circle, oriented counter-clockwise. As

$$
\left| \int_{C_R} \frac{e^{-iz\xi}}{1+z^2} dz \right| \leq \frac{R}{R^2-1} \int_0^{\pi} e^{R\xi\sin(\theta)} d\theta \leq \frac{\pi R}{R^2-1}
$$

for all $R > 0$, taking $R \to \infty$ gives

$$
\int_{\mathbb{R}} e^{-ix\xi} \frac{1}{1+x^2} dx = \pi e^{\xi}.
$$

Similarly, for $\xi \geq 0$, considering the lower half circle containing $-i$ gives

$$
\int_{\mathbb{R}} e^{-ix\xi} \frac{1}{1+x^2} dx = \pi e^{-\xi}
$$

and hence, by (4),

$$
\hat{f}(\xi) = \sqrt{\frac{\pi}{2}} e^{-|\xi|}
$$

for all $\xi \in \mathbb{R}$.

Question 1-2.

(2) For $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi(x) = 1$ for $|x| \leq 1$, define,
for $\alpha > -1$, $f(x) = \chi(x)|x|^\alpha$ for $\alpha > -1$

a) For which $k \in \mathbb{N}$ ~~f~~ does $\partial_x^k f \in L^1$? $\in L^2$?
(ie: there exists a weak der., and it belongs to L^1 or L^2)

b) Show that $\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-1} + O(|\xi|^{-N})$
for all N, as $|\xi| \to \infty$, if $\alpha \in (-1, 0)$

c) What if $\alpha \geq 0$?
Compare with the results of a): for which k does $\xi^k \hat{f} \in L^2$?

d) State an extension of a), b), c) to higher dimension, and prove it.

Solution.
ddd

Question 1-3.

(3) For χ as above, let $f(x) = \chi(x)\, e^{i x^{\alpha}}$ $\alpha \in \mathbb{R}$

For which k does $\partial^{x}_{k} f \in L^2$?

Solution.

ddd

Question 1-4.

4 (Riemann – Lebesgue lemma)
Using the density of smooth, compactly supported functions in L^1, show that, if $f \in L^1(\mathbb{R}^d)$, $\hat{f}(\xi)$ is continuous and goes to zero as $|\xi| \longrightarrow \infty$.

Solution.
The key property in this problem is that smooth and compactly supported functions on $\mathbb{R}^d$ are dense in $L^1(\mathbb{R}^d)$, and L^1 convergence gives uniform control on the Fourier domain.

Let $f \in L^1(\mathbb{R}^d), \xi \in \mathbb{R}^d$. Then,

$$
\begin{aligned}
|\hat{f}(\xi+\delta) - \hat{f}(\xi)| &= \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot(\xi+\delta)} f(x) dx - \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx \right| \\
&\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)||e^{-ix\cdot(\xi+\delta)} - e^{-ix\cdot\xi}| dx \\
&\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} 2f(x) dx \in L^1(\mathbb{R}^d)
\end{aligned}
$$

for all $\delta \in \mathbb{R}^d$. As

$$
|e^{-ix\cdot(\xi+\delta)} - e^{-ix\cdot\xi}| \to 0 \quad \text{as} \quad \delta \to 0
$$

for all $x \in \mathbb{R}^d$, by DCT,

$$
\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)||e^{-ix\cdot(\xi+\delta)} - e^{-ix\cdot\xi}| dx \to 0 \quad \text{as} \quad \delta \to 0
$$

and hence,

$$
\lim_{\delta \to 0} \hat{f}(\xi+\delta) - \hat{f}(\xi) = 0,
$$

which shows that $\hat{f}$ is continuous.
Let $f \in C_0^\infty(\mathbb{R}^d)$. Then,

$$
|\hat{f}(\xi)| = \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx \right|
$$

Therefore, the Riemann-Lebesgue lemma is true for $f \in C_0^\infty(\mathbb{R}^d)$. Now, suppose $f \in L^1(\mathbb{R}^d)$. Then, by density of $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$, we can choose $\{f_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that $f_n \to_{L^1} f$ as $n \to \infty$. Then,

$$|\hat{f}(\xi) - \hat{f}_n(\xi)| \quad = \quad dd$$

Question 1-5.

(5) (Generalized Leibniz formula)
Show that the formula $(fg)' = f'g + fg'$
remains true in the sense of weak derivatives
if $f, g \in L^\infty(\mathbb{R})$, and $f', g' \in L^1(\mathbb{R})$.
(Use an approximation argument)

Solution.
We have

$$\int_{\mathbb{R}} f\phi' \, dx \quad = \quad -\int_{\mathbb{R}} f' \phi dx$$

and

$$\int_{\mathbb{R}} g\phi' \, dx \quad = \quad -\int_{\mathbb{R}} g' \phi dx$$

for any $\phi \in \mathscr{S}$. We wish to show that

$$\int_{\mathbb{R}} (fg)\phi' \, dx \quad = \quad \int_{\mathbb{R}} (f'g + fg')\phi dx$$

for any $\phi in \mathscr{S}$.