

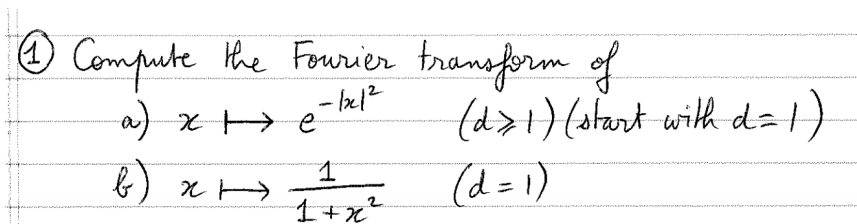
PDE II: Problem Set I

Youngduck Choi *

Abstract

This work contains solutions for the problem set I.

Question 1-1.



Solution.

(a) Set $u = x^2$ and $du = 2x dx$. Then,

$$\int_{-\infty}^{\infty} e^{-x^2} = \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du = \Gamma\left(\frac{1}{2}\right). \quad (1)$$

As

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$$

for any $s \in \mathbb{C}$, setting $s = \frac{1}{2}$ in the above and substituting to (1) gives

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}. \quad (2)$$

*Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

We now compute

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-|x|^2} \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \prod_{k=1}^d \int_{\mathbb{R}} e^{-(x_k + \frac{i\xi_k}{2})^2 - \frac{\xi_k^2}{4}} dx_k \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} (\sqrt{\pi})^d e^{-\frac{|\xi|^2}{4}} \\
&= 2^{-\frac{d}{2}} e^{-\frac{|xi|^2}{4}}
\end{aligned} \tag{3}$$

for any $\xi \in \mathbb{R}^d$, where (3) follows from (2).

(b) Firstly,

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \frac{1}{1+x^2} dx \tag{4}$$

for all $\xi \in \mathbb{R}$. Let $\xi < 0$. By Residue theroem,

$$\int_{H+C_R} \frac{e^{-iz\xi}}{1+z^2} dz = 2\pi i \text{Res}\left(\frac{e^{-iz\xi}}{1+z^2}; i\right) = \pi e^{\xi}$$

for all $R > 0$ sufficiently large, where C_R is the standard arc and H is the horizontal part of the upper half circle, oriented counter-clockwise. As

$$\left| \int_{C_R} \frac{e^{-iz\xi}}{1+z^2} dz \right| \leq \frac{R}{R^2-1} \int_0^\pi e^{R\xi \sin(\theta)} d\theta \leq \frac{\pi R}{R^2-1}$$

for all $R > 0$, taking $R \rightarrow \infty$ gives

$$\int_{\mathbb{R}} e^{-ix\xi} \frac{1}{1+x^2} dx = \pi e^{\xi}.$$

Similarly, for $\xi \geq 0$, considering the lower half circle containing $-i$ gives

$$\int_{\mathbb{R}} e^{-ix\xi} \frac{1}{1+x^2} dx = \pi e^{-\xi}$$

and hence, by (6),

$$\hat{f}(\xi) = \sqrt{\frac{\pi}{2}} e^{-|\xi|}$$

for all $\xi \in \mathbb{R}$. □

Question 1-2.

- ② For $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\chi(x) = 1$ for $|x| \leq 1$, define, for $\alpha > -1$, $f(x) = \chi(x)|x|^\alpha$ for $\alpha > -1$
- a) For which $k \in \mathbb{N}$ does $\partial_x^k f \in L^1? \in L^2?$
(ie: there exists a weak der., and it belongs to L^1 or L^2)
- b) Show that $\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-1} + O(|\xi|^{-N})$ for all N , as $|\xi| \rightarrow \infty$, if $\alpha \in (-1, 0)$
- c) What if $\alpha \geq 0$?
Compare with the results of a): for which k does $\xi^k \hat{f} \in L^2$?
- d) State an extension of a), b), c) to higher dimension, and prove it.

Solution.

Now, consider $\alpha \in \mathbb{Z}_+$. For α even, we see

$$f(x) = X(x)|x|^\alpha = X(x)x^\alpha \in C_0^\infty(\mathbb{R}).$$

Thus, $\partial_x^k f \in L^1 \cap L^2$ for all $k \in \mathbb{Z}_+$. For α odd, as $f \in C^{\alpha-1}$, and $f^{(\alpha-1)}$ has $\text{sgn}(x)(\alpha-1)!$ on $[-1, 1]$ as weak-derivative, which we know does not have a weak derivative, $\partial_x^k f \in L^1 \cap L^2$ for any $k \leq \alpha$.

Now, let $-1 < \alpha < 0$. Then, if f has a weak-derivative, then, for any $\phi \in \mathcal{S}$, and $\epsilon > 0$,

$$\begin{aligned} \int_{\mathbb{R}} X(x)|x|^\alpha \phi'(x) dx &= \int_{\epsilon}^{\infty} + \int_0^{\epsilon} + \int_{-\epsilon}^0 + \int_{-\infty}^{-\epsilon} X(x)x^\alpha \phi'(x) dx \\ &= \epsilon^\alpha (\phi(-\epsilon) - \phi(\epsilon)) + \int_0^{\epsilon} + \int_{-\epsilon}^0 - \int_{|x|>\epsilon} g(x)\phi(x) dx \end{aligned} \quad (5)$$

where (6) holds by integration by parts, and

$$g(x) = X'(x)|X|^\alpha + \text{sgn}(x)\alpha X(x)|x|^{\alpha-1}.$$

Therefore, as $g \notin L_{\text{loc}}^1$, and $P.V._{|x|>\epsilon} g(x)$ is not a function, $\partial_x^k f \in L^1 \iff k = 0$, and $\partial_x^k f \in L^2 \iff k = 0$ and $\alpha > -\frac{1}{2}$.

Now, consider $\alpha \geq 0$ and $\alpha \notin \mathbb{Z}$. Observe that $X(x)|x|^\alpha \in C^{\lceil \alpha \rceil}$ and

$$(X(x)|x|^\alpha)^{\lceil \alpha \rceil} = \binom{\alpha}{\lfloor \alpha \rfloor} \lfloor \alpha \rfloor! (\operatorname{sgn}(x))^{\lfloor \alpha \rfloor} |x|^{\alpha - \lfloor \alpha \rfloor} \quad \text{on } [0, 1].$$

Now, for any $\phi \in \mathcal{S}$ and $\epsilon > 0$,

$$\begin{aligned} \int_{\mathbb{R}} f^{(\lfloor \alpha \rfloor)}(x) \phi'(x) dx &= -f^{(\lfloor \alpha \rfloor)}(x) \phi(x)|_{-\epsilon}^{-\epsilon} + \binom{\alpha}{\lfloor \alpha \rfloor} \lfloor \alpha \rfloor! (\operatorname{sgn}(x))^{\lfloor \alpha \rfloor} \\ &\quad \int_{-\epsilon}^{\epsilon} |x|^{\alpha - \lfloor \alpha \rfloor} \phi'(x) dx - \int_{|x| > \epsilon} \phi(x) g(x) \rightarrow \int_{\mathbb{R}} \phi(x) g(x) dx \quad \text{as } \epsilon \rightarrow 0^+ \end{aligned}$$

where $g(x) = h(x)|x|^{\alpha - \lfloor \alpha \rfloor - 1}$ on $[-1, 1]$ for some h smooth and compactly supported. Therefore, by the above discussion, f only has weak derivative upto $\lfloor \alpha \rfloor$, and $\partial_x^k f \in L^1$ for $k = 0, 1, \dots, \lfloor \alpha \rfloor$ and $\partial_x^k f \in L^2$ for $k = 0, \dots, \lceil \alpha \rceil$ for $(\alpha) \leq \frac{1}{2}$ and $k = 0, \dots, \lfloor \alpha \rfloor$ for $(\alpha) > \frac{1}{2}$.

(b) We compute

$$\begin{aligned} \widehat{|x|^\alpha}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |x|^\alpha e^{-ix\xi} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |y|^\alpha |\xi|^{-\alpha-1} e^{-iy} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |y|^\alpha e^{-iy} dy |\xi|^{-\alpha-1} =: C_\alpha |\xi|^{-\alpha-1} \end{aligned} \tag{6}$$

where (6) holds by a change of variable of $y = x\xi$. Set $g(x) = f - |x|^\alpha = (X(x) - 1)|x|^\alpha$, so

$$\hat{g}(\xi) = \widehat{f - |x|^\alpha}(\xi) = \hat{f}(\xi) - C_\alpha |\xi|^{-\alpha-1}.$$

Since

$$\partial_x^N g = \sum_{k=0}^N \binom{N}{k} (X(x) - 1)^{(N-k)} \binom{\alpha}{k} k! \operatorname{sgn}(x)^k |x|^{\alpha-k}$$

for any N , we have

$$|||\xi|^N \hat{g}||_{L^\infty} \leq C ||\partial_x^N g||_{L^1} < \infty$$

and hence

$$\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-1} + O(|\xi|^{-N})$$

for any N .

(c) Let $\alpha \notin \mathbb{Z}$. Then, f has weak derivative up to $\lceil \alpha \rceil$.

$$\partial_x^{\lceil \alpha \rceil} f(x) = \sum_{k=0}^{\lceil \alpha \rceil - 1} X(x)^{\lceil \alpha \rceil - k}(x) \binom{\alpha}{k} k! |x|^{\alpha-k} (\operatorname{sgn} x)^k \binom{\lceil \alpha \rceil}{k} + C X(x) |x|^{\alpha - \lceil \alpha \rceil}.$$

Now, by (b),

$$\widehat{\partial_x^{[\alpha]} f}(\xi) \sim C_\alpha |\xi|^{-1[\alpha]-\alpha} + O(|\xi|^{-N})$$

so

$$\hat{f}(\xi) \sim C_\alpha |\xi|^{-1-\alpha} + O(|\xi|^{-N})$$

and hence $|\xi|^k \hat{f} \in L^2$ iff $k - 1 - \alpha < -\frac{1}{2}$. Now, for α even, if $f \in C_0^\infty(\mathbb{R})$, then $|||\xi|^N \hat{f}||_\infty \leq C |||\partial_x^N f||_{L^1} < \infty$, so $\hat{f} \sim O(|\xi|^{-N})$ for all N , and $|\xi|^k \hat{f} \in L^2$ for all k .

For α even, if $f \in C_0^\infty(\mathbb{R})$, then $|||\xi|^N \hat{f}||_\infty \leq C |||\partial_x^N f||_{L^1} < \infty$, so $\hat{f} \sim O(|\xi|^{-N})$ for all N , and $|\xi|^k \hat{f} \in L^2$ for all k .

For α odd, we have that f has weak derivative only up to order α . So, $|||\xi|^N \hat{f}||_{L^1} \leq C |||\partial_x^N f||_{L^1} < \infty$, and $\hat{f}(\xi) \sim O(|\xi|^{-N})$ for $N \leq \alpha + 1$. Therefore, $|\xi|^k \hat{f} \in L^2$ for $k \leq \alpha$.

(d) The proof does not change for higher dimension from the first parts.

a) For α odd, $\partial_x^k f \in L^1 \cap L^2$ for all $k \leq \alpha$. For α even, $\partial_x^k f \in L^1 \cap L^2$ for all k . For $\alpha \notin \mathbb{Z}$, if $k < \alpha + \frac{d}{2}$, $|\xi|^k \hat{f} \in L^2 \implies \partial_x^k f \in L^2 \implies \partial_x^k f \in L^1$.

b) $\alpha \in (-d, 0)$, $\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-d} + O(|\xi|^{-N})$ for all N .

c) $\alpha \notin \mathbb{Z}$. If $\alpha + \frac{d}{2} > k$, then, $\hat{f}(\xi) \sim C_\alpha |\xi|^{-\alpha-d} + O(|\xi|^{-N})$ and $|\xi|^k \hat{f} \in L^2$. α even is the same.

For α odd, $\hat{f} \sim O(|\xi|^{-\alpha-d})$ and $|\xi|^k \hat{f} \in L^2$ for $k < \alpha + \frac{d}{2}$.

Question 1-3.

③ For γ as above, let $f(x) = \chi(x) e^{i\gamma x}$ $x \in \mathbb{R}$
 For which k does $\partial_x^k f \in L^2$?

Solution.

By the same line of reasoning as above, for $\alpha \in \mathbb{Z}$, $\partial_x^k f \in L^2$ for all $k \in \mathbb{Z}_+$. Let $\alpha \notin \mathbb{Z}$. Integrating by parts, for any $\phi \in \mathcal{S}$, and $\epsilon > 0$,

$$\int_{\mathbb{R}} X(x) e^{ix^\alpha} \phi'(x) dx = -X(x) e^{ix^\alpha} \phi(x) \Big|_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{\epsilon} X(x) e^{ix^\alpha} \phi(x) dx - \int_{|x| > \epsilon} \phi(x) g(x) dx$$

where

$$g(x) = X'(x) e^{ix^\alpha} + \operatorname{sgn}(x) i \alpha X(x) x^{\alpha-1} e^{ix^\alpha}$$

for $x \in \mathbb{R}$. Taking $\epsilon \rightarrow 0^+$ $f' \sim x^{\alpha-1}$, and similarly $f^{(k)} \sim x^{\alpha-k}$. Therefore, $\partial_x^k f \in L^2 \iff 2\alpha - 2k > -1$ and $\alpha + \frac{1}{2} > k$. □

Question 1-4.

④ (Riemann-Lebesgue lemma)
Using the density of smooth, compactly supported functions in L^1 , show that, if $f \in L^1(\mathbb{R}^d)$, $\hat{f}(\xi)$ is continuous and goes to zero as $|\xi| \rightarrow \infty$.

Solution.

The key property in this problem is that smooth and compactly supported functions on \mathbb{R}^d are dense in $L^1(\mathbb{R}^d)$, and L^1 convergence gives uniform control on the Fourier domain.

Let $f \in L^1(\mathbb{R}^d)$, $\xi \in \mathbb{R}^d$. Then,

$$\begin{aligned} |\hat{f}(\xi + \delta) - \hat{f}(\xi)| &= \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot (\xi + \delta)} f(x) dx - \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx \right| \\ &\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)| |e^{-ix \cdot (\xi + \delta)} - e^{-ix \cdot \xi}| dx \\ &\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} 2|f(x)| dx \in L^1(\mathbb{R}^d) \end{aligned}$$

for all $\delta \in \mathbb{R}^d$. As

$$|e^{-ix \cdot (\xi + \delta)} - e^{-ix \cdot \xi}| \rightarrow 0 \text{ as } \delta \rightarrow 0$$

for all $x \in \mathbb{R}^d$, by DCT,

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |f(x)| |e^{-ix \cdot (\xi + \delta)} - e^{-ix \cdot \xi}| dx \rightarrow 0 \text{ as } \delta \rightarrow 0$$

and hence,

$$\lim_{\delta \rightarrow 0} \hat{f}(\xi + \delta) - \hat{f}(\xi) = 0,$$

which shows that \hat{f} is continuous.

Let $f \in C_0^\infty(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$. Then,

$$|\hat{f}(\xi)| = \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx \right| = \left| -\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla \left(\frac{e^{ix \cdot \xi}}{-|\xi|^2} \right) \right| \quad (7)$$

$$\begin{aligned} &\leq \frac{1}{(2\pi)^{\frac{d}{2}} |\xi|^2} \int_{\mathbb{R}^d} \left| \nabla f(x) \cdot (-i\xi e^{-ix \cdot \xi}) \right| dx \leq \frac{1}{(2\pi)^{\frac{d}{2}} |\xi|^2} \int_{\mathbb{R}^d} |\nabla f(x) \cdot \xi| dx \\ &\leq \frac{1}{(2\pi)^{\frac{d}{2}} |\xi|^2} \sum_{i=1}^d |\xi_i| \int_{\mathbb{R}^d} |f_i(x)| dx \leq \frac{1}{(2\pi)^{\frac{d}{2}} |\xi|^2} \max_{i \leq d} |\xi_i| \sum_{i \leq d} \int_{\mathbb{R}^d} |f_i(x)| dx \\ &\lesssim \frac{1}{|\xi|} \max_{i \leq d} \|f_i\|_{L^1} \end{aligned} \quad (8)$$

where (7) holds by integration by parts, and (8) holds by the topological equivalence of the norms in \mathbb{R}^d . By the compactness assumption, taking $|\xi| \rightarrow \infty$ sends RHS to 0. Therefore, the Riemann-Lebesgue lemma holds for $f \in C_0^\infty(\mathbb{R}^d)$. Now, suppose $f \in L^1(\mathbb{R}^d)$. Then, by density of $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mathbb{R}^d)$, we can choose $\{f_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ such that $f_n \rightarrow_{L^1} f$ as $n \rightarrow \infty$. Observe that there is some $C > 0$ such that for all $\xi \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$|\hat{f}(\xi) - \hat{f}_n(\xi)| \leq C \|f_n - f\|$$

and hence \hat{f}_n converges uniformly to \hat{f} . Fix $\epsilon > 0$. Then, we can choose f_n such that $\|\hat{f}_n - \hat{f}\|_\infty < \frac{\epsilon}{2}$ and choose M large enough that $|\hat{f}_n(\xi)| < \frac{\epsilon}{2}$ for all ξ such that $|\xi| > M$. Therefore, for any ξ with $|\xi| > M$, $|\hat{f}(\xi)| < \epsilon$, so

$$\hat{f}(\xi) \rightarrow 0 \text{ as } |\xi| \rightarrow \infty,$$

as required. □

Question 1-5.

⑤ (Generalized Leibniz formula)
 Show that the formula $(fg)' = f'g + fg'$
 remains true in the sense of weak derivatives
 if $f, g \in L^\infty(\mathbb{R})$, and $f', g' \in L^1(\mathbb{R})$.
 (Use an approximation argument)

Solution.

Consider an approximate identity $\{\phi_\epsilon\}_{\epsilon>0} \subset C_0^\infty$. Then,

$$(\phi_\epsilon * f)'(x) = \int_{\mathbb{R}} \phi'_\epsilon(x-y) f(y) dy = -(-1) \int_{\mathbb{R}} \phi_\epsilon(x-y) f'(y) dy = \phi_\epsilon * f'$$

for any $x \in \mathbb{R}, \epsilon > 0$, so $(\phi_\epsilon * f)' = \phi_\epsilon * f'$. Similarly, $(\phi_\epsilon * g)' = \phi_\epsilon * g'$. Furthermore, by a property of approximate identity

$$\phi_\epsilon * g' \rightarrow_{L^1} g' \quad \text{and} \quad \phi_\epsilon * f' \rightarrow_{L^1} f'.$$

Now, for any $\phi \in \mathcal{S}$, and $\epsilon > 0$,

$$\begin{aligned} \int_{\mathbb{R}} (\phi_\epsilon * f \phi_\epsilon * g) \phi' dx &= \int_{\mathbb{R}} (\phi_\epsilon * f \phi_\epsilon * g)' \phi dx \\ &= \int_{\mathbb{R}} (\phi_\epsilon * f' \phi_\epsilon * g + \phi_\epsilon * f \phi_\epsilon * g') \phi dx \end{aligned}$$

where (9) holds by $\phi_\epsilon * f, \phi_\epsilon * g \in C_0^\infty$. Taking $\epsilon \rightarrow 0$, by DCT,

$$\int_{\mathbb{R}} fg\phi' = \int_{\mathbb{R}} (f'g + g'f)\phi$$

for all $\phi \in \mathcal{S}$, so $(fg)' = f'g + fg'$ as required. □