ProbLimI: Pset I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A \in \mathcal{F}$ and $A_k \in \mathcal{F}$ $(k \geq 1)$.
   (i) Prove the *sub-additivity* property: $\mathbb{P}(\bigcup_k A_k) \leq \sum_k \mathbb{P}(A_k)$.
   (ii) Prove the *continuity* property: If $A_k \uparrow A$ (i.e. $A_k \subseteq A_{k+1}$ for all k and $\cup_k A_k = A$) then $\mathbb{P}(A_k) \uparrow \mathbb{P}(A)$, and if $A_k \downarrow A$ (i.e. $A_k \supseteq A_{k+1}$ for all k and $\cap_k A_k = A$) then $\mathbb{P}(A_k) \downarrow \mathbb{P}(A)$.

Solution.
(i) Note that we have finite additivity property of measure, as the emptyset belong to any σ-field by definition. We first have

$$A, B \in \mathscr{F}, A \subset B \implies \mathbb{P}(A) \leq \mathbb{P}(B) \quad (*),$$

because

$$\mathbb{P}(B) \quad = \quad \mathbb{P}((B \setminus A) \cup A) = \mathbb{P}(B \setminus A) + \mathbb{P}(A).$$

Now, define $A_0 = \emptyset$, and

$$\tilde{A}_k \quad = \quad A_k \setminus (\bigcup_{0 \leq n < k} A_n) \quad (k \geq 1).$$

It follows that $\{\tilde{A}_k\}$ is a pairwise disjoint collection such that

$$\bigcup_k \tilde{A}_k = \bigcup_k A_k \quad \text{and} \quad \tilde{A}_k \subset A_k \quad (k \geq 1).$$

The union equality holds, since if $x \in \bigcup_k A_k$, then $x \in A_{k'}$ for some k', and $x \in \tilde{A}_{k^*}$, where

$$k^* = \inf\{k; x \in A_k\},$$

as $x \notin A_k$ for $k < k^*$ and $x \in A_{k^*}$. Hence, by countable additivity,

$$\mathbb{P}(\bigcup_k A_k) = \mathbb{P}(\bigcup_k \tilde{A}_k) = \sum_k \mathbb{P}(\tilde{A}_k) \leq \sum_k \mathbb{P}(A_k),$$

where the last inequality follows from $(*)$. $\square$

(ii) Define $A_0, \tilde{A}_0 = \emptyset$ and

$$\tilde{A}_k = A_k \setminus A_{k-1} \quad (k \geq 1).$$

By finite additivity and the fact that $\{A_k\}$ is increasing, we have, for any $k \geq 1$,

$$\mathbb{P}(A_k) = \mathbb{P}(A_{k-1} \cup (A_k \setminus A_{k-1})) = \mathbb{P}(A_{k-1}) + \mathbb{P}(A_k \setminus A_{k-1}),$$

and by re-arranging

$$\mathbb{P}(\tilde{A}_k) = \mathbb{P}(A_k) - \mathbb{P}(A_{k-1}).$$

Now, $\{\tilde{A}_k\}$ are disjoint, so by countable additivity, we have

$$\begin{aligned} \mathbb{P}(A) = \mathbb{P}(\bigcup_k A_k) &= \mathbb{P}(\bigcup_k \tilde{A}_k) = \sum_k \mathbb{P}(\tilde{A}_k) = \lim_{k\to\infty} \sum_{n=1}^{k} \mathbb{P}(A_n) - \mathbb{P}(A_{n-1}) \\ &= \lim_{k\to\infty} \mathbb{P}(A_k) - \mathbb{P}(A_0) = \lim_{k\to\infty} \mathbb{P}(A_k), \end{aligned}$$

as required. Now, we show the continuity from above. Note that $\{A_k^c\}$ forms an increasing collection. By the DeMorgan's law, and continuity from below,

$$1 - \mathbb{P}(\bigcap_k A_k) = \mathbb{P}((\bigcap_k A_k)^c) = \mathbb{P}(\bigcup_k A_k^c) = \lim_{k\to\infty} \mathbb{P}(A_k^c) = 1 - \lim_{k\to\infty} \mathbb{P}(A_k),$$

so

$$\mathbb{P}(A) = \mathbb{P}(\bigcap_k A_k) = \lim_{k\to\infty} \mathbb{P}(A_k),$$

as required. $\square$

Question 2.

2. Let $\mathcal{F}$ be a field.

(i) Show that if $\{\mathcal{G}_\alpha\}$ is a (possibly uncountable) family of σ-fields then $\bigcap_\alpha \mathcal{G}_\alpha$ is also a σ-field. Conclude that $\sigma(\mathcal{F}) = \bigcap\{\mathcal{G} \supseteq \mathcal{F} : \mathcal{G} \text{ is a } \sigma\text{-field}\}$.

(ii) Prove that if $\mathcal{M}$ is a monotone class and $\mathcal{F} \subseteq \mathcal{M}$ then $\sigma(\mathcal{F}) \subseteq \mathcal{M}$. Conclude that $\sigma(\mathcal{F})$ is equal to $\mathrm{m}(\mathcal{F}) := \bigcap\{\mathcal{M} \supseteq \mathcal{F} : \mathcal{M} \text{ is a monotone class}\}$.

Solution.

(i) We just note that the index set must be non-empty. As $\emptyset$ and Ω are in $\mathscr{G}_\alpha$ for all α, by the $\sigma-$field property of each $\mathscr{G}_\alpha$, it follows that $\emptyset, \Omega \in \bigcap_\alpha \mathscr{G}_\alpha$. Now, it suffices to show that

$$A \in \bigcap_\alpha \mathscr{G}_\alpha \quad \Longrightarrow \quad A^c \in \bigcap_\alpha \mathscr{G}_\alpha,$$
$$\{A_n\} \subset \bigcap_\alpha \mathscr{G}_\alpha \quad \Longrightarrow \quad \bigcap_n A_n \in \bigcap_\alpha \mathscr{G}_\alpha.$$

If $A \in \bigcap_\alpha \mathscr{G}_\alpha$ then, $A \in \mathscr{G}_\alpha$ for all α, and by the $\sigma-$field assumption on each $\mathscr{G}_\alpha$, it follows that $A^c \in \mathscr{G}_\alpha$ for all α, so $A^c \in \bigcap_\alpha \mathscr{G}_\alpha$.

If $\{A_n\} \subset \bigcap_\alpha \mathscr{G}_\alpha$, then $\{A_n\} \subset \mathscr{G}_\alpha$ for all α, and by the $\sigma-$ field assumption on each $\mathscr{G}_\alpha$, it follows that $\bigcap_n A_n \in \mathscr{G}_\alpha$ for all α, so $\bigcap_n A_n \in \bigcap_\alpha \mathscr{G}_\alpha$.

First, note that $\{\mathscr{F} \subset \mathscr{G} \mid \mathscr{G} \text{ is a } \sigma\text{-field}\}$ is non-empty, as 2^Ω belongs to it. So by the above result $\mathscr{G} = \bigcap\{\mathscr{F} \subset \mathscr{G} \mid \mathscr{G} \text{ is a } \sigma\text{-field}\}$ is a σ-field. Now, recall that $\sigma(\mathscr{F})$ is defined to be the smallest σ-field containing $\mathscr{F}$. Consider the family of σ-field that contains $\mathscr{F}$, and denote it by $\{\mathscr{G}_\alpha\}$. The above result shows that $\bigcap_\alpha \mathscr{G}_\alpha$ is a σ-field, and it is trivial that it contains $\mathscr{F}$. Obviously, for any α, $\bigcap_\alpha \mathscr{G}_\alpha \subset \mathscr{G}_\alpha$, which tells us that any σ-algebra containing $\mathscr{F}$ contains $\bigcap_\alpha \mathscr{G}_\alpha$, so it follows that $\bigcap_\alpha \mathscr{G}_\alpha$ is the smallest σ-algebra containing $\mathscr{F}$ and notationally we have

$$\sigma(\mathscr{F}) \quad = \quad \{\mathscr{F} \subset \mathscr{G} : \mathscr{G} \text{ is a } \sigma - \text{field}\},$$

as required. $\square$

(ii) Let $\{A_k\} \subset \mathscr{F}$, and define

$$\tilde{A}_k \quad = \quad \bigcup_{n \le k} A_n \quad (k \ge 1).$$

Then, $\{\tilde{A}_k\}$ is an increasing sequence, so by a monotone class property,

$$\bigcup_k A_k) \quad = \quad \bigcup_k \tilde{A}_k \in \mathscr{M}.$$

Similarly,

Question 3.

3. Prove that if $f : \mathbb{R}^n \to [-\infty, \infty]$ is lower semi-continuous (that is, $\liminf_{\|x-x_0\|\downarrow 0} f(x) \geq f(x_0)$ for every $x_0 \in \mathbb{R}^n$) then it is a Borel function, and conclude that continuous functions are Borel measurable. (*Hint: first show every set of the form* $\{x : f(x) \leq a\}$ $(a \in \mathbb{R})$ *is closed.*)

Solution.

Question 4.

4. Let $m\mathcal{F}$ denote the set of measurable functions from $(\Omega, \mathcal{F}) \to ([-\infty, \infty], \mathcal{B}_{[-\infty,\infty]})$, where $\mathcal{B}_{[-\infty,\infty]} = \sigma([-\infty, a] : a \in \mathbb{R})$. Prove that

   (a) every simple function $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ belongs to $m\mathcal{F}$.

   (b) if $X_n \in m\mathcal{F}$ $(n \geq 1)$ then $\liminf_{n\to\infty} X_n$ and $\limsup_{n\to\infty} X_n$ also belong to $m\mathcal{F}$.

   Conclude that $m\mathcal{F}$ is the smallest class of functions satisfying properties (a) and (b).

Solution.
(a) Let f be a simple function, i.e.

$$f \quad = \quad \sum_{i=1}^{n} a_i X_{E_i},$$

where $a_i \in \mathbb{R}$, $E_i \in \mathscr{F}$ pairwise disjoint for $1 \leq i \leq n$, and $\bigcup_{i=1}^n E_i = \Omega$. For sake of completeness, we show that f is $(\mathscr{F}, \mathscr{B}_{\mathbb{R}})$ measurable. For any $a \in \mathbb{R}$, observe that $f^{-1}((-\infty, a])$ is a union of sub-collection (allowing the empty collection) of $\{E_i\}$, so it is in $\mathscr{F}$. Hence, any simple function is $(\mathscr{F}, \mathscr{B}_{\mathbb{R}})$ measurable.

Fix $a \in \mathbb{R}$. As $f^{-1}(-\infty) = \emptyset$ and f is $(\mathscr{F}, \mathscr{B}_{\mathbb{R}})$ measurable, it follows that

$$f^{-1}([-\infty, a]) \quad = \quad f^{-1}(-\infty) \cup f^{-1}((-\infty, a]) \in \mathscr{F}.$$

So, f is $(\mathscr{F}, \mathscr{B}_{[-\infty,\infty]})$ measurable, i.e. $f \in m\mathscr{F}$.

(b) Observe that

$$\begin{aligned} \liminf_{n\to\infty} X_n &= \sup_k \inf_{n\geq k} X_n \\ \limsup_{n\to\infty} X_n &= \inf_k \sup_{n\geq k} X_n \end{aligned}.$$

Hence, with symmetry of $\inf$ and $\sup$, it suffices to show that $\sup_n X_n$ is measurable.

Fix $a \in \mathbb{R}$. Then, we have

$$(\sup_n X_n)^{-1}([-\infty, a]) \quad = \quad \bigcap_n X_n^{-1}([-\infty, a]) \in \mathscr{F}. \ (*)$$

We now prove $(*)$. If $w \in \bigcap_n X_n^{-1}([-\infty, a])$, then $X_n(w) \in [-\infty, a]$ for all n, so $\sup_n X_n(w) \in [-\infty, a]$, and $w \in \sup_n^{-1}([-\infty, a])$. If $w \in \sup_n X_n^{-1}([-\infty, a])$, then $\sup_n X_n(w) \in [-\infty, a]$, which implies $X_n(w) \in [-\infty, a]$ for all n. Hence, $(*)$ is true and $\sup X_n \in m\mathscr{F}$.

Let $\mathscr{G}$ be a class of functions such that (a) and (b) are true. We wish to show that $m\mathscr{F} \subset \mathscr{G}$. By (a), we know that simple functions are in $\mathscr{G}$. Now, if $f \in m\mathscr{F}$, then by the simple approximation lemma, there exists a sequence of simple functions $\{X_n\}$ such that X_n converges pointwise to f. Then, by (b),

$$f = \limsup_{n\to\infty} X_n \in \mathscr{G},$$

so $m\mathscr{F} \subset \mathscr{G}$, and $m\mathscr{F}$ is the smallest class of functions satisfying properties (a) and (b).