ProbLimI:
Problem Set VII

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Abstract

This work contains solutions to the exercises of the problem set V. The chosen problems are 1,2, and 3.

Question 2.

2. Let (X_k) be mutually independent r.v.'s and suppose $S_n = \sum_{k=1}^{n} X_k$ has $\sigma_n^2 := \mathrm{Var}(S_n) < \infty$.

   (i) Prove that if
   $$\lim_{n\to\infty} \sigma_n^{-q} \sum_{k=1}^{n} \mathbb{E}\,|X_k - \mathbb{E}X_k|^q = 0 \qquad \text{for some } q > 2 \qquad (\star)$$
   then $(S_n - \mathbb{E}S_n)/\sigma_n \Rightarrow \mathcal{N}(0,1)$.

   (ii) Show that if $\sigma_n \to \infty$ and there exist $C > 0, q > 2$ such that $\mathbb{E}\,|X_k - \mathbb{E}X_k|^q \leq C\,\mathrm{Var}(X_k)$ for all k, then $(\star)$ holds.

   (iii) Give an example where there exist $C > 0, q > 2$ such that $\mathbb{E}\,|X_k - \mathbb{E}X_k|^q \leq C\,\mathrm{Var}(X_k)^{q/2}$ for all k and $\sigma_n \to \infty$, yet $(S_n - \mathbb{E}S_n)/\sigma_n$ does not converge in distribution.

Solution.
(i) We verify the Lindenberg condition on $\{\dfrac{S_n - \mathbb{E}[S_n]}{\sigma_n}\}$. Observe that

$$\frac{S_n - \mathbb{E}[S_n]}{\sigma_n} = \frac{1}{\sigma_n}\sum_{k=1}^{n} X_k - \mathbb{E}[X_k] = 0$$

and

$$\mathbb{E}[\frac{X_k - \mathbb{E}[X_k]}{\sigma_n}] \quad = \quad 0$$

for each $n, k \in \mathbb{N}$. Furthermore,

$$\sum_{n=1}^{\infty} \mathbb{E}[\frac{(X_k - \mathbb{E}[X_k])^2}{\sigma_n^2}] \quad = \quad \lim_{n\to\infty} \frac{1}{\sigma_n^2}\sum_{k=1}^{n} \mathrm{Var}[X_k] = \lim_{n\to\infty} \frac{1}{\sigma_n^2}\mathrm{Var}[S_n] = 1.$$

Hence, it now suffices to show the integral condition. Fix $\epsilon > 0$. With the given $q > 2$,

$$\begin{aligned}\sum_{k=1}^{n} \mathbb{E}[\frac{(X_k - \mathbb{E}[X_k])^2}{\sigma_n^2} 1_{\{|\frac{X_k - \mathbb{E}[X_k]}{\sigma_n}|>\epsilon\}}] &= \frac{1}{\sigma_n^2}\sum_{k=1}^{n} \mathbb{E}[(X_k - \mathbb{E}[X_k])^2] 1_{\{|X_k - \mathbb{E}[X_k]|>\epsilon\sigma_n\}}] \\ &\leq \frac{1}{\sigma_n^2}\sum_{k=1}^{n} \mathbb{E}[(\frac{|X_k - \mathbb{E}[X_k]|}{\epsilon\sigma_n})^{q-2}(X_k - \mathbb{E}[X_k])^2] 1_{\{|X_k - \mathbb{E}[X_k]|>\epsilon\sigma_n\}}] \\ &= \epsilon^{2-q}\frac{1}{\sigma_n^q}\sum_{k=1}^{n} \mathbb{E}[(|X_k - \mathbb{E}[X_k]|)^q 1_{\{|X_k - \mathbb{E}[X_k]|>\epsilon\sigma_n\}}] \\ &\leq \epsilon^{2-q}\frac{1}{\sigma_n^q}\sum_{k=1}^{n} \mathbb{E}[(|X_k - \mathbb{E}[X_k]|)^q]\end{aligned}$$

for any $n \in \mathbb{N}$. Now, by the assumption, the last term goes to 0 as $n \to \infty$. Since $\epsilon > 0$ was arbitrary, we have shown that the desired Lindenberg condition, Therefore, by Lindenberg CLT,

$$\frac{S_n - \mathbb{E}[S_N]}{\sigma_n} \to_D N(0,1)$$

as required.

(ii) By independence, with the given $C > 0$ and $q > 2$,

$$\sigma_n^{-q}\sum_{k=1}^{n} \mathbb{E}|X_k - \mathbb{E}[X_k]|^q \quad \leq \quad C\sigma_n^{-q}\sum_{1}^{n} \text{Var}[X_k] = C\sigma_n^{2-q}$$

for any $n \in \mathbb{N}$. Since $2 - q > 0$ and $\sigma_n \to \infty$ as $n \to \infty$, the RHS converges to 0 as $n \to \infty$. Hence, $(*)$ holds.

(iii) Let $\mathbb{P}(X_k = a_k) = \mathbb{P}(X_k = -a_k) = \dfrac{1}{2}$ for all $k \geq 1$, where $\{a_k\}$ are positive numbers to be chosen. We compute

$$\mathbb{E}[X_k] = 0,\ \ \mathbb{E}|X_k|^q = a_k^q$$

so with $C = 1$ and $q > 2$,

$$\mathbb{E}|X_k - \mathbb{E}X_k|^q = c(\text{Var}(X_k))^{\frac{q}{2}}$$

for all $k \geq 1$. Note that, by independence,

$$\sigma_n^2 = \text{Var}(S_n) = \sum_{k=1}^{n} a_k^2 \ \ \mathbb{E}|X_k|^3 < \infty.$$

By the estimate in Durrett 3.3.8,

$$\phi_{X_k}(\frac{t}{\sigma_n}) \quad = \quad 1 - \frac{a_k t^2}{2\sigma_n^2} + \Theta(\frac{1}{\sigma_n^3})$$

for all $n \geq 1$, and hence

$$\begin{aligned}\phi_{\sigma_n^{-1}S_n}(t) &= \prod_{k=1}^{n}(1 - \frac{a_k t^2}{2\sigma_n^2} + \Theta(\frac{1}{\sigma_n^3}) \\ &= \exp(\sum_{k=1}^{n}\log(1 - \frac{a_k^2 t^2}{2\sigma_n^2} + \Theta(\frac{1}{\sigma_n^3})) \\ &= \exp(\sum_{k=1}^{n} -\frac{a_k^2 t^2}{2\sigma_n^2} + \Theta(\frac{1}{\sigma_n^3})) = \exp(-\frac{t^2}{2} + \Theta(\frac{n}{\sigma_n^3}))\end{aligned}$$

for all n large enough. The theta estimate comes from the fact that we have values less than 1 in absolute value, so we can consider a geometric series to sum the remainder term. Choose $\{a_n\}$ such that $\sum_{k=1}^{n} a_k^2 = \sqrt{n}$ for all n. Then, by the above estimate, we see that the characteristic function diverges, so it does not converge in distribution, and we have the desired construction. $\square$

Question 3.

3. Let (X_k) be mutually independent r.v.'s and let $S_n = \sum_{k=1}^{n} X_k$.

(i) Suppose that for some fixed $\alpha, \beta \in \mathbb{R}$,

$$\mathbb{P}(X_k = k^\alpha) = \mathbb{P}(X_k = -k^\alpha) = \tfrac{1}{2}k^{-\beta}, \qquad \mathbb{P}(X_k = 0) = 1 - k^{-\beta}. \qquad (\star\star)$$

Show that if $\beta > 1$ then S_n converges a.s. as $n \to \infty$ in two ways: (1) via a direct application of Borel-Cantelli, and (2) via Kolmogorov's Three Series Theorem.

(ii) Suppose $(\star\star)$ holds for $0 \leq \beta < \min\{1, 2\alpha + 1\}$. Show that there exists a non-random sequence (b_n) such that $b_n^{-1} S_n \Rightarrow Z$ for some random variable Z satisfying $0 < F_Z(z) < 1$ for some $z \in \mathbb{R}$, and compute its characteristic function $\Phi_Z(t)$.

(iii) Now suppose that

$$\mathbb{P}(X_k = 2k) = \mathbb{P}(X_k = -2k) = \tfrac{1}{2k^2}, \qquad \mathbb{P}(X_k = 1) = \mathbb{P}(X_k = -1) = \tfrac{1}{2}\left(1 - \tfrac{1}{k^2}\right).$$

Show that $S_n/\sqrt{n} \Rightarrow \mathcal{N}(0,1)$.

Solution.

(i) Suppose $\beta > 1$. Choose any $0 < C < 1$. Then, as $|k^\alpha| \geq 1 > C$ for all $k \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \mathbb{P}(|X_k| \geq C) \quad = \quad \sum_{k=1}^{\infty} \mathbb{P}(|X_k| = k^\alpha) + \mathbb{P}(|X_k| = -k^{-\alpha}) = \sum_{k=1}^{\infty} k^{-\beta} < \infty.$$

Let $Y_k = X_k 1_{|X_k| \leq C}$ for each $k \in \mathbb{N}$. Then, as before,

$$\mathbb{E}[Y_k] = 0 \text{ for all } k \in \mathbb{N}$$

and hence

$$\sum_{k=1}^{\infty} \mathbb{E}[Y_k] < \infty.$$

Similarly,

$$\mathrm{Var}[Y_k] \quad = \quad \mathbb{E}[{Y_k}^2] - \mathbb{E}[Y_k]^2 = 0$$

for all $k \in \mathbb{N}$, so

$$\sum_{k=1}^{\infty} \mathrm{Var}[Y_k] \text{ converges.}$$

Therefore, by Kolmogorov's three series theorem, we have that S_n converges a.s.

Choose any $0 < C < 1$. Then,

$$\mathbb{P}(|X_k| > \epsilon) = k^{-\beta}$$

for all $k \geq 1$, and hence with $\beta > 1$,

$$\sum_{k=1}^{\infty} \mathbb{P}(|X_k| > \epsilon) \text{ is summable.}$$

Hence, by BC I,

$$\mathbb{P}(|X_k| \leq \epsilon \text{ a.a.}) \quad = \quad 1.$$

Since

$$\{|X_k| \leq \epsilon \text{ a.a.}\} \subset \{\frac{S_n}{n} \text{ converges}\},$$

we are done.

(ii) Suppose $0 \leq \beta < \min(1, 2\alpha + 1)$. Observe that

$$\sigma_k^2 \quad = \quad \mathrm{Var}[X_k] = \mathbb{E}[X_k^2] = k^{2\alpha - \beta}$$

for all $k \in \mathbb{N}$. Then, we have the third moment is 0 and the fourth moment is finite. Analogous to the solution in 2-iii,

$$\phi_{b_n^{-1}S_n}(t) = \exp(\sum_{k=1}^{n} \frac{k^{2\alpha-\beta}t^2}{2b_n} + \Theta(\frac{n}{b_n^4}))$$

Choose $\{b_n\}$ such that

$$\frac{n}{b_n^4} \to 0 \quad \sum_{k=1}^{n} k^{2\alpha-\beta} b_n^{-1} \to C$$

for some $C > 0$. Then, $b_n^{-1} S_n \to_D N(0, C)$. So we are done.

(iii) With the exact same calculation as above, we see that $\phi_{\frac{S_n}{\sqrt{n}}} \to e^{-\frac{t^2}{2}}$. Therefore, we see that it should converge to normal distribution with 0 mean, but with variance 5.

$\square$

Question 4.

4. Let $\{X_k\}$ be i.i.d. r.v.'s with $\mathbb{E}X_1 = 0$ and $\mathrm{Var}(X_1) = 1$, and let $Y_{n,k} = X_k / \sqrt{n^{-1}\sum_{i=1}^n X_i^2}$.

(i) Show that $Y_{n,1} \Rightarrow X_1$.

(ii) Show that $n^{-1/2}\sum_{k=1}^n Y_{n,k} \Rightarrow \mathcal{N}(0,1)$.

(iii) Show that if $X_1 \sim \mathcal{N}(0,1)$ then the vector $(Y_{n,1}, \ldots, Y_{n,n})$ has the uniform distribution over $\mathcal{S} = \{x \in \mathbb{R}^n : |x| = \sqrt{n}\}$ (i.e., the unique distribution over $\mathcal{S}$ that is invariant under orthogonal transformations), and interpret items (i) and (ii) in this special case.

Solution.
(i) By SLLN,

$$\frac{1}{n}\sum_{i=1}^n X_i^2 \to 1 \text{ almost surely and } (\frac{1}{n}\sum_{i=1}^n X_i^2)^{\frac{1}{2}} \to 1 \text{ almost surely}$$

which implies

$$Y_{n,1} = \frac{X_1}{(n^{-1}\sum_{i=1}^n X_i^2)^{\frac{1}{2}}} \to X_1 \text{ almost surely.}$$

Since a.s. convergence implies convergence in distribution,

$$Y_{n,1} \to_D X_1,$$

as required.

(ii) By CLT,

$$n^{-\frac{1}{2}}\sum_{k=1}^n X_k \to_D N(0,1).$$

Observe that

$$\begin{aligned} n^{-\frac{1}{2}}\sum_{k=1}^n Y_{n,k} - n^{-\frac{1}{2}}\sum_{k=1}^n X_k &= (n^{-\frac{1}{2}})\sum_{k=1}^n X_k(\frac{1}{\sqrt{n^{-1}\sum_{i=1}^n X_i^2}} - 1) \\ &= (\frac{1}{\sqrt{n^{-1}\sum_{i=1}^n X_i^2}} - 1)\sum_{k=1}^n \frac{X_k}{n^{-\frac{1}{2}}} \end{aligned}$$

for all $n \in \mathbb{N}$. By SLLN and CLT,

$$(\frac{1}{\sqrt{n^{-1}\sum_{i=1}^n X_i^2}} - 1) \to_p 0 \quad \text{and} \quad \sum_{k=1}^n \frac{X_k}{n^{-\frac{1}{2}}} \to_D N(0,1).$$

Therefore, by the result from the problem set 4 (1-iii),

$$n^{-\frac{1}{2}}\sum_{k=1}^n Y_{n,k} - n^{-\frac{1}{2}}\sum_{k=1}^n X_k \to_D 0.$$

Now, by the result from th problem set 4 (1-ii),

$$n^{-\frac{1}{2}}\sum_{k=1}^n Y_{n,k} \to_D N(0,1)$$

as required. □

(iii) Consider the distribution induced by $(X_1, ..., X_n)$, which is a multivariate Gaussian. By symmetry of the multivariate Gaussian, we know that the value of the distribution only depend on the distance away from the origin. Since $(Y_{n,1}, ..., Y_{n,n})$ are obtained by the normalization, we see that the vector gives distribution that is invariant under orthogonal transformations. The measure is unique, because we can form a product measure with a Lebesgue measure on $(0, \infty)$. and that will give rise to Lebesgue measure on $\mathbb{R}^{n+1}$ by polar decomposition. Hence, for any $A \subset S^n$ measurable, $\mu(A) = m(A \times (0,1))$, where μ is the measure we have on S^n. (i) is one dimensional case. (ii) is the avg value with $\sqrt{n}$ scaling of the uniform distribution, so by CLT goes to $N(0,1)$. □