ProbLimI:
Problem Set V

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Abstract

This work contains solutions to the exercises of the problem set V. The chosen problems are 1,2, and 3.

Question 1.

1. Let X_k be i.i.d. r.v.'s with distribution function F_X, and let $M_n = \max_{k \le n} X_k$. Establish that $(M_n - a_n)/b_n \Rightarrow M$ with the specified distribution function $F_M(x)$ in the following cases.
   (a) $F_X(x) = 1 - e^{-x}$ for $x \ge 0$, with $a_n = \log n$, $b_n = 1$ and $F_M(x) = \exp(-e^{-x})$ for $x \in \mathbb{R}$.
   (b) $F_X(x) = 1 - x^{-\alpha}$ for $x \ge 1$ and $\alpha > 0$, with $a_n = 0$, $b_n = n^{1/\alpha}$ and $F_M(x) = \exp(-x^{-\alpha})$ for $x > 0$.
   (c) $F_X(x) = 1 - |x|^{\alpha}$ for $-1 \le x \le 0$ and $\alpha > 0$, with $a_n = 0$, $b_n = n^{-1/\alpha}$ and $F_M(x) = \exp(-|x|^{\alpha})$ for $x \le 0$.

Solution.
By i.i.d. assumption on $\{X_k\}$,

$$\begin{aligned} F_{\frac{M_n - a_n}{b_n}}(x) &= \mathbb{P}(\frac{M_n - a_n}{b_n} \le x) = \mathbb{P}(M_n \le a_n + b_n x) = \mathbb{P}(\max_{k \le n} X_k \le a_n + b_n x) \\ &= \mathbb{P}(\bigcap_{k \le n} X_k \le a_n + b_n x) = \prod_{k \le n} \mathbb{P}(X_k \le a_n + b_n x) = (F_X(a_n + b_n x))^n \end{aligned} \tag{1}$$

for each $n \ge 1$ and $x \in \mathbb{R}$.

(a) Let $x \in \mathbb{R}$. Substituting the givens to (1),

$$F_{\frac{M_n - a_n}{b_n}}(x) = F_X(a_n + b_n x)^n = (1 - e^{-\log(n) - x})^n = (1 - \frac{-e^x}{n}))^n$$

for all sufficiently large $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ gives

$$\lim_{n \to \infty} F_{\frac{M_n - a_n}{b_n}}(x) = \exp(-e^{-x}) = F_M(x).$$

Therefore, $\dfrac{(M_n - a_n)}{b_n}$ converges in distribution to M.

(b) Let $x > 0$. Substituting the givens to (1),

$$F_{\frac{M_n - a_n}{b_n}}(x) = F_X(a_n + b_n x)^n = (1 - (n^{\frac{1}{\alpha}} x)^{-\alpha})^n = (1 - \frac{x^{-\alpha}}{n}))^n$$

for all sufficiently large $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ gives

$$\lim_{n \to \infty} F_{\frac{M_n - a_n}{b_n}}(x) = \exp(-x^{-\alpha}) = F_M(x).$$

Let $x \leq 0$. Then

$$a_n + b_n x \quad = \quad n^{\frac{1}{\alpha}} x \leq 0$$

and hence

$$F_{\frac{M_n - a_n}{b_n}}(x) \quad = \quad (F_X(a_n + b_n x))^n = 0$$

for each $n \geq 1$. Since $F_M(x) = 0$ on for $x \leq 0$, we have shown that

$$F_{\frac{(M_n - a_n)}{b_n}}(x) \quad \to \quad F_M(x)$$

for all $x \in \mathbb{R}$, and hence $\dfrac{M_n - a_n}{b_n}$ converges in distribution to M.

(c) Let $x < 0$. Then

$$F_{\frac{M_n - a_n}{b_n}}(x) \quad = \quad (1 - |n^{-\frac{1}{\alpha}} x|^{\alpha})^n = (1 - \frac{|x|^{\alpha}}{n})^n$$

for all sufficiently large $n \in \mathbb{N}$. Taking the limit as $n \to \infty$ gives

$$\lim_{n \to \infty} F_{\frac{M_n - a_n}{b_n}}(x) = \exp(-|x|^{\alpha}) = F_M(x).$$

Let $x \geq 0$. Then,

$$a_n + b_n x = n^{-\frac{1}{\alpha}} x \geq 0$$

and hence

$$F_{\frac{m_n - a_n}{b_n}}(x) \quad = \quad (F_X(n^{-\frac{1}{\alpha}} x))^n = 1$$

for each $n \geq 1$. Since $F_M(x) = 1$ for all $x \geq 0$, we have shown that

$$F_{\frac{(M_n - a_n)}{b_n}}(x) \quad \to \quad F_M(x)$$

for all $x \in \mathbb{R}$ and hence $\dfrac{M_n - a_n}{b_n}$ converges in distribution to M.

$\square$

Question 2.

2. (i) Let X_n, Y_n be a pair of independent r.v.'s for each $n \geq 1$, and let X, Y be independent r.v.'s such that $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$. Prove that $X_n + Y_n \Rightarrow X + Y$.

   (ii) Let X and Y be $[0,1]$-valued r.v.'s such that $\mathbb{E}[X^n] = \mathbb{E}[Y^n]$ for every integer $n \geq 0$. Show that $\mathbb{E}f(X) = \mathbb{E}f(Y)$ for every continuous function $f : [0,1] \to \mathbb{R}$ and conclude that $X \overset{d}{=} Y$. (*Hint: use the Weierstrass approximation theorem.*)

Solution.
(i) Fix $t \in \mathbb{R}$. By independence,

$$\phi_{X_n+Y_n}(t) = \phi_{X_n}(t)\phi_{Y_n}(t)$$

and

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t)$$

for each $n \geq 1$. By Levy-continuity theorem,

$$\phi_{X_n}(t) \to \phi_X(t) \text{ and } \phi_{Y_n}(t) \to \phi_Y(t)$$

so

$$\lim_{n\to\infty} \phi_{X_n+Y_n}(t) = \lim_{n\to\infty} \phi_{X_n}(t)\phi_{Y_n}(t) = \phi_X(t)\phi_Y(t) = \phi_{X+Y}(t).$$

Therefore, $\{\phi_{X_n+Y_n}\}$ converges pointwise everywhere to ϕ_{X+Y}, so again by Levy-continuity theorem, we have $X_n + Y_n$ converges in distribution to $X + Y$.

(ii) As X, Y are $[0,1]$-value random variables, by a change of variable,

$$\int_0^1 t^n \mu_X(dt) = \mathbb{E}[X^n] = \mathbb{E}[Y^n] = \int_0^1 t^n \mu_Y(dt)$$

for each $n \geq 1$. By linearity of integral,

$$\int_0^1 p(t)\mu_X(dt) = \int_0^1 p(t)\mu_Y(dt) \quad (1)$$

for any polynomial p defined on $[0,1]$. Now, fix $\epsilon > 0$, and by Weierstrass approximation theorem, choose a polynomial p_0 such that

$$||f - p_0||_{\sup} < \epsilon.$$

Then, by a change of variable and (1),

$$\begin{aligned}
|\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]| &= |\int_0^1 f(t)\mu_X(dt) - \int_0^1 f(t)\mu_Y(dt)| \\
&= |\int_0^1 f(t) - p_0(t)\mu_X(dt) - \int_0^1 f(t) - p_0(t)\mu_Y(dt)| \\
&\leq \int_0^1 |f(t) - p_0(t)|\mu_X(dt) + \int_0^1 |f(t) - p_0(t)|\mu_Y(dt) < 2\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have shown that $\mathbb{E}[f(X)] = \mathbb{E}[f(Y)]$ for any f continuous and defined on $[0,1]$. Since e^{ist} is continuous on $[0,1]$ for any fixed $s \in \mathbb{R}$,

$$\phi_X(s) = \mathbb{E}[e^{isX}] = \int_0^1 e^{ist}\mu_X(dt) = \int_0^1 e^{ist}\mu_Y(dt) = \mathbb{E}[e^{isY}] = \phi_Y(s)$$

for any $s \in \mathbb{R}$. Now, by Fourier Uniqueness, we have that $X = Y$ in distribution. $\square$

Question 3.

3. (i) Show that if $X \geq 0$ and $Y \geq 0$ satisfy $\mathbb{E}e^{-tX} = \mathbb{E}[e^{-tY}]$ for every $t > 0$ then $X \overset{d}{=} Y$.

(ii) Suppose $X_n \geq 0$ are such that $g(t) := \lim_{n\to\infty} \mathbb{E}e^{-tX_n}$ exists for every $t > 0$ and $\lim_{t\downarrow 0} g(t) = 1$. Show that the distribution functions (F_{X_n}) are uniformly tight and that there exists some r.v. $X \geq 0$ such that $X_n \Rightarrow X$ and $g(t) = \mathbb{E}e^{-tX}$ for every $t > 0$.

(iii) Let $X_n = \frac{1}{n}\sum_{j=1}^{n} jI_j$ where $I_j \in \{0,1\}$ are independent r.v.'s with $\mathbb{P}(I_j = 1) = 1/j$. Show $X_n \Rightarrow X$ for some $X \geq 0$ with $\mathbb{E}e^{-tX} = \exp\left(\int_0^1 \frac{1}{x}(e^{-xt} - 1)dx\right)$ for every $t > 0$.

Solution.
(i) Observe that e^{-X} and e^{-Y} are $[0,1]$-valued random variables. Furthermore,

$$\mathbb{E}[(e^{-X})^n] = \mathbb{E}[e^{-nx}] = \mathbb{E}[e^{-nY}] = \mathbb{E}[(e^{-Y})^n]$$

for each $n \in \mathbb{N}$. Hence, by the problem 2-(ii),

$$e^{-X} = e^{-Y} \quad \text{in distribution.}$$

It follows that, as e^{-x} is strictly decreasing,

$$\mathbb{P}(X \leq t) = \mathbb{P}(e^{-X} \geq e^{-t}) = \mathbb{P}(e^{-Y} \geq e^{-t}) = \mathbb{P}(Y \leq t)$$

for any $t \in \mathbb{R}$. Therefore, $X = Y$ in distribution.

(ii) First note that to show a sequence of distribution is uniformly tight, it suffices to show the condition holds for all sufficiently large indices in our case, by the regularity property of a Borel probability measure on a polish space.

Fix $1 > \epsilon > 0$. Choose $t_0(\epsilon) > 0$ and $N(t_0) \in \mathbb{N}$ such that

$$\mathrm{E}[e^{-t_0 X_n}] \geq 1 - \frac{\epsilon}{2}$$

for any $n \geq N$. By monotonicity of e^{-x},

$$\begin{aligned}\mathbb{E}[e^{-t_0 X_n}] &\leq 1\mathbb{P}(X_n \leq \delta) + e^{-t_0\delta}\mathbb{P}(X_n > \delta) = \mathbb{P}(X_n \leq \delta) + e^{-t_0\delta}(1 - \mathbb{P}(X_n \leq \delta)) \\ &= e^{-t_0\delta} + (1 - e^{-t_0\delta})\mathbb{P}(X_n \leq \delta)\end{aligned}$$

for any $\delta > 0$ and $n \in \mathbb{N}$, and hence

$$\frac{1 - \frac{\epsilon}{2} - e^{-t_0\delta}}{1 - e^{-t_0\delta}} \leq \mathbb{P}(X_n \leq \delta)$$

for any $\delta > 0$ and $n \geq N$. Since the LHS goes to $1 - \dfrac{\epsilon}{2}$ as $\delta \uparrow \infty$, we can choose $\delta_0 > 0$ large enough such that

$$1 - \epsilon \leq \mathbb{P}(X_n \leq \delta_0)$$

for each $n \geq N$. Since $1 > \epsilon > 0$ was arbitrary, we have shown that $\{F_{X_n}\}$ are uniformly tight. Hence, there exists a subseqeunce $\{X_{n_k}\}$ that converges in distribution to some X. By Portmanteau's theorem and the fact that $X_n \geq 0$,

$$\mathbb{P}(X < 0) \leq \liminf_{n\to\infty} \mathbb{P}(X_n < 0) = 0.$$

Therefore, we see that X, which is a weak limit of $\{X_{n_k}\}$ is non-negative as well.

To show that the full sequence converges in distribution to X, we use the following lemma: Let $\{X_n\}$ be uniformly tight. If X is the only possible weak limit, i.e. whenver $\{X_{n_k}\}$ converges in distribution to Y, $X = Y$, then $\{X_n\}$ converges in distribution to X. We supply a proof here. Suppose otherwise. Then, there exists $x \in \mathbb{R}$ $\epsilon > 0$, and a subsequence $\{n_j\}$ such that $\mathbb{P}(X = x) \neq 0$ and

$$|\mathbb{P}(X_{n_j} \leq x) - \mathbb{P}(X \leq x)| \geq \epsilon$$

for all $j \in \mathbb{N}$. Since $\{X_{n_j}\}$ is tight, we should find a subseuqence that converges in distribution, but it is not possible by the above estimate. Therefore, the lemma is proven.

Now, in view of the lemma, suppose $\{X_{n_j}\}$ converges in distribution to Y. Then, by weak convergence,

$$\mathbb{E}[e^{-tX}] = \lim_{k\to\infty} \mathbb{E}[e^{-tX_{n_k}}] \quad = \quad \lim_{j\to\infty} \mathbb{E}[e^{-tX_{n_j}}] = \mathbb{E}[e^{-tY}]$$

for all $t > 0$. Hence, by (i), it follows that $X = Y$, and we have shown that $\{X_n\}$ converges in distribution to X. Again, by weak convergence,

$$\mathbb{E}[e^{-tX}] \quad = \quad \lim_{n\to\infty} \mathbb{E}[e^{-tX_n}] = g(t)$$

for all $t > 0$ as required and we are done.

(iii) Observe that $X_n \geq 0$ for each $n \geq 1$. In view of (ii), it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}[e^{-tX_n}] = \exp(\int_0^1 \frac{1}{x}(e^{-xt} - 1)dx)$$

for any $t > 0$. Suppose the above limit is established. Observe that

$$\frac{1}{x}(e^{-xt} - 1) \quad \text{converges pointwise a.e to} \quad 0 \text{ as } t \downarrow 0 \text{ on } [0,1]$$

and

$$|\frac{1}{x}(e^{-xt} - 1)| \quad = \quad \frac{1}{x}(1 - e^{-xt}) \leq \frac{1}{x}(xt) \leq t$$

for any $t > 0$ and $x \in (0,1]$. Therefore, by DCT,

$$\begin{aligned}
\lim_{t\downarrow 0} g(t) \quad &= \quad \lim_{t\downarrow 0} \exp(\int_0^1 \frac{1}{x}(e^{-xt} - 1)dx) = \exp(\lim_{t\downarrow 0} \int_0^1 \frac{1}{x}(e^{-xt} - 1)dx) \\
&= \quad \exp(\int_0^1 \lim_{t\downarrow 0} \frac{1}{x}(e^{-xt} - 1)dx) = \exp(0) = 1.
\end{aligned}$$

We now prove the claimed limit. By independence,

$$\begin{aligned}
\mathbb{E}[e^{-tX_n}] \quad &= \quad \mathbb{E}[e^{-t\frac{1}{n}\sum_{j=1}^n jI_j}] = \prod_{j=1}^n \mathbb{E}[e^{\frac{-tj}{n}I_j}] \\
&= \quad \prod_{j=1}^n (e^{-\frac{tj}{n}}\mathbb{P}(I_j = 1) + 1\mathbb{P}(I_j = 0)) = \prod_{j=1}^n (\frac{1}{j}e^{-\frac{tj}{n}} + \frac{j-1}{j})
\end{aligned}$$

for any $t > 0$ and $n \in \mathbb{N}$. By Taylor series' approximation of $\log(1 - x) \sim -x$,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}[e^{-tX_n}] \quad &= \quad \lim_{n\to\infty} \exp(\log(\prod_{j=1}^n \frac{1}{j}e^{-\frac{tj}{n}} + \frac{j-1}{j})) = \exp(\lim_{n\to\infty} \sum_{j=1}^n \log(\frac{1}{j}e^{-\frac{tj}{n}} + \frac{j-1}{j})) \\
&= \quad \exp(\lim_{n\to\infty} \sum_{j=1}^n \log(1 - (\frac{1}{j} - \frac{1}{j}e^{-\frac{tj}{n}}))) = \exp(\lim_{n\to\infty} \sum_{j=1}^n -(\frac{1}{j} - \frac{1}{j}e^{\frac{-tj}{n}})) \\
&= \quad \exp(\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n \frac{n}{j}(e^{-\frac{tj}{n}} - 1)) = \exp(\int_0^1 \frac{1}{x}(e^{-xt} - 1)dx).
\end{aligned}$$

$\square$

Question 4.

4. In what follows, say that $X_n \xrightarrow{L^q} X$ for $q > 0$ if $X_n, X \in L^q$ and $\mathbb{E}|X_n - X|^q \to 0$, where $L^q(\Omega, \mathcal{F}, \mathbb{P})$ is the set of random variables Y on $(\Omega, \mathcal{F})$ such that $\|Y\|_q := (\mathbb{E}[|Y|^q])^{1/q} < \infty$.

   (i) Establish the following L^2 WLLN: if $X_1, \dots, X_n$ have $\mathbb{E}X_i = \mu$ and $\mathrm{Cov}(X_i, X_j) \le a_{|i-j|}$ for all i, j, where (a_k) is a bounded sequence with $\lim_{k\to\infty} a_k = 0$, then $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{L^2} \mu$.

   (ii) Establish the following WLLN: if $X_1, \dots, X_n$ are i.i.d. and $\lim_{k\to\infty} k\mathbb{P}(|X| > k) = 0$ then $\frac{1}{n}\sum_{i=1}^n X_i - \mathbb{E}[X_1 \mathbf{1}_{|X_1|\le n}] \xrightarrow{\mathbb{P}} 0$. (*Hint: establish a WLLN for the truncated variables* $X_i' := X_i \mathbf{1}_{|X_i|\le n}$ *using that* $\mathrm{Var}(X_i')/n \to 0$, *and then compare* $\sum X_i$ *to* $\sum X_i'$.)

   (iii) Let $X_1, \dots, X_n$ be i.i.d. whose law is given by $\mathbb{P}\left(X_1 = (-1)^k k\right) = 1/\left(c_0 k^2 \log k\right)$ for $k = 2, 3, \dots$, where c_0 is a normalizer. Prove that $\mathbb{E}|X_1| = \infty$ and yet there exists a constant $\mu < \infty$ such that $\frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{\mathbb{P}} \mu$.

Solution.