ProbLimI:
Problem Set IV

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Abstract

This work contains solutions to the exercises of the problem set IV. The chosen problems are 1,3, and 4.

Question 1.

1. Let $(X_n)_{n=1}^{\infty}, (Y_n)_{n=1}^{\infty}, X$ be r.v.'s on $(\Omega, \mathcal{F})$ and $c \in \mathbb{R}$ be such that $X_n \Rightarrow X$ and $Y_n \xrightarrow{p} c$.

   (i) Show that $\bar{\mathcal{C}} = \{x : F_X \text{ is discontinuous at } x\}$ is countable.

   (ii) Deduce that $X_n + Y_n \Rightarrow X + c$. Further show that if $(Z_n)_{n=1}^{\infty}$ and Z are random variables on $(\Omega, \mathcal{F})$ such that $Z_n - X_n \Rightarrow 0$ then $Z_n \Rightarrow Z$ if and only if $X_n \Rightarrow Z$.

   (iii) Prove that $X_n Y_n \Rightarrow cX$.

Solution.
(i) We characterize the discontinuity points as

$$\overline{\mathscr{C}} \quad = \quad \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{R} \, : \, \lim_{t \downarrow x} F_X(t) - F_X(x) \geq \frac{1}{n}\}$$

Since $0 \leq F_X \leq 1$, it follows that

$$|\{x \in \mathbb{R} \, : \, \lim_{t \downarrow x} F_X(t) - F_X(x) \geq \frac{1}{n}\}| \quad \leq \quad n$$

for each $n \geq 1$. To see it explicitly, suppose otherwise, then there exists some $ngeq1$ and $x, y \in \mathbb{R}$ such that $1 < \frac{n+1}{n} \leq F(x) - F(y)$, which is a contradiction. Therefore, $\overline{\mathscr{C}}$ is a countable union of finite sets, so it is countable.

(ii) Fix $\epsilon > 0$. Observe that

$$\begin{aligned} \mathbb{P}(X_n + Y_n \leq a) \quad &= \quad \mathbb{P}(X_n + Y_n \leq a, Y_n \geq c - \epsilon) + \mathbb{P}(X_n + Y_n \leq a, Y_n < c - \epsilon) \\ &\leq \quad \mathbb{P}(X_n \leq a - c + \epsilon) + \mathbb{P}(Y_n < c - \epsilon) \end{aligned}$$

and

$$\mathbb{P}(X_n + c \leq a - \epsilon) \quad \leq \quad \mathbb{P}(X_n + Y_n < a) + \mathbb{P}(Y_n \leq c - \epsilon)$$

for any $a \in \mathbb{R}$. As $Y_n \to_p c$ and X_n converges in distribution to X, by taking the limit with respect to n on both inequalities above gives

$$\limsup_n \mathbb{P}(X_n \leq a - c - \epsilon) \quad \leq \quad \liminf_n F_{X_n + Y_n}(a) \leq \limsup_n F_{X_n+Y_n}(a) \leq \liminf_n \mathbb{P}(X_n < a - c + \epsilon)$$

for any $a \in \mathbb{R}$. For a that is a F_{X+c} continuity point, we can take $\epsilon \to 0$, which gives

$$F_{X_n+Y_n}(a) \to \mathbb{P}(X + c \le a)$$

as $n \to \infty$.

(iii) Without loss of generality, assume that $c > 0$. Fix $0 < \delta < c$. Observe that

$$\mathbb{P}(X_n Y_n \le a) \quad \le \quad \mathbb{P}(X_n Y_n \le a, |Y_n - c| \le \delta) + \mathbb{P}(|Y_n - c| > \delta)$$

for any $a \in \mathbb{R}$ and $n \ge 1$. Using $Y_n \to_p c$, we obtain

$$\mathbb{P}(X_n Y_n \le a) \quad \le \quad \mathbb{P}(X_n \le \frac{a}{c - \delta} + \delta$$

for all large n. On the other hand,

$$\mathbb{P}(X_n Y_n > a) \quad \le \quad \mathbb{P}(X_n > \frac{a}{c + \delta}) + \delta.$$

Take $n \to \infty$ and $\delta \to 0$, we get the desired claim. $\square$

Question 2.

2. For some $n \geq 2$, let $U_1, \ldots, U_n$ be i.i.d. Uniform$(-1,1)$ random variables and $S_n = \sum_{j \leq n} U_j$. Prove that S_n has a probability density function $f_{S_n}(x) = \frac{1}{\pi} \int_0^\infty \cos(tx)(\frac{\sin t}{t})^n dt$, and deduce that $\int_0^\infty \cos(tx)(\frac{\sin t}{t})^n dt = 0$ for all $x > n \geq 2$.

Solution.
Fix $n \geq 1$. Let F_n be the cumulative distribution function of S_n. Then, by independence, we obtain

$$\phi_{S_n} = \phi_{U_1}^n = (\frac{\sin(t)}{t})^n.$$

By Levy's inversion formula, it follows that

$$F_n(b) - F_n(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ita} - e^{itb}}{it} (\frac{\sin(t)}{t})^n dt$$

for $a < b$ such that a, b are continuity points of F_{S_n}. Now, observe that the integrand on the RHS is continuous with respect to both b, t variables and the partial with respect to b is also continuous with respect to b, t. Hence, we can commute the partial with respect to b with the integral to obtain

$$\begin{aligned} f_n(b) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial b} (\frac{e^{-ita} - e^{itb}}{it} (\frac{\sin(t)}{t})^n) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itb} (\frac{\sin(t)}{t})^n dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(tb)(\frac{\sin(t)}{t})^n - i \sin(tb)(\frac{\sin(t)}{t})^n dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(tb)(\frac{\sin(t)}{t})^n dt = \frac{1}{\pi} \int_0^{\infty} \cos(tb)(\frac{\sin(t)}{t})^n dt \end{aligned}$$

for almost everywhere $b \in \mathbb{R}$, where the last two equalities follow from considering evenness and oddness of the integrands and the almost everywhere qualification comes from $(1-a)$. To conclude, we have $S_n \leq n$ everywhere. If $f_{S_n}(x) > 0$ for some $x > n$, then by continuity of f_{S_n} on values larger than n, we can choose some small ball B such that $B \cap (-\infty, n] = \varnothing$ and $\{w : S_n(w) \in B\} = \int_B f_{S_n}(x) dx > 0$, which contradictions $\{w : S_n(w) \leq n\} = 1$, and we are done. $\square$

Question 3.

3. Let X_n be i.i.d. random variables and let $S_n = \sum_{j \le n} X_j$.

(i) Suppose X_n has Cauchy density $f(x) = (\pi(1+x^2))^{-1}$. Show that its characteristic function is $\Phi(t) = e^{-|t|}$, and deduce that $S_n/n \overset{d}{=} X_1$.

(ii) Suppose X_n is standard normal. Prove that $S_n/\sqrt{n} \overset{d}{=} X_1$, and that for every $x > 0$,

$$\left(x^{-1} - x^{-3}\right) e^{-x^2/2} \le \int_x^\infty e^{-t^2/2} dt \le x^{-1} e^{-x^2/2}.$$

Deduce that, for some absolute constant $C > 0$ and every sufficiently large n,

$$\mathbb{P}\left(|S_n| \ge 2\sqrt{n \log n}\right) \le C n^{-2}.$$

Solution.
(i) Fix $t > 0$. We compute

$$\int_{\mathbb{R}} e^{itx} (\pi(1+x^2))^{-1} dx$$

through Cauchy integral formula. Take the circle contour γ_r in a counterclockwise orientation such that L_z consists of the horizontal line from $-r$ to r on the real axis and the C_r the half circle contour from r to $-r$. Then, for sufficiently large r, by Cauchy residue theorem,

$$e^{-t} = \int_{\gamma_r} \frac{e^{itz}}{\pi(1+z^2)} dz = \int_{-r}^{r} \frac{e^{itx}}{\pi(1+x^2)} dx + \int_{C_r} \frac{e^{itz}}{\pi(1+z^2)}$$

because the residue can be computed as

$$\lim_{z \to i} (z-i) \frac{e^{itz}}{\pi(1+z^2)} = \lim_{z \to i} \frac{e^{itz}}{\pi(z+i)} = \frac{e^{-t}}{2\pi i}.$$

The result follows now, because as $r \to \infty$, by ML inequality, the C_r integral goes to 0.

Now, for $t < 0$, by a change of variable $u = -x$,

$$\int_{\mathbb{R}} \frac{e^{itx}}{\pi(1+x^2)} = \int_{\mathbb{R}} \frac{e^{i(-t)x}}{\pi(1+u^2)} du = e^t.$$

Hence, this shows that $\phi(t) = e^{-|t|}$ for any $t \in \mathbb{R}$. Using independence and ordinary properties of characteristic functions gives

$$\Phi_{\frac{S_n}{n}}(t) = \prod_{i=1}^n \Phi_{\frac{X_i}{n}}(t) = \prod_{i=1}^n e^{-|\frac{t}{n}|} = e^{-|t|} \prod_{i=1}^n e^{-\frac{1}{n}} = e^{-|t|}$$

for any $t \in \mathbb{R}$. Therefore, by Fourier Uniqueness, we conclude that $\dfrac{S_n}{n} = X_1$ in distribution as required.

(ii) We have $\Phi_{X_1} = e^{-\frac{t^2}{2}}$. This fact is well known, and if one wishes to derive it, the complex integration method works almost identically to the case above. Then, again by independence and properties of characteristic functions, we compute

$$\Phi_{\frac{S_n}{\sqrt{n}}} = \prod_{i=1}^n \Phi_{\frac{X_i}{\sqrt{n}}} = \prod_{i=1}^n e^{-\frac{t^2}{2n}} = e^{-t^2} \prod_{i=1}^n e^{-\frac{1}{2n}} = e^{-\frac{t^2}{2}}$$

for any $t \in \mathbb{R}$. Therefore, by Fourier Uniqueness, it follows that $\dfrac{S_n}{\sqrt{n}} = X_1$ in distribution.

We now show the first estimate. Fix $x > 0$. By integration by parts,

$$\int_x^\infty e^{\frac{-t^2}{2}} dt = -\frac{1}{t} e^{-\frac{t^2}{2}} |_x^\infty - \int_x^\infty \frac{1}{t^2} e^{-\frac{t^2}{2}} dt = \frac{1}{x} e^{-\frac{x^2}{2}} - \int_x^\infty \frac{1}{t^2} e^{-\frac{t^2}{2}} dt \le x^{-1} e^{\frac{-x^2}{2}}$$

where the last inequality follows from the fact that $\frac{1}{t^2}e^{-\frac{t^2}{2}}$ is a non-negative function on $\mathbb{R}$. Now, again by integration by parts

$$\int_x^\infty e^{\frac{-t^2}{2}} \quad = \quad x^{-1}e^{-\frac{x^2}{2}} - \int_x^\infty e^{-\frac{t^2}{2}}\,dt = (x^{-1}-x^{-3})e^{-\frac{x^2}{2}} + C\int_x^\infty \frac{1}{t^4}e^{\frac{-t^2}{2}}\,dt$$

for some positive constant C. Observe that the last integral is non-negative as before. Combined with the above estimate, we finally obtain

$$(x^{-1}-x^{-3})e^{-\frac{x^2}{2}} \quad \le \quad \int_x^\infty e^{-\frac{t^2}{2}}\,dt \le x^{-1}e^{\frac{-x^2}{2}},$$

which completes the proof of the first estimate. Now, it follows that

$$\mathbb{P}(\frac{|S_N|}{\sqrt{n}} \le 2\sqrt{\log(n)}) \quad \le \quad 2((2\sqrt{\log(n)})^{-1}e^{-2\log(n)}) \le C\log(n)^{-\frac{1}{2}}n^{-2}$$

for some positive constant C and $n \ge 1$. Since $\log(n)^{-\frac{1}{2}}n^{-2} = O(n^{-2})$, the result follows. □

Question 4.

4. Let $X_1, \ldots, X_n$ be i.i.d. standard normal random variables, let $M_n := \max_{j \le n} X_j$ and a_n be such that $F_{X_j}(a_n) = (n-1)/n$.

   (a) Show that $\lim_{t\to\infty} \frac{1-F_{X_j}(t+x/t)}{1-F_{X_j}(t)} = e^{-x}$ for every $x \in \mathbb{R}$.

   (b) Prove that $a_n(M_n - a_n) \Rightarrow M$ where $F_M(x) = e^{-e^{-x}}$, and that $\lim_{n\to\infty} a_n/\sqrt{2\log n} = 1$. Deduce that $M_n/\sqrt{2\log n} \xrightarrow{p} 1$.

Solution.