ProbLimI:
Problem Set II

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Abstract

This work contains solutions to the exercises of the problem set I. The chosen problems are 1,2, and 4.

Question 1.

1. Prove that $X_n \overset{\text{a.s.}}{\to} 0$ if and only if for every $\varepsilon > 0$ there exists n such that the following holds: for every random variable $N : \Omega \to \{n, n+1, \ldots\}$, we have $\mathbb{P}(\{\omega : |X_{N(\omega)}(\omega) > \varepsilon\}) < \varepsilon$.

Solution.
Fix $\epsilon > 0$. Choose $C \in \mathscr{F}$ such that $\mathbb{P}(C) = 0$, and for any $w \in \Omega \setminus C$, there exists $n(w) \geq 1$, such that $|X_n(w)| < \epsilon$, whenever $n \geq n(w)$. Set $n(w) = \infty$ for each $w \in C$. Now, for each $n \geq 1$, define

$$A_n := \{w : n(w) > n\}.$$

It follows that $\{A_n\}$ is descending and $\bigcap_n A_n = C$. Therefore, by continuity of probability, there exists n_0 such that $\mathbb{P}(A_{n_0}) < \epsilon$. Then, it follows that, for any $N : \Omega \to \{n_0, ...\}$,

$$\mathbb{P}(\{w : |X_{N(w)}(w)| > \epsilon\} \quad \leq \quad \mathbb{P}(A_{n_0}) < \epsilon.$$

The first inequality holds, because, for all $w \in \Omega$ and $N : \Omega \to \{n_0, ...\}$,

$$X_{N(w)}(w) > \epsilon \quad \implies n_0 \leq N(w) < n(w),$$

as required.

Conversely, suppose that $\{x_n\}$ does not converge almost surely to 0. Choose $E \in \mathscr{F}$ such that $\mathbb{P}(E) > 0$, and $0 < \epsilon < \mathbb{P}(E)$ such that for any $w \in E$, there exists $\{n_k\}$ such that

$$|x_{n_k}(w)| > \epsilon, \quad \text{for any} \quad k \geq 1.$$

Fix $n \geq 1$. Define $N : \Omega \to \{n, ...\}$ by

$$w \mapsto \inf\{n_k \ : \ n_k \geq n \text{ and } |x_{n_k}(w)| > \epsilon\}$$

if $w \in E$ and $w \mapsto n$ otherwise. Then, it follows that

$$\mathbb{P}(\{w : |X_{N(w)}(w)| > \epsilon\}) \quad \geq \quad \mathbb{P}(E) > \epsilon,$$

and we are done. □

Question 2.

2. Let X be a random variable and let $f, g : \mathbb{R} \to \mathbb{R}$ be bounded and non-decreasing functions. Prove that $\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\,\mathbb{E}[g(X)]$.

Solution.

Question 3.

3. Give an example of a random variable X that has a bounded probability density function and yet its characteristic function Φ_X satisfies $\int_{\mathbb{R}} |\Phi_X(t)|dt = \infty$. Give another example of a random variable X such that $\Phi_X(t)$ is not differentiable at $t = 0$.

Solution.
Exponential distribution with density $f(x) = e^{-x}$ has a bounded density function, but

$$\int_{\mathbb{R}} |\frac{1}{1-it}|dt = \int_{\mathbb{R}} \frac{1}{\sqrt{1+t^2}}dt = \infty$$

Cauchy distribution with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ has the characteristic function $\phi(t) = e^{-|t|}$, which is not differentiable at 0. $\square$

Question 4.

4. Let X and X' be i.i.d. random variables, and let $Z = X - X'$.

(i) Show that the characteristic function of Z is nonnegative and real-valued.

(ii) Show that there do not exist $a < b$ such that $Z \sim \text{Uniform}(a, b)$.

Solution.