
ProbLimI: Problem Set XIII

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Abstract

This work contains solutions to the exercises of the problem set IX. The chosen problems are 2,3 and 4.

Question 1.

- (a) Give an example of a sub-martingale (X_n) such that (X_n^2) is a super-martingale, and explain why this not contradict the result given in class on $\Phi(X_n)$ for sub-martingale (X_n) and a convex function Φ .
- (b) Give an example of a martingale (X_n) that converges a.s. to $-\infty$, and explain why this does not contradict Doob's Convergence Theorem.

Solution.

(a) Let $X_n = 0$ for each $n \geq 1$, then $X_n^2 = 0$ for each $n \geq 1$. It follows that $\{X_n\}$ is a sub-martingale, and $\{X_n^2\}$ is a super-martingale because they are both martingales trivially, as

$$\mathbb{E}[X_n | \mathcal{F}_{n-1}] = X_n = X_{n-1} = 0$$

and

$$\mathbb{E}[X_n^2 | \mathcal{F}_{n-1}] = X_n^2 = X_{n-1}^2 = 0$$

for each $n \geq 1$. This does not contradict the given fact about the convex function, as $\{X_n^2\}$ is a sub-martingale as well, by being a martingale.

(b) Let $\{X_n\}$ independent random variables be defined by

$$\mathbb{P}(X_n = -1) = 1 - \frac{1}{2^n} \quad \text{and} \quad \mathbb{P}(X_n = 2^n - 1) = \frac{1}{2^n}$$

for each $n \geq 1$. Then,

$$\mathbb{E}[X_n] = 0$$

for each $n \geq 1$, so $\{S_n = \sum_{k=1}^n X_k\}$ is a martingale with respect to the canonical filtration. Observe that

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > -1) = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

By Borel-Cantelli,

$$\mathbb{P}(X_n > -1 \text{ i.o.}) = 0$$

and hence

$$\mathbb{P}(X_n \leq -1 \text{ a.a.}) = 1.$$

Therefore, $S_n \rightarrow -\infty$ almost surely and we are done. This does not violate the Martingale convergence theorem, as $\sup_n \mathbb{E}[|S_n|]$ is not bounded.

Question 2.

2. Let (X_n) and (Y_n) be nonnegative, integrable stochastic processes adapted to a filtration (\mathcal{F}_n) such that $\mathbb{E}[X_{n+1} | \mathcal{F}_n] \leq (1 + Y_n)X_n + Y_n$ for all n and $\sum_{n \geq 1} Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit as $n \rightarrow \infty$.

(Hint: Deduce this from the convergence of a suitable nonnegative super-martingale.)

Solution.

Let $Z_n = \frac{X_n}{\prod_{m=1}^{n-1} (1 + Y_m)}$. Then, $Z_n \in \mathcal{F}_n$ and $\mathbb{E}|Z_n| \leq \mathbb{E}|X_n| < \infty$ for all $n \geq 1$. We claim that $\{Z_n\}$ is a non-negative supermartingale. We compute

$$\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = \frac{1}{\prod_{m=1}^n (1 + Y_m)} \mathbb{E}[X_{n+1} | \mathcal{F}_n] \leq \frac{1}{\prod_{m=1}^n (1 + Y_m)} X_n = Z_n.$$

Therefore, by Martingale convergence theorem, $Z_n \rightarrow Z_\infty$ a.s. to some almost surely finite random variable Z_∞ . As $\sum_n Y_n < \infty$ a.s.,

$$\prod_{m=1}^n (1 + Y_m) \text{ converges to a finite limit a.s.}$$

and hence

$$X_n \rightarrow Z_\infty \prod_{m=1}^{\infty} (1 + Y_m) \text{ a.s.}$$

so we are done. □

Question 3.

3. Let $S_n = \sum_{k=1}^n \xi_k$ for i.i.d. random variables ξ_k and let τ be an integrable stopping time for the associated canonical filtration.

(a) Show that if ξ_1 is integrable then $\mathbb{E}S_\tau = \mathbb{E}\xi_1\mathbb{E}\tau$ (Wald's identity).

(Hint: Write $S_\tau = \sum_{k=1}^\infty \xi_k \mathbf{1}_{\{k \leq \tau\}}$.)

(b) Show that if $\mathbb{E}\xi_1^2 < \infty$ then $\mathbb{E}(S_\tau - \tau\mathbb{E}\xi_1)^2 = \text{Var}(\xi_1)\mathbb{E}\tau$ (Wald's second identity).

(Hint: argue that $\mathbb{E}\xi_1 = 0$ w.l.o.g. and apply Doob's L^2 -convergence theorem to $S_{n \wedge \tau}$.)

(c) Prove that if $\xi_1 \geq 0$ then Wald's identity holds also in case $\mathbb{E}\tau = \infty$ under the convention that $0 \times \infty = 0$.

Solution.

(a) Observe that

ξ_i and $\mathbf{1}_{\{i \leq \tau\}}$ are independent

for each $i \geq 1$. Now, we first prove for the case when $\xi_i \geq 0$ for all $i \geq 1$.

$$\begin{aligned} \mathbb{E}[S_\tau] &= \mathbb{E}[\xi_1 + \dots + \xi_\tau] = \mathbb{E}\left[\sum_{i=1}^\infty \xi_i \mathbf{1}_{\{i \leq \tau\}}\right] \\ &= \sum_{i=1}^\infty \mathbb{E}[\xi_i \mathbf{1}_{\{i \leq \tau\}}] = \sum_{i=1}^\infty \mathbb{E}[\xi_i] \mathbb{E}[\mathbf{1}_{\{i \leq \tau\}}] \end{aligned} \quad (1)$$

$$= \mathbb{E}[\xi_1] \sum_{i=1}^\infty \mathbb{P}(i \leq \tau) = \mathbb{E}[\xi_1] \mathbb{E}[\tau] \quad (2)$$

where (1) holds by MCT (or Tonelli) and independence, and (2) holds as τ being a non-negative integer valued random variable. Now consider a general $\{\xi_i\}$. From the above,

$$\mathbb{E}\left[\sum_{i=1}^\infty |\xi_i| \mathbf{1}_{\{i \leq \tau\}}\right] = \mathbb{E}[|\xi_1|] \mathbb{E}[\tau] < \infty.$$

Therefore, by Fubini,

$$\mathbb{E}[S_\tau] = \mathbb{E}\left[\sum_{i=1}^\infty \xi_i \mathbf{1}_{\{i \leq \tau\}}\right] = \sum_{i=1}^\infty \mathbb{E}[\xi_i \mathbf{1}_{\{i \leq \tau\}}] = \mathbb{E}[\xi_1] \mathbb{E}[\tau].$$

(b) Hence without loss of generality, we can assume $\mathbb{E}[\xi_1] = 0$, so it suffices to show that

$$\text{Var}[S_\tau] = \mathbb{E}[\xi_1] \mathbb{E}[\tau].$$

Suppose for a moment that τ is bounded by some constant M . Then, by (a),

$$\begin{aligned} \text{Var}(S_\tau) &= \mathbb{E}[S_\tau^2] = \mathbb{E}\left[\left(\sum_{i=1}^n \xi_i \mathbf{1}_{\{i \leq \tau\}}\right)^2\right] \\ &= \mathbb{E}\left[\sum_{i=1}^M (\xi_i^2 \mathbf{1}_{\{i \leq \tau\}}) + 2 \sum_{1 \leq i < j \leq M} \xi_i \xi_j \mathbf{1}_{\{i \leq \tau\}} \mathbf{1}_{\{j \leq \tau\}}\right] \\ &= \mathbb{E}\left[\sum_{i=1}^M (\xi_i^2 \mathbf{1}_{\{i \leq \tau\}})\right] = \mathbb{E}[\xi_1^2] \mathbb{E}[\tau] \end{aligned}$$

where the cross terms are 0 by independence. Now, for τ not bounded, let $\rho_k = \min(\tau, k)$ for all k . Then with (a) and MCT the result follows.

(c) The identity from a is still true, and with the convention given, we should have the result from a as well.