ProbLimI: Problem Set XIII

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Abstract

This work contains solutions to the exercises of the problem set IX. The chosen problems are 2,3 and 4.

Question 1.

1. (a) Give an example of a sub-martingale (X_n) such that (X_n^2) is a super-martingale, and explain why this not contradict the result given in class on $\Phi(X_n)$ for sub-martingale (X_n) and a convex function Φ.

   (b) Give an example of a martingale (X_n) that converges a.s. to $-\infty$, and explain why this does not contradict Doob's Convergence Theorem.

Solution.
(a) Let $X_n = 0$ for each $n \geq 1$, then ${X_n}^2 = 0$ for each $n \geq 1$. It follows that $\{X_n\}$ is a sub-martingale, and $\{X_n^2\}$ is a super-martingale because they are both martingales trivially, as

$$\mathbb{E}[X_n|\mathscr{F}_{n-1}] = X_n = X_{n-1} = 0$$

and

$$\mathbb{E}[X_n^2|\mathscr{F}_{n-1}] = X_n^2 = X_{n-1}^2 = 0$$

for each $n \geq 1$. This does not contradict the given fact about the convex function, as $\{X_n^2\}$ is a sub-martingale as well, by being a martingale.

(b) Let $\{X_n\}$ i.i.d random variables be defined by

$$\mathbb{P}(X_n = -1) = 1 - \frac{1}{2^n} \quad \text{and} \quad \mathbb{P}(X_n = 2^n - 1) = \frac{1}{2^n}$$

for each $n \geq 1$. Then,

$$\mathbb{E}[X_n] = 0$$

for each $n \geq 1$, so $\{S_n = \sum_{k=1}^n X_k\}$ is a martingale with respect to the canonical filteration. Observe that

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > -1) = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

By Borel-Cantelli,

$$\mathbb{P}(X_n > -1 \ i.o.) = 0$$

and hence

$$\mathbb{P}(X_n \leq -1 \ a.a.) = 1.$$

Therefore, $S_n \to -\infty$ almost surely and we are done. This does not violate the Martingale convergence theorem, as $\sup_n \mathbb{E}[|S_n|]$ is not bounded.

Question 3.

2. Let (X_n) and (Y_n) be nonnegative, integrable stochastic processes adapted to a filtration $(\mathcal{F}_n)$ such that $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \le (1+Y_n)X_n + Y_n$ for all n and $\sum_{n\ge1} Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit as $n \to \infty$.

   (*Hint: Deduce this from the convergence of a suitable nonnegative super-martingale.*)

Solution.

Question 4.

3. Let $S_n = \sum_{k=1}^{n} \xi_k$ for i.i.d. random variables ξ_k and let τ be an integrable stopping time for the associated canonical filtration.

(a) Show that if ξ_1 is integrable then $\mathbb{E}S_\tau = \mathbb{E}\xi_1 \mathbb{E}\tau$ (Wald's identity).
(*Hint: Write* $S_\tau = \sum_{k=1}^{\infty} \xi_k \mathbf{1}_{\{k \le \tau\}}$.)

(b) Show that if $\mathbb{E}\xi_1^2 < \infty$ then $\mathbb{E}(S_\tau - \tau \mathbb{E}\xi_1)^2 = \mathrm{Var}(\xi_1)\mathbb{E}\tau$ (Wald's second identity).
(*Hint: argue that* $\mathbb{E}\xi_1 = 0$ *w.l.o.g. and apply Doob's* L^2*-convergence theorem to* $S_{n \wedge \tau}$.)

(c) Prove that if $\xi_1 \ge 0$ then Wald's identity holds also in case $\mathbb{E}\tau = \infty$ under the convention that $0 \times \infty = 0$.

Solution.
(a) Observe that

$$\xi_i \text{ and } 1_{\{i \le \tau\}} \text{ are independent}$$

for each $i \ge 1$. Now, we first prove for the case when $\xi_i \ge 0$ for all $i \ge 1$.

$$\begin{aligned}
\mathbb{E}[S_\tau] &= \mathbb{E}[\xi_1 + \dots + \xi_\tau] = \mathbb{E}[\sum_{i=1}^{\infty} \xi_i 1_{\{i \le \tau\}}] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[\xi_i 1_{\{i \le \tau\}}] = \sum_{i=1}^{\infty} \mathbb{E}[\xi_i]\mathbb{E}[1_{\{i \le \tau\}}] \qquad (1) \\
&= \mathbb{E}[\xi_1] \sum_{i=1}^{\infty} \mathbb{P}(i \le \tau) = \mathbb{E}[\xi_1]\mathbb{E}[\tau] \qquad (2)
\end{aligned}$$

where (1) holds by MCT (or Tonelli) and independence, and (2) holds as τ being a non-negative integer valued random variable. Now consider a general $\{\xi_i\}$. From the above,

$$\mathbb{E}[\sum_{i=1}^{\infty} |\xi_i| 1_{\{i \le \tau\}}] = \mathbb{E}[|\xi_1|]\mathbb{E}[\tau] < \infty.$$

Therefore, by Fubini,

$$\mathbb{E}[\xi_\tau] = \mathbb{E}[\sum_{i=1}^{\infty} \xi_i 1_{\{i \le \tau\}}] = \sum_{i=1}^{\infty} \mathbb{E}[\xi_i 1_{\{i \le \tau\}}] = \mathbb{E}[\xi_1]\mathbb{E}[\tau].$$