ProbLimI: Problem Set III

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Abstract

This work contains solutions to the exercises of the problem set III. The chosen problems are 1,3, and 4.

Question 1.

1. Prove that $X_n \xrightarrow{\text{a.s.}} 0$ if and only if for every $\varepsilon > 0$ there exists n such that the following holds: for every random variable $N : \Omega \to \{n, n+1, \ldots\}$, we have $\mathbb{P}(\{\omega : |X_{N(\omega)}(\omega) > \varepsilon\}) < \varepsilon$.

Solution.
Fix $\epsilon > 0$. Choose $C \in \mathscr{F}$ such that $\mathbb{P}(C) = 0$, and for any $w \in \Omega \setminus C$, there exists $n(w) \geq 1$, such that $|X_n(w)| < \epsilon$, whenever $n \geq n(w)$. Set $n(w) = \infty$ for each $w \in C$. Now, for each $n \geq 1$, define

$$A_n := \{w : n(w) > n\}.$$

It follows that $\{A_n\}$ is descending and $\bigcap_n A_n = C$. Therefore, by continuity of probability, there exists n_0 such that $\mathbb{P}(A_{n_0}) < \epsilon$. Then, it follows that, for any $N : \Omega \to \{n_0, ...\}$,

$$\mathbb{P}(\{w : |X_{N(w)}(w)| > \epsilon\} \quad \leq \quad \mathbb{P}(A_{n_0}) < \epsilon.$$

The first inequality holds, because, for all $w \in \Omega$ and $N : \Omega \to \{n_0, ...\}$,

$$X_{N(w)}(w) > \epsilon \quad \implies n_0 \leq N(w) < n(w),$$

as required.

Conversely, suppose that $\{x_n\}$ does not converge almost surely to 0. Choose $E \in \mathscr{F}$ such that $\mathbb{P}(E) > 0$, and $0 < \epsilon < \mathbb{P}(E)$ such that for any $w \in E$, there exists $\{n_k\}$ such that

$$|x_{n_k}(w)| > \epsilon, \quad \text{for any} \quad k \geq 1.$$

Fix $n \geq 1$. Define $N : \Omega \to \{n, ...\}$ by

$$w \mapsto \inf\{n_k \ : \ n_k \geq n \text{ and } |x_{n_k}(w)| > \epsilon\}$$

if $w \in E$ and $w \mapsto n$ otherwise. Then, it follows that

$$\mathbb{P}(\{w : |X_{N(w)}(w)| > \epsilon\}) \quad \geq \quad \mathbb{P}(E) > \epsilon,$$

and we are done. $\square$

Question 2.

2. Let X be a random variable and let $f, g : \mathbb{R} \to \mathbb{R}$ be bounded and non-decreasing functions. Prove that $\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)]\,\mathbb{E}[g(X)]$.

Solution.

Question 3.

3. Give an example of a random variable X that has a bounded probability density function and yet its characteristic function Φ_X satisfies $\int_{\mathbb{R}} |\Phi_X(t)| dt = \infty$. Give another example of a random variable X such that $\Phi_X(t)$ is not differentiable at $t = 0$.

Solution.

Exponential distribution with density $f(x) = e^{-x}$ has a bounded density function, but

$$\int_{\mathbb{R}} \left|\frac{1}{1-it}\right| dt = \int_{\mathbb{R}} \frac{1}{\sqrt{1+t^2}} dt = \infty$$

Cauchy distribution with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ has the characteristic function $\phi(t) = e^{-|t|}$, which is not differentiable at 0. □

Question 4.

4. Let X and X' be i.i.d. random variables, and let $Z = X - X'$.

(i) Show that the characteristic function of Z is nonnegative and real-valued.

(ii) Show that there do not exist $a < b$ such that $Z \sim \text{Uniform}(a, b)$.

Solution.
(i) We assume basic Fourier analysis here. Fix $t \in \mathbb{R}$. By independence, we have

$$\phi_Z(t) \quad = \quad \phi_X(t)\phi_{X'}(-t) \;\; (*)$$

Now, observe that for any random variable Y and $s \in \mathbb{R}$, we have

$$\phi_Y(-s) \quad = \quad \mathbb{E}\cos(sY) - i\sin(sY) = \overline{\mathbb{E}\cos(sY) + i\sin(sY)} = \overline{\phi_Y(s)}$$

As X' and X are identically distributed, from $(*)$, and the above, we obtain

$$\phi_Z(t) = \phi_X(t)\overline{\phi_X(t)} = |\phi_X(t)|^2.$$

Since $t \in \mathbb{R}$ was arbitrary, we have shown $Z \geq 0$ as required.

(ii) Suppose otherwise. Then, Z has a density $f = \dfrac{1}{b-a}1_{[a,b]}$. Then, by a change of variable, for $t \neq 0$,

$$\phi_Z(t) \quad = \quad \int_a^b \frac{e^{itz}}{b-a}dz = \frac{e^{itb} - e^{ita}}{it(b-a)},$$

and $\phi_Z(0) = 1$. From (i), we deduce that, for any $t \neq 0$,

$$\cos(tb) \quad = \quad \cos(ta).$$

Take $t = 1, \sqrt{2}$. Then, as $a < b$, for some $k \in \mathbb{Z}^+$,

$$b \quad = \quad 2\pi k + a$$

and, for some $l \in \mathbb{Z}$,

$$\sqrt{2}(2\pi k + a) \quad = \quad 2\pi l + \sqrt{2}a,$$

which implies that l is irrational, so we have arrived at a contradiction. $\square$