ProbLimI:
Problem Set X

Youngduck Choi
CIMS
New York University
yc1104@nyu.edu

Abstract

This work contains solutions to the exercises of the problem set X. The chosen problems are 2,3,4.

Question 2.

Solution.
(a) Observe that

$$\mathbb{P}(y_{n+1} = 1 | y_1 = i, ..., y_n = i_n) \quad = \quad \frac{\int_0^1 z^{n+1}(1-z)^{n-N} dz}{\int_0^1 z^n (1-z)^{n-N} dz} = \frac{N+1}{N+2} = \frac{S_n + n + 2}{2n+4}$$

so

$$\mathbb{P}(S_{n+1} = s | \mathscr{F}_n) \in \sigma(S_n)$$

and

$$\mathbb{P}(S_{n+1} = s | \mathscr{F}_n) = \mathbb{P}(S_{n+1} = s | s_n).$$

Therefore, $\{S_n\}$ is a Markov chain.

(b) From the above formula, the transition probability is $\frac{n-x-2}{2n+4}$ if $y = x-1$, $\frac{n+x+2}{2n+4}$ if $y = x+1$, and 0 otherwise.

(c) Not homogeenous, as the transition probability depends on n. $\square$

Question 2.

3. Let X_n be a homogeneous Markov chain on a countable state space Ω with an associated σ-field $\mathcal{B}$, and let $\tau_{x,k} = \inf\{n \geq k : X_n = x\}$.

   (a) Prove that for every $x_0, x \in \Omega$, every $B \in \mathcal{B}$ and every positive integer $k \leq n$,

$$\mathbb{P}_{x_0}(X_n \in B, \tau_{x,k} \leq n) = \sum_{i=0}^{n-k} \mathbb{P}_{x_0}(\tau_{x,k} = n-i)\mathbb{P}_x(X_i \in B),$$

   where $\mathbb{P}_{x_0}(A) := \mathbb{P}(A \mid X_0 = x_0)$.

   (b) Deduce that

$$\mathbb{P}_{x_0}(X_n = x) = \sum_{i=k}^{n} \mathbb{P}_{x_0}(\tau_{x,k} = i)\mathbb{P}_x(X_{n-i} = x).$$

   (c) Conclude that for every $x \in \Omega$ and non-negative integers k, ℓ,

$$\sum_{i=0}^{\ell} \mathbb{P}_x(X_i = x) \geq \sum_{n=k}^{\ell+k} \mathbb{P}_x(X_n = x).$$

Solution.

(a) By homogeneity,

$$\mathbb{P}_x(X_i \in B) = \mathbb{P}(X_i \in B | X_0 = x) = \mathbb{P}(X_n \in B | X_{n-i} = x)$$

for any $1 \leq i \leq n$. Then,

$$\begin{aligned}
\mathbb{P}_{x_0}(X_n \in B, \tau_{x,k} \leq n) &= \mathbb{P}_{x_0}\Big(\bigcup_{i=0}^{n-k} X_n \in B, \tau_{x,k} = n-i\Big) \\
&= \sum_{i=0}^{n-k} \mathbb{P}_{x_0}(X_n \in B, \tau_{x,k} = n-i) \\
&= \sum_{i=0}^{n-k} \mathbb{P}_{x_0}(\tau_{x,k} = n-i)\mathbb{P}(X_n \in B | X_{n-i} = x) \\
&= \sum_{i=0}^{n-k} \mathbb{P}_{x_0}(\tau_{x,k} = n-i)\mathbb{P}_x(X_i \in B).
\end{aligned}$$

(b) In view of (a), setting $B = \{x\}$,

$$\begin{aligned}
\mathbb{P}_{x_0}(X_n = x) &= \mathbb{P}_{x_0}(X_n = x, \tau_{x,k} \leq n) && (1) \\
&= \sum_{i=k}^{n} \mathbb{P}_{x_0}(\tau_{x,k} = i)\mathbb{P}_x(X_{n-i} = x)
\end{aligned}$$

where (1) holds as $\{X_n = x\} \subset \{\tau_{x,k} \leq n\}$.

(c) Applying (b) with $x_0 = x$,

$$\begin{aligned}
\sum_{n=k}^{l+k} \mathbb{P}_x(X_n = x) &= \sum_{n=k}^{l+k} \sum_{i=k}^{n} \mathbb{P}_x(\tau_{x,k} = i)\mathbb{P}_x(X_{n-i} = x) \\
&= \sum_{i=0}^{l} \mathbb{P}_x(X_i = x) \sum_{i=k}^{n-j} \mathbb{P}_x(\tau_{x,k} = i) \\
&\leq \sum_{i=0}^{l} \mathbb{P}_x(X_i = x) && (2)
\end{aligned}$$

where (2) holds by disjointness. $\square$

Question 3.

4. Let X_n be a Markov chain on a countable state space $\Omega = S \cup T$ with $S \cap T = \emptyset$, where S is finite and $T = A \cup B$ for nonempty sets A, B. For $E \subset \Omega$, let $\tau_E = \inf\{n : X_n \in E\}$, and suppose that $\inf\{s \in S : \mathbb{P}_s(\tau_T < \infty)\} > 0$. Finally, call a bounded function $h : \Omega \to \mathbb{R}$ is *harmonic* at $x \in \Omega$ for the transition probabilities $P(\cdot, \cdot)$ of X_n if $h(x) = \sum_{y \in \Omega} P(x, y)h(y)$.

   (a) Prove that there exist $\varepsilon > 0$ and $m < \infty$ such that $\mathbb{P}_x(\tau_T > km) \le (1-\varepsilon)^k$ for all $k \ge 1$ and $x \in \Omega$, and conclude that $\tau_T < \infty$ a.s.

   (b) Prove that $h(x) = \mathbb{P}_x(\tau_A < \tau_B)$ satisfies

   $$h(x) = 1 \text{ for } x \in A\,, \quad h(x) = 0 \text{ for } x \in B\,, \quad h \text{ is harmonic at every } x \in S\,. \qquad (\star)$$

   (c) Prove that if $h : \Omega \to \mathbb{R}$ is a bounded function satisfying $(\star)$ then $h(x) = \mathbb{P}_x(\tau_A < \tau_B)$. To do so, first show that if $\mathcal{F}_n$ is the natural filtration associated with X_n and $g : \Omega \to \mathbb{R}$ is a bounded function that is harmonic at every $x \in S$, then the variable $Y_n = g(X_{n \wedge \tau_T})$ is a *martingale*; that is, $\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = Y_n$ for every n.

Solution.

(a) Denote θ_k as a shift operator of the chain. As S is finite, choose m such that $\epsilon = \min_{s \in S} \mathbb{P}_s(\tau_T \le m) > 0$. Then, with $k \ge 2$,

$$\begin{aligned}
\mathbb{P}_s(\tau_T > km) &= \mathbb{P}_s(\tau_T > (k-1)m, \tau_T \circ \theta_{(k-1)m} > m) \\
&= \mathbb{E}_s(\mathbb{P}_s \circ \theta_{(k-1)m} > m | \mathscr{F}_{(k-1)m} 1_{\{\tau_T > (k-1)m\}}) \\
&\le \mathbb{P}_s(\tau_T > (k-1)m)
\end{aligned}$$

where the last inequality holds by Markov property. $k = 1$ is obvious, so by induction we are done. By summability, $\tau_T m^{-1} < \infty$ almost surely, so $\tau_T < \infty$ almost surely.

(b) If $x \in A$, then

$$\tau_A = 0 \text{ and} > \ \tau_B \ge 1 \ \ \mathbb{P}_x \text{ almost surely,}$$

and hence $h(x) = \mathbb{P}_x(\tau_A < \tau_B) = 1$ for $x \in A$. Similarly, $h(x) = 0$ for $x \in B$.

for any $x \in S$. Suppose $x \notin T$. Then,

$$1_{\tau_A < \tau_B} \circ \theta_1 = 1_{\tau_B < \tau_B}$$

and hence

$$\mathbb{E}_X(\tau_A < \tau_B) = \mathbb{E}_X(\mathbb{P}_{X_1}(\tau_A < \tau_B)) = \mathbb{E}_X[h(x_1)].$$

Therefore h is harmonic.

(c) Fix $n \ge 1$, and $E \in \mathscr{F}_n$. By definition of conditional expectation, we wish to show

$$\int_E Y_{n+1} = \int_E Y_n$$

which can be rewritten as

$$\int_{E \cap \{\tau_T \le n\}} Y_{n+1} \int_{E \cap \{\tau_T > n\}} Y_{n+1} = \int_{E \cap \{\tau_T \le n\}} Y_n + \int_{E \cap \{\tau_T > n\}} Y_n$$

Observe that

$$Y_{n+1} = Y_n \quad \text{on} \quad \{\tau_T \le n\}$$

as well as

$$Y_{n+1} = g(X_{n+1}) \quad \text{and} \quad Y_n = g(X_n) \quad \text{on} \quad \{\tau_T > n\}.$$

Therefore, it suffices to show that

$$\int_{E \cap \{\tau_T > n\}} g(X_{n+1}) = \int_{E \cap \{\tau_T > n\}} g(X_n).$$

Now, by harmonic,

$$\sum_{\omega \in \Omega} \mathbb{P}(s,\omega) X_n(\omega) = X_n(s)$$

and

$$\sum_{s \in S} \int_{E \cap \{X_n = s\}} g(X_{n+1}) \quad = \quad \sum_{s \in S} \int_{E \cap \{X_n = s\}} g(X_n)$$

By harmonic

$$\sum_{s \in S} \int_{E \cap \{X_n = s\}} g(X_{n+1}) \quad = \quad \sum_{s \in S} X_n(s) \mathbb{P}(A_s)$$

and

$$\sum_{s \in S} \int_{E \cap \{X_n = s\}} g(X_{n+1}) \quad = \quad \sum_{s \in S} X_n(s) \mathbb{P}(A_s)$$