Durrett Probability: Problems

Youngduck Choi [*]

Abstract

This work contains solutions to some exercises from Durrett's probability text.

1 Chapter 6: Markov Chains

Question 6.3.3.

6.3.3. First entrance decomposition. Let $T_y = \inf\{n \geq 1 : X_n = y\}$. Show that

$$p^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m) p^{n-m}(y,y)$$

Solution.

Here we assume countable state space. Observe that

$$p^n(x,y) \;=\; P_x(X_n = y) = P_x\Big(\bigcup_{m=1}^{n} \{T_y = m \,;\, X_n = y\}\Big) = \sum_{m=1}^{n} P_x(T_y = m \,;\, X_n = y) \quad (1)$$

$$
\begin{aligned}
P_x(T_y = m \,;\, X_n = y) \;&=\; E_x(1_{\{X_n = y\}} \,;\, T_y = m)\\
&=\; E_x(E_x(1_{\{X_n = y\}}|\mathscr{F}_m); T_y = m) \quad\quad (2)\\
&=\; E_x(E_x(1_{\{X_{n-m}=y\}} \circ \theta_m|\mathscr{F}_m); T_y = m)\\
&=\; E_x(E_{X_m}(1_{\{X_{n-m}=y\}}); T_y = m) = E_x(P_y(X_{n-m} = y); T_y = m) \quad (3)\\
&=\; P_x(T_y = m)P_y(X_{n-m} = y)
\end{aligned}
$$

for any $1 \leq m \leq n$, where (4) holds by definition of conditional expectation and (5) holds by Markov property. Therefore, combining the above result with with (1) gives

$$p^n(x,y) \;=\; \sum_{m=1}^{n} P_x(T_y = m)P_y(X_{n-m} = y).$$

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[*]Department of Mathematics, Courant Institute of Mathematical Sciences, yc1104@nyu.edu; If you find an error and want to share with me, you can reach me via email.

Here is another approach using strong Markov. We compute

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\begin{aligned}
p^n(x,y) &= P_x(X_n = y) = P_x\left(\bigcup_{m=1}^{n} \{T_y = m; X_n = y\}\right) \\
&= E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y}; T_y \le n) = E_x(E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y}|\mathscr{F}_{T_y}); T_y \le n) \quad (4) \\
&= E_x(E_{X_{T_y}}(1_{\{X_{n-T_y}=y\}}); T_y \le n) = E_x(E_y(1_{\{X_{n-T_y}\}}); T_y \le n) \quad (5) \\
&= \sum_{m=1}^{n} P_x(T_y = m)E_y(1_{\{X_{n-m}=y\}}) = \sum_{m=1}^{n} P_x(T_y = m)P^{n-m}(y,y)
\end{aligned}
$$

where (4) holds by definition of conditional expectation and (5) holds by the strong Markov property. $\qquad \square$

Question 6.3.4.

6.3.4. Show that $\sum_{m=0}^{n} P_x(X_m = x) \geq \sum_{m=k}^{n+k} P_x(X_m = x)$.

Solution.

Let $k \in \mathbb{N}$, and $T_x^k = \inf\{n \geq k : X_n = x\}$. We claim that

$$P_x(X_m = x) = \sum_{l=k}^{m} P_x(T_x^k = x)p^{m-l}(x, x) \tag{6}$$

for any $m \geq k$. Fix $m \geq k$. Then,

$$P_x(X_m = x) = P_x(\bigcup_{l=k}^{m}\{T_x^k = l; X_m = x\}) = \sum_{l=k}^{m} P_x(T_x^k = l; X_m = x). \tag{7}$$

Now, we compute

$$
\begin{aligned}
P_x(T_x^k = l; X_m = x) &= E_x(1_{\{X_m=x\}}; T_x^k = l) = E_x(E_x(1_{\{X_m=x\}}|\mathscr{F}_l); T_x^k = l) \\
&= E_x(E_x(1_{\{X_{m-l}=x\}}\theta_l|\mathscr{F}_l); T_x^k = l) \\
&= E_x(E_{X_l}(1_{\{X_{m-l}=x\}}; T_x^k = l); T_x^k = l) \\
&= E_x(P_x(X_{m-l}x); T_x^k = l) = P_x(X_{m-l} = x)P_x(T_x^k = l) \\
&= P_x(T_x^k = l)p^{m-l}(x, x)
\end{aligned}
\tag{8}
$$

for any $k \leq l \leq m$, where (8) holds by Markov property. Therefore, combining the above result with (7), we have proven (6). Then,

$$
\begin{aligned}
\sum_{m=k}^{n+k} P_x(X_m = x) &= \sum_{m=k}^{n+k}\sum_{l=k}^{m} P_x(T_x^k = l)p^{m-l}(x, x) \\
&= \sum_{m=k}^{n+k} p^m(x, x)\left(\sum_{l=k}^{d} P_x(T_x^k = l)\right) \\
&\leq \sum_{m=k}^{n+k} p^m(x, x) = \sum_{m=k}^{n+k} P_x(X_m = x)
\end{aligned}
$$

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Question 6.3.5.

6.3.5. Suppose that $S - C$ is finite and for each $x \in S - C$ $P_x(\tau_C < \infty) > 0$. Then there is an $N < \infty$ and $\epsilon > 0$ so that $P_y(\tau_C > kN) \leq (1 - \epsilon)^k$.

Solution.

We assume countable state space. Observe that, for any $x \in S \setminus C$, we can choose $n(x) \in \mathbb{N}$ such that

$$P_x(\tau_C \leq n) > 0.$$

Otherwise, for some $x \in S \setminus C$, by continuity of probability,

$$P_x(\tau_C < \infty) = \lim_{k \to \infty} P_x(\tau_C \leq k) = 0,$$

which is a contradiction. Now, let

$$N = \max_{z \in S \setminus C} n(x). \quad \text{and} \quad \epsilon = \min_{z \in S \setminus C} P_z(\tau_C \leq N).$$

Trivially,

$$P_y(\tau_C > kN) = 0$$

for any $k \in \mathbb{N}$, and $y \in C$, since $y \in C$ implies $\tau_C = 0$ by definition. Therefore, it suffices to show

$$P_y(\tau_C > kN) \leq (1 - \epsilon)^k \tag{9}$$

for all $k \in \mathbb{N}$ and $y \in S \setminus C$. Fix $y \in S \setminus C$. Then,

$$P_y(\tau_C \leq N) \quad \geq \quad \epsilon.$$

and hence

$$P_y(\tau_C > N) \leq (1 - \epsilon)$$

Now, we proceed by induction to prove (9). Suppose, for some $k \in \mathbb{N}$ such that $k \geq 2$,

$$P_y(\tau_C > kN) \leq (1 - \epsilon)^k.$$

We compute

$$
\begin{aligned}
P_y(T_c > (k+1)N) &= E_y(1_{\{\tau_C > kN\}} \circ \theta_N; \tau_C > N) \\
&= E_y(E_y((1_{\{\tau_C > kN\}} \circ \theta_N | \mathscr{F}_N); \tau_C > N)) \\
&= E_y(E_{X_N}((1_{\{\tau_C > kN\}}); \tau_C > N)) \tag{10} \\
&\leq E_y(\sup_{z \in S} P_z(\tau_C > kN); \tau_C > N)) \\
&\leq (1 - \epsilon)^k E_y(1; \tau_C > N)) = (1 - \epsilon)^{k+1}
\end{aligned}
$$

where (10) holds by Markov Property, which completes the proof. $\qquad \square$

Question 6.3.6.

6.3.6. Let $h(x) = P_x(\tau_A < \tau_B)$. Suppose $A \cap B = \emptyset$, $S - (A \cup B)$ is finite, and $P_x(\tau_{A\cup B} < \infty) > 0$ for all $x \in S - (A \cup B)$. (i) Show that

$$(*) \qquad\qquad h(x) = \sum_y p(x,y) h(y) \quad \text{for } x \notin A \cup B$$

(ii) Show that if h satisfies $(*)$ then $h(X(n \wedge \tau_{A\cup B}))$ is a martingale. (iii) Use this and Exercise 6.3.5 to conclude that $h(x) = P_x(\tau_A < \tau_B)$ is the only solution of $(*)$ that is 1 on A and 0 on B.

Solution.

(i) Let $x \in S \setminus (A \cup B)$. Then,

$$1_{\{\tau_A < \tau_B\}} = 1_{\{\tau_A < \tau_B\}} \circ \theta_1.$$

It follows that

$$
\begin{aligned}
h(x) &= P_x(\tau_A < \tau_B) = E_x(1_{\{\tau_A < \tau_B\}}) = E_x(1_{\{\tau_A < \tau_B\}} \circ \theta_1) \\
&= E_x(E_x(1_{\{\tau_A < \tau_B\}} \circ \theta_1 | \mathcal{F}_1)) = E_x(E_{X_1}(1_{\{\tau_A < \tau_B\}})) \qquad\qquad (11) \\
&= \sum_y P(X_1 = y) P_y(\tau_A < \tau_B) = \sum_y p(x,y) P_y(\tau_A < \tau_B)
\end{aligned}
$$

where (11) holds by Markov property.

(ii)

(iii)

Question 6.3.7.

6.3.7. Let X_n be a Markov chain with $S = \{0, 1, \ldots, N\}$ and suppose that X_n is a martingale and $P_x(\tau_0 \wedge \tau_N < \infty) > 0$ for all x. (i) Show that 0 and N are absorbing states, i.e., $p(0,0) = p(N,N) = 1$. (ii) Show $P_x(\tau_N < \tau_0) = x/N$.

Solution.

Question 6.4.4.

Exercise 6.4.4. Use the strong Markov property to show that $\rho_{xz} \geq \rho_{xy}\rho_{yz}$.

Solution.
The key insight in this problem is that if you shift the chain by a stopping time of one state variable, then the probability of the chain coming back to another fixed state variable decreases. This relation allows one to estimate p_{xz} from below using strong Markov, which in our context is proven for a sequence space(discrete time) of a polish state space, using Monotone class theorem. Recall that to define a shift operator, indexed by ∞, by convention, we set

$$\theta_\infty(w) = \triangle$$

where $\triangle$ is the cemetery sample point we add to $S^{\mathbb{N}}$, for all $w \in S^{\mathbb{N}}$. Therefore, to extend the domain of $T_z = \inf\{n \geq 1 : X_n = z\}$ for any $z \in S$, to include $\triangle$, if necessary, we define

$$T_z(\triangle) = \infty \ \text{ so } \ 1_{\{T_z < \infty\}}(\triangle) = 0,$$

With this convention.

$$
\begin{aligned}
\{w \in S^{\mathbb{N}} : 1_{\{T_z < \infty\}} \circ \theta_{T_y}(w) = 1\} \ &= \ \{w \in S^{\mathbb{N}} : T_y(w) = n \text{ for some } n \geq 1 \\
&\qquad \text{and } T_z^n(w) = \inf\{k \geq n : X_k = z\} < \infty\} \\
&= \ \bigcup_{n=1}^{\infty} \{T_y = n \ ; \ T_z^n < \infty\} \\
&\subset \ \bigcup_{n=1}^{\infty} \{T_z^n < \infty\} = \{T_z < \infty\}
\end{aligned}
$$

for any $z, y \in S$.

Now, let $x, y, z \in S$. Then,

$$
\begin{aligned}
p_{xz} \ &= \ P_x(T_z < \infty) = E_x(1_{\{T_z < \infty\}}) \geq E_x(1_{\{T_z < \infty\}} \circ \theta_{T_y}) \\
&= \ E_x(E_x(1_{\{T_z < \infty\}} \circ \theta_{T_y} | \mathscr{F}_{T_y}); T_y < \infty) && (12) \\
&= \ E_x(E_{X_{T_y}}(1_{\{T_z < \infty\}}; T_y < \infty) = E_x(E_{X_y}(1_{\{T_z < \infty\}}; T_y < \infty) && (13) \\
&= \ E_x(P_y(T_z < \infty); T_y < \infty) = P_y(T_z < \infty)P_x(T_y < \infty) = p_{xy}p_{yz}
\end{aligned}
$$

where (12) holds by definition of conditional expectation, and (13) holds by strong Markov. $\quad\square$

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2 Chapter 2: Law of Large Numbers

3   Chapter 5: Martingales

Question 5.2.1.

> **Exercise 5.2.1.** Suppose X_n is a martingale w.r.t. $\mathcal{G}_n$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.
> Then $\mathcal{G}_n \supset \mathcal{F}_n$ and X_n is a martingale w.r.t. $\mathcal{F}_n$.

Solution.

Various properties of conditional expectations are used.

We compute

$$
\begin{align}
E[X_{n+1}|\mathscr{F}_n] &= E[X_{n+1}|\mathscr{G}_n|\mathscr{F}_n] \tag{14}\\
&= E[X_n|\mathscr{F}_n] \tag{15}\\
&= X_n \tag{16}
\end{align}
$$

for all $n \in \mathbb{N}$, where (14) holds by the Tower property, (15) holds by Martingale property of $\{G_n\}$ and (16) holds by measurability of X_n w.r.t $\mathscr{F}_n$ for all $n \in \mathbb{N}$. $\qquad \square$

Question 5.2.2.

Exercise 5.2.2. Suppose f is superharmonic on $\mathbf{R}^d$. Let $\xi_1, \xi_2, \ldots$ be i.i.d. uniform on $B(0,1)$, and define S_n by $S_n = S_{n-1} + \xi_n$ for $n \geq 1$ and $S_0 = x$. Show that $X_n = f(S_n)$ is a supermartingale.

Solution.

Question 5.2.3.

Exercise 5.2.3. Give an example of a submartingale X_n so that X_n^2 is a supermartingale. Hint: X_n does not have to be random.

Solution.