ProbLimI:
Problem Set II

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Abstract

This work contains solutions to the exercises of the problem set I. The chosen problems are 1,2, and 4.

Question 1.

1. Let X be a nonnegative random variable with $\mathbb{E}[X^2] < \infty$, and set $m_i := \mathbb{E}[X^i]$ for $i = 1, 2$.

   (i) Prove that for every $0 \le x < m_1$ we have $\mathbb{P}(X > x) \ge (m_1 - x)^2/m_2$.

   (ii) Prove that $(\mathbb{E}|X^2 - m_2|)^2 \le 4m_2(m_2 - m_1^2)$.

   (iii) Show the following inequality, and compare it to part (i) for $X = \sum_{k=1}^n \mathbf{1}_{A_k}$.

   $$\mathbb{P}\left(\bigcup_{k=1}^n A_k\right) \ge \sum_{k=1}^n \mathbb{P}(A_k) - \sum_{1\le k<\ell\le n} \mathbb{P}(A_k \cap A_\ell)\,.$$

Solution.
(1) Fix $\alpha \in [0, m_1)$. By Cauchy-Schwartz,

$$\mathbb{E}[1_{X>\alpha}X]^2 \quad\le\quad \mathbb{E}[1^2_{\{X>\alpha\}}]\mathbb{E}[X^2] = \mathbb{P}(X>\alpha)m_2.$$

On the other hand,

$$m_1 \quad=\quad \mathbb{E}[X1_{\{X>\alpha\}}] + E[X1_{\{X\le\alpha\}}],$$

which implies

$$(m_1-\alpha)^2 \quad\le\quad \mathbb{E}[X1_{\{X>\alpha\}}]^2.$$

Combining the above inequality with the first one and re-arranging yield

$$\frac{(m_1-\alpha)^2}{m_2} \quad\le\quad \mathbb{P}(X>\alpha),$$

as required. □

(2) By Cauchy-Schwartz,

$$\begin{aligned}\mathbb{E}[|(X+m_2^{\frac{1}{2}})(X-m_2^{\frac{1}{2}})|] \quad&\le\quad (\mathbb{E}[X^2+2m_2{}^{\frac{1}{2}}X+m_2]\mathbb{E}[X^2-2m_2{}^{\frac{1}{2}}X+m_2])^{\frac{1}{2}}\\ &=\quad (4m_2{}^2-4m_2m_1{}^2)^{\frac{1}{2}} = 2m_2^{\frac{1}{2}}(m_2-m_1{}^2)^{\frac{1}{2}}.\end{aligned}$$

Squaring both sides gives the desired inequality. □

(3) We show the inequality via induction. For $n = 2$,

$$\begin{aligned}
\mathbb{P}(A_1 \cup A_2) &= \mathbb{P}({A_1}^c \cap A_2) + \mathbb{P}(A_1 \cap {A_2}^c) + \mathbb{P}(A_1 \cap A_2) \\
&= \mathbb{P}({A_1}^c \cap A_2) + \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap {A_2}^c) + \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_2) \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).
\end{aligned}$$

Now, suppose the statement is true for some $n > 2$. Then, using the $n = 2$ case, the inductive hypothesis and subadditivity gives

$$\begin{aligned}
\mathbb{P}(\bigcup_{k=1}^{n+1} A_k) &= \mathbb{P}(\bigcup_{k=1}^{n} A_k \bigcup A_{n+1}) \\
&= \mathbb{P}(\bigcup_{k=1}^{n} A_k) + \mathbb{P}(A_{n+1}) - \mathbb{P}(\bigcup_{k=1}^{n} A_k \bigcap A_{n+1}) \\
&\geq \sum_{k=1}^{n} \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n} \mathbb{P}(A_k \bigcap A_l) + \mathbb{P}(A_{k+1}) - \mathbb{P}(\bigcup_{k=1}^{n} A_n \bigcap A_{n+1}) \\
&\geq \sum_{k=1}^{n+1} \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n} \mathbb{P}(A_k \bigcap A_l) - \sum_{k=1}^{n} \mathbb{P}(A_k \bigcap A_{n+1}) \\
&= \sum_{k=1}^{n+1} \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n} \mathbb{P}(A_k \bigcap A_l).
\end{aligned}$$

Therefore, the induction is complete and the inequality is true. Now, if $X = \sum_{k=1}^{n} 1_{A_k}$, then

$$\begin{aligned}
m_1 &= E[X] = \sum_{k=1}^{n} \mathbb{P}(A_k) \\
m_2 &= E[X^2] = \mathbb{E}[\sum_{1 \leq k \leq l \leq n} 1_{A_k}] = \sum_{1 \leq k \leq l \leq n} \mathbb{P}(A_k \cap A_l)
\end{aligned}$$

and

$$\mathbb{P}(X > \alpha) = \mathbb{P}(\bigcup_{k=1}^{n} A_k),$$

for any $\alpha \in [0, 1)$. Hence, to compare, if $X = \sum_{k=1}^{n} 1_{A_k}$, we can re-write (iii) as

$$P(X > \alpha) \geq 2m_1 - m_2,$$

for any $\alpha \in [0, 1)$. $\square$

Question 2.

2. Let X be a real-valued random variable.

   (a) Prove that the function $f(x) = \mathbb{E}\exp(-|X-x|)$ is continuous on $\mathbb{R}$.

   (b) Further suppose that $X \geq 0$ and $\mathbb{E}X^p < \infty$ for some $p > 0$.

      (b.1) Show that $\lim_{p\downarrow 0}(\mathbb{E}X^p - 1)/p = \mathbb{E}\log X$.

      (b.2) Conclude that $\lim_{p\downarrow 0}\log(\mathbb{E}X^p)/p = \mathbb{E}\log X$.

Solution.
(a) We first note that, for any $x \in \mathbb{R}$, $\exp(-|X-x|)$ is uniformly bounded by 1, and we have a finite measure, so the expectation is well-defined and f is well-defined everywhere. Set $\mu = L(X)$. Then, by a change of variable,

$$f(x) \quad = \quad \mathbb{E}[\exp(-|X-x|) = \int_{-\infty}^{\infty} \exp(-|t-x|)\mu(dt)$$

so, for $x, h \in \mathbb{R}$,

$$\begin{aligned} |f(x+h) - f(x)| &= |\int e^{-|t-x-h|} - e^{-|t-x|}\mu(dt)| \\ &\leq \int |e^{-|t-x-h|} - e^{-|t-x|}|\mu(dt) \quad (*). \end{aligned}$$

Observe that

$$|e^{-|t-x-h|} - e^{-|t-x|}| \leq 2$$

for any $t, x, h \in \mathbb{R}$ and

$$|e^{-|t-x-h|} - e^{-|t-x|}| \to 0 \text{ as } h \to 0$$

for any $t, x \in \mathbb{R}$. Therefore, by BCT and (*), it follows that

$$|f(x+h) - f(x)| \to 0 \text{ as } h \to 0,$$

which shows that f is continuous as required. $\square$

(b.1) With L'hopital's rule, we obtain that

$$\lim_{p\downarrow 0} \frac{X^p(w) - 1}{p} \quad = \quad \lim_{p\downarrow 0} X^p(w)\log(X(w)) = \log(X(w)),$$

for all $w \in \Omega$. Hence,

$$\{\frac{X^p - 1}{p}\}_{p \geq 0} \text{ converges almost surely to } \log(X) \text{ on } \Omega$$

By DCT,

$$\lim_{p\downarrow 0} \frac{(\mathbb{E}X^p - 1)}{p} \quad = \quad \mathbb{E}\log(X),$$

as required. $\square$

(b.2)

(Remark) Although all limit theorems are stated so far for a countable limit, they apply as well to a continuous limit. Suppose $\{X_t\}_{t\geq 0}$ is a family of L^1 dominated random variables such that $\lim_{t\downarrow 0} X_t(w) \to X_0(w)$ for all $w \in \Omega$. Then, by DCT, for any $\{t_n\} \subset (0,\infty)$ such that $t_n \downarrow 0$, we have $\lim_{n\to\infty}\mathbb{E}[X_{t_n}] = \mathbb{E}[X_0]$. Since this is true for any such sequence, it follows that $\lim_{t\downarrow 0}\mathbb{E}[X_t] = \mathbb{E}[X_0]$ in a proper continuous limit sense. We will freely use the limit theorem in the continuous setting without doing the above pass everytime.

Question 3.

3. Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be a random variable with $\mathbb{E}|X| < \infty$.

(i) Show that if $A_n \in \mathcal{F}$ are disjoint sets and $A = \bigcup_n A_n$ then $\sum_n \mathbb{E}[X\mathbf{1}_{A_n}]$ converges absolutely and $\sum_n \mathbb{E}[X\mathbf{1}_{A_n}] = \mathbb{E}[X\mathbf{1}_A]$.

(ii) Conclude that if $X \geq 0$ then $\mathbb{Q}(A) = \mathbb{E}[X\mathbf{1}_A]/\mathbb{E}X$ is a probability measure.

Solution.
We first show the case for non-negative, simple functions. Let X be simple, such that

$$X = \sum_{k=1}^{l} a_k \mathbb{1}_{E_k},$$

where $a_k \in \mathbb{R}$ for $k = 1, ..., l$ and $E_k \in \mathscr{F}$ with $\dot{\bigcup}_{k=1}^{l} E_k = \Omega$. With linearity of expectation,

$$\begin{aligned}\mathbb{E}[X\mathbb{1}_A] &= \mathbb{E}[\sum_{k=1}^{l} a_k \mathbb{1}_{E_k}\mathbb{1}_A] = \sum_{k=1}^{l} a_k \mathbb{E}[\mathbb{1}_{E_k}\mathbb{1}_A] \\ &= \sum_{k=1}^{l} a_k \mathbb{E}[\mathbb{1}_{E_k \cap A}] = \sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap A).\end{aligned}$$

Similarly,

$$\mathbb{E}[X\mathbb{1}_{A_n}] = \sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap A_n)$$

for each $n \geq 1$. Then, it follows that, for all $m \geq 1$,

$$\begin{aligned}\sum_{n=1}^{m} |\mathbb{E}[X\mathbb{1}_{A_n}]| &= \sum_{n=1}^{m}\sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap A_n) \\ &= \sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap \bigcup_{m \geq n} A_n),\end{aligned}$$

where the equality holds by disjointness of $\{A_n\}$. Since $\bigcup_n A_n = A$, we can exploit continuity of probability and obtain

$$\begin{aligned}\sum_n |\mathbb{E}[X\mathbb{1}_{A_n}]| &= \lim_{m\to\infty} \sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap \bigcup_{m\geq n} A_n) \\ &= \sum_{k=1}^{l} a_k \lim_{m\to\infty} \mathbb{P}(E_n \cap \bigcup_{m \geq n} A_n) = \sum_{k=1}^{l} a_k \mathbb{P}(E_n \cap A) = \mathbb{E}[X\mathbb{1}_A].\end{aligned}$$

Hence, we have shown that for X non-negative and simple, $\sum_n \mathbb{E}[X\mathbb{1}_{A_n}]$ converges absolutely to $\mathbb{E}[X\mathbb{1}_A]$.

We now extend the case to non-negative integrable functions. Let X be a bounded, measurable, non-negative functions. Choose $\{\phi_k\}$ simple functions such that $\phi_k \to X$. By the previous result, we observe

$$\sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}] = \mathbb{E}[\phi_k \mathbb{1}_A] \ (*)$$

for any $k \geq 1$. Since $\phi_k \to X$ uniformly, by monotone convergence theorem,

$$\mathbb{E}[\phi_k \mathbb{1}_A] \to \mathbb{E}[\phi_k \mathbb{1}_A]$$

and

$$\mathbb{E}[\phi_k \mathbb{1}_{A_n}] \to \mathbb{E}[\phi_k \mathbb{1}_{A_n}]$$

which via implies

$$\sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}] \to \sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}].$$

Combining (*) with the above limit, we see that $\sum_n \mathbb{E}[X\mathbb{1}_{A_n}]$ converges absolutely and $\sum_n \mathbb{E}[X\mathbb{1}_{A_n}] = \mathbb{E}[\mathbb{1}_A]$ as required. By considering the positive part and negative part, we can extend the result to any random variable as required.

(ii) Firstly, observe that

$$\mathbb{Q}(\Omega) \quad = \quad \frac{\mathbb{E}[\mathbb{1}_\Omega]}{\mathbb{E}[X]} = \frac{\mathbb{E}[X]}{\mathbb{E}[X]} = 1.$$

Hence, it now suffices to show that $\mathbb{Q}$ is countably additive, but from the discussion in (i), we see

$$\mathbb{Q}(\bigcup_n A_n) \quad = \quad \frac{\mathbb{E}[X\mathbb{1}_{\cup_n A_n}]}{\mathbb{E}[X]} = \frac{\sum_n \mathbb{E}[X\mathbb{1}_{A_n}]}{\mathbb{E}[X]} = \sum_n \mathbb{Q}(A_n).$$

for any $\{A_n\} \subset \mathscr{F}$ that are pairwise disjoint. So, $\mathbb{Q}$ is a probability measure, if $X \geq 0$ and we are done $\square$

Question 4.

4. Let $Y = \sum_{k=1}^{n} \mathbf{1}_{A_k}$ for some measurable sets $A_1, \dots, A_n$. Express $\mathrm{Var}(Y)$ in terms of $\mathbb{P}(A_k)$ and $\mathbb{P}(A_k \cap A_\ell)$, then calculate it for the following case: each one of m players selects, independently and uniformly, a number in $\{1, \dots, n\}$; the event A_k says that the number k was not selected by any player.

Solution.
We compute

$$
\begin{aligned}
\mathrm{Var}[Y] &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[(\sum_{k=1}^{l} \mathbb{1}_{A_k})^2] - \mathbb{E}[\sum_{k=1}^{l} \mathbb{1}_{A_k}]^2 \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k}\mathbb{1}_{A_l}] - \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k}]\mathbb{E}[\mathbb{1}_{A_l}] \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k \cap A_l}] - \mathbb{E}[\mathbb{1}_{A_k}]\mathbb{E}[\mathbb{1}_{A_l}] \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{P}(A_k \cap A_l) - \mathbb{P}(A_k)\mathbb{P}(A_l).
\end{aligned}
$$

Now, observe that, for $k = 1, \dots, n$,

$$
\mathbb{P}(A_k) = (\frac{n-1}{n})^m
$$

and for $k, l = 1, \dots, n$,

$$
\begin{aligned}
k = l &\implies \mathbb{P}(A_k \cap A_l) = (\frac{n-1}{n})^m \\
k \neq l &\implies \mathbb{P}(A_k \cap A_l) = (\frac{n-2}{n})^m.
\end{aligned}
$$

So

$$
\begin{aligned}
\mathrm{Var}[Y] &= \sum_{1 \le k, l \le n; k \neq l} (\frac{n-2}{n})^m - (\frac{n-1}{n})^{2m} + \sum_{1 \le k, l \le n; k = l} (\frac{n-1}{n})^m - (\frac{n-1}{n})^{2m} \\
&= (n^2 - n)(\frac{n-2}{n})^m + n(\frac{n-1}{n})^m - n^2(\frac{n-1}{n})^{2m}
\end{aligned}
$$

as required. □