Durrett Probability: Problems

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Abstract

This work contains solutions to some exercises from Durrett's probability text.

1 Chapter 6: Markov Chains

Question 6.3.3.

6.3.3. First entrance decomposition. Let $T_y = \inf\{n \geq 1 : X_n = y\}$. Show that

$$p^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m) p^{n-m}(y,y)$$

Solution.
Here we assume countable state space. Observe that

$$p^n(x,y) \quad = \quad P_x(X_n = y) = P_x\Big(\bigcup_{m=1}^{n} \{T_y = m \,;\, X_n = y\}\Big) = \sum_{m=1}^{n} P_x(T_y = m \,;\, X_n = y) \tag{1}$$

$$\begin{aligned} P_x(T_y = m \,;\, X_n = y) &= E_x(1_{\{X_n = y\}} \,;\, T_y = m) \\ &= E_x(E_x(1_{\{X_n=y\}} | \mathscr{F}_m); T_y = m) && (2) \\ &= E_x(E_x(1_{\{X_{n-m}=y\}} \circ \theta_m | \mathscr{F}_m); T_y = m) \\ &= E_x(E_{X_m}(1_{\{X_{n-m}=y\}}; T_y = m) = E_x(P_y(X_{n-m} = y); T_y = m) && (3) \\ &= P_x(T_y = m) P_y(X_{n-m} = y) \end{aligned}$$

for any $1 \leq m \leq n$, where (4) holds by definition of conditional expectation and (5) holds by Markov property. Therefore, combining the above result with with (1) gives

$$p^n(x,y) \quad = \quad \sum_{m=1}^{n} P_x(T_y = m) P_y(X_{n-m} = y).$$

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Here is another approach using strong Markov. We compute

$$
\begin{aligned}
p^n(x,y) &= P_x(X_n = y) = P_x\Big(\bigcup_{m=1}^{n} \{T_y = m; X_n = y\}\Big) \\
&= E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y}; T_y \le n) = E_x(E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y} | \mathscr{F}_{T_y}); T_y \le n) && (4) \\
&= E_x(E_{X_{T_y}}(1_{\{X_{n-T_y}=y\}}; T_y \le n) = E_x(E_y(1_{\{X_{n-T_y}\}}); T_y \le n) && (5) \\
&= \sum_{m=1}^{n} P_x(T_y = m) E_y(1_{\{X_{n-m}=y\}}) = \sum_{m=1}^{n} P_x(T_y = m) P^{n-m}(y,y)
\end{aligned}
$$

where (4) holds by definition of conditional expectation and (5) holds by the strong Markov property. □

Question 6.3.4.

6.3.4. Show that $\sum_{m=0}^{n} P_x(X_m = x) \geq \sum_{m=k}^{n+k} P_x(X_m = x)$.

Solution.
Let $k \in \mathbb{N}$, and $T_x^k = \inf\{n \geq k : X_n = x\}$. We claim that

$$P_x(X_m = x) \quad = \quad \sum_{l=k}^{m} P_x(T_x^k = x) p^{m-l}(x,x) \tag{6}$$

for any $m \geq k$. Fix $m \geq k$. Then,

$$P_x(X_m = x) \quad = \quad P_x(\bigcup_{l=k}^{m} \{T_x^k = l; X_m = x\}) = \sum_{l=k}^{m} P_x(T_x^k = l; X_m = x). \tag{7}$$

Now, we compute

$$\begin{aligned} P_x(T_x^k = l; X_m = x) &= E_x(1_{\{X_m = x\}}; T_x^k = l) = E_x(E_x(1_{\{X_m=x\}} | \mathscr{F}_l); T_x^k = l) \\ &= E_x(E_x(1_{\{X_{m-l}=x\}}\theta_l | \mathscr{F}_l); T_x^k = l) \\ &= E_x(E_{X_l}(1_{\{X_{m-l}=x\}}; T_x^k = l); T_x^k = l) \\ &= E_x(P_x(X_{m-l} x); T_x^k = l) = P_x(X_{m-l} = x) P_x(T_x^k = l) \\ &= P_x(T_x^k = l) p^{m-l}(x,x) \end{aligned} \tag{8}$$

for any $k \leq l \leq m$, where (8) holds by Markov property. Therefore, combining the above result with (7), we have proven (6). Then,

$$\begin{aligned} \sum_{m=k}^{n+k} P_x(X_m = x) &= \sum_{m=k}^{n+k} \sum_{l=k}^{m} P_x(T_x^k = l) p^{m-l}(x,x) \\ &= \sum_{m=k}^{n+k} p^m(x,x) \left(\sum_{l=k}^{d} P_x(T_x^k = l) \right) \\ &\leq \sum_{m=k}^{n+k} p^m(x,x) = \sum_{m=k}^{n+k} P_x(X_m = x) \end{aligned}$$

Question 6.3.5.

6.3.5. Suppose that $S - C$ is finite and for each $x \in S - C$ $P_x(\tau_C < \infty) > 0$. Then there is an $N < \infty$ and $\epsilon > 0$ so that $P_y(\tau_C > kN) \le (1-\epsilon)^k$.

Solution.
We assume countable state space. Observe that, for any $x \in S \setminus C$, we can choose $n(x) \in \mathbb{N}$ such that

$$P_x(\tau_C \le n) > 0.$$

Otherwise, for some $x \in S \setminus C$, by continuity of probability,

$$P_x(\tau_C < \infty) = \lim_{k\to\infty} P_x(\tau_C \le k) = 0,$$

which is a contradiction. Now, let

$$N = \max_{z \in S\setminus C} n(x). \quad \text{and} \quad \epsilon = \min_{z\in S\setminus C} P_z(\tau_C \le N).$$

Trivially,

$$P_y(\tau_C > kN) = 0$$

for any $k \in \mathbb{N}$, and $y \in C$, since $y \in C$ implies $\tau_C = 0$ by definition. Therefore, it suffices to show

$$P_y(\tau_C > kN) \le (1-\epsilon)^k \tag{9}$$

for all $k \in \mathbb{N}$ and $y \in S \setminus C$. Fix $y \in S \setminus C$. Then,

$$P_y(\tau_C \le N) \quad \ge \quad \epsilon.$$

and hence

$$P_y(\tau_C > N) \le (1-\epsilon)$$

Now, we proceed by induction to prove (9). Suppose, for some $k \in \mathbb{N}$ such that $k \ge 2$,

$$P_y(\tau_C > kN) \le (1-\epsilon)^k.$$

We compute

$$\begin{aligned} P_y(T_c > (k+1)N) &= E_y(1_{\{\tau_C > kN\}} \circ \theta_N; \tau_C > N) \\ &= E_y(E_y((1_{\{\tau_C > kN\}} \circ \theta_N | \mathscr{F}_N); \tau_C > N)) \\ &= E_y(E_{X_N}((1_{\{\tau_C > kN\}}); \tau_C > N)) \qquad (10) \\ &\le E_y(\sup_{z\in S} P_z(\tau_C > kN); \tau_C > N)) \\ &\le (1-\epsilon)^k E_y(1; \tau_C > N)) = (1-\epsilon)^{k+1} \end{aligned}$$

where (10) holds by Markov Property, which completes the proof. $\square$

Question 6.3.6.

6.3.6. Let $h(x) = P_x(\tau_A < \tau_B)$. Suppose $A \cap B = \emptyset$, $S - (A \cup B)$ is finite, and $P_x(\tau_{A\cup B} < \infty) > 0$ for all $x \in S - (A \cup B)$. (i) Show that

$$(*) \qquad h(x) = \sum_y p(x,y)h(y) \quad \text{for } x \notin A \cup B$$

(ii) Show that if h satisfies $(*)$ then $h(X(n \wedge \tau_{A\cup B}))$ is a martingale. (iii) Use this and Exercise 6.3.5 to conclude that $h(x) = P_x(\tau_A < \tau_B)$ is the only solution of $(*)$ that is 1 on A and 0 on B.

Solution.
(i) Let $x \in S \setminus (A \cup B)$. Then,

$$1_{\{\tau_A < \tau_B\}} = 1_{\{\tau_A < \tau_B\}} \circ \theta_1.$$

It follows that

$$\begin{aligned} h(x) &= P_x(\tau_A < \tau_B) = E_x(1_{\{\tau_A<\tau_B\}}) = E_x(1_{\{\tau_A<\tau_B\}} \circ \theta_1) \\ &= E_x(E_x(1_{\{\tau_A<\tau_B\}} \circ \theta_1 | \mathscr{F}_1)) = E_x(E_{X_1}(1_{\{\tau_A<\tau_B\}})) \qquad (11) \\ &= \sum_y P(X_1 = y)P_y(\tau_A < \tau_B) = \sum_y p(x,y)P_y(\tau_A < \tau_B) \end{aligned}$$

where (11) holds by Markov property.

(ii)

(iii)

Question 6.3.7.

6.3.7. Let X_n be a Markov chain with $S = \{0, 1, \dots, N\}$ and suppose that X_n is a martingale and $P_x(\tau_0 \wedge \tau_N < \infty) > 0$ for all x. (i) Show that 0 and N are absorbing states, i.e., $p(0,0) = p(N,N) = 1$. (ii) Show $P_x(\tau_N < \tau_0) = x/N$.

Solution.

Question 6.4.4.

Exercise 6.4.4. Use the strong Markov property to show that $\rho_{xz} \geq \rho_{xy}\rho_{yz}$.

Solution.
The key insight in this problem is that if you shift the chain by a stopping time of one state variable, then the probability of the chain coming back to another fixed state variable decreases. This relation allows one to estimate p_{xz} from below using strong Markov, which in our context is proven for a sequence space(discrete time) of a polish state space, using Monotone class theorem. Recall that to define a shift operator, indexed by ∞, by convention, we set

$$\theta_\infty(w) = \triangle$$

where $\triangle$ is the cemetery sample point we add to $S^{\mathbb{N}}$, for all $w \in S^{\mathbb{N}}$. Therefore, to extend the domain of $T_z = \inf\{n \geq 1 : X_n = z\}$ for any $z \in S$, to include $\triangle$, if necessary, we define

$$T_z(\triangle) = \infty \quad \text{so} \quad 1_{\{T_z<\infty\}}(\triangle) = 0,$$

With this convention.

$$\begin{aligned} \{w \in S^{\mathbb{N}} : 1_{\{T_z<\infty\}} \circ \theta_{T_y}(w) = 1\} &= \{w \in S^{\mathbb{N}} : T_y(w) = n \text{ for some } n \geq 1 \\ &\quad \text{and } \; T_z^n(w) = \inf\{k \geq n : X_k = z\} < \infty\} \\ &= \bigcup_{n=1}^{\infty} \{T_y = n \; ; \; T_z^n < \infty\} \\ &\subset \bigcup_{n=1}^{\infty} \{T_z^n < \infty\} = \{T_z < \infty\} \end{aligned}$$

for any $z, y \in S$.

Now, let $x, y, z \in S$. Then,

$$\begin{aligned} p_{xz} &= P_x(T_z < \infty) = E_x(1_{\{T_z<\infty\}}) \geq E_x(1_{\{T_z<\infty\}} \circ \theta_{T_y}) \\ &= E_x(E_x(1_{\{T_z<\infty\}} \circ \theta_{T_y} | \mathscr{F}_{T_y}); T_y < \infty) \qquad (12) \\ &= E_x(E_{X_{T_y}}(1_{\{T_z<\infty\}}; T_y < \infty) = E_x(E_{X_y}(1_{\{T_z<\infty\}}; T_y < \infty) \qquad (13) \\ &= E_x(P_y(T_z < \infty); T_y < \infty) = P_y(T_z < \infty)P_x(T_y < \infty) = p_{xy}p_{yz} \end{aligned}$$

where (12) holds by definition of conditional expectation, and (13) holds by strong Markov. $\square$

2 Chapter 2: Law of Large Numbers

3 Chapter 4: Random Walks

Question 4.1.1.

Exercise 4.1.1. Symmetric random walk. Let $X_1, X_2, \ldots \in \mathbf{R}$ be i.i.d. with a distribution that is symmetric about 0 and nondegenerate (i.e., $P(X_i = 0) < 1$). Show that we are in case (iv) of Theorem 4.1.2.

Question 4.1.2.

Exercise 4.1.2. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$. Use the central limit theorem to conclude that we are in case (iv) of Theorem 4.1.2. Later in Exercise 4.1.11 you will show that $EX_i = 0$ and $P(X_i = 0) < 1$ is sufficient.

Question 4.1.3.

Exercise 4.1.3. If S and T are stopping times then $S \wedge T$ and $S \vee T$ are stopping times. Since constant times are stopping times, it follows that $S \wedge n$ and $S \vee n$ are stopping times.

Question 4.1.4.

Exercise 4.1.4. Suppose S and T are stopping times. Is $S + T$ a stopping time? Give a proof or a counterexample.

Question 4.1.5.

Exercise 4.1.5. Show that if $Y_n \in \mathcal{F}_n$ and N is a stopping time, $Y_N \in \mathcal{F}_N$. As a corollary of this result we see that if $f : S \to \mathbf{R}$ is measurable, $T_n = \sum_{m \le n} f(X_m)$, and $M_n = \max_{m \le n} T_m$ then T_N and $M_N \in \mathcal{F}_N$. An important special case is $S = \mathbf{R}$, $f(x) = x$.

4 Chapter 5: Martingales

Question 5.2.1.

Exercise 5.2.1. Suppose X_n is a martingale w.r.t. $\mathcal{G}_n$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then $\mathcal{G}_n \supset \mathcal{F}_n$ and X_n is a martingale w.r.t. $\mathcal{F}_n$.

Solution.
Various properties of conditional expectations are used.

We compute

$$\begin{aligned} E[X_{n+1}|\mathscr{F}_n] &= E[X_{n+1}|\mathscr{G}_n|\mathscr{F}_n] \qquad (14) \\ &= E[X_n|\mathscr{F}_n] \qquad (15) \\ &= X_n \qquad (16) \end{aligned}$$

for all $n \in \mathbb{N}$, where (14) holds by the Tower property, (15) holds by Martingale property of $\{G_n\}$ and (16) holds by measurability of X_n w.r.t $\mathscr{F}_n$ for all $n \in \mathbb{N}$. $\square$

Question 5.2.2.

Exercise 5.2.2. Suppose f is superharmonic on $\mathbf{R}^d$. Let $\xi_1, \xi_2, \dots$ be i.i.d. uniform on $B(0,1)$, and define S_n by $S_n = S_{n-1} + \xi_n$ for $n \geq 1$ and $S_0 = x$. Show that $X_n = f(S_n)$ is a supermartingale.

Solution.

Question 5.2.3.

Exercise 5.2.3. Give an example of a submartingale X_n so that X_n^2 is a supermartingale. Hint: X_n does not have to be random.

Solution.

Consider $\{X_n = 0\}$. Then, $\{X_n^2 = 0\}$, so both are processes are martingales, we have the desired example. $\square$

Question 5.2.4.

> **Exercise 5.2.4.** Give an example of a martingale X_n with $X_n \to -\infty$ a.s. Hint: Let $X_n = \xi_1 + \cdots + \xi_n$, where the ξ_i are independent (but not identically distributed) with $E\xi_i = 0$.

Solution.

Question 5.2.5.

Exercise 5.2.5. Let $X_n = \sum_{m \leq n} 1_{B_m}$ and suppose $B_n \in \mathcal{F}_n$. What is the Doob decomposition for X_n?

Solution.

Question 5.2.6.

5.2.6. Let $\xi_1, \xi_2, \ldots$ be independent with $E\xi_i = 0$ and $\operatorname{var}(\xi_m) = \sigma_m^2 < \infty$, and let $s_n^2 = \sum_{m=1}^n \sigma_m^2$. Then $S_n^2 - s_n^2$ is a martingale.

Solution.

Question 5.2.7.

5.2.7. If $\xi_1, \xi_2, \ldots$ are independent and have $E\xi_i = 0$ then

$$X_n^{(k)} = \sum_{1 \le i_1 < \ldots < i_k \le n} \xi_{i_1} \cdots \xi_{i_k}$$

is a martingale. When $k = 2$ and $S_n = \xi_1 + \cdots + \xi_n$, $2X_n^{(2)} = S_n^2 - \sum_{m \le n} \xi_m^2$.

Solution.