ProbLimI:
Problem Set IX

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Abstract

This work contains solutions to the exercises of the problem set IX. The chosen problems are 2,3 and 4.

Question 2.

2. Let X be an r.v. in $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}X^2 < \infty$ and write $\mathrm{Var}(X \mid \mathcal{G}) := \mathbb{E}\left[(X - \mathbb{E}[X \mid \mathcal{G}])^2 \mid \mathcal{G}\right]$.

   (a) Prove that $\mathbb{E}[\mathrm{Var}(X \mid \mathcal{G})] \leq \mathbb{E}[\mathrm{Var}(X \mid \mathcal{H})]$ for every σ-fields $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$.

   (b) Prove that $\mathrm{Var}(X) = \mathbb{E}\left[\mathrm{Var}(X \mid \mathcal{G})\right] + \mathrm{Var}\left(\mathbb{E}[X \mid \mathcal{G}]\right)$ for every σ-field $\mathcal{G} \subset \mathcal{F}$.

Solution.
(a) The result intuitively makes sense, since on average knowing more information should reduce variance. We compute

$$\begin{aligned}
\mathbb{E}[\mathrm{Var}(X|H)] &= \mathbb{E}[(X - \mathbb{E}[X|H])^2] = \mathbb{E}[(X - \mathbb{E}[X|G])^2] + \mathbb{E}[(\mathbb{E}[X|G] - \mathbb{E}[X|H])^2] \\
&+ 2\mathbb{E}[(\mathbb{E}[X|G] - \mathbb{E}[X|H])(X - \mathbb{E}[X|G])] \\
&= \mathbb{E}[\mathrm{Var}(X|G)] - \mathbb{E}[(\mathbb{E}[X|G] - \mathbb{E}[X|H])^2] \geq \mathbb{E}[\mathrm{Var}(X|G)] \qquad (1)
\end{aligned}$$

where (1) holds since $\mathbb{E}[X|G] - \mathbb{E}[X|H]$ is G measurable.

(b) Let $m = \mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|\mathscr{G}]]$. We compute

$$\begin{aligned}
\mathrm{Var}[Y] &= \mathbb{E}[(Y - m)^2] = \mathbb{E}[\mathbb{E}[(Y - m)^2|\mathscr{G}]] \\
&= \mathbb{E}[\mathbb{E}[(Y - \mathbb{E}[Y|\mathscr{G}] + \mathbb{E}[Y|\mathscr{G}] - m)^2|\mathscr{G}]] \\
&= \mathbb{E}[\mathbb{E}[(Y - \mathbb{E}[Y|\mathscr{G}])^2|\mathscr{G}]] - \mathbb{E}[\mathbb{E}[(\mathbb{E}[Y|\mathscr{G}] - m)^2|\mathscr{G}]] \\
&+ \mathbb{E}[\mathbb{E}[2(Y - \mathbb{E}[Y|\mathscr{G}])(\mathbb{E}[Y\mathscr{G}] - m))|\mathscr{G}]] \qquad (2) \\
&= \mathbb{E}[\mathrm{Var}(Y|\mathscr{G}] + \mathrm{Var}[\mathbb{E}[Y|\mathscr{G}]] \\
&+ \mathbb{E}[\mathbb{E}[2(Y - \mathbb{E}[Y|\mathscr{G}])(\mathbb{E}[Y\mathscr{G}] - m))|\mathscr{G}]] \qquad (3)
\end{aligned}$$

where (2) holds by linearity of conditional expectation. Now, from ,

$$\begin{aligned}
\mathbb{E}[(Y - \mathbb{E}[Y|\mathscr{G}])(\mathbb{E}[Y|\mathscr{G}] - m)|\mathscr{G}] &= (\mathbb{E}[Y|\mathscr{G}] - m)\mathbb{E}[(Y - \mathbb{E}[Y|\mathscr{G}])|\mathscr{G}] \qquad (4) \\
&= (\mathbb{E}[Y|\mathscr{G}] - m)(\mathbb{E}[Y|\mathscr{G} - \mathbb{E}[Y|\mathscr{G}]) = 0 \qquad (5)
\end{aligned}$$

almost surely, where (4) holds by "taking out what's known." Now, combining (3) and (5),

$$\mathrm{Var}[Y] = \mathbb{E}[\mathrm{Var}(Y|\mathscr{G}] + \mathrm{Var}[\mathbb{E}[Y|\mathscr{G}]].$$

$\square$

Question 3.

3. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $Y = \mathbb{E}[X \mid \mathcal{G}]$ for a σ-field $\mathcal{G} \subset \mathcal{F}$.

(a) Prove that if $\mathbb{E}Y^2 = \mathbb{E}X^2 < \infty$ then $Y = X$ a.s.

(b) Prove that if $Y \overset{d}{=} X$ then $Y = X$ a.s., even in the case $\mathbb{E}X^2 = \infty$.

Solution.
(a) We compute

$$\begin{aligned}
\mathbb{E}[(X-Y)^2] &= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - 2\mathbb{E}[XY] \\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - 2\mathbb{E}[\mathbb{E}[XY|\mathscr{G}]] \qquad (6)\\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - 2\mathbb{E}[Y\mathbb{E}[X|\mathscr{G}]] \qquad (7)\\
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - 2\mathbb{E}[Y^2] = 0
\end{aligned}$$

where (6) holds as the expectation of the conditional expectation of any L^1 random variable is the expectation of the random variable, and (7) holds as Y is $\mathscr{G}$ measurable. Therefore, $X = Y$ almost surely.

(b) We proceed by a standard truncation argument. Let $a > 0$ and $b < 0$. By conditional Jensen,

$$\mathbb{E}[X \wedge a|\mathscr{G}] \leq \mathbb{E}[X|\mathscr{G}] \wedge a = Y \wedge a \text{ almost surely}$$

Since $X = Y$ in distribution,

$$\mathbb{E}[X \wedge a] = \mathbb{E}[Y \wedge a].$$

Therefore, the above inequality cannot be strict, which implies

$$\mathbb{E}[X \wedge a|\mathscr{G}] = Y \wedge a \text{ almost surely}$$

as otherwise by taking the expectation both sides, we get a contradiction. Similarly,

$$\mathbb{E}[(X \wedge a) \vee b|\mathscr{G}] = (Y \wedge a) \vee b \text{ almost surely} \qquad (8)$$

From $X = Y$ in distribution,

$$(X \wedge a) \vee b = (Y \wedge a) \vee b \text{ in distribution}$$

and hence

$$\mathbb{E}[((X \wedge a) \vee b)^2] = \mathbb{E}[((Y \wedge a) \vee b)^2].$$

Therefore, from part (a) and (8),

$$(X \wedge a) \vee b = (Y \wedge a) \vee b \text{ almost surely}$$

for any $a > 0$ and $b < 0$. Taking $a \to \infty$ and $b \to -\infty$,

$$X = Y \text{ almost surely.}$$

$\square$

Question 4.

4. (a) Fix $p > 0$ and let X be an r.v. in $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}|X|^p < \infty$. Show that for every σ-field $\mathcal{G} \subset \mathcal{F}$, a.s.

$$\mathbb{E}\left[|X|^p \mid \mathcal{G}\right] = \int_0^\infty p x^{p-1} \mathbb{P}(|X| > x \mid \mathcal{G}) dx \,,$$

and conclude that for every $a > 0$,

$$\mathbb{P}\left(|X| \geq a \mid \mathcal{G}\right) \leq a^{-p} \mathbb{E}\left[|X|^p \mid \mathcal{G}\right] .$$

(b) Let X, Y be r.v.'s on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}|X|^p < \infty$, $\mathbb{E}|Y|^q < \infty$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that for every σ-field $\mathcal{G} \subset \mathcal{F}$,

$$\mathbb{E}\left[|XY| \mid \mathcal{G}\right] \leq \mathbb{E}\left[|X|^p \mid \mathcal{G}\right]^{1/p} \mathbb{E}\left[|Y|^q \mid \mathcal{G}\right]^{1/q} .$$

Solution.
(a) By Fubini,

$$\begin{aligned}
\mathbb{E}[|X|^p|\mathscr{G}] &= \int_\Omega |x|^p \mathbb{P}(dx|\mathscr{G}) = \int_\Omega \int_0^\infty p x^{p-1} 1_{\{|X|>x\}} dx \mathbb{P}(dx|\mathscr{G}) \\
&= \int_0^\infty p x^{p-1} 1_{\{|X|>x\}} dx \mathbb{P}(dx|\mathscr{G}) = \int_0^\infty p x^{p-1} \mathbb{P}(|X| > x|\mathscr{G}) dx \text{ a.s.}
\end{aligned}$$

for any $\mathscr{G} \subset \mathscr{F}$. Let $a > 0$. Then,

$$\begin{aligned}
a^{-p}\mathbb{E}[|X|^p|\mathscr{G}] &= a^{-p} \int_0^a p x^{p-1} \mathbb{P}(|X| > x|\mathscr{G}) dx + a^{-p} \int_a^\infty p x^{p-1} \mathbb{P}(|X| > x|\mathscr{G}) dx \\
&\leq a^{-p} \int_0^a p x^{p-1} \mathbb{P}(|X| > x|\mathscr{G}) dx + \mathbb{P}(|X| = a|\mathscr{G}) = \mathbb{P}(|X| \geq a|\mathscr{G})
\end{aligned}$$

(b) Let $A = (\mathbb{E}[|X|^p|\mathscr{G}])^{\frac{1}{p}}$ and $B = (\mathbb{E}[|Y|^p|\mathscr{G}])^{\frac{1}{p}})$. We compute

$$\begin{aligned}
\mathbb{E}[|X|^p 1_{\{A=0\}}] &= \mathbb{E}[\mathbb{E}[|X|^p 1_{\{A=0\}}]] \\
&= \mathbb{E}[1_{\{A=0\}} \mathbb{E}[|X|^p|\mathscr{G}]] = \mathbb{E}[1_{\{A=0\}} A^p] = 0
\end{aligned}$$

and hence $|X| = 0$ a.s. on $\{A = 0\}$. By the same computation, $|Y| = 0$ a.s. on $\{B = 0\}$, which implies

$$\mathbb{E}[|XY||\mathscr{G}] = 0 \;\; a.s. \;\; \text{on} \;\; \{A = 0\} \cup \{B = 0\}.$$

Hence, it suffices to show the inequality on $\Omega_0 = \{A \neq 0\} \cap \{B \neq 0\}$. We compute

$$\begin{aligned}
\mathbb{E}[\frac{\mathbb{E}[|XY||\mathscr{G}]}{AB} 1_G] &= \mathbb{E}[\frac{|X|}{A} 1_G \frac{|Y|}{B} 1_G] \\
&\leq (\mathbb{E}[\frac{|X|^p}{A^p} 1_G])^{\frac{1}{p}})(\mathbb{E}[\frac{|Y|^q}{B^q} 1_G])^{\frac{1}{q}} \qquad (9) \\
&= \mathbb{E}[1_G]
\end{aligned}$$

for any $G \in \mathscr{G}$ where (9) holds by Holder. Therefore, the inequality holds on Ω_0, so we are done. □