# ProbLimI:
# Problem Set X

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## Abstract

This work contains solutions to the exercises of the problem set X. The chosen problems are 1,3,4.

**Question 1.**

1. Let $\{X_i\}$ be i.i.d. r.v.'s where $\mathbb{P}(X_1 = k) = q_k$ for some $(q_k)_{k=1}^{\infty}$ with $\sum_k q_k = 1$ and $\mathbb{E}X_1 < \infty$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $S_n = \sum_{i=1}^{n} X_i$, and define $(\tau_t)_{t=0}^{\infty}$ via $\tau_t = \inf\{n : S_n \geq t\}$.
   (a) Show that for every $t \geq 0$, the variable $\tau_t$ is an $\mathcal{F}_t$-*stopping time*; that is, for every $n \geq 0$, the event $\{\tau_t \leq n\}$ is $\mathcal{F}_n$-measurable. Further show that every $\mathcal{F}_n$-stopping time $\tau$ enjoys the following property: if there exists a positive integer $m$ and some $\varepsilon > 0$ such that
   $$\mathbb{P}(\tau \leq n + m \mid \mathcal{F}_n) > \varepsilon \quad \text{for all } n\,,$$
   then necessarily $\mathbb{P}(\tau > km) \leq (1-\varepsilon)^k$ for every positive integer $k$, and hence $\mathbb{E}\tau < \infty$.
   (b) Show that for any nonrandom $t$, the variable $X_{\tau_t+1}$ is independent of $\mathcal{F}_{\tau_t}$ and $X_{\tau_t+1} \stackrel{d}{=} X_1$.
   (c) Show that $D_t = S_{\tau_t} - t$ is a homogeneous Markov chain: find its transition probabilities and show that $\mu(i) = (\mathbb{E}X_1)^{-1} \sum_{k>i} q_k$ is an invariant measure for it.

**Solution.**
**(a)** Observe that

$$\begin{aligned} \{\tau_t \leq n\} &= \{\inf\{m \geq 1 \,:\, S_m \geq t\} \leq n\} \\ &= \bigcup_{m=1}^{n} \{S_m \geq t\} \in \sigma(X_1, ..., X_n) = \mathscr{F}_n \end{aligned}$$

for any $n \geq 1$ and $t \geq 0$, and hence $\tau_t$ is an $\mathscr{F}_t$ measurable.

Suppose $\tau$ is a $\mathscr{F}_n$ stopping time and there exists a positive integer $m$ and $\epsilon > 0$ such that

$$\mathbb{P}(\tau \leq n + m \,|\, \mathscr{F}_n) > \epsilon$$

for all $n \geq 1$.

**Question 2.**

3. Let $X_n$ be a homogeneous Markov chain on a countable state space $\Omega$ with an associated $\sigma$-field $\mathcal{B}$, and let $\tau_{x,k} = \inf\{n \geq k : X_n = x\}$.

   (a) Prove that for every $x_0, x \in \Omega$, every $B \in \mathcal{B}$ and every positive integer $k \leq n$,
   $$\mathbb{P}_{x_0}(X_n \in B\,, \tau_{x,k} \leq n) = \sum_{i=0}^{n-k} \mathbb{P}_{x_0}(\tau_{x,k} = n-i)\mathbb{P}_x(X_i \in B)\,,$$
   where $\mathbb{P}_{x_0}(A) := \mathbb{P}(A \mid X_0 = x_0)$.

   (b) Deduce that
   $$\mathbb{P}_{x_0}(X_n = x) = \sum_{i=k}^{n} \mathbb{P}_{x_0}(\tau_{x,k} = i)\mathbb{P}_x(X_{n-i} = x)\,.$$

   (c) Conclude that for every $x \in \Omega$ and non-negative integers $k, \ell$,
   $$\sum_{i=0}^{\ell} \mathbb{P}_x(X_i = x) \geq \sum_{n=k}^{\ell+k} \mathbb{P}_x(X_n = x)\,.$$

**Solution.**

**(a)** By homogeneity,
$$\mathbb{P}_x(X_i \in B) = \mathbb{P}(X_i \in B | X_0 = x) = \mathbb{P}(X_n \in B | X_{n-i} = x)$$
for any $1 \leq i \leq n$. Then,
$$\begin{aligned}
\mathbb{P}_{x_0}(X_n \in B, \tau_{x,k} \leq n) &= \mathbb{P}_{x_0}(\bigcup_{i=0}^{n-k} X_n \in B, \tau_{x,k} = n-i) \\
&= \sum_{i=0}^{n-k} \mathbb{P}_{x_0}(X_n \in B, \tau_{x,k} = n-i) \\
&= \sum_{i=0}^{n-k} \mathbb{P}_{x_0}(\tau_{x,k} = n-i)\mathbb{P}(X_n \in B | X_{n-i} = x) \\
&= \sum_{i=0}^{n-k} \mathbb{P}_{x_0}(\tau_{x,k} = n-i)\mathbb{P}_x(X_i \in B).
\end{aligned}$$

**(b)** In view of (a), setting $B = \{x\}$,
$$\begin{aligned}
\mathbb{P}_{x_0}(X_n = x) &= \mathbb{P}_{x_0}(X_n = x, \tau_{x,k} \leq n) \quad (1) \\
&= \sum_{i=k}^{n} \mathbb{P}_{x_0}(\tau_{x,k} = i)\mathbb{P}_x(X_{n-i} = x)
\end{aligned}$$
where (1) holds as $\{X_n = x\} \subset \{\tau_{x,k} \leq n\}$.

**(c)** Applying (b) with $x_0 = x$,
$$\begin{aligned}
\sum_{n=k}^{l+k} \mathbb{P}_x(X_n = x) &= \sum_{n=k}^{l+k} \sum_{i=k}^{n} \mathbb{P}_x(\tau_{x,k} = i)\mathbb{P}_x(X_{n-i} = x) \\
&= \sum_{i=0}^{l} \mathbb{P}_x(X_i = x) \sum_{i=k}^{n-j} \mathbb{P}_x(\tau_{x,k} = i) \\
&\leq \sum_{i=0}^{l} \mathbb{P}_x(X_i = x) \quad (2)
\end{aligned}$$
where (2) holds by disjointness.

## Question 3.

4. Let $X_n$ be a Markov chain on a countable state space $\Omega = S \cup T$ with $S \cap T = \emptyset$, where $S$ is finite and $T = A \cup B$ for nonempty sets $A, B$. For $E \subset \Omega$, let $\tau_E = \inf\{n : X_n \in E\}$, and suppose that $\inf\{s \in S : \mathbb{P}_s(\tau_T < \infty)\} > 0$. Finally, call a bounded function $h : \Omega \to \mathbb{R}$ is *harmonic* at $x \in \Omega$ for the transition probabilities $P(\cdot,\cdot)$ of $X_n$ if $h(x) = \sum_{y\in\Omega} P(x,y)h(y)$.

   (a) Prove that there exist $\varepsilon > 0$ and $m < \infty$ such that $\mathbb{P}_x(\tau_T > km) \leq (1-\varepsilon)^k$ for all $k \geq 1$ and $x \in \Omega$, and conclude that $\tau_T < \infty$ a.s.

   (b) Prove that $h(x) = \mathbb{P}_x(\tau_A < \tau_B)$ satisfies

   $$h(x) = 1 \text{ for } x \in A\,, \quad h(x) = 0 \text{ for } x \in B\,, \quad h \text{ is harmonic at every } x \in S\,. \qquad (\star)$$

   (c) Prove that if $h : \Omega \to \mathbb{R}$ is a bounded function satisfying $(\star)$ then $h(x) = \mathbb{P}_x(\tau_A < \tau_B)$. To do so, first show that if $\mathcal{F}_n$ is the natural filtration associated with $X_n$ and $g : \Omega \to \mathbb{R}$ is a bounded function that is harmonic at every $x \in S$, then the variable $Y_n = g(X_{n\wedge\tau_T})$ is a *martingale*; that is, $\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = Y_n$ for every $n$.

**Solution.**
**(a)** We claim that

$$\mathbb{P}(X_i \in B | X_0 = x) = \mathbb{P}(X_n \in B | X_{n-i} = x)$$

for any $n \geq 1$ and $n \geq i \geq 1$. We proceed by induction on $i$. Then,

$$\mathbb{P}_{x_0}(X_n \in B, \tau_{x,k} \leq n) \quad = \quad \mathbb{P}_{x_0}\Big(\bigcup_{i=0}^{n-k} X_n \in B, \tau_{x,k} = n - i\Big)$$

**(b)**