ProbLimI:
Problem Set XIII

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Abstract

This work contains solutions to the exercises of the problem set IX. The chosen problems are 2,3 and 4.

Question 1.

1. (a) Give an example of a sub-martingale (X_n) such that (X_n^2) is a super-martingale, and explain why this not contradict the result given in class on $\Phi(X_n)$ for sub-martingale (X_n) and a convex function Φ.

   (b) Give an example of a martingale (X_n) that converges a.s. to $-\infty$, and explain why this does not contradict Doob's Convergence Theorem.

Solution.
(a) Let $X_n = 0$ for each $n \geq 1$, then ${X_n}^2 = 0$ for each $n \geq 1$. It follows that $\{X_n\}$ is a sub-martingale, and $\{X_n^2\}$ is a super-martingale because they are both martingales trivially, as

$$\mathbb{E}[X_n|\mathscr{F}_{n-1}] = X_n = X_{n-1} = 0$$

and

$$\mathbb{E}[X_n^2|\mathscr{F}_{n-1}] = X_n^2 = X_{n-1}^2 = 0$$

for each $n \geq 1$. This does not contradict the given fact about the convex function, as $\{X_n^2\}$ is a sub-martingale as well, by being a martingale.

(b) Let $\{X_n\}$ independent random variables be defined by

$$\mathbb{P}(X_n = -1) = 1 - \frac{1}{2^n} \quad \text{and} \quad \mathbb{P}(X_n = 2^n - 1) = \frac{1}{2^n}$$

for each $n \geq 1$. Then,

$$\mathbb{E}[X_n] = 0$$

for each $n \geq 1$, so $\{S_n = \sum_{k=1}^n X_k\}$ is a martingale with respect to the canonical filteration. Observe that

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n > -1) = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

By Borel-Cantelli,

$$\mathbb{P}(X_n > -1 \ i.o.) = 0$$

and hence

$$\mathbb{P}(X_n \leq -1 \ a.a.) = 1.$$

Therefore, $S_n \to -\infty$ almost surely and we are done. This does not violate the Martingale convergence theorem, as $\sup_n \mathbb{E}[|S_n|]$ is not bounded.

Question 2.

2. Let (X_n) and (Y_n) be nonnegative, integrable stochastic processes adapted to a filtration $(\mathcal{F}_n)$ such that $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq (1+Y_n)X_n + Y_n$ for all n and $\sum_{n\geq 1} Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit as $n \to \infty$.

(*Hint: Deduce this from the convergence of a suitable nonnegative super-martingale.*)

Solution.

Let $Z_n = \dfrac{X_n}{\prod_{m=1}^{n-1}(1+Y_m)}$. Then, $Z_n \in \mathscr{F}_n$ and $\mathbb{E}|Z_n| \leq \mathbb{E}|X_n| < \infty$ for all $n \geq 1$. We claim that $\{Z_n\}$ is a non-negative supermartingale. We compute

$$\mathbb{E}[Z_{n-1}|\mathscr{F}_n] = \frac{1}{\prod_{m=1}^{n}(1+Y_m)}\mathbb{E}[X_{n+1}|\mathscr{F}_n] \leq \frac{1}{\prod_{m=1}^{n}(1+Y_m)}X_n = Z_n.$$

Therefore, by Martingale convergence theorem, $Z_n \to Z_\infty$ a.s. to some almost surely finite random variable Z_∞. As $\sum_n Y_n < \infty$ a.s.,

$$\prod_{m=1}^{n}(1+Y_m) \text{ converges to a finite limit a.s.}$$

and hence

$$X_n \to Z_\infty \prod_{m=1}^{\infty}(1+Y_m) a.s.$$

so we are done. $\square$

Question 3.

3. Let $S_n = \sum_{k=1}^{n} \xi_k$ for i.i.d. random variables ξ_k and let τ be an integrable stopping time for the associated canonical filtration.

   (a) Show that if ξ_1 is integrable then $\mathbb{E}S_\tau = \mathbb{E}\xi_1 \mathbb{E}\tau$ (Wald's identity).
   (*Hint: Write* $S_\tau = \sum_{k=1}^{\infty} \xi_k \mathbf{1}_{\{k \le \tau\}}$.)

   (b) Show that if $\mathbb{E}\xi_1^2 < \infty$ then $\mathbb{E}(S_\tau - \tau\mathbb{E}\xi_1)^2 = \mathrm{Var}(\xi_1)\mathbb{E}\tau$ (Wald's second identity).
   (*Hint: argue that* $\mathbb{E}\xi_1 = 0$ *w.l.o.g. and apply Doob's* L^2*-convergence theorem to* $S_{n\wedge\tau}$.)

   (c) Prove that if $\xi_1 \ge 0$ then Wald's identity holds also in case $\mathbb{E}\tau = \infty$ under the convention that $0 \times \infty = 0$.

Solution.
(a) Observe that

$$\xi_i \text{ and } 1_{\{i \le \tau\}} \text{ are independent}$$

for each $i \ge 1$. Now, we first prove for the case when $\xi_i \ge 0$ for all $i \ge 1$.

$$\begin{aligned}
\mathbb{E}[S_\tau] &= \mathbb{E}[\xi_1 + \ldots + \xi_\tau] = \mathbb{E}[\sum_{i=1}^{\infty} \xi_i 1_{\{i\le\tau\}}] \\
&= \sum_{i=1}^{\infty} \mathbb{E}[\xi_i 1_{\{i\le\tau\}}] = \sum_{i=1}^{\infty} \mathbb{E}[\xi_i]\mathbb{E}[1_{\{i\le\tau\}}] \qquad (1)\\
&= \mathbb{E}[\xi_1] \sum_{i=1}^{\infty} \mathbb{P}(i \le \tau) = \mathbb{E}[\xi_1]\mathbb{E}[\tau] \qquad (2)
\end{aligned}$$

where (1) holds by MCT (or Tonelli) and independence, and (2) holds as τ being a non-negative integer valued random variable. Now consider a general $\{\xi_i\}$. From the above,

$$\mathbb{E}[\sum_{i=1}^{\infty} |\xi_i| 1_{\{i\le\tau\}}] = \mathbb{E}[|\xi_1|]\mathbb{E}[\tau] < \infty.$$

Therefore, by Fubini,

$$\mathbb{E}[S_\tau] = \mathbb{E}[\sum_{i=1}^{\infty} \xi_i 1_{\{i\le\tau\}}] \quad = \quad \sum_{i=1}^{\infty} \mathbb{E}[\xi_i 1_{\{i\le\tau\}}] = \mathbb{E}[\xi_1]\mathbb{E}[\tau].$$

(b) Hence without loss of generality, we can assume $\mathbb{E}[\xi_1] = 0$, so it suffices to show that

$$\mathrm{Var}[S_\tau] \quad = \quad \mathbb{E}[\xi_1]\mathbb{E}[\tau].$$

Suppose for a moment that τ is bounded by some constant M. Then, by (a),

$$\begin{aligned}
\mathrm{Var}(S_\tau) &= \mathbb{E}[S_\tau^2] = \mathbb{E}[\sum_{i=1}^{n} \xi_i 1_{\{i\le\tau\}})^2] \\
&= \mathbb{E}[\sum_{i=1}^{M} (\xi_i^2 1_{\{i\le\tau\}}) = \mathbb{E}[\xi_1^2]\mathbb{E}[\tau]
\end{aligned}$$

where the cross terms are 0 by independence. Now, for τ not bounded, let $\rho_k = \min(\tau, k)$ for all k. Then with (a) and MCT the result follows.

(c) The identity from a is still true, and with the convention given, we should have the result from a as well.

Question 4.

4. Let $S_n = \sum_{k=1}^{n} \xi_k$ for i.i.d. ξ_k's with $\mathbb{P}(\xi_1 = 1) = p$ and $\mathbb{P}(\xi_1 = -1) = 1-p$. Let z be a positive integer, write $\tau_z = \inf\{n \geq 0 : S_n = z\}$, and let $M_n = e^{\lambda S_n} M(\lambda)^{-n}$ where $M(\lambda) = \mathbb{E}[e^{\lambda \xi_1}]$.

   (a) Suppose $\frac{1}{2} \leq p < 1$. Prove that $\mathbb{E}[M(\lambda)^{-\tau_z}] = e^{-\lambda z}$ for every $\lambda > 0$. Conclude that for every $0 < \alpha < 1$,
$$\mathbb{E}[\alpha^{\tau_1}] = \frac{1 - \sqrt{1 - 4p(1-p)\alpha^2}}{2(1-p)\alpha}$$
   and that $\mathbb{E}[\alpha^{\tau_z}] = (\mathbb{E}[\alpha^{\tau_1}])^z$.

   (b) Suppose $0 < p < \frac{1}{2}$. Prove that $\mathbb{P}(\tau_z < \infty) = \exp(-\lambda_0 z)$ where $\lambda_0 = \log(\frac{1-p}{p}) > 0$. Conclude that the r.v. $Z = 1 + \max_{n \geq 0} S_n$ is Geometric with success probability $1 - e^{-\lambda_0}$.

Solution.

(a) Let $\lambda > 0$. First of all, $\{M_n\}$ is a martingale, since

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathscr{F}_n] &= \mathbb{E}[e^{\lambda S_{n+1}} M(\lambda)^{-(n+1)}|\mathscr{F}_n] = M(\lambda)^{-(n+1)}\mathbb{E}[e^{\lambda S_n} e^{\lambda \xi_{n+1}}|\mathscr{F}_n] \\
&= M(\lambda)^{-(n+1)} e^{\lambda S_n} \mathbb{E}[e^{\lambda \xi_{n+1}}] = e^{\lambda S_n} M(\lambda)^{-n}
\end{aligned}$$

for each $n \geq 1$. Now, $\{\tau_z \wedge n\}$ are bounded stopping times, so by theorem 5.4.1 in Durrett,

$$\mathbb{E}[M_{n \wedge \tau_z}] = \mathbb{E}[M_1] = \mathbb{E}[e^{-\lambda \xi_1} e^{\lambda \xi_1}] = 1$$

for each $n \geq 1$. Observe that $\mathbb{E}|M_{n \wedge \tau_z}| \leq e^{\lambda z} M(\lambda)^{-n} < \infty$ for all $n \geq 1$, so by DCT

$$\mathbb{E}[M_{n \wedge \tau_z} \to \mathbb{E}[M_{\tau_z}] \quad \text{as} \quad n \to \infty$$

and hence

$$\mathbb{E}[e^{\lambda S_{\tau_z}} M(\lambda)^{-\tau_z}] = 1.$$

As z is positive, and $\dfrac{1}{2} \leq p < 1$, $\mathbb{P}(\tau_z < \infty) = 1$, we have $\mathbb{P}(S_{\tau_z} = z) = 1$. Therefore,

$$\mathbb{E}[e^{\lambda z} M(\lambda)^{-\tau_z}] = 1$$

and hence

$$\mathbb{E}[M(\lambda)^{-\tau_z}] = e^{-\lambda z}.$$

Now, let $0 < \alpha < 1$. Then, choose $\lambda > 0$ such that