
ProbLimI: Problem Set II

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Abstract

This work contains solutions to the exercises of the problem set I. The chosen problems are 1, 2, and 4.

Question 1.

1. Let X be a nonnegative random variable with $\mathbb{E}[X^2] < \infty$, and set $m_i := \mathbb{E}[X^i]$ for $i = 1, 2$.

(i) Prove that for every $0 \leq x < m_1$ we have $\mathbb{P}(X > x) \geq (m_1 - x)^2/m_2$.

(ii) Prove that $(\mathbb{E}|X^2 - m_2|)^2 \leq 4m_2(m_2 - m_1^2)$.

(iii) Show the following inequality, and compare it to part (i) for $X = \sum_{k=1}^n \mathbf{1}_{A_k}$.

$$\mathbb{P}(\bigcup_{k=1}^n A_k) \geq \sum_{k=1}^n \mathbb{P}(A_k) - \sum_{1 \leq k < \ell \leq n} \mathbb{P}(A_k \cap A_\ell).$$

Solution.

(1) Fix $\alpha \in [0, m_1]$. By Cauchy-Schwartz,

$$\mathbb{E}[1_{X>\alpha} X]^2 \leq \mathbb{E}[1_{\{X>\alpha\}}^2] \mathbb{E}[X^2] = \mathbb{P}(X > \alpha) m_2.$$

On the other hand,

$$m_1 = \mathbb{E}[X 1_{\{X>\alpha\}}] + \mathbb{E}[X 1_{\{X \leq \alpha\}}],$$

which implies

$$(m_1 - \alpha)^2 \leq \mathbb{E}[X 1_{\{X>\alpha\}}]^2.$$

Combining the above inequality with the first one and re-arranging yield

$$\frac{(m_1 - \alpha)^2}{m_2} \leq \mathbb{P}(X > \alpha),$$

as required. □

(2) By Cauchy-Schwartz,

$$\begin{aligned} \mathbb{E}[|(X + m_2^{\frac{1}{2}})(X - m_2^{\frac{1}{2}})|] &\leq (\mathbb{E}[X^2 + 2m_2^{\frac{1}{2}}X + m_2] \mathbb{E}[X^2 - 2m_2^{\frac{1}{2}}X + m_2])^{\frac{1}{2}} \\ &= (4m_2^2 - 4m_2m_1^2)^{\frac{1}{2}} = 2m_2^{\frac{1}{2}}(m_2 - m_1^2)^{\frac{1}{2}}. \end{aligned}$$

Squaring both sides gives the desired inequality. □

(3) We show the inequality via induction. For $n = 2$,

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2) &= \mathbb{P}(A_1^c \cap A_2) + \mathbb{P}(A_1 \cap A_2^c) + \mathbb{P}(A_1 \cap A_2) \\ &= \mathbb{P}(A_1^c \cap A_2) + \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap A_2^c) + \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_2) \\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).\end{aligned}$$

Now, suppose the statement is true for some $n > 2$. Then, using the $n = 2$ case, the inductive hypothesis and subadditivity gives

$$\begin{aligned}\mathbb{P}\left(\bigcup_{k=1}^{n+1} A_k\right) &= \mathbb{P}\left(\bigcup_{k=1}^n A_k \cup A_{n+1}\right) \\ &= \mathbb{P}\left(\bigcup_{k=1}^n A_k\right) + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\bigcup_{k=1}^n A_k \cap A_{n+1}\right) \\ &\geq \sum_{k=1}^n \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n} \mathbb{P}(A_k \cap A_l) + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\bigcup_{k=1}^n A_k \cap A_{n+1}\right) \\ &\geq \sum_{k=1}^{n+1} \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n} \mathbb{P}(A_k \cap A_l) - \sum_{k=1}^n \mathbb{P}(A_k \cap A_{n+1}) \\ &= \sum_{k=1}^{n+1} \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n+1} \mathbb{P}(A_k \cap A_l).\end{aligned}$$

Therefore, the induction is complete and the inequality is true. Now, if $X = \sum_{k=1}^n 1_{A_k}$, then

$$\begin{aligned}m_1 &= E[X] = \sum_{k=1}^n \mathbb{P}(A_k) \\ m_2 &= E[X^2] = \mathbb{E}\left[\sum_{1 \leq k \leq l \leq n} 1_{A_k}\right] = \sum_{1 \leq k \leq l \leq n} \mathbb{P}(A_k \cap A_l)\end{aligned}$$

and

$$\mathbb{P}(X > \alpha) = \mathbb{P}\left(\bigcup_{k=1}^n A_k\right),$$

for any $\alpha \in [0, 1)$. Hence, to compare, if $X = \sum_{k=1}^n 1_{A_k}$, we can re-write (iii) as

$$P(X > \alpha) \geq 2m_1 - m_2,$$

for any $\alpha \in [0, 1)$. □

Question 2.

2. Let X be a real-valued random variable.

(a) Prove that the function $f(x) = \mathbb{E} \exp(-|X - x|)$ is continuous on \mathbb{R} .

(b) Further suppose that $X \geq 0$ and $\mathbb{E}X^p < \infty$ for some $p > 0$.

(b.1) Show that $\lim_{p \downarrow 0} (\mathbb{E}X^p - 1)/p = \mathbb{E} \log X$.

(b.2) Conclude that $\lim_{p \downarrow 0} \log(\mathbb{E}X^p)/p = \mathbb{E} \log X$.

Solution.

(a) We first note that, for any $x \in \mathbb{R}$, $\exp(-|X - x|)$ is uniformly bounded by 1, and we have a finite measure, so the expectation is well-defined and f is well-defined everywhere. Set $\mu = L(X)$. Then, by a change of variable,

$$f(x) = \mathbb{E}[\exp(-|X - x|)] = \int_{-\infty}^{\infty} \exp(-|t - x|) \mu(dt)$$

so, for $x, h \in \mathbb{R}$,

$$\begin{aligned} |f(x+h) - f(x)| &= \left| \int e^{-|t-x-h|} - e^{-|t-x|} \mu(dt) \right| \\ &\leq \int |e^{-|t-x-h|} - e^{-|t-x|}| \mu(dt) \quad (*). \end{aligned}$$

Observe that

$$|e^{-|t-x-h|} - e^{-|t-x|}| \leq 2$$

for any $t, x, h \in \mathbb{R}$ and

$$|e^{-|t-x-h|} - e^{-|t-x|}| \rightarrow 0 \text{ as } h \rightarrow 0$$

for any $t, x \in \mathbb{R}$. Therefore, by BCT and (*), it follows that

$$|f(x+h) - f(x)| \rightarrow 0 \text{ as } h \rightarrow 0,$$

which shows that f is continuous as required. \square

(b.1) With L'hospital's rule, we obtain that

$$\lim_{p \downarrow 0} \frac{X^p(w) - 1}{p} = \lim_{p \downarrow 0} X^p(w) \log(X(w)) = \log(X(w)),$$

for all $w \in \Omega$. Hence,

$$\left\{ \frac{X^p - 1}{p} \right\}_{p>0} \text{ converges almost surely to } \log(X) \text{ as } p \downarrow 0 \text{ on } \Omega$$

By DCT,

$$\lim_{p \downarrow 0} \frac{(\mathbb{E}X^p - 1)}{p} = \mathbb{E} \log(X),$$

as required. \square

(b.2)

(Remark) Although all limit theorems are stated so far for a countable limit, they apply as well to a continuous limit. Suppose $\{X_t\}_{t \geq 0}$ is a family of L^1 dominated random variables such that $\lim_{t \downarrow 0} X_t(w) \rightarrow X_0(w)$ for all $w \in \Omega$. Then, by DCT, for any $\{t_n\} \subset (0, \infty)$ such that $t_n \downarrow 0$, we have $\lim_{n \rightarrow \infty} \mathbb{E}[X_{t_n}] = \mathbb{E}[X_0]$. Since this is true for any such sequence, it follows that $\lim_{t \downarrow 0} \mathbb{E}[X_t] = \mathbb{E}[X_0]$ in a proper continuous limit sense. We will freely use the limit theorem in the continuous setting without doing the above pass everytime.

Question 3.

3. Let $X : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be a random variable with $\mathbb{E}|X| < \infty$.

- (i) Show that if $A_n \in \mathcal{F}$ are disjoint sets and $A = \bigcup_n A_n$ then $\sum_n \mathbb{E}[X \mathbb{1}_{A_n}]$ converges absolutely and $\sum_n \mathbb{E}[X \mathbb{1}_{A_n}] = \mathbb{E}[X \mathbb{1}_A]$.
- (ii) Conclude that if $X \geq 0$ then $\mathbb{Q}(A) = \mathbb{E}[X \mathbb{1}_A] / \mathbb{E}X$ is a probability measure.

Solution.

We first show the case for non-negative, simple functions. Let X be simple, such that

$$X = \sum_{k=1}^l a_k \mathbb{1}_{E_k},$$

where $a_k \in \mathbb{R}$ for $k = 1, \dots, l$ and $E_k \in \mathcal{F}$ with $\bigcup_{k=1}^l E_k = \Omega$. With linearity of expectation,

$$\begin{aligned} \mathbb{E}[X \mathbb{1}_A] &= \mathbb{E}\left[\sum_{k=1}^l a_k \mathbb{1}_{E_k} \mathbb{1}_A\right] = \sum_{k=1}^l a_k \mathbb{E}[\mathbb{1}_{E_k} \mathbb{1}_A] \\ &= \sum_{k=1}^l a_k \mathbb{E}[\mathbb{1}_{E_k \cap A}] = \sum_{k=1}^l a_k \mathbb{P}(E_k \cap A). \end{aligned}$$

Similarly,

$$\mathbb{E}[X \mathbb{1}_{A_n}] = \sum_{k=1}^l a_k \mathbb{P}(E_k \cap A_n)$$

for each $n \geq 1$. Then, it follows that, for all $m \geq 1$,

$$\begin{aligned} \sum_{n=1}^m |\mathbb{E}[X \mathbb{1}_{A_n}]| &= \sum_{n=1}^m \sum_{k=1}^l a_k \mathbb{P}(E_k \cap A_n) \\ &= \sum_{k=1}^l a_k \mathbb{P}(E_k \cap \bigcup_{m \geq n} A_n), \end{aligned}$$

where the equality holds by disjointness of $\{A_n\}$. Since $\bigcup_n A_n = A$, we can exploit continuity of probability and obtain

$$\begin{aligned} \sum_n |\mathbb{E}[X \mathbb{1}_{A_n}]| &= \lim_{m \rightarrow \infty} \sum_{k=1}^l a_k \mathbb{P}(E_k \cap \bigcup_{m \geq n} A_n) \\ &= \sum_{k=1}^l a_k \lim_{m \rightarrow \infty} \mathbb{P}(E_k \cap \bigcup_{m \geq n} A_n) = \sum_{k=1}^l a_k \mathbb{P}(E_k \cap A) = \mathbb{E}[X \mathbb{1}_A]. \end{aligned}$$

Hence, we have shown that for X non-negative and simple, $\sum_n \mathbb{E}[X \mathbb{1}_{A_n}]$ converges absolutely to $\mathbb{E}[X \mathbb{1}_A]$.

We now extend the case to non-negative integrable functions. Let X be a bounded, measurable, non-negative function. Choose $\{\phi_k\}$ simple functions such that $\phi_k \rightarrow X$. By the previous result, we observe

$$\sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}] = \mathbb{E}[\phi_k \mathbb{1}_A] \quad (*)$$

for any $k \geq 1$. Since $\phi_k \rightarrow X$ uniformly, by monotone convergence theorem,

$$\mathbb{E}[\phi_k \mathbb{1}_A] \rightarrow \mathbb{E}[X \mathbb{1}_A]$$

and

$$\mathbb{E}[\phi_k \mathbb{1}_{A_n}] \rightarrow \mathbb{E}[X \mathbb{1}_{A_n}]$$

which via implies

$$\sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}] \rightarrow \sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}].$$

Combining (*) with the above limit, we see that $\sum_n \mathbb{E}[X \mathbb{1}_{A_n}]$ converges absolutely and $\sum_n \mathbb{E}[X \mathbb{1}_{A_n}] = \mathbb{E}[\mathbb{1}_A]$ as required. By considering the positive part and negative part, we can extend the result to any random variable as required.

(ii) Firstly, observe that

$$\mathbb{Q}(\Omega) = \frac{\mathbb{E}[\mathbb{1}_\Omega]}{\mathbb{E}[X]} = \frac{\mathbb{E}[X]}{\mathbb{E}[X]} = 1.$$

Hence, it now suffices to show that \mathbb{Q} is countably additive, but from the discussion in (i), we see

$$\mathbb{Q}\left(\bigcup_n A_n\right) = \frac{\mathbb{E}[X \mathbb{1}_{\bigcup_n A_n}]}{\mathbb{E}[X]} = \frac{\sum_n \mathbb{E}[X \mathbb{1}_{A_n}]}{\mathbb{E}[X]} = \sum_n \mathbb{Q}(A_n).$$

for any $\{A_n\} \subset \mathcal{F}$ that are pairwise disjoint. So, \mathbb{Q} is a probability measure, if $X \geq 0$ and we are done \square

Question 4.

4. Let $Y = \sum_{k=1}^n \mathbf{1}_{A_k}$ for some measurable sets A_1, \dots, A_n . Express $\text{Var}(Y)$ in terms of $\mathbb{P}(A_k)$ and $\mathbb{P}(A_k \cap A_l)$, then calculate it for the following case: each one of m players selects, independently and uniformly, a number in $\{1, \dots, n\}$; the event A_k says that the number k was not selected by any player.

Solution.

We compute

$$\begin{aligned}
 \text{Var}[Y] &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
 &= \mathbb{E}\left[\left(\sum_{k=1}^n \mathbf{1}_{A_k}\right)^2\right] - \mathbb{E}\left[\sum_{k=1}^n \mathbf{1}_{A_k}\right]^2 \\
 &= \sum_{k=1}^n \sum_{l=1}^n \mathbb{E}[\mathbf{1}_{A_k} \mathbf{1}_{A_l}] - \sum_{k=1}^n \sum_{l=1}^n \mathbb{E}[\mathbf{1}_{A_k}] \mathbb{E}[\mathbf{1}_{A_l}] \\
 &= \sum_{k=1}^n \sum_{l=1}^n \mathbb{E}[\mathbf{1}_{A_k \cap A_l}] - \mathbb{E}[\mathbf{1}_{A_k}] \mathbb{E}[\mathbf{1}_{A_l}] \\
 &= \sum_{k=1}^n \sum_{l=1}^n \mathbb{P}(A_k \cap A_l) - \mathbb{P}(A_k) \mathbb{P}(A_l).
 \end{aligned}$$

Now, observe that, for $k = 1, \dots, n$,

$$\mathbb{P}(A_k) = \left(\frac{n-1}{n}\right)^m$$

and for $k, l = 1, \dots, n$,

$$\begin{aligned}
 k = l &\implies \mathbb{P}(A_k \cap A_l) = \left(\frac{n-1}{n}\right)^m \\
 k \neq l &\implies \mathbb{P}(A_k \cap A_l) = \left(\frac{n-2}{n}\right)^m.
 \end{aligned}$$

So

$$\begin{aligned}
 \text{Var}[Y] &= \sum_{1 \leq k, l \leq n; k \neq l} \left(\frac{n-2}{n}\right)^m - \left(\frac{n-1}{n}\right)^{2m} + \sum_{1 \leq k, l \leq n; k = l} \left(\frac{n-1}{n}\right)^m - \left(\frac{n-1}{n}\right)^{2m} \\
 &= (n^2 - n) \left(\frac{n-2}{n}\right)^m + n \left(\frac{n-1}{n}\right)^m - n^2 \left(\frac{n-1}{n}\right)^{2m}
 \end{aligned}$$

as required. □