ProbLimI:
Problem Set II

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Abstract

This work contains solutions to the exercises of the problem set I. The chosen problems are 1,2, and 4.

Question 1.

1. Let X be a nonnegative random variable with $\mathbb{E}[X^2] < \infty$, and set $m_i := \mathbb{E}[X^i]$ for $i = 1, 2$.
   (i) Prove that for every $0 \leq x < m_1$ we have $\mathbb{P}(X > x) \geq (m_1 - x)^2/m_2$.
   (ii) Prove that $(\mathbb{E}|X^2 - m_2|)^2 \leq 4m_2(m_2 - m_1^2)$.
   (iii) Show the following inequality, and compare it to part (i) for $X = \sum_{k=1}^n \mathbf{1}_{A_k}$.

$$\mathbb{P}\left(\bigcup_{k=1}^n A_k\right) \geq \sum_{k=1}^n \mathbb{P}(A_k) - \sum_{1 \leq k < \ell \leq n} \mathbb{P}(A_k \cap A_\ell).$$

Solution.

Question 2.

2. Let X be a real-valued random variable.

   (a) Prove that the function $f(x) = \mathbb{E}\exp(-|X - x|)$ is continuous on $\mathbb{R}$.

   (b) Further suppose that $X \geq 0$ and $\mathbb{E}X^p < \infty$ for some $p > 0$.

      (b.1) Show that $\lim_{p \downarrow 0}(\mathbb{E}X^p - 1)/p = \mathbb{E}\log X$.

      (b.2) Conclude that $\lim_{p \downarrow 0}\log(\mathbb{E}X^p)/p = \mathbb{E}\log X$.

Solution.

Question 3.

3. Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be a random variable with $\mathbb{E}|X| < \infty$.

(i) Show that if $A_n \in \mathcal{F}$ are disjoint sets and $A = \bigcup_n A_n$ then $\sum_n \mathbb{E}[X\mathbf{1}_{A_n}]$ converges absolutely and $\sum_n \mathbb{E}[X\mathbf{1}_{A_n}] = \mathbb{E}[X\mathbf{1}_A]$.

(ii) Conclude that if $X \geq 0$ then $\mathbb{Q}(A) = \mathbb{E}[X\mathbf{1}_A]/\mathbb{E}X$ is a probability measure.

Solution.
We first show the case for non-negative, simple functions. Let X be simple, such that

$$X = \sum_{k=1}^{l} a_k \mathbb{1}_{E_k},$$

where $a_k \in \mathbb{R}$ for $k = 1, \ldots, l$ and $E_k \in \mathscr{F}$ with $\dot\bigcup_{k=1}^{l} E_k = \Omega$. With linearity of expectation,

$$\begin{aligned} \mathbb{E}[X\mathbb{1}_A] &= \mathbb{E}[\sum_{k=1}^{l} a_k \mathbb{1}_{E_k}\mathbb{1}_A] = \sum_{k=1}^{l} a_k \mathbb{E}[\mathbb{1}_{E_k}\mathbb{1}_A] \\ &= \sum_{k=1}^{l} a_k \mathbb{E}[\mathbb{1}_{E_k \cap A}] = \sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap A). \end{aligned}$$

Similarly,

$$\mathbb{E}[X\mathbb{1}_{A_n}] = \sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap A_n)$$

for each $n \geq 1$. Then, it follows that, for all $m \geq 1$,

$$\begin{aligned} \sum_{n=1}^{m} |\mathbb{E}[X\mathbb{1}_{A_n}]| &= \sum_{n=1}^{m}\sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap A_n) \\ &= \sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap \bigcup_{m \geq n} A_n), \end{aligned}$$

where the equality holds by disjointness of $\{A_n\}$. Since $\bigcup_n A_n = A$, we can exploit continuity of probability and obtain

$$\begin{aligned} \sum_n |\mathbb{E}[X\mathbb{1}_{A_n}]| &= \lim_{m\to\infty} \sum_{k=1}^{l} a_k \mathbb{P}(E_k \cap \bigcup_{m\geq n} A_n) \\ &= \sum_{k=1}^{l} a_k \lim_{m\to\infty} \mathbb{P}(E_n \cap \bigcup_{m\geq n} A_n) = \sum_{k=1}^{l} a_k \mathbb{P}(E_n \cap A) = \mathbb{E}[X\mathbb{1}_A]. \end{aligned}$$

Hence, we have shown that for X non-negative and simple, $\sum_n \mathbb{E}[X\mathbb{1}_{A_n}]$ converges absolutely to $\mathbb{E}[X\mathbb{1}_A]$.

We now extend the case to non-negative integrable functions. Let X be a bounded, measurable, non-negative functions. Choose $\{\phi_k\}$ simple functions such that $\phi_k \to X$. By the previous result, we observe

$$\sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}] = \mathbb{E}[\phi_k \mathbb{1}_A] \ (*)$$

for any $k \geq 1$. Since $\phi_k \to X$ uniformly, by monotone convergence theorem,

$$\mathbb{E}[\phi_k \mathbb{1}_A] \to \mathbb{E}[\phi_k \mathbb{1}_A]$$

and

$$\mathbb{E}[\phi_k \mathbb{1}_{A_n}] \to \mathbb{E}[\phi_k \mathbb{1}_{A_n}]$$

which via implies

$$\sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}] \to \sum_n \mathbb{E}[\phi_k \mathbb{1}_{A_n}].$$

Combining (*) with the above limit, we see that $\sum_n \mathbb{E}[X\mathbb{1}_{A_n}]$ converges absolutely and $\sum_n \mathbb{E}[X\mathbb{1}_{A_n}] = \mathbb{E}[\mathbb{1}_A]$ as required. By considering the positive part and negative part, we can extend the result to any random variable as required.

(ii) Firstly, observe that

$$\mathbb{Q}(\Omega) \quad = \quad \frac{\mathbb{E}[\mathbb{1}_\Omega]}{\mathbb{E}[X]} = \frac{\mathbb{E}[X]}{\mathbb{E}[X]} = 1.$$

Hence, it now suffices to show that $\mathbb{Q}$ is countably additive, but from the discussion in (i), we see

$$\mathbb{Q}(\bigcup_n A_n) \quad = \quad \frac{\mathbb{E}[X\mathbb{1}_{\cup_n A_n}]}{\mathbb{E}[X]} = \frac{\sum_n \mathbb{E}[X\mathbb{1}_{A_n}]}{\mathbb{E}[X]} = \sum_n \mathbb{Q}(A_n).$$

for any $\{A_n\} \subset \mathscr{F}$ that are pairwise disjoint. So, $\mathbb{Q}$ is a probability measure, if $X \geq 0$ and we are done $\square$

Question 4.

4. Let $Y = \sum_{k=1}^{n} \mathbf{1}_{A_k}$ for some measurable sets $A_1, \ldots, A_n$. Express $\mathrm{Var}(Y)$ in terms of $\mathbb{P}(A_k)$ and $\mathbb{P}(A_k \cap A_\ell)$, then calculate it for the following case: each one of m players selects, independently and uniformly, a number in $\{1, \ldots, n\}$; the event A_k says that the number k was not selected by any player.

Solution.
We compute

$$
\begin{aligned}
\mathrm{Var}[Y] &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[(\sum_{k=1}^{l} \mathbb{1}_{A_k})^2] - \mathbb{E}[\sum_{k=1}^{l} \mathbb{1}_{A_k}]^2 \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k}\mathbb{1}_{A_l}] - \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k}]\mathbb{E}[\mathbb{1}_{A_l}] \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k \cap A_l}] - \mathbb{E}[\mathbb{1}_{A_k}]\mathbb{E}[\mathbb{1}_{A_l}] \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{P}(A_k \cap A_l) - \mathbb{P}(A_k)\mathbb{P}(A_l).
\end{aligned}
$$

Now, observe that, for $k = 1, \ldots, n$,

$$\mathbb{P}(A_k) = (\frac{n-1}{n})^m$$

and for $k, l = 1, \ldots, n$,

$$
\begin{aligned}
k = l &\implies \mathbb{P}(A_k \cap A_l) = (\frac{n-1}{n})^m \\
k \neq l &\implies \mathbb{P}(A_k \cap A_l) = (\frac{n-2}{n})^m.
\end{aligned}
$$

So

$$\mathrm{Var}[Y] = \sum_{1 \le k, l \le n; k \neq l} (\frac{n-2}{n})^m - (\frac{n-1}{n})^{2m},$$

as required.