Durrett Probability: Problems

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Abstract

This work contains solutions to some exercises from Durrett's probability text.

1 Chapter 6: Markov Chains

Question 6.3.3.

6.3.3. First entrance decomposition. Let $T_y = \inf\{n \geq 1 : X_n = y\}$. Show that

$$p^n(x, y) = \sum_{m=1}^{n} P_x(T_y = m) p^{n-m}(y, y)$$

Solution.
Here we assume countable state space. Observe that

$$
\begin{aligned}
p^n(x, y) &= P_x(X_n = y) = P_x\big(\bigcup_{m=1}^{n} \{T_y = m; X_n = y\}\big) \\
&= E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y}; T_y \leq n) = E_x(E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y} | \mathscr{F}_{T_y}); T_y \leq n) \quad (1) \\
&= E_x(E_{X_{T_y}}(1_{\{X_{n-T_y}=y\}}); T_y \leq n) = E_x(E_y(1_{\{X_{n-T_y}\}}); T_y \leq n) \quad (2) \\
&= \sum_{m=1}^{n} P_x(T_y = m) E_y(1_{\{X_{n-m}=y\}}) = \sum_{m=1}^{n} P_x(T_y = m) P^{n-m}(y, y)
\end{aligned}
$$

where (1) holds by definition of conditional expectation and (2) holds by the strong Markov property.

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Question 6.3.4.

6.3.4. Show that $\sum_{m=0}^{n} P_x(X_m = x) \geq \sum_{m=k}^{n+k} P_x(X_m = x)$.

Solution.

Question 6.3.5.

6.3.5. Suppose that $S - C$ is finite and for each $x \in S - C$ $P_x(\tau_C < \infty) > 0$. Then there is an $N < \infty$ and $\epsilon > 0$ so that $P_y(\tau_C > kN) \leq (1 - \epsilon)^k$.

Solution.

We assume countable state space. Observe that, for any $x \in S \setminus C$, we can choose $n(x) \in \mathbb{N}$ such that

$$P_x(\tau_C \leq n) > 0.$$

Otherwise, for some $x \in S \setminus C$, by continuity of probability,

$$P_x(\tau_C < \infty) = \lim_{k \to \infty} P_x(\tau_C \leq k) = 0,$$

which is a contradiction. Now, let

$$\epsilon = \min_{z \in S \setminus C} P_z(\tau_C < \infty) \quad \text{and} \quad N = \max_{z \in S \setminus C} n(x).$$

Trivially,

$$P_y(\tau_C > kN) = 0$$

for any $k \in \mathbb{N}$, and $y \in C$, since $y \in C$ implies $\tau_C = 0$ by definition. Therefore, it suffices to show

$$P_y(\tau_C > kN) \leq (1 - \epsilon)^k \tag{3}$$

for all $k \in \mathbb{N}$ and $y \in S \setminus C$. Fix $y \in S \setminus C$. Then,

$$P_y(\tau_C \leq N) \geq P_y(\tau_C < \infty) \geq \epsilon$$

and hence

$$P_y(\tau_C > N) \leq (1 - \epsilon)$$

Now, we proceed by induction to prove (3). Suppose, for some $k \in \mathbb{N}$ such that $k \geq 2$,

$$P_y(\tau_C > kN) \leq (1 - \epsilon)^k.$$

We compute

$$
\begin{aligned}
P_y(T_c > (k+1)N) &= E_y(1_{\{\tau_C > kN\}} \circ \theta_N; \tau_C > N) \\
&= E_y(E_y((1_{\{\tau_C > kN\}} \circ \theta_N | \mathscr{F}_N); \tau_C > N)) \\
&= E_y(E_{X_N}((1_{\{\tau_C > kN\}}); \tau_C > N)) \tag{4} \\
&\leq E_y(\sup_{z \in S} P_z(\tau_C > kN); \tau_C > N)) \\
&\leq (1 - \epsilon)^k E_y(1; \tau_C > N)) = (1 - \epsilon)^{k+1}
\end{aligned}
$$

where (4) holds by Markov Property, which completes the proof. $\qquad \square$

Question 6.3.6.

6.3.6. Let $h(x) = P_x(\tau_A < \tau_B)$. Suppose $A \cap B = \emptyset$, $S - (A \cup B)$ is finite, and $P_x(\tau_{A \cup B} < \infty) > 0$ for all $x \in S - (A \cup B)$. (i) Show that

$(*)$ $$h(x) = \sum_y p(x, y) h(y) \quad \text{for } x \notin A \cup B$$

(ii) Show that if h satisfies $(*)$ then $h(X(n \wedge \tau_{A \cup B}))$ is a martingale. (iii) Use this and Exercise 6.3.5 to conclude that $h(x) = P_x(\tau_A < \tau_B)$ is the only solution of $(*)$ that is 1 on A and 0 on B.

Solution.
(i) Let $x \in S \setminus (A \cup B)$. Then,

$$1_{\{\tau_A < \tau_B\}} = 1_{\{\tau_A < \tau_B\}} \circ \theta_1.$$

It follows that

$$
\begin{aligned}
h(x) &= P_x(\tau_A < \tau_B) = E_x(1_{\{\tau_A < \tau_B\}}) = E_x(1_{\{\tau_A < \tau_B\}} \circ \theta_1) \\
&= E_x(E_x(1_{\{\tau_A < \tau_B\}} \circ \theta_1 | \mathscr{F}_1)) = E_x(E_{X_1}(1_{\{\tau_A < \tau_B\}})) \\
&= \sum_y P(X_1 = y) P_y(\tau_A < \tau_B) = \sum_y p(x, y) P_y(\tau_A < \tau_B)
\end{aligned}
\tag{5}
$$

where (5) holds by Markov property.

(ii)

(iii)

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Question 6.3.7.

6.3.7. Let X_n be a Markov chain with $S = \{0, 1, \ldots, N\}$ and suppose that X_n is a martingale and $P_x(\tau_0 \wedge \tau_N < \infty) > 0$ for all x. (i) Show that 0 and N are absorbing states, i.e., $p(0, 0) = p(N, N) = 1$. (ii) Show $P_x(\tau_N < \tau_0) = x/N$.

Solution.