Durrett Probability:
Problems

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Abstract

This work contains solutions to the exercises of Durrett's probability book.

Question 6.3.3.

6.3.3. First entrance decomposition. Let $T_y = \inf\{n \geq 1 : X_n = y\}$. Show that

$$p^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m) p^{n-m}(y,y)$$

Solution.
Here we assume countable state space. Observe that

$$\begin{aligned}
p^n(x,y) &= P_x(X_n = y) = P_x\Big(\bigcup_{m=1}^{n} \{T_y = m; X_n = y\}\Big) \\
&= E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y}; T_y \leq n) = E_x(E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y} | \mathscr{F}_{T_y}); T_y \leq n) \quad (1) \\
&= E_x(E_{X_{T_y}}(1_{\{X_{n-T_y}=y\}}; T_y \leq n) = E_x(E_y(1_{\{X_{n-T_y}\}}); T_y \leq n) \quad (2) \\
&= \sum_{m=1}^{n} P_x(T_y = m) E_y(1_{\{X_{n-m}=y\}}) = \sum_{m=1}^{n} P_x(T_y = m) P^{n-m}(y,y)
\end{aligned}$$

where (1) holds by definition of conditional expectation and (2) holds by the strong Markov property.

Question 6.3.4.

6.3.4. Show that $\sum_{m=0}^{n} P_x(X_m = x) \geq \sum_{m=k}^{n+k} P_x(X_m = x)$.

Solution.

Question 6.3.5.

6.3.5. Suppose that $S - C$ is finite and for each $x \in S - C$ $P_x(\tau_C < \infty) > 0$. Then there is an $N < \infty$ and $\epsilon > 0$ so that $P_y(\tau_C > kN) \le (1-\epsilon)^k$.

Solution.
We assume countable state space. Observe that, for any $x \in S \setminus C$, we can choose $n(x) \in \mathbb{N}$ such that

$$P(\tau_C \le n) > 0,$$

as otherwise, by continuity of probability

$$P(\tau_C < \infty) = \lim_{k \to \infty} P(\tau_C \le k) = 0,$$

which is a contradiction. Now, let

$$\epsilon = \min_{z \in S \setminus C} P_z(\tau_C < \infty) \quad \text{and} \quad N = \max_{z \in S \setminus C} n(x).$$

Trivially,

$$P_y(\tau_C > kN) = 0$$

for any $k \in \mathbb{N}$, and $y \in C$, since $y \in C$ implies $\tau_C = 0$ by definition. Therefore, it suffices to show

$$P_y(\tau_C > kN) \le (1-\epsilon)^k \tag{3}$$

for all $k \in \mathbb{N}$ and $y \in S \setminus C$. Fix $y \in S \setminus C$. Then,

$$P_y(\tau_C \le N) \ge P_y(\tau_C < \infty) \ge \epsilon$$

and hence

$$P_y(\tau_C > N) \le (1-\epsilon)$$

Now, we proceed by induction to prove (3). Suppose, for some $k \in \mathbb{N}$,

$$P_y(\tau_C > kN) \le (1-\epsilon)^k.$$

We compute

$$\begin{aligned}
P_y(T_c > (k+1)N) &= E_y(1_{\{\tau_C > kN\}} \circ \theta_N; \tau_C > N) \\
&= E_y(E_y((1_{\{\tau_C > kN\}} \circ \theta_N | \mathscr{F}_N); \tau_C > N)) \\
&= E_{y}(E_{X_N}((1_{\{\tau_C > kN\}}); \tau_C > N)) \qquad (4) \\
&\le E_y(\sup_{z \in S} P_z(\tau_C > kN); \tau_C > N)) \\
&\le (1-\epsilon)^k E_y(1; \tau_C > N)) = (1-\epsilon)^{k+1}
\end{aligned}$$

where (4) holds by Markov Property, which completes the proof. ☐

Question 6.3.6.

6.3.6. Let $h(x) = P_x(\tau_A < \tau_B)$. Suppose $A \cap B = \emptyset$, $S - (A \cup B)$ is finite, and $P_x(\tau_{A\cup B} < \infty) > 0$ for all $x \in S - (A \cup B)$. (i) Show that

$$(*) \qquad h(x) = \sum_y p(x,y)h(y) \quad \text{for } x \notin A \cup B$$

(ii) Show that if h satisfies $(*)$ then $h(X(n \wedge \tau_{A\cup B}))$ is a martingale. (iii) Use this and Exercise 6.3.5 to conclude that $h(x) = P_x(\tau_A < \tau_B)$ is the only solution of $(*)$ that is 1 on A and 0 on B.

Solution.

Question 6.3.7.

6.3.7. Let X_n be a Markov chain with $S = \{0, 1, \ldots, N\}$ and suppose that X_n is a martingale and $P_x(\tau_0 \wedge \tau_N < \infty) > 0$ for all x. (i) Show that 0 and N are absorbing states, i.e., $p(0,0) = p(N,N) = 1$. (ii) Show $P_x(\tau_N < \tau_0) = x/N$.

Solution.