ProbLimI:
Problem Set XI

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Abstract

This work contains solutions to the exercises of the problem set XI. The chosen problems are 2,3,4.

Question 2.

2. Let X_n be an irreducible homogeneous Markov chain on a countable state set $\mathbb{S}$ with transition kernel P (i.e., $P(x,y) = \mathbb{P}_x(X_1 = y)$). We call a bounded from below or above function $f : \mathbb{S} \to \mathbb{R}$ *super-harmonic* w.r.t. P if $f(x) \geq (Pf)(x)$ for all x. Prove that X_n is recurrent if and only if the only nonnegative super-harmonic functions for it are the constant functions.

Solution.
Observe that $f(X_n)$ is a super-martingale. By martingale convergence, $f(X_n)$ converges a.s. to some RV Y. By recurrence, $Y = f(x)$ $\mathbb{P}_x$ a.s. so f is constant.

Conversely, suppose the chain is transient. Define

$$\tau = \inf\{n \geq 0 : X_n = x_0\}$$

for some $x_0 \in \mathbb{S}$, and

$$f(x) = \mathbb{P}(\tau < \infty \mid X_0 = x)$$

By definition, $f(x) \in [0,1]$ for all x and $f(x_0) = 1$. By transience, $f(y) < 1$ for some $y \in \mathbb{S}$. Observe that

$$f(x) = \sum_{z \in \mathbb{S}} p(x,y) f(y)$$

for all $x \in \mathbb{S}$, by strong markov property. Hence, we have constructed a non-constant, super-harmonic function, and we are done. □

Question 3.

3. Let X_n be an irreducible Markov chain on a countable state set $\mathbb{S}$ with transition kernel P and let $\mu : \mathbb{S} \to (0, \infty)$ be a positive invariant measure for it (i.e., $\mu^\mathsf{T} = \mu^\mathsf{T} P = \sum_{x \in \mathbb{S}} \mu(x) P(x, \cdot)$).

(a) Show that $\tilde{P}(x, y) = P(y, x)\mu(y)/\mu(x)$ is a transition kernel on $\mathbb{S}$.

(b) Show that if a non-zero $\nu : \mathbb{S} \to [0, \infty)$ satisfies $\nu \geq \nu^\mathsf{T} P$ then the function $h = \nu/\mu$ is super-harmonic w.r.t. $\tilde{P}$.

(c) Prove that if X_n is recurrent then so is the Markov chain corresponding to $\tilde{P}$. Deduce that h is a constant function, that is, $\nu(x) = \alpha\mu(x)$ holds for some $\alpha > 0$ for every $x \in \mathbb{S}$.

Solution.
(a) As the space is discrete, we canonically equip it with the full sigma algebra, so

$$\tilde{P}(\cdot, A)$$

is measurable for any $A \in 2^{\mathbb{S}}$. Furthermore,

$$\tilde{P}(x, \cdot)$$

is a probability measure for any $x \in \mathbb{S}$, as

$$\tilde{P}(x, \mathbb{S}) = \sum_{y \in \mathbb{S}} P(y, x)\mu(y)\mu(x)^{-1} = \mu(x)\mu(x)^{-1} = 1$$

for any $x \in \mathbb{S}$, as P is a transition kernel. Countable additivity for each $x \in \mathbb{S}$ follows in the same way.

(b) As $\nu \leq \nu^T P$,

$$h(y) \leq \sum_{s \in \mathbb{S}} h(x)\mu(x)P(x, y) = \sum_{s \in \mathbb{S}} h(x)\mu(y)\tilde{P}(y, x)$$

for all $y \in \mathbb{S}$, so dividing both sides by $\mu(y)$, shows that h is super-harmonic w.r.t $\tilde{P}$.

(c) From the same computation as $4 - a$, we see that

$$\tilde{P}^n(x, y) = \mu(y)\mu(x)^{-1}P^n(y, x)$$

for all $x, y \in \mathbb{S}$. Hence, irreducibility and recurrency of P implies that $\tilde{P}$ is irreducible and recurrent, since

$$\sum_{n=1}^{\infty} P^n(x, x) = \infty = \sum_{n=1}^{\infty} \tilde{P}(x, x)$$

for any $x \in \mathbb{S}$. Therefore, by problem 2, h is a constant function, and $\nu = \alpha\mu$ for some $\alpha > 0$.

$\square$

Question 4.

4. Let X_n be a Markov chain on a countable state set $\mathbb{S}$ and μ be an invariant measure for it.

(a) Show that $\mu^{\mathrm{T}} = \mu^{\mathrm{T}} P^k$ where $P^k(x, y) = \mathbb{P}_x(X_k = y)$ is the k-step transition kernel, and deduce that if $\mu(x) > 0$ for some $x \in \mathbb{S}$ then $\mu(y) > 0$ for every y accessible from x.

(b) Let $\mathcal{R} \subset \mathbb{S}$ be an accessibility ($\leftrightarrow$) equivalence class that is recurrent. Show that $\mu(x)P(x, y) = 0$ for every $x \notin \mathcal{R}$ and $y \in \mathcal{R}$.

(c) Conclude that if $\mathcal{R}$ as above is accessible from $x \notin \mathcal{R}$ then $\mu(x) = 0$.

Solution.
(a) When $n = 1$, the statement is true by definition of invariant measure. Suppose the statement is true for some $n \geq 2$. Then, by fubini,

$$\begin{aligned} \mu(x) &= \sum_{s \in \mathbb{S}} \mu(s) P^n(s, x) = \sum_{t \in \mathbb{S}} \sum_{s \in \mathbb{S}} P^n(s, x) P(t, s) \\ &= \sum_{t \in \mathbb{S}} P^k(t, x) \mu(t) \end{aligned}$$

for any $x \in \mathbb{S}$. Threrefore, by induction, we have the statement.

Suppose $\mu(x) > 0$, and let y be accessible from x. Then, $p_{xy} > 0$, so $P^n(x, y) > 0$ for some n, as otherwise, by countable subadditivity we have $p_{xy} = 0$, which is a contradiction. Then, the above result,

$$\mu(y) = \sum_{z \in \mathbb{S}} \mu(z) P^n(z, y) \geq \mu(x) P^n(x, y) > 0$$

as required.

(b) We provide the proof for the case when it's a stationary distribution. Suppose x is transient. Then, by the contrapositive of theorem 6.5.4 in Durrett, $\mu(x) > 0$. Suppose x is recurrent. Then, by theorem 6.4.3 in Durrett, $p_{xy} = 0$, as otherwise $p_{yx} = 1$ and x and y communicate, which contradicts that $x \notin R$. As $P(x, y) \leq p_{xy}$, $P(x, y) = 0$, and we are done.

(c) If R is accessible from $x \notin R$, then x must be transient. Otherwise, by theorem 6.4.3, $p_{yx} = 1$, so x and y communicates, which is a contradiction. Therefore, as before x being transient implies $\mu(x) > 0$.

□