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# ProbLimI: Pset I

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## Abstract

This work contains solutions to the exercises of the problem set I.

### Question 1.

1. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, and let  $A \in \mathcal{F}$  and  $A_k \in \mathcal{F}$  ( $k \geq 1$ ).
  - (i) Prove the *sub-additivity* property:  $\mathbb{P}(\bigcup_k A_k) \leq \sum_k \mathbb{P}(A_k)$ .
  - (ii) Prove the *continuity* property: If  $A_k \uparrow A$  (i.e.  $A_k \subseteq A_{k+1}$  for all  $k$  and  $\bigcup_k A_k = A$ ) then  $\mathbb{P}(A_k) \uparrow \mathbb{P}(A)$ , and if  $A_k \downarrow A$  (i.e.  $A_k \supseteq A_{k+1}$  for all  $k$  and  $\bigcap_k A_k = A$ ) then  $\mathbb{P}(A_k) \downarrow \mathbb{P}(A)$ .

### Solution.

(i) Note that we have finite additivity property of measure, as the empty set belong to any  $\sigma$ -field by definition. We first have

$$A, B \in \mathcal{F}, A \subset B \implies \mathbb{P}(A) \leq \mathbb{P}(B) \quad (*),$$

because

$$\mathbb{P}(B) = \mathbb{P}((B \setminus A) \cup A) = \mathbb{P}(B \setminus A) + \mathbb{P}(A).$$

Now, define  $A_0 = \emptyset$ , and

$$\tilde{A}_k = A_k \setminus \left( \bigcup_{0 \leq n < k} A_n \right) \quad (k \geq 1).$$

It follows that  $\{\tilde{A}_k\}$  is a pairwise disjoint collection such that

$$\bigcup_k \tilde{A}_k = \bigcup_k A_k \quad \text{and} \quad \tilde{A}_k \subset A_k \quad (k \geq 1).$$

The union equality holds, since if  $x \in \bigcup_k A_k$ , then  $x \in A_{k'}$  for some  $k'$ , and  $x \in \tilde{A}_{k^*}$ , where

$$k^* = \inf\{k; x \in A_k\},$$

as  $x \notin A_k$  for  $k < k^*$  and  $x \in A_{k^*}$ . Hence, by countable additivity,

$$\mathbb{P}\left(\bigcup_k A_k\right) = \mathbb{P}\left(\bigcup_k \tilde{A}_k\right) = \sum_k \mathbb{P}(\tilde{A}_k) \leq \sum_k \mathbb{P}(A_k),$$

where the last inequality follows from (\*). □

(ii) Define  $A_0, \tilde{A}_0 = \emptyset$  and

$$\tilde{A}_k = A_k \setminus A_{k-1} \quad (k \geq 1).$$

By finite additivity and the fact that  $\{A_k\}$  is increasing, we have, for any  $k \geq 1$ ,

$$\mathbb{P}(A_k) = \mathbb{P}(A_{k-1} \cup (A_k \setminus A_{k-1})) = \mathbb{P}(A_{k-1}) + \mathbb{P}(A_k \setminus A_{k-1}),$$

and by re-arranging

$$\mathbb{P}(\tilde{A}_k) = \mathbb{P}(A_k) - \mathbb{P}(A_{k-1}).$$

Now,  $\{\tilde{A}_k\}$  are disjoint, so by countable additivity, we have

$$\begin{aligned} \mathbb{P}(A) = \mathbb{P}\left(\bigcup_k A_k\right) &= \mathbb{P}\left(\bigcup_k \tilde{A}_k\right) = \sum_k \mathbb{P}(\tilde{A}_k) = \lim_{k \rightarrow \infty} \sum_{n=1}^k \mathbb{P}(A_n) - \mathbb{P}(A_{n-1}) \\ &= \lim_{k \rightarrow \infty} \mathbb{P}(A_k) - \mathbb{P}(A_0) = \lim_{k \rightarrow \infty} \mathbb{P}(A_k), \end{aligned}$$

as required. Now, we show the continuity from above. Note that  $\{A_k^c\}$  forms an increasing collection. By the DeMorgan's law, and continuity from below,

$$1 - \mathbb{P}\left(\bigcap_k A_k\right) = \mathbb{P}\left(\left(\bigcap_k A_k\right)^c\right) = \mathbb{P}\left(\bigcup_k A_k^c\right) = \lim_{k \rightarrow \infty} \mathbb{P}(A_k^c) = 1 - \lim_{k \rightarrow \infty} \mathbb{P}(A_k),$$

so

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcap_k A_k\right) = \lim_{k \rightarrow \infty} \mathbb{P}(A_k),$$

as required. □

## Question 2.

2. Let  $\mathcal{F}$  be a field.

- (i) Show that if  $\{\mathcal{G}_\alpha\}$  is a (possibly uncountable) family of  $\sigma$ -fields then  $\bigcap_\alpha \mathcal{G}_\alpha$  is also a  $\sigma$ -field. Conclude that  $\sigma(\mathcal{F}) = \bigcap \{\mathcal{G} \supseteq \mathcal{F} : \mathcal{G} \text{ is a } \sigma\text{-field}\}$ .
- (ii) Prove that if  $\mathcal{M}$  is a monotone class and  $\mathcal{F} \subseteq \mathcal{M}$  then  $\sigma(\mathcal{F}) \subseteq \mathcal{M}$ . Conclude that  $\sigma(\mathcal{F})$  is equal to  $m(\mathcal{F}) := \bigcap \{\mathcal{M} \supseteq \mathcal{F} : \mathcal{M} \text{ is a monotone class}\}$ .

### Solution.

(i) We just note that the index set must be non-empty. As  $\emptyset$  and  $\Omega$  are in  $\mathcal{G}_\alpha$  for all  $\alpha$ , by the  $\sigma$ -field property of each  $\mathcal{G}_\alpha$ , it follows that  $\emptyset, \Omega \in \bigcap_\alpha \mathcal{G}_\alpha$ . Now, it suffices to show that

$$\begin{aligned} A \in \bigcap_\alpha \mathcal{G}_\alpha &\implies A^c \in \bigcap_\alpha \mathcal{G}_\alpha, \\ \{A_n\} \subset \bigcap_\alpha \mathcal{G}_\alpha &\implies \bigcup_n A_n \in \bigcap_\alpha \mathcal{G}_\alpha. \end{aligned}$$

If  $A \in \bigcap_\alpha \mathcal{G}_\alpha$  then,  $A \in \mathcal{G}_\alpha$  for all  $\alpha$ , and by the  $\sigma$ -field assumption on each  $\mathcal{G}_\alpha$ , it follows that  $A^c \in \mathcal{G}_\alpha$  for all  $\alpha$ , so  $A^c \in \bigcap_\alpha \mathcal{G}_\alpha$ .

If  $\{A_n\} \subset \bigcap_\alpha \mathcal{G}_\alpha$ , then  $\{A_n\} \subset \mathcal{G}_\alpha$  for all  $\alpha$ , and by the  $\sigma$ -field assumption on each  $\mathcal{G}_\alpha$ , it follows that  $\bigcup_n A_n \in \mathcal{G}_\alpha$  for all  $\alpha$ , so  $\bigcup_n A_n \in \bigcap_\alpha \mathcal{G}_\alpha$ .

First, note that  $\{\mathcal{F} \subset \mathcal{G} \mid \mathcal{G} \text{ is a } \sigma\text{-field}\}$  is non-empty, as  $2^\Omega$  belongs to it. So by the above result  $\mathcal{G}^* = \bigcap \{\mathcal{F} \subset \mathcal{G} \mid \mathcal{G} \text{ is a } \sigma\text{-field}\}$  is a  $\sigma$ -field. From construction, it is trivial that for any  $\sigma$ -field such that  $\mathcal{F} \subset \mathcal{G}$ , we have

$$\mathcal{G}^* \subset \mathcal{G},$$

so  $\mathcal{G}^*$  is the smallest  $\sigma$ -algebra and notationally

$$\sigma(\mathcal{F}) = \mathcal{G}^* = \{\mathcal{F} \subset \mathcal{G} : \mathcal{G} \text{ is a } \sigma\text{-field}\},$$

as required. □

(ii) Let  $\{A_k\} \subset \mathcal{F}$ , and define

$$\tilde{A}_k = \bigcup_{n \leq k} A_n \quad (k \geq 1).$$

Then,  $\{\tilde{A}_k\}$  is an increasing sequence, so by a monotone class property,

$$\bigcup_k \tilde{A}_k = \bigcup_k A_k \in \mathcal{M}.$$

Therefore,  $\sigma(\mathcal{F}) \subset \mathcal{M}$ .

From the above result, it follows that, for any monotone class  $\mathcal{M}$  such that  $\mathcal{F} \subset \mathcal{M}$ ,

$$\sigma(\mathcal{F}) \subset \mathcal{M},$$

so

$$\sigma(\mathcal{F}) \subset \bigcap \{\mathcal{F} \subset \mathcal{M} : \mathcal{M} \text{ is a monotone class}\} = m(\mathcal{F}).$$

Conversely, as a  $\sigma$ -field is a monotone class, we have that

$$m(\mathcal{F}) = \bigcap \{\mathcal{F} \subset \mathcal{M} : \mathcal{M} \text{ is a monotone class}\} \subset \bigcap \{\mathcal{F} \subset \mathcal{G} : \mathcal{G} \text{ is a } \sigma\text{-field}\} = \sigma(\mathcal{F}),$$

so

$$\sigma(\mathcal{F}) = m(\mathcal{F}),$$

as required. In passing, we mention that the intersection of any family of monotone class is a monotone class and the proven result is known as the monotone class lemma. □

**Question 3.**

3. Prove that if  $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$  is lower semi-continuous (that is,  $\liminf_{\|x-x_0\| \downarrow 0} f(x) \geq f(x_0)$  for every  $x_0 \in \mathbb{R}^n$ ) then it is a Borel function, and conclude that continuous functions are Borel measurable. (*Hint: first show every set of the form  $\{x : f(x) \leq a\}$  ( $a \in \mathbb{R}$ ) is closed.*)

**Solution.**

We first note that, as closed sets belong to the Borel  $\sigma$ -field, to show that  $f$  is a Borel function, it suffices to show that

Now, as Now, as inverse images of closed sets are closed for continuous functions, from the above result, it follows that

**Question 4.**

4. Let  $m\mathcal{F}$  denote the set of measurable functions from  $(\Omega, \mathcal{F}) \rightarrow ([-\infty, \infty], \mathcal{B}_{[-\infty, \infty]})$ , where  $\mathcal{B}_{[-\infty, \infty]} = \sigma([-\infty, a] : a \in \mathbb{R})$ . Prove that
- (a) every simple function  $f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$  belongs to  $m\mathcal{F}$ .
  - (b) if  $X_n \in m\mathcal{F}$  ( $n \geq 1$ ) then  $\liminf_{n \rightarrow \infty} X_n$  and  $\limsup_{n \rightarrow \infty} X_n$  also belong to  $m\mathcal{F}$ .
- Conclude that  $m\mathcal{F}$  is the smallest class of functions satisfying properties (a) and (b).

**Solution.**

(a) Let  $f$  be a simple function, i.e.

$$f = \sum_{i=1}^n a_i X_{E_i},$$

where  $a_i \in \mathbb{R}$ ,  $E_i \in \mathcal{F}$  pairwise disjoint for  $1 \leq i \leq n$ , and  $\bigcup_{i=1}^n E_i = \Omega$ . For sake of completeness, we show that  $f$  is  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$  measurable. For any  $a \in \mathbb{R}$ , observe that  $f^{-1}((-\infty, a])$  is a union of sub-collection (allowing the empty collection) of  $\{E_i\}$ , so it is in  $\mathcal{F}$ . As it is sufficient to check the measurability condition on the generators, we conclude that any simple function is  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$  measurable.

Fix  $a \in \mathbb{R}$ . As  $f^{-1}(-\infty) = \emptyset$  and  $f$  is  $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$  measurable, it follows that

$$f^{-1}([-\infty, a]) = f^{-1}(-\infty) \cup f^{-1}((-\infty, a]) \in \mathcal{F}.$$

So,  $f$  is  $(\mathcal{F}, \mathcal{B}_{[-\infty, \infty]})$  measurable, i.e.  $f \in m\mathcal{F}$ . □

(b) Observe that

$$\begin{aligned} \liminf_{n \rightarrow \infty} X_n &= \sup_k \inf_{n \geq k} X_n \\ \limsup_{n \rightarrow \infty} X_n &= \inf_k \sup_{n \geq k} X_n \end{aligned}$$

Hence, combined with the fact that  $\inf_n X_n = -\sup_n -X_n$ , it suffices to show that  $\sup_n X_n$  is measurable.

Fix  $a \in \mathbb{R}$ . Then, we have

$$(\sup_n X_n)^{-1}([-\infty, a]) = \bigcap_n X_n^{-1}([-\infty, a]) \in \mathcal{F}. \quad (*)$$

We now prove (\*). If  $w \in \bigcap_n X_n^{-1}([-\infty, a])$ , then  $X_n(w) \in [-\infty, a]$  for all  $n$ , so  $\sup_n X_n(w) \in [-\infty, a]$ , and  $w \in \sup_n X_n^{-1}([-\infty, a])$ . If  $w \in \sup_n X_n^{-1}([-\infty, a])$ , then  $\sup_n X_n(w) \in [-\infty, a]$ , which implies  $X_n(w) \in [-\infty, a]$  for all  $n$ . Hence, (\*) is true and  $\sup_n X_n \in m\mathcal{F}$ .

Let  $\mathcal{G}$  be a class of functions such that (a) and (b) are true. We wish to show that  $m\mathcal{F} \subset \mathcal{G}$ . By (a), we know that simple functions are in  $\mathcal{G}$ . Now, if  $f \in m\mathcal{F}$ , then by the simple approximation lemma, there exists a sequence of simple functions  $\{X_n\}$  such that  $X_n$  converges pointwise to  $f$ . Then, by (b),

$$f = \limsup_{n \rightarrow \infty} X_n \in \mathcal{G},$$

so  $m\mathcal{F} \subset \mathcal{G}$ , and  $m\mathcal{F}$  is the smallest class of functions satisfying properties (a) and (b). □