# ProbLimI:
# Problem Set XII


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## Abstract

This work contains solutions to the exercises of the problem set XII. The chosen problems are 2,3,4.


**Question 2.**

2. (a) Let $M_n$ be a submartingale. Show that for every $p \geq 1$, every $n \geq 0$ and every $m > 0$,

$$\mathbb{P}\left(\max_{1\leq k\leq n} M_k \geq m\right) \leq \frac{\mathbb{E}(M_n \vee 0)^p}{m^p}.$$

(b) Let $M_n$ be a martingale. Show that for every $p \geq 1$, every $n \geq 0$ and every $m > 0$,

$$\mathbb{P}\left(\max_{1\leq k\leq n} |M_k| \geq m\right) \leq \frac{\mathbb{E}|M_n|^p}{m^p}.$$

(c) Let $M_n$ be a martingale started at $M_0 = 0$. Show that for every $m > 0$,

$$\mathbb{P}\left(\max_{1\leq k\leq n} M_k \geq m\right) \leq \frac{\mathbb{E}M_n^2}{\mathbb{E}M_n^2 + m^2}.$$

(*Hint: use the fact that* $(M_n + c)^2$ *is a submartingale.*)

**Solution.**
**(a)** By convexity, $\{(M_n{}^+)^p\}$ is a submartingale. Then, by Doob's inequality,

$$\mathbb{P}(\max_{1\leq k\leq n} M_k \geq m) = \mathbb{P}(\max_{1\leq k\leq n}(M_k{}^+)^p \geq m^p) \quad \leq \quad \frac{\mathbb{E}(M_k{}^+)^p}{m^p}.$$

**(b)** By convexity $\{|M_n|\}$ is a martingale. By (a),

$$\mathbb{P}(\max_{1\leq k\leq n}|M_k| \geq m) \leq \frac{\mathbb{E}[|M_k|^p}{m^p}.$$

**(c)** Let $-c < m$. Then, by hint, and Doob's inequality,

$$\mathbb{P}(\max_{1\leq k\leq m} M_k \leq m) \leq \mathbb{P}(\max_{1\leq k\leq m}(M_k + c)^2 \geq (m+c)^2) \leq \frac{\mathbb{E}[M_n^2 + c^2]}{(m+c)^2}.$$

Viewing the RHS as a function of $c$, denoted as $f$, taking the derivative, and setting it equal to 0 give

$$-2(\mathbb{E}[M_n{}^2] + c^2) + 2c(c+m) = 0$$

so minimum happens at $c = \dfrac{\mathbb{E}M_n{}^2}{m} > 0$. Plugging in the value of $c$ at the minimum gives the desired inequality. $\square$

**Question 3.**

3. Let $M_n = \prod_{k=1}^{n} X_k$ where $\{X_k\}$ are i.i.d. nonnegative r.v.'s with $\mathbb{E}X_1 = 1$ and $\mathbb{P}(X_1 = 1) < 1$.

(a) Argue that $\mathbb{E}(\log X_1)_+ < \infty$ and infer from the SLLN that $\lim_{n\to\infty} n^{-1} \log M_n \overset{a.s.}{=} m < 0$.

(b) Deduce that $M_n \xrightarrow{a.s.} 0$ as $n \to \infty$ and that $M_n$ is not uniformly integrable.

(c) Conclude that Doob's $L^p$ maximal inequality cannot be extended to the case $p = 1$; that is, there is no $q < \infty$ such that $\mathbb{E}\left[(\max_{1\le k\le n} M_k)_+\right] \le q\mathbb{E}(M_n)_+$ holds for every submartingale $M_n$ and every $n$.

**Solution.**
Define $M_0 = 1$. Observe that $\{M_n\}$ is a martingale, because, by independence,

$$\mathbb{E}[M_n|\mathscr{F}_{n-1}] = \mathbb{E}[\prod_{k=1}^{n} X_k|\mathscr{F}_{n-1}] = M_{n-1}\mathbb{E}[X_n|\mathscr{F}_{n-1}] = M_{n-1}\mathbb{E}[X_n] = M_{n-1}$$

for any $n \ge 1$.

**(a)** Suppose $\mu = 0$. By chebyshev,

$$\mathbb{E}[\log(X)_+] = \int_0^\infty \mathbb{P}((\log(X) > \lambda)d\lambda = \int_0^\infty \mathbb{P}(X > e^\lambda)d\lambda \le \int_0^\infty \frac{\mathbb{E}[X]}{e^\lambda}d\lambda = e^{-\lambda}|_0^\infty = 1.$$

Hence, by SLLN, $n^{-1}\sum_{k=1}^{n}\log(X_k) = n^{-1}\log(M_n) \to \mu$ almost surely, where $\mu = \log(X_1)$.

If $\mu = 0$, then it follows that $X_1$ will be constant, which contradicts the assumption.

**(b)** From the convergence in (a),

$$n^{-1}\log(M_n) \le 2^{-1}\mu$$

and hence

$$M_n \le \exp(2^{-1}\mu n)$$

almost surely, for all $n$ sufficiently large. Taking $n \to \infty$, shows that $M_n \to 0$ almost surely. Now, suppose $\{M_n\}$ is uniformly integrable. Then, $M_n \to 0$ in $L^1$. However, $\mathbb{E}[M_n] = 1$ for all $n \ge 1$, so we have a contradiction, and $\{M_n\}$ is not integrable.

**(c)** Suppose the $L^p$ maximal inequlaity holds for $p = 1$. Then,

$$\mathbb{E}[\max_{k\le n} M_k] \le q\mathbb{E}[M_n] = q\mathbb{E}[M_0] < \infty$$

for some $q < \infty$. Taking $n \to \infty$ shows that

$$\mathbb{E}[\sup_k M_k] < \infty$$

which implies that $\{M_n\}$ is U.I, which contradicts $(b)$. So, the maximal inequality cannot be extended to $p = 1$. □

**Question 4.**

4. Let $S_n = \sum_{k=1}^{n} X_k$ where the $X_k$ are mutually independent.

(a) Show that for every $n$ and $s, t \geq 0$, if $Z_n = \max_{1\leq k\leq n} |S_k|$ then

$$\mathbb{P}(Z_n \geq t+s) \leq \mathbb{P}(|S_n| \geq t) + \mathbb{P}(Z_n \geq t+s) \max_{1\leq k\leq n} \mathbb{P}(|S_n - S_k| > s)\,.$$

(b) Suppose that $\mathbb{E}|X_k| < \infty$ for all $k$ and that $\sup_n \mathbb{E}|S_n| < \infty$. Show that $\mathbb{E}[\sup_n |S_n|] < \infty$.

**Solution.**

**(a)** The idea is to mimic the proof of Levy's maximal inequality. Set $E_k = \{|S_1| < t+s, |S_2| < t+s, ..., |S_{k-1}| < t+s, |S_k| \geq t+s\}$. Then,

$$\begin{aligned}
\mathbb{P}(Z_n \geq t_s) &= \mathbb{P}(Z_n \geq t+s, |S_n| \geq t) + \mathbb{P}(Z_n \geq t_s, |S_n| < t) \\
&\leq \mathbb{P}(|S_n| \geq t) + \sum_{k=1}^{n-1} \mathbb{P}(E_k \cap \{|S_n - S_k| > s\}) \\
&\leq \mathbb{P}(|S_n| \geq t) + \sum_{k=1}^{n-1} \mathbb{P}(E_k) \text{max}_{1\leq k\leq n} \mathbb{P}(|S_n - S_k| > s) \\
&\leq \mathbb{P}(|S_n| \geq t) + \mathbb{P}(Z_n \geq t+s) \text{max}_{1\leq k\leq n} \mathbb{P}(|S_n - S_k| > s).
\end{aligned}$$

**(b)** Observe that

$$\begin{aligned}
\mathbb{E}[\sup_n |S_n|] &\leq \sum_{k=0}^{\infty} \mathbb{P}(\sup_n |S_n| \geq k) = \sum_{k=0}^{\infty} \mathbb{P}(\sup_n Z_n \geq k) \\
&\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}(Z_n \geq k) \leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}(|S_n| \geq k - \frac{1}{k}) + \mathbb{P}(Z_n \geq k) \max_{1\leq i\leq n} \mathbb{P}(|S_n - S_i| > \frac{1}{k}) \\
&\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}(|S_n| \geq k - \frac{1}{k}) + \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}(Z_n \geq k) \max_{1\leq i\leq n} \mathbb{P}(|S_n - S_i| > \frac{1}{k}) \\
&\leq \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \mathbb{P}(|S_n| \geq k - \frac{1}{k}) + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}(Z_n \geq k) \max_{1\leq i\leq n} \mathbb{P}(|S_n - S_i| > \frac{1}{k}).
\end{aligned}$$

For $k = 0$, just take $\frac{1}{k}$ in the above expression. Then, from the fact that $\sup_n \mathbb{E}S_n]$ is bounded, maybe we can deduce the summability of the RHS.