ProbLimI:
Problem Set VI

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Abstract

This work contains solutions to the exercises of the problem set V. The chosen problems are 1,2, and 3.

Question 1.

1. Let $\{A_n\}$ be pairwise independent events with $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, and let $S_n = \sum_{k=1}^{n} \mathbf{1}_{A_k}$.
   (a) Show that $\mathrm{Var}(S_n) \leq \mathbb{E}S_n$ and deduce that $S_n/\mathbb{E}S_n \xrightarrow{P} 1$.
   (b) Show that if $n_k = \inf\{n : \mathbb{E}S_n \geq k^2\}$ then $S_{n_k}/\mathbb{E}S_{n_k} \xrightarrow{a.s.} 1$. (*Hint: use Borel-Cantelli I.*)
   (c) Prove that $\mathbb{E}[S_{n_{k+1}}]/\mathbb{E}[S_{n_k}] \to 1$ and deduce that $S_n/\mathbb{E}S_n \xrightarrow{a.s.} 1$.

Solution.
Observe that

$$\sum_{k=1}^{n} \mathbb{P}(A_k) \quad = \quad \mathbb{E}[S_n]$$

for any $n \in \mathbb{N}$. As the LHS tends to ∞ as $n \to \infty$, we can choose N large enough such that $\mathbb{E}[S_n] > 0$ for all $n \geq N$. We relabel the indices to start from N so that the random variables $\{\frac{S_n}{\mathbb{E}[S_n]}\}$ are well-defined for the problem.

(i) By independence,

$$\begin{aligned} \mathrm{Var}(S_n) &= \sum_{k=1}^{n} \mathrm{Var}(1_{A_k}) = \sum_{k=1}^{n} \mathbb{E}[1_{A_k}^2] - \mathbb{E}[1_{A_k}]^2 = \sum_{k=1}^{n} \mathbb{P}(1_{A_k}) - \mathbb{P}(1_{A_k})^2 \\ &\leq \sum_{k=1}^{n} \mathbb{P}(1_{A_k}) = \mathbb{E}[S_n] \end{aligned}$$

for each $n \geq 1$. Now, we prove the claimed convergence in probability. Let $\epsilon > 0$. By Chebyshev's inequality and the above result,

$$\begin{aligned} \mathbb{P}(|\frac{S_n}{\mathbb{E}[S_n]} - 1| > \epsilon) &= \mathbb{P}(|S_n - \mathbb{E}[S_n]| > \epsilon\mathbb{E}[S_n]) \\ &\leq \frac{\mathrm{Var}(S_n)}{\epsilon^2\mathbb{E}[S_n]^2} \leq \frac{1}{\epsilon^2\mathbb{E}[S_n]} \end{aligned}$$

for any $n \in \mathbb{N}$. Therefore, taking $n \to \infty$,

$$\lim_{n\to\infty} \mathbb{P}(|\frac{S_n}{\mathrm{E}[S_n]} - 1| > \epsilon) \quad = \quad 0.$$

Since $\epsilon > 0$ was arbitrary, $\dfrac{S_n}{\mathrm{E}[S_n]} \to 1$ in probability.

(ii) As $\mathbb{E}[S_n]$ tends to ∞ as $n \to \infty$, we can find a subsequence with the given property. Let $\epsilon > 0$. By the same argument as above, and the property of the chosen subsequence,

$$\mathbb{P}(|\frac{S_n}{\mathbb{E}[S_n]} - 1| > \epsilon) \quad \le \quad \frac{1}{\epsilon^2 \mathbb{E}[S_n]} \le \frac{1}{\epsilon^2 k^2}$$

for all $k \in \mathbb{N}$, which implies

$$\sum_{k=1}^{\infty} \mathbb{P}(|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| > \epsilon) < \infty.$$

By Borel-Cantelli I,

$$\mathbb{P}(|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| > \epsilon \quad \text{i.o.}) \quad = \quad 0$$

for any $\epsilon > 0$. Now, by definition of pointwise convergence,

$$\mathbb{P}(\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} \to 1) \quad = \quad \mathbb{P}(\bigcap_{\epsilon>0} \{|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| \le \epsilon \text{ a.a.}\}) = 1 - \mathbb{P}(\bigcup_{\epsilon>0} \{|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| > \epsilon \text{ i.o.}\})$$

By density of rationals and the above result,

$$\begin{aligned} \mathbb{P}(\bigcup_{\epsilon>0} \{|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| > \epsilon \text{ i.o.}\}) \quad &= \quad \mathbb{P}(\bigcup_{\epsilon>0; \epsilon \in \mathbb{Q}} \{|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| > \epsilon \text{ i.o.}\}) \\ &\le \quad \sum_{\epsilon>0; \epsilon\in\mathbb{Q}} \mathbb{P}(|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| > \epsilon \text{ i.o.}) = 0 \end{aligned}$$

and hence

$$\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} \quad \to \quad 1 \text{ almost surely.}$$

(iii) Observe that

$$|\mathbb{E}[S_{n+1}] - \mathbb{E}[S_n]| = \mathbb{P}(A_{n+1}) \le 1$$

for all $n \ge 1$, which implies that $\{n_k\}$ chosen is strictly increasing as a function k and

$$\mathbb{E}[S_{n_k}] \quad < \quad (k+1)^2$$

for all $k \ge 1$. Therefore,

$$1 \quad \le \quad \frac{\mathbb{E}[S_{n_{k+1}}]}{\mathbb{E}[S_{n_k}]} \le \frac{(k+2)^2}{k^2}$$

for all $k \ge 1$, and hence, taking $k \to \infty$,

$$\lim_{k\to\infty} \frac{\mathbb{E}[S_{n_{k+1}}]}{\mathbb{E}[S_{n_k}]} \quad = \quad 1.$$

Now, let $w \in \Omega$ such that $\dfrac{S_{n_k(w)}}{\mathbb{E}[S_{n_k}]} \to 1$. Recall that

$$S_n(w) \le S_{n+1}(w) \qquad \text{and} \qquad \mathbb{E}[S_n] \le \mathbb{E}[S_{n+1}]$$

for all $n \ge 1$, and hence

$$\frac{\mathbb{E}[S_{n_k}]}{\mathbb{E}[S_{n_{k+1}}]} \frac{S_{n_k}(w)}{\mathbb{E}[S_{n_k}]} = \frac{S_{n_k}(w)}{\mathbb{E}[S_{n_{k+1}}]} \quad \le \quad \frac{S_n(w)}{\mathbb{E}[S_n]} \le \frac{S_{n_{k+1}}(w)}{\mathbb{E}[S_{n_k}]} = \frac{S_{n_{k+1}}(w)}{\mathbb{E}[S_{n_{k+1}}]} \frac{\mathbb{E}[S_{n_{k+1}}]}{\mathbb{E}[S_{n_k}]}$$

for any $k \in \mathbb{N}$ and $n_k \leq n < n_{k+1}$. Set

$$L_n = \frac{\mathbb{E}[S_l]}{\mathbb{E}[S_u]}\frac{S_l(w)}{\mathbb{E}[S_l]} \quad \text{and} \quad U_n = \frac{\mathbb{E}[S_u]}{\mathbb{E}[S_l]}\frac{S_u(w)}{\mathbb{E}[S_u]}$$

where $l = \sup\{n_k : n_k \leq n; k \in \mathbb{N}\}$ and $u = \inf\{n_k : n_k > n; k \in \mathbb{N}\}$, for any $n \in \mathbb{N}$. Then,

$$1 = \lim_{n\to\infty} L_n \leq \limsup_{n\to\infty} \frac{S_n(w)}{\mathbb{E}[S_n]} \leq \liminf_{n\to\infty} \frac{S_n(w)}{\mathbb{E}[S_n]} \leq \lim_{n\to\infty} U_n = 1$$

and hence

$$\{\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} \to 1\} \quad \subset \quad \{\frac{S_n}{\mathbb{E}[S_n]} \to 1\}$$

which implies

$$\frac{S_n}{\mathbb{E}[S_n]} \quad \to \quad 1 \;\; \text{almost surely.}$$

□

Question 2.

2. (a) Let X be a nonnegative random variable. Show that $Y = \lfloor X \rfloor$ satisfies $Y = \sum_{n=1}^{\infty} \mathbf{1}_{\{X \geq n\}}$, and deduce that $\mathbb{E}X - 1 \leq \sum_{n=1}^{\infty} \mathbb{P}(X \geq n) \leq \mathbb{E}X$.

(b) Let $X_1, \ldots, X_n, \ldots$ be i.i.d. r.v.'s with $\mathbb{E}|X_1|^\alpha = \infty$ for $\alpha > 0$. Show that for every $\beta > 0$ one has $\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n| > \beta n^{1/\alpha}\right) = \infty$, and deduce that $\limsup_{n\to\infty} n^{-1/\alpha}|X_n| = \infty$, a.s.

(c) Conclude that $S_n := \sum_{k=1}^{n} X_k$ satisfies $\limsup_{n\to\infty} n^{-1/\alpha}|S_n| = \infty$, a.s.

Solution.
(a) As X is non-negative real-valued RV and $1_{\{X \geq n\}}(w) = 0$ for each $n > \max\{k \in \mathbb{N} : k \leq X(w)\}$.

$$\lfloor X(w) \rfloor \quad = \quad \max\{k \in \mathbb{N} : k \leq X(w)\} = \sum_{n=1}^{\max\{k\in\mathbb{N} : k \leq X(w)\}} 1_{\{X\geq n\}}(w) = \sum_{n=1}^{\infty} 1_{\{X \geq n\}}(w)$$

for any $w \in \Omega$, and hence

$$\lfloor X \rfloor \quad = \quad \sum_{n=1}^{\infty} 1_{\{X \geq n\}} = Y.$$

Observe that $\{\sum_{n=1}^{k} 1_{\{X\geq n\}}\}_k$ is a pointwise non-decreasing and non-negative sequence of RVs, which converges pointwise everywhere to $Y = \lfloor X \rfloor$. Hence, by MCT,

$$\mathbb{E}Y \quad = \quad \sum_{n=1}^{\infty} \mathbb{E}1_{\{X\geq n\}} = \sum_{n=1}^{\infty} \mathbb{P}(X \geq n).$$

Since $X - 1 \leq \lfloor X \rfloor \leq X$, if X is integrable, by monotonicity of integration,

$$\mathbb{E}[X] - 1 \quad \leq \quad \sum_{n=1}^{\infty} \mathbb{P}(X \geq n) \leq \mathbb{E}[X].$$

If $\mathbb{E}[X] = \mathbb{E}[X] - 1 = \infty$, then $X - 1$ is not integrable, as otherwise it will contradict the non-integrability of X by linearity. Therefore, $\mathbb{E}[Y] = \mathbb{E}[X-1] = \infty$, so the inequality holds trivially.

(b) Let $\beta > 0$. Observe that

$$\sum_{n=0}^{\infty} 1_{\{\beta^{-\alpha}|X_1|^\alpha > n\}} \quad = \quad \lceil \beta^{-\alpha}|X_1|^\alpha \rceil.$$

Similar to (a), by MCT,

$$\sum_{n=0}^{\infty} \mathbb{P}(\beta^{-\alpha}|X_1|^\alpha > n) \quad = \quad \mathbb{E}\lceil \beta^{-\alpha}|X_1|^\alpha \rceil.$$

We now have the following pointwise estimate:

$$\beta^{-\alpha}|X_1|^\alpha \quad \leq \quad \lceil \beta^{-\alpha}|X_1|^\alpha \rceil.$$

As $\mathbb{E}|X_1|^\alpha = \infty$, we see $\mathbb{E}[\beta^{-\alpha}|X_1|^\alpha] = \infty$ and combined with the above estimate

$$\sum_{n=0}^{\infty} \mathbb{P}(|X_n| > \beta n^{\frac{1}{\alpha}}) = \sum_{n=0}^{\infty} \mathbb{P}(\beta^{-\alpha}|X_1|^\alpha > n) \quad = \quad \infty$$

which implies

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \beta n^{\frac{1}{\alpha}}) \quad = \quad \infty.$$

Since $\beta > 0$ was arbitrary, we have the result for all $\beta > 0$.

Set

$$A_k \quad = \quad \{n^{-\frac{1}{\alpha}}|X_n| > k \quad \text{i.o.}\}$$

for each $k \in \mathbb{N}$. By Borel-Cantelli II, combined with the above result,

$$\mathbb{P}(A_k) = 1$$

for each $k \in \mathbb{N}$. Since $\{A_k\}$ is descending, by continuity of probability,

$$\mathbb{P}(\bigcap_{k=1}^{\infty} A_k) \quad = \quad 1.$$

Suppose $w \in \bigcap_{k=1}^{\infty} A_k$. By induction, we construct a subsequence, which diverges to ∞. Choose n_1 such that

$$(n_1)^{-\frac{1}{\alpha}}|X_{n_1}(w)| > 1.$$

Given $\{n_i\}_{i=1}^{l}$, choose n_{l+1} larger than all previous indices such that

$$(n_{l+1})^{-\frac{1}{\alpha}}|X_{n_{l+1}}(w)| > l + 1.$$

By induction, we have constructed a subsequence $\{n_l\}$ such that

$$(n_l)^{-\frac{1}{\alpha}}|X_{n_l}(w)| > l$$

for each $l \in \mathbb{N}$, and hence

$$\bigcap_{k=1}^{\infty} A_k \quad \subset \quad \{\limsup_{n\to\infty} n^{-\frac{1}{\alpha}}|X_n| = \infty\}.$$

Therefore,

$$\limsup_{n\to\infty} n^{-\frac{1}{\alpha}}|X_n| = \infty \quad \text{a.s.}$$

(c) Firstly, by reverse triangle inequality,

$$|n^{-\frac{1}{\alpha}}|S_{n-1}| - n^{-\frac{1}{\alpha}}|X_n|| \quad \leq \quad n^{-\frac{1}{\alpha}}|S_n|$$

for all $n \geq 2$, and hence, by elementary properties of $\limsup$

$$\begin{aligned}
\limsup n^{-\frac{1}{\alpha}}|X_n| \quad &\leq \quad \limsup(n^{-\frac{1}{\alpha}}|X_n| - n^{-\frac{1}{\alpha}}|S_{n-1}|) + \limsup n^{-\frac{1}{\alpha}}|S_{n-1}| \\
&= \quad \limsup(n^{-\frac{1}{\alpha}}|X_n| - n^{-\frac{1}{\alpha}}|S_{n-1}|) + \limsup(n-1)^{-\frac{1}{\alpha}}|S_{n-1}| \limsup(\frac{n}{n-1})^{-\frac{1}{\alpha}} \\
&= \quad \limsup(n^{-\frac{1}{\alpha}}|X_n| - n^{-\frac{1}{\alpha}}|S_{n-1}|) + \limsup n^{-\frac{1}{\alpha}}|S_n| \\
&\leq \quad 2\limsup n^{-\frac{1}{\alpha}}|S_n|.
\end{aligned}$$

By the above estimate,

$$\{\limsup n^{-\frac{1}{\alpha}}|X_n| = \infty\} \quad \subset \quad \{\limsup n^{-\frac{1}{\alpha}}|S_n| = \infty\}$$

and hence

$$\limsup n^{-\frac{1}{\alpha}}|S_n| = \infty \qquad \text{a.s.}$$

Question 3.

3. Let (X_k) be i.i.d. r.v.'s taking values in $\overline{\mathbb{R}}$ and let $M_n = \max_{k \le n} X_k$.

(a) Show that $\mathbb{P}(\{|X_n| > n\} \text{ i.o.}) = 0$ if and only if $\mathbb{E}|X_1| < \infty$.

(b) Show that $n^{-1} X_n \xrightarrow{a.s.} 0$ if and only if $\mathbb{E}|X_1| < \infty$.

(c) Show that $n^{-1} M_n \xrightarrow{a.s.} 0$ if and only if $\mathbb{E}(X_1)_+ < \infty$ and $\mathbb{P}(X_1 > -\infty) > 0$. Further show that $n^{-1} M_n \xrightarrow{P} 0$ if and only if $n\mathbb{P}(X_1 > n) \to 0$ and $\mathbb{P}(X_1 > -\infty) > 0$.

(d) Show that $n^{-1} X_n \xrightarrow{P} 0$ if and only if $\mathbb{P}(|X_1| < \infty) = 1$.

Solution.

The argument in $3 - c$ is not complete. Verbatim repeat the argument given in the problem 2 to have

$$\mathbb{E}|X| < \infty \quad \iff \quad \sum_n \mathbb{P}(|X| > n) \text{ converges}$$

and

$$\mathbb{E}X_+ < \infty \quad \iff \quad \sum_n \mathbb{P}(X > n) \text{ converges.}$$

(a) As (X_n) are identically distributed,

$$\sum_n \mathbb{P}(|X_1| > n) \quad = \quad \sum_n \mathbb{P}(|X_n| > n).$$

Therefore, by Borel-Cantelli I,II and the above equivalences

$$\mathbb{P}(|X_n| > n \text{ i.o.}) = 0 \quad \iff \quad \mathbb{E}|X_1| < \infty.$$

(b) Suppose $\mathbb{E}|X_1| < \infty$. Then,

$$\mathbb{E}|\frac{X_1}{\alpha}| < \infty$$

so, by (a)

$$\mathbb{P}(n^{-1}|X_n| > \alpha \text{ i.o.}) = 0$$

for any $\alpha > 0$. By the same argument given in $1 - b$,

$$n^{-1}|X_n| \to 0 \text{ a.s.}$$

Conversely, suppose $n^{-1} X_n \to 1$ a.s. Observe that

$$\{n^{-1}X_n \to 0\} \quad = \quad \bigcap_{\alpha > 0} \{n^{-1}X_n \le \alpha \text{ a.a.}\}.$$

With $\alpha = 1$,

$$\mathbb{P}(n^{-1}|X_n| \le 1 \text{ a.a.}) = 1$$

so

$$\mathbb{P}(|X_1| > n \text{ i.o.}) = 0.$$

which by (a) implies $\mathbb{E}|X_1| < \infty$.

(c) We first show the forward direction. Assume $n^{-1}M_n \to 0$ a.s. Suppose for sake of contradiction that $\mathbb{P}(X_1 > -\infty) = 0$. Then, $M_n = -\infty$ a.s. for all $n \ge 1$. Therefore, $n^{-1}M_n \to -\infty$ a.s, which is a contradiction, and $\mathbb{P}(X_1 > -\infty) > 0$. Now, suppose again for sake of contradiction that $\mathbb{E}(X_1)_+ = \infty$, then by the equivalence established before, and Borel Cantelli II, we have $\mathbb{P}(\{X_n > n \text{ i.o.}\}) = 1$. Then, it follows that for a.s. $w \in \Omega$, there exists a subsequence $X_{n_k}(w) > 1$ for all k, and

$$\limsup M_n(w) \ge \limsup X_n(w) \ge 1 \text{ a.s.}$$

which is a contradiction. Now, conversely, suppose there exists a set A with positive probability, where $n^{-1}M_n \not\to 0$, and $\mathbb{E}(X_1)_+ < \infty$. Then, we wish to show that $X_1 = -\infty$ a.s. Since negative values do not affect the limit behavior of M_n, we can further assume that Xs take values that are non-negative or $-\infty$. Therefore,

$$\mathbb{P}(X_1 \geq 0) = 0.$$

Since $\mathbb{E}(X_1)_+ < \infty$. by Borel Cantelli I,

$$\mathbb{P}(X_n > n \ \text{ i.o.}) = 0$$

Hence, there almost everywhere$-w$, we have $X_n(w) \leq n$ for all sufficiently large n, and

$$n^{-1} \limsup_n M_n(w) \leq 1 \ \text{ a.s.}$$

Assume $n^{-1}M_n \to_p 0$. If $\mathbb{P}(X_1 > -\infty) = 0$, then $n^{-1}M_n \to -\infty$ a.s. so $n^{-1}M_n \to_p -\infty$, a contradiction. Hence, $\mathbb{P}(X_1 > -\infty) > 0$. Now, by iid assumption,

$$\begin{aligned} \mathbb{P}(n^{-1}M_n > \epsilon) &= 1 - \mathbb{P}(n^{-1}M_n \leq \epsilon) = 1 - \mathbb{P}(|n^{-1}X_1| \leq \epsilon)^n \\ &= 1 - (1 - \mathbb{P}(|n^{-1}X_1| > \epsilon)^n = 1 - e^{-n\mathbb{P}(|X_1|>n\epsilon)} \end{aligned}$$

for any $\epsilon > 0$ and $n \geq 1$. Hence, by convergence in probability, with $\epsilon = 1$,

$$n\mathbb{P}(|X_1| > \epsilon) \to 0$$

so

$$n\mathbb{P}(|X_1| > \epsilon) \to 0$$

(d) Since

$$\{|X_1| < \infty\} = \{\frac{|X_1|}{\epsilon} < \infty\} = \bigcup_{n=1}^{\infty} \{\frac{|X_1|}{\epsilon} \leq n\}$$

by continuity of probability,

$$\mathbb{P}(|X_1| < \infty) = 1 - \lim_n \mathbb{P}(|n^{-1}X_n| > \epsilon)$$

for any $\epsilon > 0$. Therefore,

$$\mathbb{P}(|X_1| < \infty) = 1 \quad \Longleftrightarrow \quad \lim_n \mathbb{P}(|n^{-1}X_n| > \epsilon) = 0 \ \ \forall \epsilon > 0.$$

By definition $n^{-1}X_n \to_p 0$ iff

$$\lim_n \mathbb{P}(|n^{-1}X_n| > \epsilon) = 0$$

for any $\epsilon > 0$, so we are done. □