

# Durrett Probability: Problems

Youngduck Choi \*

## Abstract

This work contains solutions to some exercises from Durrett's probability text.

## 1 Chapter 6: Markov Chains

### Question 6.3.3.

**6.3.3. First entrance decomposition.** Let  $T_y = \inf\{n \geq 1 : X_n = y\}$ . Show that

$$p^n(x, y) = \sum_{m=1}^n P_x(T_y = m) p^{n-m}(y, y)$$

### Solution.

Here we assume countable state space. Observe that

$$\begin{aligned} p^n(x, y) &= P_x(X_n = y) = P_x\left(\bigcup_{m=1}^n \{T_y = m; X_n = y\}\right) \\ &= E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y}; T_y \leq n) = E_x(E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y} | \mathcal{F}_{T_y}); T_y \leq n) \quad (1) \\ &= E_x(E_{X_{T_y}}(1_{\{X_{n-T_y}=y\}}; T_y \leq n) = E_x(E_y(1_{\{X_{n-T_y}=y\}}); T_y \leq n) \quad (2) \\ &= \sum_{m=1}^n P_x(T_y = m) E_y(1_{\{X_{n-m}=y\}}) = \sum_{m=1}^n P_x(T_y = m) p^{n-m}(y, y) \end{aligned}$$

where (1) holds by definition of conditional expectation and (2) holds by the strong Markov property.

---

\*Department of Mathematics,

**Question 6.3.4.**

**6.3.4.** Show that  $\sum_{m=0}^n P_x(X_m = x) \geq \sum_{m=k}^{n+k} P_x(X_m = x)$ .

**Solution.**

**Question 6.3.5.**

**6.3.5.** Suppose that  $S - C$  is finite and for each  $x \in S - C$   $P_x(\tau_C < \infty) > 0$ . Then there is an  $N < \infty$  and  $\epsilon > 0$  so that  $P_y(\tau_C > kN) \leq (1 - \epsilon)^k$ .

**Solution.**

We assume countable state space. Observe that, for any  $x \in S \setminus C$ , we can choose  $n(x) \in \mathbb{N}$  such that

$$P_x(\tau_C \leq n) > 0.$$

Otherwise, for some  $x \in S \setminus C$ , by continuity of probability,

$$P_x(\tau_C < \infty) = \lim_{k \rightarrow \infty} P_x(\tau_C \leq k) = 0,$$

which is a contradiction. Now, let

$$N = \max_{z \in S \setminus C} n(z). \text{ and } \epsilon = \min_{z \in S \setminus C} P_z(\tau_C \leq N).$$

Trivially,

$$P_y(\tau_C > kN) = 0$$

for any  $k \in \mathbb{N}$ , and  $y \in C$ , since  $y \in C$  implies  $\tau_C = 0$  by definition. Therefore, it suffices to show

$$P_y(\tau_C > kN) \leq (1 - \epsilon)^k \tag{3}$$

for all  $k \in \mathbb{N}$  and  $y \in S \setminus C$ . Fix  $y \in S \setminus C$ . Then,

$$P_y(\tau_C \leq N) \geq \epsilon.$$

and hence

$$P_y(\tau_C > N) \leq (1 - \epsilon)$$

Now, we proceed by induction to prove (3). Suppose, for some  $k \in \mathbb{N}$  such that  $k \geq 2$ ,

$$P_y(\tau_C > kN) \leq (1 - \epsilon)^k.$$

We compute

$$\begin{aligned} P_y(\tau_C > (k+1)N) &= E_y(1_{\{\tau_C > kN\}} \circ \theta_N; \tau_C > N) \\ &= E_y(E_y(1_{\{\tau_C > kN\}} \circ \theta_N | \mathcal{F}_N); \tau_C > N) \\ &= E_y(E_{X_N}(1_{\{\tau_C > kN\}}); \tau_C > N) \\ &\leq E_y(\sup_{z \in S} P_z(\tau_C > kN); \tau_C > N) \\ &\leq (1 - \epsilon)^k E_y(1; \tau_C > N) = (1 - \epsilon)^{k+1} \end{aligned} \tag{4}$$

where (4) holds by Markov Property, which completes the proof.  $\square$

**Question 6.3.6.**

**6.3.6.** Let  $h(x) = P_x(\tau_A < \tau_B)$ . Suppose  $A \cap B = \emptyset$ ,  $S - (A \cup B)$  is finite, and  $P_x(\tau_{A \cup B} < \infty) > 0$  for all  $x \in S - (A \cup B)$ . (i) Show that

$$(*) \quad h(x) = \sum_y p(x, y)h(y) \quad \text{for } x \notin A \cup B$$

(ii) Show that if  $h$  satisfies  $(*)$  then  $h(X(n \wedge \tau_{A \cup B}))$  is a martingale. (iii) Use this and Exercise 6.3.5 to conclude that  $h(x) = P_x(\tau_A < \tau_B)$  is the only solution of  $(*)$  that is 1 on  $A$  and 0 on  $B$ .

**Solution.**

(i) Let  $x \in S \setminus (A \cup B)$ . Then,

$$1_{\{\tau_A < \tau_B\}} = 1_{\{\tau_A < \tau_B\}} \circ \theta_1.$$

It follows that

$$\begin{aligned} h(x) &= P_x(\tau_A < \tau_B) = E_x(1_{\{\tau_A < \tau_B\}}) = E_x(1_{\{\tau_A < \tau_B\}} \circ \theta_1) \\ &= E_x(E_x(1_{\{\tau_A < \tau_B\}} \circ \theta_1 | \mathcal{F}_1)) = E_x(E_{X_1}(1_{\{\tau_A < \tau_B\}})) \\ &= \sum_y P(X_1 = y)P_y(\tau_A < \tau_B) = \sum_y p(x, y)P_y(\tau_A < \tau_B) \end{aligned} \tag{5}$$

where (5) holds by Markov property.

(ii)

(iii)

**Question 6.3.7.**

**6.3.7.** Let  $X_n$  be a Markov chain with  $S = \{0, 1, \dots, N\}$  and suppose that  $X_n$  is a martingale and  $P_x(\tau_0 \wedge \tau_N < \infty) > 0$  for all  $x$ . (i) Show that 0 and  $N$  are absorbing states, i.e.,  $p(0, 0) = p(N, N) = 1$ . (ii) Show  $P_x(\tau_N < \tau_0) = x/N$ .

**Solution.**

## 2 Chapter 2: Law of Large Numbers

### 3 Chapter 5: Martingales

#### Question 5.2.1.

**Exercise 5.2.1.** Suppose  $X_n$  is a martingale w.r.t.  $\mathcal{G}_n$  and let  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ . Then  $\mathcal{G}_n \supset \mathcal{F}_n$  and  $X_n$  is a martingale w.r.t.  $\mathcal{F}_n$ .

#### Solution.

Various properties of conditional expectations are used.

We compute

$$E[X_{n+1}|\mathcal{F}_n] = E[X_{n+1}|\mathcal{G}_n|\mathcal{F}_n] \tag{6}$$

$$= E[X_n|\mathcal{F}_n] \tag{7}$$

$$= X_n \tag{8}$$

for all  $n \in \mathbb{N}$ , where (6) holds by the Tower property, (7) holds by Martingale property of  $\{G_n\}$  and (8) holds by measurability of  $X_n$  w.r.t  $\mathcal{F}_n$  for all  $n \in \mathbb{N}$ .  $\square$

**Question 5.2.2.**

**Exercise 5.2.2.** Suppose  $f$  is superharmonic on  $\mathbf{R}^d$ . Let  $\xi_1, \xi_2, \dots$  be i.i.d. uniform on  $B(0, 1)$ , and define  $S_n$  by  $S_n = S_{n-1} + \xi_n$  for  $n \geq 1$  and  $S_0 = x$ . Show that  $X_n = f(S_n)$  is a supermartingale.

**Solution.**



**Question 5.2.3.**

**Exercise 5.2.3.** Give an example of a submartingale  $X_n$  so that  $X_n^2$  is a supermartingale. Hint:  $X_n$  does not have to be random.

**Solution.**