ProbLimI:
Pset I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A \in \mathcal{F}$ and $A_k \in \mathcal{F}$ $(k \geq 1)$.

   (i) Prove the *sub-additivity* property: $\mathbb{P}(\bigcup_k A_k) \leq \sum_k \mathbb{P}(A_k)$.

   (ii) Prove the *continuity* property: If $A_k \uparrow A$ (i.e. $A_k \subseteq A_{k+1}$ for all k and $\cup_k A_k = A$) then $\mathbb{P}(A_k) \uparrow \mathbb{P}(A)$, and if $A_k \downarrow A$ (i.e. $A_k \supseteq A_{k+1}$ for all k and $\cap_k A_k = A$) then $\mathbb{P}(A_k) \downarrow \mathbb{P}(A)$.

Solution.

Question 2.

2. Let $\mathcal{F}$ be a field.

   (i) Show that if $\{\mathcal{G}_\alpha\}$ is a (possibly uncountable) family of σ-fields then $\bigcap_\alpha \mathcal{G}_\alpha$ is also a σ-field. Conclude that $\sigma(\mathcal{F}) = \bigcap\{\mathcal{G} \supseteq \mathcal{F} : \mathcal{G} \text{ is a } \sigma\text{-field}\}$.

   (ii) Prove that if $\mathcal{M}$ is a monotone class and $\mathcal{F} \subseteq \mathcal{M}$ then $\sigma(\mathcal{F}) \subseteq \mathcal{M}$. Conclude that $\sigma(\mathcal{F})$ is equal to $\mathrm{m}(\mathcal{F}) := \bigcap\{\mathcal{M} \supseteq \mathcal{F} : \mathcal{M} \text{ is a monotone class}\}$.

Solution.
(i) As $\emptyset$ and Ω are in $\mathscr{G}_\alpha$ for all α, by the $\sigma-$field property of each $\mathscr{G}_\alpha$, it follows that $\emptyset, \Omega \in \bigcap_\alpha \mathscr{G}_\alpha$. Now, it suffices to show that

$$A \in \bigcap_\alpha \mathscr{G}_\alpha \quad \Longrightarrow \quad A^c \in \mathscr{G}_\alpha,$$
$$\{A_n\} \subset \bigcap_\alpha \mathscr{G}_\alpha \quad \Longrightarrow \quad \bigcap_n A_n \in \bigcap_\alpha \mathscr{G}_\alpha.$$

If $A \in \bigcap_\alpha \mathscr{G}_\alpha$ then, $A \in \mathscr{G}_\alpha$ for all α, and by the $\sigma-$field assumption on each $\mathscr{G}_\alpha$, it follows that $A^c \in \mathscr{G}_\alpha$ for all α, so $A^c \in \bigcap_\alpha \mathscr{G}_\alpha$.

If $\{A_n\} \subset \bigcap_\alpha \mathscr{G}_\alpha$, then $\{A_n\} \subset \mathscr{G}_\alpha$ for all α, and by the $\sigma-$ field assumption on each $\mathscr{G}_\alpha$, it follows that $\bigcap_n A_n \in \mathscr{G}_\alpha$ for all α, so $\bigcap_n A_n \in \bigcap_\alpha \mathscr{G}_\alpha$.

Now, recall that $\sigma(\mathscr{F})$ is defined to be the smallest σ-field containing $\mathscr{F}$. Consider the family of σ-field that contains $\mathscr{F}$, and denote it by $\{\mathscr{G}_\alpha\}$. The above result shows that $\bigcap_\alpha \mathscr{G}_\alpha$ is a σ-field, and it is trivial that it contains $\mathscr{F}$. Obviously, for any α, $\bigcap_\alpha \mathscr{G}_\alpha \subset \mathscr{G}_\alpha$, which tells us that any σ-algebra containing $\mathscr{F}$ contains $\mathscr{G}_\alpha$, so it follows that $\bigcap_\alpha \mathscr{G}_\alpha$ is the smallest σ-algebra containing $\mathscr{F}$ and notationally we have

$$\sigma(\mathscr{F}) \quad = \quad \{\mathscr{F} \subset \mathscr{G} : \mathscr{G} \text{ is a } \sigma - \text{field}\},$$

as required. $\square$

(ii)

Question 3.

3. Prove that if $f : \mathbb{R}^n \to [-\infty, \infty]$ is lower semi-continuous (that is, $\liminf_{\|x-x_0\|\downarrow 0} f(x) \geq f(x_0)$ for every $x_0 \in \mathbb{R}^n$) then it is a Borel function, and conclude that continuous functions are Borel measurable. (*Hint: first show every set of the form* $\{x : f(x) \leq a\}$ $(a \in \mathbb{R})$ *is closed.*)

Solution.

Question 4.

4. Let $m\mathcal{F}$ denote the set of measurable functions from $(\Omega, \mathcal{F}) \to ([-\infty, \infty], \mathcal{B}_{[-\infty,\infty]})$, where $\mathcal{B}_{[-\infty,\infty]} = \sigma([-\infty, a] : a \in \mathbb{R})$. Prove that

   (a) every simple function $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ belongs to $m\mathcal{F}$.

   (b) if $X_n \in m\mathcal{F}$ $(n \geq 1)$ then $\liminf_{n\to\infty} X_n$ and $\limsup_{n\to\infty} X_n$ also belong to $m\mathcal{F}$.

   Conclude that $m\mathcal{F}$ is the smallest class of functions satisfying properties (a) and (b).

Solution.