
ProbLimI: Pset I

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Abstract

This work contains solutions to the exercises of the problem set I.

Question 1.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A \in \mathcal{F}$ and $A_k \in \mathcal{F}$ ($k \geq 1$).
 - (i) Prove the *sub-additivity* property: $\mathbb{P}(\bigcup_k A_k) \leq \sum_k \mathbb{P}(A_k)$.
 - (ii) Prove the *continuity* property: If $A_k \uparrow A$ (i.e. $A_k \subseteq A_{k+1}$ for all k and $\bigcup_k A_k = A$) then $\mathbb{P}(A_k) \uparrow \mathbb{P}(A)$, and if $A_k \downarrow A$ (i.e. $A_k \supseteq A_{k+1}$ for all k and $\bigcap_k A_k = A$) then $\mathbb{P}(A_k) \downarrow \mathbb{P}(A)$.

Solution.

(ii) Define $A_0 = \emptyset$ and

$$\tilde{A}_k = A_k \setminus A_{k-1} \quad (k \in \mathbb{N}).$$

Then, by countable additivity, we have

$$\mathbb{P}\left(\bigcup_k\right)$$

Question 2.

2. Let \mathcal{F} be a field.

- (i) Show that if $\{\mathcal{G}_\alpha\}$ is a (possibly uncountable) family of σ -fields then $\bigcap_\alpha \mathcal{G}_\alpha$ is also a σ -field. Conclude that $\sigma(\mathcal{F}) = \bigcap \{\mathcal{G} \supseteq \mathcal{F} : \mathcal{G} \text{ is a } \sigma\text{-field}\}$.
- (ii) Prove that if \mathcal{M} is a monotone class and $\mathcal{F} \subseteq \mathcal{M}$ then $\sigma(\mathcal{F}) \subseteq \mathcal{M}$. Conclude that $\sigma(\mathcal{F})$ is equal to $m(\mathcal{F}) := \bigcap \{\mathcal{M} \supseteq \mathcal{F} : \mathcal{M} \text{ is a monotone class}\}$.

Solution.

(i) As \emptyset and Ω are in \mathcal{G}_α for all α , by the σ -field property of each \mathcal{G}_α , it follows that $\emptyset, \Omega \in \bigcap_\alpha \mathcal{G}_\alpha$. Now, it suffices to show that

$$\begin{aligned} A \in \bigcap_\alpha \mathcal{G}_\alpha &\implies A^c \in \bigcap_\alpha \mathcal{G}_\alpha, \\ \{A_n\} \subset \bigcap_\alpha \mathcal{G}_\alpha &\implies \bigcap_n A_n \in \bigcap_\alpha \mathcal{G}_\alpha. \end{aligned}$$

If $A \in \bigcap_\alpha \mathcal{G}_\alpha$ then, $A \in \mathcal{G}_\alpha$ for all α , and by the σ -field assumption on each \mathcal{G}_α , it follows that $A^c \in \mathcal{G}_\alpha$ for all α , so $A^c \in \bigcap_\alpha \mathcal{G}_\alpha$.

If $\{A_n\} \subset \bigcap_\alpha \mathcal{G}_\alpha$, then $\{A_n\} \subset \mathcal{G}_\alpha$ for all α , and by the σ -field assumption on each \mathcal{G}_α , it follows that $\bigcap_n A_n \in \mathcal{G}_\alpha$ for all α , so $\bigcap_n A_n \in \bigcap_\alpha \mathcal{G}_\alpha$.

Now, recall that $\sigma(\mathcal{F})$ is defined to be the smallest σ -field containing \mathcal{F} . Consider the family of σ -field that contains \mathcal{F} , and denote it by $\{\mathcal{G}_\alpha\}$. The above result shows that $\bigcap_\alpha \mathcal{G}_\alpha$ is a σ -field, and it is trivial that it contains \mathcal{F} . Obviously, for any α , $\bigcap_\alpha \mathcal{G}_\alpha \subset \mathcal{G}_\alpha$, which tells us that any σ -algebra containing \mathcal{F} contains $\bigcap_\alpha \mathcal{G}_\alpha$, so it follows that $\bigcap_\alpha \mathcal{G}_\alpha$ is the smallest σ -algebra containing \mathcal{F} and notationally we have

$$\sigma(\mathcal{F}) = \{\mathcal{F} \subset \mathcal{G} : \mathcal{G} \text{ is a } \sigma\text{-field}\},$$

as required. □

(ii)

Question 3.

3. Prove that if $f : \mathbb{R}^n \rightarrow [-\infty, \infty]$ is lower semi-continuous (that is, $\liminf_{\|x-x_0\| \downarrow 0} f(x) \geq f(x_0)$ for every $x_0 \in \mathbb{R}^n$) then it is a Borel function, and conclude that continuous functions are Borel measurable. (*Hint: first show every set of the form $\{x : f(x) \leq a\}$ ($a \in \mathbb{R}$) is closed.*)

Solution.

Question 4.

4. Let $m\mathcal{F}$ denote the set of measurable functions from $(\Omega, \mathcal{F}) \rightarrow ([-\infty, \infty], \mathcal{B}_{[-\infty, \infty]})$, where $\mathcal{B}_{[-\infty, \infty]} = \sigma([-\infty, a] : a \in \mathbb{R})$. Prove that
- (a) every simple function $f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ belongs to $m\mathcal{F}$.
 - (b) if $X_n \in m\mathcal{F}$ ($n \geq 1$) then $\liminf_{n \rightarrow \infty} X_n$ and $\limsup_{n \rightarrow \infty} X_n$ also belong to $m\mathcal{F}$.
- Conclude that $m\mathcal{F}$ is the smallest class of functions satisfying properties (a) and (b).

Solution.

(a) Let f be a simple function, e.g.

$$f = \sum_{i=1}^n a_i X_{E_i},$$

where $a_i \in \mathbb{R}$, $E_i \in \mathcal{F}$ pairwise disjoint for $1 \leq i \leq n$, and $\bigcup_{i=1}^n E_i = \Omega$. For sake of completeness, we show that f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ measurable. For any $a \in \mathbb{R}$, observe that $f^{-1}((-\infty, a])$ is a union of sub-collection (allowing the empty collection) of $\{E_i\}$, so it is in \mathcal{F} . Hence, any simple function is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ measurable.

Fix $a \in \mathbb{R}$. As $f^{-1}(-\infty) = \emptyset$ and f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}})$ measurable, it follows that

$$f^{-1}([-\infty, a]) = f^{-1}((-\infty, a]) \in \mathcal{F}.$$

So, f is $(\mathcal{F}, \mathcal{B}_{[-\infty, \infty]})$ measurable.

(b) Observe that

$$\begin{aligned} \liminf_{n \rightarrow \infty} X_n &= \inf_k \sup_{n \geq k} X_n \\ \limsup_{n \rightarrow \infty} X_n &= \sup_k \inf_{n \geq k} X_n \end{aligned}$$

Hence, with symmetry of inf and sup, it suffices to show that $\sup_n X_n$ is measurable.

Fix $a \in \mathbb{R}$. Then, we have

$$(\sup_n X_n)^{-1}([-\infty, a]) = \bigcap_n X_n^{-1}([-\infty, a]) \in \mathcal{F}.$$

So, $\sup_n X_n$ is measurable.

Let \mathcal{G} be a class of functions such that (a) and (b) are true. We wish to show that $m\mathcal{F} \subset \mathcal{G}$. By (a), we know that simple functions are in \mathcal{G} . Now, if $f \in m\mathcal{F}$, then by the simple approximation lemma, there exists a sequence of simple functions $\{X_n\}$ such that X_n converges pointwise to f . Then, by (b),

$$f = \limsup_{n \rightarrow \infty} X_n \in \mathcal{G},$$

so $m\mathcal{F} \subset \mathcal{G}$, and $m\mathcal{F}$ is the smallest class of functions satisfying properties (a) and (b).