ProbLimI:
Problem Set VI

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Abstract

This work contains solutions to the exercises of the problem set V. The chosen problems are 1,2, and 3.

Question 1.

1. Let X_k be i.i.d. random variables and let $S_n = \sum_{k=1}^{n} X_k$. Show that if S_n/n converges a.s. as $n \to \infty$, then X_1 is necessarily integrable.

Solution.
We first prove the following lemma: if $Y \geq 0$ and $p > 0$, then $\mathbb{E}(Y^p) = \int_0^\infty py^{p-1}P(Y > y)dy$.

By Tonelli,

$$\int_0^\infty py^{p-1}\mathbb{P}(Y > y)dy \quad = \quad \int_0^\infty \int_\Omega py^{p-1}1_{\{Y>y\}}dPdy =$$

We first prove the following lemma: if $\{X_n\}$ are i.i.d. with $\mathbb{E}|X_1| = \infty$, then $\mathbb{P}(\frac{S_n}{n} \text{ converges}) = 0$. From 2.2.

$$\mathbb{E}|X_1| = \int_0^\infty \mathbb{P}(|X_1| > x)dx \quad \leq \quad \sum_{n=0}^{\infty} \mathbb{P}(|X_1| > n).$$

As $\mathbb{E}|X_1| = \infty$, $\sum_{n=0}^{\infty} \mathbb{P}(|X_1| > n) = \infty$, and by Borel Cantelli II,

$$\mathbb{P}(|X_n| \geq n \text{ i.o.}) \quad = \quad 1.$$

Now, it suffices to show that

$$\{\frac{S_n}{n} \text{ converges}\} \quad \subset \quad \{|X_n \geq n \text{ i.o.}\}.$$

Suppose for sake of contradiction that $\mathbb{E}|X_1| = \infty$. Then, by the lemma,

$$\mathbb{P}(\frac{S_n}{n} \text{ converges}) \quad = \quad 0$$

which is a contradiction. Hence, $\mathbb{E}|X_1| < \infty$, i.e. X_1 is integrable.

Question 2.

Solution.

2. Let $S_n = \sum_{k=1}^n X_k$ for i.i.d. r.v.'s X_k.

(a) Prove that if $\frac{d}{dt}\Phi_{X_1}(0) = a + ib \in \mathbb{C}$ then $a = 0$ and $S_n/n \xrightarrow{p} b$ as $n \to \infty$.

(b) Prove that $S_n/n \xrightarrow{p} b \in \mathbb{R}$ implies that $\Phi_{X_1}(x_k)^{\lfloor t/x_k \rfloor} \to \exp(ibt)$ as $x_k \downarrow 0$ and $\Phi_{X_1}(x_k)^{\lfloor t/x_k \rfloor} \to \exp(-ibt)$ as $x_k \uparrow 0$ for all $t > 0$. Deduce that $\frac{d}{dt}\Phi_{X_1}(0) = ib$, and conclude that $S_n/n \xrightarrow{p} b \in \mathbb{R}$ if and only if $\Phi_{X_1}(t)$ is differentiable at $t = 0$.

(c) Give an example of a random variable X with $\Phi_X(t)$ differentiable at $t = 0$ yet $\mathbb{E}|X| = \infty$.

Question 2.

3. Let $(X_{n,k})_{n\geq 1; 1\leq k\leq n}$ be mutually independent r.v.'s with $\mathbb{E}X_{n,k} = 0$ and $\mathbb{E}X_{n,k}^2 = \sigma_{n,k}^2$ such that $S_n := \sum_{k=1}^n X_{n,k}$ satisfies $\mathrm{Var}(S_n) \to 1$ as $n \to \infty$.

(i) Show that $\Phi_{S_n}(t) = \prod_{k=1}^n (1 + a_{n,k}(t))$ for every $t \in \mathbb{R}$, where $a_{n,k}(t) = \Phi_{X_{n,k}}(t) - 1$, and further, $|a_{n,k}(t)| \leq 2t^2\sigma_{n,k}^2$.

(ii) With $g_n(\varepsilon)$ the function from Lindeberg's CLT, show that, for every $t \in \mathbb{R}$,

$$\sum_{k=1}^n \left|a_{n,k}(t) + \tfrac{1}{2}t^2\sigma_{n,k}^2\right| \leq t^2 g_n(\varepsilon) + \tfrac{1}{6}|t|^3\varepsilon\,\mathrm{Var}(S_n)\,.$$

(iii) Prove that if $g_n(\varepsilon) \to 0$ as $n \to \infty$ then $\sum_{k=1}^n a_{n,k}(t) \to -t^2/2$ and $\sum_{k=1}^n |a_{n,k}(t)|^2 \to 0$ as $n \to \infty$ for every $t \in \mathbb{R}$, and deduce in that case that $S_n \Rightarrow \mathcal{N}(0,1)$.

Solution.

Question 4.

4. Define $f_n(x) = e^{ix} - \sum_{k=0}^{n}(ix)^k/k!$ for $x \in \mathbb{R}$ and $n \geq 0$.

   (a) Show that $|f_n(x)| \leq \min\left\{2|x|^n/n!\ ,\ |x|^{n+1}/(n+1)!\right\}$ for all x and n.

   (b) Use this to show that if $\mathbb{E}|X|^n < \infty$ then

$$\left|\Phi_X(t) - \sum_{k=0}^{n}(it)^k\mathbb{E}[X^k]/k!\right| \leq |t|^n\mathbb{E}\left[\min\left\{2|X|^n/n!\ ,\ |t||X|^{n+1}/(n+1)!\right\}\right].$$

   Explain the implication this has for a CLT for i.i.d. r.v.'s X_k with $\mathbb{E}|X_k|^n < \infty$.

Solution.

Let $x \in \mathbb{R}$. By integration by parts,

$$\int_0^x (x-s)^n e^{is}ds \quad = \quad \frac{x^{n+1}}{n+1} + \frac{i}{n+1}\int_0^x (x-s)^{n+1}e^{is}ds \tag{1}$$

for each $n \geq 0$. If $n = 0$, then

$$x + i\int_0^x (x-s)e^{is}ds \quad = \quad \int_0^x e^{is}ds = \frac{e^{ix}-1}{i}$$

and hence

$$e^{ix} = 1 + ix + i^2\int_0^x (x-s)e^{is}ds.$$

Suppose for some $n > 0$

$$e^{ix} - \sum_{k=0}^{n}\frac{(ix)^k}{k!} \quad = \quad \frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds \tag{2}$$

Then, combined with (),

$$\begin{aligned} e^{ix} - \sum_{k=0}^{n+1}\frac{(ix)^k}{k!} &= \frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds - \frac{(ix)^{n+1}}{(n+1)!} \\ &= \frac{i^{n+1}}{n!}\Big(\int_0^x (x-s)^n e^{is}ds - \frac{x^{n+1}}{(n+1)} \\ &= \frac{i^{n+2}}{(n+1)!}\int_0^x (x-s)^{n+1}e^{is}ds. \end{aligned}$$

Hence, by induction, () holds for all $n \geq 0$. If $x \geq 0$, then

$$\Big|\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds\Big| \quad \leq \quad \frac{1}{n!}\int_0^x |(x-s)^n|ds = \frac{1}{n!}\int_0^x (x-s)^n ds = \frac{1}{(n+1)!}|x|^{n+1}.$$

If $x < 0$, then

$$\begin{aligned} \Big|\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds\Big| &= \Big|\frac{i^{n+1}}{n!}\int_x^0 (x-s)^n e^{is}ds\Big| \leq \frac{1}{n!}\int_x^0 |(x-s)^n e^{is}|ds \\ &\leq \frac{1}{n!}\int_x^0 (s-x)^n ds = \frac{1}{(n+1)!}(-x)^{n+1} = \frac{1}{(n+1)!}|x|^{n+1}. \end{aligned}$$

Therefore,

$$|f_n(x)| \quad \leq \quad \frac{1}{(n+1)!}|x|^{n+1}$$

for any $n \geq 0$. Now, again by integration by parts,

$$\frac{i}{n}\int_0^x (x-s)^n e^{is}ds \quad = \quad -\frac{x^n}{n} + \int_0^x (x-s)^{n-1}e^{is}ds$$

and hence

$$\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds = \frac{i^n}{(n-1)!}\int_0^x (x-s)^{n-1}(e^{is}-1)ds$$

for any $n \geq 1$. If $x \geq 0$, then

$$\begin{aligned} |\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds| &= |\frac{i^n}{(n-1)!}\int_0^x (x-s)^{n-1}(e^{is}-1)ds| \\ &\leq \frac{1}{(n-1)!}\int_0^x |(x-s)^{n-1}||e^{is}-1|ds \\ &\leq \frac{2}{(n-1)!}\int_0^x (x-s)^{n-1}ds = \frac{2}{n!}|x|^n. \end{aligned}$$

If $x < 0$, then

$$\begin{aligned} |\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds| &= |\frac{i^n}{(n-1)!}\int_0^x (x-s)^{n-1}(e^{is}-1)ds| \\ &\leq \frac{1}{(n-1)!}\int_0^x |(x-s)^{n-1}||e^{is}-1|ds \\ &\leq \frac{2}{(n-1)!}\int_0^x (x-s)^{n-1}ds = \frac{2}{n!}|x|^n. \end{aligned}$$