ProbLimI:
Problem Set VIII

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Abstract

This work contains solutions to the exercises of the problem set VIII. The chosen problems are 1,2, and 3.

Question 1.

1. Let X_k be i.i.d. random variables and let $S_n = \sum_{k=1}^{n} X_k$. Show that if S_n/n converges a.s. as $n \to \infty$, then X_1 is necessarily integrable.

Solution.
Suppose for sake of contradiction that $\mathbb{E}|X_1| = \infty$. By Tonelli,

$$\begin{aligned} \mathbb{E}|X_1| &= \int_\Omega |X_1| d\mathbb{P} = \int_\Omega \int_0^{|X_1|} 1 dy d\mathbb{P} = \int_\Omega \int_0^\infty 1_{\{|X_1|>y\}} dy d\mathbb{P} \\ &= \int_0^\infty \int_\Omega 1_{\{|X_1|>y\}} \mathbb{P} dy = \int_0^\infty \mathbb{P}(|X_1| > y) dy. \end{aligned}$$

Since

$$\int_0^n \mathbb{P}(|X_1| > y) \leq \sum_{k=0}^{n} \mathbb{P}(|X_1| > k)$$

for all $n \geq 1$, from (**??**),

$$\sum_{k=0}^{\infty} \mathbb{P}(|X_1| > k) = \infty.$$

Therefore, by Borel Cantelli II,

$$\mathbb{P}(|X_n| \geq n \text{ i.o.}) = 1.$$

Now, observe that

$$\frac{S_n}{n} - \frac{S_{n+1}}{n+1} = \frac{S_n}{n(n+1)} - \frac{X_{n+1}}{n+1}$$

and hence, by reverse triangle inequality,

$$|\frac{S_n}{n} - \frac{S_{n+1}}{n+1}| \geq ||\frac{S_n}{n(n+1)}| - |\frac{X_{n+1}}{n+1}||$$

for all $n \geq 1$. Suppose

$$w \in \{\frac{S_n}{n} \text{ converges}\} \cap \{|X_n| \geq n \text{ i.o}\}.$$

Then, by the convergence of $\frac{S_n(w)}{n}$, there exists $0 < \delta < 1$ such that

$$|\frac{S_n(w)}{n(n+1)}| < \delta$$

for all n large enough, and hence

$$|\frac{S_n(w)}{n} - \frac{S_{n+1}(w)}{n+1}| \quad > \quad 1 - \delta \ \text{ i.o.},$$

which contradicts the fact that $\frac{S_n(w)}{n}$ converges. Hence,

$$\{\frac{S_n}{n} \ \text{ converges}\} \cap \{|X_n| \geq n \ \text{ i.o}\} = \varnothing$$

so

$$\mathbb{P}(\frac{S_n}{n} \ \text{ converges}) \quad = \quad 0$$

which contradicts that $\frac{S_n}{n}$ converges a.s. Hence, $\mathbb{E}|X_1| < \infty$, i.e. X_1 is integrable. $\square$

Question 2.

2. Let $S_n = \sum_{k=1}^{n} X_k$ for i.i.d. r.v.'s X_k.

   (a) Prove that if $\frac{d}{dt}\Phi_{X_1}(0) = a + ib \in \mathbb{C}$ then $a = 0$ and $S_n/n \xrightarrow{p} b$ as $n \to \infty$.

   (b) Prove that $S_n/n \xrightarrow{p} b \in \mathbb{R}$ implies that $\Phi_{X_1}(x_k)^{\lfloor t/x_k \rfloor} \to \exp(ibt)$ as $x_k \downarrow 0$ and $\Phi_{X_1}(x_k)^{\lfloor t/x_k \rfloor} \to \exp(-ibt)$ as $x_k \uparrow 0$ for all $t > 0$. Deduce that $\frac{d}{dt}\Phi_{X_1}(0) = ib$, and conclude that $S_n/n \xrightarrow{p} b \in \mathbb{R}$ if and only if $\Phi_{X_1}(t)$ is differentiable at $t = 0$.

   (c) Give an example of a random variable X with $\Phi_X(t)$ differentiable at $t = 0$ yet $\mathbb{E}|X| = \infty$.

Solution.
(a) Let $\dfrac{d}{dt}\Phi_{X_1}(0) = a + ib$. Then, there exists $t_0 > 0$ small enough such that

$$\Phi_{X_1}(t_0) = 1 + (a+ib)t_0 + o(|t_0|) \quad \text{and} \quad \Phi_{X_1}(-t_0) = 1 - (a+ib)t_0 + o(|t_0|). \tag{1}$$

As $\Phi_{X_1}(t) = \overline{\Phi_{X_1}(-t)}$ for any $t \in \mathbb{R}$, from (**??**),

$$(1 + at_0) + bt_0 i = (1 - at_0) + bt_0 i$$

which implies $a = 0$. Observe that

$$\Phi_{n^{-1}S_n}(t) = \Phi_{X_1}(\frac{t}{n})^n = (1 + \frac{bti}{n} + o(\frac{1}{n}))^n$$

for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we see that

$$\Phi_{n^{-1}S_n}(t) = e^{ibt}$$

so by continuity theorem, $\dfrac{S_n}{n} \to_D b$ as $n \to \infty$. From pset 4-1, $\dfrac{S_n}{n} \to_p b$ to $n \to \infty$.

(b) Since convergence in probability implies convergence in distribution, and by continuity theorem,

$$\Phi_{X_1}(\frac{t}{n})^n \to e^{ibt}$$

for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$, and $\epsilon > 0$. By uniform continuity of Φ_{X_1}, there exists $\delta > 0$,

$$|\frac{t}{n_k} - x_k| < \delta \implies |\Phi_{X_1}(x_k) - \Phi_{X_1}(\frac{t}{n_k})| < \epsilon.$$

where $\{n_k\}$ is any subsequence of $\{n\}$. Since $x_k \to 0$ and $n_k \to \infty$,

$$|\Phi_{X_1}(x_k) - \Phi_{X_1}(\frac{t}{n_k})| < \epsilon$$

and

$$(\Phi_{X_1}(\frac{t}{n_k}) - \epsilon)^{n_k - n_k + \lfloor \frac{t}{x_k} \rfloor} \leq \Phi_{X_1}(x_k)^{\lfloor \frac{t}{x_k} \rfloor} \leq (\Phi_{X_1}(+\frac{t}{n_k}) + \epsilon)^{n_k - n_k + \lfloor \frac{t}{x_k} \rfloor}$$

for all k large enough. Since $\epsilon > 0$ was arbitrary, we can take $\{\epsilon_k\}$ such that $\epsilon_k \to 0$. Then, by the givens,

$$\Phi_{X_1}(x_k)^{\lfloor \frac{t}{x_k} \rfloor} \to e^{ibt}$$

as $k \to \infty$. For the negative part, consider the conjugate relation as part a, then carry out the exact same limit process.

To conclude, conversely, suppose $\Phi(\frac{t}{n})^n \to e^{ibt}$, then by taking logs $n\log(\Phi(\frac{t}{n})) \to ibt$. By differenability of $\log(z)$ for $z = 1$ on the complex plane, $n(\Phi(\frac{t}{n}) - 1) \to ibt$. Hence, by the above result and (a), we have that $\Phi(t)$ is differentiable at $t = 0$.

(c) Define each X such that

$$\mathbb{P}(X = (-1)^k k) = C\frac{k}{k^2 \log(k)}$$

for all $k \geq 2$, where C is the normalization constant. Then,

$$\mathbb{E}|X| \quad = \quad \sum_{k=2}^{\infty} \frac{C}{k \log(k)} = \infty.$$

Furthermore,

$$n\mathbb{P}(|X| > n) \quad = \quad n \sum_{k=n+1}^{\infty} \frac{C}{k^2 \log(k)} \to 0$$

and

$$\mathbb{E}(X 1_{\{|X| \leq n\}}) = \sum_{k=2}^{n} (-1)^k \frac{C}{k \log(k)} \quad \text{converges}$$

as $n \to \infty$. Hence, from thm 2.2.7 in Durrett,

$$\frac{S_n}{n} \to_P \sum_{k=2}^{\infty} (-1)^k \frac{C}{k \log(k)}$$

and by (b), Φ_X is differentiable at $t = 0$. Therefore, X is the desired construction. □

Question 4.

4. Define $f_n(x) = e^{ix} - \sum_{k=0}^n (ix)^k/k!$ for $x \in \mathbb{R}$ and $n \geq 0$.

(a) Show that $|f_n(x)| \leq \min\left\{2|x|^n/n!\ ,\ |x|^{n+1}/(n+1)!\right\}$ for all x and n.

(b) Use this to show that if $\mathbb{E}|X|^n < \infty$ then

$$\left|\Phi_X(t) - \sum_{k=0}^n (it)^k \mathbb{E}[X^k]/k!\right| \leq |t|^n \mathbb{E}\left[\min\left\{2|X|^n/n!\ ,\ |t||X|^{n+1}/(n+1)!\right\}\right].$$

Explain the implication this has for a CLT for i.i.d. r.v.'s X_k with $\mathbb{E}|X_k|^n < \infty$.

Solution.
Let $x \in \mathbb{R}$. By integration by parts,

$$\int_0^x (x-s)^n e^{is} ds = \frac{x^{n+1}}{n+1} + \frac{i}{n+1}\int_0^x (x-s)^{n+1} e^{is} ds \tag{2}$$

for each $n \geq 0$. If $n = 0$, then

$$x + i\int_0^x (x-s) e^{is} ds = \int_0^x e^{is} ds = \frac{e^{ix} - 1}{i}$$

and hence

$$e^{ix} = 1 + ix + i^2 \int_0^x (x-s) e^{is} ds.$$

Suppose for some $n > 0$

$$e^{ix} - \sum_{k=0}^n \frac{(ix)^k}{k!} = \frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is} ds \tag{3}$$

Then, combined with (**??**),

$$\begin{aligned} e^{ix} - \sum_{k=0}^{n+1} \frac{(ix)^k}{k!} &= \frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is} ds - \frac{(ix)^{n+1}}{(n+1)!} \\ &= \frac{i^{n+1}}{n!}\Big(\int_0^x (x-s)^n e^{is} ds - \frac{x^{n+1}}{(n+1)} \\ &= \frac{i^{n+2}}{(n+1)!}\int_0^x (x-s)^{n+1} e^{is} ds. \end{aligned}$$

Hence, by induction, (**??**) holds for all $n \geq 0$. If $x \geq 0$, then

$$\Big|\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is} ds\Big| \leq \frac{1}{n!}\int_0^x |(x-s)^n| ds = \frac{1}{n!}\int_0^x (x-s)^n ds = \frac{1}{(n+1)!}|x|^{n+1}.$$

If $x < 0$, then

$$\begin{aligned} \Big|\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is} ds\Big| &= \Big|\frac{i^{n+1}}{n!}\int_x^0 (x-s)^n e^{is} ds\Big| \leq \frac{1}{n!}\int_x^0 |(x-s)^n e^{is}| ds \\ &\leq \frac{1}{n!}\int_x^0 (s-x)^n ds = \frac{1}{(n+1)!}(-x)^{n+1} = \frac{1}{(n+1)!}|x|^{n+1}. \end{aligned}$$

Therefore,

$$|f_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1} \tag{4}$$

for any $n \geq 0$. Now, again by integration by parts,

$$\frac{i}{n}\int_0^x (x-s)^n e^{is} ds = -\frac{x^n}{n} + \int_0^x (x-s)^{n-1} e^{is} ds$$

and hence

$$\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds = \frac{i^n}{(n-1)!}\int_0^x (x-s)^{n-1}(e^{is}-1)ds$$

for any $n \geq 1$. If $x \geq 0$, then

$$\begin{aligned} |\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds| &= |\frac{i^n}{(n-1)!}\int_0^x (x-s)^{n-1}(e^{is}-1)ds| \\ &\leq \frac{1}{(n-1)!}\int_0^x |(x-s)^{n-1}||e^{is}-1|ds \\ &\leq \frac{2}{(n-1)!}\int_0^x (x-s)^{n-1}ds = \frac{2}{n!}|x|^n. \end{aligned}$$

If $x < 0$, then

$$\begin{aligned} |\frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is}ds| &= |\frac{i^n}{(n-1)!}\int_0^x (x-s)^{n-1}(e^{is}-1)ds| \\ &\leq \frac{1}{(n-1)!}\int_0^x |(x-s)^{n-1}||e^{is}-1|ds \\ &\leq \frac{2}{(n-1)!}\int_0^x (x-s)^{n-1}ds = \frac{2}{n!}|x|^n. \end{aligned}$$

Therefore, combined with (**??**),

$$|f_n(x)| \leq \min(\frac{2|x|^n}{n!}, \frac{|x|^{n+1}}{(n+1)!})$$

for all $x \in \mathbb{R}$ and $n \geq 1$.

(ii) From the above estimate,

$$|e^{itX} - \sum_{k=0}^{n}\frac{(itX)^k}{k!}| \leq \min(\frac{2|tX|^n}{n!}, \frac{|tX|^{n+1}}{(n+1)!})$$

and hence, by Jensen's inequality, and linearity of expectation, which is granted by $\mathbb{E}|X|^n < \infty$,

$$\begin{aligned} |\Phi_X(t) - \sum_{k=0}^{n}\frac{\mathbb{E}(itX)^k}{k!}| &\leq \mathbb{E}|e^{itX} - \sum_{k=0}^{n}\frac{(itX)^k}{k!}| \\ &\leq \mathbb{E}\left[\min(\frac{2|tX|^n}{n!}, \frac{|tX|^{n+1}}{(n+1)!})\right] \\ &= |t|^n\mathbb{E}\left[\min(\frac{2|X|^n}{n!}, \frac{|t||X|^{n+1}}{(n+1)!})\right] \end{aligned}$$

for all $t \in \mathbb{R}$.

Suppose $\mathbb{E}|X|^2 < \infty$. Let $t \in \mathbb{R}$. Then,

$$\Phi_X(t) = 1 + it\mathbb{E}X - t^2\mathbb{E}\frac{X^2}{2} + \Psi(t)$$

for some constant Ψ, dependent on t. By the established result,

$$|\Psi(t)| \leq t^2\mathbb{E}(|t||X|^3 \wedge 2|X|^2)$$

where denotes the minimum operation. Observe that as $t \to 0$,

$$|t||X|^3 \wedge 2|X|^2 \to 0 \text{ everywhere and } |t||X|^3 \wedge 2|X|^2 \leq 2|X|^2 \text{everywhere.}$$

Since $\mathbb{E}|X|^2 < \infty$, by DCT,

$$\frac{|\Psi(t)|}{t^2} \to 0$$

so

$$\Phi_X(t) \quad = \quad 1 + it\mathbb{E}X - t^2\mathbb{E}\frac{X^2}{2} + o(t^2).$$

This shows that we can prove the CLT for the iid case without assuming the finite third moment.
□