ProbLimI:
Problem Sset I

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Abstract

This work contains solutions to the exercises of the problem set I. The chosen problems are 1,2, and 4.

Question 1.

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A \in \mathcal{F}$ and $A_k \in \mathcal{F}$ $(k \geq 1)$.
   (i) Prove the *sub-additivity* property: $\mathbb{P}(\bigcup_k A_k) \leq \sum_k \mathbb{P}(A_k)$.
   (ii) Prove the *continuity* property: If $A_k \uparrow A$ (i.e. $A_k \subseteq A_{k+1}$ for all k and $\cup_k A_k = A$) then $\mathbb{P}(A_k) \uparrow \mathbb{P}(A)$, and if $A_k \downarrow A$ (i.e. $A_k \supseteq A_{k+1}$ for all k and $\cap_k A_k = A$) then $\mathbb{P}(A_k) \downarrow \mathbb{P}(A)$.

Solution.
(i) Note that we have finite additivity property of measure, as the emptyset belong to any σ-field by definition. We first have

$$A, B \in \mathscr{F}, A \subset B \implies \mathbb{P}(A) \leq \mathbb{P}(B) \quad (*),$$

because

$$\mathbb{P}(B) \quad = \quad \mathbb{P}((B \setminus A) \cup A) = \mathbb{P}(B \setminus A) + \mathbb{P}(A).$$

Now, define $A_0 = \emptyset$, and

$$\tilde{A}_k \quad = \quad A_k \setminus (\bigcup_{0 \leq n < k} A_n) \quad (k \geq 1).$$

It follows that $\{\tilde{A}_k\}$ is a pairwise disjoint collection such that

$$\bigcup_k \tilde{A}_k = \bigcup_k A_k \quad \text{and} \quad \tilde{A}_k \subset A_k \quad (k \geq 1).$$

The union equality holds, since if $x \in \bigcup_k A_k$, then $x \in A_{k'}$ for some k', and $x \in \tilde{A}_{k^*}$, where

$$k^* = \inf\{k; x \in A_k\},$$

as $x \notin A_k$ for $k < k^*$ and $x \in A_{k^*}$. Hence, by countable additivity,

$$\mathbb{P}(\bigcup_k A_k) = \mathbb{P}(\bigcup_k \tilde{A}_k) = \sum_k \mathbb{P}(\tilde{A}_k) \leq \sum_k \mathbb{P}(A_k),$$

where the last inequality follows from $(*)$. □

(ii) Define $A_0, \tilde{A}_0 = \emptyset$ and

$$\tilde{A}_k = A_k \setminus A_{k-1} \quad (k \geq 1).$$

By finite additivity and the fact that $\{A_k\}$ is increasing, we have, for any $k \geq 1$,

$$\mathbb{P}(A_k) = \mathbb{P}(A_{k-1} \cup (A_k \setminus A_{k-1})) = \mathbb{P}(A_{k-1}) + \mathbb{P}(A_k \setminus A_{k-1}),$$

and by re-arranging

$$\mathbb{P}(\tilde{A}_k) = \mathbb{P}(A_k) - \mathbb{P}(A_{k-1}).$$

Now, $\{\tilde{A}_k\}$ are disjoint, so by countable additivity, we have

$$\begin{aligned} \mathbb{P}(A) = \mathbb{P}(\bigcup_k A_k) &= \mathbb{P}(\bigcup_k \tilde{A}_k) = \sum_k \mathbb{P}(\tilde{A}_k) = \lim_{k\to\infty} \sum_{n=1}^{k} \mathbb{P}(A_n) - \mathbb{P}(A_{n-1}) \\ &= \lim_{k\to\infty} \mathbb{P}(A_k) - \mathbb{P}(A_0) = \lim_{k\to\infty} \mathbb{P}(A_k), \end{aligned}$$

as required. Now, we show the continuity from above. Note that $\{A_k^c\}$ forms an increasing collection. By the DeMorgan's law, and continuity from below,

$$1 - \mathbb{P}(\bigcap_k A_k) = \mathbb{P}((\bigcap_k A_k)^c) = \mathbb{P}(\bigcup_k A_k^c) = \lim_{k\to\infty} \mathbb{P}(A_k^c) = 1 - \lim_{k\to\infty} \mathbb{P}(A_k),$$

so

$$\mathbb{P}(A) = \mathbb{P}(\bigcap_k A_k) = \lim_{k\to\infty} \mathbb{P}(A_k),$$

as required. $\square$

Question 2.

2. Let $\mathcal{F}$ be a field.

(i) Show that if $\{\mathcal{G}_\alpha\}$ is a (possibly uncountable) family of σ-fields then $\bigcap_\alpha \mathcal{G}_\alpha$ is also a σ-field. Conclude that $\sigma(\mathcal{F}) = \bigcap\{\mathcal{G} \supseteq \mathcal{F} : \mathcal{G} \text{ is a } \sigma\text{-field}\}$.

(ii) Prove that if $\mathcal{M}$ is a monotone class and $\mathcal{F} \subseteq \mathcal{M}$ then $\sigma(\mathcal{F}) \subseteq \mathcal{M}$. Conclude that $\sigma(\mathcal{F})$ is equal to $\mathrm{m}(\mathcal{F}) := \bigcap\{\mathcal{M} \supseteq \mathcal{F} : \mathcal{M} \text{ is a monotone class}\}$.

Solution.

(i) We just note that the index set must be non-empty. As $\emptyset$ and Ω are in $\mathscr{G}_\alpha$ for all α, by the $\sigma-$field property of each $\mathscr{G}_\alpha$, it follows that $\emptyset, \Omega \in \bigcap_\alpha \mathscr{G}_\alpha$. Now, it suffices to show that

$$A \in \bigcap_\alpha \mathscr{G}_\alpha \quad \Longrightarrow \quad A^c \in \bigcap_\alpha \mathscr{G}_\alpha,$$
$$\{A_n\} \subset \bigcap_\alpha \mathscr{G}_\alpha \quad \Longrightarrow \quad \bigcap_n A_n \in \bigcap_\alpha \mathscr{G}_\alpha.$$

If $A \in \bigcap_\alpha \mathscr{G}_\alpha$ then, $A \in \mathscr{G}_\alpha$ for all α, and by the $\sigma-$field assumption on each $\mathscr{G}_\alpha$, it follows that $A^c \in \mathscr{G}_\alpha$ for all α, so $A^c \in \bigcap_\alpha \mathscr{G}_\alpha$.

If $\{A_n\} \subset \bigcap_\alpha \mathscr{G}_\alpha$, then $\{A_n\} \subset \mathscr{G}_\alpha$ for all α, and by the $\sigma-$ field assumption on each $\mathscr{G}_\alpha$, it follows that $\bigcap_n A_n \in \mathscr{G}_\alpha$ for all α, so $\bigcap_n A_n \in \bigcap_\alpha \mathscr{G}_\alpha$.

First, note that $\{\mathscr{F} \subset \mathscr{G} \mid \mathscr{G} \text{ is a } \sigma\text{-field}\}$ is non-empty, as 2^Ω belongs to it. So by the above result $\mathscr{G}^* = \bigcap\{\mathscr{F} \subset \mathscr{G} \mid \mathscr{G} \text{ is a } \sigma\text{-field}\}$ is a σ-field, and we see that $\mathscr{F} \subset \mathscr{G}^*$. So far, we have shown that there exists a σ-field that contains $\mathscr{F}$. From construction, it is trivial that for any σ-field such that $\mathscr{F} \subset \mathscr{G}$, we have

$$\mathscr{G}^* \subset \mathscr{G}$$

so this shows that there exists a smallest σ-field that contains $\mathscr{F}$. The uniqueness follows as well, because if $\mathscr{G}_1$ and $\mathscr{G}_1$ are both smallest σ-field, then by definition

$$\mathscr{G}_1 \subset \mathscr{G}_2 \text{ and } \mathscr{G}_2 \subset \mathscr{G}_1,$$

so

$$\mathscr{G}_1 = \mathscr{G}_2.$$

Hence, we have shown precisely that for $\mathscr{F}$ (obviously the proof will go through for any collection), there exists a unique σ-algebra that contains $\mathscr{F}$ and notationally

$$\sigma(\mathscr{F}) \quad = \quad \mathscr{G}^* = \{\mathscr{F} \subset \mathscr{G} : \mathscr{G} \text{ is a } \sigma - \text{field}\},$$

as required. □

(ii) We first establish that as above the intersection of a collection montone classes $\{\mathscr{M}_\alpha\}$ is a montone class. It suffices to show that

$$\{A_n\} \subset \bigcap_\alpha \mathscr{M}_\alpha \text{ and } A_n \subset A_{n+1} \ \forall n \geq 1 \quad \Longrightarrow \quad \bigcup_n A_n \in \bigcap_\alpha \mathscr{M}_\alpha,$$

which holds, because by montone class property of each $\mathscr{M}_\alpha$, $\bigcup_n A_n \in \mathscr{M}_\alpha$ for each α, so $\bigcup_n A_n \in \bigcap_\alpha \mathscr{M}_\alpha$. Now, as above 2^Ω is a monotone class, we deduce that there exists a unique smallest monotone class containing any subset of 2^Ω, which we call the generated montone class.

Given that $\mathscr{F}$ is a field, we contend that the montone class generated by $\mathscr{F}$, $m(\mathscr{F})$ is a σ-field. Then, by definition of the generated σ-field, we would get the desired conclusion that $\sigma(\mathscr{F}) \subset \mathscr{M}$ for any $\mathscr{M}$ that contains $\mathscr{F}$, because

$$\sigma(\mathscr{F}) \subset m(\mathscr{F}) \subset \mathscr{M}.$$

It suffices to show that

$$m(\mathscr{F}) \quad \text{is a field },$$

since for $\{A_n\} \subset m(\mathscr{F})$, we have

$$\bigcup_n A_n \quad = \quad \bigcup_n \bigcup_{k=1}^{n} A_n \in m(\mathscr{F}),$$

where the last inclusion holds by the field, and monotone class property of $m(\mathscr{F})$. As $X \in \mathscr{F} \subset m(\mathscr{F})$, it again suffices to show

$$A, B \in m(\mathscr{F}) \quad \Longrightarrow \quad A \setminus B, A \cap B \in m(\mathscr{F}).$$

Fix $A \in m(\mathscr{F})$, and consider

$$m(A) \quad = \quad \{B \in m(\mathscr{F})\,;\; A \setminus B, B \setminus A, A \cap B \in m(\mathscr{F})\}.$$

One should note that $m(A)$ is a monotone class and

$$A \in m(B) \quad \Longleftrightarrow \quad B \in m(A) \;\; (*),$$

by the symmetry in the definition. It suffices to show that $m(\mathscr{F}) \subset m(A)$. First, we prove the case when $A \in \mathscr{F}$. Then, by definition of field, it follows that

$$A \subset m(A) \quad \text{and } m(\mathscr{F}) \subset m(A),$$

where the last set inclusion holds as $m(A)$ is a monotone class. Now, we extend to the case when $A \in m(\mathscr{F})$. By the above result and the $(*)$ equivalence,

$$A \in m(B) \;\text{ and }\; B \in m(A),$$

for any $B \in \mathscr{F}$. Hence, it follows that

$$\mathscr{F} \subset m(A) \;\text{ and }\; m(\mathscr{F}) \subset m(A),$$

and we are done. □

For sake of completeness, we use the above statement to conclude the remaining statement. From the statement, it follows that, for any monotone class $\mathscr{M}$ such that $\mathscr{F} \subset \mathscr{M}$,

$$\sigma(\mathscr{F}) \quad \subset \quad \mathscr{M},$$

so

$$\sigma(\mathscr{F}) \quad \subset \quad \bigcap \{\mathscr{F} \subset \mathscr{M} : \mathscr{M} \;\text{ is a monotone class}\} = m(\mathscr{F}).$$

Conversely, as a σ-field is a monotone class, we have that

$$m(\mathscr{F}) \quad = \quad \bigcap \{\mathscr{F} \subset \mathscr{M} : \mathscr{M} \;\text{ is a monotone class}\} \subset \bigcap \{\mathscr{F} \subset \mathscr{G} : \mathscr{G} \;\text{ is a } \sigma - \text{field}\} = \sigma(\mathscr{F}),$$

so

$$\sigma(\mathscr{F}) \quad = \quad m(\mathscr{F}),$$

as required. In passing, we mention that the intersection of any family of monotone class is a monotone class and the proven result is known as the monotone class lemma. □

Question 3.

3. Prove that if $f : \mathbb{R}^n \to [-\infty, \infty]$ is lower semi-continuous (that is, $\liminf_{\|x-x_0\|\downarrow 0} f(x) \geq f(x_0)$ for every $x_0 \in \mathbb{R}^n$) then it is a Borel function, and conclude that continuous functions are Borel measurable. (*Hint: first show every set of the form* $\{x : f(x) \leq a\}$ $(a \in \mathbb{R})$ *is closed.*)

Solution.
We first establish an important observation follows: Let X, Y be topological spaces, and $\mathscr{B}_X$, $\mathscr{B}_Y$ are respective Borel σ-fields. Then, for any $f : X \to Y$,

$$f \text{ is continuous} \quad \Longrightarrow \quad f \text{ is } (B_X, B_Y) \text{ measurable.}$$

This holds, because open sets are generators of any Borel σ-field, and by continuity, we obtain that the generators of the image are mapped to the σ-field on the domain. This problem gives a stronger result of this nature in this concrete setting, where the domain is $\mathbb{R}^n$ and the image is $[-\infty, \infty]$, because it shows that the lower semi-continuity is sufficient for measurability. In summary, the conclusion holds, as continuity implies lower continuity, which by the stated result implies measurability.

Now, we proceed with the proof. It suffices to show that

$$A = \{x \in \mathbb{R}^n : \ f(x) \leq \alpha\},$$

is closed for any $a \in \mathbb{R}$, because closed sets belong to the Borel σ-field, and $\{[-\infty, a]\}_a$ form a generator of the image σ-field. We claim that A is closed. Suppose that $x \in \mathbb{R}^n$ such that there exists x_n from A that $x_n \to x$. As $||x_n - x|| \to 0$, by the lower-semicontinuity, we obtain

$$f(x) \leq \liminf f(x_n) \leq a,$$

so $x \in A$. Hence, A is closed and we are done. $\square$

Question 4.

4. Let $m\mathcal{F}$ denote the set of measurable functions from $(\Omega, \mathcal{F}) \to ([-\infty, \infty], \mathcal{B}_{[-\infty,\infty]})$, where $\mathcal{B}_{[-\infty,\infty]} = \sigma([-\infty, a] : a \in \mathbb{R})$. Prove that

(a) every simple function $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ belongs to $m\mathcal{F}$.

(b) if $X_n \in m\mathcal{F}$ $(n \geq 1)$ then $\liminf_{n\to\infty} X_n$ and $\limsup_{n\to\infty} X_n$ also belong to $m\mathcal{F}$.

Conclude that $m\mathcal{F}$ is the smallest class of functions satisfying properties (a) and (b).

Solution.
(a) Let f be a simple function, i.e.

$$f \;=\; \sum_{i=1}^{n} a_i X_{E_i},$$

where $a_i \in \mathbb{R}$, $E_i \in \mathscr{F}$ pairwise disjoint for $1 \leq i \leq n$, and $\bigcup_{i=1}^{n} E_i = \Omega$. For sake of completeness, we show that f is $(\mathscr{F}, \mathscr{B}_{\mathbb{R}})$ measurable. For any $a \in \mathbb{R}$, observe that $f^{-1}((-\infty, a])$ is a union of sub-collection (allowing the empty collection) of $\{E_i\}$, so it is in $\mathscr{F}$. As it is sufficient to check the measurability condition on the generators, we conclude that any simple function is $(\mathscr{F}, \mathscr{B}_{\mathbb{R}})$ measurable.

Fix $a \in \mathbb{R}$. As $f^{-1}(-\infty) = \emptyset$ and f is $(\mathscr{F}, \mathscr{B}_{\mathbb{R}})$ measurable, it follows that

$$f^{-1}([-\infty, a]) \;=\; f^{-1}(-\infty) \cup f^{-1}((-\infty, a]) \in \mathscr{F}.$$

So, f is $(\mathscr{F}, \mathscr{B}_{[-\infty,\infty]})$ measurable, i.e. $f \in m\mathscr{F}$. $\square$

(b) Observe that

$$\begin{aligned} \liminf_{n\to\infty} X_n &= \sup_k \inf_{n\geq k} X_n \\ \limsup_{n\to\infty} X_n &= \inf_k \sup_{n\geq k} X_n \end{aligned}.$$

Hence, combined with the fact that $\inf_n X_n = -\sup_n -X_n$, it suffices to show that $\sup_n X_n$ is measurable.

Fix $a \in \mathbb{R}$. Then, we have

$$(\sup_n X_n)^{-1}([-\infty, a]) \;=\; \bigcap_n X_n^{-1}([-\infty, a]) \in \mathscr{F}. \quad (*)$$

We now prove $(*)$. If $w \in \bigcap_n X_n^{-1}([-\infty, a])$, then $X_n(w) \in [-\infty, a]$ for all n, so $\sup_n X_n(w) \in [-\infty, a]$, and $w \in \sup_n X_n^{-1}([-\infty, a])$. If $w \in \sup_n X_n^{-1}([-\infty, a])$, then $\sup_n X_n(w) \in [-\infty, a]$, which implies $X_n(w) \in [-\infty, a]$ for all n. Hence, $(*)$ is true and $\sup X_n \in m\mathscr{F}$.

Let $\mathscr{G}$ be a class of functions such that (a) and (b) are true. We wish to show that $m\mathscr{F} \subset \mathscr{G}$. By (a), we know that simple functions are in $\mathscr{G}$. Now, if $f \in m\mathscr{F}$, then by the simple approximation lemma, there exists a sequence of simple functions $\{X_n\}$ such that X_n converges pointwise to f. Then, by (b),

$$f = \limsup_{n\to\infty} X_n \in \mathscr{G},$$

so $m\mathscr{F} \subset \mathscr{G}$, and $m\mathscr{F}$ is the smallest class of functions satisfying properties (a) and (b). $\square$