ProbLimI:
Problem Set VI

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Abstract

This work contains solutions to the exercises of the problem set V. The chosen problems are 1,2, and 3.

Question 1.

1. Let $\{A_n\}$ be pairwise independent events with $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, and let $S_n = \sum_{k=1}^{n} \mathbf{1}_{A_k}$.
   (a) Show that $\mathrm{Var}(S_n) \leq \mathbb{E}S_n$ and deduce that $S_n/\mathbb{E}S_n \xrightarrow{P} 1$.
   (b) Show that if $n_k = \inf\{n : \mathbb{E}S_n \geq k^2\}$ then $S_{n_k}/\mathbb{E}S_{n_k} \xrightarrow{a.s.} 1$. (*Hint: use Borel-Cantelli I.*)
   (c) Prove that $\mathbb{E}[S_{n_{k+1}}]/\mathbb{E}[S_{n_k}] \to 1$ and deduce that $S_n/\mathbb{E}S_n \xrightarrow{a.s.} 1$.

Solution.
Observe that

$$\sum_{k=1}^{n} \mathbb{P}(A_k) = \mathbb{E}[S_n]$$

for any $n \in \mathbb{N}$. As the LHS tends to ∞ as $n \to \infty$, we can choose N large enough such that $\mathbb{E}[S_n] > 0$ for all $n \geq N$. We relabel the indices to start from N so that the random variables $\{\frac{S_n}{\mathbb{E}[S_n]}\}$ are well-defined for the problem.

(i) By independence,

$$\begin{aligned} \mathrm{Var}(S_n) &= \sum_{k=1}^{n} \mathrm{Var}(1_{A_k}) = \sum_{k=1}^{n} \mathbb{E}[1_{A_k}^2] - \mathbb{E}[1_{A_k}]^2 = \sum_{k=1}^{n} \mathbb{P}(1_{A_k}) - \mathbb{P}(1_{A_k})^2 \\ &\leq \sum_{k=1}^{n} \mathbb{P}(1_{A_k}) = \mathbb{E}[S_n] \end{aligned}$$

for each $n \geq 1$. Now, we prove the claimed convergence in probability. Let $\epsilon > 0$. By Chebyshev's inequality and the above result,

$$\begin{aligned} \mathbb{P}(|\frac{S_n}{\mathbb{E}[S_n]} - 1| > \epsilon) &= \mathbb{P}(|S_n - \mathbb{E}[S_n]| > \epsilon\mathbb{E}[S_n]) \\ &\leq \frac{\mathrm{Var}(S_n)}{\epsilon^2 \mathbb{E}[S_n]^2} \leq \frac{1}{\epsilon^2 \mathbb{E}[S_n]} \end{aligned}$$

for any $n \in \mathbb{N}$. Therefore, taking $n \to \infty$,

$$\lim_{n\to\infty} \mathbb{P}(|\frac{S_n}{\mathrm{E}[S_n]} - 1| > \epsilon) = 0.$$

Since $\epsilon > 0$ was arbitrary, $\frac{S_n}{\mathrm{E}[S_n]} \to 1$ in probability.

(ii) As $\mathbb{E}[S_n]$ tends to ∞ as $n \to \infty$, we can find a subsequence with the given property. Let $\epsilon > 0$. By the same argument as above, and the property of the chosen subsequence,

$$\mathbb{P}(|\frac{S_n}{\mathbb{E}[S_n]} - 1| > \epsilon) \quad \leq \quad \frac{1}{\epsilon^2 \mathbb{E}[S_n]} \leq \frac{1}{\epsilon^2 k^2}$$

for all $k \in \mathbb{N}$, which implies

$$\sum_{k=}^{\infty} \mathbb{P}(|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| > \epsilon) < \infty.$$

By Borel-Cantelli I,

$$\mathbb{P}(|\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} - 1| > \epsilon \quad i.o) \quad = \quad 0$$

Since $\epsilon > 0$ was arbitrary,

$$\frac{S_{n_k}}{\mathbb{E}[S_{n_k}]} \quad \to \quad 1 \;\; \text{almost surely.}$$

Question 2.

2. (a) Let X be a nonnegative random variable. Show that $Y = \lfloor X \rfloor$ satisfies $Y = \sum_{n=1}^{\infty} \mathbf{1}_{\{X \geq n\}}$, and deduce that $\mathbb{E}X - 1 \leq \sum_{n=1}^{\infty} \mathbb{P}(X \geq n) \leq \mathbb{E}X$.

   (b) Let $X_1, \ldots, X_n, \ldots$ be i.i.d. r.v.'s with $\mathbb{E}|X_1|^{\alpha} = \infty$ for $\alpha > 0$. Show that for every $\beta > 0$ one has $\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n| > \beta n^{1/\alpha}\right) = \infty$, and deduce that $\limsup_{n\to\infty} n^{-1/\alpha}|X_n| = \infty$, a.s.

   (c) Conclude that $S_n := \sum_{k=1}^{n} X_k$ satisfies $\limsup_{n\to\infty} n^{-1/\alpha}|S_n| = \infty$, a.s.

Solution.

Question 3.

3. Let (X_k) be i.i.d. r.v.'s taking values in $\overline{\mathbb{R}}$ and let $M_n = \max_{k\le n} X_k$.

(a) Show that $\mathbb{P}(\{|X_n| > n\} \text{ i.o.}) = 0$ if and only if $\mathbb{E}|X_1| < \infty$.

(b) Show that $n^{-1}X_n \xrightarrow{a.s.} 0$ if and only if $\mathbb{E}|X_1| < \infty$.

(c) Show that $n^{-1}M_n \xrightarrow{a.s.} 0$ if and only if $\mathbb{E}(X_1)_+ < \infty$ and $\mathbb{P}(X_1 > -\infty) > 0$. Further show that $n^{-1}M_n \xrightarrow{p} 0$ if and only if $n\mathbb{P}(X_1 > n) \to 0$ and $\mathbb{P}(X_1 > -\infty) > 0$.

(d) Show that $n^{-1}X_n \xrightarrow{p} 0$ if and only if $\mathbb{P}(|X_1| < \infty) = 1$.

Solution.

Question 4.

4. Let (X_k) be integrable i.i.d. r.v.'s with $\mathbb{E}X_k = 0$.

(a) Let $\{a_n\}$ and $\{b_n\}$ are to sequences of real numbers such that $b_n > 0$ and $b_n \uparrow \infty$. Show that if $\sum_n a_n/b_n$ converges then $b_n^{-1} \sum_{k=1}^n a_k \to 0$.

(b) Show that $\sum_{k=1}^\infty k^{-2} \operatorname{Var}(X_k \mathbf{1}_{\{|X_k| \le k\}}) \le 2\mathbb{E}|X_1|$.

(c) Conclude from parts (a),(b) and Kolmogorov's One Series Theorem that if c_n is a bounded sequence of non-random constants, then $n^{-1} \sum_{k=1}^n c_k X_k \stackrel{a.s.}{\to} 0$ as $n \to \infty$.

Solution.