ProbLimI:
Problem Set II

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Abstract

This work contains solutions to the exercises of the problem set I. The chosen problems are 1,2, and 4.

Question 1.

1. Let X be a nonnegative random variable with $\mathbb{E}[X^2] < \infty$, and set $m_i := \mathbb{E}[X^i]$ for $i = 1, 2$.
   (i) Prove that for every $0 \le x < m_1$ we have $\mathbb{P}(X > x) \ge (m_1 - x)^2/m_2$.
   (ii) Prove that $(\mathbb{E}|X^2 - m_2|)^2 \le 4m_2(m_2 - m_1^2)$.
   (iii) Show the following inequality, and compare it to part (i) for $X = \sum_{k=1}^n \mathbf{1}_{A_k}$.

$$\mathbb{P}\left(\bigcup_{k=1}^n A_k\right) \ge \sum_{k=1}^n \mathbb{P}(A_k) - \sum_{1 \le k < \ell \le n} \mathbb{P}(A_k \cap A_\ell).$$

Solution.
(1) Fix $\alpha \in [0, m_1)$. By Cauchy-Schwartz,

$$\mathbb{E}[1_{X>\alpha} X]^2 \quad \le \quad \mathbb{E}[1_{\{X>\alpha\}}^2]\mathbb{E}[X^2] = \mathbb{P}(X > \alpha) m_2.$$

On the other hand,

$$m_1 \quad = \quad \mathbb{E}[X 1_{\{X>\alpha\}}] + E[X 1_{\{X \le \alpha\}}],$$

which implies

$$(m_1 - \alpha)^2 \quad \le \quad \mathbb{E}[X 1_{\{X>\alpha\}}]^2.$$

Combining the above inequality with the first one and re-arranging yield

$$\frac{(m_1 - \alpha)^2}{m_2} \quad \le \quad \mathbb{P}(X > \alpha),$$

as required. $\square$

(2) By Cauchy-Schwartz,

$$\begin{aligned} \mathbb{E}[|(X + m_2^{\frac{1}{2}})(X - m_2^{\frac{1}{2}})|] &\le (\mathbb{E}[X^2 + 2{m_2}^{\frac{1}{2}} X + m_2]\mathbb{E}[X^2 - 2{m_2}^{\frac{1}{2}} X + m_2])^{\frac{1}{2}} \\ &= (4{m_2}^2 - 4m_2{m_1}^2)^{\frac{1}{2}} = 2m_2^{\frac{1}{2}}(m_2 - {m_1}^2)^{\frac{1}{2}}. \end{aligned}$$

Squaring both sides gives the desired inequality. $\square$

(3) We show the inequality via induction. For $n = 2$,

$$\begin{aligned}
\mathbb{P}(A_1 \cup A_2) &= \mathbb{P}({A_1}^c \cap A_2) + \mathbb{P}(A_1 \cap {A_2}^c) + \mathbb{P}(A_1 \cap A_2) \\
&= \mathbb{P}({A_1}^c \cap A_2) + \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap {A_2}^c) + \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1 \cap A_2) \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).
\end{aligned}$$

Now, suppose the statement is true for some $n > 2$. Then, using the $n = 2$ case, the inductive hypothesis and subadditivity gives

$$\begin{aligned}
\mathbb{P}(\bigcup_{k=1}^{n+1} A_k) &= \mathbb{P}(\bigcup_{k=1}^{n} A_k \bigcup A_{n+1}) \\
&= \mathbb{P}(\bigcup_{k=1}^{n} A_k) + \mathbb{P}(A_{n+1}) - \mathbb{P}(\bigcup_{k=1}^{n} A_k \bigcap A_{n+1}) \\
&\geq \sum_{k=1}^{n} \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n} \mathbb{P}(A_k \bigcap A_l) + \mathbb{P}(A_{k+1}) - \mathbb{P}(\bigcup_{k=1}^{n} A_n \bigcap A_{n+1}) \\
&\geq \sum_{k=1}^{n+1} \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n} \mathbb{P}(A_k \bigcap A_l) - \sum_{k=1}^{n} \mathbb{P}(A_k \bigcap A_{n+1}) \\
&= \sum_{k=1}^{n+1} \mathbb{P}(A_k) - \sum_{1 \leq k < l \leq n} \mathbb{P}(A_k \bigcap A_l).
\end{aligned}$$

Therefore, the induction is complete and the inequality is true. Now, if $X = \sum_{k=1}^{n} 1_{A_k}$, then

$$\begin{aligned}
m_1 &= E[X] = \sum_{k=1}^{n} \mathbb{P}(A_k) \\
m_2 &= E[X^2] = \mathbb{E}[\sum_{1 \leq k \leq l \leq n} 1_{A_k}] = \sum_{1 \leq k \leq l \leq n} \mathbb{P}(A_k \cap A_l)
\end{aligned}$$

and

$$\mathbb{P}(X > \alpha) = \mathbb{P}(\bigcup_{k=1}^{n} A_k),$$

for any $\alpha \in [0, 1)$. Hence, to compare, if $X = \sum_{k=1}^{n} 1_{A_k}$, we can re-write (iii) as

$$P(X > \alpha) \geq 2m_1 - m_2,$$

for any $\alpha \in [0, 1)$. $\square$

Question 2.

2. Let X be a real-valued random variable.

(a) Prove that the function $f(x) = \mathbb{E}\exp(-|X-x|)$ is continuous on $\mathbb{R}$.

(b) Further suppose that $X \geq 0$ and $\mathbb{E}X^p < \infty$ for some $p > 0$.

(b.1) Show that $\lim_{p\downarrow 0}(\mathbb{E}X^p - 1)/p = \mathbb{E}\log X$.

(b.2) Conclude that $\lim_{p\downarrow 0}\log(\mathbb{E}X^p)/p = \mathbb{E}\log X$.

Solution.
(a) We first note that, for any $x \in \mathbb{R}$, $\exp(-|X-x|)$ is uniformly bounded by 1, and we have a finite measure, so the expectation is well-defined and f is well-defined everywhere. Set $\mu = L(X)$. Then, by a change of variable,

$$f(x) = \mathbb{E}[\exp(-|X-x|) = \int_{-\infty}^{\infty}\exp(-|t-x|)\mu(dt)$$

so, for $x, h \in \mathbb{R}$,

$$\begin{aligned}|f(x+h)-f(x)| &= |\int e^{-|t-x-h|} - e^{-|t-x|}\mu(dt)| \\ &\leq \int |e^{-|t-x-h|} - e^{-|t-x|}|\mu(dt) \quad (*).\end{aligned}$$

Observe that

$$|e^{-|t-x-h|} - e^{-|t-x|}| \leq 2$$

for any $t, x, h \in \mathbb{R}$ and

$$|e^{-|t-x-h|} - e^{-|t-x|}| \to 0 \text{ as } h \to 0$$

for any $t, x \in \mathbb{R}$. Therefore, by BCT and (*), it follows that

$$|f(x+h)-f(x)| \to 0 \text{ as } h \to 0,$$

which shows that f is continuous as required. □

(b.1) Let $q > 0$ be a constant such that $X^q \in L^1$. With L'hopital's rule, we obtain that

$$\lim_{p\downarrow 0}\frac{X^p(w)-1}{p} = \lim_{p\downarrow 0}X^p(w)\log(X(w)) = \log(X(w)),$$

for all $w \in \Omega$. By taking the derivatives, one can show that the convergence is in fact monotonic. Hence, $\{\frac{X^p-1}{p}1_{\{X\geq 1\}}\}_{q\geq p>0}$ is a family of non-negative RVs monotonically decreasing almost surely to $\log(X)1_{\{X\geq 1\}}$ as $p \downarrow 0$ and $\{\frac{X^p-1}{p}1_{\{X<1\}}\}_{q\geq p>0}$ is a family of non-positive RVs monotonically decreasing almost surely to $\log(X)1_{\{X<1\}}$ as $p \downarrow 0$. Hence, by MCT(formally, the variants used here can be deduced from the standard non-negative, increasing version; the non-negative, decreasing case requires L^1 integrability for the first RV in the sequence and we choose it to be q to achieve this)

$$\begin{aligned}\lim_{p\downarrow 0}\frac{(\mathbb{E}X^p-1)}{p} &= \lim_{p\downarrow 0}\mathbb{E}[\frac{X^p-1}{p}1_{\{X\geq 1\}}] + \mathbb{E}[\frac{X^p-1}{p}1_{\{X<1\}}] \\ &= \mathbb{E}[\log(X)1_{\{X\geq 1\}}] + \mathbb{E}[\log(X)1_{\{X<1\}}] \\ &= \mathbb{E}[\log(X)],\end{aligned}$$

as required. □

(b.2) Observe that $X^p \to 1$ almost surely as $p \downarrow 0$, and $\max(X^q, 1)$ dominates $\{X^p\}_{0<p\leq q}$. Therefore, by DCT, and the fact that $\lim_{x\to 1}\frac{\log x}{x-1} = 1$ by L'hopital, combined with (b), we obtain

$$\lim_{p\downarrow 0}\frac{\log(\mathbb{E}X^p)}{p} = \lim_{p\downarrow 0}\frac{\log(\mathbb{E}X^p)}{\mathbb{E}X^p-1}\frac{\mathbb{E}X^p-1}{p} = \mathbb{E}\log(X)$$

(Remark) Although all limit theorems are stated so far for a countable limit, they apply as well to a continuous limit. Suppose $\{X_t\}_{t\geq 0}$ is a family of L^1 dominated random variables such that $\lim_{t\downarrow 0} X_t(w) \to X_0(w)$ for all $w \in \Omega$. Then, by DCT, for any $\{t_n\} \subset (0, \infty)$ such that $t_n \downarrow 0$, we have $\lim_{n\to\infty} \mathbb{E}[X_{t_n}] = \mathbb{E}[X_0]$. Since this is true for any such sequence, it follows that $\lim_{t\downarrow 0} \mathbb{E}[X_t] = \mathbb{E}[X_0]$ in a proper continuous limit sense. We will freely use the limit theorem in the continuous setting without doing the above pass everytime.

Question 3.

3. Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ be a random variable with $\mathbb{E}|X| < \infty$.

   (i) Show that if $A_n \in \mathcal{F}$ are disjoint sets and $A = \bigcup_n A_n$ then $\sum_n \mathbb{E}[X\mathbf{1}_{A_n}]$ converges absolutely and $\sum_n \mathbb{E}[X\mathbf{1}_{A_n}] = \mathbb{E}[X\mathbf{1}_A]$.

   (ii) Conclude that if $X \geq 0$ then $\mathbb{Q}(A) = \mathbb{E}[X\mathbf{1}_A]/\mathbb{E}X$ is a probability measure.

Solution.

Question 4.

4. Let $Y = \sum_{k=1}^{n} \mathbf{1}_{A_k}$ for some measurable sets $A_1, \ldots, A_n$. Express $\mathrm{Var}(Y)$ in terms of $\mathbb{P}(A_k)$ and $\mathbb{P}(A_k \cap A_\ell)$, then calculate it for the following case: each one of m players selects, independently and uniformly, a number in $\{1, \ldots, n\}$; the event A_k says that the number k was not selected by any player.

Solution.
We compute

$$\begin{aligned}
\mathrm{Var}[Y] &= \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \\
&= \mathbb{E}[(\sum_{k=1}^{l} \mathbb{1}_{A_k})^2] - \mathbb{E}[\sum_{k=1}^{l} \mathbb{1}_{A_k}]^2 \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k}\mathbb{1}_{A_l}] - \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k}]\mathbb{E}[\mathbb{1}_{A_l}] \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{E}[\mathbb{1}_{A_k \cap A_l}] - \mathbb{E}[\mathbb{1}_{A_k}]\mathbb{E}[\mathbb{1}_{A_l}] \\
&= \sum_{k=1}^{n}\sum_{l=1}^{n} \mathbb{P}(A_k \cap A_l) - \mathbb{P}(A_k)\mathbb{P}(A_l).
\end{aligned}$$

Now, observe that, for $k = 1, ..., n$,

$$\mathbb{P}(A_k) = (\frac{n-1}{n})^m$$

and for $k, l = 1, ..., n$,

$$\begin{aligned}
k = l &\implies \mathbb{P}(A_k \cap A_l) = (\frac{n-1}{n})^m \\
k \neq l &\implies \mathbb{P}(A_k \cap A_l) = (\frac{n-2}{n})^m.
\end{aligned}$$

So

$$\begin{aligned}
\mathrm{Var}[Y] &= \sum_{1 \le k, l \le n; k \neq l} (\frac{n-2}{n})^m - (\frac{n-1}{n})^{2m} + \sum_{1 \le k, l \le n; k = l} (\frac{n-1}{n})^m - (\frac{n-1}{n})^{2m} \\
&= (n^2 - n)(\frac{n-2}{n})^m + n(\frac{n-1}{n})^m - n^2(\frac{n-1}{n})^{2m}
\end{aligned}$$

as required. $\square$