Durrett Probability: Problems

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Abstract

This work contains solutions to some exercises from Durrett's probability text.

1 Chapter 6: Markov Chains

Question 6.3.3.

6.3.3. First entrance decomposition. Let $T_y = \inf\{n \geq 1 : X_n = y\}$. Show that

$$p^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m) p^{n-m}(y,y)$$

Solution.
Here we assume countable state space. Observe that

$$p^n(x,y) = P_x(X_n = y) = P_x(\bigcup_{m=1}^{n} \{T_y = m\,;\, X_n = y\}) = \sum_{m=1}^{n} P_x(T_y = m\,;\, X_n = y) \tag{1}$$

$$\begin{aligned} P_x(T_y = m\,;\, X_n = y) &= E_x(1_{\{X_n = y\}}\,;\, T_y = m) \\ &= E_x(E_x(1_{\{X_n=y\}}|\mathscr{F}_m); T_y = m) \qquad (2) \\ &= E_x(E_x(1_{\{X_{n-m}=y\}} \circ \theta_m|\mathscr{F}_m); T_y = m) \\ &= E_{x}(E_{X_m}(1_{\{X_{n-m}=y\}}; T_y = m) = E_x(P_y(X_{n-m} = y); T_y = m) \qquad (3) \\ &= P_x(T_y = m)P_y(X_{n-m} = y) \end{aligned}$$

for any $1 \leq m \leq n$, where (4) holds by definition of conditional expectation and (5) holds by Markov property. Therefore, combining the above result with with (1) gives

$$p^n(x,y) = \sum_{m=1}^{n} P_x(T_y = m)P_y(X_{n-m} = y).$$

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Here is another approach using strong Markov. We compute

$$
\begin{aligned}
p^n(x,y) &= P_x(X_n = y) = P_x\Big(\bigcup_{m=1}^{n} \{T_y = m; X_n = y\}\Big) \\
&= E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y}; T_y \le n) = E_x(E_x(1_{\{X_{n-T_y}=y\}} \circ \theta_{T_y} | \mathscr{F}_{T_y}); T_y \le n) \qquad (4) \\
&= E_x(E_{X_{T_y}}(1_{\{X_{n-T_y}=y\}}; T_y \le n) = E_x(E_y(1_{\{X_{n-T_y}\}}); T_y \le n) \qquad (5) \\
&= \sum_{m=1}^{n} P_x(T_y = m) E_y(1_{\{X_{n-m}=y\}}) = \sum_{m=1}^{n} P_x(T_y = m) P^{n-m}(y,y)
\end{aligned}
$$

where (4) holds by definition of conditional expectation and (5) holds by the strong Markov property. $\square$

Question 6.3.4.

6.3.4. Show that $\sum_{m=0}^{n} P_x(X_m = x) \geq \sum_{m=k}^{n+k} P_x(X_m = x)$.

Solution.
Let $k \in \mathbb{N}$, and $T_x^k = \inf\{n \geq k : X_n = x\}$. We claim that

$$P_x(X_m = x) = \sum_{l=k}^{m} P_x(T_x^k = x) p^{m-l}(x,x) \tag{6}$$

for any $m \geq k$. Fix $m \geq k$. Then,

$$P_x(X_m = x) = P_x(\bigcup_{l=k}^{m} \{T_x^k = l; X_m = x\}) = \sum_{l=k}^{m} P_x(T_x^k = l; X_m = x). \tag{7}$$

Now, we compute

$$\begin{aligned} P_x(T_x^k = l; X_m = x) &= E_x(1_{\{X_m=x\}}; T_x^k = l) = E_x(E_x(1_{\{X_m=x\}} | \mathscr{F}_l); T_x^k = l) \\ &= E_x(E_x(1_{\{X_{m-l}=x\}} \theta_l | \mathscr{F}_l); T_x^k = l) \\ &= E_x(E_{X_l}(1_{\{X_{m-l}=x\}}; T_x^k = l); T_x^k = l) \\ &= E_x(P_x(X_{m-l} x); T_x^k = l) = P_x(X_{m-l} = x) P_x(T_x^k = l) \\ &= P_x(T_x^k = l) p^{m-l}(x,x) \end{aligned} \tag{8}$$

for any $k \leq l \leq m$, where (8) holds by Markov property. Therefore, combining the above result with (7), we have proven (6). Then,

$$\begin{aligned} \sum_{m=k}^{n+k} P_x(X_m = x) &= \sum_{m=k}^{n+k} \sum_{l=k}^{m} P_x(T_x^k = l) p^{m-l}(x,x) \\ &= \sum_{l=k}^{n+k} \sum_{m=l}^{n+k} P_x(T_x^k = l) p^{m-l}(x,x) \\ &= \sum_{m=0}^{n} p^m(x,x) \left(\sum_{l=k}^{d} P_x(T_x^k = l) \right) \\ &\leq \sum_{m=0}^{n} p^m(x,x) = \sum_{m=0}^{n} P_x(X_m = x) \end{aligned}$$

Question 6.3.5.

6.3.5. Suppose that $S - C$ is finite and for each $x \in S - C$ $P_x(\tau_C < \infty) > 0$. Then there is an $N < \infty$ and $\epsilon > 0$ so that $P_y(\tau_C > kN) \leq (1-\epsilon)^k$.

Solution.
We assume countable state space. Observe that, for any $x \in S \setminus C$, we can choose $n(x) \in \mathbb{N}$ such that

$$P_x(\tau_C \leq n) > 0.$$

Otherwise, for some $x \in S \setminus C$, by continuity of probability,

$$P_x(\tau_C < \infty) = \lim_{k\to\infty} P_x(\tau_C \leq k) = 0,$$

which is a contradiction. Now, let

$$N = \max_{z \in S\setminus C} n(x). \ \text{ and } \ \epsilon = \min_{z \in S\setminus C} P_z(\tau_C \leq N).$$

Trivially,

$$P_y(\tau_C > kN) = 0$$

for any $k \in \mathbb{N}$, and $y \in C$, since $y \in C$ implies $\tau_C = 0$ by definition. Therefore, it suffices to show

$$P_y(\tau_C > kN) \leq (1-\epsilon)^k \tag{9}$$

for all $k \in \mathbb{N}$ and $y \in S \setminus C$. Fix $y \in S \setminus C$. Then,

$$P_y(\tau_C \leq N) \quad \geq \quad \epsilon.$$

and hence

$$P_y(\tau_C > N) \leq (1-\epsilon)$$

Now, we proceed by induction to prove (9). Suppose, for some $k \in \mathbb{N}$ such that $k \geq 2$,

$$P_y(\tau_C > kN) \leq (1-\epsilon)^k.$$

We compute

$$\begin{aligned}
P_y(T_c > (k+1)N) &= E_y(1_{\{\tau_C > kN\}} \circ \theta_N; \tau_C > N) \\
&= E_y(E_y((1_{\{\tau_C > kN\}} \circ \theta_N | \mathscr{F}_N); \tau_C > N)) \\
&= E_y(E_{X_N}((1_{\{\tau_C > kN\}}); \tau_C > N)) \qquad (10) \\
&\leq E_y(\sup_{z\in S} P_z(\tau_C > kN); \tau_C > N)) \\
&\leq (1-\epsilon)^k E_y(1; \tau_C > N)) = (1-\epsilon)^{k+1}
\end{aligned}$$

where (10) holds by Markov Property, which completes the proof. □

Question 6.3.6.

6.3.6. Let $h(x) = P_x(\tau_A < \tau_B)$. Suppose $A \cap B = \emptyset$, $S - (A \cup B)$ is finite, and $P_x(\tau_{A\cup B} < \infty) > 0$ for all $x \in S - (A \cup B)$. (i) Show that

$$(*) \qquad h(x) = \sum_y p(x,y)h(y) \quad \text{for } x \notin A \cup B$$

(ii) Show that if h satisfies $(*)$ then $h(X(n \wedge \tau_{A\cup B}))$ is a martingale. (iii) Use this and Exercise 6.3.5 to conclude that $h(x) = P_x(\tau_A < \tau_B)$ is the only solution of $(*)$ that is 1 on A and 0 on B.

Solution.
(i) Let $x \in S \setminus (A \cup B)$. Then,

$$1_{\{\tau_A < \tau_B\}} = 1_{\{\tau_A < \tau_B\}} \circ \theta_1.$$

It follows that

$$\begin{aligned} h(x) &= P_x(\tau_A < \tau_B) = E_x(1_{\{\tau_A < \tau_B\}}) = E_x(1_{\{\tau_A < \tau_B\}} \circ \theta_1) \\ &= E_x(E_x(1_{\{\tau_A < \tau_B\}} \circ \theta_1 | \mathscr{F}_1)) = E_x(E_{X_1}(1_{\{\tau_A < \tau_B\}})) \qquad (11) \\ &= \sum_y P(X_1 = y) P_y(\tau_A < \tau_B) = \sum_y p(x,y) P_y(\tau_A < \tau_B) \end{aligned}$$

where (11) holds by Markov property.

(ii)

(iii)

Question 6.3.7.

6.3.7. Let X_n be a Markov chain with $S = \{0, 1, \ldots, N\}$ and suppose that X_n is a martingale and $P_x(\tau_0 \wedge \tau_N < \infty) > 0$ for all x. (i) Show that 0 and N are absorbing states, i.e., $p(0,0) = p(N,N) = 1$. (ii) Show $P_x(\tau_N < \tau_0) = x/N$.

Solution.

Question 6.4.1.

Exercise 6.4.1. Suppose y is recurrent and for $k \geq 0$, let $R_k = T_y^k$ be the time of the kth return to y, and for $k \geq 1$ let $r_k = R_k - R_{k-1}$ be the kth interarrival time. Use the strong Markov property to conclude that under P_y, the vectors $v_k = (r_k, X_{R_{k-1}}, \dots, X_{R_k - 1})$, $k \geq 1$ are i.i.d.

Solution.

We wish to show that for all $k, l \in \mathbb{N}$

Question 6.4.2.

Exercise 6.4.2. Let $a \in S$, $f_n = P_a(T_a = n)$, and $u_n = P_a(X_n = a)$. (i) Show that $u_n = \sum_{1 \le m \le n} f_m u_{n-m}$. (ii) Let $u(s) = \sum_{n \ge 0} u_n s^n$, $f(s) = \sum_{n \ge 1} f_n s^n$, and show $u(s) = 1/(1 - f(s))$. Setting $s = 1$ gives (6.4.1) for $x = y = a$.

Solution.

Question 6.4.3.

Exercise 6.4.3. Consider asymmetric simple random walk on $\mathbf{Z}$, i.e., we have $p(i, i+1) = p$, $p(i, i-1) = q = 1 - p$. In this case,

$$p^{2m}(0,0) = \binom{2m}{m} p^m q^m \quad \text{and} \quad p^{2m+1}(0,0) = 0$$

(i) Use the Taylor series expansion for $h(x) = (1-x)^{-1/2}$ to show $u(s) = (1 - 4pqs^2)^{-1/2}$ and use the last exercise to conclude $f(s) = 1 - (1 - 4pqs^2)^{1/2}$. (ii) Set $s = 1$ to get the probability the random walk will return to 0 and check that this is the same as the answer given in part (c) of Theorem 5.7.7.

Solution.

Question 6.4.4.

Exercise 6.4.4. Use the strong Markov property to show that $\rho_{xz} \geq \rho_{xy}\rho_{yz}$.

Solution.
The key insight in this problem is that if you shift the chain by a stopping time of one state variable, then the probability of the chain coming back to another fixed state variable decreases. This relation allows one to estimate p_{xz} from below using strong Markov, which in our context is proven for a sequence space(discrete time) of a polish state space, using Monotone class theorem. Recall that to define a shift operator, indexed by ∞, by convention, we set

$$\theta_\infty(w) = \triangle$$

where $\triangle$ is the cemetery sample point we add to $S^{\mathbb{N}}$, for all $w \in S^{\mathbb{N}}$. Therefore, to extend the domain of $T_z = \inf\{n \geq 1 : X_n = z\}$ for any $z \in S$, to include $\triangle$, if necessary, we define

$$T_z(\triangle) = \infty \;\text{ so }\; 1_{\{T_z<\infty\}}(\triangle) = 0,$$

With this convention.

$$\begin{aligned}
\{w \in S^{\mathbb{N}} : 1_{\{T_z<\infty\}} \circ \theta_{T_y}(w) = 1\} &= \{w \in S^{\mathbb{N}} : T_y(w) = n \text{ for some } n \geq 1 \\
&\quad \text{and } \; T_z^n(w) = \inf\{k \geq n : X_k = z\} < \infty\} \\
&= \bigcup_{n=1}^{\infty} \{T_y = n \;;\; T_z^n < \infty\} \\
&\subset \bigcup_{n=1}^{\infty} \{T_z^n < \infty\} = \{T_z < \infty\}
\end{aligned}$$

for any $z, y \in S$.

Now, let $x, y, z \in S$. Then,

$$\begin{aligned}
p_{xz} &= P_x(T_z < \infty) = E_x(1_{\{T_z<\infty\}}) \geq E_x(1_{\{T_z<\infty\}} \circ \theta_{T_y}) \\
&= E_x(E_x(1_{\{T_z<\infty\}} \circ \theta_{T_y} | \mathscr{F}_{T_y}); T_y < \infty) \qquad (12) \\
&= E_x(E_{X_{T_y}}(1_{\{T_z<\infty\}}; T_y < \infty) = E_x(E_{X_y}(1_{\{T_z<\infty\}}; T_y < \infty) \qquad (13) \\
&= E_x(P_y(T_z < \infty); T_y < \infty) = P_y(T_z < \infty)P_x(T_y < \infty) = p_{xy}p_{yz}
\end{aligned}$$

where (12) holds by definition of conditional expectation, and (13) holds by strong Markov. $\square$

2 Chapter 2: Law of Large Numbers

3 Chapter 4: Random Walks

Question 4.1.1.

Exercise 4.1.1. Symmetric random walk. Let $X_1, X_2, \ldots \in \mathbf{R}$ be i.i.d. with a distribution that is symmetric about 0 and nondegenerate (i.e., $P(X_i = 0) < 1$). Show that we are in case (iv) of Theorem 4.1.2.

Question 4.1.2.

Exercise 4.1.2. Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$ and $EX_i^2 = \sigma^2 \in (0, \infty)$. Use the central limit theorem to conclude that we are in case (iv) of Theorem 4.1.2. Later in Exercise 4.1.11 you will show that $EX_i = 0$ and $P(X_i = 0) < 1$ is sufficient.

Question 4.1.3.

Exercise 4.1.3. If S and T are stopping times then $S \wedge T$ and $S \vee T$ are stopping times. Since constant times are stopping times, it follows that $S \wedge n$ and $S \vee n$ are stopping times.

Question 4.1.4.

Exercise 4.1.4. Suppose S and T are stopping times. Is $S + T$ a stopping time? Give a proof or a counterexample.

Question 4.1.5.

Exercise 4.1.5. Show that if $Y_n \in \mathcal{F}_n$ and N is a stopping time, $Y_N \in \mathcal{F}_N$. As a corollary of this result we see that if $f : S \to \mathbf{R}$ is measurable, $T_n = \sum_{m \le n} f(X_m)$, and $M_n = \max_{m \le n} T_m$ then T_N and $M_N \in \mathcal{F}_N$. An important special case is $S = \mathbf{R}$, $f(x) = x$.

4 Chapter 5: Martingales

Question 5.2.1.

Exercise 5.2.1. Suppose X_n is a martingale w.r.t. $\mathcal{G}_n$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then $\mathcal{G}_n \supset \mathcal{F}_n$ and X_n is a martingale w.r.t. $\mathcal{F}_n$.

Solution.
Various properties of conditional expectations are used.

We compute

$$
\begin{aligned}
E[X_{n+1}|\mathscr{F}_n] &= E[X_{n+1}|\mathscr{G}_n|\mathscr{F}_n] && (14) \\
&= E[X_n|\mathscr{F}_n] && (15) \\
&= X_n && (16)
\end{aligned}
$$

for all $n \in \mathbb{N}$, where (14) holds by the Tower property, (15) holds by Martingale property of $\{G_n\}$ and (16) holds by measurability of X_n w.r.t $\mathscr{F}_n$ for all $n \in \mathbb{N}$. $\square$

Question 5.2.2.

Exercise 5.2.2. Suppose f is superharmonic on $\mathbf{R}^d$. Let $\xi_1, \xi_2, \ldots$ be i.i.d. uniform on $B(0,1)$, and define S_n by $S_n = S_{n-1} + \xi_n$ for $n \geq 1$ and $S_0 = x$. Show that $X_n = f(S_n)$ is a supermartingale.

Solution.

Question 5.2.3.

> **Exercise 5.2.3.** Give an example of a submartingale X_n so that X_n^2 is a supermartingale. Hint: X_n does not have to be random.

Solution.
Consider $\{X_n = 0\}$. Then, $\{X_n^2 = 0\}$, so both are processes are martingales, we have the desired example. $\square$

Question 5.2.4.

Exercise 5.2.4. Give an example of a martingale X_n with $X_n \to -\infty$ a.s. Hint: Let $X_n = \xi_1 + \cdots + \xi_n$, where the ξ_i are independent (but not identically distributed) with $E\xi_i = 0$.

Solution.
Set $\xi_n = -1$ with probability 2^{-1} and $\xi_n = 2^n$ with probability $2^{-(n+1)}$ for each $n \in \mathbb{N}$, such that they are independent. Then, by construction,

$$E[X_{n+1}|\mathscr{F}_n] \quad = \quad E[\xi_{n+1}] + E[X_n|\mathscr{F}_n] = X_n$$

for all $n \in \mathbb{N}$, so $\{X_n\}$ is a martingale. Now, as

$$\sum_{n=1}^{\infty} P(\xi_n \leq -1) = \infty$$

by Borel-Cantelli II,

$$P(\xi_n \leq -1 \ \text{ i.o}) = 1 \ \text{ and } \ P(X_n \to -\infty) = 1,$$

as required. $\square$

Question 5.2.5.

Exercise 5.2.5. Let $X_n = \sum_{m \le n} 1_{B_m}$ and suppose $B_n \in \mathcal{F}_n$. What is the Doob decomposition for X_n?

Solution.

Question 5.2.6.

5.2.6. Let $\xi_1, \xi_2, \ldots$ be independent with $E\xi_i = 0$ and $\text{var}(\xi_m) = \sigma_m^2 < \infty$, and let $s_n^2 = \sum_{m=1}^n \sigma_m^2$. Then $S_n^2 - s_n^2$ is a martingale.

Solution.

We compute

$$
\begin{aligned}
E(S_{n+1}^2 - s_{n+1}^2 | \mathscr{F}_n) &= E(S_n^2|\mathscr{F}_n) + E(\xi_{n+1}^2|\mathscr{F}_n) + 2E(\xi_{n+1}S_n|\mathscr{F}_n) - s_{n+1}^2 && (17) \\
&= S_n^2 + E(\xi_{n+1}^2) + && (18)
\end{aligned}
$$

Question 5.2.7.

5.2.7. If $\xi_1, \xi_2, \ldots$ are independent and have $E\xi_i = 0$ then

$$X_n^{(k)} = \sum_{1 \le i_1 < \ldots < i_k \le n} \xi_{i_1} \cdots \xi_{i_k}$$

is a martingale. When $k = 2$ and $S_n = \xi_1 + \cdots + \xi_n$, $2X_n^{(2)} = S_n^2 - \sum_{m \le n} \xi_m^2$.

Solution.
Observe that

$$1 + y \le e^y$$

and hence

$$log(1 + y) \le y$$

for all $y \in \mathbb{R}$. Now, fix $|y| \le 2^{-1}$. Then,

$$\begin{aligned} \log(1 + y) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y^n}{n}. \\ &\ge y - |\sum_{n=2}^{\infty} (-1)^{n-1} \frac{y^n}{n}| \\ &\le y - \frac{y^2}{2} (\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}) = y + y^2. \end{aligned}$$

Therefore,

$$|y| \le 2^{-1} \implies y - y^2 \le (1 + y)$$

Question 5.2.8.

5.2.8. Generalize (i) of Theorem 5.2.4 by showing that if X_n and Y_n are submartingales w.r.t. $\mathcal{F}_n$ then $X_n \vee Y_n$ is also.

Solution.

Question 5.2.9.

5.2.9. Let $Y_1, Y_2, \ldots$ be nonnegative i.i.d. random variables with $EY_m = 1$ and $P(Y_m = 1) < 1$. (i) Show that $X_n = \prod_{m \le n} Y_m$ defines a martingale. (ii) Use Theorem 5.2.9 and an argument by contradiction to show $X_n \to 0$ a.s. (iii) Use the strong law of large numbers to conclude $(1/n) \log X_n \to c < 0$.

Solution.
(i) As $\{Y_n\}$ are non-negative and independent,

$$E(|X_n|) = E(|\prod_{m \le n} Y_n|) = E(\prod_{m \le n} Y_m) = \prod_{m \le n} E(Y_m) = 1 \tag{19}$$

for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} E(X_{n+1}|\mathscr{F}_n) &= E(\prod_{m \le n+1} Y_n|\mathscr{F}_n) = X_n E(\prod_{m \le n} Y_{n+1}|\mathscr{F}_n) && (20) \\ &= X_n E(Y_{n+1}) = X_n && (21) \end{aligned}$$

for all $n \in \mathbb{N}$, where (20) holds by theorem 5.1.7, and (19), and (21) holds by independence. Therefore, $\{X_n\}$ is a martingale. We remark that since $\{X_n\}$ is a non-negative martingale, it converges almost surely to some $X_\infty \in L^1$ by Martingale convergence theorem.

(ii) Fix $n \in \mathbb{N}$. Suppose there does not exists $\epsilon > 0$ such that

$$P(|Y_n - 1| > \epsilon) > 0.$$

Then, by continuity of probability,

$$P(Y_n = 1) = P(|Y_n - 1| = 0) = P(\bigcap_{k=1}^{\infty} |Y_n - 1| \le k^{-1}) = \lim_{k \to \infty} P(|Y_n - 1| \le k^{-1}) = 1,$$

which contradicts that $P(Y_n = 1) < 1$. Hence, as Y_n is identically distributed, we can choose $\epsilon > 0$, such that

$$P(|Y_n - 1| > \epsilon) > 0$$

for all $n \in \mathbb{N}$. Now,

$$\begin{aligned} P(|X_{n+1} - X_n| \ge \epsilon\delta) &= P(X_n|Y_{n+1} - 1| \ge \epsilon\delta) \\ &\ge P(X_n \ge \delta; |Y_{n+1} - 1| > \epsilon) P(X_n \ge \delta) P(|Y_{n+1} - 1| > \epsilon) && (22) \end{aligned}$$

for any $\delta > 0$, where (22) holds by independence. As X_n converges almost surely,

$$\lim_{n \to \infty} P(|X_{n+1} - X_n| \ge \epsilon\delta) = 0$$

and hence, by (22),

$$\lim_{n\to\infty} P(X_n \geq \delta) = 0$$

for all $\delta > 0$. Therefore, $X_n \to_p 0$. Since, we have $X_n \to X_\infty$ almost surely, which implies $X_n \to_p X_\infty$, we have $X_\infty = 0$ almost surely, and $X_n \to 0$ almost surely.

(iii)

Question 5.2.10.

5.2.10. Suppose $y_n > -1$ for all n and $\sum |y_n| < \infty$. Show that $\prod_{m=1}^{\infty}(1+y_m)$ exists.

Solution.
The key idea in this problem is that one can make a use of Taylor estimates of exponential and log, to prove convergence of a product($x \mapsto x^2$ is is below $x \mapsto x$ for any $|x| \leq 1$!).

Observe that

$$1 + y \leq e^y$$

and hence

$$\log(1+y) \leq y$$

for all $y > -1$. Now, fix $|y| \leq 2^{-1}$. Then,

$$\begin{aligned} \log(1+y) &= y - \frac{y^2}{2} + \frac{y^3}{3} + \cdots \\ &\geq y - \left| -\frac{y^2}{2} + \frac{y^3}{3} + \cdots \right| \\ &\geq y - \frac{y^2}{2}\left|1 + \frac{1}{2} + \cdots\right| = y - y^2 \end{aligned}$$

Therefore,

$$|y| \leq 2^{-1} \quad \Longrightarrow \quad y - y^2 \leq \log(1+y) \leq y.$$

Now, as $\sum_{n=1}^{\infty} |y_n| < \infty$, we can choose M large enough such that

$$|y_n| \quad \leq \quad 2^{-1} \tag{23}$$

for all $n \geq M$. Since $\sum_{n=1}^{\infty} |y_n| < \infty$ and (23), $\sum_{n=M}^{\infty} y_n$ and $\sum_{n=M}^{\infty} y_n^2$ converge, and hence $\sum_{n=M}^{\infty} y_n - y_n^2$ converges. By comparison,

$$\sum_{n=k}^{\infty} y_n - y_n^2 \leq \sum_{n=k}^{\infty} \log(1+y_n) \leq \sum_{n=k}^{\infty} y_n$$

for all $k \geq M$. Letting $k \to \infty$,

$$\sum_{n=k}^{\infty} \log(1+y_n) \to 0$$

and hence

$$\sum_{n=1}^{m} \log(1+y_n) = \log(\prod_{n=1}^{m}(1+y_n)) \quad \text{converges.}$$

By continuity of log, $\prod_{n=1}^{\infty}(1+y_n)$ exists. $\square$

Question 5.2.11.

5.2.11. Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose

$$E(X_{n+1}|\mathcal{F}_n) \leq (1+Y_n)X_n$$

with $\sum Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit by finding a closely related supermartingale to which Theorem 5.2.9 can be applied.

Solution.
Let Z_n be defined by

$$Z_n = \frac{X_n}{\prod_{m=1}^{n-1}(1+Y_m)}$$

for each $n \geq 2$. It is clear that Z_n is a non-negative process. We claim that it is a super-martingale with respect to $\{\mathscr{F}_n\}_{n\geq 2}$. Additionally, Observe that $Z_n \leq X_n$, so $Z_n \in L^1$ for all $n \geq 2$, and $\left(\prod_{m=1}^{n-1}(1+Y_m)\right)^{-1} \leq 1$, so $\left(\prod_{m=1}^{n-1}(1+Y_m)\right)^{-1} \in L^1$ for all $n \geq 2$. We now compute

$$\begin{aligned}
E(Z_{n+1}|\mathscr{F}_n) &= E\Big(\frac{X_{n+1}}{\prod_{m=1}^{n}(1+Y_m)}|\mathscr{F}_n\Big) \\
&= \left(\prod_{m=1}^{n}(1+Y_m)\right)^{-1} E(X_{n+1}|\mathscr{F}_n) && (24) \\
&\leq \left(\prod_{m=1}^{n}(1+Y_m)\right)^{-1} (1+Y_n)X_n && (25) \\
&= \frac{X_n}{\left(\prod_{m=1}^{n-1}(1+Y_m)\right)^{-1}} = Z_n \quad \text{almost surely}
\end{aligned}$$

for each $n \geq 2$, where (24) holds by the integrability conditions computed above, and adaptedness of $\{Y_n\}$ w.r.t to $\mathscr{F}_n$, and (25) holds by the given estimate. Hence, $\{Z_n\}_{n\geq 2}$ is a non-negative super-martingale, so it converges to a finite value almost surely. Now, as $\sum_n Y_n < \infty$ and $Y_n \geq 0$ for all $n \geq 1$, by 5.2.10, we have

$$\prod_{m=1}^{n}(1+Y_m) \quad \text{converges almost surely.}$$

Therefore, we see

$$X_n = Z_n \cdot \left(\prod_{m=1}^{n-1}(1+y_m)\right)^{-1} \quad \text{converges almost surely as} \quad n \to \infty,$$

as required. □

Question 5.2.12.

5.2.12. Use the random walks in Exercise 5.2.2 to conclude that in $d \leq 2$, nonnegative superharmonic functions must be constant. The example $f(x) = |x|^{2-d}$ shows this is false in $d > 2$.

Solution.

Question 5.2.13.

5.2.13. The switching principle. Suppose X_n^1 and X_n^2 are supermartingales with respect to $\mathcal{F}_n$, and N is a stopping time so that $X_N^1 \geq X_N^2$. Then

$$Y_n = X_n^1 1_{(N>n)} + X_n^2 1_{(N \leq n)} \text{ is a supermartingale.}$$
$$Z_n = X_n^1 1_{(N \geq n)} + X_n^2 1_{(N<n)} \text{ is a supermartingale.}$$

Solution.

Question 5.3.1.

Exercise 5.3.1. Let X_n, $n \geq 0$, be a submartingale with $\sup X_n < \infty$. Let $\xi_n = X_n - X_{n-1}$ and suppose $E(\sup \xi_n^+) < \infty$. Show that X_n converges a.s.

Solution.

Question 5.3.2.

Exercise 5.3.2. Give an example of a martingale X_n with $\sup_n |X_n| < \infty$ and $P(X_n = a \text{ i.o.}) = 1$ for $a = -1, 0, 1$. This example shows that it is not enough to have $\sup |X_{n+1} - X_n| < \infty$ in Theorem 5.3.1.

Solution.

Question 5.3.3.

Exercise 5.3.3. (Assumes familiarity with finite state Markov chains.) Fine tune the example for the previous problem so that $P(X_n = 0) \to 1 - 2p$ and $P(X_n = -1)$, $P(X_n = 1) \to p$, where p is your favorite number in $(0, 1)$, i.e., you are asked to do this for one value of p that you may choose. This example shows that a martingale can converge in distribution without converging a.s. (or in probability).

Solution.

Question 5.3.4.

Exercise 5.3.4. Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose $E(X_{n+1}|\mathcal{F}_n) \leq X_n + Y_n$, with $\sum Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit. Hint: Let $N = \inf_k \sum_{m=1}^{k} Y_m > M$, and stop your supermartingale at time N.

Solution.