Basic Probability:
Problem Set II

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1. Countability and σ-algebra.

Solution. Suppose Ω is a infinite set. Let $\mathscr{A}$ be a sub-collection of $\mathbb{P}(\Omega)$, defined by

$$\mathscr{A} \quad = \quad \{X \subseteq \Omega \mid X \text{ is finite or } X^c \text{ is finite } \}.$$

As Ω is infinite, there exists an injective map $\phi : \mathbb{N} \to \Omega$. Consider the two images of ϕ: $\phi(2\mathbb{N})$ and $\phi(2\mathbb{N}+1)$, where $2\mathbb{N}$ denotes the set of evens and $2\mathbb{N}+1$ denotes the set of odds. Observe that

Question 2. Limit of Probabilities of Disjoint Events.

Solution. Let $(A_n)_{n\geq 0}$ be a set of disjoint events and $\mathbb{P}$ be a probability. Suppose for sake of contradiction that $\lim\limits_{n\to\infty} \mathbb{P}(A_n)$ does not converge to 0. Then, there exists $\epsilon > 0$ such that for all $n \geq 0$, such that $\mathbb{P}(A_n) \geq \epsilon$. Then, as the events are disjoint, by the countable additivity of probability, we have

$$\mathbb{P}(\cup_{n=0}^{\infty} A_n) \quad = \quad \sum_{n=0}^{\infty} \mathbb{P}(A_n) \geq \sum_{n=0}^{\infty} \epsilon = \infty.$$

Hence, we obtain that $\mathbb{P}(\cup_{n=0}^{\infty} A_n) \geq \infty$. This is a contradiction, as a probability measure assigns any event in the σ-algebra to some real number in $[0,1]$. Hence, $\lim\limits_{n\to\infty} \mathbb{P}(A_n) = 0$. $\square$

Question 3..

Solution.

Question 4..

Solution. A pair of dice is rolled until a sum of either 5 or 7 appears. We wish to compute the probability that a 5 occurs first. Let E_n denote the event, where a 5 occurs on the nth roll, and no 5 or 7 occurs on the first $(n-1)$ roll. The probability that we wish to compute can be written as

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n),$$

as each E_n are pairwise disjoint events. Let F_n denote the event, where no 5 or 7 occurs on the nth roll, and let G_n denote the event, where 5 occurs on the nth roll.

$$\mathbb{P}(E_n) \quad = \quad \mathbb{P}((\cup_{k=1}^{n-1} F_k) \cup G_n).$$

Now, with the independence assumption on each throw, we can factorize the RHS, and obtain

$$\mathbb{P}(E_n) = \mathbb{P}(G_n) \prod_{k=1}^{n-1} \mathbb{P}(F_k), \quad (1)$$

where $n - 1 = 0$ case for the product term is defined to be 1. Assuming that the two dices are both fair dices, through a simple combinatorial argument(just count!), we can obtain that

$$\begin{aligned} \mathbb{P}(G_n) &= \frac{4}{36} = \frac{1}{9}, \\ \mathbb{P}(F_n) &= 1 - \frac{4}{36} - \frac{6}{36} = \frac{26}{36} = \frac{13}{18}. \end{aligned}$$

Substituting the above equations into (1), we have

$$\begin{aligned} \mathbb{P}(E_n) &= \frac{1}{9} \prod_{k=1}^{n-1} \frac{13}{18} \\ &= \frac{1}{9} (\frac{13}{18})^{n-1} \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}(E_n) &= \sum_{n=1}^{\infty} \frac{1}{9} (\frac{13}{18})^{n-1} \\ &= \frac{1}{9} (\frac{1}{1 - \frac{13}{18}}) \\ &= \frac{2}{5}. \end{aligned}$$

Therefore, the probability that a 5 occurs first is $\frac{2}{5}$. $\square$

Question 5. $\limsup$ **and** $\liminf$ **.**

Solution. Let $\mathbb{P}$ be a probability measure on Ω endowed with a σ-algebra $\mathscr{A}$. We then define $\limsup$ and $\liminf$ of a sequence of events $\{A_n\}$, chosen from $\mathscr{A}$ as follow:

$$\begin{aligned} \limsup_{n \to \infty} \{A_n\} &= \cap_{n=1}^{\infty} \cup_{m \geq n} A_n, \\ \liminf_{n \to \infty} \{A_n\} &= \cup_{n=1}^{\infty} \cap_{m \geq n} A_n. \end{aligned}$$

(i) The meaning of the above events can be written as

$$\begin{aligned} \limsup_{n \to \infty} \{A_n\} &= \cap_{n=1}^{\infty} \cup_{m \geq n} A_n \\ &= \{x \in \Omega \mid \forall n, \exists m \geq n \text{ such that } x \in A_m\} \\ &= \{x \in \Omega \mid \text{the outcome } x \text{ appears infinitely often in the event sequence}\}, \\ \liminf_{n \to \infty} \{A_n\} &= \cup_{n=1}^{\infty} \cap_{m \geq n} A_n \\ &= \{x \in \Omega \mid \exists n, \forall m \geq n, x \in A_m\}. \\ &= \{x \in \Omega \mid \text{the outcome } x \text{ does not appear only for a} \\ &\quad \text{finite number of events in the sequence}\}. \end{aligned}$$

(ii)

(iii)

Question 6..

Solution.