Basic Probability: Problem Set II

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1. Countability and σ-algebra.

Solution. Suppose Ω is a infinite set. Let $\mathscr{A}$ be a sub-collection of $\mathbb{P}(\Omega)$, defined by

$$\mathscr{A} \quad = \quad \{X \subseteq \Omega \mid X \text{ is finite or } X^c \text{ is finite } \}.$$

As Ω is infinite, there exists an injective map $\phi : \mathbb{N} \to \Omega$. Consider the two images of ϕ: $\phi(2\mathbb{N})$ and $\phi(2\mathbb{N}+1)$, where $2\mathbb{N}$ denotes the set of evens and $2\mathbb{N}+1$ denotes the set of odds. Observe that $\phi(2\mathbb{N})$ is infinite and $\phi(2\mathbb{N})^c$ is infinite, as ϕ is injective. Hence, $\phi(2\mathbb{N}) \notin \mathscr{A}$. The image, however, can be expressed in the following way:

$$\phi(2\mathbb{N}) \quad = \quad \bigcup_{k \in 2\mathbb{N}} \{\phi(k)\},$$

which gives that $\phi(2\mathbb{N})$ is a countable union of sets in $\mathscr{A}$, as each point set $\{\phi(k)\}$ is finite and in $\mathscr{A}$. Therefore, by the definition of σ-algebra, we obtain $\phi(2\mathbb{N}) \in \mathscr{A}$. This is a contradiction. Hence, $\mathscr{A}$, defined by the given characterization, is not a σ-algebra, when Ω is infinite. $\square$

Question 2. Limit of Probabilities of Disjoint Events.

Solution. Let $(A_n)_{n \geq 0}$ be a set of disjoint events and $\mathbb{P}$ be a probability. Suppose for sake of contradiction that $\lim_{n \to \infty} \mathbb{P}(A_n)$ does not converge to 0. Then, there exists $\epsilon > 0$ such that for all $n \geq 0$, such that $\mathbb{P}(A_n) \geq \epsilon$. Then, as the events are disjoint, by the countable additivity of probability, we have

$$\mathbb{P}(\cup_{n=0}^{\infty} A_n) \quad = \quad \sum_{n=0}^{\infty} \mathbb{P}(A_n) \geq \sum_{n=0}^{\infty} \epsilon = \infty.$$

Hence, we obtain that $\mathbb{P}(\cup_{n=0}^{\infty} A_n) \geq \infty$. This is a contradiction, as a probability measure assigns any event in the σ-algebra to some real number in $[0, 1]$. Hence, $\lim_{n \to \infty} \mathbb{P}(A_n) = 0$. $\square$

Question 3.Bonferroni Inequalities.

Solution. Let $\{A_i\}$ be a sequence of events. We wish to prove the following probalistic inequality:

$$\mathbb{P}(\bigcup_{i=1}^{n} A_i) \quad \geq \quad \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j).$$

We proceed by induction. Let A_1 and A_2 be two events. Then, by the finite additivity of probability measure, we have

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).$$

Hence, the inequality holds for $n = 2$ case. Now, assume that the inequality holds for some k.

Question 4..

Solution. A pair of dice is rolled until a sum of either 5 or 7 appears. We wish to compute the probability that a 5 occurs first. Let E_n denote the event, where a 5 occurs on the nth roll, and no 5 or 7 occurs on the first $(n-1)$ roll. The probability that we wish to compute can be written as

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n),$$

as each E_n are pairwise disjoint events. Let F_n denote the event, where no 5 or 7 occurs on the nth roll, and let G_n denote the event, where 5 occurs on the nth roll.

$$\mathbb{P}(E_n) \quad = \quad \mathbb{P}((\cup_{k=1}^{n-1} F_k) \cup G_n).$$

Now, with the independence assumption on each throw, we can factorize the RHS, and obtain

$$\mathbb{P}(E_n) \quad = \quad \mathbb{P}(G_n) \prod_{k=1}^{n-1} \mathbb{P}(F_k), \tag{1}$$

where $n - 1 = 0$ case for the product term is defined to be 1. Assuming that the two dices are both fair dices, through a simple combinatorial argument(just count!), we can obtain that

$$\begin{aligned} \mathbb{P}(G_n) &= \frac{4}{36} = \frac{1}{9}, \\ \mathbb{P}(F_n) &= 1 - \frac{4}{36} - \frac{6}{36} = \frac{26}{36} = \frac{13}{18}. \end{aligned}$$

Substituting the above equations into (1), we have

$$\begin{aligned} \mathbb{P}(E_n) &= \frac{1}{9} \prod_{k=1}^{n-1} \frac{13}{18} \\ &= \frac{1}{9} (\frac{13}{18})^{n-1} \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}(E_n) &= \sum_{n=1}^{\infty} \frac{1}{9} (\frac{13}{18})^{n-1} \\ &= \frac{1}{9} (\frac{1}{1 - \frac{13}{18}}) \\ &= \frac{2}{5}. \end{aligned}$$

Therefore, the probability that a 5 occurs first is $\frac{2}{5}$. $\square$

Question 5. $\limsup$ **and** $\liminf$ **.**

Solution. Let $\mathbb{P}$ be a probability measure on Ω endowed with a σ-algebra $\mathscr{A}$. We then define $\limsup$ and $\liminf$ of a sequence of events $\{A_n\}$, chosen from $\mathscr{A}$ as follow:

$$\begin{aligned}\limsup_{n\to\infty}\{A_n\} &= \cap_{n=1}^{\infty} \cup_{m\geq n} A_n,\\ \liminf_{n\to\infty}\{A_n\} &= \cup_{n=1}^{\infty} \cap_{m\geq n} A_n.\end{aligned}$$

(i) The meaning of the above events can be written as

$$\begin{aligned}\limsup_{n\to\infty}\{A_n\} &= \bigcap_{n=1}^{\infty} \bigcup_{m\geq n} A_n\\ &= \{x\in\Omega \mid \forall n, \exists m\geq n \text{ such that } x\in A_m\}\\ &= \{x\in\Omega \mid \text{the outcome } x \text{ appears infinitely often in the event sequence}\},\\ \liminf_{n\to\infty}\{A_n\} &= \bigcup_{n=1}^{\infty} \bigcap_{m\geq n} A_n\\ &= \{x\in\Omega \mid \exists n, \forall m\geq n, x\in A_m\}.\\ &= \{x\in\Omega \mid \text{the outcome } x \text{ does not appear only for a}\\ &\quad \text{finite number of events in the sequence}\}.\end{aligned}$$

(ii) Let $\Omega=\mathbb{R}$, and $\mathscr{A}$ is the Borel $\sigma-$algebra. For any $p\leq 1$, we define

$$A_{2p}=\Big[-1,2+\frac{1}{2p}\Big),\ \ A_{2p+1}=\Big(-2-\frac{1}{2p+1},1\Big].$$

We first denote the $\limsup$ set as S and $\liminf$ set as I. Then, we claim that

$$\begin{aligned}S &= \limsup_{n\to\infty}\{A_n\}=\Big[-2,2\Big],\\ I &= \liminf_{n\to\infty}\{A_n\}=\Big[-1,1\Big],\end{aligned}$$

which we denote as S and I respectively.

We first prove that $S=[-2,2]$. Consider $x\in[0,2]$. Observe that for any m, $x\in A_{2\lceil\frac{m}{2}\rceil}$, as $[0,2]\subseteq[-1,2+\dfrac{1}{2\lceil\frac{m}{2}\rceil})$. As $2\lceil\frac{m}{2}\rceil\geq m$, we have $[0,2]\subseteq S$. Similarly, consider $x\in[-2,0]$. Observe that for any m, $x\in A_{2\lfloor\frac{m}{2}\rfloor+1}$, as $[-2,0]\subseteq(-2-\dfrac{1}{2\lfloor\frac{m}{2}\rfloor+1},1]$. As $2\lfloor\frac{m}{2}\rfloor\geq m$, we have $[-2,0]\subseteq S$. Suppose now that $x>2$. Then, by the Archemedian property of real number, there exists an integer p such that $2+\dfrac{1}{2p}<x$. Hence, $x\notin A_k$ for all $k\geq 2p$. Therefore, for $x>2$, $x\notin S$. Analogously, for $x<-2$, $x\notin S$. We have shown that $S=[-2,2]$.

We now show that $I=[-1,1]$. Observe that $[-1,1]\subseteq A_k$ for all positive integer k greater than 1. Therefore, $[-1,1]\subseteq I$. For $x>1$, the odd terms do not contain x. For $x<-1$, the even terms do not contain x. Hence, we have shown that $I=[-1,1]$ as desired.

(iii)

Question 6.Zeta Function in Probability.

Solution. Let n and m be random numbers chosen independently and uniformly on $[[1,N]]$. Then, we can characterize Ω, and $\mathscr{A}$, which all implicitly depend on N as follow:

$$\begin{aligned}\Omega &= [[1,N]]\times[[1,N]],\\ \mathscr{A} &= 2^{\Omega},\end{aligned}$$

where 2^Ω denotes the power set of Ω. Furthermore, with the uniform probability assumption, we can define the probability measure on the above measurable space $\mathbb{P} : \mathscr{A} \to [0,1]$ by

$$\mathbb{P}(A) = \frac{|A|}{N^2} \text{ for } A \in \mathscr{A}.$$

Now, define set of events A_p as follow:

$$\{x \in \Omega \mid \text{both n and m have a prime } p \text{ as its factor}\}.$$

Notice that the $(n,m) = 1$ event, for a fixed N, can be expressed in terms of A_ps as

$$\mathbb{P}((n,m) = 1) = \mathbb{P}(\bigcap_{p \in \bar{P}} A_p^c),$$

where $\bar{P}$ denotes the set of primes not greater than N. As $\{A_p\}$ collection forms an independent collection, arising from the uniform probability assumption and independence, the probability can be factorized into

$$\mathbb{P}((n,m) = 1) \quad = \quad \prod_{p \in \bar{P}} \mathbb{P}(A_p^c),$$

with $\mathbb{P}(A_p^c) = 1 - \frac{\lfloor \frac{N}{p} \rfloor}{p}^2$. Hence, we can re-write the above equation as

$$\mathbb{P}((n,m) = 1) \quad = \quad \prod_{p \in \bar{P}} 1 - \frac{\lfloor \frac{N}{p} \rfloor}{p}^2.$$

Now, as $N \to \infty$, we have

$$\begin{aligned} \mathbb{P}((n,m) = 1)_{N \to \infty} \quad &= \quad \lim_{n \to \infty} \prod_{p \in \bar{P}} 1 - \frac{\lfloor \frac{N}{p} \rfloor}{p}^2 \\ &= \quad \prod_{p \in P} 1 - \frac{1}{p^2} \\ &= \quad \zeta(2)^{-1} \\ &= \quad \frac{6}{\pi^2}, \end{aligned}$$

where P denotes the set of primes in the second equation, as desired. $\square$