Basic Probability:
Problem Set I

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1. Countability.

Solution. (i) Let $X = \{(a,b) | a \in \mathbb{Q}, b \in \mathbb{Q}\}$. In particular, $X = \mathbb{Q} \times \mathbb{Q}$. Notice that by the Cantor diagonalization argument, there exists a bijective map $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$. Now, consider a map $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\psi(m,n) = (m+n)^2 + n$. With some algebra, we can show that if $\psi(m,n) = \psi(m',n')$, then $m = m'$ and $n = n'$. Hence, ψ is injective, and $\mathbb{N} \times \mathbb{N}$ is equipotent to $\psi(\mathbb{N} \times \mathbb{N})$, which is a subset of the countable set $\mathbb{N}$. Therefore, we have shown that $\mathbb{N} \times \mathbb{N}$ is countable, and consequently X is countable, due to the existence of the bijective map ϕ.

(ii) Let $X = \{B((x,y),r) | x = y\}$. Consider the set Y such that $Y = \{B((x,y),1) | x = y\}$. Y is in fact equipotent with $\mathbb{R}$, as you can arbitrarily pick the center from $\mathbb{R}$ and that determines the circle in Y. Hence, Y is uncountable. Furthermore, notice that $Y \subset X$. Hence, X is uncountable.

(iii) Uncountable

Question 2. Basic Combinatorics.

Solution. (i) The total number of possible ordering of age is 5!, as we treat each children to be distinct. The total number of possible ordering of age, given that the eldest three children all have to girls is 3!2!. Hence, the probability of the event, where the eldest three children are girls, is $\dfrac{3!2!}{5!} = \dfrac{1}{\binom{5}{2}}$, given the assumption that all orderings are equally likely.

(ii) The total number of possible team arrangements, given the constraints is $\binom{11}{5}\binom{6}{4}$. A combinatorial interpretation is that we first pick the team with 5 members, then pick the team with 4 members from the remaining people. As 5, 4, and 2 are all distinct numbers, we do not have to consider over-counting in this case.

Question 4. Geometric Distribution and its Variance.

Solution. We have that $\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. We first compute $\mathrm{E}[X]$.

$$\mathrm{E}[X] = 3232 \quad (1)$$

Question 5. σ-algebra generated.

Solution. Assume $\emptyset \subsetneq A \subsetneq B \subsetneq \Omega$. The σ-algebra generated by $\{A, B\}$ can be written as

$$\sigma(\{A,B\}) = \{\emptyset,\ A,\ B,\ A^C,\ B^C,\ A \cup B^C,\ A^C \cap B,\ \Omega\}.$$