Self-test Questions on Prerequisites

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

The following is a collection of solutions of the self-test questions for Real Variables at the Courant Institute.

Question.

Solution. Two sets are said to be equipotent provided there is a bijective map from one to another. Hence, to show that the sets $(0,1]$ and $[0,1]$ are equipotent, it suffices to construct a bijective map from $(0,1]$ to $[0,1]$.

Question 1-2. Equipotence is an RST relation.

Solution. We prove that equipotence is an equivalence relation on sets, denoted as R. First, a set is equipotent with itself, as the identity map establishes an equipotence. Second, let $(A, B) \in R$. Then, by the definition of equipotence, there exists a map $f : A \to B$ such that f is a one-to-one correpondence. Now, the inverse relation of the map f, f^{-1}, is also a one-to-one map from B to A. Hence, $(B, A) \in R$, and R is reflexive. Now, let (A, B) and (B, C) be elements in R. Then, there exists two bijective maps f_{AB} and f_{BC}. Consider the composition of the two maps $f_{AC} : A \to C$. The map f_{AC} is a bijective map from A to C. Hence, there exists a one-to-one correspondence between A and C. Hence, R is transitive. Therefore, R is an equivalence relation. $\square$

Question 1-3.

Solution. Let E be a nonempty subset of the real numbers. We want to show that $\inf E = \sup E$ iff E contains a single point. Assume that E is a single point, thus $E = \{x\}$. As, $x \geq x$ and $x \leq x$, x is both $\sup E$ and $\inf E$. Hence, $\inf E = \sup E$. Assume that $\inf E = \sup E$. By the definition of supremum and infimum, we have that for all $x \in E$, we have $\inf E \leq x \leq \sup E$. Combined with $\inf E = \sup E$, we have $\inf E = x = \sup E$. Hence, E is a single point set.

Question The Cauchy Convergence Criterion for Real Sequences.

Solution. Let $\{a_n\}$ be a sequence of real numbers. First, assume that $\{a_n\} \to a$. Then, for all natural numbers n and m, by the triangle inequality, we have

$$|a_n - a_m| = |(a_n - a) - (a_m - a)| \leq |a_n - a| - |a_m - a|.$$

As $\{a_n\}$ is convergent, for any $\epsilon > 0$, we have N such that $|a_k - a| < \frac{\epsilon}{2}$, for $k \geq N$. Hence, there exists N, such that for $n, m \geq N$, we have N such that $|a_n - a| < \frac{\epsilon}{2}$ and $|a_m - a| < \frac{\epsilon}{2}$, thus $|a_n - a_m| < \frac{\epsilon}{2}$. $\{a_n\}$ is cauchy.

Question 1-4. σ**-algebra.**

Solution. Let F be a collection of subsets of X, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a collection of collections of subsets of X that contains F. Consider $\cap_{\lambda\in\Lambda}A_\lambda$. Clearly, $F \in \cap_{\lambda\in\Lambda}A_\lambda$. We now want to show that $\cap_{\lambda\in\Lambda}A_\lambda$ is indeed a σ-algebra. $\emptyset$ and X are in $\cap_{\lambda\in\Lambda}$, as they are in every σ-algebra. It remains to show that it is "closed" under countable union and complement. Let $E \in \cap_{\lambda\in\Lambda}A_\lambda$. Then, E is in A_λ for all $\lambda \in \Lambda$. As each A_λs are σ-algebra, E^C is in A_λ for all $\lambda \in \Lambda$.

Question 1.5. Further subsequence.

Solution. Suppose for sake of contradiction that $\{x_n\}$ does not converge to x. Then, for some $\epsilon > 0$, for all $N \in \mathbb{N}$, there exists x_n with $n \geq N$, such that $|x_n - x| \geq \epsilon$. Then, for each $N \in \mathbb{N}$, pick an element that satisfies $|x_N - x| \geq \epsilon$; this is a subsequence of $\{x_n\}$, which we denote as $\{x_{n_k}\}$. We now have that $|x_{n_k} - x| \geq \epsilon$ for all k. Hence, this particular subsequence of $\{x_n\}$ cannot have a further subsequence that converges to x, which is a contradiction. Hence, $\{x_n\} \to x$. $\square$

Question 1.4. $\limsup$ **is** $\sup C$**.**

Solution. Let $\{a_n\}$ be a sequence of real numbers, C be a set of cluster points of $\{a_n\}$. First, we simply denote $\limsup\{a_n\}$ as s, which can be written as

$$s = \lim_{n\to\infty}[\,\sup\{a_k \mid k \geq n\}\,].$$

We first show that $s \in C$. We have two cases. First, assume $|s| = \infty$. Then, the sequence is divergent, and any subsequence of the sequence converges to either ∞ or $-\infty$, depending on the situation. Hence, s is a cluster point. Second, assume that $|s| < \infty$. We would like to construct a subsequence $\{a_{n_k}\}$, which converges to s. Let $s_n = \sup\{a_k \mid k \geq n\}$. By the approximation property of supremum, there exists an index n_1, such that $s_1 - \dfrac{1}{2} < a_{n_1} < s_1$ holds. Now, consider an inductive selection process, where given the choice of a_{n_k}, by using the approximation property of supremum, we pick $a_{n_{k+1}}$ such that

$$s_{a_{n_k}+1} - \frac{1}{2^{a_{n_k}+1}} < a_{n_{k+1}} < s_{a_{n_k}+1},$$

holds. Now, the sequences on the right hand side and the left hand side both converge to s, and by squeeze theorem, the constructed sequence $\{a_{n_k}\}$ converges to s. Hence, $\limsup\{a_n\}$ is a cluster point.

Now, we show that $\limsup\{a_n\}$ is the largest cluster point. When $\limsup\{a_n\}$ is unbounded, it trivially holds. Assume that $\limsup\{a_n\}$ is a real number. Let x be any cluster point of $\{a_n\}$. By the definition of a cluster point, we have a subsequence $\{a_{n_k}\}$ such that converges to x. Notice that, by the definition of $\limsup$, we have $a_{n_k} \leq s_{n_k}$ for all k, as a_{n_k} is an element of the set considered for the s_{n_k} term. As $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$, it also converges to s and hence, $s \geq x$. We have shown that $\limsup\{a_n\}$ is the largest cluster point.

Question $\limsup$**.**

Solution. Let $\{a_n\}$, and $\{b_n\}$ be real valued sequences.

Question 2. Continous functions.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and assume that $f(0) > 0$. By the $\epsilon - \delta$ criterion of continuity at 0, we have that for any $\epsilon > 0$, there exists $\delta > 0$ such that for $x \in \mathbb{R}$, if $|x - 0| < \delta$, then $|f(x) - f(0)| < \epsilon$. Set $\epsilon = \dfrac{f(0)}{2}$. Then, we have there exists $\delta > 0$ such that for $x \in B(0, \delta)$, $|f(x) - f(0)| < \dfrac{f(0)}{2}$, thus $f(x) > 0$. Hence, we have shown that there exists a nonempty interval (δ, δ), where δ is chosen from the continuity criterion with respect to $\dfrac{f(0)}{2}$, that all elements inside is strictly positive. $\square$

Question.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Assume that f is continuous.