Basic Probability: Problem Set III

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1. Limit and Conditional Probability.

Solution. We first claim that if $A_n \to A$ and $B_n \to B$, then $A_n \cap B_n \to A \cap B$. Let $w \in \Omega$. As $A_n \to A$, there exists N_A such that

$$1_{A_n}(w) \quad = \quad 1_A(w) \quad \text{for} \quad n \geq N_A.$$

Similarly, as $B_n \to B$, there exists N_B such that

$$1_{B_n}(w) \quad = \quad 1_B(w) \quad \text{for} \quad n \geq N_B.$$

Take $N^* = \max(N_A, N_B)$. Then, for $n \geq N^*$, we have

$$\begin{aligned} 1_{A_n \cap B_n}(w) &= 1_{A_n}(w) 1_{B_n}(w) \\ &= 1_A(w) 1_B(w) \\ &= 1_{A \cap B}(w). \end{aligned}$$

Since w was arbitrary, we have shown that $A_n \cap B_n \to A \cap B$.

Recall the following theorem established earlier in the class: let P be a probability measure, and A_n be a sequence of events in $\mathscr{A}$ which converge to A. Then, $A \in \mathscr{A}$ and $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$. Combining the theorem with the above result we have that if $A_n \to A$ and $B_n \to B$, we have

$$\lim_{n\to\infty} \mathbb{P}(A_n \cap B_n) = \mathbb{P}(A \cap B).$$

From this point on, we assume that $\mathbb{P}(B) > 0$ and $\mathbb{P}(B_n) > 0$ for all n, so that the conditional probabilities are well-defined.

(i) As $A_n \to A$ and $B \to B$, by the established result, we obtain

$$\lim_{n\to\infty} \mathbb{P}(A_n \cap B) = \mathbb{P}(A \cap B) \text{ and } \lim_{n\to\infty} \mathbb{P}(B) = \mathbb{P}(B).$$

Hence, by the limit rule, we have that the limit of the given term exists and

$$\lim_{n\to\infty} \frac{\mathbb{P}(A_n \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B).$$

(ii) As $A \to A$ and $B_n \to B$, by the established result, we obtain

$$\lim_{n\to\infty} \mathbb{P}(A \cap B_n) = \mathbb{P}(A \cap B) \text{ and } \lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(B).$$

Hence, by the limit rule, we have that the limit of the given term exists and

$$\lim_{n\to\infty} \frac{\mathbb{P}(A \cap B_n)}{\mathbb{P}(B_n)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B).$$

(iii) As $A_n \to A$ and $B_n \to B$, by the established result, we obtain

$$\lim_{n\to\infty} \mathbb{P}(A_n \cap B_n) = \mathbb{P}(A \cap B) \text{ and } \lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(B).$$

Hence, by the limit rule, we have that the limit of the given term exists and

$$\lim_{n\to\infty} \frac{\mathbb{P}(A_n \cap B_n)}{\mathbb{P}(B_n)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B),$$

which completes the proof. $\square$

Question 2. Conditional Probability.

Solution. As $\mathbb{P}(B \cap C) \neq 0$, by the definition of conditional probability, we have

$$\mathbb{P}(A|B \cap C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)}.$$

As $\{A, B, C\}$ are mutually independent events, we obtain

$$\begin{aligned} \mathbb{P}(A|B \cap C) &= \frac{\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)}{\mathbb{P}(B)\mathbb{P}(C)} \\ &= \mathbb{P}(A), \end{aligned}$$

as desired. $\square$

Question 3. Conditional Probability.

Solution. Given the assumption, we can compute the probability of $\mathbb{P}(A)$ by using the conditional probability as

$$\mathbb{P}(A) = \mathbb{P}(A|G = M)\mathbb{P}(G = M) + \mathbb{P}(A|G = F)\mathbb{P}(G = F).$$

Substituting the givens, we have

$$\begin{aligned} \mathbb{P}(A) &= 0.3 \cdot 0.50.2 \cdot 0.5 \\ &= 0.25. \end{aligned}$$

Similarly, we can express $\mathbb{P}(B)$ as

$$\mathbb{P}(B) = \mathbb{P}(B|G = M)\mathbb{P}(G = M) + \mathbb{P}(B|G = F)\mathbb{P}(G = F).$$

As the claims are made independently each year, by substituting the givens, we have

$$\begin{aligned} \mathbb{P}(B) &= 0.3^2 \cdot 0.5 + 0.2^2 \cdot 0.5 \\ &= 0.065. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \mathbb{P}(B|A) &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(B)}{\mathbb{P}(C)} \\ &= \frac{0.0065}{0.25} \\ &= 0.26. \end{aligned}$$

The independence assumption was used to compute the probability of the event B. It does not give an explicit information about the conditional probability $\mathbb{P}(B|A)$ without the computation. As the head of the insurance company, one should minimize the probability of filling a complaint. Hence, one would prefer one who did not have a claim in the previous year, as the computation shows. $\square$

Question 4. Genetics.

Solution. We are given that a parent passes on one of its alleles, chosen at random uniformly to its child, and the genotype of the child combines alleles from both parents. Furthermore, the probability applies homogenously to the entire population. We then can write a probability of a parent passing on A allele as

$$\mathbb{P}(A) \quad = \quad P.$$

Then, by the definition of probability, we have

$$\mathbb{P}(a) \quad = \quad 1 - P = Q,$$

where $P + Q = 1$, as there are only two options. Hence, the probabilities that two parents passing on particular alleles are

$$\begin{aligned} \mathbb{P}(AA) &= \mathbb{P}(A)\mathbb{P}(A) = P^2 \\ \mathbb{P}(Aa) &= 2\mathbb{P}(A)\mathbb{P}(a) = 2PQ \\ \mathbb{P}(aa) &= \mathbb{P}(a)\mathbb{P}(a) = Q^2, \end{aligned}$$

as desired. Recall that $P + Q = 1$. Hence, the p, q, r terms, which are the probabilities computed above, only depend on one parameter P. $\square$

Question 5. The Moments of Poisson Distribution.

Solution. Let X be a random variable with Poisson distribution with parameter $\lambda > 0$. We proceed to compute the expectation $\mathbb{E}[X]$. By the definition of expectation, we have

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} \\ &= \sum_{k=1}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!}, \end{aligned}$$

as the 0-th term vanishes. Pulling out the $\lambda e^{-\lambda}$, we obtain

$$\begin{aligned} \mathbb{E}[X] &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \\ &= \lambda e^{-\lambda} \sum_{p=0}^{\infty} \frac{\lambda^p}{p!}, \end{aligned}$$

where $p = k - 1$. As the series converges to e^{λ}, we have

$$\mathbb{E}[X] \quad = \quad \lambda.$$

We now compute the variance $\mathrm{Var}[X]$. By the definition of variance, we have

$$\mathrm{Var}[X] \quad = \quad \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Again by the definition of expectation, we have

$$\begin{aligned} \mathbb{E}[X^2] &= \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} \\ &= \sum_{k=1}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!}, \end{aligned}$$

as the 0 term vanishes. With a series of simple algebraic manipulation, including the taylor series expansion for exponential as above, we have

$$\begin{aligned}
\mathbb{E}[X^2] &= \sum_{k=1}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!} \\
&= \lambda e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda} (\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} (k-1) \frac{\lambda^{k-1}}{(k-1)!}) \\
&= \lambda e^{-\lambda} (\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} + \lambda \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}) \\
&= \lambda e^{-\lambda} (e^{\lambda} + \lambda e^{\lambda}) \\
&= \lambda^2 + \lambda.
\end{aligned}$$

Consequently, it follows that

$$\mathrm{Var}[X] = \lambda.$$

Now, we compute the term $\mathbb{E}[X(X-1)...(X-K+1)]$ for any $k \in \mathbb{N}$. Note that the above term can be written as $\mathbb{E}\left[\frac{X!}{(X-K)!}\right]$. Now, we can write the expectation of the $\frac{X!}{(X-K)!}$ random variable as

$$\begin{aligned}
\mathbb{E}\left[\frac{X!}{(X-K)!}\right] &= \sum_{p=K}^{\infty} \frac{p!}{(p-K)!} \frac{\lambda^p e^{-\lambda}}{p!} \\
&= e^{-\lambda} \lambda^K \sum_{p=K}^{\infty} \frac{\lambda^{(p-K)}}{(p-K)!} \\
&= \lambda^K.
\end{aligned}$$

The kth moment of the distribution can be written as

$$\mathbb{E}[X^k] = \sum_{i=1}^{K} \lambda^i S(K, i),$$

where $S(K, i)$ denotes the Striling number of second kind, which computes the number of unlabelled partitions out of labeled objects. We esssentially get the number of partions for the falling factorial of size i. $\square$

Question 6. Memoryless Property of the Geometric Distribution.

Solution. We first compute an explicit formula for the $\mathbb{P}(X > k)$ term with an arbitrary $k \geq 0$. By the definition of the geometric random variable and geometric series formula, we have

$$\begin{aligned}
\mathbb{P}(X > k) &= 1 - \sum_{i=0}^{k} \mathbb{P}(X = i) \\
&= 1 - \sum_{i=0}^{k} (1-q)^i q \\
&= 1 - q \sum_{i=0}^{k} (1-q)^i \\
&= 1 - q \frac{1 - (1-q)^{k+1}}{q} \\
&= (1-q)^{k+1}
\end{aligned}$$

By the above formula, we can write the LHS as

$$\begin{aligned}
\mathbb{P}(X > i + j | X \geq i) &= \frac{\mathbb{P}(X > i + j, X > i)}{\mathbb{P}(X \geq i)} \\
&= \frac{\mathbb{P}(X > i + j)}{\mathbb{P}(X > i - 1)} \\
&= \frac{(1-q)^{i+j+1}}{(1-q)^{i}} \\
&= (1-q)^{j+1},
\end{aligned}$$

for $i, j > 0$. Again, by the above formula, we can write the RHS as

$$\mathbb{P}(X > j) = (1-q)^{j+1},$$

for $i, j > 0$. Hence, we have shown that the given memoryless property hold for $i, j > 0$. $\square$