Basic Probability:
Problem Set III

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1. Limit and Conditional Probability.

Solution. We first claim that if $A_n \to A$ and $B_n \to B$, then $A_n \cap B_n \to A \cap B$. Let $w \in \Omega$. As $A_n \to A$, there exists N_A such that

$$1_{A_n}(w) \quad = \quad 1_A(w) \quad \text{for} \quad n \geq N_A.$$

Similarly, as $B_n \to B$, there exists N_B such that

$$1_{B_n}(w) \quad = \quad 1_B(w) \quad \text{for} \quad n \geq N_B.$$

Take $N^* = \max(N_A, N_B)$. Then, for $n \geq N^*$, we have

$$\begin{aligned} 1_{A_n \cap B_n}(w) &= 1_{A_n}(w) + 1_{B_n}(w) + 1_{A_n}(w)1_{B_n}(w) \\ &= 1_A(w) + 1_B(w) + 1_A(w)1_B(w) \\ &= 1_{A \cap B}(w). \end{aligned}$$

Since w was arbitrary, we have shown that $A_n \cap B_n \to A \cap B$.

Recall the following theorem established earlier in the class: let P be a probability measure, and A_n be a sequence of events in $\mathscr{A}$ which converge to A. Then, $A \in \mathscr{A}$ and $\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$. Combining the theorem with the above result we have that if $A_n \to A$ and $B_n \to B$, we have

$$\lim_{n\to\infty} \mathbb{P}(A_n \cap B_n) = \mathbb{P}(A \cap B).$$

From this point on, we assume that $\mathbb{P}(B) > 0$ and $\mathbb{P}(B_n) > 0$ for all n, so that the conditional probabilities are well-defined.

(i) As $A_n \to A$ and $B \to B$, by the established result, we obtain

$$\lim_{n\to\infty} \mathbb{P}(A_n \cap B) = \mathbb{P}(A \cap B) \text{ and } \lim_{n\to\infty} \mathbb{P}(B) = \mathbb{P}(B).$$

Hence, by the limit rule, we have that the limit of the given term exists and

$$\lim_{n\to\infty} \frac{\mathbb{P}(A_n \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B).$$

(ii) As $A \to A$ and $B_n \to B$, by the established result, we obtain

$$\lim_{n\to\infty} \mathbb{P}(A \cap B_n) = \mathbb{P}(A \cap B) \text{ and } \lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(B).$$

Hence, by the limit rule, we have that the limit of the given term exists and

$$\lim_{n\to\infty} \frac{\mathbb{P}(A \cap B_n)}{\mathbb{P}(B_n)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B).$$

(iii) As $A_n \to A$ and $B_n \to B$, by the established result, we obtain

$$\lim_{n\to\infty} \mathbb{P}(A_n \cap B_n) = \mathbb{P}(A \cap B) \text{ and } \lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(B).$$

Hence, by the limit rule, we have that the limit of the given term exists and

$$\lim_{n\to\infty} \frac{\mathbb{P}(A_n \cap B_n)}{\mathbb{P}(B_n)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A|B),$$

which completes the proof. $\square$

Question 2. Conditional Probability.

Solution. As $\mathbb{P}(B \cap C) \neq 0$, by the definition of conditional probability, we have

$$\mathbb{P}(A|B \cap C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(B \cap C)}.$$

As $\{A, B, C\}$ are mutually independent events, we obtain

$$\begin{aligned} \mathbb{P}(A|B \cap C) &= \frac{\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)}{\mathbb{P}(B)\mathbb{P}(C)} \\ &= \mathbb{P}(A), \end{aligned}$$

as desired. $\square$

Question 3. Conditional Probability.

Solution. Given the assumption, we can compute the probability of $\mathbb{P}(A)$ by using the conditional probability as

$$\mathbb{P}(A) = \mathbb{P}(A|G = M)\mathbb{P}(G = M) + \mathbb{P}(A|G = F)\mathbb{P}(G = F).$$

Substituting the givens, we have

$$\begin{aligned} \mathbb{P}(A) &= 0.3 \cdot 0.50.2 \cdot 0.5 \\ &= 0.25. \end{aligned}$$

Similarly, we can express $\mathbb{P}(B)$ as

$$\mathbb{P}(B) = \mathbb{P}(B|G = M)\mathbb{P}(G = M) + \mathbb{P}(B|G = F)\mathbb{P}(G = F).$$

As the claims are made independently each year, by substituting the givens, we have

$$\begin{aligned} \mathbb{P}(B) &= 0.3^2 \cdot 0.5 + 0.2^2 \cdot 0.5 \\ &= 0.065. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \mathbb{P}(B|A) &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(B)}{\mathbb{P}(C)} \\ &= \frac{0.0065}{0.25} \\ &= 0.26. \end{aligned}$$

The independence assumption was used to compute the probability of the event B. It does not give an explicit information about the conditional probability $\mathbb{P}(B|A)$ without the computation. As the head of the insurance company, one should minimize the probability of filling a complaint. Hence, one would prefer one who did not have a claim in the previous year, as the computation shows. $\square$

Question 4. Genetics.
Solution.

Question 5. The Moments of Poisson Distribution.
Solution. Let X be a random variable with Poisson distribution with parameter $\lambda > 0$. We proceed to compute the expectation $\mathbb{E}[X]$. By the definition of expectation, we have

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=0}^{\infty} k\frac{\lambda^k e^{-\lambda}}{k!}\\ &= \sum_{k=1}^{\infty} k\frac{\lambda^k e^{-\lambda}}{k!},\end{aligned}$$

as the 0-th term vanishes. Pulling out the $\lambda e^{-\lambda}$, we obtain

$$\begin{aligned}\mathbb{E}[X] &= \lambda e^{-\lambda}\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}\\ &= \lambda e^{-\lambda}\sum_{p=0}^{\infty} \frac{\lambda^{p}}{p!},\end{aligned}$$

where $p = k-1$. As the series converges to e^{λ}, we have

$$\mathbb{E}[X] = \lambda.$$

We now compute the variance $\mathrm{Var}[X]$. By the definition of variance, we have

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Again by the definition of expectation, we have

$$\mathbb{E}[X^2] =$$

Question 6. Memoryless Property of the Geometric Distribution.
Solution. We first compute an explicit formula for the $\mathbb{P}(X > k)$ term with an arbitrary $k \geq 0$. By the definition of the geometric random variable and geometric series formula, we have

$$\begin{aligned}\mathbb{P}(X > k) &= 1 - \sum_{i=0}^{k}\mathbb{P}(X = i)\\ &= 1 - \sum_{i=0}^{k}(1-q)^i q\\ &= 1 - q\sum_{i=0}^{k}(1-q)^i\\ &= 1 - q\frac{1-(1-q)^{k+1}}{q}\\ &= (1-q)^{k+1}\end{aligned}$$

By the above formula, we can write the LHS as

$$\begin{aligned}\mathbb{P}(X > i+j | X \geq i) &= \frac{\mathbb{P}(X > i+j, X > i)}{\mathbb{P}(X \geq i)}\\ &= \frac{\mathbb{P}(X > i+j)}{\mathbb{P}(X > i-1)}\\ &= \frac{(1-q)^{i+j+1}}{(1-q)^{i}}\\ &= (1-q)^{j+1},\end{aligned}$$

for $i, j > 0$. Again, by the above formula, we can write the RHS as

$$\mathbb{P}(X > j) \quad = \quad (1 - q)^{j+1},$$

for $i, j > 0$. Hence, we have shown that the given memoryless property hold for $i, j > 0$. $\square$