Basic Probability:
Problem Set IV

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1.

Solution. We are given that

$$\begin{aligned} P(M) &= 0.5 \quad \text{and} \quad P(F) = 0.5 \\ P(C|M) &= 0.03 \quad \text{and} \quad P(C|F) = 0.05, \end{aligned}$$

where C, M and F are events corresponding to a chosen person being colorblind, male and female respectively. Note that M and F form a partition of the sample space. Now, by the Bayes' theorem, we have

$$P(M|C) = \frac{P(C|M)P(M)}{P(C)}.$$

As M and F is a partition of the sample space, we have

$$P(M|C) = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(F|M)P(M)}.$$

Therefore, substituting the givens yields

$$\begin{aligned} P(M|C) &= \frac{0.03 \cdot 0.5}{0.03 \cdot 0.5 + 0.05 \cdot 0.5} \\ &= 0.375. \end{aligned}$$

If there are twice many males over females, we have

$$\begin{aligned} P(M|C) &= \frac{0.03 \cdot \frac{2}{3}}{0.03 \cdot \frac{2}{3} + 0.05 \cdot \frac{1}{3}} \\ &\approx 0.545. \end{aligned}$$

This completes the computation. $\square$

Question 2.

Solution. We have an experiment of selecting a coin from a box of three coins at random and flipping the selected coin. Let C_1, C_2 and C_3 be events corresponding to a chosen coin being the coin 1, coin 2 and coin 3. Let H be an event that a head shows. Then, we are given that

$$\begin{aligned} P(C_1) = \frac{1}{3} \qquad &\text{and} \qquad P(H|C_1) = 1 \\ P(C_2) = \frac{1}{3} \qquad &\text{and} \qquad P(H|C_2) = 0.5 \\ P(C_3) = \frac{1}{3} \qquad &\text{and} \qquad P(H|C_3) = 0.65. \end{aligned}$$

By the Bayes' theorem, we have

$$P(C_1|H) \quad = \quad \frac{P(H|C_1)P(C_1)}{P(H)}.$$

As C_1, C_2 and C_3 form a partition of the sample space, we have

$$\begin{aligned} P(C_1|H) &= \frac{P(H|C_1)P(C_1)}{P(H|C_1)P(C_1) + P(H|C_2)P(C_2) + P(H|C_3)P(C_3)} \\ &= \frac{1 \cdot \frac{1}{3}}{1 \cdot \frac{1}{3} + 0.5 \cdot \frac{1}{3} + 0.65 \cdot \frac{1}{3}} \\ &\approx 0.465. \end{aligned}$$

Therefore, the probability that it was the two-headed coin, given that the throw resulted in a head is approximately 0.465. $\square$

Question 3.

Solution. Given the distribution function F, by definition, we can write the probability of an event (a, b) as

$$P((a, b)) = F(b) - F(a).$$

Therefore, it follows that

$$\begin{aligned} P((-1/2, 1/2)) &= F(1/2) - F(-1/2) \\ &= \frac{1}{4} - 0 \\ &= \frac{1}{4} \\ P((1/2, 3/2)) &= F(3/2) - F(-1/2) \\ &= \frac{3}{4} - 0 \\ &= \frac{3}{4} \\ P((2/3, 5/2)) &= F(5/2) - F(2/3) \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \\ P((3, \infty)) &= F(\infty) - F(3) \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

This completes the computations. $\square$

Question 4.

Solution. We wish to compute the factorial moment of the geometric distribution. Writing out definition of the factorial moment and simplifying with geometric series and rth derivative, we have

$$\begin{aligned}
\mathbb{E}\left[\frac{X!}{(X-r)!}\right] &= \sum_{k=r}^{\infty}\frac{k!}{(k-r)!}(1-p)^k p \\
&= p(1-p)^r\sum_{k=r}^{\infty}\frac{k!}{(k-r)!}(1-p)^{k-r} \\
&= p(1-p)^r(-1)^r\frac{d^r}{d_p^r}\sum_{k=0}^{\infty}(1-p)^k \\
&= p(1-p)^r(-1)^r\frac{d^r}{d_p^r}\frac{1}{p} \\
&= p(1-p)^r(-1)^r(-1)^r r!\frac{1}{p^{r+1}} \\
&= \frac{r!(1-p)^r}{p^r},
\end{aligned}$$

as desired. $\square$

Question 5.

Solution. Let X have a binomial distribution with parameters (p, n). Let $p \in (0, 1)$. We proceed by mathematical induction. For the $n = 1$, we have

$$\begin{aligned}
\frac{1}{2}(1+(1-2p)^1) &= \frac{1}{2}(2-2p) \\
&= (1-p) \\
&= \binom{1}{0}p^0(1-p)^1
\end{aligned}$$

Therefore the base case holds. Assume that the formula holds for n. Let X_n and X_{n+1} be the binomial distribution with n and $n+1$ parameters respectively. Observe the following recurrence relation:

$$P(X_{n+1} = even) = P(X_n = even)(1-p) + P(X_n = odd)p,$$

which can be seen by partitioning the probability by the outcome of the $n+1$ trial. If the last trial is a success, then the number of successes upto n must be odd. Similarly, if the last trial is a failure, then the number of successes upto n must be even. Substituting the inductive hypothesis into the above recurrence relation yields

$$\begin{aligned}
&= \frac{1}{2}(1+(1-2p)^n)(1-p) + (1-\frac{1}{2}(1+(1-2p)^n)p \\
&= (\frac{1}{2}+\frac{1}{2}(1-2p)^n)(1-p) + (\frac{1}{2}-\frac{1}{2}(1-2p)^n)p \\
&= \frac{1}{2}+\frac{1}{2}(1-2p)^n(1-2p) \\
&= \frac{1}{2}(1+(1-2p)^{n+1}),
\end{aligned}$$

which completes the induction. Therefore, we have shown that X is even with probability $\frac{1}{2}(1 + (1-2p)^n)$. $\square$

Question 6.

Solution. Let $g : [0, \infty) \to [0, \infty)$ be strictly increasing and non-negative. Let X be a real-valued random variable on a countable space Ω, and P be the law of X. Let A denote the following set : $A = \{\omega \in \Omega \mid |X(\omega)| \geq a\}$. By the construction of the set A, we have

$$|X(\omega)| \geq a 1_A(\omega),$$

for all $\omega \in \Omega$. Since g is strictly increasing and defined on non-negative reals, we have

$$\begin{aligned} g(|X(\omega)|) &\geq g(a 1_A(\omega)) \\ &\geq g(a) 1_A(\omega), \end{aligned}$$

for all $\omega \in \Omega$. Now, as the inequality holds for all $\omega \in \Omega$, summing over all $\omega \in \Omega$, we have

$$\begin{aligned} \sum_{\omega \in \Omega} g(|X(w)|) &\geq \sum_{\omega \in \Omega} g(a) 1_A(w) \\ &= g(a) \sum_{\omega \in \Omega} 1_A(w) \\ &= g(a) P(A). \end{aligned}$$

Observe that LHS is an expectation of $g(|X|)$ and $g(a) > 0$ as $a > 0$. Therefore, rewriting and rearranging the terms yield

$$P(|X| \geq a) \leq \frac{\mathbb{E}[g(|X|)]}{g(a)},$$

as desired. $\square$