Basic Probability: Problem Set I

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1. Countability.

Solution. (i) Let $X = \{(a,b) | a \in \mathbb{Q}, b \in \mathbb{Q}\}$. In particular, $X = \mathbb{Q} \times \mathbb{Q}$. Notice that by the Cantor diagonalization argument, there exists a bijective map $\phi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$. Now, consider a map $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $\psi(m,n) = (m+n)^2 + n$. With some algebra, we can show that if $\psi(m,n) = \psi(m',n')$, then $m = m'$ and $n = n'$. Hence, ψ is injective, and $\mathbb{N} \times \mathbb{N}$ is equipotent to $\psi(\mathbb{N} \times \mathbb{N})$, which is a subset of the countable set $\mathbb{N}$. Therefore, we have shown that $\mathbb{N} \times \mathbb{N}$ is countable, and consequently X is countable, due to the existence of the bijective map ϕ.

(ii) Let $X = \{B((x,y),r) | x = y\}$. Consider the set Y such that $Y = \{B((x,y),1) | x = y\}$. Y is in fact equipotent with $\mathbb{R}$, as you can arbitrarily pick the center from $\mathbb{R}$ and that determines the circle in Y. Hence, Y is uncountable. Furthermore, notice that $Y \subset X$. Hence, X is uncountable.

(iii) We can prove that the set of all sequences of integers whose term are either 0 or 1 is uncountable, by the diagnonalization argument. Assume that the set is countable. Then, let $\{s_i\}_{i=1}^{\infty}$ be an enumeration of such set, such that each s_i is a sequence, further denoted by $\{s_{i,k}\}_{k=1}^{\infty}$. Construct a sequence $\{x_i\}_{i=1}^{\infty}$ such that $x_i = s_{i,i}$. Then, $\{x_i\}$ sequence does not belong to the set of all sequences integers whose term are either 0 or 1. Contradiction. The set is uncountable.

Question 2. Basic Combinatorics I.

Solution. (i) The total number of possible ordering of age is 5!, as we treat each children to be distinct. The total number of possible ordering of age, given that the eldest three children all have to girls is 3!2!. Hence, the probability of the event, where the eldest three children are girls, is $\frac{3!2!}{5!} = \frac{1}{\binom{5}{2}}$, given the assumption that all orderings are equally likely.

(ii) The total number of possible team arrangements, given the constraints is $\binom{11}{5}\binom{6}{4}$. A combinatorial interpretation is that we first pick the team with 5 members, then pick the team with 4 members from the remaining people. As 5, 4, and 2 are all distinct numbers, we do not have to consider over-counting in this case.

Question 3. Basic Combinatorics II.

Solution. As there are 30 distinct classes, we have $\binom{30}{7}$ for the numerator. For the denominator, the total number of possible configuration, with which we have a class everyday, we first have 6^5 to choose a class for Monday through Friday. Now, for the remaining two classes, we can freely choose from the free 2 slots. Hence, we have $\binom{25}{2}$, and $6^5\binom{25}{2}$ for the total number. Therefore, the probability, under uniform randomness, is $\frac{6^5\binom{25}{2}}{\binom{30}{7}}$.

Question 4. Geometric Distribution and its Variance.

Solution. We have that $\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. We proceed to compute $\mathrm{E}[X]$, for the given geometric distribution, $\mathbb{P}(X = k) = (1-q)^k q$.

$$\begin{aligned} \mathrm{E}[X] &= \sum_{k=0}^{\infty} k(1-q)^k q \\ &= q(1-q)\sum_{k=1}^{\infty} k(p)^{k-1}, \end{aligned}$$

where $p = 1 - q$. Notice that the summation is a derivative of the power series $\sum p^k$. Hence, we can simplify the RHS and obtain

$$\begin{aligned} \mathrm{E}[X] &= q(1-q)\frac{1}{(1-p)^2} \\ &= \frac{1-q}{q}. \end{aligned}$$

We now proceed to compute $\mathrm{E}[X^2]$. We first have

$$\begin{aligned} \mathrm{E}[X^2] &= \sum_{k=0}^{\infty} k^2(1-q)^k q \\ &= q(1-q)^2\sum_{k=2}^{\infty} k(k-1)(p)^{k-1} - q(1-q)\sum_{k=1}^{\infty} k(p)^{k-1}, \end{aligned}$$

where $p = 1 - q$. Again using the derivative of the power series $\sum p_k$ we have,

$$\begin{aligned} \mathrm{E}[X^2] &= q(1-q)^2(\frac{2}{(1-p)^3}) - q(1-q)(\frac{1}{(1-p)^2}) \\ &= \frac{2(1-q)^2}{q^2} - \frac{1-q}{q} \\ &= \frac{(2-q)(1-q)}{q^2}. \end{aligned}$$

Substituting the computed equality into the variance equation, we obtain

$$\mathrm{Var}(X) = \frac{(2-q)(1-q)}{q^2} - (\frac{1-q}{q})^2$$

Hence, we have that

$$\mathrm{Var}(X) = \frac{1-q}{q^2}.$$

Question 5. σ-algebra generated.

Solution. Assume $\emptyset \subsetneq A \subsetneq B \subsetneq \Omega$. The σ-algebra generated by $\{A, B\}$ can be written as

$$\sigma(\{A, B\}) = \{\emptyset,\ A,\ B,\ A^C,\ B^C,\ A \cup B^C,\ A^C \cap B,\ \Omega\}.$$

Question 6. Stirling's Formula.

Solution. We want to compute $\lim_{n\to\infty} \sum_{k=1}^{n} ln(k)$. This can be re-written as $\lim_{n\to\infty} ln(k!)$. Using the trapezoid approximation, we have that

$$ln(n!) \approx \int_1^n ln(x)dx = nln(n) - n + 1.$$