Basic Probability: Problem Set V

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1.

Solution. **(i)** By the definition of Dirac Distribution and the linearity, we have

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_{-\infty}^{\infty} x(p\delta_a(x)) + q\delta_b(x)) dx \\ &= p\int_{-\infty}^{\infty} x\delta_a(x)) dx + q\int_{-\infty}^{\infty} x\delta_b(x) dx \\ &= pa + qb, \end{aligned}$$

as expected.

(ii) By definition of Poisson Distribution, we have

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} k\frac{1}{k!}\lambda^k e^{-\lambda} \\ &= \lambda e^{-\lambda} \sum_{k=0}^{\infty} k\frac{1}{k!}\lambda^{k-1} \\ &= \lambda, \end{aligned}$$

as expected. $\square$

Question 2.

Solution. Let X be uniformly distributed on $[0,1]$. Then, the distribution of X can be written as

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x & \text{for } x \in (0,1) \\ 1 & \text{for } x \geq 1. \end{cases}$$

Now, consider the random variable $-\lambda \log X$, for $\lambda > 0$. Observe that for $x > 0$,

$$-\lambda \log(y) \leq x \iff y \geq e^{-\frac{x}{\lambda}}.$$

Consequently, the distribution of $-\lambda \log X$ can be written as

$$F_{-\lambda \log X}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \\ 1 - e^{-\lambda x} & \text{for } x \in [0, \infty) \end{cases},$$

which is precisely the exponential distribution as desired. $\square$

Question 3.

Solution. Let X be a standard Gaussian random variable. The density then can be written as

$$f_X(x) = \frac{1}{2\pi} e^{-x^2}.$$

Notice that for the $\frac{1}{X^2}$ random variable, the density can be written as

$$\begin{aligned} f_{\frac{1}{X^2}}(x) &= f_X(\frac{1}{\sqrt{x}}) + f_X(-\frac{1}{\sqrt{x}}) \\ &= \frac{1}{\pi} e^{\frac{1}{x}}, \end{aligned}$$

as desired. $\square$

Question 4.

Solution. (i) We have $\Omega = \{H, T\}^N$, $\mathscr{A} = 2^\Omega$, and $\mathbb{P} : \mathscr{A} \to [0, 1]$ such that $\mathbb{P}(A) = \frac{|A|}{2^N}$, for any $A \in \mathscr{A}$.

(ii) Define A_n be a sequence of events such that the pattern (H, H, T, H, T, H, H) occurs with the n toss experiment, and $p(A_n)$ be the probability of A_n. Observe that we can identify an event L_n where (H, H, T, H, T, H, H) occurring from the 1st index to the 7th index with the n toss experiment precisely. The probability of such event is $\frac{2^{n-7}}{2^n}$ for $n \geq 7$. Observe that $\lim_{n\to\infty} p(L_n) = 1$. Since L_n is a subset of A_n, by the monotonicity property of probability measure, we have $p(L_n) \leq p(A_n) \leq 1$ for all n. Therefore, by the Squeeze Theorem, we have that $\lim_{n\to\infty} p(A_n) = 1$. $\square$

Question 5.

Solution.

Question 6.

Solution. As $c > 0$ and $\delta > 0$, by the Markov inequality, we have

$$P(|X| > \delta) \leq \frac{E[e^{\lambda |X|}]}{e^{\lambda \delta}}$$