Basic Probability:
Problem Set V

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1.

Solution. (i) By the definition of Dirac Distribution and the linearity, we have

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= \int_{-\infty}^{\infty} x(p\delta_a(x)) + q\delta_b(x)) dx \\
&= p \int_{-\infty}^{\infty} x\delta_a(x)) dx + q \int_{-\infty}^{\infty} x\delta_b(x) dx \\
&= pa + qb,
\end{aligned}
$$

as expected.

(ii) By definition of Poisson Distribution, we have

$$
\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} k \frac{1}{k!} \lambda^k e^{-\lambda} \\
&= \lambda e^{-\lambda} \sum_{k=0}^{\infty} k \frac{1}{k!} \lambda^{k-1} \\
&= \lambda,
\end{aligned}
$$

as expected. $\square$

Question 2.

Solution. Let X be uniformly distributed on $[0, 1]$. Then, the distribution of X can be written as

$$
F_X(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x & \text{for } x \in (0, 1) \\ 1 & \text{for } x \geq 1. \end{cases}
$$

Now, consider the random variable $-\lambda \log X$, for $\lambda > 0$. Observe that for $x > 0$,

$$
-\lambda \log(y) \leq x \iff y \geq e^{-\frac{x}{\lambda}}.
$$

Consequently, the distribution of $-\lambda \log X$ can be written as

$$F_{-\lambda \log X}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \\ 1 - e^{-\lambda x} & \text{for } x \in [0, \infty) \end{cases},$$

which is precisely the exponential distribution as desired. $\quad\square$

Question 3.

Solution. Let X be a standard Gaussian random variable. The density then can be written as

$$f_X(x) = \frac{1}{2\pi} e^{-x^2}.$$

Notice that for the $\dfrac{1}{X^2}$ random variable, the density can be written as

$$\begin{aligned} f_{\frac{1}{X^2}}(x) &= f_X(\frac{1}{\sqrt{x}}) + f_X(-\frac{1}{\sqrt{x}}) \\ &= \frac{1}{\pi} e^{\frac{1}{x}}, \end{aligned}$$

as desired. $\quad\square$

Question 4.

Solution. (i) We have $\Omega = \{H, T\}^N$, $\mathscr{A} = 2^\Omega$, and $\mathbb{P} : \mathscr{A} \to [0, 1]$ such that $\mathbb{P}(A) = \dfrac{|A|}{2^N}$, for any $A \in \mathscr{A}$.

(ii) Define A_n be a sequence of events such that the pattern (H, H, T, H, T, H, H) occurs with the n toss experiment, and $p(A_n)$ be the probability of A_n. Observe that we can identify an event L_n where (H, H, T, H, T, H, H) occurring from the 1st index to the 7th index with the n toss experiment precisely. The probability of such event is $\dfrac{2^{n-7}}{2^n}$ for $n \geq 7$. Observe that $\lim_{n \to \infty} p(L_n) = 1$. Since L_n is a subset of A_n, by the monotonicity property of probability measure, we have $p(L_n) \leq p(A_n) \leq 1$ for all n. Therefore, by the Squeeze Theorem, we have that $\lim_{n \to \infty} p(A_n) = 1$. $\quad\square$

Question 5.

Solution. We have that X is in $\mathscr{L}^1(\Omega, \mathscr{A}, P)$. Observe that X dominates $\{X1_{A_n}\}$. Fix $\epsilon > 0$. It follows that

$$0 \leq P(\{w \in \Omega | |X1_{A_n}(w) - 0(w)| > \epsilon\}) \leq P(A_n),$$

for all n. Since $P(A_n) \to 0$, by the Squeeze Theorem, we have

$$P(\{|X1_{A_n} - 0|\} > \epsilon) \to 0.$$

Since ϵ was arbitrary, $X1_{A_n} \to 0$ in measure. Therefore, by the Dominated Convergence Theorem, we have

$$E[X1_{A_n}] \to 0,$$

as $n \to \infty$. $\quad\square$

Question 6.

Solution. As $c > 0$ and $\delta > 0$, by the Markov inequality, we have

$$
\begin{aligned}
P(|X| > \delta) &= P(e^{\lambda|X|} > e^{\lambda\delta}) \\
&\leq \frac{E[e^{\lambda|X|}]}{e^{\lambda\delta}}.
\end{aligned}
$$

As $e^{\lambda|X|} \leq e^{\lambda X} + e^{-\lambda X}$, it follows that

$$
\begin{aligned}
P(|X| > \delta) &\leq \frac{E[e^{\lambda X} + e^{-\lambda X}]}{e^{\lambda\delta}} \\
&\leq \frac{E[e^{\lambda X}] + E[e^{-\lambda X}]}{e^{\lambda\delta}}.
\end{aligned}
$$

From the given, we obtain that

$$
P(|X| > \delta) \leq 2e^{c\lambda^2/4 - \lambda\delta},
$$

for all $\lambda \in \mathbb{R}$. Since the bound is minimized at $\lambda = \dfrac{2\delta}{c}$, which can be reasoned through quadratic, we have that

$$
P(|X| > \delta) \leq 2e^{-\frac{\delta^2}{c}},
$$

as desired. $\quad\square$