Basic Probability:
Problem Set V

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Abstract

This work contains a collection of solutions for selected problems of the Basic Probability course of Fall 2015.

Question 1.
Solution. (ii) By definition of Poisson Distribution, we have

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} k \frac{1}{k!} \lambda^k e^{-\lambda} \\ &= \lambda e^{-\lambda} \sum_{k=0}^{\infty} k \frac{1}{k!} \lambda^{k-1} \\ &= \lambda, \end{aligned}$$

as expected. □

Question 2.
Solution. Let X be uniformly distributed on $[0, 1]$. Then, the distribution of X can be written as

$$F_X(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x & \text{for } x \in (0, 1) \\ 1 & \text{for } x \geq 1. \end{cases}$$

Now, consider the random variable $logX$. Then, the distribution of $logX$ can be written as

$$F_{logX}(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \\ log(x) & \text{for } x \in (0, e) \\ 1 & \text{for } x \geq e. \end{cases}$$

Question 3.
Solution. Let X be a standard Gaussian random variable. The density then can be written as

$$f_X(x) = \frac{1}{2\pi} e^{-x^2}.$$

Notice that for the $\frac{1}{X^2}$ random variable, the density can be written as

$$\begin{aligned} f_{\frac{1}{X^2}}(x) &= f_X\left(\frac{1}{\sqrt{x}}\right) + f_X\left(-\frac{1}{\sqrt{x}}\right) \\ &= \frac{1}{\pi} e^{\frac{1}{x}}, \end{aligned}$$

as desired. □

Question 4.

Solution. (i) We have $\Omega = \{H, T\}^N$, $\mathscr{A} = 2^\Omega$, and $\mathbb{P} : \mathscr{A} \to [0, 1]$ such that $\mathbb{P}(A) = \dfrac{|A|}{2^N}$, for any $A \in \mathscr{A}$.

(ii) Define A_n be a sequence of events such that the pattern (H, H, T, H, T, H, H) occurs with the n toss experiment, and $p(A_n)$ be the probability of A_n. Observe that we can identify an event L_n where (H, H, T, H, T, H, H) occurring from the 1st index to the 7th index with the n toss experiment precisely. The probability of such event is $\dfrac{2^{n-7}}{2^n}$ for $n \geq 7$. Observe that $\lim_{n\to\infty} p(L_n) = 1$. Since L_n is a subset of A_n, by the monotonicity property of probability measure, we have $p(L_n) \leq p(A_n) \leq 1$ for all n. Therefore, by the Squeeze Theorem, we have that $\lim_{n\to\infty} p(A_n) = 1$. $\square$