Putnam: Assignment I

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Abstract

Analysis problems

1 Solutions to the problems

Question 1.1. Convergence of an Improper Integral.

Solution. We wish to show that the given improper integral $\lim_{B\to\infty}\int_0^B sin(x)sin(x^2)dx$ is convergent. Observe that $\int_0^1 sin(x)sin(x^2)dx$ achieves a finite value, as the absolute values of the integrand over the interval $[0,1]$ is bounded by 1. Hence, it suffices to show that $\int_1^B sin(x)sin(x^2)dx$ is convergent. We integrate $\int_1^B sin(x)sin(x^2)dx$ by parts. To this end, we set $u = \frac{sin(x)}{2x}$ and $dv = sin(x^2)2xdx$, so that $du = \frac{1}{2}(\frac{cos(x)}{x} - \frac{sin(x)}{x^2})dx$ and $v = -cos(x^2)$. It follows that

$$\int_1^B sin(x)sin(x^2)dx \quad = \quad -\frac{sin(x)}{2x}cos(x^2)|_1^B + \frac{1}{2}\int_1^B cos(x^2)(\frac{cos(x)}{x} - \frac{sin(x)}{x^2})dx.$$

Simplifying the second integral, we obtain

$$\begin{aligned}\int_1^B sin(x)sin(x^2)dx \quad &= \quad -\frac{sin(x)}{2x}cos(x^2)|_1^B - \frac{1}{2}\int_1^B \frac{sin(x)cos(x^2)}{x^2})dx \\ &+ \quad \frac{1}{2}\int_1^B \frac{cos(x)cos(x^2)}{x}dx.\end{aligned}$$

As $B \to \infty$ we see that $-\frac{sin(x)}{2x}cos(x^2)|_1^B$ tends to 0 and the $\int_1^B \frac{sin(x)cos(x^2)}{x^2}dx$ can be shown to be absolutely convergent, thus convergent via the comparison test for improper integrals with $\frac{1}{2}\int_1^B \frac{1}{x^2}dx$, it only remains to show that $\int_1^B \frac{cos(x)(cos(x^2)}{x}dx$ term is convergent. We now integrate $\int_1^B \frac{cos(x)cos(x^2)}{x}dx$ by parts. We set $u = \frac{cos(x)}{2x^2}$ and $dv = cos(x^2)2xdx$, so that

$du = -\frac{1}{2}(\frac{sin(x)}{x^2} + \frac{2cos(x)}{x^3})dx$ and $v = sin(x^2)$. It follows that

$$\int_1^B \frac{cos(x)cos(x^2)}{x}dx = \frac{cos(x)}{2x^2}sin(x^2)|_1^B + \frac{1}{2}\int_1^B sin(x^2)(\frac{sin(x)}{x^2} + \frac{2cos(x)}{x^3})dx.$$

Simplifying the second integral, we have

$$\begin{aligned}\int_1^B \frac{cos(x)cos(x^2)}{x}dx &= \frac{cos(x)}{2x^2}sin(x^2)|_1^B + \frac{1}{2}\int_1^B \frac{sin(x^2)sin(x)}{x^2} \\ &+ \int_1^B \frac{sin(x^2)cos(x)}{x^3}dx.\end{aligned}$$

As $B \to \infty$, we see that $\frac{cos(x)}{2x^2}sin(x^2)|_1^B$ tends to 0, and both integrals can be shown to be absolutely convergent, thus convergent via the comparison test with $\int_1^B \frac{1}{x^2}dx$ and $\int_1^B \frac{1}{x^3}dx$ respectively. Therefore, we have proven that the improper integral $\lim_{B\to\infty}\int_0^B sin(x)sin(x^2)dx$ is convergent. $\square$

Question.

Solution. We want to show that, for all integers $n > 1$, the following inequality holds:

$$\frac{1}{2ne} < \frac{1}{e} - (1 - \frac{1}{n})^n < \frac{1}{ne}.$$

Multiplying by e and subtracting 1 from all sides of the inequality we can rewrite the inequality as

$$\frac{1}{2n} - 1 < -e(1 - \frac{1}{n})^n < \frac{1}{n} - 1.$$

Multiplying by -1 and taking the log to all sides, we can again rewrite the inequality as

$$log(1 - \frac{1}{2n}) > -1 + nlog(1 - \frac{1}{n}) > log(1 - \frac{1}{n}).$$

Now, we can rewrite the above inequality with the terms of the Taylor expansion of $log(1 - x)$ as

$$-\sum_{i=1}^{\infty}\frac{1}{i2^in^i} > -\sum_{i=1}^{\infty}\frac{1}{(i+1)n^i} > -\sum_{i=1}^{\infty}\frac{1}{in^i},$$

which can be seen to hold as the inequality holds term by term. $\square$