Putnam: Assignment II

Youngduck Choi
Courant Institute of Mathematical Sciences
New York University
yc1104@nyu.edu

Abstract

Analysis problems

1 Solutions to the problems

Question.
Solution. We wish to find the minimum value of the given expression:

$$\frac{(x+\frac{1}{x})^6-(x^6+\frac{1}{x^6})-2}{(x+\frac{1}{x})^3+(x^3+\frac{1}{x^3})},$$

where $x > 0$. Observe that the following identity which follows from the difference of squares:

$$\begin{aligned}((x+\frac{1}{x})^3+(x^3+\frac{1}{x^3}))((x+\frac{1}{x})^3-(x^3+\frac{1}{x^3})) &= (x+\frac{1}{x})^6-(x^3+\frac{1}{x^3})^2 \\ &= (x+\frac{1}{x})^6-(x^6+\frac{1}{x^6})-2.\end{aligned}$$

Dividing the both side of the identity by $(x+\frac{1}{x})^3+(x^3+\frac{1}{x^3})$, we see that the given expression is equivalent to:

$$(x+\frac{1}{x})^3-(x^3+\frac{1}{x^3}),$$

which can be further simplified to $3(x+\frac{1}{x})$. As $x > 0$, by the AM-GM inequality we see that the minimum happens at $x = 1$, thereby showing that the minimum of the given expression with $x > 0$ is 3. $\square$

Question.
Solution. We wish to show that $(\prod_{i=1}^n a_i)^{\frac{1}{n}} + (\prod_{i=1}^n b_i)^{\frac{1}{n}} \le (\prod_{i=1}^n (a_i+b_i))^{\frac{1}{n}}$ Observe that the AM-GM inequality yields

$$\begin{aligned}(\prod_{i=1}^n \frac{a_i}{a_i+b_i})^{\frac{1}{n}} &\le \frac{1}{n}\prod_{i=1}^n \frac{a_i}{a_i+b_i}, \\ (\prod_{i=1}^n \frac{b_i}{a_i+b_i})^{\frac{1}{n}} &\le \frac{1}{n}\prod_{i=1}^n \frac{b_i}{a_i+b_i}.\end{aligned}$$

Adding the two inequalities, we obtain

$$(\prod_{i=1}^n \frac{a_i}{a_i+b_i})^{\frac{1}{n}} + (\prod_{i=1}^n \frac{b_i}{a_i+b_i})^{\frac{1}{n}} \le \frac{1}{n}\prod_{i=1}^n \frac{a_i+b_i}{a_i+b_i}.$$

We observe that the RHS of the inequality simplifies to 1. Multiplying the both sides by $(a_i+b_i)^{\frac{1}{n}}$, we get

$$(\prod_{i=1}^{n} a_i)^{\frac{1}{n}} + (\prod_{i=1}^{n} b_i)^{\frac{1}{n}} \leq (\prod_{i=1}^{n}(a_i+b_i))^{\frac{1}{n}},$$

as desired. $\square$

Question.

Solution. We wish to integrate $\int_2^4 \frac{\sqrt{ln(9-x)}dx}{\sqrt{ln(9-x)}+\sqrt{ln(x+3)}}$. We can rewrite the given integral as

$$\int_{-1}^{1} \frac{\sqrt{ln(6-x)}dx}{\sqrt{ln(6-x)}+\sqrt{ln(x+6)}}.$$

We can now separate the integral to two parts as

$$\int_{0}^{1} \frac{\sqrt{ln(6-x)}dx}{\sqrt{ln(6-x)}+\sqrt{ln(x+6)}} + \int_{-1}^{0} \frac{\sqrt{ln(6-x)}dx}{\sqrt{ln(6-x)}+\sqrt{ln(x+6)}}.$$

We now use the substitution $u = -x$, so that $du = -dx$ on the second integral. The integral becomes

$$\int_{0}^{1} \frac{\sqrt{ln(6-x)}dx}{\sqrt{ln(6-x)}+\sqrt{ln(x+6)}} + \int_{0}^{1} \frac{\sqrt{ln(x+6)}dx}{\sqrt{ln(6+x)}+\sqrt{ln(x-6)}}.$$

which can be simplifed to $\int_0^1 1dx = 1$. $\square$

Question.

Solution. We want to show the following inequality:

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

such that m and n are positive integers. Multiplying both sides by $\frac{(m+n)^{m+n}m^m n^n}{m!n!}$, we see that the given inequality is equivalent to

$$\frac{m^m n^n (m+n)!}{m!n!} < (m+n)^{m+n}.$$

We can rewrite the inequality in terms of combinations as

$$\binom{m+n}{m} m^m n^n < (m+n)^{m+n}.$$

We can see from the binomial identity that the RHS contains the LHS term as one of its terms in the binomial expansion. As m and n are restricted to be positive integers, the RHS will always contain another term with the $\binom{m+n}{0}$ coffecient, hence acheiving a strict inequality. Hence, we have shown that the given equality holds.

Question 1999-A3.

Solution.