
Putnam: Assignment I

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Abstract

Analysis problems

1 Solutions to the problems

Question 1.1. Convergence of an Improper Integral.

Solution. We wish to show that the given improper integral $\lim_{B \rightarrow \infty} \int_0^B \sin(x) \sin(x^2) dx$ is convergent. Observe that $\int_0^1 \sin(x) \sin(x^2) dx$ achieves a finite value, as the absolute values of the integrand over the interval $[0, 1]$ is bounded by 1. Hence, it suffices to show that $\int_1^B \sin(x) \sin(x^2) dx$ is convergent. We integrate $\int_1^B \sin(x) \sin(x^2) dx$ by parts. To this end, we set $u = \frac{\sin(x)}{2x}$ and $dv = \sin(x^2) 2x dx$, so that $du = \frac{1}{2} \left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \right) dx$ and $v = -\cos(x^2)$. It follows that

$$\int_1^B \sin(x) \sin(x^2) dx = -\frac{\sin(x)}{2x} \cos(x^2) \Big|_1^B + \frac{1}{2} \int_1^B \cos(x^2) \left(\frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \right) dx.$$

Simplifying the second integral, we obtain

$$\begin{aligned} \int_1^B \sin(x) \sin(x^2) dx &= -\frac{\sin(x)}{2x} \cos(x^2) \Big|_1^B - \frac{1}{2} \int_1^B \frac{\sin(x) \cos(x^2)}{x^2} dx \\ &\quad + \frac{1}{2} \int_1^B \frac{\cos(x) \cos(x^2)}{x} dx. \end{aligned}$$

As $B \rightarrow \infty$ we see that $-\frac{\sin(x)}{2x} \cos(x^2) \Big|_1^B$ tends to 0 and the $\int_1^B \frac{\sin(x) \cos(x^2)}{x^2} dx$ can be shown to be absolutely convergent, thus convergent via the comparison test for improper integrals with $\frac{1}{2} \int_1^B \frac{1}{x^2} dx$, it only remains to show that $\int_1^B \frac{\cos(x) \cos(x^2)}{x} dx$ term is convergent. We now integrate $\int_1^B \frac{\cos(x) \cos(x^2)}{x} dx$ by parts. We set $u = \frac{\cos(x)}{2x^2}$ and $dv = \cos(x^2) 2x dx$, so that

$du = -\frac{1}{2}(\frac{\sin(x)}{x^2} + \frac{2\cos(x)}{x^3})dx$ and $v = \sin(x^2)$. It follows that

$$\int_1^B \frac{\cos(x)\cos(x^2)}{x} dx = \frac{\cos(x)}{2x^2} \sin(x^2) \Big|_1^B + \frac{1}{2} \int_1^B \sin(x^2) \left(\frac{\sin(x)}{x^2} + \frac{2\cos(x)}{x^3} \right) dx.$$

Simplifying the second integral, we have

$$\begin{aligned} \int_1^B \frac{\cos(x)\cos(x^2)}{x} dx &= \frac{\cos(x)}{2x^2} \sin(x^2) \Big|_1^B + \frac{1}{2} \int_1^B \frac{\sin(x^2)\sin(x)}{x^2} \\ &\quad + \int_1^B \frac{\sin(x^2)\cos(x)}{x^3} dx. \end{aligned}$$

As $B \rightarrow \infty$, we see that $\frac{\cos(x)}{2x^2} \sin(x^2) \Big|_1^B$ tends to 0, and both integrals can be shown to be absolutely convergent, thus convergent via the comparison test with $\int_1^B \frac{1}{x^2} dx$ and $\int_1^B \frac{1}{x^3} dx$ respectively. Therefore, we have proven that the improper integral $\lim_{B \rightarrow \infty} \int_0^B \sin(x)\sin(x^2) dx$ is convergent. \square

Question.

Solution. We want to show that, for all integers $n > 1$, the following inequality holds:

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

Multiplying by e and subtracting 1 from all sides of the inequality we can rewrite the inequality as

$$\frac{1}{2n} - 1 < -e\left(1 - \frac{1}{n}\right)^n < \frac{1}{n} - 1.$$

Multiplying by -1 and taking the log to all sides, we can again rewrite the inequality as

$$\log\left(1 - \frac{1}{2n}\right) > -1 + n\log\left(1 - \frac{1}{n}\right) > \log\left(1 - \frac{1}{n}\right).$$

Now, we can rewrite the above inequality with the terms of the Taylor expansion of $\log(1 - x)$ as

$$-\sum_{i=1}^{\infty} \frac{1}{i2^i n^i} > -\sum_{i=1}^{\infty} \frac{1}{(i+1)n^i} > -\sum_{i=1}^{\infty} \frac{1}{in^i},$$

which can be seen to hold as the inequality holds term by term. \square