Improved Taylor Expansion for Wilson Fermions

The Wilson Dirac operator with chemical potential can be written as

$$D_{ij}[U](\mu) = (m+4)\delta_{ij} - \sum_{\nu=1}^{4}\left(U_\nu(i)\frac{1-\gamma_\nu}{2}\delta_{i+\hat{\nu},j} + U_\nu^\dagger(i-\hat{\nu})\frac{1+\gamma_\nu}{2}\delta_{i-\hat{\nu},j}\right) - \rho M_{ij} - \overline{\rho}\,\overline{M}_{ij}$$

The parameters ρ and $\overline{\rho}$ (that will be used later as expansion parameters) and the matrices M and $\overline{M}$ can be defined in two different ways:

- **Expansion 1**

  $$\rho = e^{\mu} - 1\ ; \overline{\rho} = e^{-\mu} - 1\ .$$

  The matrices M and $\overline{M}$ are:

  $$M_{ij} = U_4(i)\frac{1-\gamma_4}{2}\delta_{i+\hat{4},j}$$

  $$\overline{M}_{ij} = U_4^\dagger(i-\hat{4})\frac{1+\gamma_4}{2}\delta_{i-\hat{4},j}$$

- **Expansion 2**

  $$\rho = e^{N_T\mu} - 1\ ; \overline{\rho} = e^{-N_T\mu} - 1\ .$$

  The matrices M and $\overline{M}$ are:

  $$M_{ij} = \frac{1-\gamma_4}{2}\delta_{\overline{i},\overline{j}}\delta_{i_4,N_T}\delta_{j_4,1}$$

  $$\overline{M}_{ij} = \frac{1+\gamma_4}{2}\delta_{\overline{i},\overline{j}}\delta_{i_4,1}\delta_{j_4,N_T}$$

The partition sum is

$$Z(\mu) = \int D[\psi,\overline{\psi},U]e^{-S_G-S_F+R} \tag{1}$$

where S_G is the usual Wilson plaquette action, S_F is the fermionic action and the quantity R is defined as $\rho\overline{\psi}M\psi + \overline{\rho}\overline{\psi}\overline{M}\psi$.

e^R is complex and cannot be used as part of the probability weights. However, it is possible to perform a Taylor expansion of the exponential for small values of ρ and $\overline{\rho}$ and compute all expectation values at zero chemical potential. We obtain the following expresion for the partition sum (up to fourth order or quantities with four propagators):

$$Z(\mu) \approx Z(0)\left[1 + \langle R\rangle + \frac{1}{2}\langle R^2\rangle + \frac{1}{6}\langle R^3\rangle + \frac{1}{24}\langle R^4\rangle\right] \tag{2}$$

To compute the observables we need the logarithm of the partition sum. Therefore, we also expand the logarithm up to fourth order (in R or/and number of propagators)

$$\ln Z(\mu) \approx \ln Z(0) + \langle R\rangle + \frac{1}{2}\langle R^2\rangle - \frac{1}{2}\langle R\rangle^2 + \frac{1}{6}\langle R^3\rangle - \frac{1}{2}\langle R\rangle\langle R^2\rangle + \frac{1}{3}\langle R\rangle^3$$
$$-\frac{1}{8}\langle R^2\rangle^2 - \frac{1}{6}\langle R\rangle\langle R^3\rangle + \frac{1}{2}\langle R\rangle^2\langle R^2\rangle - \frac{1}{4}\langle R\rangle^4 + \frac{1}{24}\langle R^4\rangle \quad (3)$$

We will compute the first and second derivatives with respect to the chemical potential and the mass for the expansions 1 and 2. In the following I will write down explicitly the expressions only for the expansion 1. It is straight forward to get the expressions for expansion 2 (replace ρ and $\overline{\rho}$ and multiply the derivatives of R with respect to the chemical potential by N_T).

Derivatives with respect to μ

We need the following derivatives:

$$\frac{\partial R}{\partial \mu} = e^{\mu}\overline{\psi}M\psi - e^{-\mu}\overline{\psi M}\psi := D_- \quad (4)$$
$$\frac{\partial^2 R}{\partial \mu^2} = e^{\mu}\overline{\psi}M\psi + e^{-\mu}\overline{\psi M}\psi := D_+ \quad (5)$$

We compute two observables, the particle number n and its susceptibility χ_n. The particle number up to fourth order (in number of propagators) is given by:

$$\begin{aligned} n = \frac{\partial \ln Z}{\partial \mu} = {} & \langle D_-\rangle + \langle RD_-\rangle - \langle R\rangle\langle D_-\rangle + \frac{1}{2}\langle R^2 D_-\rangle - \frac{1}{2}\langle D_-\rangle\langle R^2\rangle - \langle R\rangle\langle RD_-\rangle \\ & +\langle R\rangle^2\langle D_-\rangle + \frac{1}{6}\langle R^3 D_-\rangle - \frac{1}{2}\langle R^2\rangle\langle RD_-\rangle - \frac{1}{6}\langle D_-\rangle\langle R^3\rangle - \frac{1}{2}\langle R\rangle\langle R^2 D_-\rangle \\ & +\langle R\rangle\langle D_-\rangle\langle R^2\rangle + \langle R\rangle^2\langle RD_-\rangle - \langle R\rangle^3\langle D_-\rangle \end{aligned} \quad (6)$$

The susceptibility

$$\begin{aligned} \chi_n = \frac{\partial^2 \ln Z}{\partial \mu^2} = {} & \langle D_+\rangle + \langle D_-^2\rangle + \langle RD_+\rangle - \langle D_-\rangle^2 - \langle R\rangle\langle D_+\rangle + \langle RD_-^2\rangle + \frac{1}{2}\langle R^2 D_+\rangle \\ & -\frac{1}{2}\langle D_+\rangle\langle R^2\rangle - 2\langle D_-\rangle\langle RD_-\rangle - \langle R\rangle\langle D_-^2\rangle - \langle R\rangle\langle RD_+\rangle + 2\langle R\rangle\langle D_-\rangle^2 \\ & +\langle R\rangle^2\langle D_+\rangle + \frac{1}{2}\langle R^2 D_-^2\rangle + \frac{1}{6}\langle R^3 D_+\rangle - \langle RD_-\rangle^2 - \frac{1}{2}\langle R^2\rangle\langle D_-^2\rangle - \frac{1}{2}\langle R^2\rangle\langle RD_+\rangle \\ & -\frac{1}{6}\langle D_+\rangle\langle R^3\rangle - \langle D_-\rangle\langle R^2 D_-\rangle - \langle R\rangle\langle RD_-^2\rangle - \frac{1}{2}\langle R\rangle\langle R^2 D_+\rangle + \langle D_-\rangle^2\langle R^2\rangle \\ & +\langle R\rangle\langle D_+\rangle\langle R^2\rangle + 4\langle R\rangle\langle D_-\rangle\langle RD_-\rangle + \langle R\rangle^2\langle D_-^2\rangle + \langle R\rangle^2\langle RD_+\rangle \\ & -3\langle R\rangle^2\langle D_-\rangle^2 - \langle R\rangle^3\langle D_+\rangle \end{aligned} \quad (7)$$

Derivatives with respect to m

We need the following derivative

$$
\begin{aligned}
\frac{\partial\langle O\rangle}{\partial m} &= \frac{\partial}{\partial m}\frac{1}{Z_0}\int D[\psi,\overline{\psi},U]\exp(-S_G-\overline{\psi}_i D_{ij}\psi_j)O \\
&= \langle\psi_i\overline{\psi}_i O\rangle - \langle\psi_i\overline{\psi}_i\rangle\langle O\rangle \qquad (8)
\end{aligned}
$$

We compute the observables $\partial \ln Z/\partial m$ (defining $\Psi = \psi_i\overline{\psi}_i$)

$$
\begin{aligned}
\frac{\partial \ln Z}{\partial m} &= \langle\Psi\rangle - \langle\Psi\rangle\langle R\rangle + \langle\Psi R\rangle - \frac{1}{2}\langle\Psi\rangle\langle R^2\rangle + \frac{1}{2}\langle\Psi R^2\rangle + \langle R\rangle^2\langle\Psi\rangle \\
&\quad -\langle R\rangle\langle\Psi R\rangle - \frac{1}{6}\langle\Psi\rangle\langle R^3\rangle + \frac{1}{6}\langle\Psi R^3\rangle + \langle R^2\rangle\langle\Psi\rangle\langle R\rangle - \frac{1}{2}\langle R^2\rangle\langle\Psi R\rangle \\
&\quad -\frac{1}{2}\langle R\rangle\langle\Psi R^2\rangle - \langle R^3\rangle\langle\Psi\rangle + \langle R\rangle^2\langle\Psi R\rangle \qquad (9)
\end{aligned}
$$

and its susceptibility

$$
\begin{aligned}
\frac{\partial^2 \ln Z}{\partial m^2} &= \langle\Psi^2\rangle - \langle\Psi\rangle^2 + 2\langle\Psi\rangle^2\langle R\rangle - \langle\Psi^2\rangle\langle R\rangle - 2\langle\Psi\rangle\langle\Psi R\rangle + \langle\Psi^2 R\rangle \\
&\quad +\langle\Psi\rangle^2\langle R^2\rangle - \frac{1}{2}\langle\Psi^2\rangle\langle R^2\rangle - \langle\Psi\rangle\langle\Psi R^2\rangle + \frac{1}{2}\langle\Psi^2 R^2\rangle - 3\langle\Psi\rangle^2\langle R\rangle^2 \\
&\quad +4\langle\Psi\rangle\langle R\rangle\langle\Psi R\rangle + \langle R\rangle^2\langle\Psi^2\rangle - \langle\Psi R\rangle^2 - \langle R\rangle\langle\Psi^2 R\rangle \qquad (10)
\end{aligned}
$$

Explicit expressions for each expectation value

Terms with one propagator

To compute the observables we need the following expectation values:

$$
\begin{aligned}
\langle R\rangle &= -\rho\langle M_{ij}\overline{\psi}_j\psi_i\rangle - \overline{\rho}\langle\overline{M}_{ij}\overline{\psi}_j\psi_i\rangle \qquad (11) \\
\langle D_-\rangle &= -e^{\mu}\langle M_{ij}\overline{\psi}_j\psi_i\rangle + e^{-\mu}\langle\overline{M}_{ij}\overline{\psi}_j\psi_i\rangle \qquad (12) \\
\langle D_+\rangle &= -e^{\mu}\langle M_{ij}\overline{\psi}_j\psi_i\rangle - e^{-\mu}\langle\overline{M}_{ij}\overline{\psi}_j\psi_i\rangle \qquad (13) \\
\langle\Psi\rangle &= -\langle\overline{\psi}_i\psi_i\rangle \qquad (14)
\end{aligned}
$$

From the Wick theorem

$$\overline{\psi}_j\psi_i = D^{-1}_{ji}$$

Then

$$\langle M_{ij}\overline{\psi}_j\psi_i\rangle = Tr[MD^{-1}]$$

$$\langle M_{ij}\overline{\psi}_j\psi_i\rangle = Tr[\overline{M}D^{-1}]$$

$$\langle\overline{\psi}_i\psi_i\rangle = -Tr[D^{-1}]$$

Terms with two propagators

To compute the observables we need the following expectation values:

$$\begin{aligned}\langle R^2\rangle &= \rho^2\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle + 2\overline{\rho}\rho\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \\ &\quad +\overline{\rho}^2\langle\overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \end{aligned} \tag{15}$$

$$\begin{aligned}\langle D_-^2\rangle &= e^{2\mu}\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle - 2\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \\ &\quad +e^{-2\mu}\langle\overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \end{aligned} \tag{16}$$

$$\begin{aligned}\langle D_+^2\rangle &= e^{2\mu}\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle + 2\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \\ &\quad +e^{-2\mu}\langle\overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \end{aligned} \tag{17}$$

$$\begin{aligned}\langle RD_-\rangle &= \rho e^{\mu}\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle + (\overline{\rho}e^{\mu} - \rho e^{-\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \\ &\quad -\overline{\rho}e^{-\mu}\langle\overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \end{aligned} \tag{18}$$

$$\begin{aligned}\langle RD_+\rangle &= \rho e^{\mu}\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle + (\rho e^{-\mu} + \overline{\rho}e^{\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \\ &\quad +\overline{\rho}e^{-\mu}\langle\overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle \end{aligned} \tag{19}$$

$$\langle\Psi^2\rangle = \langle\overline{\psi}\psi_{i_1i_1i_2i_2}\rangle \tag{20}$$

$$\langle\Psi R\rangle = -\rho\langle M_{i_2j_2}\overline{\psi}\psi_{i_1i_1j_2i_2}\rangle - \overline{\rho}\langle\overline{M}_{i_2j_2}\overline{\psi}\psi_{i_1i_1j_2i_2}\rangle \tag{21}$$

From the Wick theorem

$$\overline{\psi}\psi_{j_1i_1j_2i_2} = D^{-1}_{j_1i_1}D^{-1}_{j_2i_2} - D^{-1}_{j_1i_2}D^{-1}_{j_2i_1}$$

Then

$$\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle = Tr[MD^{-1}]^2 - Tr[MD^{-1}MD^{-1}]$$

$$\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle = Tr[MD^{-1}]Tr[\overline{M}D^{-1}] - Tr[MD^{-1}\overline{M}D^{-1}]$$

$$\langle\overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1j_2i_2}\rangle = Tr[\overline{M}D^{-1}]^2 - Tr[\overline{M}D^{-1}\overline{M}D^{-1}]$$

$$\langle M_{i_2j_2}\overline{\psi}\psi_{i_1i_1j_2i_2}\rangle = Tr[D^{-1}]Tr[MD^{-1}] - Tr[D^{-1}MD^{-1}]$$

$$\langle\overline{M}_{i_2j_2}\overline{\psi}\psi_{i_1i_1j_2i_2}\rangle = Tr[D^{-1}]Tr[\overline{M}D^{-1}] - Tr[D^{-1}\overline{M}D^{-1}]$$

$$\langle\overline{\psi}\psi_{i_1i_1i_2i_2}\rangle = Tr[D^{-1}]^2 - Tr[D^{-1}D^{-1}]$$

Terms with three propagators

To compute the observables we need the following expectation values:

$$\begin{aligned}\langle R^3\rangle &= -\rho^3\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle - \overline{\rho}^3\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad -3\rho^2\overline{\rho}\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad -3\overline{\rho}^2\rho\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \end{aligned} \tag{22}$$

$$\begin{aligned}\langle R^2D_-\rangle &= -\rho^2e^{\mu}\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad +\overline{\rho}^2e^{-\mu}\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad +(e^{-\mu}\rho^2-2\rho\overline{\rho}e^{\mu})\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad +(-e^{\mu}\overline{\rho}^2+2\rho\overline{\rho}e^{-\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \end{aligned} \tag{23}$$

$$\begin{aligned}\langle R^2D_+\rangle &= -\rho^2e^{\mu}\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad -\overline{\rho}^2e^{-\mu}\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad -(e^{-\mu}\rho^2+2\rho\overline{\rho}e^{\mu})\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad -(e^{\mu}\overline{\rho}^2+2\rho\overline{\rho}e^{-\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \end{aligned} \tag{24}$$

$$\begin{aligned}\langle RD_-^2\rangle &= -\rho e^{2\mu}\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad -\overline{\rho}e^{-2\mu}\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad +(2\rho-\overline{\rho}e^{2\mu})\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \\ &\quad +(2\overline{\rho}-\rho e^{-2\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3}\rangle \end{aligned} \tag{25}$$

$$\begin{aligned}\langle \Psi R^2\rangle &= \rho^2\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1\ldots i_3i_3}\rangle \\ &\quad +\overline{\rho}\rho\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1\ldots i_3i_3}\rangle+\overline{\rho}\rho\langle \overline{M}_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1\ldots i_3i_3}\rangle \\ &\quad +\overline{\rho}^2\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1\ldots i_3i_3}\rangle \end{aligned} \tag{26}$$

$$\langle \Psi^2 R\rangle = -\rho\langle M_{i_1j_1}\overline{\psi}\psi_{j_1i_1\ldots i_3i_3}\rangle - \overline{\rho}\langle \overline{M}_{i_1j_1}\overline{\psi}\psi_{j_1i_1\ldots i_3i_3}\rangle \tag{27}$$

From the Wick theorem

$$\begin{aligned}\overline{\psi}\psi_{j_1i_1\ldots j_3i_3} &= D^{-1}_{j_1i_1}D^{-1}_{j_2i_2}D^{-1}_{j_3i_3}+D^{-1}_{j_1i_2}D^{-1}_{j_2i_3}D^{-1}_{j_3i_1}+D^{-1}_{j_1i_3}D^{-1}_{j_2i_1}D^{-1}_{j_3i_2} \\ &\quad -D^{-1}_{j_1i_3}D^{-1}_{j_2i_2}D^{-1}_{j_3i_1}-D^{-1}_{j_1i_1}D^{-1}_{j_2i_3}D^{-1}_{j_3i_2}-D^{-1}_{j_1i_2}D^{-1}_{j_2i_1}D^{-1}_{j_3i_3}\end{aligned}$$

Then

$$
\begin{aligned}
\langle M_{i_1 j_1} M_{i_2 j_2} M_{i_3 j_3} \overline{\psi} \psi_{j_1 i_1 \ldots j_3 i_3} \rangle &= Tr[MD^{-1}]^3 - 3Tr[MD^{-1}]Tr[(MD^{-1})^2] + 2Tr[(MD^{-1})^3] \\
\langle M_{i_1 j_1} M_{i_2 j_2} \overline{M}_{i_3 j_3} \overline{\psi} \psi_{j_1 i_1 \ldots j_3 i_3} \rangle &= Tr[MD^{-1}]^2 Tr[\overline{M}D^{-1}] - Tr[\overline{M}D^{-1}]Tr[(MD^{-1})^2] \\
&\quad -2Tr[MD^{-1}]Tr[MD^{-1}\overline{M}D^{-1}] + 2Tr[\overline{M}D^{-1}(MD^{-1})^2] \\
\langle M_{i_1 j_1} \overline{M}_{i_2 j_2} \overline{M}_{i_3 j_3} \overline{\psi} \psi_{j_1 i_1 \ldots j_3 i_3} \rangle &= Tr[\overline{M}D^{-1}]^2 Tr[MD^{-1}] - Tr[MD^{-1}]Tr[(\overline{M}D^{-1})^2] \\
&\quad -2Tr[\overline{M}D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}] + 2Tr[MD^{-1}(\overline{M}D^{-1})^2] \\
\langle \overline{M}_{i_1 j_1} \overline{M}_{i_2 j_2} \overline{M}_{i_3 j_3} \overline{\psi} \psi_{j_1 i_1 \ldots j_3 i_3} \rangle &= Tr[\overline{M}D^{-1}]^3 - 3Tr[\overline{M}D^{-1}]Tr[(\overline{M}D^{-1})^2] + 2Tr[(\overline{M}D^{-1})^3] \\
\langle M_{i_1 j_1} M_{i_2 j_2} \overline{\psi} \psi_{j_1 i_1 \ldots i_3 i_3} \rangle &= Tr[MD^{-1}]^2 Tr[D^{-1}] - Tr[D^{-1}]Tr[(MD^{-1})^2] \\
&\quad -2Tr[MD^{-1}]Tr[D^{-1}MD^{-1}] + 2Tr[D^{-1}(MD^{-1})^2] \\
\langle M_{i_1 j_1} \overline{M}_{i_2 j_2} \overline{\psi} \psi_{j_1 i_1 \ldots i_3 i_3} \rangle &= Tr[D^{-1}]Tr[MD^{-1}]Tr[\overline{M}D^{-1}] - Tr[D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}] \\
&\quad -Tr[MD^{-1}]Tr[D^{-1}\overline{M}D^{-1}] - Tr[\overline{M}D^{-1}]Tr[D^{-1}MD^{-1}] \\
&\quad +2Tr[D^{-1}MD^{-1}\overline{M}D^{-1}] \\
\langle \overline{M}_{i_1 j_1} M_{i_2 j_2} \overline{\psi} \psi_{j_1 i_1 \ldots i_3 i_3} \rangle &= Tr[D^{-1}]Tr[MD^{-1}]Tr[\overline{M}D^{-1}] - Tr[D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}] \\
&\quad -Tr[MD^{-1}]Tr[D^{-1}\overline{M}D^{-1}] - Tr[\overline{M}D^{-1}]Tr[D^{-1}MD^{-1}] \\
&\quad +2Tr[D^{-1}\overline{M}D^{-1}MD^{-1}] \\
\langle \overline{M}_{i_1 j_1} \overline{M}_{i_2 j_2} \overline{\psi} \psi_{j_1 i_1 \ldots i_3 i_3} \rangle &= Tr[\overline{M}D^{-1}]^2 Tr[D^{-1}] - Tr[D^{-1}]Tr[(\overline{M}D^{-1})^2] \\
&\quad -2Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}] + 2Tr[D^{-1}(\overline{M}D^{-1})^2] \\
\langle M_{i_1 j_1} \overline{\psi} \psi_{j_1 i_1 \ldots i_3 i_3} \rangle &= Tr[MD^{-1}]Tr[D^{-1}]^2 - Tr[MD^{-1}]Tr[D^{-1}D^{-1}] \\
&\quad -2Tr[D^{-1}]Tr[D^{-1}MD^{-1}] + 2Tr[D^{-1}MD^{-1}D^{-1}] \\
\langle \overline{M}_{i_1 j_1} \overline{\psi} \psi_{j_1 i_1 \ldots i_3 i_3} \rangle &= Tr[\overline{M}D^{-1}]Tr[D^{-1}]^2 - Tr[\overline{M}D^{-1}]Tr[D^{-1}D^{-1}] \\
&\quad -2Tr[D^{-1}]Tr[D^{-1}\overline{M}D^{-1}] + 2Tr[D^{-1}\overline{M}D^{-1}D^{-1}]
\end{aligned}
$$

Terms with four propagators

To compute the observables we need the following expectation values:

$$\begin{aligned}
\langle R^3 D_- \rangle &= \rho^3 e^{\mu}\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}M_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+(3\overline{\rho}\rho^2 e^{\mu}-\rho^3 e^{-\mu})\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+2(\rho\overline{\rho}^2 e^{\mu}-\overline{\rho}\rho^2 e^{-\mu})\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+(\rho\overline{\rho}^2 e^{\mu}-\overline{\rho}\rho^2 e^{-\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}M_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+(\overline{\rho}^3 e^{\mu}-3\rho\overline{\rho}^2 e^{-\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&-\overline{\rho}^3 e^{-\mu}\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle
\end{aligned} \tag{28}$$

$$\begin{aligned}
\langle R^3 D_+ \rangle &= \rho^3 e^{\mu}\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}M_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+(3\overline{\rho}\rho^2 e^{\mu}+\rho^3 e^{-\mu})\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+2(\rho\overline{\rho}^2 e^{\mu}+\overline{\rho}\rho^2 e^{-\mu})\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+(\rho\overline{\rho}^2 e^{\mu}+\overline{\rho}\rho^2 e^{-\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}M_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+(\overline{\rho}^3 e^{\mu}+3\rho\overline{\rho}^2 e^{-\mu})\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+\overline{\rho}^3 e^{-\mu}\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle
\end{aligned} \tag{29}$$

$$\begin{aligned}
\langle R^2 D_-^2 \rangle &= \rho^2 e^{2\mu}\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}M_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+2(\overline{\rho}\rho e^{2\mu}-\rho^2)\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+(\rho^2 e^{-2\mu}+\overline{\rho}^2 e^{2\mu}-2\rho\overline{\rho})\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&-2\rho\overline{\rho}\langle M_{i_1j_1}\overline{M}_{i_2j_2}M_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+2(\rho\overline{\rho}e^{-2\mu}-\overline{\rho}^2)\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle \\
&+\overline{\rho}^2 e^{-2\mu}\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\dots j_4i_4}\rangle
\end{aligned} \tag{30}$$

$$\begin{aligned}
\langle \Psi^2 R^2 \rangle &= \rho^2\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle+\overline{\rho}\rho\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle \\
&+\rho\overline{\rho}\langle \overline{M}_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle+\overline{\rho}^2\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle
\end{aligned} \tag{31}$$

$$\begin{aligned}
\langle \Psi R^3 \rangle &= -\rho^3\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle-\rho^2\overline{\rho}\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle \\
&-\rho^2\overline{\rho}\langle M_{i_1j_1}\overline{M}_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle-\rho^2\overline{\rho}\langle \overline{M}_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle \\
&-\overline{\rho}^2\rho\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle-\overline{\rho}^2\rho\langle \overline{M}_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle \\
&-\overline{\rho}^2\rho\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle-\overline{\rho}^3\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\dots i_4i_4}\rangle
\end{aligned} \tag{32}$$

From the Wick theorem

$$\begin{aligned}
\overline{\psi}\psi_{j_1i_1\dots j_4i_4} &= D^{-1}_{j_1i_1}D^{-1}_{j_2i_2}D^{-1}_{j_3i_3}D^{-1}_{j_4i_4}+D^{-1}_{j_1i_1}D^{-1}_{j_2i_3}D^{-1}_{j_3i_4}D^{-1}_{j_4i_2}+D^{-1}_{j_1i_1}D^{-1}_{j_2i_4}D^{-1}_{j_3i_2}D^{-1}_{j_4i_3} \\
&+D^{-1}_{j_1i_3}D^{-1}_{j_2i_2}D^{-1}_{j_3i_4}D^{-1}_{j_4i_1}+D^{-1}_{j_1i_4}D^{-1}_{j_2i_2}D^{-1}_{j_3i_1}D^{-1}_{j_4i_3}+D^{-1}_{j_1i_2}D^{-1}_{j_2i_4}D^{-1}_{j_3i_3}D^{-1}_{j_4i_1} \\
&+D^{-1}_{j_1i_4}D^{-1}_{j_2i_1}D^{-1}_{j_3i_3}D^{-1}_{j_4i_2}+D^{-1}_{j_1i_2}D^{-1}_{j_2i_3}D^{-1}_{j_3i_1}D^{-1}_{j_4i_4}+D^{-1}_{j_1i_3}D^{-1}_{j_2i_1}D^{-1}_{j_3i_2}D^{-1}_{j_4i_4} \\
&-D^{-1}_{j_1i_1}D^{-1}_{j_2i_2}D^{-1}_{j_3i_4}D^{-1}_{j_4i_3}-D^{-1}_{j_1i_1}D^{-1}_{j_2i_4}D^{-1}_{j_3i_3}D^{-1}_{j_4i_2}-D^{-1}_{j_1i_1}D^{-1}_{j_2i_3}D^{-1}_{j_3i_2}D^{-1}_{j_4i_4} \\
&-D^{-1}_{j_1i_4}D^{-1}_{j_2i_2}D^{-1}_{j_3i_3}D^{-1}_{j_4i_1}-D^{-1}_{j_1i_3}D^{-1}_{j_2i_2}D^{-1}_{j_3i_1}D^{-1}_{j_4i_4}-D^{-1}_{j_1i_2}D^{-1}_{j_2i_1}D^{-1}_{j_3i_3}D^{-1}_{j_4i_4} \\
&+D^{-1}_{j_1i_2}D^{-1}_{j_2i_1}D^{-1}_{j_3i_4}D^{-1}_{j_4i_3}+D^{-1}_{j_1i_3}D^{-1}_{j_2i_4}D^{-1}_{j_3i_1}D^{-1}_{j_4i_2}+D^{-1}_{j_1i_4}D^{-1}_{j_2i_3}D^{-1}_{j_3i_2}D^{-1}_{j_4i_1}
\end{aligned}$$

$$-D^{-1}_{j_1i_2}D^{-1}_{j_2i_3}D^{-1}_{j_3i_4}D^{-1}_{j_4i_1} - D^{-1}_{j_1i_2}D^{-1}_{j_2i_4}D^{-1}_{j_3i_1}D^{-1}_{j_4i_3} - D^{-1}_{j_1i_3}D^{-1}_{j_2i_1}D^{-1}_{j_3i_4}D^{-1}_{j_4i_2}$$
$$-D^{-1}_{j_1i_3}D^{-1}_{j_2i_4}D^{-1}_{j_3i_2}D^{-1}_{j_4i_1} - D^{-1}_{j_1i_4}D^{-1}_{j_2i_3}D^{-1}_{j_3i_1}D^{-1}_{j_4i_2} - D^{-1}_{j_1i_4}D^{-1}_{j_2i_1}D^{-1}_{j_3i_2}D^{-1}_{j_4i_3}$$

Then

$$\begin{aligned}
\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}M_{i_4j_4}\overline{\psi}\psi_{j_1i_1\ldots j_4i_4}\rangle &= Tr[MD^{-1}]^4 - 6Tr[MD^{-1}]^2Tr[(MD^{-1})^2] \\
&\quad +3Tr[(MD^{-1})^2]^2 + 8Tr[MD^{-1}]Tr[(MD^{-1})^3] - 6Tr[(MD^{-1})^4]
\end{aligned}$$

$$\begin{aligned}
\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\ldots j_4i_4}\rangle &= Tr[MD^{-1}]^3Tr[\overline{M}D^{-1}] - 3Tr[\overline{M}D^{-1}]Tr[MD^{-1}]Tr[(MD^{-1})^2] \\
&\quad -3Tr[MD^{-1}]^2Tr[MD^{-1}\overline{M}D^{-1}] + 3Tr[MD^{-1}\overline{M}D^{-1}]Tr[(MD^{-1})^2] \\
&\quad +2Tr[\overline{M}D^{-1}]Tr[(MD^{-1})^3] + 6Tr[MD^{-1}]Tr[(MD^{-1})^2\overline{M}D^{-1}] \\
&\quad -6Tr[(MD^{-1})^3\overline{M}D^{-1}]
\end{aligned}$$

$$\begin{aligned}
\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\ldots j_4i_4}\rangle &= Tr[MD^{-1}]^2Tr[\overline{M}D^{-1}]^2 - Tr[\overline{M}D^{-1}]^2Tr[(MD^{-1})^2] \\
&\quad -4Tr[MD^{-1}]Tr[\overline{M}D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}] - Tr[MD^{-1}]^2Tr[(\overline{M}D^{-1})^2] \\
&\quad +Tr[(\overline{M}D^{-1})^2]Tr[(MD^{-1})^2] + 2Tr[MD^{-1}\overline{M}D^{-1}]^2 \\
&\quad +4Tr[MD^{-1}]Tr[MD^{-1}(\overline{M}D^{-1})^2] + 4Tr[\overline{M}D^{-1}]Tr[\overline{M}D^{-1}(MD^{-1})^2] \\
&\quad -6Tr[(MD^{-1})^2(\overline{M}D^{-1})^2]
\end{aligned}$$

$$\begin{aligned}
\langle M_{i_1j_1}\overline{M}_{i_2j_2}M_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\ldots j_4i_4}\rangle &= Tr[MD^{-1}]^2Tr[\overline{M}D^{-1}]^2 - Tr[\overline{M}D^{-1}]^2Tr[(MD^{-1})^2] \\
&\quad -4Tr[MD^{-1}]Tr[\overline{M}D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}] - Tr[MD^{-1}]^2Tr[(\overline{M}D^{-1})^2] \\
&\quad +Tr[(\overline{M}D^{-1})^2]Tr[(MD^{-1})^2] + 2Tr[MD^{-1}\overline{M}D^{-1}]^2 \\
&\quad +4Tr[MD^{-1}]Tr[MD^{-1}(\overline{M}D^{-1})^2] + 4Tr[\overline{M}D^{-1}]Tr[\overline{M}D^{-1}(MD^{-1})^2] \\
&\quad -6Tr[(MD^{-1}\overline{M}D^{-1})^2]
\end{aligned}$$

$$\begin{aligned}
\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\ldots j_4i_4}\rangle &= Tr[\overline{M}D^{-1}]^3Tr[MD^{-1}] - 3Tr[MD^{-1}]Tr[\overline{M}D^{-1}]Tr[(\overline{M}D^{-1})^2] \\
&\quad -3Tr[\overline{M}D^{-1}]^2Tr[\overline{M}D^{-1}MD^{-1}] + 3Tr[\overline{M}D^{-1}MD^{-1}]Tr[(\overline{M}D^{-1})^2] \\
&\quad +2Tr[MD^{-1}]Tr[(\overline{M}D^{-1})^3] + 6Tr[\overline{M}D^{-1}]Tr[(\overline{M}D^{-1})^2MD^{-1}] \\
&\quad -6Tr[(\overline{M}D^{-1})^3MD^{-1}]
\end{aligned}$$

$$\begin{aligned}
\langle\overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{M}_{i_4j_4}\overline{\psi}\psi_{j_1i_1\ldots j_4i_4}\rangle &= Tr[\overline{M}D^{-1}]^4 - 6Tr[\overline{M}D^{-1}]^2Tr[(\overline{M}D^{-1})^2] \\
&\quad +3Tr[(\overline{M}D^{-1})^2]^2 + 8Tr[\overline{M}D^{-1}]Tr[(\overline{M}D^{-1})^3] - 6Tr[(\overline{M}D^{-1})^4]
\end{aligned}$$

$$\begin{aligned}
\langle M_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle &= Tr[D^{-1}]Tr[MD^{-1}]^3 - 3Tr[D^{-1}]Tr[MD^{-1}]Tr[(MD^{-1})^2] \\
&\quad -3Tr[MD^{-1}]^2Tr[MD^{-1}D^{-1}] + 3Tr[D^{-1}MD^{-1}]Tr[(MD^{-1})^2] \\
&\quad +2Tr[D^{-1}]Tr[(MD^{-1})^3] + 6Tr[MD^{-1}]Tr[D^{-1}MD^{-1}MD^{-1}] \\
&\quad -6Tr[D^{-1}(MD^{-1})^3]
\end{aligned}$$

$$\begin{aligned}
\langle M_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle &= Tr[MD^{-1}]^2Tr[\overline{M}D^{-1}]Tr[D^{-1}] - Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[(MD^{-1})^2] \\
&\quad -2Tr[D^{-1}]Tr[MD^{-1}]Tr[MD^{-1}\overline{M}D^{-1}] + Tr[D^{-1}\overline{M}D^{-1}]Tr[(MD^{-1})^2] \\
&\quad -Tr[MD^{-1}]^2Tr[D^{-1}\overline{M}D^{-1}] - 2Tr[\overline{M}D^{-1}]Tr[MD^{-1}]Tr[D^{-1}MD^{-1}]
\end{aligned}$$

$$
\begin{aligned}
& +2Tr[D^{-1}MD^{-1}]Tr[MD^{-1}\overline{M}D^{-1}]+4Tr[MD^{-1}]Tr[D^{-1}MD^{-1}\overline{M}D^{-1}] \\
& +2Tr[D^{-1}]Tr[\overline{M}D^{-1}(MD^{-1})^2]+2Tr[\overline{M}D^{-1}]Tr[D^{-1}(MD^{-1})^2] \\
& -6Tr[D^{-1}(MD^{-1})^2\overline{M}D^{-1}]
\end{aligned}
$$

$$
\begin{aligned}
\langle M_{i_1j_1}\overline{M}_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[MD^{-1}]^2Tr[\overline{M}D^{-1}]Tr[D^{-1}]-Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[(MD^{-1})^2] \\
& -2Tr[D^{-1}]Tr[MD^{-1}]Tr[MD^{-1}\overline{M}D^{-1}]+Tr[D^{-1}\overline{M}D^{-1}]Tr[(MD^{-1})^2] \\
& -Tr[MD^{-1}]^2Tr[D^{-1}\overline{M}D^{-1}]-2Tr[\overline{M}D^{-1}]Tr[MD^{-1}]Tr[D^{-1}MD^{-1}] \\
& +2Tr[D^{-1}MD^{-1}]Tr[MD^{-1}\overline{M}D^{-1}]+4Tr[MD^{-1}]Tr[D^{-1}MD^{-1}\overline{M}D^{-1}] \\
& +2Tr[D^{-1}]Tr[\overline{M}D^{-1}(MD^{-1})^2]+2Tr[\overline{M}D^{-1}]Tr[D^{-1}(MD^{-1})^2] \\
& -6Tr[D^{-1}MD^{-1}\overline{M}D^{-1}MD^{-1}]
\end{aligned}
$$

$$
\begin{aligned}
\langle \overline{M}_{i_1j_1}M_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[MD^{-1}]^2Tr[\overline{M}D^{-1}]Tr[D^{-1}]-Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[(MD^{-1})^2] \\
& -2Tr[D^{-1}]Tr[MD^{-1}]Tr[MD^{-1}\overline{M}D^{-1}]+Tr[D^{-1}\overline{M}D^{-1}]Tr[(MD^{-1})^2] \\
& -Tr[MD^{-1}]^2Tr[D^{-1}\overline{M}D^{-1}]-2Tr[\overline{M}D^{-1}]Tr[MD^{-1}]Tr[D^{-1}MD^{-1}] \\
& +2Tr[D^{-1}MD^{-1}]Tr[MD^{-1}\overline{M}D^{-1}]+4Tr[MD^{-1}]Tr[D^{-1}MD^{-1}\overline{M}D^{-1}] \\
& +2Tr[D^{-1}]Tr[\overline{M}D^{-1}(MD^{-1})^2]+2Tr[\overline{M}D^{-1}]Tr[D^{-1}(MD^{-1})^2] \\
& -6Tr[D^{-1}\overline{M}D^{-1}(MD^{-1})^2]
\end{aligned}
$$

$$
\begin{aligned}
\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[\overline{M}D^{-1}]^2Tr[MD^{-1}]Tr[D^{-1}]-Tr[D^{-1}]Tr[MD^{-1}]Tr[(\overline{M}D^{-1})^2] \\
& -2Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[\overline{M}D^{-1}MD^{-1}]+Tr[D^{-1}MD^{-1}]Tr[(\overline{M}D^{-1})^2] \\
& -Tr[\overline{M}D^{-1}]^2Tr[D^{-1}MD^{-1}]-2Tr[MD^{-1}]Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}] \\
& +2Tr[D^{-1}\overline{M}D^{-1}]Tr[\overline{M}D^{-1}MD^{-1}]+4Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}MD^{-1}] \\
& +2Tr[D^{-1}]Tr[MD^{-1}(\overline{M}D^{-1})^2]+2Tr[MD^{-1}]Tr[D^{-1}(\overline{M}D^{-1})^2] \\
& -6Tr[D^{-1}MD^{-1}(\overline{M}D^{-1})^2]
\end{aligned}
$$

$$
\begin{aligned}
\langle \overline{M}_{i_1j_1}M_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[\overline{M}D^{-1}]^2Tr[MD^{-1}]Tr[D^{-1}]-Tr[D^{-1}]Tr[MD^{-1}]Tr[(\overline{M}D^{-1})^2] \\
& -2Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[\overline{M}D^{-1}MD^{-1}]+Tr[D^{-1}MD^{-1}]Tr[(\overline{M}D^{-1})^2] \\
& -Tr[\overline{M}D^{-1}]^2Tr[D^{-1}MD^{-1}]-2Tr[MD^{-1}]Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}] \\
& +2Tr[D^{-1}\overline{M}D^{-1}]Tr[\overline{M}D^{-1}MD^{-1}]+4Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}MD^{-1}] \\
& +2Tr[D^{-1}]Tr[MD^{-1}(\overline{M}D^{-1})^2]+2Tr[MD^{-1}]Tr[D^{-1}(\overline{M}D^{-1})^2] \\
& -6Tr[D^{-1}\overline{M}D^{-1}MD^{-1}\overline{M}D^{-1}]
\end{aligned}
$$

$$
\begin{aligned}
\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}M_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[\overline{M}D^{-1}]^2Tr[MD^{-1}]Tr[D^{-1}]-Tr[D^{-1}]Tr[MD^{-1}]Tr[(\overline{M}D^{-1})^2] \\
& -2Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[\overline{M}D^{-1}MD^{-1}]+Tr[D^{-1}MD^{-1}]Tr[(\overline{M}D^{-1})^2] \\
& -Tr[\overline{M}D^{-1}]^2Tr[D^{-1}MD^{-1}]-2Tr[MD^{-1}]Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}] \\
& +2Tr[D^{-1}\overline{M}D^{-1}]Tr[\overline{M}D^{-1}MD^{-1}]+4Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}MD^{-1}] \\
& +2Tr[D^{-1}]Tr[MD^{-1}(\overline{M}D^{-1})^2]+2Tr[MD^{-1}]Tr[D^{-1}(\overline{M}D^{-1})^2] \\
& -6Tr[D^{-1}(\overline{M}D^{-1})^2MD^{-1}]
\end{aligned}
$$

$$
\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{M}_{i_3j_3}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; Tr[D^{-1}]Tr[\overline{M}D^{-1}]^3-3Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[(\overline{M}D^{-1})^2]
$$

$$
\begin{aligned}
&-3Tr[\overline{M}D^{-1}]^2Tr[\overline{M}D^{-1}D^{-1}]+3Tr[D^{-1}\overline{M}D^{-1}]Tr[(\overline{M}D^{-1})^2]\\
&+2Tr[D^{-1}]Tr[(\overline{M}D^{-1})^3]+6Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}\overline{M}D^{-1}]\\
&-6Tr[D^{-1}(\overline{M}D^{-1})^3]
\end{aligned}
$$

$$
\begin{aligned}
\langle M_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[MD^{-1}]^2Tr[D^{-1}]^2-Tr[D^{-1}]^2Tr[(MD^{-1})^2]\\
&-4Tr[D^{-1}]Tr[MD^{-1}]Tr[D^{-1}MD^{-1}]-Tr[MD^{-1}]^2Tr[D^{-1}D^{-1}]\\
&+Tr[D^{-1}D^{-1}]Tr[(MD^{-1})^2]+2Tr[D^{-1}MD^{-1}]^2\\
&+4Tr[MD^{-1}]Tr[MD^{-1}D^{-1}D^{-1}]+4Tr[D^{-1}]Tr[D^{-1}(MD^{-1})^2]\\
&-6Tr[D^{-1}D^{-1}(MD^{-1})^2]
\end{aligned}
$$

$$
\begin{aligned}
\langle M_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[D^{-1}]^2Tr[MD^{-1}]Tr[\overline{M}D^{-1}]-Tr[D^{-1}]^2Tr[MD^{-1}\overline{M}D^{-1}]\\
&-2Tr[D^{-1}]Tr[MD^{-1}]Tr[D^{-1}\overline{M}D^{-1}]+Tr[D^{-1}D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}]\\
&-Tr[MD^{-1}]Tr[\overline{M}D^{-1}]Tr[D^{-1}D^{-1}]-2Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[D^{-1}MD^{-1}]\\
&+2Tr[D^{-1}MD^{-1}]Tr[D^{-1}\overline{M}D^{-1}]+2Tr[MD^{-1}]Tr[\overline{M}D^{-1}D^{-1}D^{-1}]\\
&+2Tr[\overline{M}D^{-1}]Tr[MD^{-1}D^{-1}D^{-1}]+4Tr[D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}D^{-1}]\\
&-6Tr[MD^{-1}\overline{M}D^{-1}D^{-1}D^{-1}]
\end{aligned}
$$

$$
\begin{aligned}
\langle \overline{M}_{i_1j_1}M_{i_2j_2}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[D^{-1}]^2Tr[MD^{-1}]Tr[\overline{M}D^{-1}]-Tr[D^{-1}]^2Tr[MD^{-1}\overline{M}D^{-1}]\\
&-2Tr[D^{-1}]Tr[MD^{-1}]Tr[D^{-1}\overline{M}D^{-1}]+Tr[D^{-1}D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}]\\
&-Tr[MD^{-1}]Tr[\overline{M}D^{-1}]Tr[D^{-1}D^{-1}]-2Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[D^{-1}MD^{-1}]\\
&+2Tr[D^{-1}MD^{-1}]Tr[D^{-1}\overline{M}D^{-1}]+2Tr[MD^{-1}]Tr[\overline{M}D^{-1}D^{-1}D^{-1}]\\
&+2Tr[\overline{M}D^{-1}]Tr[MD^{-1}D^{-1}D^{-1}]+4Tr[D^{-1}]Tr[MD^{-1}\overline{M}D^{-1}D^{-1}]\\
&-6Tr[\overline{M}D^{-1}MD^{-1}D^{-1}D^{-1}]
\end{aligned}
$$

$$
\begin{aligned}
\langle \overline{M}_{i_1j_1}\overline{M}_{i_2j_2}\overline{\psi}\psi_{j_1i_1\ldots i_4i_4}\rangle \;=\; & Tr[\overline{M}D^{-1}]^2Tr[D^{-1}]^2-Tr[D^{-1}]^2Tr[(\overline{M}D^{-1})^2]\\
&-4Tr[D^{-1}]Tr[\overline{M}D^{-1}]Tr[D^{-1}\overline{M}D^{-1}]-Tr[\overline{M}D^{-1}]^2Tr[D^{-1}D^{-1}]\\
&+Tr[D^{-1}D^{-1}]Tr[(\overline{M}D^{-1})^2]+2Tr[D^{-1}\overline{M}D^{-1}]^2\\
&+4Tr[\overline{M}D^{-1}]Tr[\overline{M}D^{-1}D^{-1}D^{-1}]+4Tr[D^{-1}]Tr[D^{-1}(\overline{M}D^{-1})^2]\\
&-6Tr[D^{-1}D^{-1}(\overline{M}D^{-1})^2]
\end{aligned}
$$

Comparison with the Fourier transform of the free case

From the free case we have

$$\ln Z(\mu) = 2\sum_p \ln R_p + 2\sum_p \ln\left(1 - \rho b_p - \overline{\rho} b_p^\star + \frac{\rho\overline{\rho}}{R_p}\right) \tag{33}$$

with:

$$\begin{aligned}
R_p &= c_p^2 - 2c_p\cos p_4 + \sum_{i=1}^{3}\sin^2 p_i + 1 \\
c_p &= m + 4 - \sum_{i=1}^{3}\cos p_i \\
b_p &= a_p - \frac{1}{R_p} \\
a_p &= \frac{c_p e^{ip_4}}{R_p}
\end{aligned}$$

Warning!!!!

$$\rho b_p + \overline{\rho} b_p^\star - \frac{\rho\overline{\rho}}{R_p} = \rho a_p + \overline{\rho} a_p^\star$$

- The square of the term inside the second logarithm of Eq.(33) is the **exact** expression for e^R
$$(1 - \rho a_p - \overline{\rho} a_p^\star)^2 = e^R$$
- We cannot compute e^R exactly. Therefore we have to expand in number of propagators. It is feasible only up to the fourth order.
- See Eq.(2). Expanding in number of propagators is equivalent to expanding in R.
- Expanding in "orders of R" is equivalent to expanding in orders of ρ and $\overline{\rho}$. **But** treated as independent variables. So that:
  - $R = \rho A_1 + \overline{\rho} B_1$ has one propagator and first order in ρ and $\overline{\rho}$
  - $R^2 = \rho^2 A_2 + 2\rho\overline{\rho} AB_2 + \overline{\rho}^2 B_2$ has 2 propagators and second order terms in ρ and $\overline{\rho}$. The tricky part: if we replace $\rho\overline{\rho} = -\rho - \overline{\rho}$ then a 2 propagator term contributes to the first oder term.
- Replacing $\rho\overline{\rho}$ by $-\rho - \overline{\rho}$ leads to contributions from all terms to the first order term.
- In the full case, as Christof said, it is better to cut the expansion in orders of propagators (the same as cutting in orders of R). The confusion was: If one expands Eq.(33) and treats ρ and $\overline{\rho}$ as independent, then $\rho b_p + b_p^\star + \rho\overline{\rho}/R_p$ does not simplify to $\rho a_p + \overline{\rho} a_p^\star$ oder by order!!
- First I tried expanding $\rho a_p + \overline{\rho} a_p^\star$ and the results did not agree with the results from the CG+random vectors. Only if we expand $\rho b_p + b_p^\star + \rho\overline{\rho}/R_p$,

both methods agree (Fourier Transform and CG+random vectors). For example: The first order contribution to the observable n is:

CG+random vectors:

$$\frac{1}{3}(-e^{\mu}Tr[MD^{-1}] + e^{-\mu}Tr[\overline{M}D^{-1}])$$

Fourier transform: $-2\sum_p[e^{\mu}a_p - e^{-\mu}a_p^\star]$ or $-2\sum_p[e^{\mu}b_p - e^{-\mu}b_p^\star]$

But only:

$$-2\sum_p[e^{\mu}b_p - e^{-\mu}b_p^\star] = \frac{1}{3}(-e^{\mu}Tr[MD^{-1}] + e^{-\mu}Tr[\overline{M}D^{-1}])$$

The term $\propto \rho\overline{\rho}$ contributes to the second order. Then: $2\sum_p b_p = Tr[MD^{-1}]/3$ and $2\sum_p b_p^\star = Tr[\overline{M}D^{-1}]/3$. I checked this for different masses and with other methods (ie. no possible bug in the code). Order by order in R: