MATH 135: AMC I

Contents

Macalester College
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Preface

This is the class handbook for Math 135 AMC I at Macalester College. The content here was made by Andrew Beveridge and draws upon:

- Material prepared by Kristin Heysse, Lori Ziegelmeier and other faculty in the Department of Mathematics, Statistics and Computer Science at Macalester College.
- The open textbook Applied Calculus by Shana Calaway, Dale Hoffman, and David Lippman.

Chapter 1

R Studio Primer

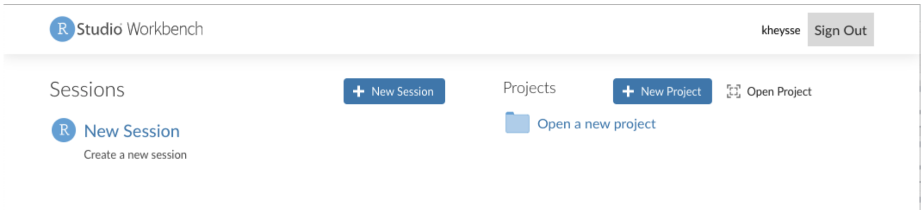
In our class, we will be using RStudio, a developing environment commonly used for modeling and statistics. In real world mathematics, you will need to be able to work with data and apply your calculus concepts directly. That's what we'll do in RStudio, as it provides an excellent sandbox for us to play in. R (what RStudio is based on) is the programming language used by professional statisticians, so these skills will be useful no matter what field you go into!

However, just like any programming language, there is a startup cost. The learning curve for RStudio can be pretty steep, particularly if you've never done any coding before. Don't worry, that's where this guide comes in! In here you'll find all sorts of explanation and examples for some of the coding we'll be doing in (and out of) class. I promise, it'll be worth in the end, and you'll be a coding master.

1.1 Getting Oriented

1.1.1 First login

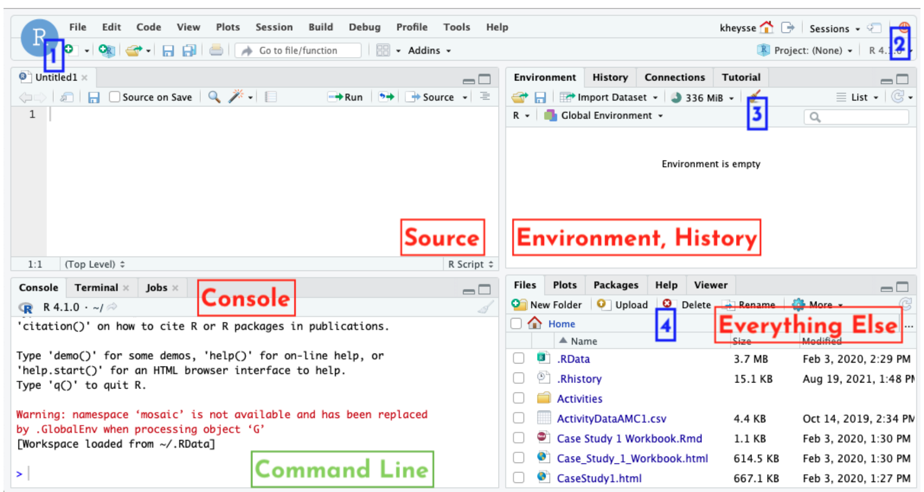
You'll get into RStudio via the url https://rstudio.macalester.edu. It's linked in our Moodle page if you're feeling a little lazy. Your login is your Macalester username and password. You'll be given the following page (with your username in the uppper right corner):



I tend to ignore the projects and just activate a new session by clicking that panel.

1.1.2 The RStudio window

Once you've got an active session, you'll be shown the following window. We'll talk about its parts, starting with the four panels you'll see. This picture is from a previous version of RStudio, but everything should work the same!



1.1.2.1 Panels

- Source: This is where you will edit RMarkdown notebooks, which are described much more in the RMarkdown section.
- Console: This is RStudio's command line. It's a great place to test code and serves as a basic calculator.
- Environment, history: this panel keeps track of your work. The environment tab shows all the variables you have named in the session (including data, functions, and lists) and gives you a preview of them. The history tab tracks the pieces of code you run.
- Everything else: this is where you'll find a lot of stuff, including the files you build and the plots you create of data/functions. It also has the packages

list (described in the next subsection) and the help window, which can show you documentation of commands.

Now that we know what we're looking at, let's talk specifics. We'll talk a lot more about the source panel in the RMarkdown section.

1.1.2.2 Specific functions

- The new document button. This is how you'll make new files in your source window.
- The quit session button. Somtimes it's best to just start over if you can't figure out why something it not working. You'll be kicked back to that first page from the previous section.
- The clean environment button. Use this when you want to clear all you variables, including data sets.
- The files buttons. You'll use these to create, delete, and organize RMarkdown files and data files. See the data section for more information on how to upload data.

1.1.2.3 Command Line Console

The command line console is the most basic way to run code. It's great for simple calculations and to test out code before you run it. We'll run longer blocks of code in our notebooks. You can try out a few calculations here.

Notice! Once you run a command (by pressing enter) it is gone to history. You don't go back and edit previous lines if you find a mistake. Instead, you run the command again with the fixed piece in a new line. Thankfully, RStudio makes this not so terrible. By pushing the UP button on your keyboard, you can scroll through previous commands that you've used.

1.1.3 Packages

RStudio is open source, which means many people write packages to augment what it can do by default. Their work is our gain, and we'll need a few packages to be able to access what we want. In the lower right hand panel click on the packages tab. Then click on the check boxes next to the following packages:

- fetch: this package knows how to load in data files
- mosaic, mosaicCore, mosaicData: these contain the specific built-ins which create and plot functions for us. By checking the box by them, we import their functions into the session. Easy enough.
- mosaicCalc: Some of the function commands from the mosaic family of packages don't interact well with the built in derivative and antiderivative functions in RStudio. mosaicCalc fixes this

  - If you don't see mosaicCalc, then you need to download it. Run the following line of code in your console: `install.packages("mosaicCalc")` Once you do this, make sure that the mosaicCal box is checked in the packages list

1.2 RMarkdown

In AMC1, we'll be working with code a lot: writing it, running it, analyzing it, and commenting on it. We'll also be going back to code we've already used and tweaking it to solve new but related questions. For all of these reasons, we're going to use RMarkdown notebooks. The notebook will run large chunks of code at a time and show the output right next to the code. It's also easier for you to comment on the code you've written and go back to it later.

The best way to get working on this is to upload the template that I've put in the course Moodle page and just start experimenting with what's there. I'm sure you'll get it sorted!

Chapter 2

RStudio Quick Referrence

2.1 Common Functions

function	R command	comments
x^2	`x^2`	
$4x^2 - 7x + 3$	`4*x^2 - 7*x + 3`	You must **always** include the `*` when you multiply
$\sqrt{x}$	`sqrt(x)` or `x^(1/2)`	
$\sqrt[3]{x}$	`x^(1/3)`	
$\sin(x)$	`sin(x)`	
$\cos(x)$	`cos(x)`	
e^x	`exp(x)`	`exp` is a **function** (like `sin` and `cos`), so you do NOT need a `^`
$\ln(x)$	`log(x)`	RStudio uses the natural logarithm as the default
$\log(x) = \log_{10}(x)$	`log10(x)`	Humans default to "log base 10." RStudio does not!
$\log_b(x)$	`log(x,b)`	You can use any number b as the base for your logarithm

2.2 Defining your own function

The `makeFun` command is part of the mosaic package.

desired function	R command	comments
$f(x) = x^2 + 5x + 6$	`f = makeFun(x^2 + 5*x + 6 ~ x)`	remember your `*` signs!
$g(x) = \sin^2(x) - \frac{1}{2}$	`g = makeFun(sin(x)^2 - 1/2 ~ x)`	note the placement of `^2`
$P(t) = 5e^{.25t}$	`P = makeFun(5 * exp(0.25 * t) ~ t)`	`exp` does not need a `^`
$Q(t) = 12.38(1.041)^t$	`Q = makeFun(12.38 * (1.041)^t ~ t)`	

Notes:

- The general syntax is `makeFun( OUTPUT ~ INPUT )` where the OUTPUT is a function of the INPUT.
- You must **assign this to a variable** so that you can use it later. So your command must **start with** `f = ...` so that you get a function named `f`.
- The variable (`x` or `t`, etc) that appears in the OUTPUT **must match** the INPUT variable that appears after the tilde `~`. The the command `makeFun(x^2 ~ t)` won't work because the `x^2` is not a function of `t`.

2.3 Creating Lists of Data

Use `c()` to "combine" some values into a list. Assign the list to a variable so that you can use it later.

```
my_primes = c(2, 3, 5, 7, 11, 13, 17, 19, 23, 29)
my_primes
```

```
##  [1]  2  3  5  7 11 13 17 19 23 29
```

```
my_data = c(5, -2, 7, 3, -10, 15)
my_data
```

```
## [1]   5  -2   7   3 -10  15
```

Use `seq()` to generate a sequential list of values in a specified range. The optional third parameter tells RStudio to increment by that value instead of incrementing by 1.

```
my_seq1 = seq(1,10)
my_seq1
```

```
##  [1]  1  2  3  4  5  6  7  8  9 10
```

```
my_seq2 = seq(5, 6, 0.25)
my_seq2
```

```
## [1] 5.00 5.25 5.50 5.75 6.00
```

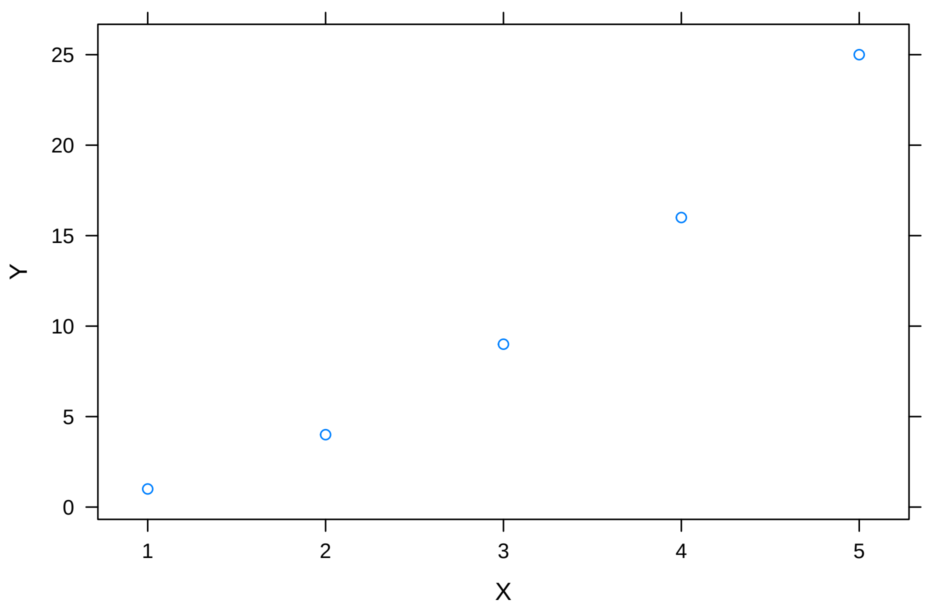
```
my_seq3 = seq(5, 6, 0.1)
my_seq3
```

```
##  [1] 5.0 5.1 5.2 5.3 5.4 5.5 5.6 5.7 5.8 5.9 6.0
```

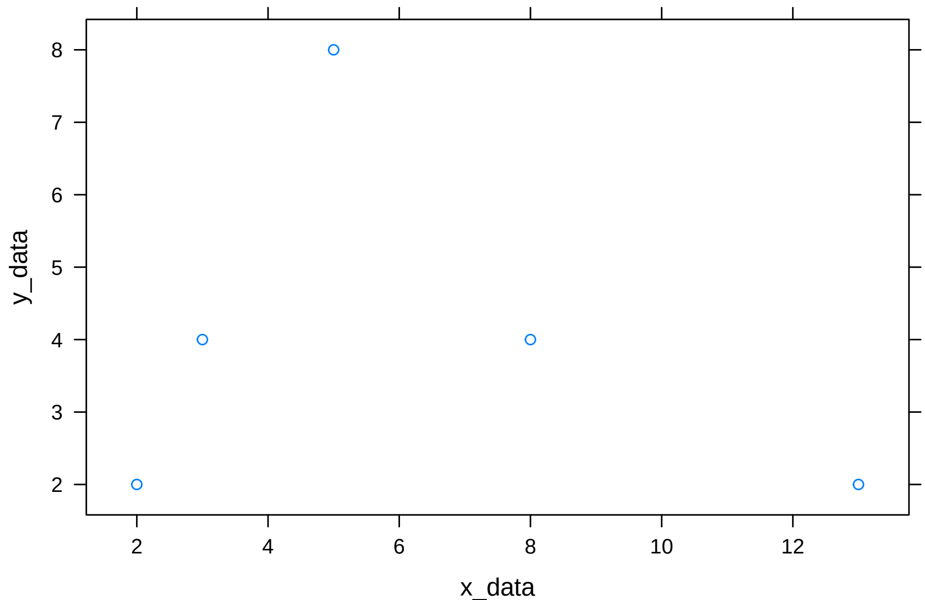
2.4 Plotting Data

Use `plotPoints` to plot a list of y-axis data versus a list of x-axis data. Separate these lists with a tilde `~`.

```
X = seq(1:5)
Y = X^2
plotPoints(Y ~ X)
```




```
x_data = c(2,3,5,8,13)
y_data = c(2,4,8,4,2)
plotPoints(y_data ~ x_data)
```

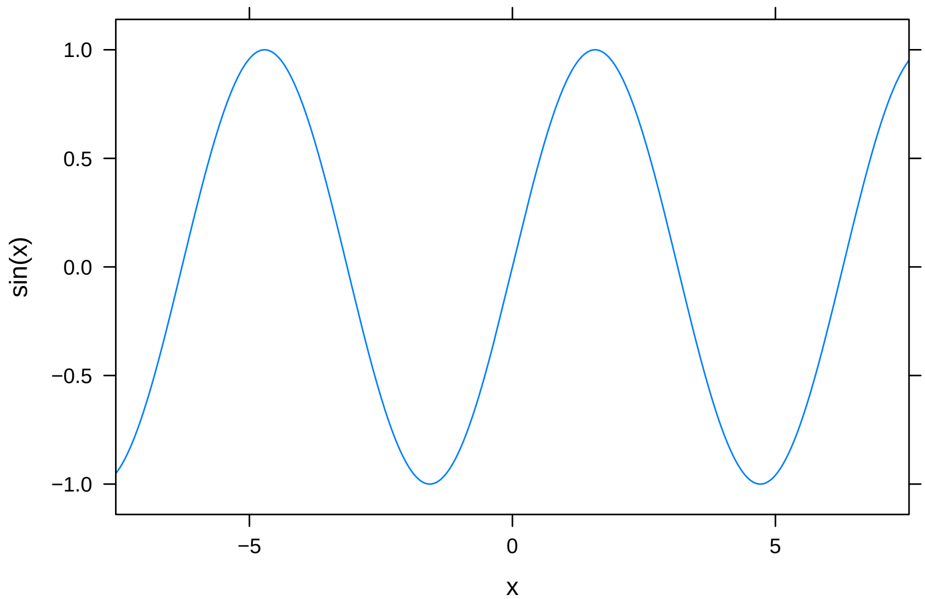


2.5 Plotting Functions

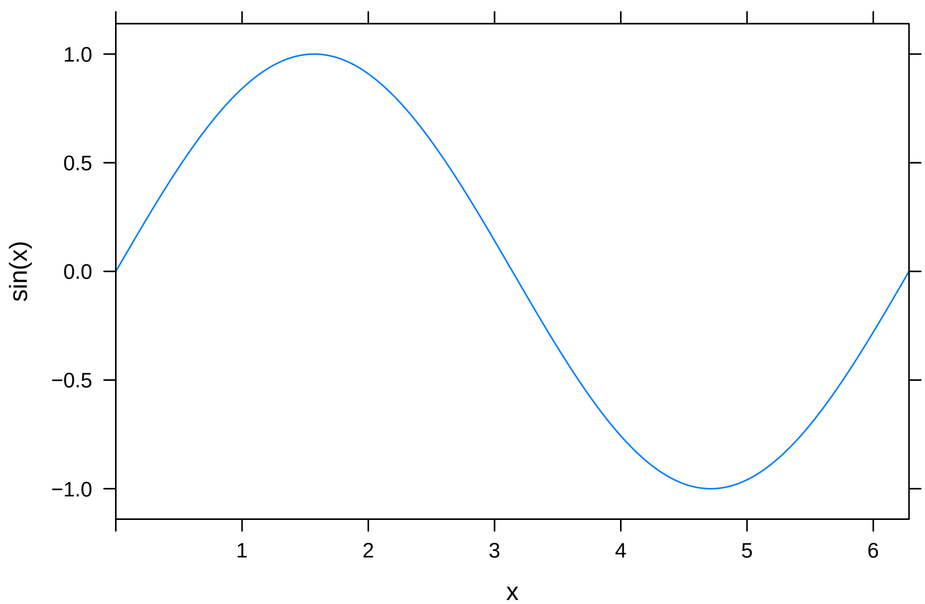
Use `plotFun` to plot a function.

- As with `plotPoints`, we use a tilde `~` to separate the output variable (the function) from the input variable.
- The optional `xlim` argument specifies the domain for the plot.

```
plotFun(sin(x) ~ x)
```



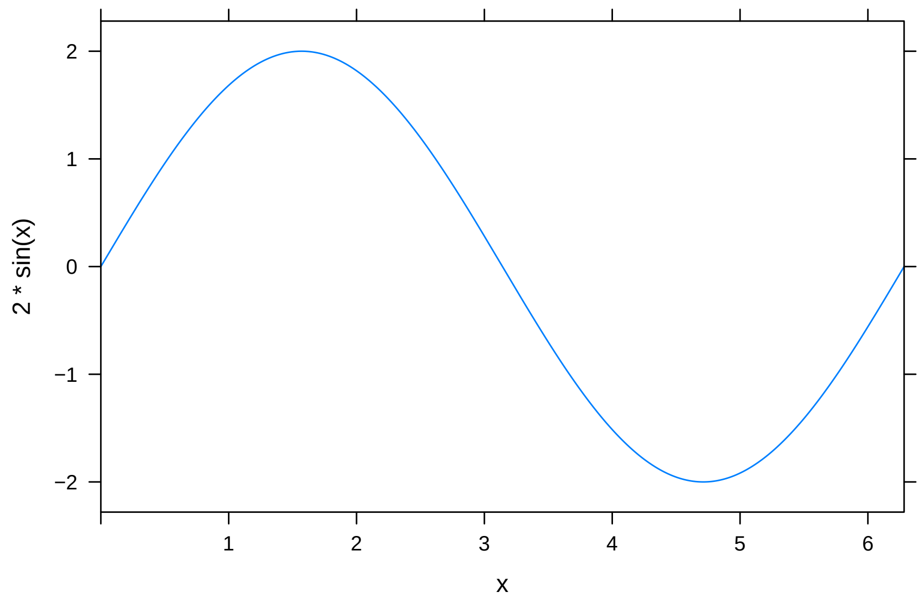

```
plotFun(sin(x) ~ x, xlim=c(0, 2*pi))
```



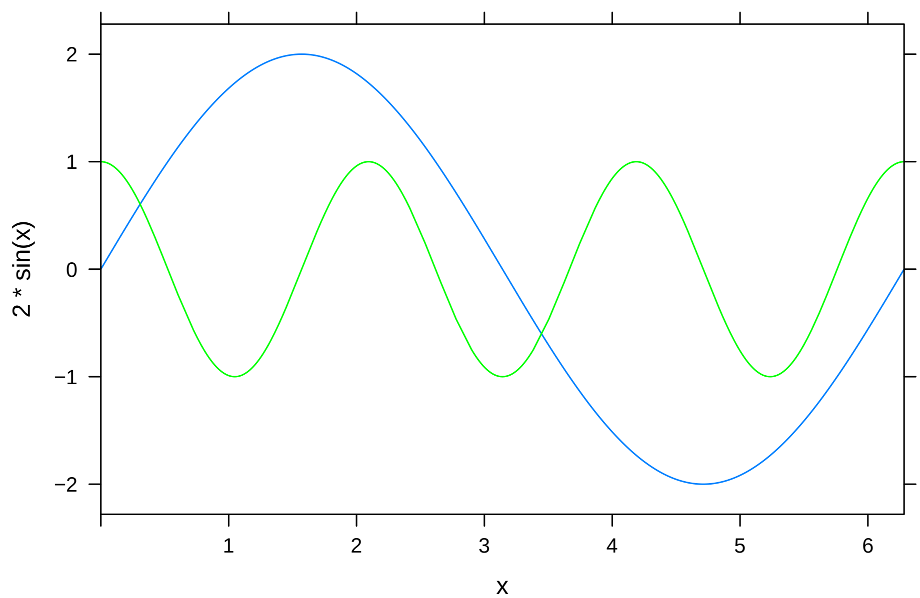
We can layer one plot on top of another by using the `add=TRUE` argument.

- You only need to specify the `xlim` for the **first plot** because the second one is added to the first.
- It's helpful to change the color for subsequent plots. Use the argument `col="red"` using any color name you like (RStudio knows a lot of colors!)

```
plotFun(2 * sin(x) ~ x, xlim=c(0, 2*pi))
```

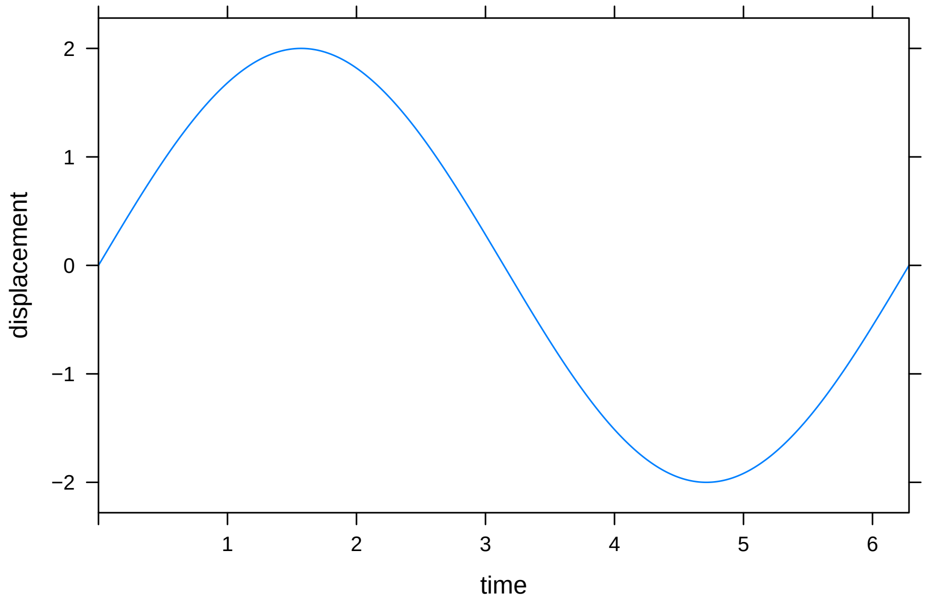



```
plotFun(cos(3*x) ~ x, add=TRUE, col='green')
```

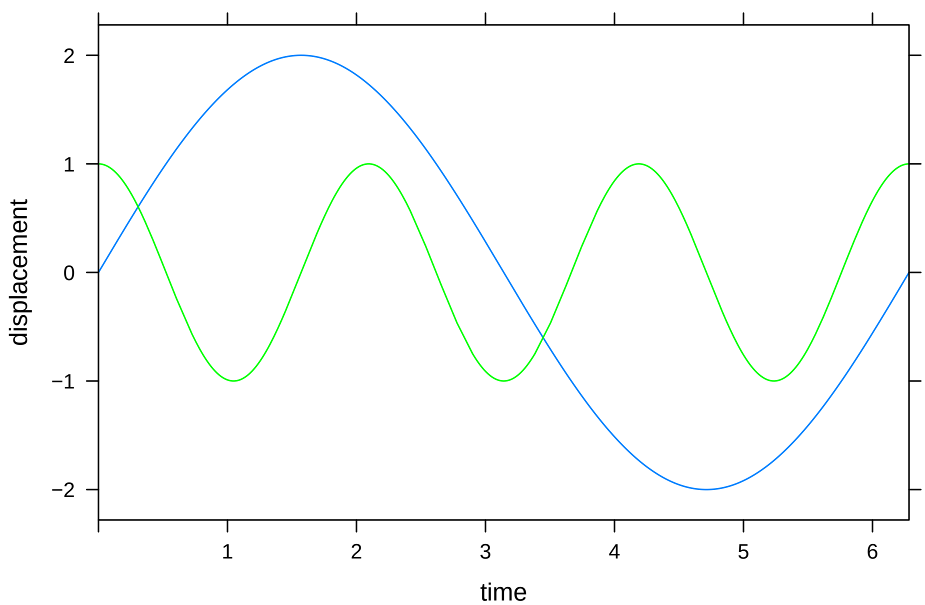


We can label the axes using `xlab` and `ylab`. When you are layering two plots, you must specify these labels in the first function that you plot!

```
plotFun(2 * sin(x) ~ x, xlim=c(0, 2*pi), xlab='time', ylab='displacement')
```




```
plotFun(cos(3*x) ~ x, add=TRUE, col='green')
```



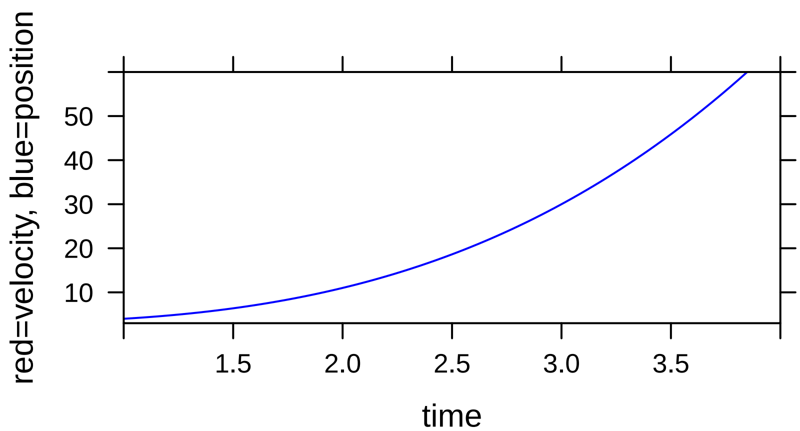
The `plotFun` command takes a whole variety of parameters, here are a few useful ones:

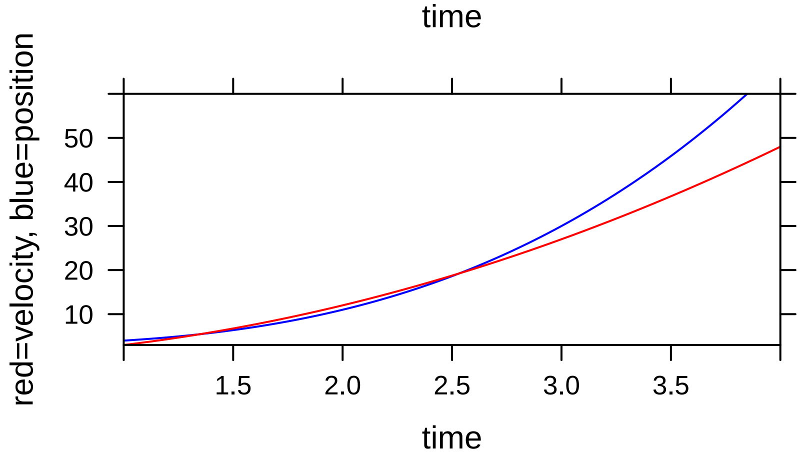
- `xlim`, `ylim`: sets the range of your x-axis. Example: `xlim=range(1,4)`, `ylim=range(3,60)`

- `col`: sets the color of the function you plot. RStudio will recognize most basic colors, which you will pass as strings. Example: `col="red"`
- `add`: you'll use this when you want to add multiple functions to the same plot. We'll see an example of it below. Example: `add=TRUE`
- `xlab`, `ylab`: for labeling your axes. You'll pass this as a string as well. Example: `xlab="time", ylab="red=velocity, blue=position"`

Let's try some of this out. I broke the third line so it would all fit, you don't need to!

```
f=makeFun(x^3+3~x)
g=makeFun(3*x^2~x)
plotFun(f(x)~x, xlim=range(1,4), ylim=range(3,60),
        col="blue",xlab="time", ylab="red=velocity, blue=position")
plotFun(g(x)~x, col="red", add=TRUE)
```



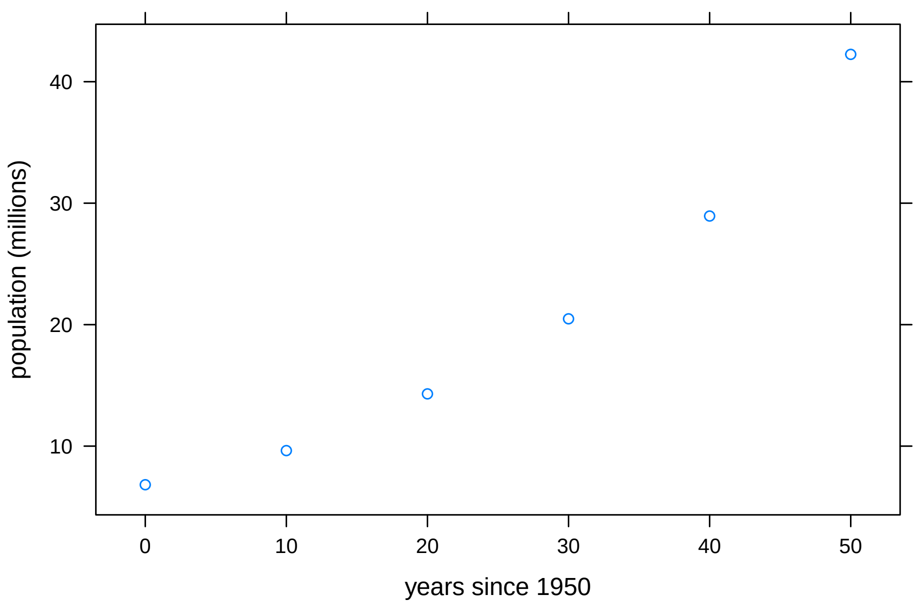


RStudio is a little weird about plots in the RMarkdown in that you might get multiples if you're adding plots together. If you want the chunk to hold all plots until the end, add `fig.show='hold'` into the script braces at the top of the chunk.

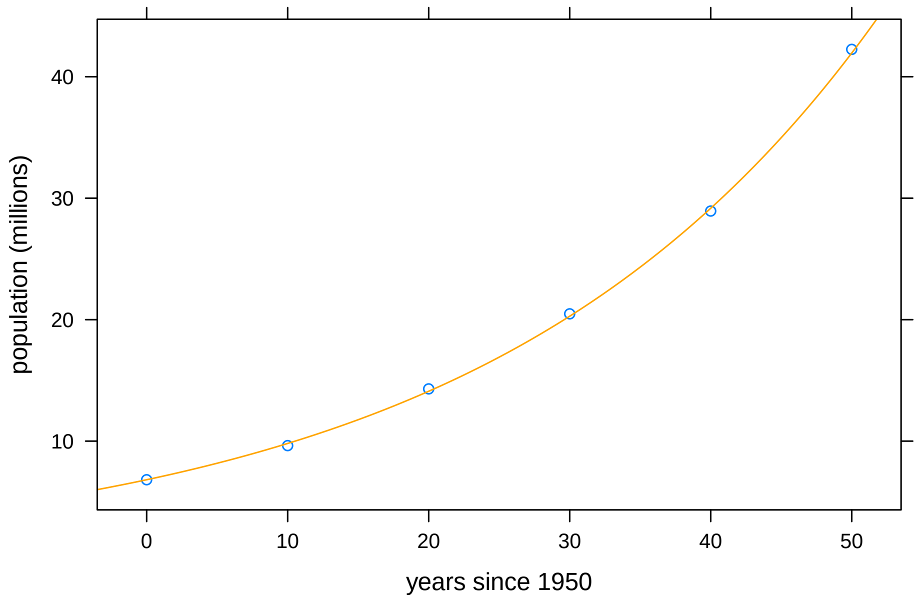
2.6 Plotting Data and Functions

We can also combine plots of data and functions using `add=TRUE`.

```
time_data = seq(0,50, 10)
pop_data = c(6.82, 9.63, 14.3, 20.48, 28.94, 42.25)

plotPoints(pop_data ~ time_data, xlab='years since 1950', ylab='population (millions)')
```




```
P = makeFun( 6.82 * (1.037)^t ~ t)

plotFun(P(t) ~ t, add=TRUE, col='orange')
```



2.7 Fitting a Function to Data

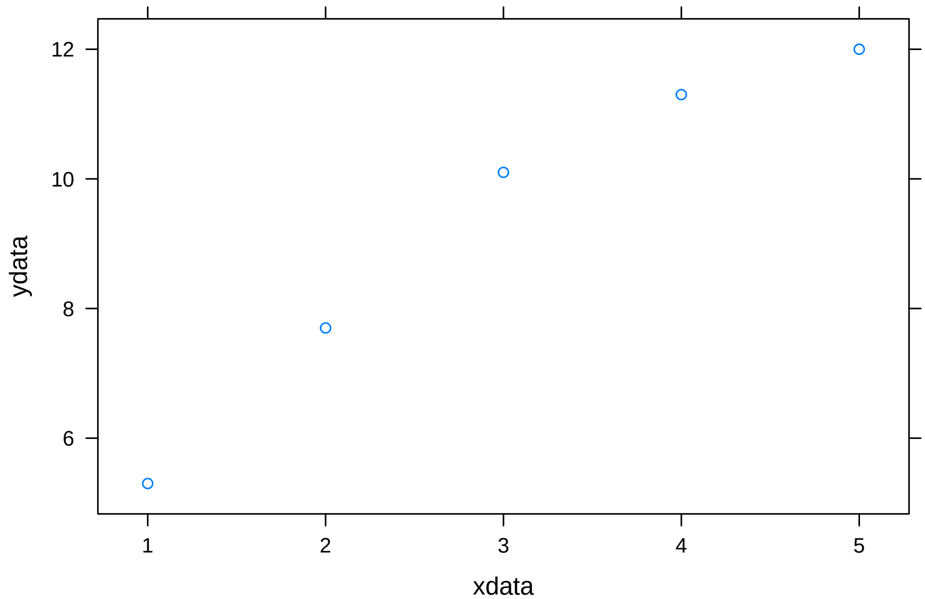
The `fitModel` command is part of the mosaic package. Here is how you would fit a linear model to data

```
xdata = c(1,2,3,4,5)
ydata = c(5.3, 7.7, 10.1, 11.3, 12.0    )

fitModel(ydata  ~ A* xdata  + B)
```

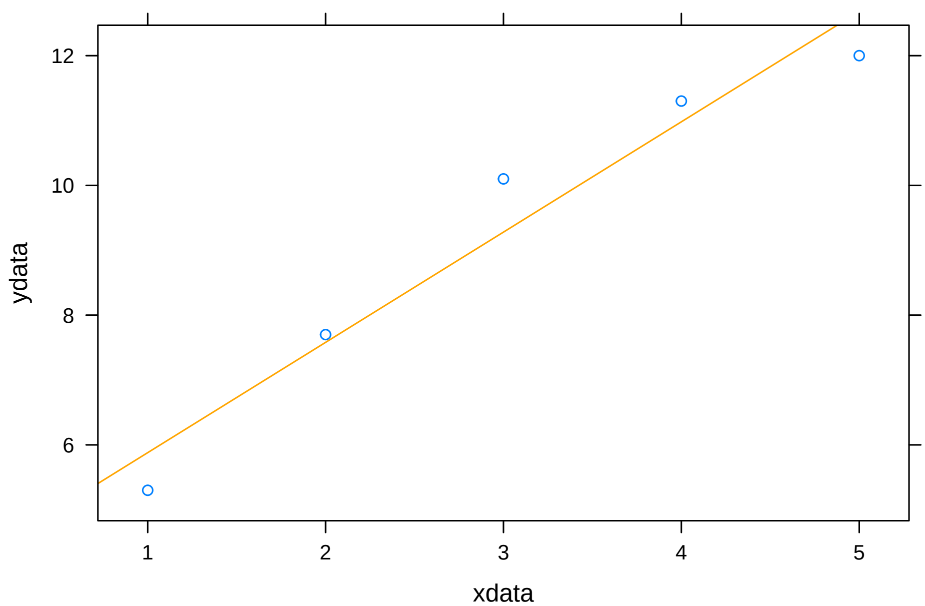
```
## function (xdata, ..., transformation = function (x)
## x)
## return(transformation(predict(model, newdata = data.frame(xdata = xdata),
##     ...)))
## <environment: 0x7f826e760e10>
## attr(,"coefficients")
##    A    B
## 1.70 4.18
## attr(,"class")
## [1] "nlsfunction" "function"
```

This output says that the best fitting linear function has slope $A = 1.70$ and y-intercept $B = 4.18$. So let's create the corresponding linear function and plot it with the data.

```
f = makeFun(1.70 *  x + 4.18 ~ x)

plotPoints(ydata ~ xdata)
```




```
plotFun(f(x) ~ x,  add=TRUE, col='orange')
```



2.7.1 Fitting a Power Function to Data

Here are the steps to fit `xdata` and `ydata` to a power function $f(x) = cx^p$.

1. Take the log of both `xdata` and `ydata`
2. Fit a linear function $y = Ax + B$ to the resulting data
3. The original constants are `p = A` and `c = exp(B)`

Here is an example

```
f = makeFun(4.3 * x^(1.7)~x)

xdata = c(1, 3, 5, 7, 9)
ydata = c(7, 33, 65, 110, 180)

logxdata = log(xdata)
logydata = log(ydata)

fitModel(logydata ~ A * logxdata + B)
```

```
## function (logxdata, ..., transformation = function (x)
## x)
## return(transformation(predict(model, newdata = data.frame(logxdata = logxdata),
##     ...)))
## <environment: 0x7f8251be5bd0>
## attr(,"coefficients")
##        A        B
## 1.449432 1.915983
## attr(,"class")
## [1] "nlsfunction" "function"
```

This tells us that $A = 1.45$ and $B = 1.92$. We solve for c and plot the function with the data

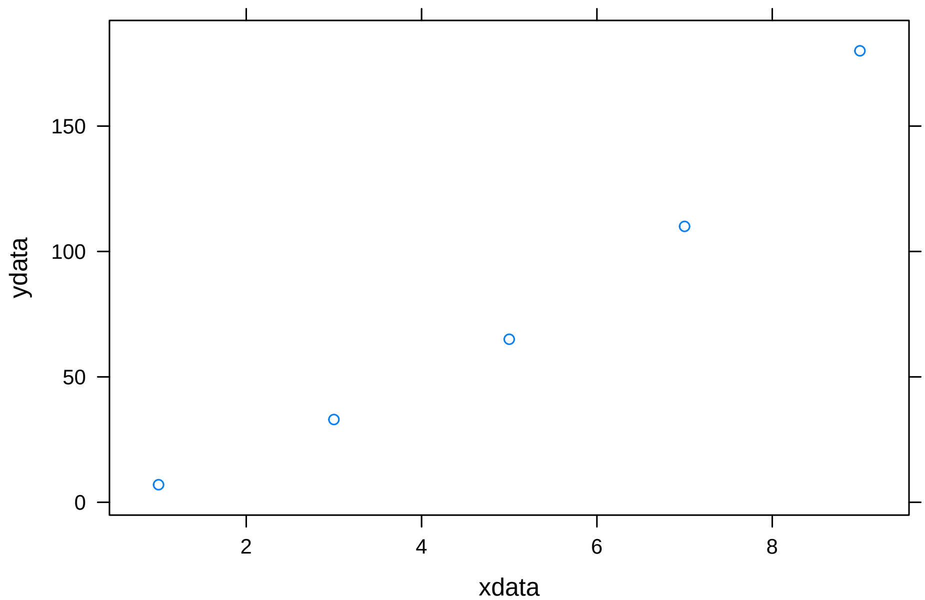
```
p = 1.45
p
```

```
## [1] 1.45
```

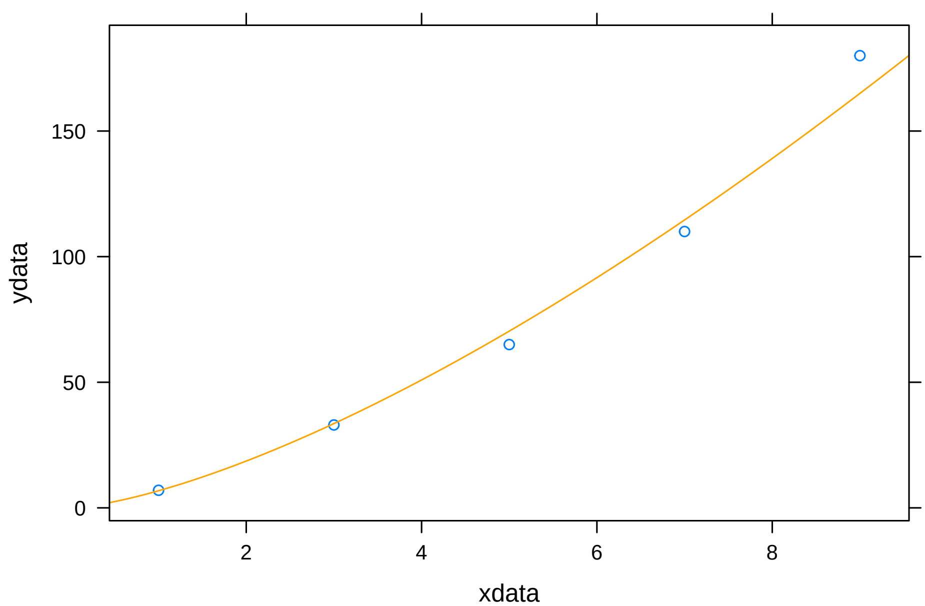
```
c = exp(1.92)
c
```

```
## [1] 6.820958
```

```
plotPoints(ydata ~ xdata)
```




```
plotFun(6.82 * x^1.45 ~ x, add=TRUE, col='orange')
```



2.7.2 Fitting a Model by Specifying Starting Values

Sometimes `fitModel` fails to find a good model for the data. Here are two things that can happen.

- The command `fitModel` fails, 7.complaining about a "singular gradient." This means that `fitModel` picked a bad starting point, and can't figure out how to improve it's initial guess.
- The command `fitModel` does find a function, but when we plot it, we realize that it's not what we want.

Here is an example of `fitModel` returning a function that we don't like. Let's load in the 'utilities.csv' data set and look at the first few rows.

```
utilitydata = fetch::fetchData("utilities.csv")
```

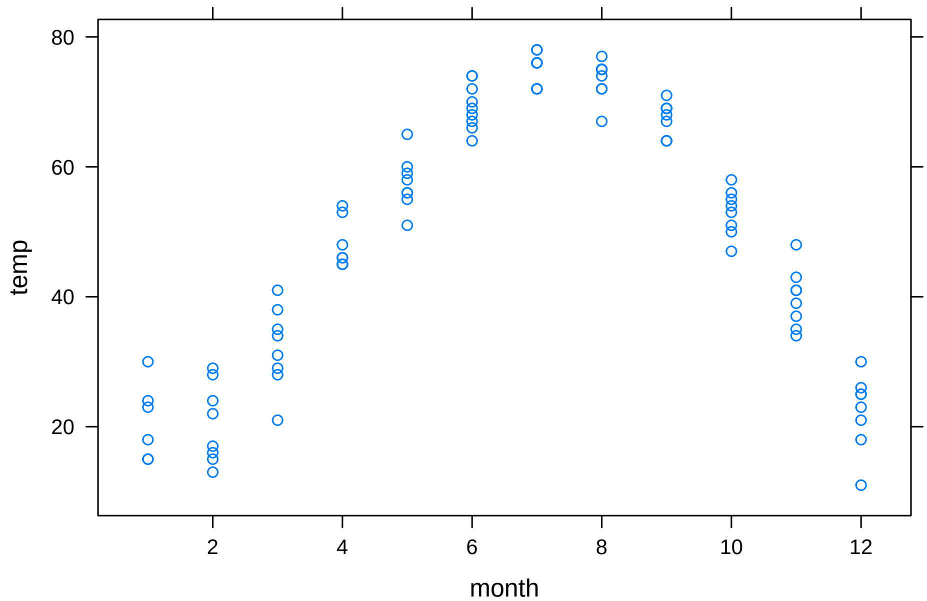
```
## Retrieving from http://www.mosaic-web.org/go/datasets/utilities.csv
```

```
head(utilitydata)
```

```
##   month day year temp  kwh ccf thermsPerDay dur totalbill gasbill
## 1     2  24 2005   29  557 166          6.0  28    213.71  166.63
## 2     3  29 2005   31  772 179          5.5  33    239.85  117.05
## 3     1  27 2005   15  891 224          7.5  30    294.96  223.92
## 4    11  23 2004   43  860  82          2.8  29    160.26   88.51
## 5    12  28 2004   23 1160 208          6.0  35    317.47  224.18
## 6     9  26 2004   71  922  15          0.5  32    117.46   21.25
##   elecbill notes
## 1    47.08
## 2    62.80
## 3    71.04
## 4    71.75
## 5    93.29
## 6    96.21
```

Now let's plot temperature versus month

```
plotPoints(temp ~ month, data=utilitydata)
```

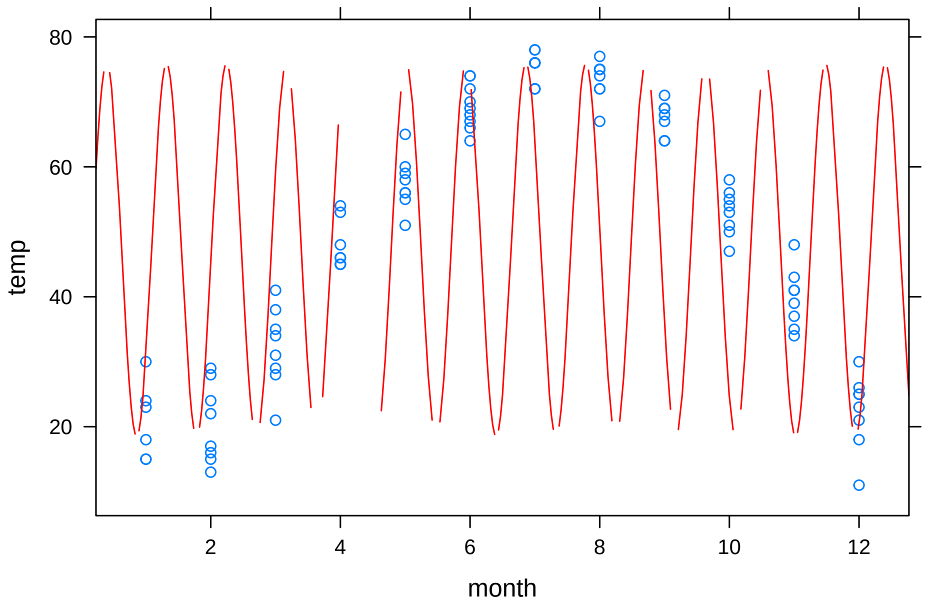


We have ploted monthly temperature data for multiple years. Let's try to fit a periodic function.

```
M = fitModel(temp ~ A * sin(w*(month + p))+C, data=utilitydata)
coef(M)
```

```
##          A          w          p          C
## -28.67993  -6.78753  56.15894  47.20908
```

```
f = makeFun(-28.7 * sin(-6.8*(t + 56.2))+47.2 ~ t)
plotFun(f(t)~t, add=TRUE, col='red')
```

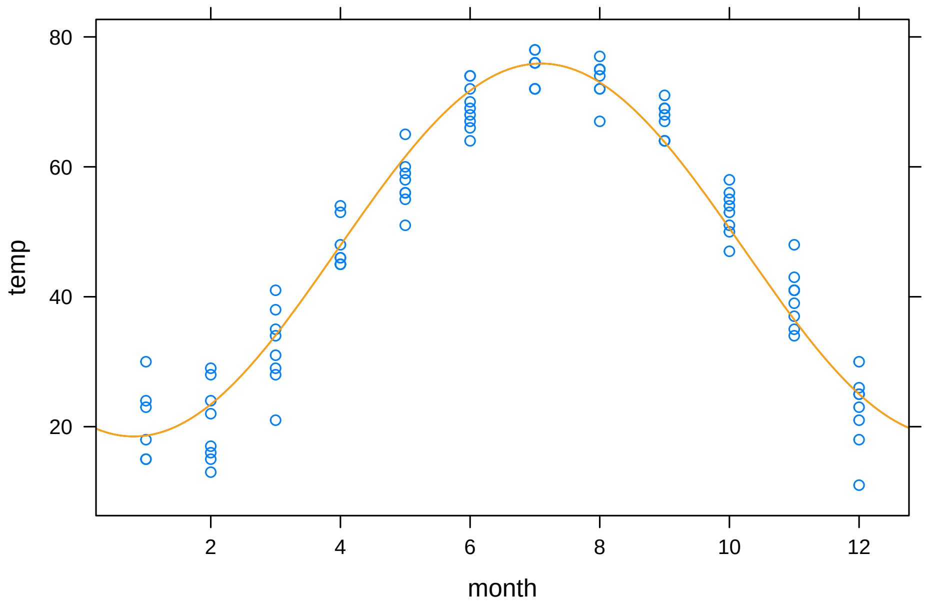


This is not what we wanted! It looks like `fitModel` was confused by the multiple data points for each month. So the frequency `w` is completely wrong. We can correct this by specifying an initial guess for `w`. We think that the period should be 12 months, so we want to start with `w=2*pi/12`.

```
M = fitModel(temp ~ A * sin(w*(month + p))+C, data=utilitydata, start=list(w=2*pi/12))
coef(M)
```

```
##          w          A          p          C
##   0.5043435 -28.6799747 -22.8387987  47.2090049
```

```
f = makeFun(-28.7 * sin(0.5*(t -22.8))+47.2 ~ t)
plotFun(f(t)~t, add=TRUE, col='orange')
```

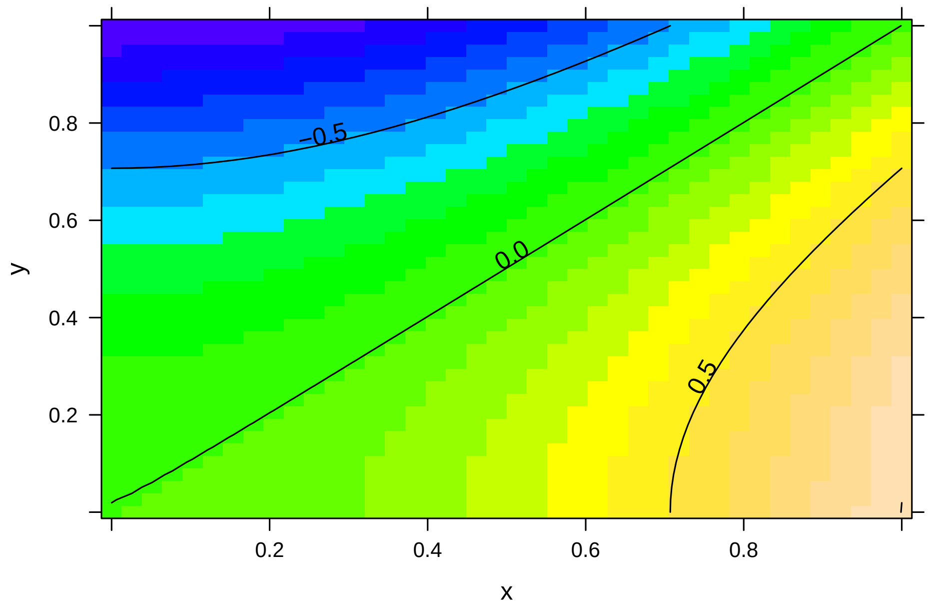


That looks much better!

2.8 Plotting Surfaces in 3D

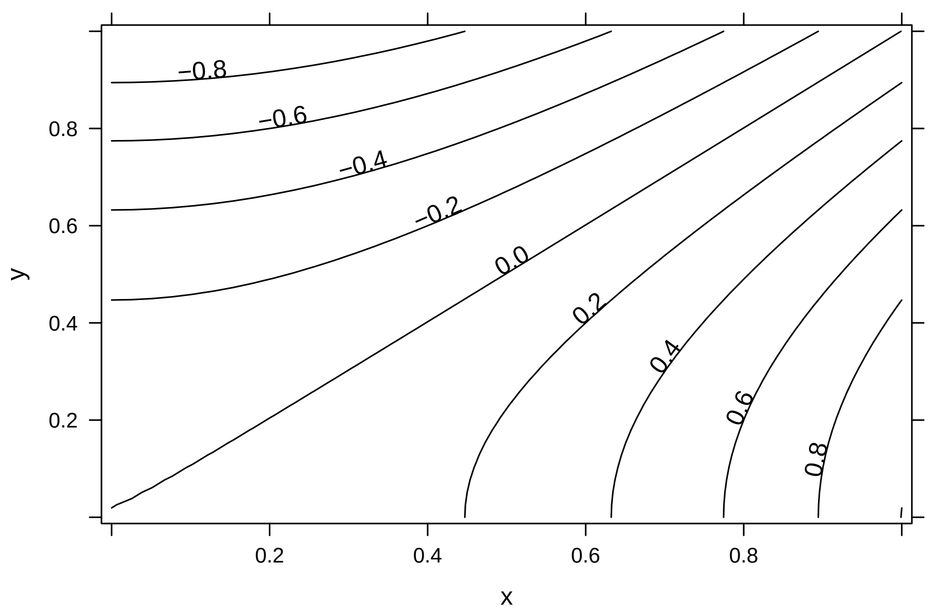
We plot functions $z = f(x, y)$ using the `plotFun()` command. We specify the independent variables with the syntax `~ x&y`. By default, RStudio will create a **contour plot** of this function and include **shading** to help visualize the values of the function.

```
f = makeFun(x^2 - y^2 ~ x&y)

plotFun(f(x,y) ~ x&y)
```



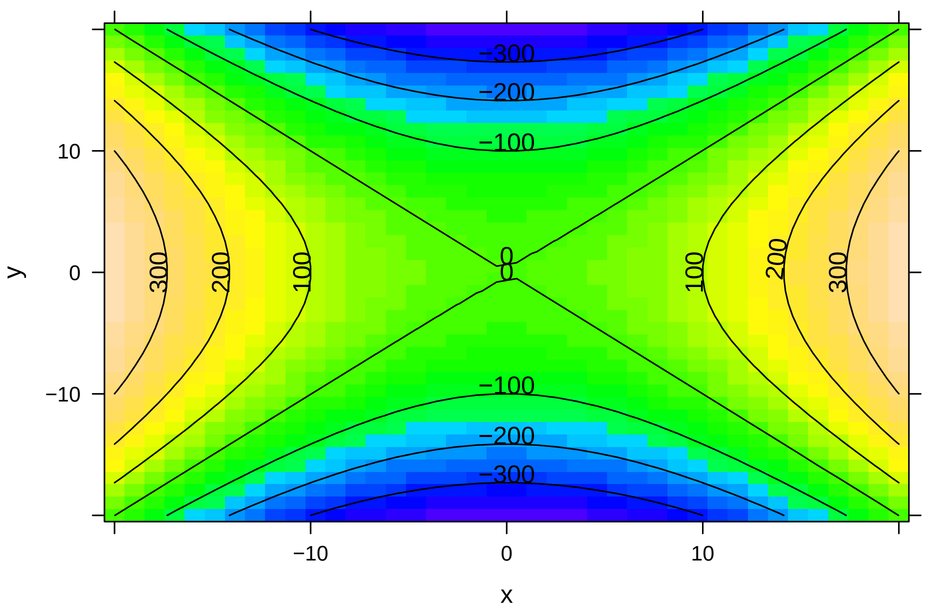
You can add more contours using the `levels` parameter, and turn off the shading with the `filled` parameter.

```
plotFun(f(x,y) ~ x&y, levels = seq(-1 ,1, .2), filled = FALSE)
```



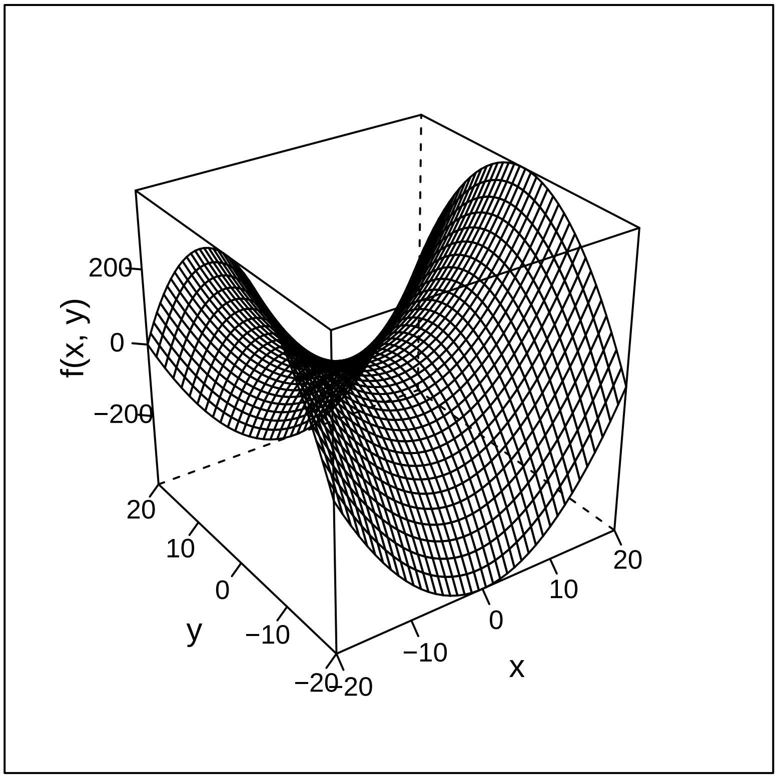
You can change the plot range using `xlim` and `ylim`.

```
plotFun(f(x,y) ~ x&y, xlim=c(-20,20), ylim=c(-20,20))
```



You can create a 3D plot by setting `surface=TRUE`.

```
plotFun(f(x,y) ~ x&y, xlim=c(-20,20), ylim=c(-20,20), surface=TRUE)
```



2.9 Estimating the Derivative

We can estimate the derivative of $f'(x)$ by using the average rate of change

$$f'(x) \approx \frac{f(x+\alpha) - f(x)}{\alpha}$$

for some small number $\alpha > 0$. Smaller and smaller α's lead to better and better approximations.

So we can estimate the derivative by using smaller and smaller α values until our estimates stabilize. Here is an example that estimates $f'(4)$ for the function $f(x) = x^2 + 10 * \sin(x)$.

```
f=makeFun(x^2+10*sin(x)~x)
alpha=c(1, 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001, 0.0000001)
d=(f(4+alpha)-f(4))/alpha
d
```

```
## [1] 6.978782 1.952538 1.511513 1.468349 1.464042 1.463612 1.463569
## [8] 1.463564
```

The estimates stabilize to the value 1.46356. So our estimate is $f'(4) \approx 1.46456$

We can use the same method to estimate the partial derivates of $g(x, y)$. We have

$$\frac{\partial g}{\partial x} \approx \frac{g(x+\alpha, y) - f(x, y)}{\alpha} \quad \text{and} \quad \frac{\partial g}{\partial y} \approx \frac{g(x, y+\alpha) - f(x, y)}{\alpha}$$

Let's estimate the partial derivatives of $g(x, y) = e^{x \sin(y)}$ at the point $(2, 7)$. First, let's estimate $g_x(2, 7) = \frac{\partial g}{\partial x}\Big|_{(2,7)}$. The code is very similar to the code we used for a single variable function.

```
g=makeFun(exp(x*sin(y))~x&y)
alpha=c(1, 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001, 0.0000001)
d=(g(2+alpha,7)-g(2,7))/alpha
d
```

```
## [1] 3.456634 2.526691 2.452648 2.445403 2.444680 2.444608 2.444601
## [8] 2.444600
```

We conclude that $g_x(2, 7) \approx 2.445$.

Now let's estimate $g_y(2, 7) = \frac{\partial g}{\partial y}\Big|_{(2,7)}$. All we need to do is move `alpha` into the y-variable.

```
g=makeFun(exp(x*sin(y))~x&y)
alpha=c(1, 0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001, 0.0000001)
d=(g(2,7+alpha)-g(2,7))/alpha
d
```

```
## [1] 3.512524 5.761618 5.628032 5.612215 5.610611 5.610450 5.610434
## [8] 5.610433
```

We conclude that $g_y(2, 7) \approx 5.610$.

2.10 Gradient Search

Here is code that will use gradient search to find the maximum of the function

$$f(x, y) = -x^4 - x^3 + 10xy + 2y - 8y^2$$

whose partial derivatives are

$$\frac{\partial f}{\partial x} = -4x^3 - 3x^2 + 10y \quad \text{and} \quad \frac{\partial f}{\partial x} = 10x + 2 - 16y$$

First, you define the partial derivatives and then choose your starting point `(newx, newy)`. In this case, we start at `(1,1)`.

```
partialx=makeFun( -4*x^3 -3*x^2 + 10*y~x&y)
partialy=makeFun(10*x + 2 - 16*y~x&y)
newx = 1
newy = 1
```

Next, you repeatedly run the following code block, which updates the current point by moving `0.1` times the gradient vector. This is equivilant to taking a small step in the uphill direction.

Repeatedly to run this code block until the partial derivatives are essentially zero (at least two zeros after the decimal point). Congrats! You have found your local maximum.

```
newx=newx+0.1*partialx(newx,newy)
newy=newy+0.1*partialy(newx,newy)
# print new partial derivatives
c(partialx(newx,newy), partialy(newx,newy))
```

```
## [1] -4.858  0.600
```

```
# print new point
c(newx, newy)
```

```
## [1] 1.3 0.9
```

Repeatedly running this code block will take you to the point `(1.04,0.77)`. But note that starting at another initial point might take you to a different local maximum.

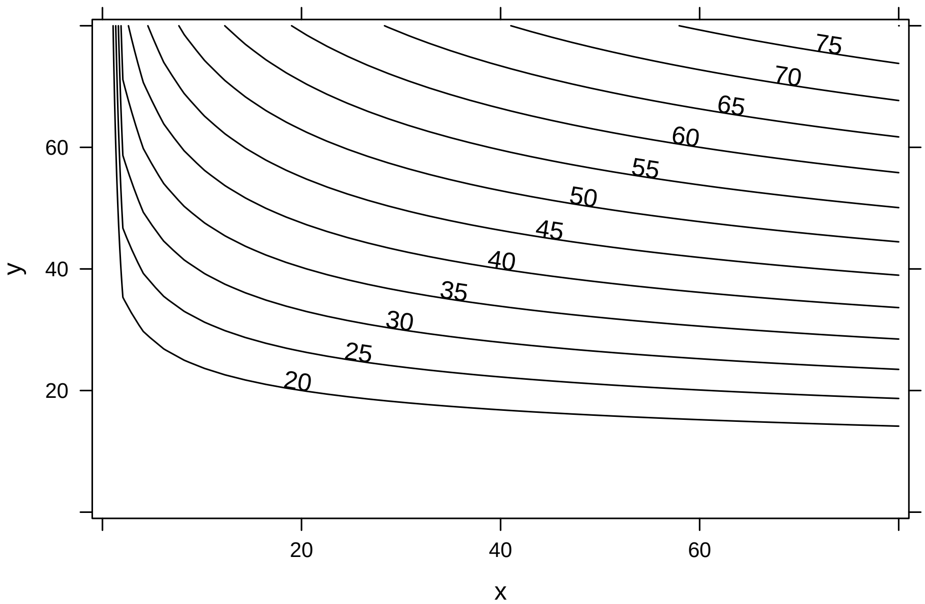
If you want to find a local minimum, then you should multiply the partials by `-0.1` instead. This is equivalent to taking a small step downhill. Try this on the function $f(x, y) = x^2 + 2xy + 3x + 4y + 5y^2$.

2.11 Constrained Optimization

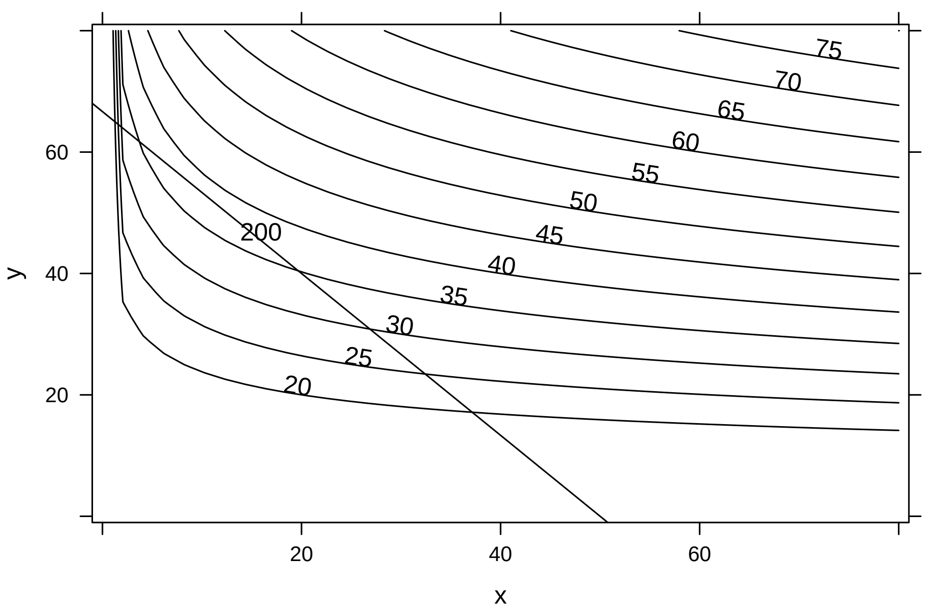
Suppose that we want to find the maximum of the production function $P(x, y) = 3000x^{0.2}y^{0.8}$ given the constraint $Q(x, y) = 4x + 3y = 200$.

We want to make a contour plot of $P(x, y)$ and then add the constraint. The maximum value is achieved where the constraint $Q(x, y) = 300$ is tangent to the contour curve of $P(x, y)$.

```
P = makeFun( x^(0.2) * y^(0.8) ~ x&y)
Q = makeFun( 4*x + 3*y ~ x&y)

plotFun(P(x,y)~x&y, xlim=c(0,80), ylim=c(0,80), filled=FALSE, levels=seq(20,80,5))
```

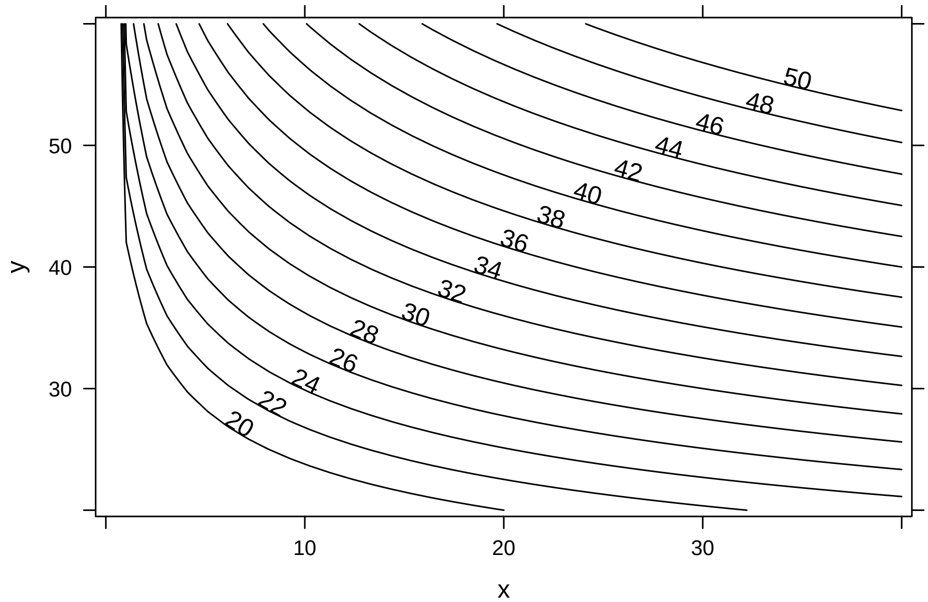



```
plotFun(Q(x,y)~x&y, add=TRUE, filled=FALSE, levels=200)
```

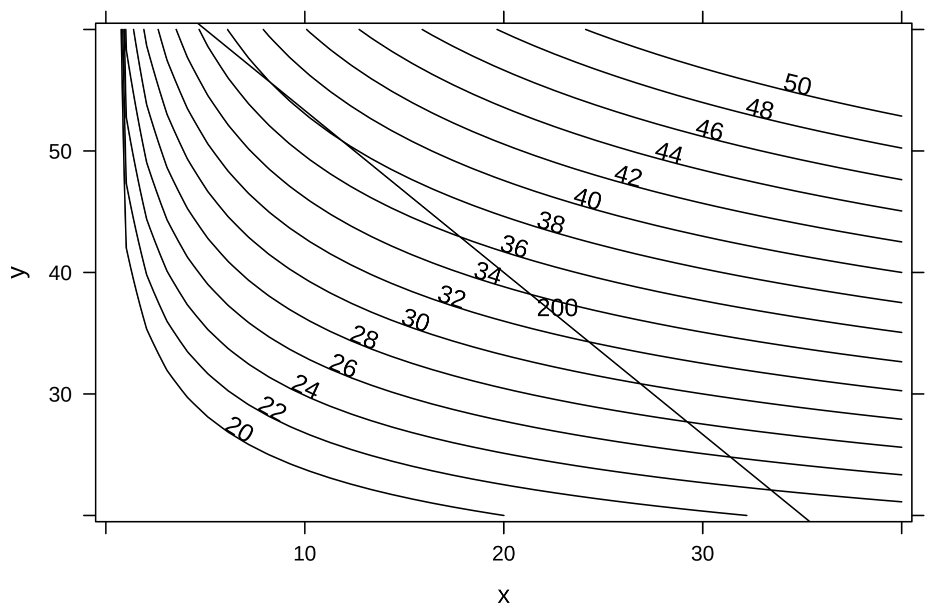


In this case, the optimal value of 38 is achieved at approximately $(10, 55)$. Let's make another plot to get a better estimate.

```
plotFun(P(x,y)~x&y, xlim=c(0,40), ylim=c(20,60), filled=FALSE, levels=seq(20,50,2))
```



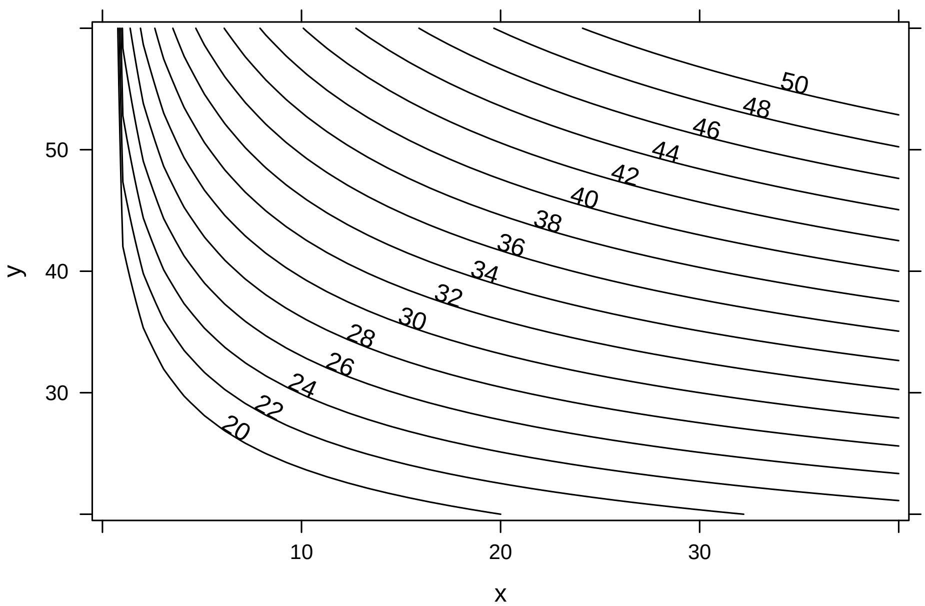

```
plotFun(Q(x,y)~x&y, add=TRUE, filled=FALSE, levels=200)
```



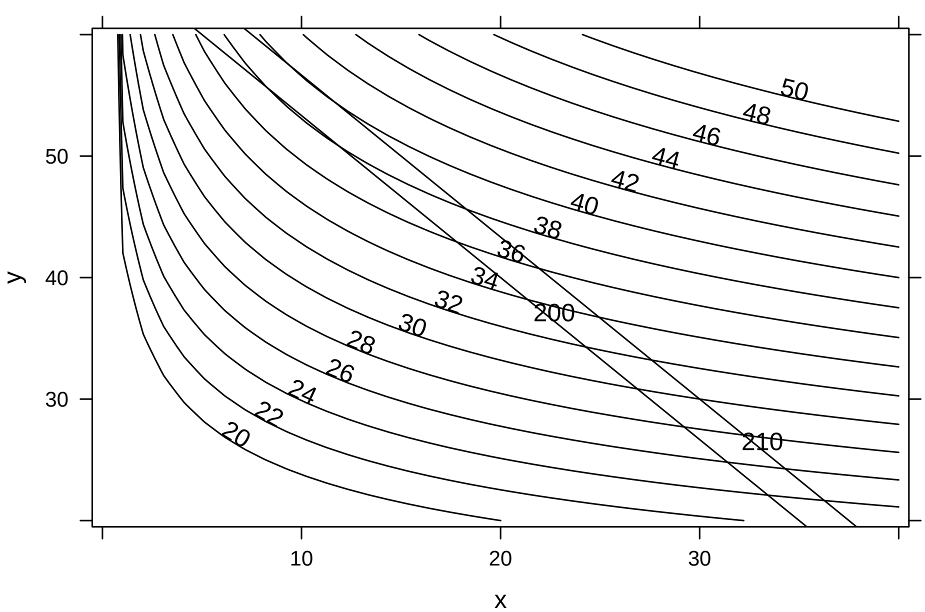
On this zoomed in plot, we can refine our estimate a bit. The maximum $f(10, 53) = 38$ is achieved at $x = 10$ and $y = 53$.

Now, if I want to estimate the Lagrange multiplier λ, I would change my constraint a little bit (say to 210) and then look at the change in the optimal value.

```
plotFun(P(x,y)~x&y, xlim=c(0,40), ylim=c(20,60), filled=FALSE, levels=seq(20,50,2))
```




```
plotFun(Q(x,y)~x&y, add=TRUE, filled=FALSE, levels=c(200,210))
```



We have

$$\lambda = \frac{\triangle\text{optimal value}}{\triangle\text{constraint}} = \frac{40-38}{210-200} = \frac{2}{10} = 0.2.$$

2.12 Estimating the Definite Integral

The define integral $\int_a^b f(x)dx$ is the (signed) area between the curve $y = f(x)$ and the x-axis. We approximate this area by partitioning the interval $[a, b]$ into smaller intervals of width $\triangle x$ and then summing the areas $f(x)\triangle x$ of the corresponding rectangles.

Here is code that makes this approximation for the function $f(x) = x + \sin(x^2)$ over the interval $[5, 10]$. The code splits this interval into 100 subintervals. You can get a better approximation by increasing the denominator of `base=(b-a)/100` to a much larger number, for example: `base=(b-a)/1000000`.

```
f=makeFun(x + sin(x^2)~x)
a=5
b=10
base=(b-a)/100
points=seq(from=a+base, to=b, by=base)
heights=f(points)
areas=base*heights
sum(areas)
```

```
## [1] 37.67298
```

2.13 Troubleshooting

Without fail, you will run into problems. Sometimes RStudio will give you helpful error messages, but sometimes they are obtuse. Here's a list of some common errors and what might be wrong:

- *Nothing ran, all I got was a plus sign...* You probably forgot some parentheses. Try the following:
  - Enter a parenthesis on that line to see if this closes it out.
  - Press the Escape key to get to a new line.
- *Unexpected symbol in...* You may have forgotten a multiplication sign or otherwise gave RStudio a symbol it cannot parse. Check your line of code again to see if you have all your symbols.
- *Object not found...* You may not have defined that variable (or function) yet. Check your environment tab to see if it has the object you're looking for (or if you've called it something different than you remember!). You also

may be trying to use something from a data set that you have not included as a parameter. See if you need to add in the parameter `data=...`.

- *Could not find function...* RStudio doesn't know what function you're using. Check your capitalization and that you have the right packages installed.
- *... number of items to replace is not a multiple of replacement length ...* Your input variable doesn't match the name variable you used in the output when you called `makeFun`. Maybe you used `x` on one side of the tilde `~` and `t` on the other side?
- *Argument "name" is missing, with no default...* You're probably trying to take a derivative or antiderivative without the mosaicCalc package. Go back to the packages section of the introduction for how to fix this.
- *I can't find my data!* You may have fetched it but did not give it a name. Go back to your fetch command and make sure you have assigned it a variable.

Chapter 3

Intro to RStudio

Welcome to our RStudio Introduction! RStudio is a front end (user friendly) interface for R, a programming language for statistics. We'll be using it to work with data and our various forms of functions.

RStudio is a **long term learning goal** for us. We will be adding new functions and learning new things throughout the semester.

3.1 Getting Started

Follow my lead as we get ourselves going with RStudio. We will

- Log in to RStudio
- Add some *packages* to the R environment
- Create a math135 directory to work in
- Create our first Rmd ('R Markdown') file
- Knit that Rmd file to create a PDF

You will need to replace the auto-generated code block at the top of your Rmd file with this one, which loads the mosaic functionality into the R environment when we knit to PDF.

````
```{r setup, include=FALSE, warning=FALSE}
knitr::opts_chunk$set(echo = TRUE)
suppressPackageStartupMessages(library(mosaic))
```
````

Pro Tip: Be kind to future you.

- Save all of your work for this class in the `math135` subdirectory that we create today.

- Give your files meaningful names. Include the unit, the topic, the date, etc, or a helpful combination of these identifiers.

3.2 R Markdown files

An R Markdown file is a hybrid file that has both text and code. You can then **knit** this Rmd file to make a lovely PDF.

3.2.1 Formatting Regular Text

The regular text uses Markdown, which has simple commands to create formatting such has sections, bold, italic, links, itemized lists, and much more. Here are is some example Markdown syntax.

```
# Heading

This is a regular sentence.

## Sub Heading

You get *italics like this.*

Skipping a line starts a new paragraph.
But if you don't skip a line then it's part of the same paragraph

### Sub Sub Heading

You get **bold text like this.**

Here is how to make an itemized list

* Make sure to skip a line before your first item
* The asterisk must be the **first** character on the line (no spaces!)
* If you want a sublist
   + Start the sublist on the line below the item
   + Indent by three spaces.
   + Use a plus sign
```

Here is a nice reference page for common R Markdown syntax.

3.2.2 Code Chunks

Your R code must be placed inside **code chunks.** Here is what a code chuck looks like

````
```{r}
1 + 1

sin(pi/4)
```
````

When you knit your Rmd file, the code chunks are executed in order. You can also run an individual chunk by clicking on the green arrow in its upper right corner.

Adding a code chunk is easy!

- In the top bar of the Markdown window, there's a small, green plus C button.
- When you click this button, it'll bring up options for code blocks to insert
- Select "R" which is the first option. (It's the only one we will use.)

3.3 Writing Code

Add a code chunk and let's get cracking!

3.3.1 Basic Calculations

We can use that code chunk to do some basic calculations.

3.3.1.1 You Try

Create a code block and try to calculate each of the following. Try some other calculations, too!

- $8 + 13$
- $7 * 9$
- $\sin(\pi/2)$
- $\sqrt{121}$
- $(2.25)^3$
- $8\cos(\pi/6)$

Which ones work? If you got an error, can you fix it? Can you guess the name of the square root function?

3.3.2 Variables, Lists and Sequences

When we do a calculation, we can store the result for later. Here are some examples:

```
a = 15
b = 4 * a
```

R is particularly good at doing calculations on lists of data. So let's make some lists!

RStudio makes a list with the `c()` command. The "c" in short for "combine".

```
P=c(3, 5, 11, -1, 4)
P
```

```
## [1]  3  5 11 -1  4
```

If you want to make a sequence, you use the `seq()` command. For example:

```
S=seq(1, 10)
S
```

```
##  [1]  1  2  3  4  5  6  7  8  9 10
```

Notice that the values increment by 1. You can use a different step size by adding a third argument. For exmaple, if I wanted a sequence 1, 1.1, 1.2, 1.3, etc… all the way up to 2, I would write

```
T=seq(1, 2, 0.1)
T
```

```
##  [1] 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0
```

Now look at the **Environment** tab in the upper right window. What do you notice?

Pro Tips: Be kind to future you as well as current collaborators.

- Always write `0.1` instead of `.1` in your code. It is much easier to read and to debug later!
- Consider adding a space after your commas to separate parameters. It's easier to read!

3.3.2.1 You Try

Try to do the following:

- Create a list `X` containing the values 8, 6, 7, 5, 3, 0, 9.
- Create a list `Y` containing the sequence of odd integers between (and including) 1 and 19
- What is the output of the command 'X^2'? Can you see what R is doing?

- Try making another lists. Apply some functions to that list.

3.3.3 Functions

Now it's time to write your own function. Here is how you create $f(x) = x^2 - 3$.

```
f = makeFun(x^2 - 3 ~ x)
```

Much of this is pretty intuitive, except for maybe the final `~ x`. This tilde symbol comes up a lot in RStudio and it indicates *dependence*. We need to tell RStudio that f is in terms of this variable x, so it knows where to put an input when I ask it to:

Notice! In the right upper window, you now have a function called `f`. We can do a lot with this function, but to start, let's evaluate it at 1.

```
f(1)
```

```
## [1] -2
```

What do you think this code is doing?

```
f(S)
```

```
##  [1] -2  1  6 13 22 33 46 61 78 97
```

3.3.3.1 You Try

- Create a new function g where $g(t) = \sqrt{t-4}$
- Evaluate this function at $t = 8$, and $t = 12.34$ and $t = 100$
- What happens when you try to evaluate this function at $t = 2$? What do you think that means?

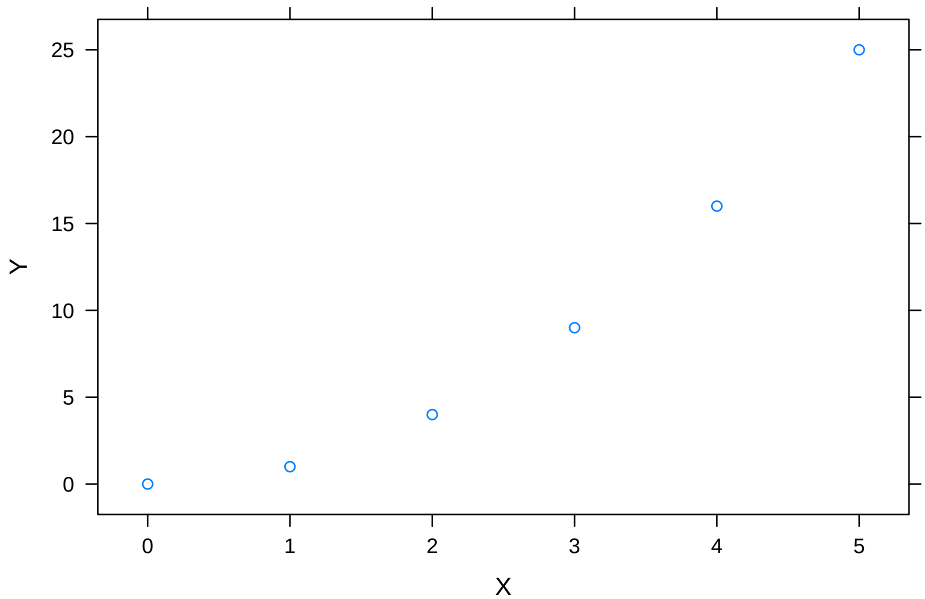
3.3.4 Plotting

Let's see how to create plots using RStudio. First, let's create a plot of this data

x	0	1	2	3	4	5
h(x)	0	1	4	9	16	25

from the function $h(x) = x^2$.

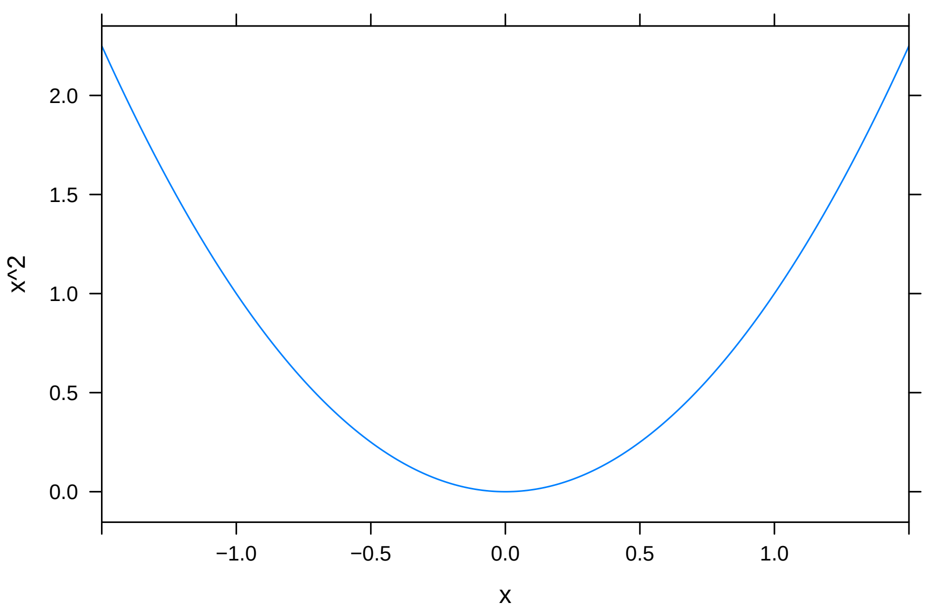
```
X = seq(0,5)
Y = X^2
plotPoints(Y ~ X)
```



Notice! We list the **dependent** variable `Y` first and then the **independent** variable `X` after the tidle `~` (which matches the syntax of `makeFun`).

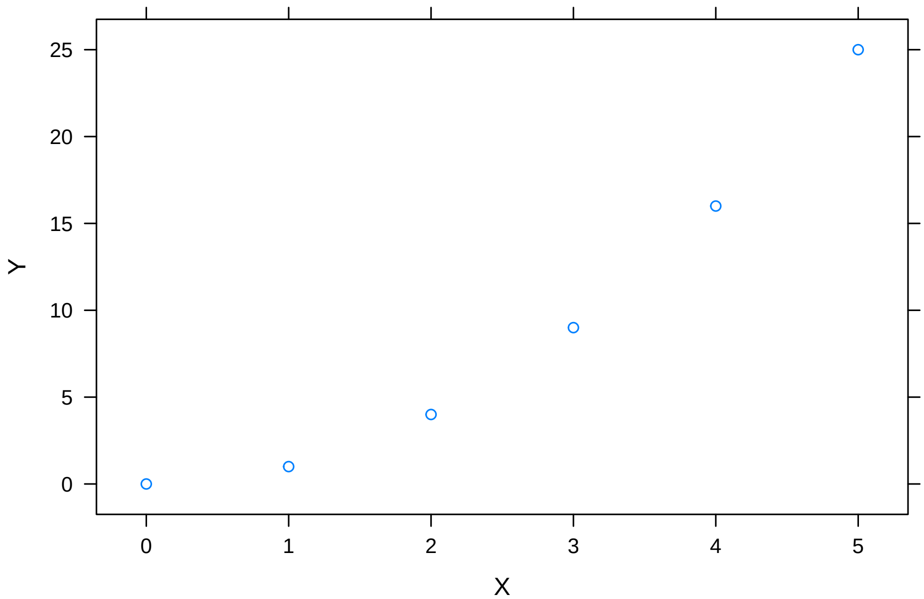
We can also plot a function. Once again the dependent variable appears after the tilde.

```
plotFun(x^2 ~ x)
```

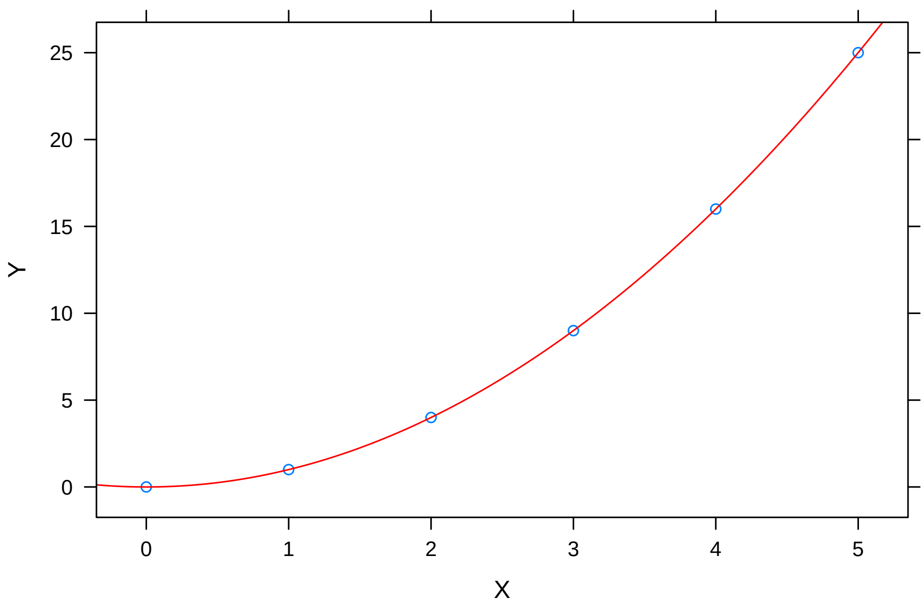


We can actually plot the data and the function on the same graph. We will plot the data and then add the function by using the `add=TRUE` parameter for `plotFun`. I will also use the optional `col` parameter to change the color of the function curve.

```
plotPoints(Y ~ X)
```




```
plotFun(x^2 ~ x, add=TRUE, col='red')
```



3.3.4.1 You Try

- Create a sequence of integer points `P` from 1 to 10.
- Define `Q = sin(P)`
- Create a plot of `Q` versus `P` and then add the function plot of `sin(x)`
- Try plotting some other data and fuunctions.

3.4 Troubleshooting

Without fail, you will make mistakes. Things will break and you'll get error messages. That's ok! That's how we learn.

3.4.1 You Try

Each of the following lines of code will break when run. Try them out, notice the error, and see if you can fix it.

- `2 sin(pi/2)`
- `sqrt(3*(7+5)`
- `Sqrt(3*(7+5))`
- `makeFun(x^2-4, x)`
- `makeFun(x+1 ~ t)`
- `makefun(2x+1 ~ x)`

Part I

Functions

Chapter 4

Functions

4.1 Steps of the Modeling Cycle

1. Ask a question about reality.
2. Make some observations and collect the corresponding data.
3. Conjecture a model or modify a known model based on the data.
4. Test the model against known data (from step 2.) and modify the model as needed.
5. Repeat steps 2, 3 and 4 to improve the model

4.2 Goals

- Explain what a function is
- Identify and explain independent and dependent variables
- Recognize functions in a variety of forms
- Find a function's intercept(s)
- Identify regions where a function is increasing/decreasing
- Compute the average rate of change between two points on a function
- Determine where a function is concave up/down
- Compute the relative change between two points on a function

4.3 Activities

4.3.1 Life insurance

A family needs to purchase life insurance. What possible inputs might be used to determine the premium (monthly payment) for life insurance?

4.3.2 Bat species

In the Andes mountains in Peru, the number N of species of bats is a *function* of the elevation h in feet above sea level.

1. What is the independent variable?
2. What is the dependent variable?
3. Interpret the statement $f(500) = 200$ if f represents this function.
4. Do you expect this function to be increasing or decreasing? Why?

4.3.3 Slope and concavity

Sketch each of the following:

1. An increasing function that is concave up
2. An increasing function that is concave down
3. A decreasing function that is concave up
4. A decreasing function that is concave down

4.3.4 Relative change of the Dow Jones average

The *relative change* of a function $f(x)$ on interval $[a, b]$ is

$$\frac{f(b) - f(a)}{f(a)}.$$

1. What is the *relative change* in the Dow Jones average from 169.84 to 77.90 (from 1 January 1931 to 31 December 1931)?
2. Compare this to the *relative change* in the Dow Jones average from 35,443.82 to 35369.09 (from 2 September 2021 to 3 September 2021).

4.3.5 Measuring rainfall

One rainy summer day, measurements were recorded (in inches) from a rain gauge every hour.

Time	8 am	9	10	11	noon	1 pm	2	3	4	5 pm
Amount	0.15	0.17	0.2	0.45	0.48	0.75	1.03	1.20	1.45	1.60

1. What was the average rate of rainfall from 8 a.m. to 5 p.m.?
2. What are the units of this rate of change?

Chapter 5

Linear Functions

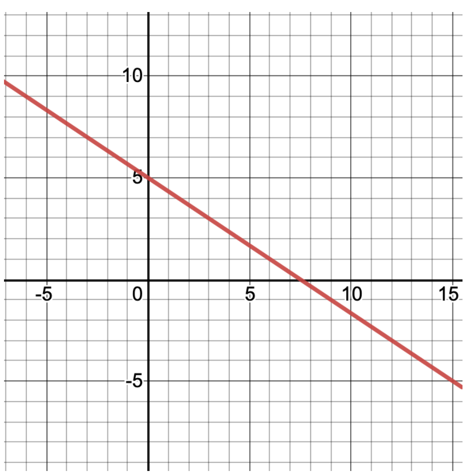
5.1 Goals

- Explain linear functions in words, tables, equations, and graphs
- Describe the effects of varying parameters in linear functions
- Assess whether given data is linear and, if so, model it with a function

5.2 Activities

5.2.1 Graph of a linear function

Find the equation $y = mx + b$ for the graph shown below.



5.2.2 Height and weight of American men

The table below lists the average weight, w, in pounds of American men in their sixties for height, h, in inches.

Height (inches)	68	69	70	71	72	73	74	75
Weight (pounds)	166	171	176	181	186	191	196	201

1. Which is the dependent variable and which is the independent variable?

2. Is this an increasing function or a decreasing function?
3. Can this be represented by a linear function? If so, find an equation to describe this data.

5.2.3 Scooter rates

A scooter company charges for monthly service according to the formula:

$$C(t) = 5.99 + 0.23t$$

where t is the number of minutes minutes ridden.

Find and interpret the rate of change and the vertical intercept.

5.2.4 Oil reserves

There were 1650 billion barrels of proven oil reserves in the world as of 2016. At constant consumption levels (and excluding unproven reserves), the world had enough oil to last about 47 more years beyond 2016.

1. Create a linear function $R(t)$ that models the oil reserves (in *billions* of barrels) where t is the number of years since 2016.
2. Find and interpret the rate of change and the vertical intercept of your function.

1. Explain why a linear model of this data is reasonable.
2. Come up with an equation that (approximately) describes this data.
3. What decision(s) did you make in part (b)? What are the strengths and weaknesses of your choice?
4. Create a plot in RStudio that shows both the data points and the linear function that you found in part (b).
   - Start by doing a cut-and-paste the R code above
   - You can add your linear function to this plot using `plotFun(m*t+b~t,add=TRUE)` where `m` and `b` are the slope and the vertical intercept of the function you found in part (b)

–>

Chapter 6

Average Rate of Change

6.1 Goals

- Calculate average rate of change (AROC) on an interval
- Recognize when a function is increasing or decreasing
- Recognize when a function is concave up or concave down

6.2 Activities

6.2.1 AROC of a function

Find the average rate of change (AROC) of the function $f(x) = 2x^2 - 3x + 1$ on the interval between $x = -1$ and $x = 3$.

6.2.2 Measuring daily rainfall

One rainy summer day, hourly measurements of the total rainfall were recorded (in inches) by a rain gauge.

Time	8 am	9	10	11	noon	1 pm	2	3	4	5 pm
Amount	0.15	0.17	0.2	0.45	0.48	0.75	1.03	1.20	1.45	1.60

1. What was the average rate of rainfall from 8 a.m. to 5 p.m.?
2. What are the units of this rate of change?

6.2.3 Sketching a function

1. Sketch a *nonlinear* function that has an average rate of change of 1 on the interval $[0, 2]$.
2. Compare sketches at your table. Are they similar?
3. Sketch a function with an AROC of 1 on $[0, 2]$ that is *very different* from everyone else's function.

The take away: AROC is an estimation of the behavior of the function on the interval. But there is no guarantee about the quality of the estimate.

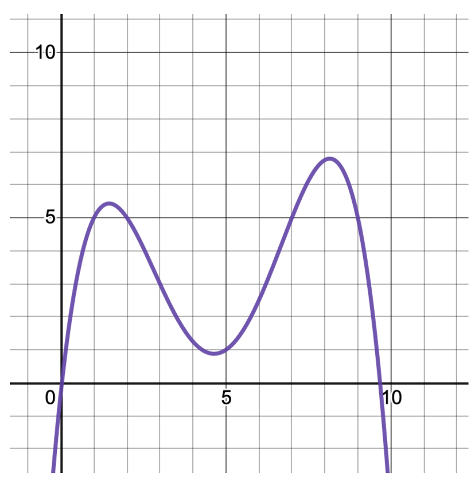
6.2.4 Slope and Concavity

Sketch an example of a function that is

1. Increasing and concave up
2. Increasing and concave down
3. Decreasing and concave up
4. Decreasing and concave down

6.2.5 Example Function

Here is the graph of a function.



Find the subintervals of $[0, 10]$ where the function is

1. Increasing
2. Decreasing
3. Concave up
4. Concave down

6.2.6 Vinyl LPs

Let $V(t)$ denote the number of vinyl LP records sold in the US (in millions) in year t.

t	1995	2000	2005	2010	2015	2020
$V(t)$	0.8	1.5	0.9	2.8	11.9	27.5

1. Calculate the average rate of change for each of the five year intervals 1995-2000, 2000-2005, 2005-2010, 2010-15, 2015-2020.
2. Use your answers from part 1 to determine whether the function $V(t)$ is increasing or decreasing on these five year intervals.
3. Use your answers from part 1 to estimate whether the function $V(t)$ is concave up or concave down at each of the points 2000, 2005, 2010 and 2015.

6.2.7 Relative change of the Dow Jones average

The *relative change* of a function $f(x)$ on interval $[a, b]$ is

$$\frac{f(b) - f(a)}{f(a)}.$$

1. What is the *relative change* in the Dow Jones average from 169.84 to 77.90 (from 1 January 1931 to 31 December 1931)?
2. Compare this to the *relative change* in the Dow Jones average from 35,443.82 to 35369.09 (from 2 September 2021 to 3 September 2021).

Chapter 7

Exponential and Logarithmic Functions

7.1 Goals

- Explain exponential functions in words, equations, and graphs
- Describe the effects of varying parameters in exponential functions
- Assess whether given data is exponential, and if so, model it
- Solve for unknowns in an exponent using the logarithm
- Write exponential functions in equivalent forms
- Determine half-lives and doubling times

7.2 Activities

7.2.1 Growing population

A population starts with $50,000$ organisms and grows by 6.7% each year. Find an exponential model for the population.

7.2.2 Consumer price index

The consumer price index (CPI) for a given year is the amount of money in that year that has the same purchasing power as \$100 in 1983. At the start of 2009, the CPI was 211. Write a formula for the CPI as a function of t years after 2009, assuming the CPI increases by a rate of 2.8% every year.

7.2.3 Two data sets

Here are approximate values of two functions. For each one, decide whether it could be a linear function or an exponentialfunction. Give a reason and then find an approximate formula.

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	26.00	26.15	26.30	26.45	26.60	26.75	26.90
$g(x)$	190	217	247	282	321	366	418

7.2.4 Interest rate for an investment

Suppose that an investment triples in 15 years.

1. What is the interest rate r if the interest is compounded annually: $I(t) = I_0(1+r)^t$.
2. What is the interest rate r if interest is compounded continuously: $I(t) = I_0e^{rt}$.
3. Which rate is larger? Why does that make sense?

7.2.5 Exponential decay

Radioactive cobalt-60 has a half-life of 5.3 years. If the initial amount is 60 grams, find a formula for the amount of cobalt-60 left after t years.

7.2.6 Doubling time

The doubling time of an exponentially increasing fuunction is the amount of time it takes for the quantity to double. Find the doubling time for the function $f(t) = 10e^{1.2t}$.

7.2.7 People and vehicles

In 1980, there were about 170 million vehicles (cars and trucks) and about 227 million people in the United States. The number of vehicles has been growing at an annual rate of 4% a year, while the population has been growing at an annual rate of 1% a year.

1. According to this model, when was there, on average, one vehicle per person?
2. In desmos, make a plot of the number of vehicles and the number of people over time on the same graph with a different color for each curve.

3. Use your models to predict the number of vehicles and the number of people in 2020. Then use Google to see how accurate these models are over time.

Chapter 8

Power Functions

8.1 Goals

- Describe proportional and inversely proportional relationships
- Recognize power functions by creating a log-log plot.
- Fit a power function to data

When you have data that you suspect comes from a power function

$$y = kx^p,$$

here is how you can find the constants p and k

1. Take the natural log of both the x-axis data and the y-axis data
2. Fit a line to the resulting data
3. Use the slope and y-intercept of this line to find the values of the power p and the constant k.

8.2 Activities

8.2.1 Examples and non-examples

Determine whether each of the following is a power function $y = kx^p$. If so, then find the leading constant k and the power p.

1. $y = 5 \cdot 3^x$
2. $y = 10\sqrt{x}$
3. $y = \dfrac{7}{2x}$
4. $y = 9x^2 - 2$

5. $y = (4x)^3$

8.2.2 Descriptions of power functions

Write a formula representing the function.

1. The energy E expended by a swimming dolphin is proportional to the cube of its speed v.
2. The number N of animal species with body length L is inversely proportional to the square of the body length.

8.2.3 Allometry of pectoral fins

Allometry is the study of the relative sizes of parts of an animal. Often, as an animal grows, the size of a body part will be a power function of its length. Here is a table of fish body lengths L and pectoral fin lengths F.

Body Length (cm)	6.35	7.03	8.17	9.03	9.21	9.39	9.58	11.59
Pectoral Fin Length (cm)	2.72	3.49	4.06	4.95	4.86	5.00	5.05	6.05

1. Create a log-log plot of this data and explain why this provides evidence that there is a proportional relationship between fin length and body length.
2. Find the power function $F = kL^p$ that fits this data.
3. Do the fin lengths increase faster, slower or at the same rate as the body lengths? How do you know?

8.2.4 Specific heat

The specific heat s of an element is the number of calories of heat required to raise the temperature of one gram of the element by one degree Celcius. Use the following data to decide if s is proportional or inversely proportional to the atomic weight w. Then find the function $s(w)$.

Element	Li	Mg	Al	Fe	Ag	Pb	Hg
weight w (g/mol)	6.9	24.3	27.0	55.8	107.9	207.2	200.6
specific heat s (cal)	0.92	0.25	0.21	0.11	0.056	0.031	0.033

Chapter 9

Periodic Functions

9.1 Goals

- Describe periodic phenomena using words, equations and graphs.
- Explain the effect of changing the four parameters of sine and cosine functions.
- Recognize and model periodic phenomena.

9.2 Activities

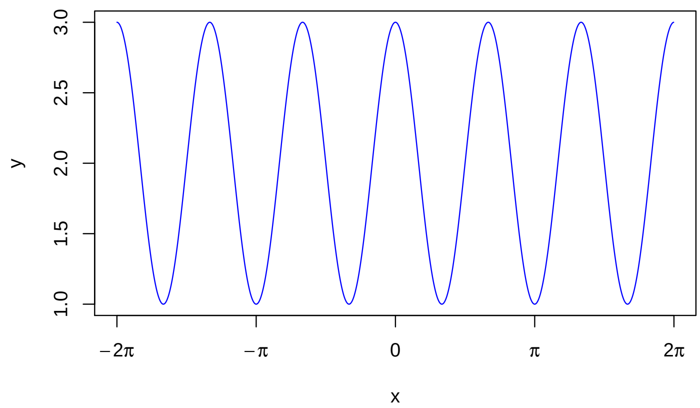
9.2.1 Periodic function examples

Brainstorm at your table! Give four examples of periodic phenomena. Can you think of a periodic phenomenon that doesn't "look like" a sine function. (We've already seen one: the electrocardiagram pattern for a heartbeat),

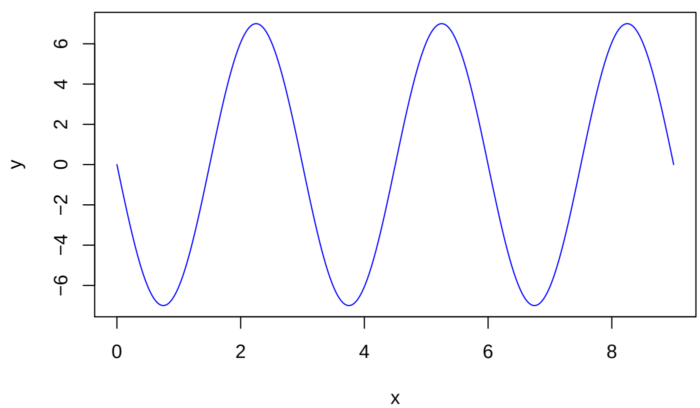
9.2.2 Find a periodic formula

For each graph shown, write down a corresponding periodic formula of the form $A\sin(\omega x) + C$ or $A\cos(\omega x) + C$.

1.



2.



9.2.3 Variable star

Mira is a red giant star in the constellation Cetus. Its brightness (as seen from Earth) fluctuates as it expands and contracts. The apparent magnitude ranges from a maximum of 10th magnitude to a minimum of 2nd magnitude. This apparent brightness has a period of 332 days.

Find a formula that models the brightness of Mira as a function of time, with $t = 0$ at peak brightness.

9.2.4 Tides and currents

Here is a plot of the water level (in red) and the current speed (in blue) of the St. Johns River. Each of them go through two cycles in 24.83 hours. This is the length of one **lunar day**: the time it takes for a point on the Earth to rotate from an exact point under the Moon to that same point under the Moon. (This is slightly longer than one day because the Moon revolves around the Earth in the same direction as the Earth's rotation.) So you can assume that the period for each is 12.415 hours.

Model the water level $w(t)$ and the current speed $c(t)$ using periodic functions, where t is the number of hours since midnight April 28, 1998.

- Be sure to use the appropriate vertical axis for each one!
- Decide which one will be easier to model: $w(t)$ or $c(t)$? Do that one first, and then adjust that function to get the other one.

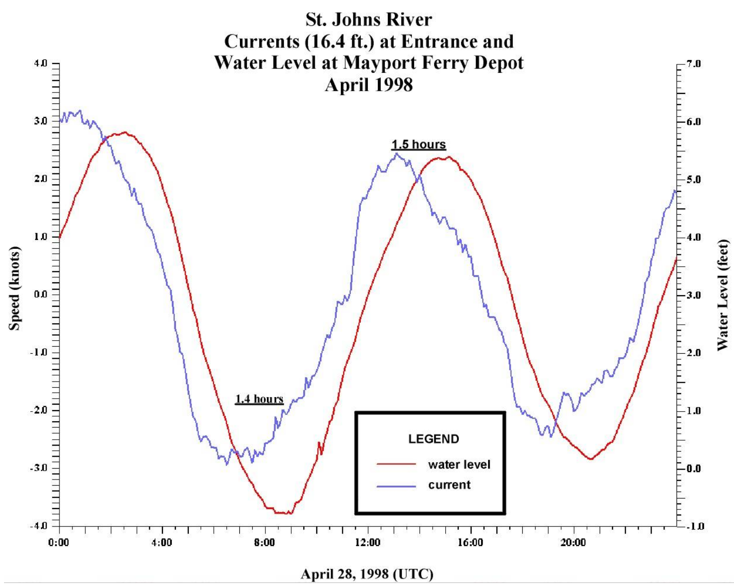


Image Source: https://tidesandcurrents.noaa.gov/

9.2.5 Rainfall in Seattle

Here is the average rainfall in Seattle.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Rainfall (inches)	5.1	3.9	4	2.8	1.9	1.5	0.9	1.2	1.9	3.2	5.4	6.1

Model the monthly rainfall with a periodic function $r(t)$ where $t = 0$ corresponds to January. Your function must attain its maximum in December, so you will need a horizontal shift!

Chapter 10

Multivariable Functions

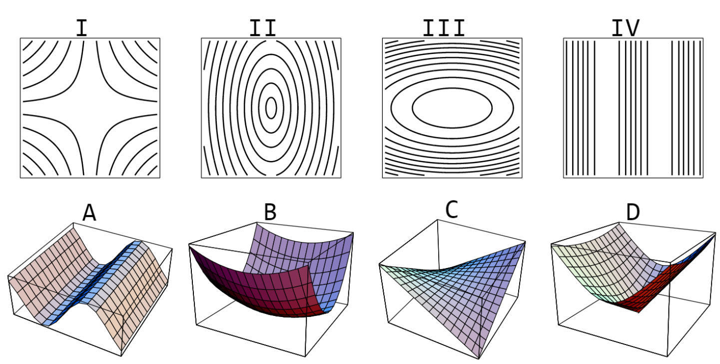
10.1 Goals

- Identify and explain independent and dependent variables in multivariable functions
- Interpret functions represented as tables and equations
- Interpret contour diagrams and make function value estimates
- Interpret cross sections and connect them with contour diagrams
- Use RStudio to create a contour plot and a surface plot of a function $z = f(x, y)$.

10.2 Activities

10.2.1 Matching Contours

Match contours I - IV to surfaces A - D.



10.2.2 Plotting in RStudio

Create these plots using RStudio.

You will need to import the mosaic package. Cut and paste this code chunk into your RMD file and run it. This will load in the `plotFun` command to create surface plots and contour plots.

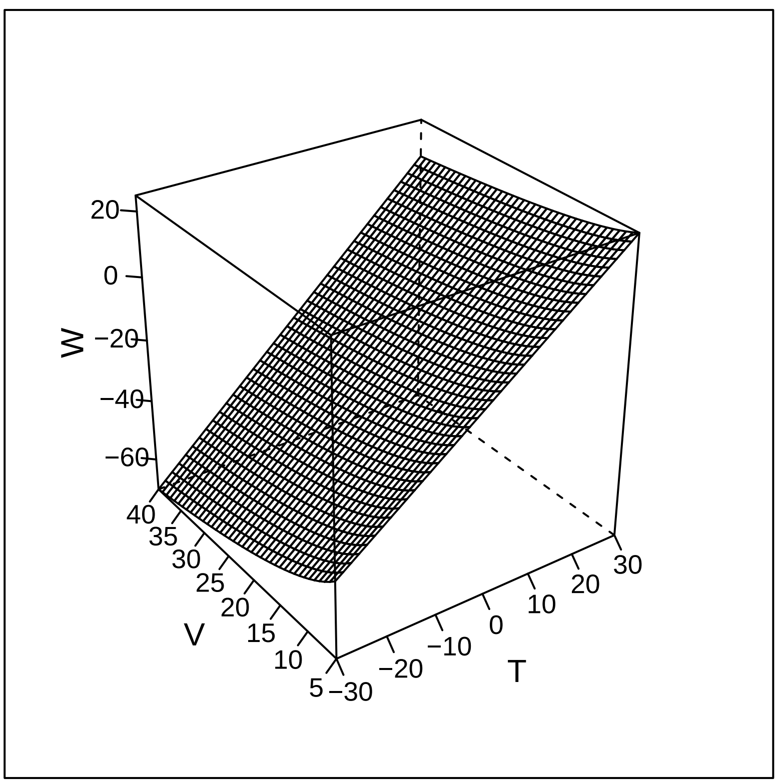
````
```{r setup, include=FALSE, warning=FALSE}
knitr::opts_chunk$set(echo = TRUE)
suppressPackageStartupMessages(library(mosaic))
```
````

1. Windchill Function

$$W(T,V) = 35.74 + 0.6215\,T - 35.75\,V^{0.16} + 0.4275\,T\,V^{0.16}$$

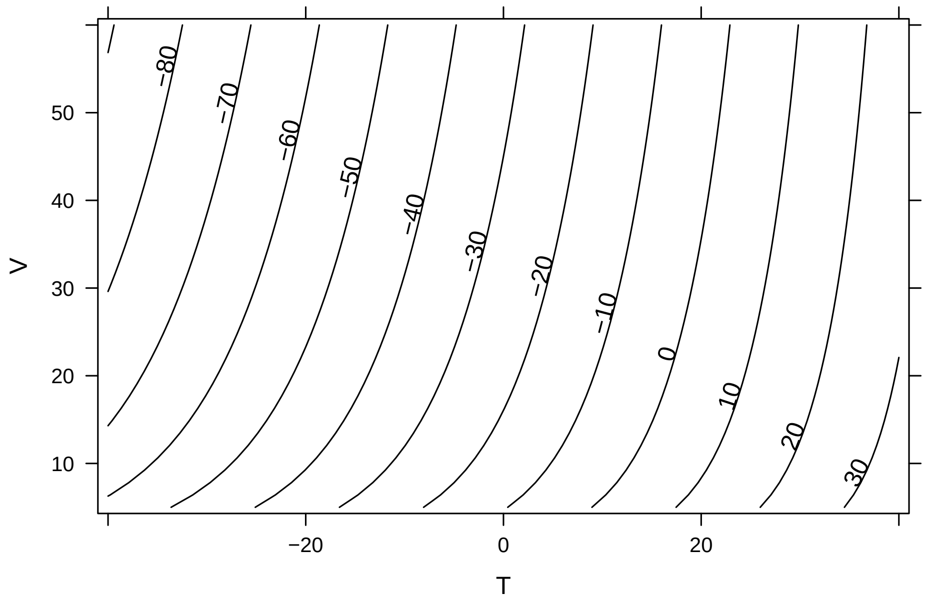
Surface plot

```
plotFun(35.74+0.6215*T-35.75*V^(0.16)+0.4275*T*V^(0.16)~T & V, T.lim=range(-30,30),
V.lim=range(5,40),surface=TRUE,zlab="W",xlab="T",ylab="V")
```



Contour Plot

```
plotFun(35.74+0.6215*T-35.75*V^(0.16)+0.4275*T*V^(0.16)~T & V, T.lim=range(-40,40), filled=FALSE,
V.lim=range(5,60),zlab="V",xlab="T",ylab="V", levels = seq(-100,40,10))
```

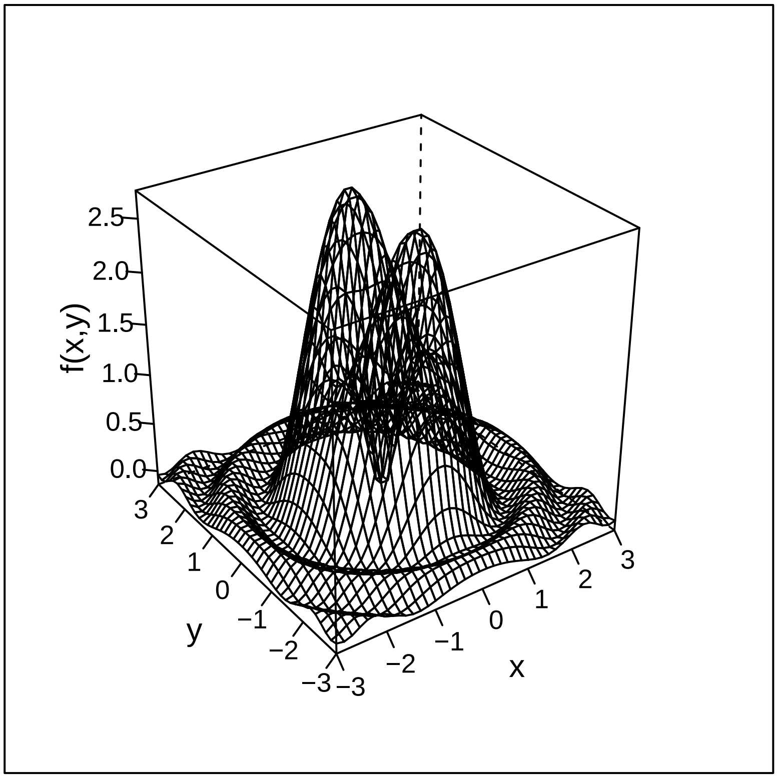


2. Another Function

$$f(x,y) = \frac{\sin(x^2+y^2)}{0.1+x^2+y^2} + \frac{1}{2}(x^2+4y^2)e^{1-x^2-y^2}$$

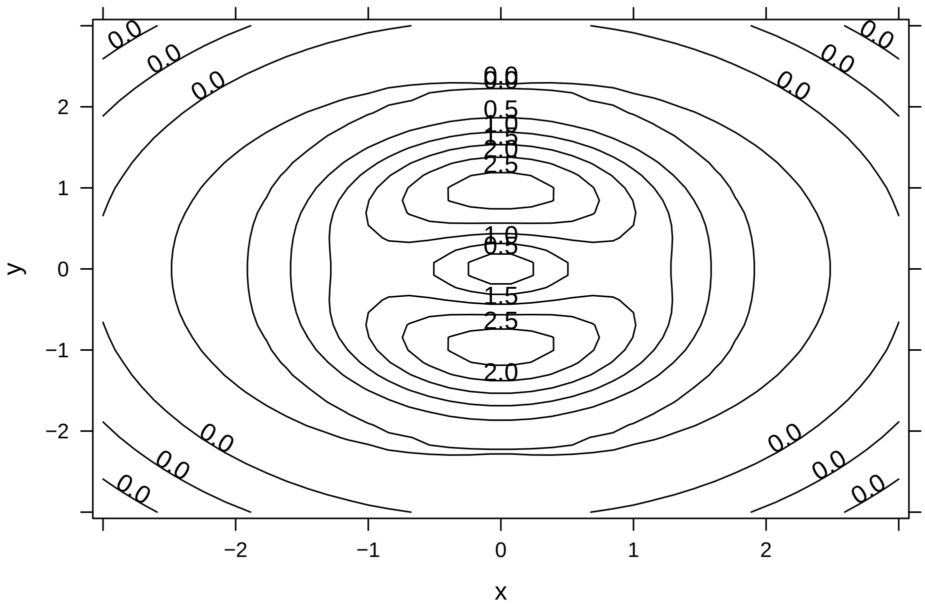
Surface Plot

```
plotFun(sin(x^2+y^2)/(0.1+x^2+y^2)+(x^2+4*y^2)*exp(1-x^2-y^2)/2~x&y,
surface=TRUE,zlab="f(x,y)",x.lim=range(-3,3),y.lim=range(-3,3))
```



Contour Plot

```
plotFun(sin(x^2+y^2)/(0.1+x^2+y^2)+(x^2+4*y^2)*exp(1-x^2-y^2)/2~x&y,
filled=FALSE,zlab="f(x,y)",x.lim=range(-3,3),y.lim=range(-3,3))
```



10.2.3 Cross Sections

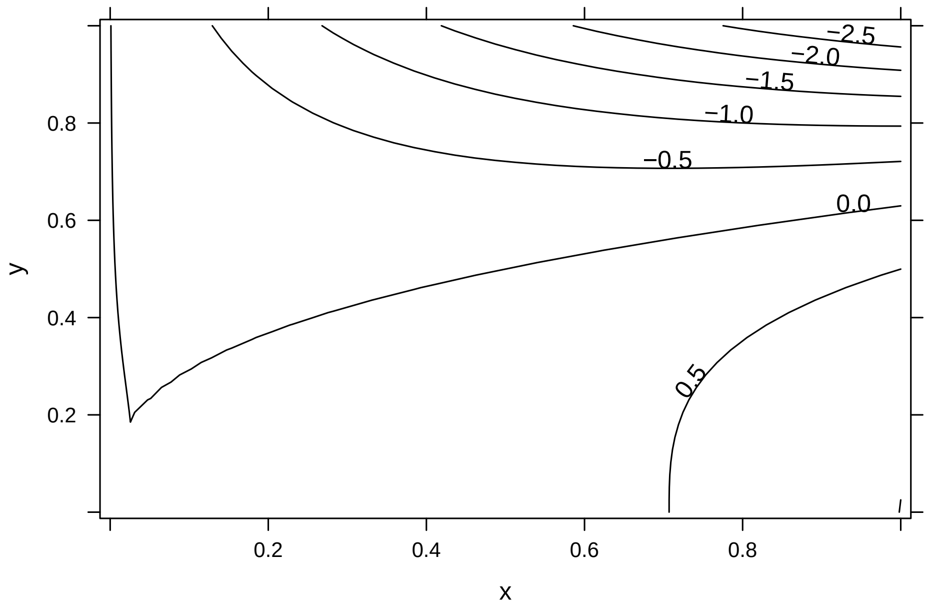
For each of the the two functions in the previous problem:

- Sketch two or three horizontal cross sections
- Sketch two or three vertical cross sections
- Then use RStudio to create these cross sections and compare to your sketches. How well did you do?

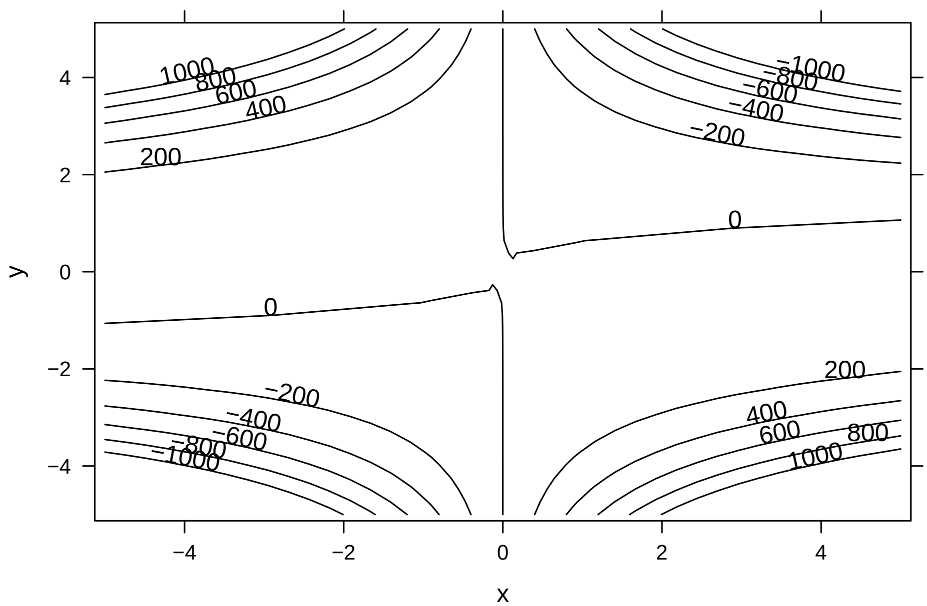
10.2.4 Plotting in RStudio

Here is some sample RStudio code that defines a function `P(x,y) = x^2 - 4x y^3` and then creates two contour plots. The first plot uses the default domain and default level curves (chosen by RStudio). The second plot specifies both of these using additional arguments.

```
P = makeFun(x^2 - 4*x*y^3 ~ x&y)

plotFun(P(x,y) ~ x&y, filled=FALSE)
```




```
plotFun(P(x,y) ~ x&y, filled=FALSE, x.lim=range(-5,5),y.lim=range(-5,5), levels = seq(-
```



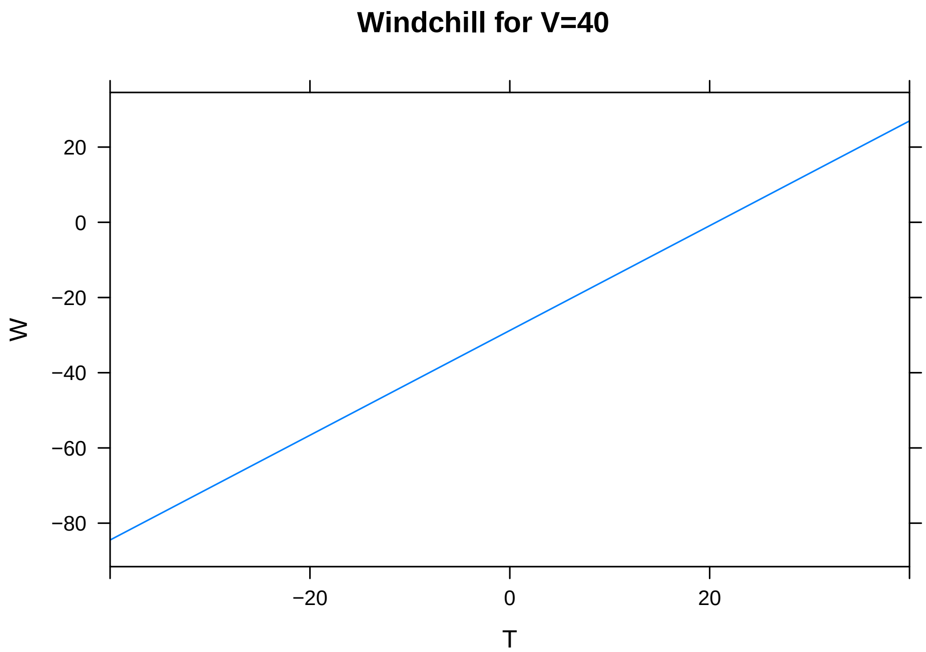
Using RStudio,

- Use `makeFun` to create each of these functions.
- Then use `plotFun` to create a contour plot.

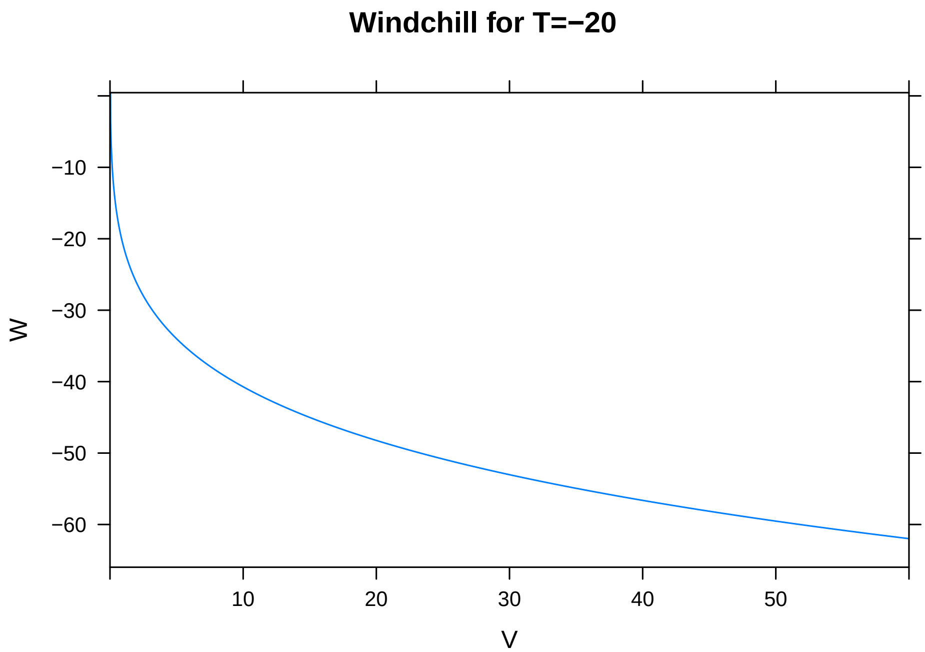
Try out different horizontal and vertical domains that are centered around the origin. Change the level curves.

1. $f(x,y) = \sin(\sqrt{x^2+y^2})$
2. $g(x,y) = 100x^2y^2e^{-x^2-y^2}$
3. $h(x,y) = \sin^2 x + \frac{1}{4}y^2$

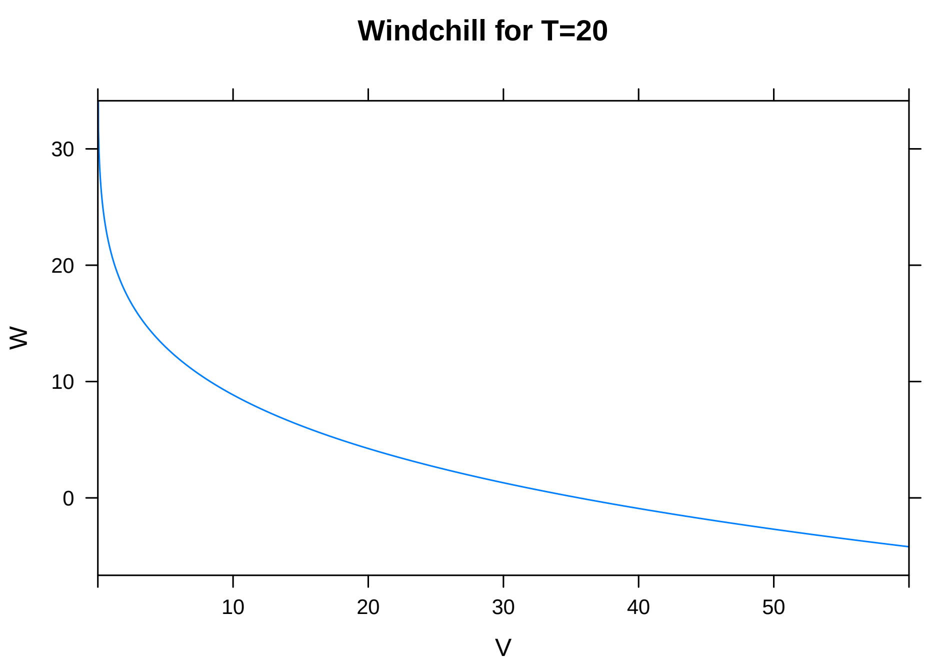
```
plotFun(W(T,40)~T, T.lim=range(-40,40), xlab="T",ylab="W",  main="Windchill for V=40",levels = se
```




```
plotFun(W(-20,V)~V, V.lim=range(0,60), xlab="V",ylab="W",  main="Windchill for T=-20")
```




```
plotFun(W(20,V)~V, V.lim=range(0,60), xlab="V",ylab="W",  main="Windchill for T=20")
```

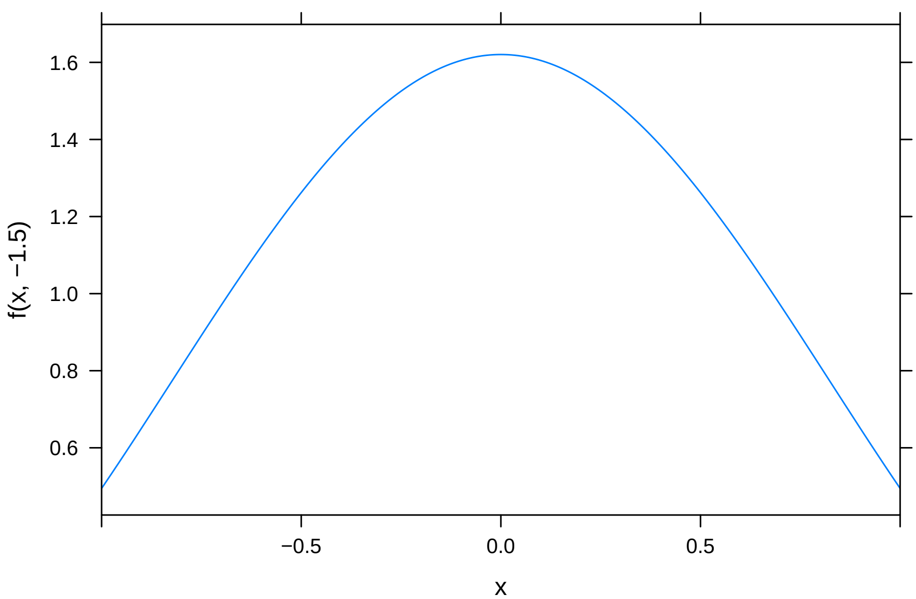


2. Here are some cross sections for the second function.

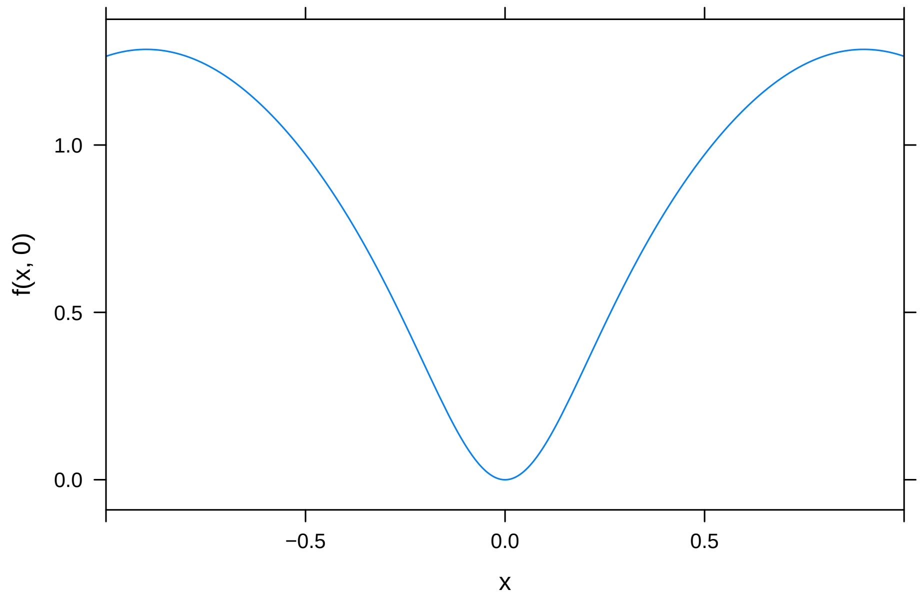
$$f(x,y) = \frac{\sin(x^2+y^2)}{0.1+x^2+y^2} + \frac{1}{2}(x^2+4y^2)e^{1-x^2-y^2}$$

```
f = makeFun(sin(x^2+y^2)/(0.1+x^2+y^2)+(x^2+4*y^2)*exp(1-x^2-y^2)/2~x&y)
```

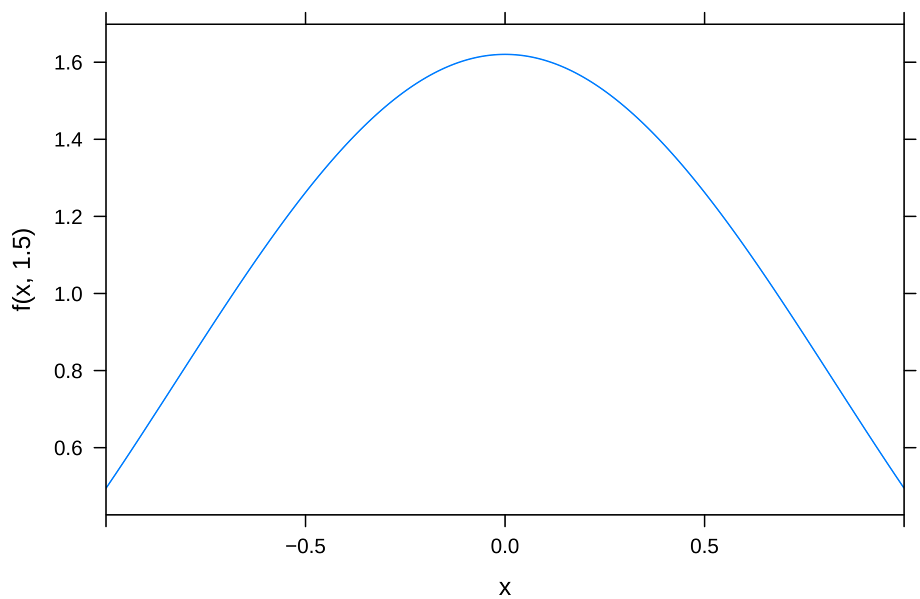
```
plotFun(f(x,-1.5)~x,x.lim=range(-1,1))
```



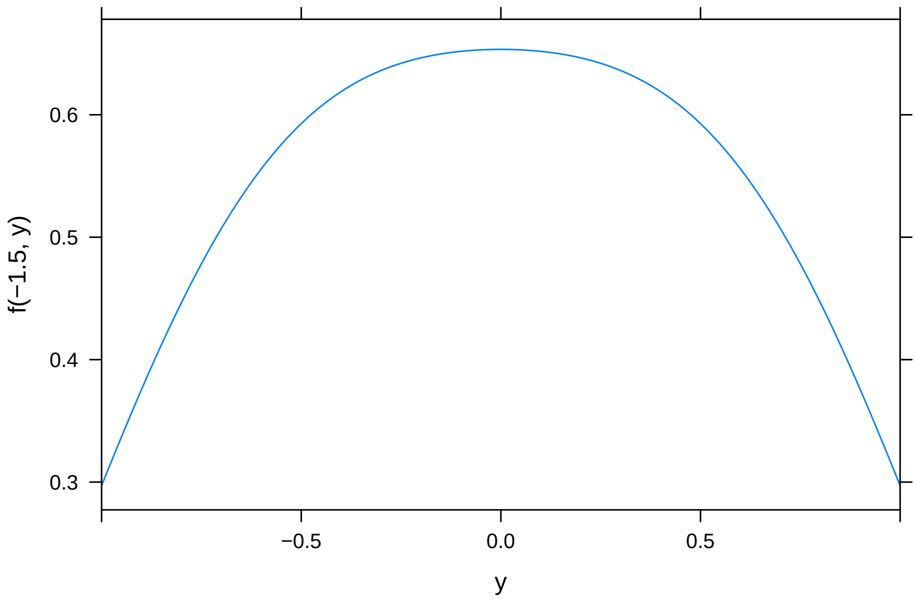

```
plotFun(f(x,0)~x,x.lim=range(-1,1))
```



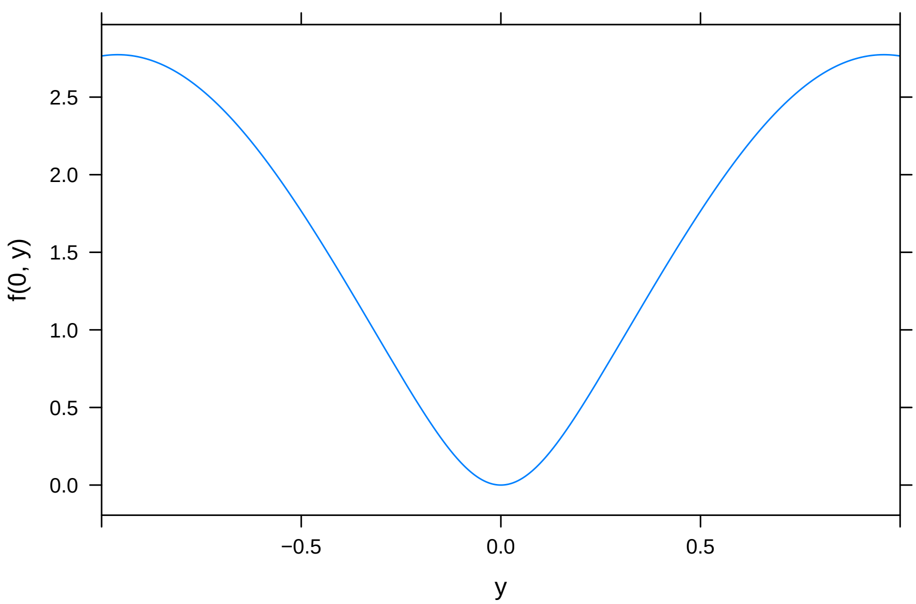

```
plotFun(f(x,1.5)~x,x.lim=range(-1,1))
```



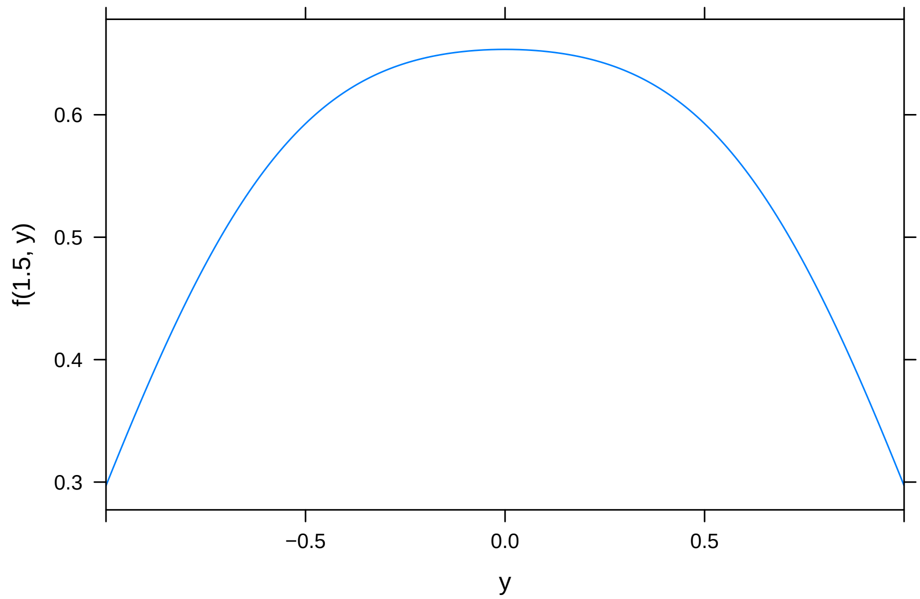

```
plotFun(f(-1.5,y)~y,y.lim=range(-1,1))
```




```
plotFun(f(0,y)~y,y.lim=range(-1,1))
```



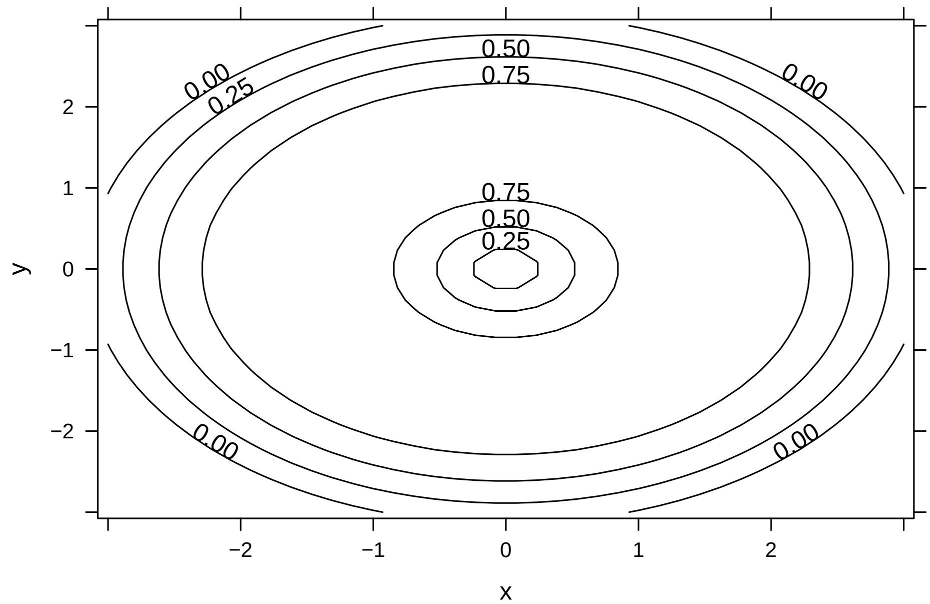

```
plotFun(f(1.5,y)~y,y.lim=range(-1,1))
```



10.2.5 Plotting in RStudio

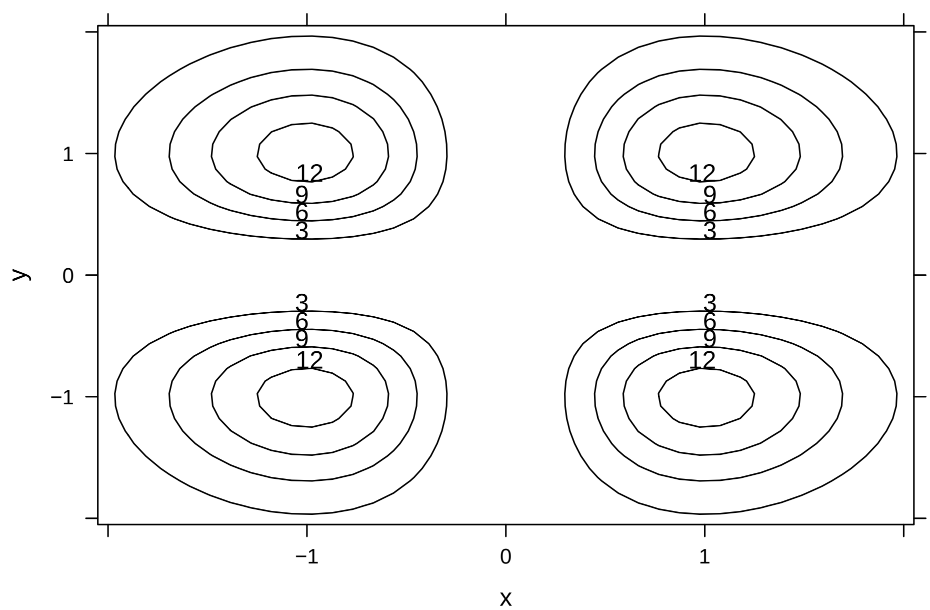
1. $f(x, y) = \sin(\sqrt{x^2 + y^2})$

```
f = makeFun(sin( sqrt((x^2+y^2))) ~ x&y)

plotFun(f(x,y) ~ x&y, filled=FALSE, x.lim=range(-3,3),y.lim=range(-3,3), levels = seq(
```



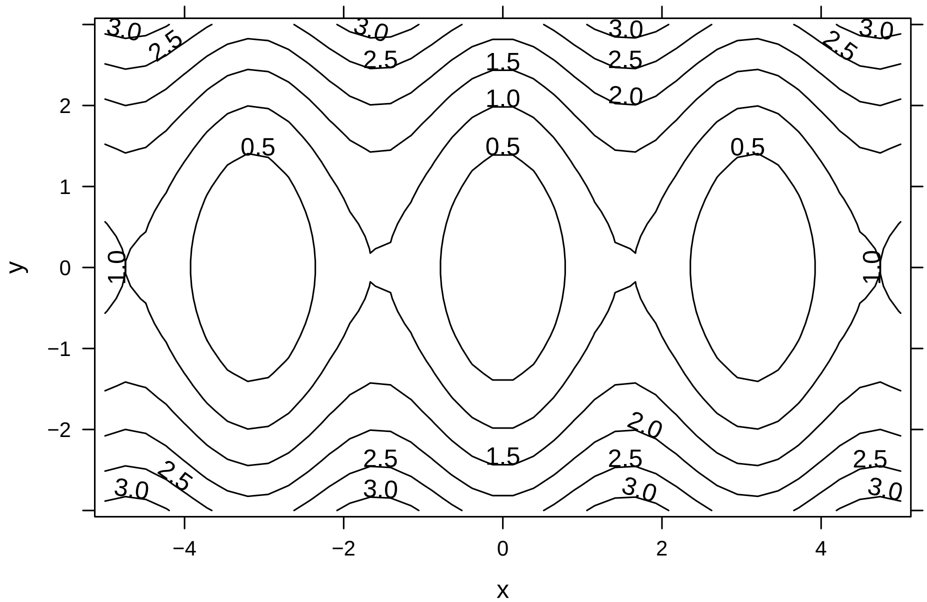
2. $g(x, y) = x^2 y^2 e^{-x^2 - y^2}$

```
g = makeFun(100 * x^2 *  y^2 * exp(-x^2-y^2)~ x&y)

plotFun(g(x,y) ~ x&y, filled=FALSE, x.lim=range(-2,2), y.lim=range(-2,2), levels=seq(0,15,3))
```



3. $h(x, y) = \sin^2 x + \frac{1}{4} y^2$

```
g = makeFun((sin(x))^2 + y^2/4 ~ x&y)

plotFun(g(x,y) ~ x&y, filled=FALSE, x.lim=range(-5,5), y.lim=range(-3,3), levels=seq(0
```



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Part II

Units and Estimation

Chapter 11

Dimensions and Units

There are 7 **fundamental dimensions**.

Table 11.1: Fundamental Dimensions

Dimension	Symbol	Sample Units
Length	L	meter (m), foot (ft), mile (mi)
Time	T	second (s), day, hour (hr)
Mass	M	gram (g), kilogram (kg)
Amount	N	mole (mol)
Temperature	θ	Kelvin (K), Celsius (°C)
Electric Current	I	Ampere (amp)
Luminosity	J	Candela (cd)

Everything else is a **derived dimension**. Here are some examples.

Table 11.2: Derived Dimensions

Dimension	Symbol	Intuition	Sample Units
Area	L^2	length × width	square foot (ft^2), square meter (m^2), acre
Volume	L^3	length × width × height	cubic meter (m^3), liter(L), gallon (gal)
Velocity	$\frac{L}{T}$	distance/time	meter/second, miles/hour
Acceleration	$\frac{L}{T^2}$	velocity/time	meters/second^2
Force	$\frac{ML}{T^2}$	mass × acceleration	Newton (N), pound (lb)

Dimension	Symbol	Intuition	Sample Units
Energy (Work)	$\frac{ML^2}{T^2}$	force $\times$ displacement	Joule (J), calorie, kilowatt hour (kwh), foot-pound
Power	$\frac{ML^2}{T^3}$	energy/time	Watt (W), horsepower
Density	$\frac{M}{L^3}$	mass/volume	kilogram/meter^3
Pressure	$\frac{M}{LT^2}$	force/area	newton/meter^2, pascal (Pa), psi, atm
Electric Charge	IT	current $\times$ time	coulomb

11.1 Activities

11.1.1 Find the Dimension

Find the dimension of the following quantities.

1. Momentum is the product of mass and velocity.
2. Jerk is the rate at which acceleration changes with respect to time.
3. Voltage is the work needed per unit charge to move a test charge between two points.

11.1.2 Dimensionally Feasible

A formula is **dimensionally feasible** when the units for the two sides match. Decide whether each of the following formulas is dimensionally feasible or infeasible.

1. $s = vt + \frac{1}{2}at^2$ where s is displacement, v is velocity and a is acceleration.
2. $P = E/A$ where P is pressure, E is energy and A is area.

11.1.3 Geometric Formulas for a Cylinder

A student is trying to remember the formulas for the surface area A and the volume V of a cylinder of radius r and height h. They come across the following two formulas:

$$\pi r^2 h$$

and

$$2\pi r(r+h).$$

What are the dimensions for each of these formulas? Why does this tell you which one is A and which one is V?

11.1.4 Unit Conversions

Solve the following problems by multiplying by the appropriate unit conversions. This unit conversion chart will be helpful, or you can use Google to find a particular conversion, for example just search for "km to miles".

1. Convert a density of 3 g/mL into pounds/in^3.
2. A car is going 35 miles per hour. How many feet per second is it traveling?
3. Sunlight deposits energy at a rate of 250 W/m^2 on a 10 m^2 garden. How much energy (in Joules) is deposited in a week? Note that 1 Watt is 1 Joule per second)

Chapter 12

Dimensional Analysis

12.1 Performing a Dimensional Analysis

Given independent variables x, y, z and dependent variable w:

1. Write down the hypothesized "generalized product" $w = kx^a y^b z^c$
2. Write & simplify the dimensional version $[w] = [x]^a [y]^b [z]^c$
3. List the dimensions of all quantities
4. Invoke dimensional compatibility to solve.
5. Rewrite your equation, marvel at your awesomeness!

12.2 Activities

12.2.1 Vocal Chord Frequency

The frequency f of the sound of an organism's vocal chords depend on their length ℓ, tension s (a force) and mass density μ (mass per unit length). Find a formula for the frequency $f = f(\ell, s, \mu)$.

12.2.2 Mixer Power

A mixer with power P turns a mixer with wings of length D through a liquid of viscosity H. Find a formula for the angular velocity $A = A(P, D, H)$. Note that since angles are dimensionless, we have $[A] = 1/T$. The dimensions of viscosity are $[H] = M/(LT)$.

12.2.3 Velocity of an Ocean Wave

The velocity V of a large ocean wave depends upon the period P, the acceleration of gravity g, and the density μ of the water. Find a formula for the velocity $V = V(P, g, \mu)$.

Chapter 13

Fermi Estimation

13.1 Working Fermi Problems

Here's a general game plan for Fermi Problems:

1. Write down what you know.
2. Estimate what you don't know!
   a. Ask a series of smaller questions.
   b. Estimate the answers to the smaller questions.
3. Multiply and pay attention to units.

Remember, **we're trying to estimate!**

- Do not obsess over details, but also,
- Do not guess blindly.
- Find a middle ground!

Pro Tip #1: Identify smaller problems that you can estimate using your everyday common sense. For example, rather than estimating the height of of a building, I would estimate the height of one floor and estimate the number of floors, and then multiply.

Pro Tip #2: When estimating, determine the order of magnitude (10, 100, 1000, etc). Use the geometric mean when you think the answer is somewhere in between. For example, if you think 1,000 is too small but that 10,000 is too big, then use 3,000.

13.2 Some Useful Data

The following approximate data may be useful in various Fermi problems. Each has been stated with one or two significant figures.

- Population of the world: 7×10^9 people
- Population of the United States: 3×10^8 people
- Average human life span worldwide (in years): 71 years

- United States national debt (in dollars): $\$1.8 \times 10^{13}$
- United States federal spending (in dollars per year): $\$3.8 \times 10^{12}$.
- Fraction of United States federal spending devoted to defense (as a decimal): 0.16
- Fraction of United States federal spending devoted to Medicare/Medicaid (as a decimal): 0.27
- Radius of earth (in km): 6400 km
- Acceleration due to gravity on the surface of the Earth: 9.8 m/s^2
- Fraction of Earth's surface covered by water (as a decimal): 0.7
- Density of water at sea level: 1 g/cm^3

13.3 Activities

1. How many blades of grass are there on a football field?

2. How many babies are born each day? It will be helpful to know that (a) there are 7.8 billion people on the planet, (b) the average life expectancy is 73 years, and (c) the population doesn't change that much (relatively speakiing) in one year.

3. If all 7.8 billion humans in the world were crammed together, how much area would we require? Is this closer to the size of a city, a state, the U.S., or North America? Start by figuring out how many people would fit in 1 square meter. Then figure out how many square kilometers we would need.

Part III

Computational Derivatives

Chapter 14

The Derivative

14.1 The Derivative

The derivative is the **instantaneous rate of change**.

- The derivative of $f(x)$ is denoted by $f'(x)$ and by $\dfrac{df}{dx}$.
- We approximate this the derivative $f'(x)$ at the point $x = a$ with the formula

  $$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

  where h is a **very, very small number**.
- The **derivative** measures the **sensitivity** of outputs to a small change in input.

  $$\text{the derivative} \approx \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x}$$

- The **derivative of a function** f at a point a is the slope of the tangent line at $x = a$.
- **Economics.** Given a cost function $C(x)$, the **marginal cost** at $x = a$ is the cost of producing one additional unit of a good. The formula is for the marginal cost at $x = a$ is $C(a+1) - C(a)$.
- **Physics.** Given a displacement function $s(t)$, the velocity is $v(t) = s'(t)$, and the acceleration is $a(t) = v'(t)$.

14.2 Activities

14.2.1 Studying

Suppose that spending 3 extra hours on studying increases our percent grade on an exam by 14 percent.

1. Calculate the derivative, give the units, and give an interpretation of the number you get.
2. Is this rate of change constant? Explain.

14.2.2 Marginal Cost

Economists use the term "marginal cost" for the derivative of the cost of production. Here is the definition:

The **marginal cost** is the cost to produce 1 additional good.

Suppose that it costs \$250,000 to produce 1000 iPhones.

1. If it costs \$250,440 to produce 1002 iPhones, what's the marginal cost of 1 iPhone?
2. Using the marginal cost from part 1, estimate how much it will cost to produce 1030 iPhones.

14.2.3 Farmland in the United States

The total acreage of farmland has been shrinking in the United States since 1980.

Year	1980	1985	1990	1995	2000
Farmland (millions of acres)	1039	1012	987	963	945

1. Let $F(t)$ denote the amount of farmland at a time t. Use the data to estimate $F'(1995)$, the derivative at $t = 1995$.
2. Did you use 1990 or 2000 as your estmation point? Did it matter? What does that mean?
3. You got a negative number for $F'(1995)$. Does that make sense with the situation? Why?
4. Give an example of a year that, if you had the data for it, would better help us estimate $F'(1995)$. Why?

14.2.4 A Moving Particle

A particle's position over time is given by the formula

$$s(t) = 2 + 3\sin(4t) + 1.2t^2$$

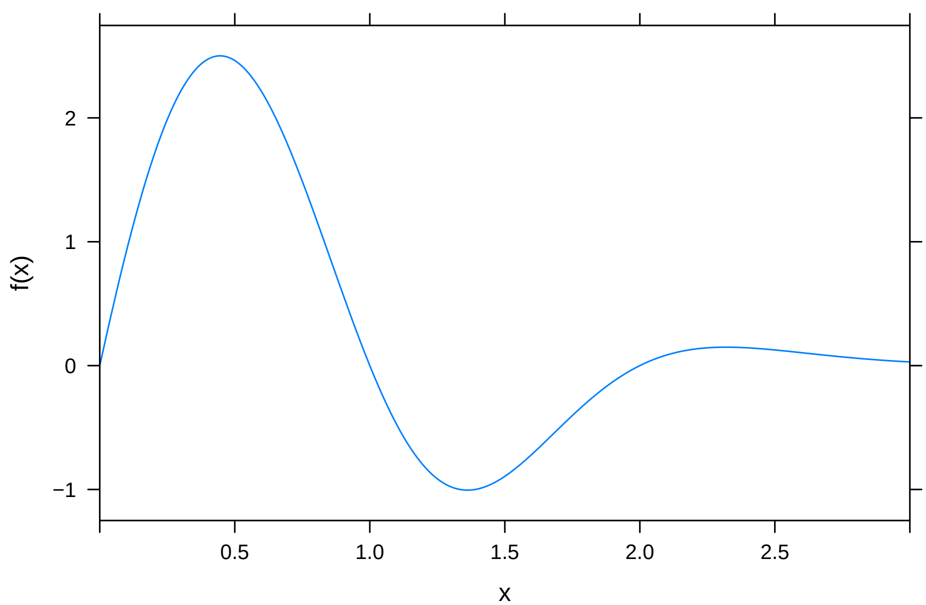
1. Find the velocity at $t = 1$ and $t = 2$ numerically (using small subintervals). Remember that the velocity is the derivative of the position.
2. Use desmos to graph the position curve $s(t)$ and the tangent lines at both $t = 1$ and $t = 2$.

Chapter 15

The Second Derivative

15.1 Activities

15.1.1 Estimating the first and second derivatives from a graph

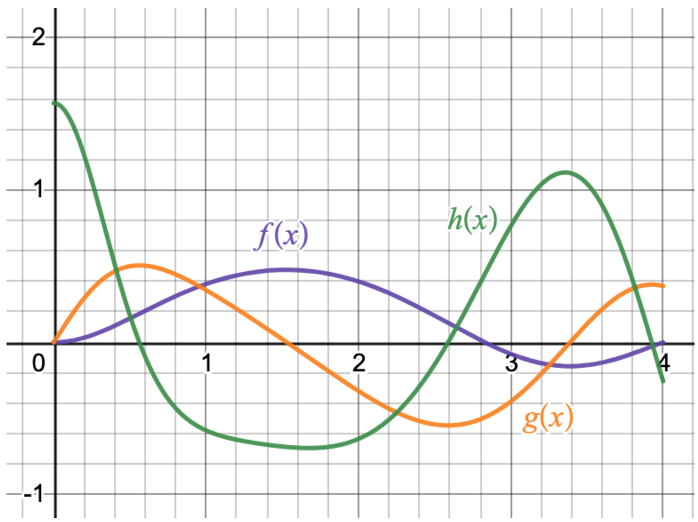


Here is a plot of the function $f(x)$.

1. Estimate the intervals on which the derivative $f'(x)$ is positive and the intervals on which the derivative $f'(x)$ is negative
2. Estimate the intervals on which the second derivative $f''(x)$ is positive and the intervals on which the second derivative $f''(x)$ is negative

15.1.2 Function Matching

Three curves $f(x), g(x), h(x)$ are plotted below. These curves are actually a function, its derivative, and its second derivative. Which is which? How do you know?

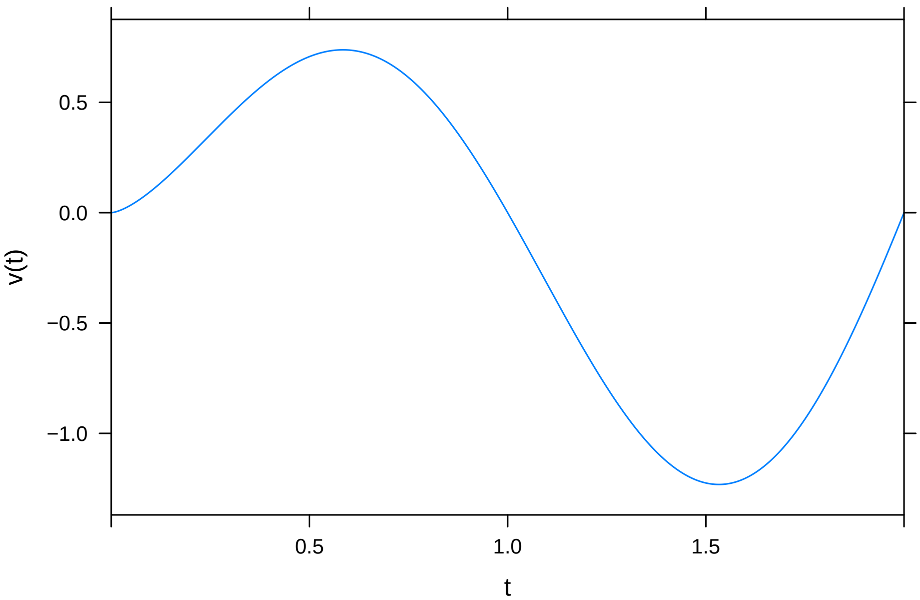


15.1.3 A Plot of Velocity

Recall that displacement $x(t)$, velocity $v(t)$ and acceleration $a(t)$ are related by the equations

$$v = \frac{\partial x}{\partial t} \quad \text{and} \quad a = \frac{\partial v}{\partial t} = \frac{\partial^2 x}{\partial t^2}.$$

Here is a plot of **velocity** $v(t)$ of an object.



Identify the time interval(s) where:

1. The object has positive acceleration.
2. The position of the object is increasing.

15.1.4 Characterize the Derivatives

Each of the scenarios below describes a quantity that is changing over time. For each one, (a) determine whether the first derivative is positive, negative or zero and (b) determine whether the second derivative is positive, negative or zero.

1. The number of views of a video that is going viral.
2. The altitude of a bird that is about to have a gentle landing on the ground.
3. The location of a car that is cruising at the speed limit on a straight highway.

Chapter 16

Partial Derivatives

16.1 Activities

16.1.1 Estimating From Data

The maximum duration of a scuba dive $T(V, D)$ (in minutes) depends on the volume V of air (at sea-level pressure) in the tank and the depth D (in feet) of the dive. The following table shows the estimated dive times for various combinations of air volume and dive depth.

Dive Time (minutes)

Air Volume

20 ft3

40 ft3

60 ft3

80 ft3

Depth

20 ft

12.5

25.0

37.5

50.0

40 ft

9.0

18.0

27.0

36.0

60 ft

7.1

14.2

21.3

28.4

80 ft

5.8

11.6

17.4

23.2

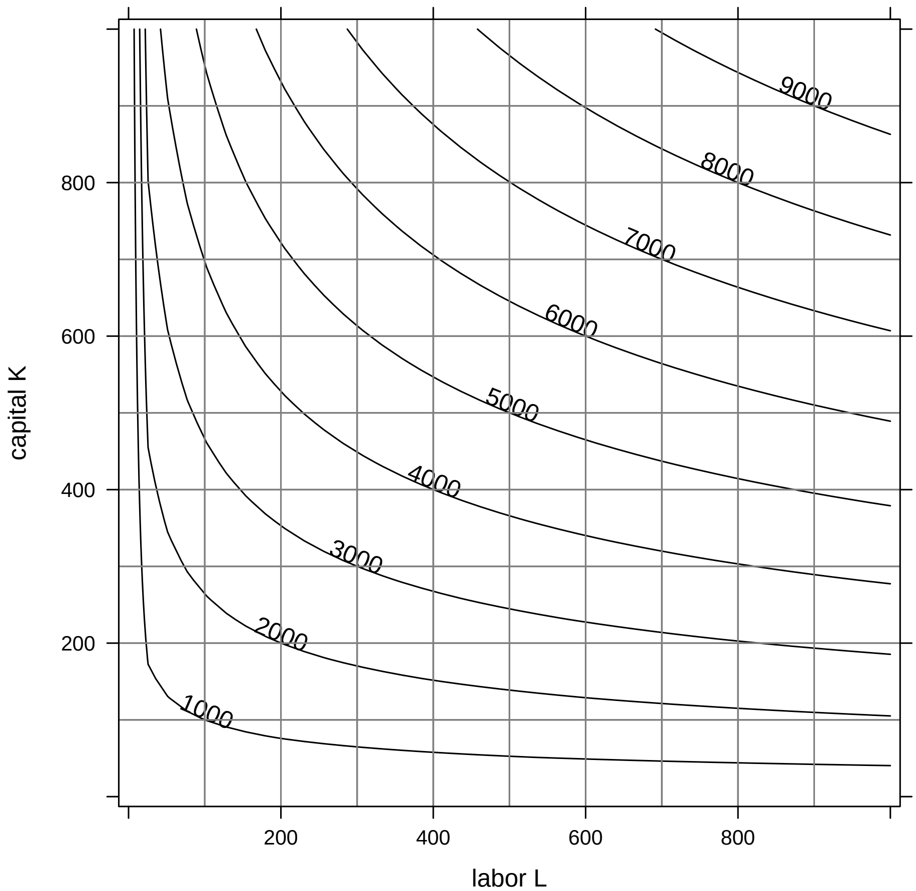
Use this table to estimate the derivatives $\frac{\partial T}{\partial V}$ and $\frac{\partial T}{\partial D}$ at the point $(V = 60, T = 40)$. Give units for each of these derivatives and then interpret what these numbers mean.

16.1.2 Estimating From a Contour Plot

The production function $F(L, K)$ for a factory gives the number of units produced by labor L and capital K (both measured in dollars). This production function is given by

$$F(L, K) = 10L^{2/7}K^{5/7}.$$

Here is a contour plot of the production function.



1. Estimate the production output for $L = 400$ and $K = 600$.
2. Estimate the partial derivative $\dfrac{\partial F}{\partial L}$ at $(400, 600)$.
3. Estimate the partial derivative $\dfrac{\partial F}{\partial K}$ at $(400, 600)$.

16.1.3 Estimating Using Desmos

Use desmos to define the prodution function $F(L,K) = L^{2/7}K^{5/7}$ from the previous problem and to answer the analogous questions

1. Calculate the production outpuut $F(400, 600)$.
2. Estimate the partial derivative $\dfrac{\partial F}{\partial L}$ at $(400, 600)$. Your answer must be correct to two decimal places.
3. Estimate the partial derivative $\dfrac{\partial F}{\partial K}$ at $(400, 600)$. Your answer must be correct to two decimal places.

Compare your three answers to the estimates from the contour plot in the previous problem.

Chapter 17

Local Linear Approximation

The linearization of $f(x)$ at $x = a$ is

$$L_a(x) = f(a) + f'(a)(x - a).$$

17.1 Activities

17.1.1 Estimating Sugar Solution Temperature

We are boiling a sugar solution to make salt water taffy. When we start, the temperature of the solution is 80°F, after 5 minutes it's 115°F, and after 10 minutes it's 165°F.

1. Estimate the temperature after 17 minutes.
2. The sugar solution needs to reach 289°F to be the consistancy of taffy. Estimate when this will happen.

17.1.2 Comparing Linearizations

1. Draw two (nonlinear) functions $f(x)$ and $g(x)$ and their linearizations at $x = 1$. One function should be "well estimated" by the linearization around $x = 1$ and the other should be "poorly estimated" by the linearization around $x = 1$.
2. What feature(s) of the curve $y = f(x)$ result in a good approximation?
3. What feature(s) of the curve $y = g(x)$ result in a bad approximation?

17.1.3 Estimating a Function

1. Find the linearization $L_2(x)$ the function $g(t) = t^3 + e^t - 4$.
   - Remember that $L_2(x)$ means "the local linear approximation at $x = 2$."
   - You will need to use desmos to approximate $f'(2)$.
2. Use your formula for $L_2(x)$ to estimate $g(2.1)$, $g(1.98)$, $g(3)$ and $g(0.7)$.
3. Check your estimations with the original function.
4. What do you notice about your approximations as you get further from the spot where you built the linearization?

17.1.4 Estimating the Windchill

We've talked about windchill $w(t, v)$ as a function of temperature t and wind speed v. Here is the table that we've used

Wind Chill Chart

Temperature (°F)

Wind (mph) / Calm	40	35	30	25	20	15	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45
5	36	31	25	19	13	7	1	-5	-11	-16	-22	-28	-34	-40	-46	-52	-57	-63
10	34	27	21	15	9	3	-4	-10	-16	-22	-28	-35	-41	-47	-53	-59	-66	-72
15	32	25	19	13	6	0	-7	-13	-19	-26	-32	-39	-45	-51	-58	-64	-71	-77
20	30	24	17	11	4	-2	-9	-15	-22	-29	-35	-42	-48	-55	-61	-68	-74	-81
25	29	23	16	9	3	-4	-11	-17	-24	-31	-37	-44	-51	-58	-64	-71	-78	-84
30	28	22	15	8	1	-5	-12	-19	-26	-33	-39	-46	-53	-60	-67	-73	-80	-87
35	28	21	14	7	0	-7	-14	-21	-27	-34	-41	-48	-55	-62	-69	-76	-82	-89
40	27	20	13	6	-1	-8	-15	-22	-29	-36	-43	-50	-57	-64	-71	-78	-84	-91
45	26	19	12	5	-2	-9	-16	-23	-30	-37	-44	-51	-58	-65	-72	-79	-86	-93
50	26	19	12	4	-3	-10	-17	-24	-31	-38	-45	-52	-60	-67	-74	-81	-88	-95
55	25	18	11	4	-3	-11	-18	-25	-32	-39	-46	-54	-61	-68	-75	-82	-89	-97
60	25	17	10	3	-4	-11	-19	-26	-33	-40	-48	-55	-62	-69	-76	-84	-91	-98

Frostbite Times: 30 minutes, 10 minutes, 5 minutes

Wind Chill (°F) = $35.74 + 0.6215T - 35.75(V^{0.16}) + 0.4275T(V^{0.16})$

Where, T= Air Temperature (°F) V= Wind Speed (mph)

Effective 11/01/01

1. The perceived temperature when $t = -10$°F and $v = 40$ mph is $w(-10, 40) = -43$. I have used desmos to estimate that

$$w_t(-10, -40) = 1.39, \qquad w_v(-10, 40) = 0.28.$$

   Use these values to find the linearization $L_{(-10,40)}(t, v)$ at the point $(-10, 40)$.

2. Use your linearization (and desmos!) to estimate the four windchill values $w(-5, 40)$, $w(-15, 40)$, $w(-10, 45)$ and $w(-10, 35)$. Are your estimates close to the actual values?

3. Find the linearization of $w(t, v)$ at $t = -10$ and $v = 40$ using the the partial derivatives of the formula $w(t, v) = 35.74 + 0.6215t - 35.75v^{0.16} + 0.4275t\,v^{0.16}$. Here is a link to my desmos calculator for the partial derivatives of this function.

Chapter 18

Unit 3 Summary

18.1 The Derivative

I can...

- Define what a derivative is.
- Interpret the derivative $f'(x)$ as
  - a rate of change
  - a measure of the sensitivity of output to changes in input
  - the slope of a tangent line.
- Estimate the derivative using a table of data.
- Estimate the derivatives from the plot of a function and its tangent line.
- Estimate the derivative $f'(x)$ at $x = a$ using the average rate of change.

$$f'(a) = \frac{f(a+h) - f(a)}{h}$$

  where h is a very small number
- Use desmos to find an approximation of the derivative of a function to a desired accuracy.
- Use the derivative to determine whether a function is increasing or decreasing.
- Define marginal cost.
- Calculate the marginal cost of $C(x)$ at $x = a$ using the formula $C(a+1) - C(a)$
- Explain why marginal cost is an example of an approximate derivative
- Explain how the derivative connects displacement, velocity and acceleration.

18.2 The Second Derivative

I can. . .

- Give the definition of the kth derivative $\dfrac{d^k f}{dx^k} = f^{(k)}(x)$.
- Explain why acceleration $a(t)$ is the second derivative of displacement $s(t)$. That is, $a(t) = s''(t)$.
- Use the second derivative to determine the concavity of a function.

18.3 Partial Derivatives

- Define what a partial derivative is.
- Explain the difference between $f_x = \dfrac{df}{dx}$ and $f_y = \dfrac{df}{dy}$
- Estimate the partial derivatives from a table of data
- Estimate the partial derivatives from a contour plot.
- Use desmos to approximate the partial derivatives using

$$\frac{\partial f}{\partial x} \approx \frac{f(x+h,y) - f(x,y)}{h} \quad \text{and} \quad \frac{\partial f}{\partial y} \approx \frac{f(x,y+h) - f(x,y)}{h}$$

where h is a very, very small number.

18.4 Local Linear Approximation

I can. . .

- Create the linearization $L_a(x)$ for function $f(x)$ at $x = a$ using the formula

$$L_a(x) = f(a) + f'(a)(x - a).$$

- Approximate $f(x)$ using the linearization $L_a(x)$.
- Create the linearization $L_{(a,b)}(x,y)$ for function $f(x,y)$ at point (a,b) using the formula

$$L_{(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

- Approximate $f(x,y)$ near the point (x,y) using the linearization $L_{(a,b)}(x)$.

Part IV

Symbolic Derivatives

Chapter 19

Rules of Differentiation

19.1 Derivatives of Familiar Functions

19.1.1 Constant Rule

The derivative of a constant is 0.

$$\frac{d}{dx}(c) = 0$$

19.1.2 Linear Function Rule

If $f(x) = mx + b$, then the derivative is the slope m.

$$\frac{d}{dx}(mx + b) = m$$

19.1.3 Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

19.1.4 Trig Rules

$$\frac{d}{dx}\sin(x) = \cos(x)$$

and

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

19.1.5 Exponential Rules

$$\frac{d}{dx}(e^x) = e^x$$

and for any positive number $a > 0$,

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

19.1.6 Logarithmic Rules

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

and for any positive number $a > 0$,

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln(a)} \cdot \frac{1}{x}$$

19.2 Arithmetic Rules for Derivatives

19.2.1 Constant Multiple Rule

If c is a constant, then

$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}f(x) = cf'(x)$$

19.2.2 Sum and Difference Rules

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x)$$

and

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}f(x) - \frac{d}{dx}g(x) = f'(x) - g'(x)$$

19.3 Rules for Combinations of Functions

19.3.1 Product Rule

If $f(x)$ and $g(x)$ are functions then

$$(fg)' = f'g + fg' \qquad \text{or} \qquad (1st \cdot 2nd)' = D1st \cdot 2nd + 1st \cdot D2nd$$

19.3.2 Quotient Rule

If $f(x)$ and $g(x)$ are functions then

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \qquad \text{or} \qquad \left(\frac{hi}{lo}\right)' = \frac{loDhi - hiDlo}{lo^2}$$

19.3.3 Chain Rule

If $y = f(t)$ and $t = g(x)$ are functions, then the derivative of $y = f(g(x))$ is given by

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} \qquad \text{or} \qquad \frac{d}{dx}(f(g(x))) = f'(g(x))g'(x) = D(outside)D(inside)$$

Chapter 20

Symbolic Differentiation

20.1 Activities

20.1.1 Basic Derivative Practice

Find the derivative for each of the following functions.

1. $W(r) = r^3 + 5r - 12$
2. $f(x) = x^2 - 3\ln x$
3. $P(t) = 4t^2 + 7\sin t$
4. $f(x) = 1/x^2 + 5\sqrt{x} - 7$
5. $s(x) = 2e^x + x^2$
6. $h(\theta) = 1/\sqrt[3]{\theta}$
7. $b(t) = t^2 + 5\cos t$

20.1.2 More Basic Derivative Practice

Find the derivative for each of the following functions.

1. $g(x) = -\frac{1}{2}\left(x^5 + 2x - 9\right)$
2. $s(t) = 6t^{-2} + 3t^3 - 4t^{1/2}$
3. $q(x) = 3x - 2 \cdot 4^x$
4. $y(x) = \sqrt{x}(x+1)$
5. $P(t) = 3000(1.02)^t$
6. $f(x) = 2^x + x^2 + 1$
7. $d(r) = Ae^r - Br^2 + C$
8. $s(t) = t^2 + 2\ln t$
9. $g(x) = 2x - \frac{1}{\sqrt[3]{x}} + 3^x - e$
10. $y(g) = 5\sin g - 5g + 4$

20.1.3 Product Rule and Quotient Rule Practice

Find the derivative for each of the following functions.

1. $s(t) = (t^2 + 4)(5t - 1)$
2. $y(x) = x^2 \ln x$
3. $g(x) = \frac{25x^2}{e^x}$
4. $y(x) = x^2 \cos x$
5. $h(t) = \frac{t+4}{t-4}$

Chapter 21

The Chain Rule

21.1 Activities

21.1.1 Chain Rule Practice

Find the derivative for each of the following functions.

1. $g(z) = (z^2 + 5)^3$
2. $g(m) = 200e^{0.12m}$
3. $h(x) = e^{5x^2+1}$
4. $g(x) = \sqrt{1 + x^3}$
5. $f(t) = \cos(2t^3 + 5t^2)$
6. $k(x) = \ln(4x^2)$

21.1.2 More Chain Rule Practice

Find the derivative for each of the following functions.

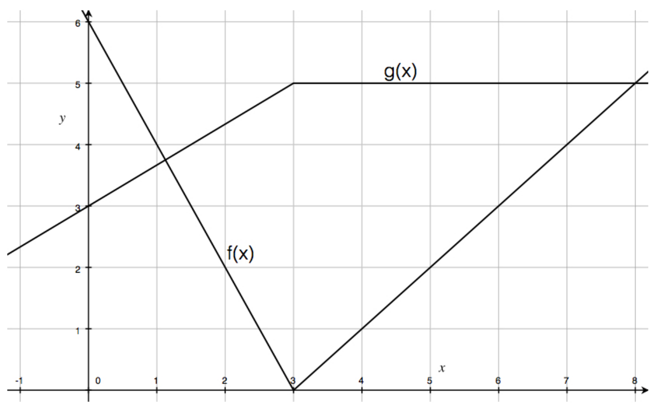
1. $f(x) = \sin(2x)$
2. $P(t) = e^{2t}$
3. $y(x) = xe^{3x}$
4. $f(z) = \ln(z^2 + 1)$
5. $R(t) = (\sin t)^5$
6. $f(t) = \sin\sqrt{e^t + 1}$
7. $g(y) = e^{2e^{y^3}}$

21.1.3 Conceptual Chain Rule Problems

1. The amount of gasoline G, in gallons, used by a car is a function of the distance traveled s, in miles. The distance traveled s depends on the time t, in hours, spent driving. If the rate at which gas is consumed is 0.06 gallons per mile, and if the car is traveling at 40 miles per hour, how fast is the gas being used?
2. The radius of a circular oil spill is increasing at a rate of 2 meters per minute. Find how fast the area is changing when $r = 50$ meters

21.1.4 Visual Chain Rule Problem

Below are the graphs of two functions, $f(x)$ and $g(x)$. Use the chain rule to calculate the derivatives specified below



1. $(f(g(x))'$ when $x = 3$.
2. $(g(f(x)))'$ when $x = 1$.
3. $(g(f(x)))'$ when $x = 2$.

Chapter 22

The Gradient

22.1 Activities

22.1.1 Partial Derivative Practice

Find the partial derivatives f_x and f_y for each of these functions.

1. $f(x, y) = x^5 + x^2y^3 + y^4$
2. $f(x, y) = e^x \sin(y) + \ln(y) \cos(x)$
3. $f(x, y) = \ln(x^2 - y^2)$
4. $f(x, y) = x^2y^3e^x \sin(y)$

22.1.2 Second Partial Derivative Practice

For each of these functions, find the second partial derivatives f_{xx}, f_{yy}, f_{xy} and f_{yx}. Confirm that $f_{xy} = f_{yx}$

1. $f(x, y) = 7x^3 + \cos(4y)$
2. $f(x, y) = e^{2x+3y}$

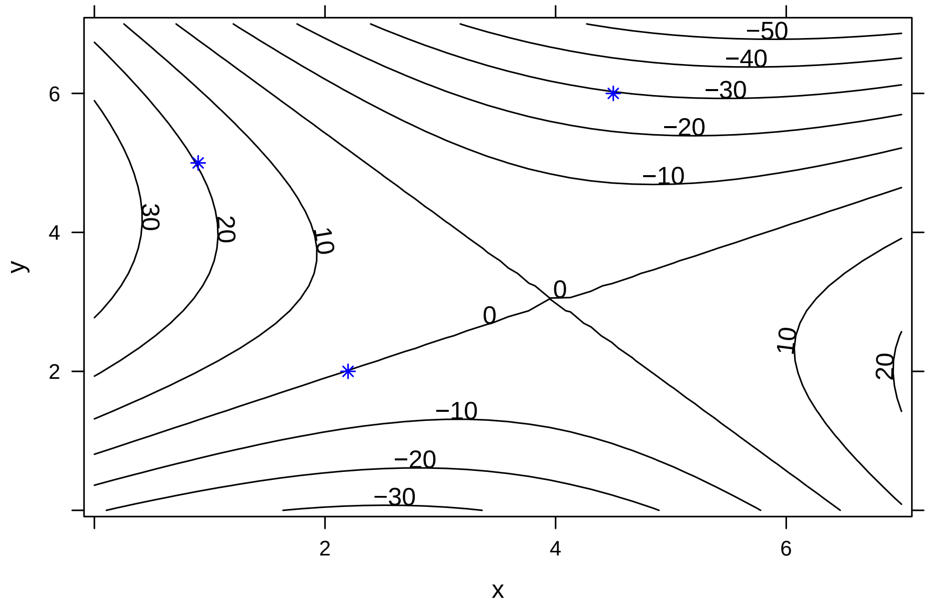
22.1.3 Gradient Practice

Calculate the gradient $\nabla f(x, y)$ for each of these functions

1. $f(x, y) = 8x^2y^4$
2. $f(x, y) = \dfrac{x}{y}$
3. $f(x, y) = \ln(xy)$

22.1.4 Gradient on a Contour Diagram

Here is a contour plot of $f(x, y)$. At each of the three points indicated by $*$, draw a vector in the direction of the gradient ∇f.



22.1.5 Gradient and the Direction of Greatest Increase

Let $f(x, y) = 3x^2 - 4xy + y^2$.

1. Find the gradient vector $\nabla f(2, 1)$ the at the point $(2, 1)$.
2. Find the rate of change of $f(x, y)$ at $(2, 1)$ in the direction of greatest increase.

22.1.6 Gradient and the Direction of Greatest Increase

1. $\nabla f(x, y) = \langle 6x - 4y, -4x + 2y\rangle$ so $\nabla f(x, y) = \langle 12 - 4, -8 + 2\rangle = \langle 8, -6\rangle$.
2. $\|\nabla f(2, 1)\| = \sqrt{8^2 + (-6)^2} = \sqrt{64 + 36} = \sqrt{100} = 10$.

–>

Chapter 23

Unit 4 Summary

23.1 Symbolic Derivatives

I can ...

- Find the derivative of x^n, $\sin(x)$, $\cos(x)$, e^x, a^x, $\ln(x)$, and $\log_a(x)$.
- Find the derivative ...
  - of the sum of function
  - using the product rule
  - using the quotient rule
  - using the chain rule
- Explain why the derivative is not defined for a cusp point.
- Solve a "word problem" about rates of change by using derivatives.

23.2 Partial Derivatives

I can ...

- Interpret the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as rates of change.
- Find the partial derivatives of a function $f(x, y)$ by holding one variable fixed and then applying the rules of differentiation.
- Find the second partial derivatives $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.
- Solve a "word problem" about rates of change of a multivariable function using partial derivatives.

23.3 The Gradient

I can ...

- Explain what a vector $\langle a, b\rangle$ is, draw it as an arrrow, and find its magnitude $\|\langle a, b\rangle\| = \sqrt{a^2 + b^2}$.
- Find the gradient vector $\nabla f(x, y) = \langle f_x, f_y\rangle$
- Evaluate the gradient vector at a point (a, b)
- Use the gradient to find the direction of greatest increase, and the magnitude of the gradient to find the rate of change (slope) in that direction
- Draw a vector in the direction of greatest increase on a contour diagram.

Part V

Optimization

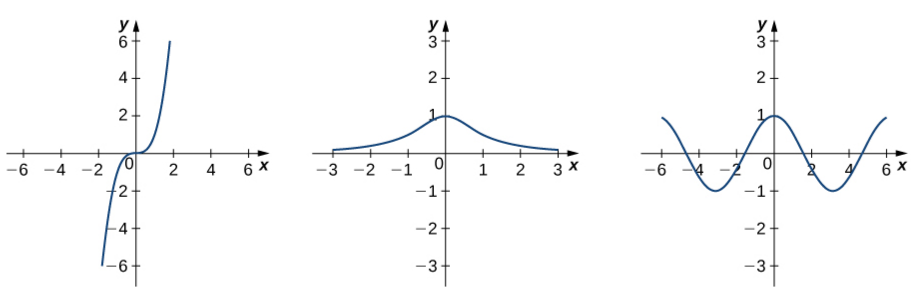
Chapter 24

1D Optimization

24.1 Activities

24.1.1 Characterize the Extrema

1. Each of the functions below is defined on $(-\infty, \infty)$. Find all the local extrema and global extrema.



$f(x) = x^3$ on $(-\infty, \infty)$ $\quad$ $f(x) = \frac{1}{x^2+1}$ on $(-\infty, \infty)$ $\quad$ $f(x) = \cos(x)$ on $(-\infty, \infty)$

image: https://openstax.org/books/calculus-volume-1/pages/4-3-maxima-and-minima

2. Each of the functions below is defined on a finite interval. Find all the local extrema and global extrema.

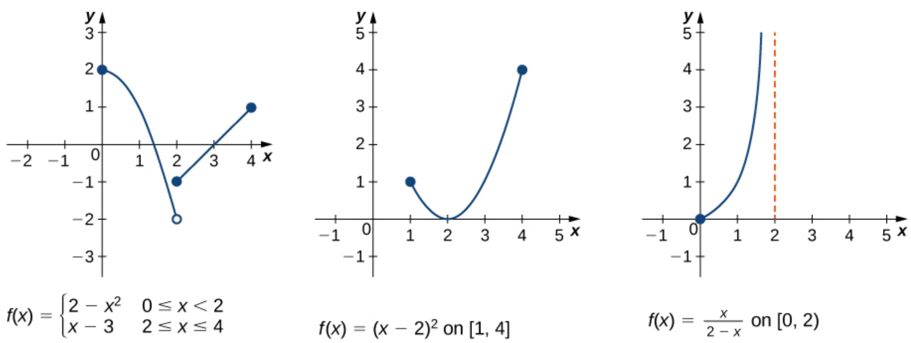


image: https://openstax.org/books/calculus-volume-1/pages/4-3-maxima-and-minima

24.1.2 First Derivative Test

Find the critical point(s) of the following functions. Then use the first derivative test to determine whether it is a local minimum, a local maximum, or neither

1. $f(x) = x - e^x$
2. $g(x) = x + \sin(x)$ on $[0, 2\pi]$

24.1.3 Second Derivative Test

The function $f(x) = x^4 - 8x^3 + 22x^2 - 24x$ has critical points at $x = 1$ and $x = 2$ and $x = 3$.

1. Find $f''(x)$.
2. Use the second derivative test to determine whether each of the critical points is a minimum or a maximum.

24.1.4 Cupcake Store

A cupcake store finds that at a price of \$4.00, the demand is 400 cupcakes. For every \$.25 decrease in price, the demand increases by 20 cupcakes.

1. Define a function $q(x)$ that gives the demand for cupcakes at a price of x dollars.
2. The revenue for a price of x dollars is $r(x) = x \cdot q(x)$. Use your function from part (a) to find the price that maximizes revenue.

Find the price and quantity that maximize the revenue.

Chapter 25

2D Optimization

25.1 Summary

The **critical points** of $f(x, y)$ are the points (a, b) where both $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

There are three ways to determine whether a critical point (a, b) is a minimum, a maximum, or a saddle point.

1. The second derivative test relies on the value

$$D = f_{xx} f_{yy} - (f_{xy})^2.$$

   - If $D > 0$ and $f_{xx} > 0$ then (a, b) is a local minimum.
   - If $D > 0$ and $f_{xx} < 0$ then (a, b) is a local maximum
   - If $D < 0$ then (a, b) is a saddle point
   - If $D = 0$ then the test fails

2. Create a contour plot for $f(x, y)$ on a very small neighborhood of (a, b). Use the contours to figure out whether it is a local max, a local min, or a saddle point.

3. Find the values of $f(a, b) - f(x, y)$ for points (x, y) in a small circle around (a, b).

   - If these values are ALL positive then (a, b) is a local maximum
   - If these values are ALL negative then (a, b) is a local minimum
   - If some values are positive and some values are negative, then (a, b) is a saddle point.

Here is some R code that you can use to check values of $f(x, y)$ in a small circle around a point (a, b).

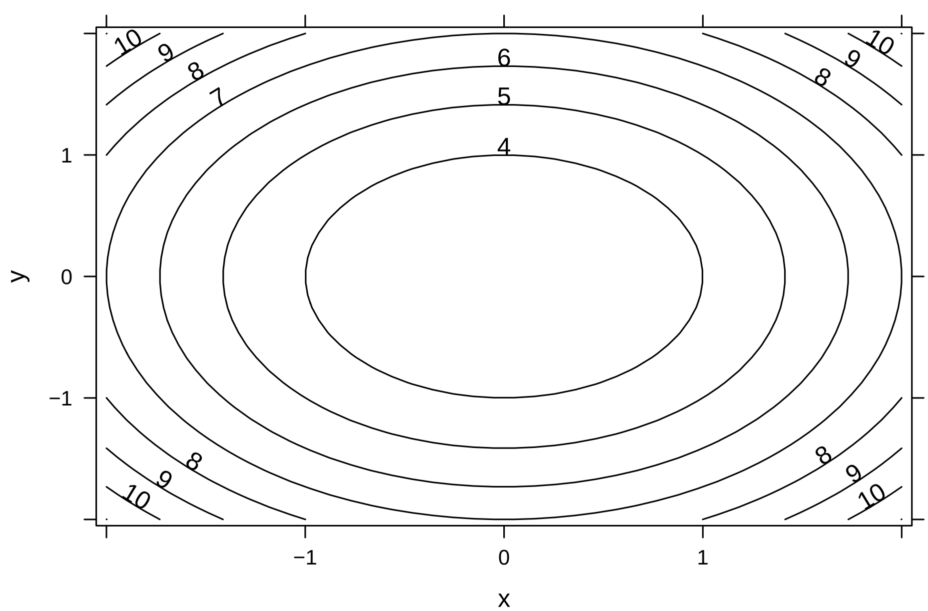
```
r = 0.1
theta = seq(0,2*pi,pi/10)
f(a,b) - f(a+r*cos(theta), b+r*sin(theta))
```

25.2 Activities

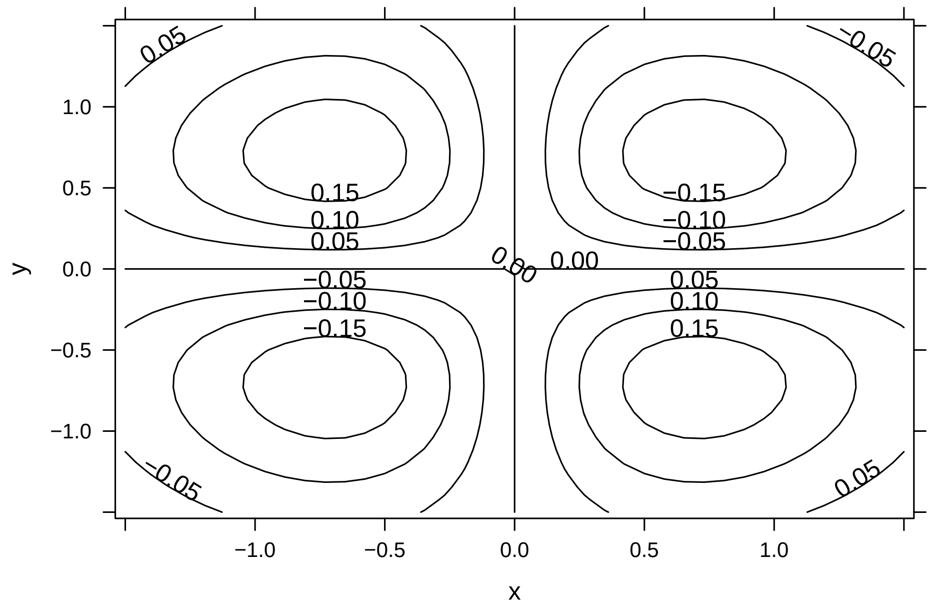
25.2.1 Characterize the Extrema

Multiple 2D functions have been (contour) plotted below. For each, identify the critical points and determine if they are maximums, minimums, or saddle points.

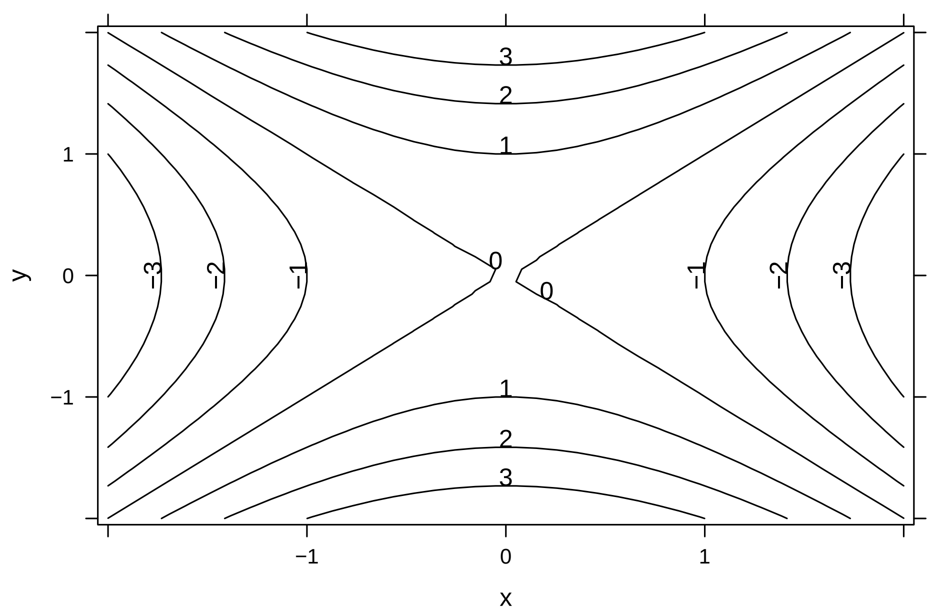
```
f = makeFun(x^2+y^2+3~x&y)
plotFun(f(x,y)~x&y, xlim=range(-2,2), ylim=range(-2,2), filled=FALSE)
```



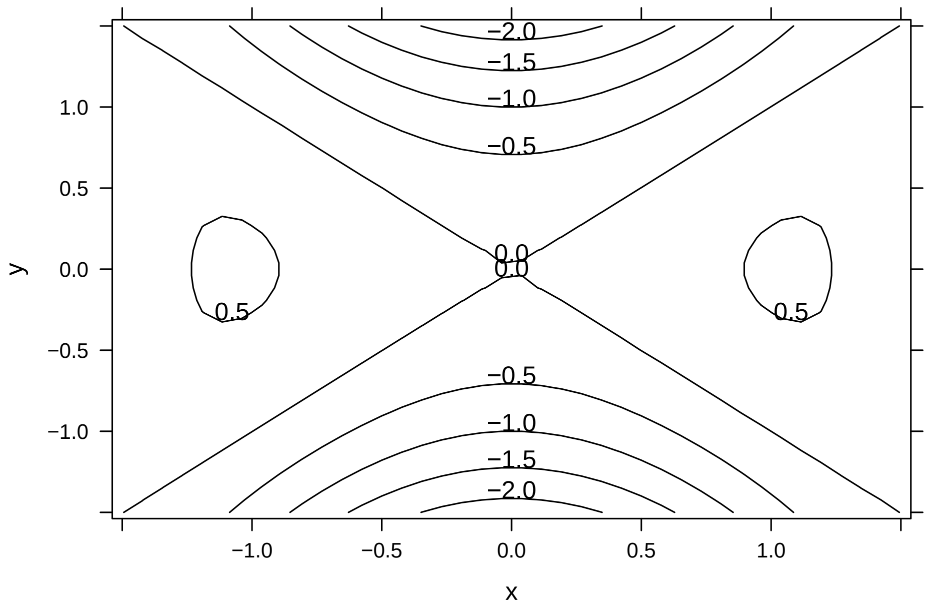

```
f = makeFun(-x*y*exp(-x^2-y^2)~x&y)
plotFun(f(x,y)~x&y, xlim=range(-1.5,1.5), ylim=range(-1.5,1.5), filled=FALSE)
```




```
f = makeFun(y^2-x^2~x&y)
plotFun(f(x,y)~x&y, xlim=range(-2,2), ylim=range(-2,2), filled=FALSE)
```




```
f = makeFun(cos(x)*(x^2-y^2)~x&y)
plotFun(f(x,y)~x&y, xlim=range(-1.5,1.5), ylim=range(-1.5,1.5), filled=FALSE)
```



25.2.2 Classifying a critical point using a small circle of values

The function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 3$ has four critical points:

$$(0,0) \qquad (0,2) \qquad (1,1) \qquad (1,,-1).$$

For each critical point (a, b), evaluate $f(a, b) - f(x, y)$ in a small circle centered at (a, b) to determine whether it is a local minimum, a local maximum, or a saddle point

25.2.3 The 2D Second Derivative Test

The function $f(x, y) = 10 + x^3 + 8y^3 - 3xy$ has two critical points: $(0, 0)$ and $(1/2, 1/4)$.

1. Use the second derivative test to classify each point as a local minimum, local maximum, or saddle point. For convenience, here are the second derivatives of $f(x, y)$.

$$f_{xx} = 6x, \qquad f_{yy} = 48y, \qquad f_{xy} = -3.$$

2. Confirm that your answer is correct by creating a contour plot on a very small neighborhood around each critical point.

25.2.4 Classify the Critical Points

Find the critical points of $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2$. Then use any method that you like to classify each one as a local minimum, a local maximum or a saddle point.

25.2.5 Optimizing Flight Control

The range over which a flying drone can be guided is related to the atmospheric temperature and humidity. The range function is

$$R(t, h) = 27800 - 5t^2 - 6ht - 3h^2 + 400t + 300h$$

What are the optimal atmospheric conditions?

Chapter 26

Gradient Search

26.1 Gradient Search Example

Let's use gradient search to find the maximum of the function

$$f(x,y) = -x^4 - x^3 + 10xy + 2y - 8y^2$$

whose partial derivatives are

$$\frac{\partial f}{\partial x} = -4x^3 - 3x^2 + 10y \quad \text{and} \quad \frac{\partial f}{\partial x} = 10x + 2 - 16y$$

First, you define the partial derivatives and then choose your starting point `(newx, newy)`. In this case, we start at `(1,1)`.

```
partialx = makeFun( -4*x^3 -3*x^2 + 10*y ~ x&y)
partialy = makeFun(10*x + 2 - 16*y ~ x&y)
newx = 1
newy = 1
```

Next, you **repeatedly** run the following code block, which updates the current point by moving `0.1` times the gradient vector. This takes a small step in the uphill direction.

```
newx = newx + 0.1*partialx(newx, newy)
newy = newy + 0.1*partialy(newx, newy)
# print new partial derivatives
c(partialx(newx, newy), partialy(newx, newy))
# print new point
c(newx, newy)
```

Repeatedly run this code block until the partial derivatives are essentially zero (at least two zeros after the decimal point). Congrats! You have found your local maximum.

You will end at the point `(1.04,0.77)`. But note that starting at another initial point might take you to a different local maximum!

26.2 Activities

26.2.1 Finding a Local Minimum

1. How should I change to the code if I want to find a **local minimum** instead of a local maximum? (Hint: we want to take a small step downhill.)
2. Use gradient search to find a **local minimum** of

$$f(x,y) = x^2 + 2xy + 3x + 4y + 5y^2.$$

Chapter 27

Constrained 2D Optimization

27.1 Creating a Contour Plot for 2D Optimization

Here is some example code to create a plot of an objective function $P(x, y) = x^{0.2}y^{0.8}$ and a constraint function $Q(x, y) = 4x + 3y = 200$.

```
P = makeFun( x^(0.2) * y^(0.8) ~ x&y)
Q = makeFun( 4*x + 3*y ~ x&y)

plotFun(P(x,y)~x&y, xlim=c(0,40), ylim=c(20,60), filled=FALSE, levels=seq(20,50,4))

plotFun(Q(x,y)~x&y, add=TRUE, filled=FALSE, levels=200)
```

27.2 Activities

27.2.1 Constrained Production

A company produces $P(x, y) = 100x^{4/11}y^{7/11}$ units of its product from raw materials x and y. The cost of each unit of x is \$5 and the cost of each unit of y is \$4.

1. Find the maximal output for a budget of \$500, along with how much input x and y is used to create them. (Hint: plot your contour diagram on the

domain $0 \le x \le 100$ and $0 \le y \le 100$. Start with the **default** contours by removing the `levels` input.)

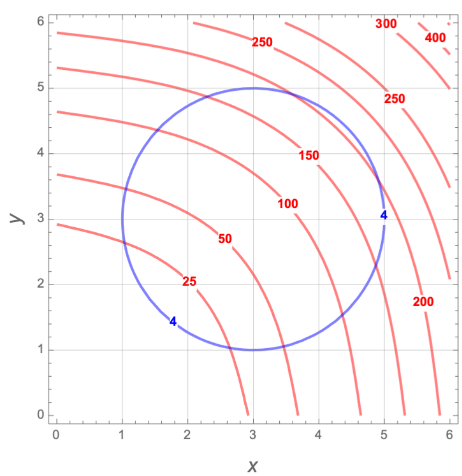
2. What is the maximal output if the unit cost of x increases to \$7? How much of inputs x and y are used?

3. What is the maximal output if the unit cost of x returns to \$5 and the budget is decreased to \$300? How much of inputs x and y are used?

27.2.2 Constrained Optimization on a Circle

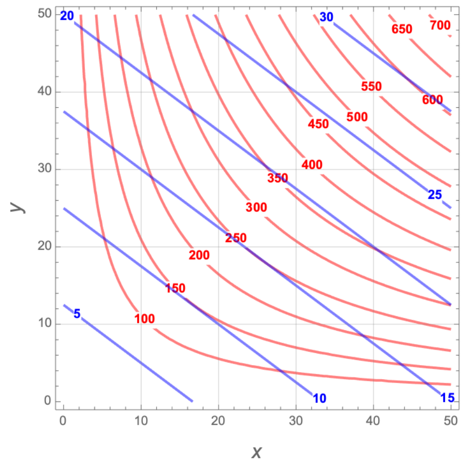
Below is a plot of an objective function $f(x, y)$ in red, and its constraint function $g(x, y) = 9$ in blue.

1. Estimate the **minimum** value of $f(x, y)$ constrained to $g(x, y) = 9$. What point achieves this minimum?
2. Estimate the **maximum** value of $f(x, y)$ constrained to $g(x, y) = 9$. What point achieves this maximum?



27.2.3 Estimating the Lagrange Multiplier

Below is a plot of an objective function $f(x, y)$ in red, and its constraint function $g(x, y)$ for various budgets in blue (in thousands of dollars).



1. Estimate the point (x, y) that maximizes production for a budget of $g(x, y) = 15$. What is the production level?
2. Estimate the point (x, y) that maximizes production for a budget of $g(x, y) = 20$. What is the production level?
3. What is the value of the Lagrange multiplier for the budget $g(x, y) = 15$? What are the units? What does this number mean?
4. What price would make this increase useful to the company? (Hint: the units for your answer are "dollars per unit produced.")

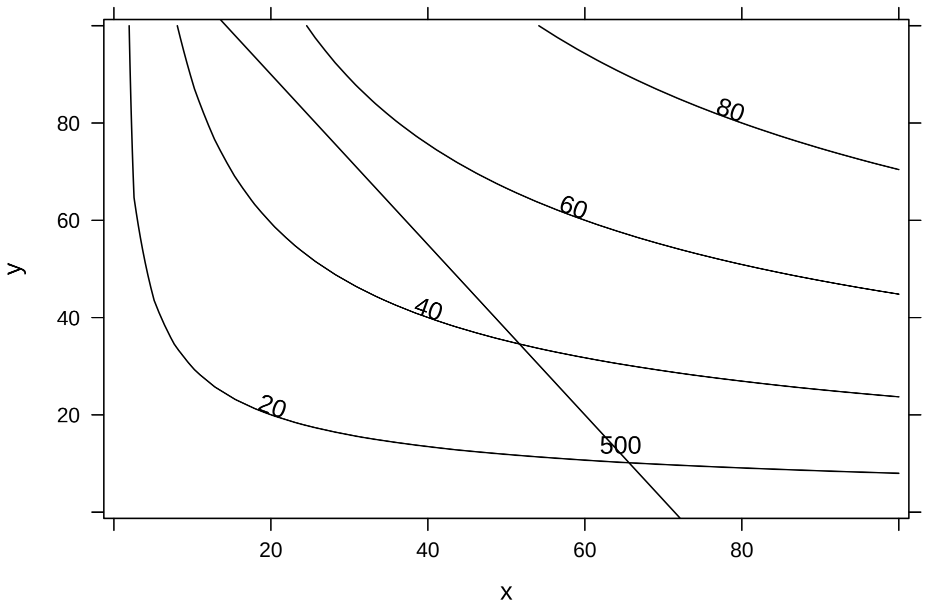
The maximum output for our budget occurs where the $Q(x, y) = 500$ constraint is tangent to a contour of $P(x, y)$. This occurs at (approximately) the point $(38, 78)$ and we estimate that the maximum production is $P(38, 78) = 60$.

2. We need to make a new plot using an updated constraint function.

```
P = makeFun( x^(4/11) * y^(7/11) ~ x&y)
Q = makeFun( 7*x + 4*y ~ x&y)

plotFun(P(x,y)~x&y, xlim=c(0,100), ylim=c(0,100), filled=FALSE)
```

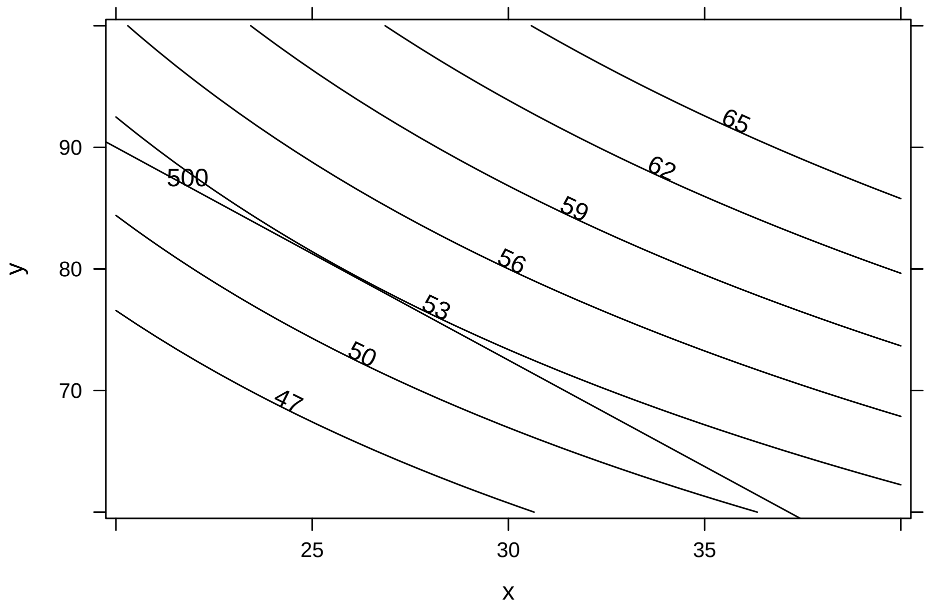
```
plotFun(Q(x,y)~x&y, add=TRUE, filled=FALSE, levels=500)
```



Now, the constraint isn't tangent to any of the shown contours, but it IS tangent to a contour that we haven't drawn! That point is somewhere around (25, 80), and the optimal productions probably around 52 or so. Let's make another plot on a smaller domain and pick some contours to draw.

```
plotFun(P(x,y)~x&y, xlim=c(20,40), ylim=c(60,100), filled=FALSE, levels=seq(47,66,3))
```

```
plotFun(Q(x,y)~x&y, add=TRUE, filled=FALSE, levels=500)
```



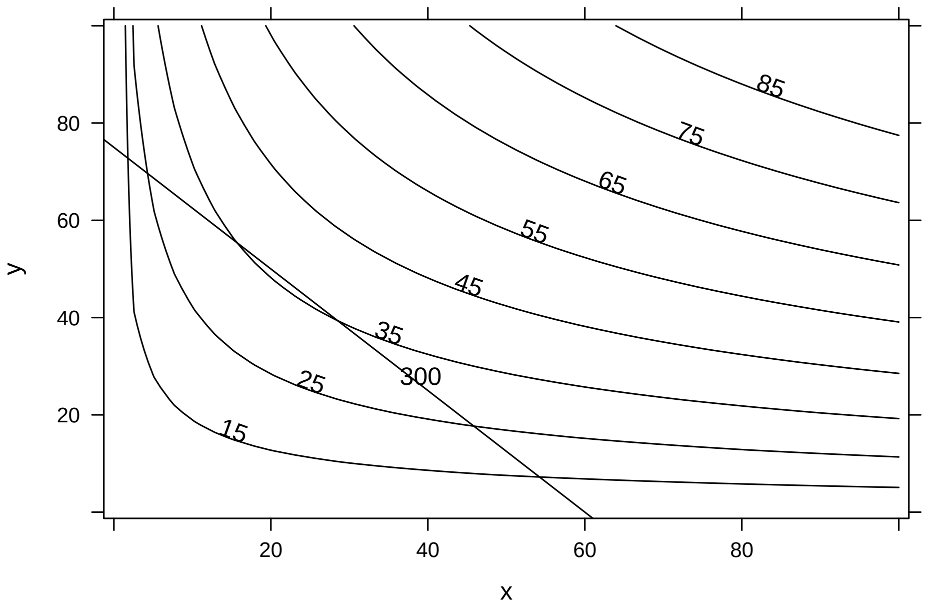
After some experimentation, I realized that $P(x,y)=53$ is the contour that is tangent \$ Q(x,y)=500\$. So the optimal point is $(26,80)$ and the optimal output is 53.

3. Now we return to the original constraint function, but decrease the budget to 300.

```
P = makeFun( x^(4/11) * y^(7/11) ~ x&y)
Q = makeFun( 5*x + 4*y ~ x&y)

plotFun(P(x,y)~x&y, xlim=c(0,100), ylim=c(0,100), filled=FALSE, levels=seq(15,85,10))
```

```
plotFun(Q(x,y)~x&y, add=TRUE, filled=FALSE, levels=300)
```



The optimal point is at $(21, 40)$ and the optimal production is approximately $f(21, 40) = 35$. (This is a slight under-estimate, but it's pretty close.)

27.2.4 Constrained Optimization on a Circle

1. The minimum is approximately $f(1.5, 1.5) = 20$.
2. The maximum is approximately $f(4.5, 4.5) = 210$.

27.2.5 Estimating the Lagrange Multiplier

1. For a budget of 15, the maximal production is $f(25, 18) = 250$.
2. For a budget of 20, the maximal production is $f(32, 26) = 360$.
3. The Lagrange multiplier is

$$\lambda = \frac{\triangle\text{production}}{\triangle\text{constraint}} \approx \frac{360 - 250}{20 - 15} = \frac{110}{5} = 22 \frac{\text{units}}{\text{dollar}}$$

   So if we increase our budget by \$1, then we will produce 22 more units.

4. The Lagrange multiplier is approximately 22 units/dollar. We will break even if we can sell one unit for $1/22 = 0.045$ dollars/unit. So if the sale price is at least \$0.045, we will decide to increase production.

–>

Chapter 28

Unit 5 Summary

28.1 1D Optimization

I can ...

- Find the local maxima, local minima, global maximum and global minimum for the graph of a function.
- Find the critical points of $f(x)$ by solving $f'(x) = 0$.
- Use the first derivative test to determine whether a critical point is a local minimum, a local maximum or a point of inflection.
- Use the second derivative test to determine whether a critical point is a local minimum or a local maximum.

28.2 2D Optimization

I can ...

- Find the 2D critical points of $f(x, y)$ by (simultaneously) solving $f_x(x, y) = 0$ and $f_y(x, y) = 0$.
- Characterize a 2D critical point (a, b) as a local minimimum, local maximum or saddle point by:
  - Using the 2D second derivative test at (a, b)
  - Making a contour plot near (a, b)
  - Looking at the values of $f(a, b) - f(x, y)$ on a small circle around (a, b).

28.3 Gradient Search

I can ...

- Find a local maximum of a 2D function $f(x, y)$ using gradient search in *RStudio*.
- Find a local minimum of a 2D function $f(x, y)$ using gradient search in *RStudio*.
- Simulate gradient search using the contour plot of a 2D function to find the trajectory that gradient search follows to reach an extreme point.
- Explain why the extreme point found by gradient search depends upon the starting point that we use.

28.4 Constrained Optimization

I can ...

- Identify the objective function and the constaint function for a constrained optimization problem.
- Explain why the extreme point for a constrained optimization problem occurs where the constraint contour is tangent to a contour of the objective function.
- Use RStudio to create a contour plot to solve a constrained optimization problem.
- Estimate the value of the Lagrange multiplier λ using a contour plot.
- Interpret the Lagrange multiplier λ as a rate of change.

Part VI

Integration

Chapter 29

Accumulating Change

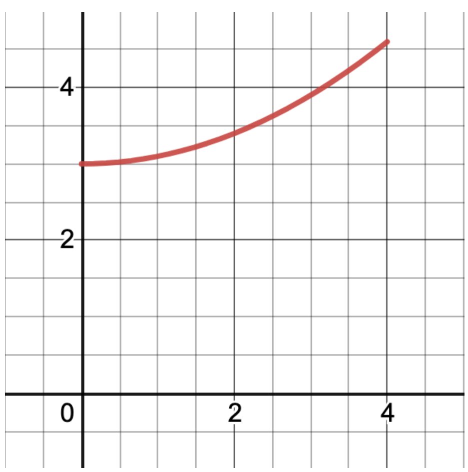
29.1 Activities

29.1.1 Growing Bacteria

A bacteria population is growing at a rate given by

$$f(t) = 3 + 0.1t^2 \quad \text{millions of bacteria per hour}$$

A plot of this growth rate is shown below.



We have already estimated the change in population from $t = 0$ to $t = 3$ using intervals of width 1. Find a better estimate of the increase in population by

1. Using intervals of width 1/2 and evaluating $f(t)$ at the **right endpoint** of each interval.

2. Using intervals of width 1/2 and evaluating $f(t)$ at the **left endpoint** of each interval.

You can do this by hand, or use Desmos to help out. Decide whether your value is an over-estimate or an under-estimate.

29.1.2 The Change in World Population

The rate of change of the world's population, in millions of people per year, is in the table below.

Year	1950	1960	1970	1980	1990	2000
Rate of Change	37	41	78	77	86	79

Using this information, estimate the **total change** in the world's population between 1950 and 2000 by using

1. A left endpoint estimate
2. A right endpoint estimate

29.1.3 Speed Tests

The velocity of a new electric car is measured every two seconds. The data is shown below.

Time (seconds)	0	2	4	6	8
Velocity (m/s)	0	8	16.7	27.3	36.2

Estimate the total distance traveled by this car in the first eight seconds by using

1. A left endpoint estimate
2. A right endpoint estimate

29.1.4 Estimating Area with RStudio

Here is some RStudio code that will estimate the area under the curve $f(x) = 2 + 4x^2 - x^3$ on the interval $0 \leq x \leq 4$ using a **right endpoint estimation** on intervals of size 0.5.

```
f=makeFun(2+4*x^2-x^3~x)
a=0
b=4
```

```r
base = 0.5
points = seq(from=a+base, to=b, by=base)
heights = f(points)
areas = base*heights
sum(areas)
```

```
## [1] 29
```

1. Update this code to estimate this area using
   a. A **right endpoint estimation** on intervals of size 0.1
   b. A **right endpoint estimation** on intervals of size 0.01
   c. A **left endpoint estimation** on intervals of size 0.5
   d. A **left endpoint estimation** on intervals of size 0.01
   e. A **left endpoint estimation** on intervals of size 0.00001
2. Make a plot of $f(x)$ on $[0, 4]$. Is is clear whether each of the values about is an over-estimate? an under-estimate? or neither?

It is **not** clear whether any of these values is an over-estimate or an under-estimate. There is a maximum at $x = 2.75$.

- A left endpoint estimate will be an under-estimate on $[0, 2.75]$ and an over-estimate on $[2.75, 4]$.
- A right endpoint estimate will be an over-estimate on $[0, 2.75]$ and an under-estimate on $[2.75, 4]$.

So we can't easily tell whether it's an under-estimate or an over-estimate.

–>

Chapter 30

The Definite Integral

30.1 Introduction

Here is some code that approximates the area under $f(x) = x^2$ on the interval $[a, b]$ using **right hand endpoints** on 100 subintervals. (How would you change this code to use **left hand endpoints** instead?)

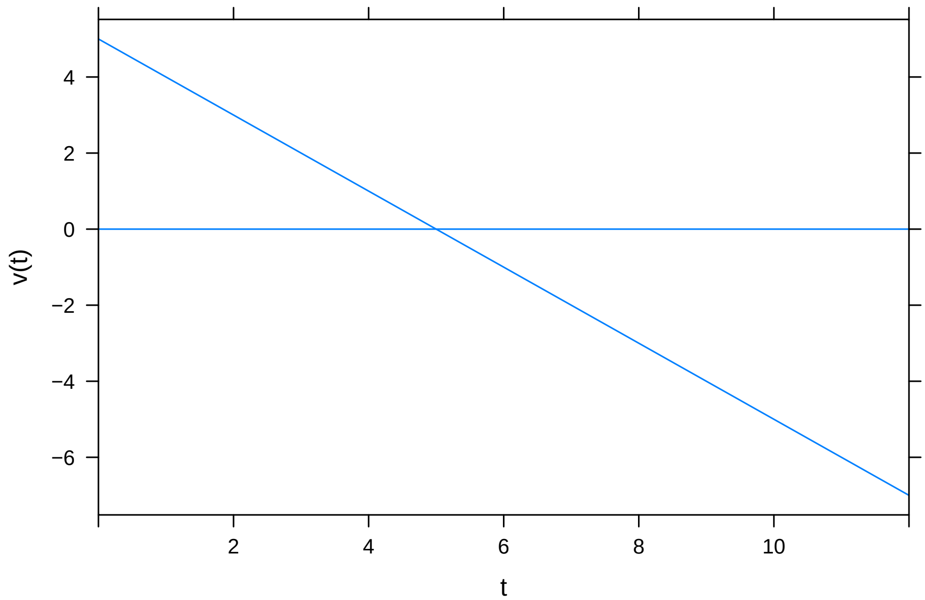
```
f=makeFun(x^2 ~ x)
a = 0
b = 3
num = 100
### the approximation of f(x) on interval [a,b]
base = (b-a)/num
points = seq(from=a+base, to=b, by=base)
heights = f(points)
areas = base*heights
sum(areas)
```

```
## [1] 9.13545
```

30.2 Activities

30.2.1 Drive My Car

A car is driving at a rate of $v(t) = -t + 5$. Here is a plot of the velocity.



1. Use RStudio to approximate these definite integrals using 1000 subintervals.

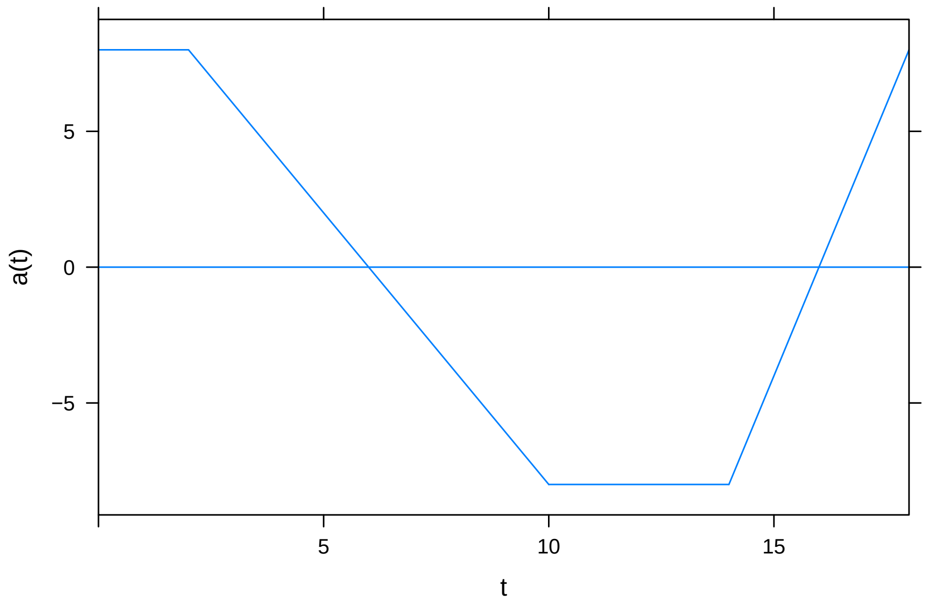
a. $\int_0^5 v(x)dx$
b. $\int_0^8 v(x)dx$
c. $\int_0^{12} v(x)dx$

2. When does the car **turn around**?

3. When does the car **return to where it started**?

30.2.2 Accelerate My Car

A car is traveling at 55 feet per second at time $t = 0$. The plot below shows the car's **acceleration** on the interval $[0, 18]$.



1. Is the velocity increasing or decreasing at $t = 2$? $t = 15$?
2. What happens when $t = 6$?
3. When else (besides $t = 0$) is the velocity 55 feet per second?
4. When is velocity at a **global maximum**?
5. When is velocity at a **global minimum**?

30.2.3 Making Cupcakes

The marginal cost of making cupcakes is given by

$$K'(q) = \frac{1}{100}q^2 + \frac{1}{5}q + 1,$$

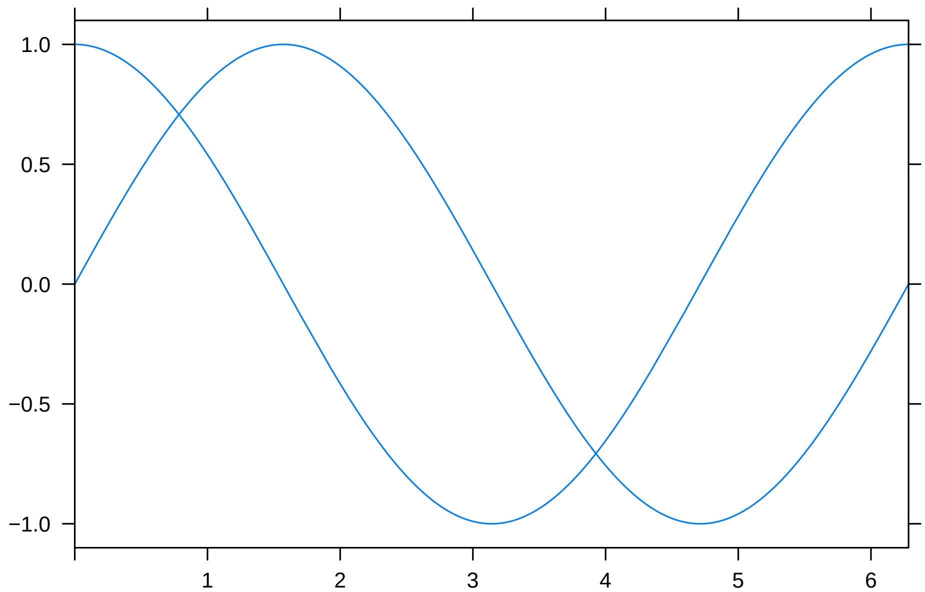
where q is in dozens of cupcakes.

1. Set up a definite integral to calculate $K(30) - K(0)$, which is the change in costs when we create 30 dozen cupcakes.
2. Estimate this definite integral using *RStudio*.
3. If our fixed costs are \$1500, what is the **total cost** to make 30 dozen cupcakes?

30.2.4 Area Between Curves

Below is a graph of both $\sin(x)$ and $\cos(x)$ on the interval $[0, 2\pi]$.

```
plotFun(sin(x)~x,xlim=range(0,2*pi),ylim=range(-1.1,1.1),xlab="",ylab="")
plotFun(cos(x)~x,add=TRUE)
```



1. Write a **sum of integrals** which represents the *area* enclosed by the two curves.
   Hint: $\sin(x) = \cos(x)$ at $x = \pi/4$ and $x = 5\pi/4$.
2. Use *RStudio* to estimate the area between these two curves.

Chapter 31

Rules of Integration

31.1 The Fundamental Theorem of Calculus

$$\int f'(x)\,dx = f(x) + C$$

and

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

31.2 Integrals of Familiar Functions

31.2.1 Constant Rule

The integral of a constant a is $ax + C$.

$$\int a\,dx = ax + C$$

31.2.2 Power Rules

For $n \neq -1$,

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

and

$$\int \frac{1}{x}\,dx = \ln(x) + C$$

31.2.3 Trig Rules

$$\int \sin(x)\,dx = -\cos(x) + C$$

and

$$\int \cos(x)\,dx = \sin(x) + C$$

31.2.4 Exponential Rules

$$\int e^x\,dx = e^x + C$$

and for any positive number $a > 0$,

$$\int a^x\,dx = \frac{1}{\ln(a)}a^x + C$$

31.2.5 Logarithmic Rules

$$\int \ln(x)\,dx = x\ln(x) - x + C$$

and for any positive number $a > 0$,

$$\int \log_a(x)\,dx = x\log_a(x) - \frac{1}{\ln(a)}x + C$$

31.3 Arithmetic Rules for Integrals

31.3.1 Constant Multiple Rule

If c is a constant, then

$$\int cf(x)\,dx = c\int f(x)\,dx$$

31.3.2 Sum and Difference Rules

$$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

and

$$\int (f(x) - g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$$

31.3.3 Substitution Rule

For any constant k,

$$\int f'(kx) = \frac{1}{k} f(kx) + C$$

31.4 Rules for Endpoints

31.4.1 Decomposition Rule

For $a < c < b$, we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

31.4.2 Zero Integral Rule

$$\int_c^c f(x)dx = 0$$

Chapter 32

The Indefinite Integral

32.1 Activities

32.1.1 Indefinite Integral Practice

Find the indefinite integral.

1. $\int 3x^4 \, dx$
2. $\int r^5 - \frac{7}{r^3} + r^{3/4} \, dr$
3. $\int 8t^2 - t^{13} \, dt$
4. $\int 5e^{4x} \, dx$
5. $\int (x+2)^2 \, dx$
6. $\int 4\sqrt{x} - 9x + e \, dx$
7. $\int 3^t - 7t^4 + \frac{9}{t} \, dt$
8. $\int \cos(x) + 2e^x + 7 \, dx$
9. $\int \sin(3x) - 2\cos(x) dx$
10. $\int 7e^{-0.02t} + \frac{4}{t} \, dt$

32.1.2 Definite Integral Practice

Evaluate the definite integral using antiderivates.

1. $\int_{-2}^{5} x^2 \, dx$
2. $\int_{0}^{\pi/4} \sin(x) + e^x \, dx$
3. $\int_{1}^{e} 4t + \frac{1}{4t} \, dt$

32.1.3 Area Between Curves

Find the area enclosed by the curves $f(x) = 5 - x^2$ and $g(x) = x^2 - 3$ by following the steps below

1. Find the points of intersection of $f(x)$ and $g(x)$.
2. Which curve is above the other?
3. Set up the definite integral.
4. Evaluate (symbolically or with RStudio)

32.1.4 Average Value

The average value of a function $f(x)$ on interval $[a, b]$ is given by

$$\frac{1}{b-a} \int_{a}^{b} f(x) \, dx.$$

Find the average value of function on the given interval.

1. The average value of $\sin(x)$ on $[0, \pi]$.
2. The average value of e^{2x} on $[0, 5]$.
3. The average value of $x^2 + 3$ on $[4, 6]$

32.1.5 Modeling Population Growth

The second phase of bacteria growth is called the *log phase*. The rate of population growth is proportional to the number of bacteria at that time with proportionality constant 0.003. In other words

$$\frac{dP}{dt} = 0.003P$$

This is equivalent to the integration formula

$$\int \frac{1}{P}\,dP = \int 0.003\,dt.$$

1. Integrate both sides of this equation and find a general equation for P.
2. Suppose there are 1000 bacteria initially. Find the specific equation for P.

32.1.6 Zombie Apocalypse

The **rate** $r(t)$ at which people become zombies during the inevitable zombie apocalypse can be approximated by

$$r(t) = 1000e^{-0.5t}$$

where t is the number of days since the start of the outbreak.

1. Write the number of infected people $I(T)$ as an indefinite integral.
2. How many total people get infected by time T where T is a very large number?

Chapter 33

Unit 6 Summary

33.1 Accumulating Change

I can ...

- Estimate the accumulated change of $f(x)$ using rectangles under the curve of $y = f(x)$ and
  - Using a left endpoint approximation (LEP)
  - Using a right endpoint approximation (REP)
- Evaluate when a LEP or REP approximation is an under-estimate or an over-estimate.
- Estimate the accumulated change for a table of data
- Estimate the accumulated change of $f(x)$ using RStudio

33.2 The Definite Integral

I can ...

- Calculate or estimate a definite integral using a sum of signed areas.
- Interpret a definite integral of a rate of change as the total accumulated change.
- Find the change in distance (or displacement) using a velocity versus time graph.
- Find the change in velocity using an acceleration versus time graph.
- Find the total cost by finding the definite integral of a marginal cost function.
- Find the area between two curves.

33.3 The Indefinite Integral

I can …

- Find the antiderivative(s) of our familiar functions.
- Explain why we need "$+C$" when finding antiderivatives.
- Use the rules of integration to write a complicated indefinite integral as the sum of multiple indefinite integrals.
- Use substitution to evaluate and integral of the form $\int f(kx)\,dx$ wherre k is a constant.
- Calculate the average value of $f(x)$ on interval $[a, b]$ using the formula

$$\text{average value} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$