1. The Basic Problem

Consider real-valued input functions with domain $\mathcal{X} = [a, b]$, where a, b be two fixed real numbers with $a < b$. We want to do the function recovery and let S be a solution operator. Here we consider:

$$S : f \mapsto f, \qquad S : \mathcal{W}^{2,\infty}[a, b] \to \mathcal{L}_\infty[a, b].$$

where for all real numbers α, β with $\alpha < \beta$,

$$\mathcal{W}^{2,\infty} := \mathcal{W}^{2,\infty}[a, b], \qquad \mathcal{W}^{2,\infty}[\alpha, \beta] := \{f \in C[\alpha, \beta] : \left\| f'' \right\|_{\infty,[\alpha,\beta]} < \infty\},$$
$$\mathcal{L}_\infty := \mathcal{L}_\infty[a, b], \qquad \mathcal{L}_\infty[\alpha, \beta] := \mathcal{W}^{0,\infty}[\alpha, \beta],$$
$$\left\| f'' \right\|_{[\alpha,\beta]} := \left\| f'' \right\|_{\infty,[\alpha,\beta]}, \qquad \left\| f \right\|_{[\alpha,\beta]} := \left\| f \right\|_{\infty,[\alpha,\beta]} = \max_{\alpha \le x \le \beta} |f(x)| \, .$$

Here we only consider infinity norm, so we omit sub index ∞.

2. Solving the Problems on Partitions

For any interval $[\alpha, \ \beta] \subset [a, \ b]$, let let $f_{[\alpha, \ \beta]}$ denote $\mathcal{W}^{2,\infty}$ restricted to the interval $[\alpha, \ \beta]$, i.e.,

$$f_{[\alpha, \ \beta]} : [\alpha, \ \beta] \to \mathbb{R}, \qquad f_{[\alpha, \ \beta]} : \boldsymbol{x} \mapsto f(\boldsymbol{x}).$$

Moreover, let $\mathcal{W}^{2,\infty}[\alpha, \beta]$ denote the space of functions in $\mathcal{W}^{2,\infty}$ restricted to the set interval $[\alpha, \ \beta]$.

For function recovery, these partitions take the form of subintervals of $[a, b]$:

$$\mathcal{T} = \{[\alpha, \beta] : a \le \alpha < \beta \le b\}$$
$$\mathcal{P} = \{[t_0, t_1], [t_1, t_2], \ldots, [t_{L-1}, t_L]\}, \qquad a = t_0 < t_1 < \cdots < t_L = b.$$

For each $[\alpha, \beta] \in \mathcal{T}$, we let

$$\left\| f'' \right\|_{[\alpha,\beta]} = \left\| f''_{[\alpha,\beta]} \right\|_{\infty,[\alpha,\beta]}, \qquad S_{[\alpha,\beta]} : f \mapsto f \cdot \mathbb{1}_{[\alpha,\beta]}.$$

For partitions $\mathcal{P}$, there exists a function $\Phi(\cdot; \mathcal{P}) : \mathcal{L}_\infty^L \to \mathcal{L}_\infty$ that combines the solutions defined on the subsets to reconstruct the true solution:

$$S(f) = \Phi(\boldsymbol{S}_\mathcal{P}(f)), \quad \boldsymbol{S}_\mathcal{P}(f) := \{S(f; [\alpha, \beta])\}_{[\alpha,\beta] \in \mathcal{P}}, \qquad \forall f \in \mathcal{W}^{2,\infty}.$$

Here $L = |\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$.

- There exists a pair of functions $(\widetilde{\Phi}, \mathrm{err})(\cdot, \cdot; \mathcal{P}) : \mathcal{H}^{|\mathcal{P}|} \times [0, \infty)^{|\mathcal{P}|} \to \mathcal{H} \times [0, \infty)$ that combine approximate solutions defined on the subsets with error bounds to reconstruct an approximation to the true solution. If $\left\| S(f; \mathcal{Y}) - \widetilde{S}_\mathcal{Y} \right\|_\mathcal{H} \le \varepsilon_\mathcal{Y}$ for all $\mathcal{Y} \in \mathcal{P}$, $\widetilde{\boldsymbol{S}}_\mathcal{P} := \{\widetilde{S}_\mathcal{Y}\}_{\mathcal{Y} \in \mathcal{P}}$, and $\tilde{\boldsymbol{\varepsilon}}_\mathcal{P} := \{\tilde{\varepsilon}_\mathcal{Y}\}_{\mathcal{Y} \in \mathcal{P}}$, then

$$\left\| S(f) - \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_\mathcal{P}, \tilde{\boldsymbol{\varepsilon}}_\mathcal{P}; \mathcal{P}) \right\|_\mathcal{H} \le \mathrm{err}(\widetilde{\boldsymbol{S}}_\mathcal{P}, \tilde{\boldsymbol{\varepsilon}}_\mathcal{P}; \mathcal{P}) \qquad \forall f \in \mathcal{F}.$$

  Here $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$. Moreover,

$$\mathrm{err}(\widetilde{\boldsymbol{S}}_\mathcal{P}, \mathbf{0}; \mathcal{P}) = 0, \qquad \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_\mathcal{P}, \mathbf{0}; \mathcal{P}) = \Phi(\boldsymbol{S}_\mathcal{P}(f)) = S(f).$$

$$
\begin{aligned}
\Phi(\widetilde{\boldsymbol{S}}_{\mathcal{P}};\mathcal{P}) &= \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}},\boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) = \sum_{l=1}^{L} \widetilde{S}_{[t_{l-1},t_l]},\\
\operatorname{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}},\boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) &= \left\|\boldsymbol{\varepsilon}_{\mathcal{P}}\right\|_\infty
\end{aligned}
$$

3. Algorithms

Now we consider numerical algorithms for solving the problems on a subinterval $[\alpha, \beta]$ of $[a, b]$.

We use piecewise linear interpolation. The number of possible function values is $n \in \mathbb{N}$, then we have

$$
x_j = \alpha + (\beta - \alpha)\frac{j-1}{n-1}, \qquad j = 1, \ldots, n, \tag{1}
$$

$$
\begin{aligned}
&A_n(f,[\alpha,\beta]) := \\
&\qquad \frac{n-1}{\beta-\alpha}\left[f(x_j)(x_{j+1}-x) + f(x_{j+1})(x-x_j)\right], \qquad x_j \le x \le x_{j+1}
\end{aligned} \tag{2}
$$

The difference between f and its linear spline can be bounded in terms of an integral involving the second derivative using integration by parts. For $x \in [\alpha, \beta]$ it follows that

$$
\begin{aligned}
f(x) - A_n(f,[\alpha,\beta])(x) &= f(x) - \frac{n-1}{\beta-\alpha}\left[f(x_j)(x_{j+1}-x) + f(x_{j+1})(x-x_j)\right]\\
&= \frac{n-1}{\beta-\alpha}\int_{x_j}^{x_{j+1}} v_j(t,x) f''(t)\,\mathrm{d}t, \qquad (3)\\
f'(x) - A_n(f,[\alpha,\beta])'(x) &= \frac{n-1}{\beta-\alpha}\int_{x_j}^{x_{j+1}} \frac{\partial v_j}{\partial x}(t,x) f''(t)\,\mathrm{d}t, \qquad (4)
\end{aligned}
$$

where the continuous, piecewise differentiable kernel v is defined as

$$
v_j(t,x) := \begin{cases} (x_{j+1}-x)(x_j-t), & x_j \le t \le x, \\ (x-x_j)(t-x_{j+1}), & x < t \le x_{j+1}, \end{cases}.
$$

To derive the error bounds for $A_n(f,[\alpha,\beta])$ we have:

$$
\begin{aligned}
\|f - A_n(f,[\alpha,\beta])\|_\infty &\le \sup_{\substack{x_j \le x \le x_{j+1}\\ j=1,\ldots,n_i-1}} |f(x) - A_n(f,[\alpha,\beta])(x)| \\
&= \frac{n-1}{\beta-\alpha} \sup_{\substack{x_j \le x \le x_{j+1}\\ j=1,\ldots,n-1}} \int_{x_j}^{x_{j+1}} |v_j(t,x) f''(t)| \, \mathrm{d}t \\
&\le \frac{n-1}{\beta-\alpha} \|f''\|_{[\alpha,\beta]} \sup_{\substack{x_j \le x \le x_{j+1}\\ j=1,\ldots,n-1}} \int_{x_j}^{x_{j+1}} |v_j(t,x)| \, \mathrm{d}t \\
&= \|f''\|_\infty \sup_{\substack{x_j \le x \le x_{j+1}\\ j=1,\ldots,n-1}} \frac{(x-x_j)(x_{j+1}-x)}{2} \\
&= h(n,[\alpha,\beta]) \, \|f''\|_{[\alpha,\beta]}, \qquad h(n,[\alpha,\beta]) := \frac{(\beta-\alpha)^2}{8(n-1)^2}.
\end{aligned}
$$

4. Functions in Cones

The challenge in applying the error bounds for the non-adaptive function recovery algorithm is that $f''\|_{[\alpha,\beta]}$ is not known a priori. Our approach is to assume that the input functions lie inside cones.

For each $[\alpha,\ \beta] \in \mathcal{T}$, there exists a norm $\left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_{[\alpha,\beta]}$ on space $\mathcal{W}^{2,\infty}[\alpha,\beta]$ weaker than $\|f''\|_{[\alpha,\beta]}$. Here we define our cone condition

$$
\mathcal{C}_\tau := \left\{ f \in \mathcal{W}^{2,\infty} : \|f''\|_{[\alpha,\beta]} \le \frac{\tau_{[\alpha,\beta]}}{\beta-\alpha} \left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_{[\alpha,\beta]} \right\}. \tag{5}
$$

where $\tau_{[\alpha,\beta]}$ is a parameter depend on the length of interval $[\alpha,\beta]$, and $\tau_{[\alpha,\beta]} : [0,\infty) \to [0,\infty)$.

Lower bound on $\|f''\|_{\infty,[\alpha,\beta]}$ can be derived similarly to the previous section using a centered difference. Specifically, for $n_i \ge 3$,

$$
F_n(f;[\alpha,\beta]) := \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} |f(x_j) - 2f(x_{j+1}) + f(x_{j+2})|. \tag{6}
$$

It follows using the Hölder's inequality that

$$
\begin{aligned}
F_n(f;[\alpha,\beta]) &= \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} \left| \int_{x_j}^{x_{j+2}} \left[\frac{\beta-\alpha}{n-1} - |x - x_{j+1}| \right] f''(x) \, \mathrm{d}x \right| \\
&\le \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} \|f''\|_\infty \int_{x_j}^{x_{j+2}} \left| \frac{\beta-\alpha}{n-1} - |x - x_{j+1}| \right| \mathrm{d}x = \|f''\|_{\infty,[\alpha,\beta]}.
\end{aligned}
$$

Define

$$
\begin{aligned}
G_n(f;[\alpha,\beta]) &:= \|A_n(f,[\alpha,\beta])' - A_2(f,[\alpha,\beta])'\|_\infty \\
&= \sup_{j=1,\ldots,n-1} \left| \frac{n-1}{\beta-\alpha} [f(x_{j+1}) - f(x_j)] - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right|.
\end{aligned} \tag{7}
$$

Note $G_n(f;[\alpha,\beta])$ never overestimates $\left\|f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right\|_{[\alpha,\beta]}$ because

$$\begin{aligned}
\left\|f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right\|_{[\alpha,\beta]} &= \|f' - A_2(f,[\alpha,\beta])'\|_{[\alpha,\beta]} = \sup_{\substack{x_j\le x\le x_{j+1}\\ j=1,\dots,n-1}} |f'(x) - A_2(f,[\alpha,\beta])'(x)| \\
&\geq \sup_{j=1,\dots,n-1} \frac{n-1}{\beta-\alpha} \int_{x_j}^{x_{j+1}} \left| f'(x) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right| \mathrm{d}x \\
&\geq \sup_{j=1,\dots,n-1} \frac{n-1}{\beta-\alpha} \left| \int_{x_j}^{x_{j+1}} \left[f'(x) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right] \mathrm{d}x \right| \\
&= \sup_{j=1,\dots,n-1} \frac{n-1}{\beta-\alpha} \left| f(x_{j+1}) - f(x_j) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right| = G_n(f;[\alpha,\beta]).
\end{aligned}$$

Since $(f - A_n(f,[\alpha,\beta]))(x)$ vanishes for $x=\alpha,\beta$. Using (4) it then follows that

$$\begin{aligned}
&\left\|f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right\|_{[\alpha,\beta]} - G_n(f;[\alpha,\beta]) \\
&\leq \left\|f' - A_n(f,[\alpha,\beta])'\right\|_\infty \\
&= \sup_{\substack{x_j\le x\le x_{j+1}\\ j=1,\dots,n-1}} \left| f'(x) - \frac{n-1}{\beta-\alpha}[f(x_{j+1}) - f(x_j)]\right| \\
&= \frac{n-1}{\beta-\alpha} \sup_{\substack{x_j\le x\le x_{j+1}\\ j=1,\dots,n_i-1}} \left| \int_{x_j}^{x_{j+1}} \frac{\partial v_j}{\partial x}(t,x) f''(t)\,\mathrm{d}t\right| \\
&\leq \frac{n-1}{\beta-\alpha} \|f''\|_{[\alpha,\beta]} \sup_{\substack{x_j\le x\le x_{j+1}\\ j=1,\dots,n_i-1}} \int_{x_j}^{x_{j+1}} \left|\frac{\partial v_j}{\partial x}(t,x)\right| \mathrm{d}t \\
&= \frac{n-1}{\beta-\alpha} \|f''\|_{[\alpha,\beta]} \sup_{\substack{x_j\le x\le x_{j+1}\\ j=1,\dots,n-1}} \left\{ \frac{(\beta-\alpha)^2}{2(n-1)^2} - (x-x_j)(x_{j+1}-x)\right\} \\
&= \frac{\beta-\alpha}{2(n-1)} \|f''\|_{[\alpha,\beta]}.
\end{aligned}$$

Thus, we obtain

$$G_n(f;[\alpha,\beta]) \leq \left\|f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right\|_{[\alpha,\beta]} \leq \frac{G_n(f;[\alpha,\beta])}{1-\tau_{[\alpha,\beta]}/(2n-2)}, \quad \forall f\in\mathcal{C}_\tau,$$

Therefore,

$$\begin{aligned}
\|f - A_n(f,[\alpha,\beta])\|_\infty &\leq \frac{(\beta-\alpha)^2}{8(n-1)^2}\|f''\|_\infty, \\
&\leq \frac{(\beta-\alpha)^2}{8(n-1)^2} \cdot \frac{\tau(\mathrm{vol}([\alpha,\beta]))G_n(f;[\alpha,\beta])}{1-\tau(\mathrm{vol}([\alpha,\beta]))(\beta-\alpha)/(2n-2)} \\
&\leq \frac{\tau(\mathrm{vol}([\alpha,\beta]))(\beta-\alpha)^2 G_n(f;[\alpha,\beta])}{4(n-1)(2n-2-\tau(\mathrm{vol}([\alpha,\beta]))(\beta-\alpha))} = \varepsilon_{[\alpha,\beta]}.
\end{aligned}$$

Algorithm 1 (Locally Adaptive Univariate Function Recovery)**.** Let the sequence of algorithms $\{A_n\}$, $\{G_n\}$, $\{F_n\}$, and n be as described above. Set $L = 1,\ l = 1, \ldots, L$. Choose integer $\tau_{\text{lo}}, \tau_{\text{hi}}$, where $\tau_{\text{hi}} \geq \tau_{\text{lo}}$ such that

$$\tau_{[t_{l-1},t_l]} = \max\left\{\left\lceil \tau_{\text{hi}} \left(\frac{\tau_{\text{lo}}}{\tau_{\text{hi}}}\right)^{\frac{1}{1+t_i-t_{i-1}}} \right\rceil, 3\right\}.$$

Then let $n_i = \left\lceil \frac{\tau_{[t_{l-1},t_l]}+1}{2} \right\rceil + 1,\ \varepsilon_{[t_{l-1},t_l]} = \infty$. For any error tolerance ε and input function f, do the following:

Stage 1. Find the maximum error If $\max\limits_{l=1,\ldots,L} \varepsilon_{[t_{l-1},t_l]} < \varepsilon$, stop. Otherwise, find

$$k = \arg \max_{l=1,\ldots,L} \varepsilon_{[t_{l-1},t_l]}.$$

Stage 2. Compute $\left\| f' - \frac{f(t_k)-f(t_{k-1})}{t_k - t_{k-1}} \right\|_\infty$ **and bound** $\|f''\|_{\infty,[t_{k-1},t_k]}$**.** Compute $G_{n_k}(f)$ in (7) and $F_{n_k}(f)$ in (6).

Stage 3. Check the necessary condition for $f \in \mathcal{C}_{\tau_{[t_{k-1},t_k]}}$**.** Compute

$$\tau_{\min,n_k} = \frac{F_{n_k}(f)}{G_{n_k}(f)/(t_k - t_{k-1}) + F_{n_k}(f)/(2n_k - 2)}.$$

If $\tau_{[t_{k-1},t_k]} \geq \tau_{\min,n_k}$, then go to stage 4. Otherwise, set $\tau_{[t_{k-1},t_k]} = 2\tau_{[t_{k-1},t_k]}$. If $n_k \geq (\tau_{[t_{k-1},t_k]} + 1)/2$, then go to stage 4. Otherwise, go to Stage 5.

Stage 4. Check for convergence. Estimate $\varepsilon_{[t_{k-1},t_k]}$.

$$\varepsilon_{[t_{l-1},t_l]} = \frac{\tau_{[t_{k-1},t_k]}(t_k - t_{k-1})G_{n_k}(f)}{4(n_k - 1)(2n_k - 2 - \tau_{[t_{k-1},t_k]})}.$$

If $\varepsilon_{[t_{l-1},t_l]} < \varepsilon$, go to Stage 1. Otherwise go to Stage 5.

Stage 5. Double the initial number of points and split the interval Let $L = L + 1$. Then

$$t_l = t_{l-1},\ n_l = n_{l-1}, \varepsilon_{[t_{l-1},t_l]} = \varepsilon_{[t_{l-2},t_{l-1}]} \text{ when } l > k.$$
$$t_k = \frac{t_{k-1} + t_k}{2},\ \varepsilon_{[t_{k-1},t_k]} = \infty,\ \varepsilon_{[t_k,t_{k+1}]} = \infty.$$

Go to Stage 1.