1. The Basic Problem

Consider real-valued input functions with domain $\mathcal{X} = [a, b]$, where a, b be two fixed real numbers with $a < b$. We want to do the function recovery and let S be a solution operator. Here we consider:

$$S : f \mapsto f, \qquad S : \mathcal{W}^{2,\infty}[a, b] \to \mathcal{L}_\infty[a, b].$$

where for all real numbers α, β with $\alpha < \beta$,

$$\mathcal{W}^{2,\infty} := \mathcal{W}^{2,\infty}[a, b], \qquad \mathcal{W}^{2,\infty}[\alpha, \beta] := \{f \in C[\alpha, \beta] : \left\| f'' \right\|_{\infty,[\alpha,\beta]} < \infty\},$$
$$\mathcal{L}_\infty := \mathcal{L}_\infty[a, b], \qquad \mathcal{L}_\infty[\alpha, \beta] := \mathcal{W}^{0,\infty}[\alpha, \beta],$$
$$\left\| f'' \right\|_{[\alpha,\beta]} := \left\| f'' \right\|_{\infty,[\alpha,\beta]}, \qquad \left\| f \right\|_{[\alpha,\beta]} := \left\| f \right\|_{\infty,[\alpha,\beta]} = \max_{\alpha \le x \le \beta} |f(x)| \, .$$

Here we only consider infinity norm, so we omit sub index ∞.

2. Solving the Problems on Partitions

For any interval $[\alpha,\ \beta] \subset [a,\ b]$, let let $f_{[\alpha,\ \beta]}$ denote $\mathcal{W}^{2,\infty}$ restricted to the interval $[\alpha,\ \beta]$, i.e.,

$$f_{[\alpha,\ \beta]} : [\alpha,\ \beta] \to \mathbb{R}, \qquad f_{[\alpha,\ \beta]} : \boldsymbol{x} \mapsto f(\boldsymbol{x}).$$

Moreover, let $\mathcal{W}^{2,\infty}[\alpha, \beta]$ denote the space of functions in $\mathcal{W}^{2,\infty}$ restricted to the set interval $[\alpha,\ \beta]$.

For function recovery, these partitions take the form of subintervals of $[a, b]$:

$$\mathcal{T} = \{[\alpha, \beta] : a \le \alpha < \beta \le b\}$$
$$\mathcal{P} = \{[t_0, t_1], [t_1, t_2], \dots, [t_{L-1}, t_L]\}, \qquad a = t_0 < t_1 < \cdots < t_L = b.$$

For each $[\alpha, \beta] \in \mathcal{T}$, we let $\left\| f'' \right\|_{[\alpha,\beta]} = \left\| f''_{[\alpha,\beta]} \right\|_{\infty,[\alpha,\beta]}$.

For partitions $\mathcal{P}$, define algorithm and error

$$\boldsymbol{A}(f; \mathcal{P}) = \{A(f; [\alpha, \beta])\}_{[\alpha,\beta]\in\mathcal{P}}, \qquad \boldsymbol{\varepsilon}_{\mathcal{P}} = \{\varepsilon_{[\alpha,\beta]}\}_{[\alpha,\beta]\in\mathcal{P}}.$$

$$\boldsymbol{A}(f; \mathcal{P}) = \begin{cases} A(f; [t_0, t_1]) & t_0 \le x \le t_1, \\ A(f; [t_1, t_2]) & t_1 \le x \le t_2, \\ \vdots & \vdots, \\ A(f; [t_{i-1}, t_i]) & t_{i-1} \le x \le t_i, \\ \vdots & \vdots, \\ A(f; [t_{L-1}, t_L]) & t_{L-1} \le x \le t_L. \end{cases}$$

Then our goal is

$$\left\| f - \boldsymbol{A}(f; \mathcal{P}) \right\| \le \left\| \boldsymbol{\varepsilon}_{\mathcal{P}} \right\| < \varepsilon.$$

i.e.

$$\left\| f - A(f; [\alpha, \beta]) \right\|_{[\alpha,\beta]} \le \varepsilon_{[\alpha,\beta]} < \varepsilon \qquad \forall [\alpha, \beta] \in \mathcal{P}.$$

3. Algorithms

Now we consider numerical algorithms for solving the problems on a subinterval $[\alpha, \beta]$ of $[a, b]$.

We use piecewise linear interpolation. The number of possible function values is $n \in \mathbb{N}$, then we have

$$x_j = \alpha + (\beta - \alpha)\frac{j-1}{n-1}, \qquad j = 1, \ldots, n, \tag{1}$$

$$\begin{aligned} &A_n(f; [\alpha, \beta]) := \\ &\qquad \frac{n-1}{\beta - \alpha}\left[f(x_j)(x_{j+1} - x) + f(x_{j+1})(x - x_j)\right], \qquad x_j \le x \le x_{j+1} \end{aligned} \tag{2}$$

The difference between f and its linear spline can be bounded in terms of an integral involving the second derivative using integration by parts. For $x \in [\alpha, \beta]$ it follows that

$$\begin{aligned} f(x) - A_n(f; [\alpha, \beta])(x) &= f(x) - \frac{n-1}{\beta - \alpha}\left[f(x_j)(x_{j+1} - x) + f(x_{j+1})(x - x_j)\right] \\ &= \frac{n-1}{\beta - \alpha}\int_{x_j}^{x_{j+1}} v_j(t, x) f''(t)\,\mathrm{d}t, \end{aligned} \tag{3}$$

$$f'(x) - A_n(f; [\alpha, \beta])'(x) = \frac{n-1}{\beta - \alpha}\int_{x_j}^{x_{j+1}} \frac{\partial v_j}{\partial x}(t, x) f''(t)\,\mathrm{d}t, \tag{4}$$

where the continuous, piecewise differentiable kernel v is defined as

$$v_j(t, x) := \begin{cases} (x_{j+1} - x)(x_j - t), & x_j \le t \le x, \\ (x - x_j)(t - x_{j+1}), & x < t \le x_{j+1}, \end{cases}.$$

To derive the error bounds for $A_n(f; [\alpha, \beta])$ we have:

$$\begin{aligned} \|f - A_n(f; [\alpha, \beta])\|_{[\alpha, \beta]} &\le \sup_{\substack{x_j \le x \le x_{j+1} \\ j = 1, \ldots, n_i - 1}} |f(x) - A_n(f; [\alpha, \beta])(x)| \\ &= \frac{n-1}{\beta - \alpha} \sup_{\substack{x_j \le x \le x_{j+1} \\ j = 1, \ldots, n-1}} \int_{x_j}^{x_{j+1}} |v_j(t, x) f''(t)|\,\mathrm{d}t \\ &\le \frac{n-1}{\beta - \alpha}\|f''\|_{[\alpha, \beta]} \sup_{\substack{x_j \le x \le x_{j+1} \\ j = 1, \ldots, n-1}} \int_{x_j}^{x_{j+1}} |v_j(t, x)|\,\mathrm{d}t \\ &= \|f''\|_{[\alpha, \beta]} \sup_{\substack{x_j \le x \le x_{j+1} \\ j = 1, \ldots, n-1}} \frac{(x - x_j)(x_{j+1} - x)}{2} \\ &= \frac{(\beta - \alpha)^2}{8(n-1)^2}\|f''\|_{[\alpha, \beta]}. \end{aligned} \tag{5}$$

4. Functions in Cones

The challenge in applying the error bounds for the non-adaptive function recovery algorithm is that $\left\|f''\right\|_{[\alpha,\beta]}$ is not known a priori. Our approach is to assume that the input functions lie inside cones.

Let $\boldsymbol{\tau}_{\mathcal{P}} = \{\tau_{[\alpha,\beta]}\}_{[\alpha,\beta]\in\mathcal{P}}$. For each $[\alpha,\ \beta] \in \mathcal{T}$, there exists a norm $\left\|f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right\|_{[\alpha,\beta]}$ on space $\mathcal{W}^{2,\infty}[\alpha,\beta]$ weaker than $\|f''\|_{[\alpha,\beta]}$. Here we define our cone condition

$$\mathcal{C}_{\boldsymbol{\tau}} := \left\{ f \in \mathcal{W}^{2,\infty} : \|f''\|_{[\alpha,\beta]} \leq \frac{\tau_{[\alpha,\beta]}}{\beta-\alpha} \left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_{[\alpha,\beta]} \right\}. \tag{6}$$

where $\tau_{[\alpha,\beta]}$ is a parameter depend on the length of interval $[\alpha,\beta]$.

Lower bound on $\|f''\|_{[\alpha,\beta]}$ can be derived similarly to the previous section using a centered difference. Specifically, for $n_i \geq 3$,

$$F_n(f;[\alpha,\beta]) := \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} |f(x_j) - 2f(x_{j+1}) + f(x_{j+2})|\,. \tag{7}$$

It follows using the Hölder's inequality that

$$\begin{aligned} F_n(f;[\alpha,\beta]) &= \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} \left| \int_{x_j}^{x_{j+2}} \left[\frac{\beta-\alpha}{n-1} - |x - x_{j+1}|\right] f''(x)\,\mathrm{d}x \right| \\ &\leq \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} \|f''\|_{[\alpha,\beta]} \int_{x_j}^{x_{j+2}} \left| \frac{\beta-\alpha}{n-1} - |x-x_{j+1}| \right| \mathrm{d}x = \|f''\|_{[\alpha,\beta]}\,. \end{aligned}$$

Define

$$\begin{aligned} G_n(f;[\alpha,\beta]) &:= \|A_n(f;[\alpha,\beta])' - A_2(f;[\alpha,\beta])'\|_\infty \\ &= \sup_{j=1,\ldots,n-1} \left| \frac{n-1}{\beta-\alpha}[f(x_{j+1}) - f(x_j)] - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right|. \end{aligned} \tag{8}$$

Note $G_n(f;[\alpha,\beta])$ never overestimates $\left\|f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}\right\|_{[\alpha,\beta]}$ because

$$
\begin{aligned}
\left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_{[\alpha,\beta]} &= \| f' - A_2(f;[\alpha,\beta])' \|_{[\alpha,\beta]} \\
&= \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} |f'(x) - A_2(f;[\alpha,\beta])'(x)| \\
&\ge \sup_{j=1,\ldots,n-1} \frac{n-1}{\beta-\alpha} \int_{x_j}^{x_{j+1}} \left| f'(x) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right| \mathrm{d}x \\
&\ge \sup_{j=1,\ldots,n-1} \frac{n-1}{\beta-\alpha} \left| \int_{x_j}^{x_{j+1}} \left[f'(x) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right] \mathrm{d}x \right| \\
&= \sup_{j=1,\ldots,n-1} \frac{n-1}{\beta-\alpha} \left| f(x_{j+1}) - f(x_j) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right| \\
&= G_n(f;[\alpha,\beta]).
\end{aligned}
$$

Since $(f - A_n(f;[\alpha,\beta]))(x)$ vanishes for $x = \alpha, \beta$. Using (4) it then follows that

$$
\begin{aligned}
&\left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_{[\alpha,\beta]} - G_n(f;[\alpha,\beta]) \\
&\le \left\| f' - A_n(f;[\alpha,\beta])' \right\|_{\infty} \\
&= \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} \left| f'(x) - \frac{n-1}{\beta-\alpha} [f(x_{j+1}) - f(x_j)] \right| \\
&= \frac{n-1}{\beta-\alpha} \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n_i-1}} \left| \int_{x_j}^{x_{j+1}} \frac{\partial v_j}{\partial x}(t,x) f''(t) \, \mathrm{d}t \right| \\
&\le \frac{n-1}{\beta-\alpha} \|f''\|_{[\alpha,\beta]} \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n_i-1}} \int_{x_j}^{x_{j+1}} \left| \frac{\partial v_j}{\partial x}(t,x) \right| \mathrm{d}t \\
&= \frac{n-1}{\beta-\alpha} \|f''\|_{[\alpha,\beta]} \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} \left\{ \frac{(\beta-\alpha)^2}{2(n-1)^2} - (x-x_j)(x_{j+1}-x) \right\} \\
&= \frac{\beta-\alpha}{2(n-1)} \|f''\|_{[\alpha,\beta]} .
\end{aligned}
$$

Thus, we obtain if $n > 1 + \tau_{[\alpha,\beta]}/2$

$$
G_n(f;[\alpha,\beta]) \le \left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_{[\alpha,\beta]} \le \frac{G_n(f;[\alpha,\beta])}{1 - \tau_{[\alpha,\beta]}/(2n-2)} \quad \forall f \in \mathcal{C}_{\boldsymbol{\tau}}.
$$

Therefore, we obtain

$$F_n(f;[\alpha,\beta]) \le \|f''\|_{[\alpha,\beta]} \le \frac{\tau_{[\alpha,\beta]}}{(\beta-\alpha)} \cdot \frac{G_n(f;[\alpha,\beta])}{1-\tau_{[\alpha,\beta]}/(2n-2)}$$
$$\Rightarrow \tau_{\min,n} = \frac{F_n(f;[\alpha,\beta])}{G_n(f;[\alpha,\beta])/(\beta-\alpha)+F_n(f;[\alpha,\beta])/(2n_i-2)} \le \tau_{[\alpha,\beta]}$$

And by (5), we obtain

$$\|f - A_n(f;[\alpha,\beta])\|_{[\alpha,\beta]} \le \frac{\tau_{[\alpha,\beta]}(\beta-\alpha)G_n(f;[\alpha,\beta])}{4(n-1)(2n-2-\tau_{[\alpha,\beta]})} = \varepsilon_{[\alpha,\beta]}.$$

Let $\boldsymbol{n} = \{n_{[\alpha,\beta]}\}_{[\alpha,\beta]\in\mathcal{P}}$ denote the number of points. Define

$$\boldsymbol{A_n}(f;\mathcal{P}) = \{A_{n_{[\alpha,\beta]}}(f;[\alpha,\beta])\}_{[\alpha,\beta]\in\mathcal{P}}$$
$$\boldsymbol{G_n}(f;\mathcal{P}) = \{G_{n_{[\alpha,\beta]}}(f;[\alpha,\beta])\}_{[\alpha,\beta]\in\mathcal{P}}$$
$$\boldsymbol{F_n}(f;\mathcal{P}) = \{F_{n_{[\alpha,\beta]}}(f;[\alpha,\beta])\}_{[\alpha,\beta]\in\mathcal{P}}$$

For simplification, we can denote

$$\tau_{[t_{l-1},t_l]} = \tau_l, \qquad n_{[t_{l-1},t_l]} = n_l, \qquad \varepsilon_{[t_{l-1},t_l]} = \varepsilon_l, \qquad l = 1,\ldots,L$$

where $L = |\mathcal{P}|$ is the cardinality of $\mathcal{P}$.

Algorithm 1 (Locally Adaptive Univariate Function Recovery)**.** Let the sequence of algorithms $\{\boldsymbol{A_n}(\cdot;\mathcal{P})\}$, $\{\boldsymbol{G_n}(\cdot;\mathcal{P})\}$, $\{\boldsymbol{F_n}(\cdot;\mathcal{P})\}$, and $\boldsymbol{n},\mathcal{P}$ be as described above. Set $L=1,\ l=1,\ldots,L$. Choose integer $\tau_{\text{lo}},\tau_{\text{hi}}$, where $\tau_{\text{hi}} \ge \tau_{\text{lo}}$ such that

$$\tau_l = \max\left\{\left\lceil \tau_{\text{hi}}\left(\frac{\tau_{\text{lo}}}{\tau_{\text{hi}}}\right)^{\frac{1}{1+t_i-t_{i-1}}}\right\rceil, 3\right\}.$$

Then let $n_l = \left\lceil \frac{\tau_{[t_{l-1},t_l]}+1}{2}\right\rceil + 1,\ \varepsilon_l = \infty$. For any error tolerance ε and input function f, do the following:

Stage 1. Find the maximum error If $\max\limits_{l=1,\ldots,L} \varepsilon_l < \varepsilon$, stop. Otherwise, find

$$k = \arg\max_{l=1,\ldots,L} \varepsilon_l.$$

Stage 2. Compute $\left\|f' - \frac{f(t_k)-f(t_{k-1})}{t_k-t_{k-1}}\right\|_{[t_{k-1},t_k]}$ **and bound** $\|f''\|_{[t_{k-1},t_k]}$**.** Compute $G_{n_k}(f;[t_{k-1},t_k])$ in (8) and $F_{n_k}(f;[t_{k-1},t_k])$ in (7).

Stage 3. Check the necessary condition for $f \in \mathcal{C}_{\tau_k}$. Compute

$$\tau_{\min,n_k} = \frac{F_{n_k}(f;[t_{k-1},t_k])}{G_{n_k}(f;[t_{k-1},t_k])/(t_k - t_{k-1}) + F_{n_k}(f;[t_{k-1},t_k])/(2n_k-2)}.$$

If $\tau_k \geq \tau_{\min,n_k}$, then go to stage 4. Otherwise, set $\tau_k = 2\tau_k$. If $n_k \geq (\tau_k+1)/2$, then go to stage 4. Otherwise, go to Stage 5.

Stage 4. Check for convergence. Estimate ε_k.

$$\varepsilon_k = \frac{\tau_k(t_k - t_{k-1})G_{n_k}(f)}{4(n_k-1)(2n_k-2-\tau_k)}.$$

If $\varepsilon_{[t_{l-1},t_l]} < \varepsilon$, go to Stage 1. Otherwise go to Stage 5.

Stage 5. Double the initial number of points and split the interval Let $L = L+1$. Then

$$t_l = t_{l-1}, \qquad n_l = n_{l-1}, \qquad \varepsilon_l = \varepsilon_{l-1}, \qquad \tau_l = \tau_{l-1}, \qquad \text{when } l > k,$$

$$t_k = \frac{t_{k-1}+t_k}{2}, \qquad \varepsilon_k = \varepsilon_{k+1} = \infty,$$

$$\tau_k = \tau_{k+1} = \max\left\{ \left\lceil \tau_{\text{hi}} \left(\frac{\tau_{\text{lo}}}{\tau_{\text{hi}}}\right)^{\frac{1}{1+t_k-t_{k-1}}} \right\rceil, 3 \right\}.$$

Go to Stage 1.

Theorem 1. *Let $\{\boldsymbol{A}(\cdot;\mathcal{P})\}$ be the adaptive linear spline defined by Algorithm 1, and let L, $\boldsymbol{\tau}$, and ε be the inputs and parameters described there. Let $\mathcal{C}_{\boldsymbol{\tau}}$ be the cone of functions defined in* (6)*. Then it follows that Algorithm 1 is successful for all functions in $\mathcal{C}_{\boldsymbol{\tau}}$, i.e., $\|f - \boldsymbol{A}(f,\varepsilon;\mathcal{P})\|_\infty \leq \varepsilon$. Moreover, the cost of this algorithm is bounded below and above as follows:*

$$\operatorname{cost}(\boldsymbol{A}, f; \mathcal{P}, \varepsilon) \leq \sqrt{\frac{\|\boldsymbol{\tau}\|_\infty (b-a)^2 \|f''\|_{[a,b]}}{4\varepsilon}} + \|\boldsymbol{\tau}\|_1 + 4L.$$

The algorithm is computationally stable, meaning that maximum costs for all functions, f, with fixed $\|f''\|_\infty$ are an ε-independent constant of each other.

Proof. For each subinterval $[t_{k-1},t_k] \subset \mathcal{P}$, we want

$$\frac{\tau_k(t_k - t_{k-1})G_{n_k}(f;[t_{k-1},t_k])}{4(n_k-1)(2n_k-2-\tau_k)} \leq \varepsilon.$$

And we know that

$$\frac{\tau_k(t_k - t_{k-1})G_{n_k}(f;[t_{k-1},t_k])}{4(n_k-1)(2n_k-2-\tau_k)} \leq \frac{\tau_k(t_k - t_{k-1})^2 \|f''\|_{[t_{k-1},t_k]}}{8(n_k-1)(2n_k-2-\tau_k)}$$

i.e.

$$n_k \le \sqrt{\frac{\tau_k (t_k - t_{k-1})^2 \left\| f'' \right\|_{[t_{k-1}, t_k]}}{4\varepsilon}} + \tau_k + 4.$$

Hence,

$$\begin{aligned}
\mathrm{cost}(\boldsymbol{A}, f; \mathcal{P}, \varepsilon) = \sum_{k=1}^{L} n_k &\le \sum_{k=1}^{L} \left[\sqrt{\frac{\tau_k (t_k - t_{k-1})^2 \left\| f'' \right\|_{[t_{k-1}, t_k]}}{4\varepsilon}} + \tau_k + 4 \right] \\
&\le \sqrt{\frac{\left\| \boldsymbol{\tau} \right\|_\infty \left\| f'' \right\|_{[a,b]}}{4\varepsilon}} \sum_{k=1}^{L} (t_k - t_{k-1}) + \sum_{k=1}^{L} \tau_k + 4L \\
&= \sqrt{\frac{\left\| \boldsymbol{\tau} \right\|_\infty (b-a)^2 \left\| f'' \right\|_{[a,b]}}{4\varepsilon}} + \left\| \boldsymbol{\tau} \right\|_1 + 4L.
\end{aligned}$$

□