1. The Basic Problem

Let $\mathcal{F}$ be a separable Banach space of real-valued input functions with domain $\mathcal{X}$, and let $\mathcal{F}$ have a semi-norm $|\cdot|_{\mathcal{F}}$. Let $\mathcal{H}$ be a separable Banach space of output functions with norm $\|\cdot\|_H$, and let S be a solution operator, $S : \mathcal{F} \to \mathcal{H}$. Let a, b be two fixed real numbers with $a < b$, and $\mathcal{X} = [a, b]$. Here we consider:

$$S : f \mapsto f, \qquad S : \mathcal{W}^{2,\infty}[a, b] \to \mathcal{L}_\infty[a, b].$$

The Sobolev and Lebesgue spaces and their (semi-)norms are defined as follows. For all real numbers α, β with $\alpha < \beta$,

$$\mathcal{W}^{2,\infty} := \mathcal{W}^{2,\infty}[a, b], \qquad \mathcal{W}^{2,\infty}[\alpha, \beta] := \{f \in C[\alpha, \beta] : \|f''\|_{\infty,[\alpha,\beta]} < \infty\},$$
$$\mathcal{L}_\infty := \mathcal{L}_\infty[a, b], \qquad \mathcal{L}_\infty[\alpha, \beta] := \mathcal{W}^{0,\infty}[\alpha, \beta],$$
$$|f|_{\mathcal{W}^{2,\infty}[\alpha,\beta]} := \|f''\|_{\infty,[\alpha,\beta]}, \qquad \|f\|_{\mathcal{L}_\infty[\alpha,\beta]} := \|f\|_{\infty,[\alpha,\beta]},$$
$$\|f\|_{\infty,[\alpha,\beta]} := \max_{\alpha\le x\le\beta} |f(x)| .$$

2. Solving the Problems on Partitions

For any $\mathcal{Y} \subset \mathcal{X}$, let let $f_{\mathcal{Y}}$ denote f restricted to the set $\mathcal{Y}$, i.e.,

$$f_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{R}, \qquad f_{\mathcal{Y}} : \boldsymbol{x} \mapsto f(\boldsymbol{x}).$$

Moreover, let $\mathcal{F}_{\mathcal{Y}}$ denote the space of functions in $\mathcal{F}$ restricted to the set $\mathcal{Y}$, i.e., $\mathcal{F}_{\mathcal{Y}} = \{f_{\mathcal{Y}} : f \in \mathcal{F}\}$.

It is also assumed that their exists, $\mathcal{T}$, some sets of measurable subsets of $\mathcal{X}$ for which one can define norms, solution operators, and approximation operators. It is assumed that

- For each $\mathcal{Y} \in \mathcal{T}$, the subspace $\mathcal{F}_{\mathcal{Y}}$ has a semi-norm, $|\cdot|_{\mathcal{F}_{\mathcal{Y}}}$ satisfying $|f_{\mathcal{Y}}|_{\mathcal{F}_{\mathcal{Y}}} \le |f|_{\mathcal{F}}$. For simplicity of notation, we let $|f|_{\mathcal{F}_{\mathcal{Y}}} = |f_{\mathcal{Y}}|_{\mathcal{F}_{\mathcal{Y}}}$
- For each $\mathcal{Y} \in \mathcal{T}$ there exists a solution operator $S(\cdot; \mathcal{Y}) : \mathcal{F} \to \mathcal{H}$, for which $S(f, \mathcal{Y})$ is actually only a function of $f_{\mathcal{Y}}$.

Partitions of $\mathcal{X}$ are finite subsets of $\mathcal{T}$ such that the following conditions hold:

- There exists a function $\Phi(\cdot; \mathcal{P}) : \mathcal{H}^{|\mathcal{P}|} \to \mathcal{H}$ that combines the solutions defined on the subsets to reconstruct the true solution:

  $$S(f) = \Phi(\boldsymbol{S}_{\mathcal{P}}(f)), \quad \boldsymbol{S}_{\mathcal{P}}(f) := \{S(f; \mathcal{Y})\}_{\mathcal{Y}\in\mathcal{P}}, \qquad \forall f \in \mathcal{F}.$$

  Here $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$.
- There exists a pair of functions $(\widetilde{\Phi}, \mathrm{err})(\cdot, \cdot; \mathcal{P}) : \mathcal{H}^{|\mathcal{P}|} \times [0, \infty)^{|\mathcal{P}|} \to \mathcal{H} \times [0, \infty)$ that combine approximate solutions defined on the subsets with error bounds to reconstruct an approximation to the true solution. If

$\left\|S(f;\mathcal{Y}) - \widetilde{S}_{\mathcal{Y}}\right\|_{\mathcal{H}} \le \varepsilon_{\mathcal{Y}}$ for all $\mathcal{Y} \in \mathcal{P}$, $\widetilde{\boldsymbol{S}}_{\mathcal{P}} := \{\widetilde{S}_{\mathcal{Y}}\}_{\mathcal{Y}\in\mathcal{P}}$, and $\tilde{\boldsymbol{\varepsilon}}_{\mathcal{P}} := \{\tilde{\varepsilon}_{\mathcal{Y}}\}_{\mathcal{Y}\in\mathcal{P}}$, then

$$\left\|S(f) - \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \tilde{\boldsymbol{\varepsilon}}_{\mathcal{P}}; \mathcal{P})\right\|_{\mathcal{H}} \le \operatorname{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \tilde{\boldsymbol{\varepsilon}}_{\mathcal{P}}; \mathcal{P}) \qquad \forall f \in \mathcal{F}.$$

Here $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$. Moreover,

$$\operatorname{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \mathbf{0}; \mathcal{P}) = 0, \qquad \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \mathbf{0}; \mathcal{P}) = \Phi(\boldsymbol{S}_{\mathcal{P}}(f)) = S(f).$$

Note that the subsets of $\mathcal{X}$ comprising the partition $\mathcal{P}$ need not have nonempty intersection.

For function recovery, these partitions take the form of subintervals of $[a, b]$:

$$\mathcal{T} = \{[\alpha, \beta] : a \le \alpha < \beta \le b\}$$
$$\mathcal{P} = \{[t_0, t_1], [t_1, t_2], \dots, [t_{L-1}, t_L]\}, \qquad a = t_0 < t_1 < \cdots < t_L = b.$$

The semi-norms and solution operators defined on the elements of $\mathcal{T}$, and the functions Φ, $\widetilde{\Phi}$, and err that combine the solutions on the sets in the partition into the full solution, the approximate, and the upper error bound are the following:

$$\begin{aligned}
&\left\|f_{[\alpha,\beta]}\right\|_{\mathcal{F}_{[\alpha,\beta]}} = \left\|f''\right\|_{\infty,[\alpha,\beta]}, \qquad S_{[\alpha,\beta]} : f \mapsto f\mathbb{1}_{[\alpha,\beta]},\\
&\Phi(\widetilde{\boldsymbol{S}}_{\mathcal{P}}; \mathcal{P}) = \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{\varepsilon}_{\mathcal{P}}; \mathcal{P}) = \sum_{l=1}^{L} \widetilde{S}_{[t_{l-1},t_l]},\\
&\operatorname{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}}, \boldsymbol{\varepsilon}_{\mathcal{P}}; \mathcal{P}) = \|\boldsymbol{\varepsilon}_{\mathcal{P}}\|_{\infty}
\end{aligned}$$

3. Algorithms

Now we consider numerical algorithms for solving the problems on a subset of the whole domain. Suppose that

- For each $\mathcal{Y} \in \mathcal{T}$ and each $n \in \mathcal{J}$ there exists a non-adaptive approximation operator $A_n(\cdot; \mathcal{Y}) : \mathcal{F} \to \mathcal{H}$ that uses n function values sampled only in $\mathcal{Y}$.
- There exists an error bound function $h : \mathcal{J} \times \mathcal{T} \to [0, \infty)$ such that $h(\cdot, \mathcal{Y})$ is non-increasing, and

$$\|S(f; \mathcal{Y}) - A_n(f; \mathcal{Y})\|_{\mathcal{H}} \le h(n, \mathcal{Y})\, |f|_{\mathcal{F}_{\mathcal{Y}}}, \qquad \forall f \in \mathcal{F}.$$

For approximation, we use piecewise linear interpolation. The number of possible function values $\mathcal{J} = \{j : j \in \mathbb{N}\}$

$$x_j = \alpha + (\beta - \alpha)\frac{j-1}{n-1}, \qquad j = 1, \dots, n, \tag{1}$$

$$A_n(f,[\alpha,\beta]) := \\ \frac{n-1}{\beta-\alpha}\left[f(x_j)(x_{j+1}-x)+f(x_{j+1})(x-x_j)\right], \qquad x_j \le x \le x_{j+1} \tag{2}$$

The difference between f and its linear spline can be bounded in terms of an integral involving the second derivative using integration by parts. For $x \in [\alpha, \beta]$ it follows that

$$\begin{aligned} f(x) - A_n(f,[\alpha,\beta])(x) &= f(x) - \frac{n-1}{\beta-\alpha}\left[f(x_j)(x_{j+1}-x)+f(x_{j+1})(x-x_j)\right] \\ &= \frac{n-1}{\beta-\alpha}\int_{x_j}^{x_{j+1}} v_j(t,x) f''(t)\,\mathrm{d}t, \qquad (3) \\ f'(x) - A_n(f,[\alpha,\beta])'(x) &= \frac{n-1}{\beta-\alpha}\int_{x_j}^{x_{j+1}} \frac{\partial v_j}{\partial x}(t,x) f''(t)\,\mathrm{d}t, \qquad (4) \end{aligned}$$

where the continuous, piecewise differentiable kernel v is defined as

$$v_j(t,x) := \begin{cases} (x_{j+1}-x)(x_j-t), & x_j \le t \le x, \\ (x-x_j)(t-x_{j+1}), & x < t \le x_{j+1}, \end{cases}.$$

To derive the error bounds for $A_n(f,[\alpha,\beta])$ we have:

$$\begin{aligned} \|f - A_n(f,[\alpha,\beta])\|_\infty &\le \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n_i-1}} |f(x) - A_n(f,[\alpha,\beta])(x)| \\ &= \frac{n-1}{\beta-\alpha} \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} \int_{x_j}^{x_{j+1}} |v_j(t,x) f''(t)|\,\mathrm{d}t \\ &\le \frac{n-1}{\beta-\alpha} \|f''\|_\infty \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} \int_{x_j}^{x_{j+1}} |v_j(t,x)|\,\mathrm{d}t \\ &= \|f''\|_\infty \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} \frac{(x-x_j)(x_{j+1}-x)}{2} \\ &= h(n,[\alpha,\beta])\,\|f''\|_\infty, \qquad h(n,[\alpha,\beta]) := \frac{(\beta-\alpha)^2}{8(n-1)^2}. \end{aligned}$$

4. Functions in Cones

The challenge in applying the error bounds for the non-adaptive function recovery algorithm is that $|f|_{\mathcal{F}_\mathcal{Y}}$ is not known a priori. Our approach is to assume that the input functions lie inside cones. For partitions $\mathcal{P}$ suppose that

- For each $\mathcal{Y} \in \mathcal{T}$ there exists a semi-norm $|\cdot|_{\mathcal{G}_\mathcal{Y}}$ defined on the space $\mathcal{F}_\mathcal{Y}$ that is weaker than $|\cdot|_{\mathcal{F}_\mathcal{Y}}$.

- For a fixed function non-increasing $\tau : (0,1) \to (0,\infty)$, define the cone

$$\mathcal{C}_\tau = \{f \in \mathcal{F} : |f|_{\mathcal{F}_\mathcal{Y}} \le \tau(\mathrm{vol}(\mathcal{Y}))\, |f|_{\mathcal{G}_\mathcal{Y}}\}, \tag{5}$$

  where $\mathrm{vol}(\mathcal{Y})$ denotes relative volume of $\mathcal{Y}$, i.e., the Lebesgue measure $\mathcal{Y}$ divided by the Lebesgue measure of $\mathcal{X}$.

- For each $\mathcal{Y} \in \mathcal{T}$ and each $n \in \mathcal{J}$ there exists a non-adaptive approximation operator $G_n(\cdot;\mathcal{Y}) : \mathcal{F} \to \mathcal{H}$ that uses n function values sampled only in $\mathcal{Y}$.

- There exists error bound functions $g_\pm : \mathcal{J} \times \mathcal{T} \to [0,\infty)$ such that $g(\cdot,\mathcal{Y})$ is non-increasing, and

$$-g_-(n,\mathcal{Y})\, |f|_{\mathcal{F}_\mathcal{Y}} \le |f|_{\mathcal{G}_\mathcal{Y}} - G_n(f;\mathcal{Y}) \le g_+(n,\mathcal{Y})\, |f|_{\mathcal{F}_\mathcal{Y}}\,, \qquad \forall f \in \mathcal{F}.$$

  Invoking the definition of the cone implies a two sided bound for $|f|_{\mathcal{G}_\mathcal{Y}}$ and $|f|_{\mathcal{G}_\mathcal{Y}}$ in terms of $G_n(f;\mathcal{Y})$:

$$-\tau(\mathrm{vol}(\mathcal{Y}))g_-(n,\mathcal{Y})\, |f|_{\mathcal{G}_\mathcal{Y}} \le |f|_{\mathcal{G}_\mathcal{Y}} - G_n(f;\mathcal{Y}) \le \tau(\mathrm{vol}(\mathcal{Y}))g_+(n,\mathcal{Y})\, |f|_{\mathcal{G}_\mathcal{Y}}\,, \qquad \forall f \in \mathcal{C}_\tau.$$

$$\begin{aligned}
\frac{G_n(f;\mathcal{Y})}{1+\tau(\mathrm{vol}(\mathcal{Y}))g_-(n,\mathcal{Y})} &\le |f|_{\mathcal{G}_\mathcal{Y}} \le \frac{G_n(f;\mathcal{Y})}{1-\tau(\mathrm{vol}(\mathcal{Y}))g_+(n,\mathcal{Y})}, \quad \forall f \in \mathcal{C}_\tau, \\
\frac{\tau(\mathrm{vol}(\mathcal{Y}))G_n(f;\mathcal{Y})}{1+\tau(\mathrm{vol}(\mathcal{Y}))g_-(n,\mathcal{Y})} &\le |f|_{\mathcal{F}_\mathcal{Y}} \le \frac{\tau(\mathrm{vol}(\mathcal{Y}))G_n(f;\mathcal{Y})}{1-\tau(\mathrm{vol}(\mathcal{Y}))g_+(n,\mathcal{Y})}, \quad \forall f \in \mathcal{C}_\tau.
\end{aligned}$$

Here we define our cone condition

$$|f|_{\mathcal{G}_\mathcal{Y}} := \left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_\infty, \tag{6a}$$

$$\mathcal{C}_\tau := \left\{ f \in \mathcal{W}^{2,\infty} : \|f''\|_\infty \le \frac{\tau_{[\alpha,\beta]}}{\beta-\alpha} \left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_\infty \right\}. \tag{6b}$$

where $\tau_{[a,b]}$ is a parameter depend on the length of interval $[a,b]$, and $\tau_{[a,b]} : [0,\infty) \to [0,\infty)$.

Lower bound on $\|f''\|_{\infty,[\alpha,\beta]}$ can be derived similarly to the previous section using a centered difference. Specifically, for $n_i \ge 3$,

$$F_n(f;[\alpha,\beta]) := \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} |f(x_j) - 2f(x_{j+1}) + f(x_{j+2})|\,. \tag{7}$$

It follows using the Hölder's inequality that

$$\begin{aligned}
F_n(f;[\alpha,\beta]) &= \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} \left| \int_{x_j}^{x_{j+2}} \left[\frac{\beta-\alpha}{n-1} - |x - x_{j+1}| \right] f''(x)\, \mathrm{d}x \right| \\
&\le \frac{(n-1)^2}{(\beta-\alpha)^2} \sup_{j=1,\ldots,n-2} \|f''\|_\infty \int_{x_j}^{x_{j+2}} \left| \frac{\beta-\alpha}{n-1} - |x - x_{j+1}| \right| \mathrm{d}x = \|f''\|_{\infty,[\alpha,\beta]}\,.
\end{aligned}$$

Define

$$
\begin{aligned}
G_n(f;[\alpha,\beta]) &:= \|A_n(f,[\alpha,\beta])' - A_2(f,[\alpha,\beta])'\|_\infty \\
&= \sup_{j=1,\ldots,n-1} \left| \frac{n-1}{\beta-\alpha}[f(x_{j+1}) - f(x_j)] - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right|. \qquad (8)
\end{aligned}
$$

Note $G_n(f;[\alpha,\beta])$ never overestimates $|f|_{\mathcal{G_Y}}$ because

$$
\begin{aligned}
|f|_{\mathcal{G_Y}} = \|f' - A_2(f,[\alpha,\beta])'\|_\infty &= \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} |f'(x) - A_2(f,[\alpha,\beta])'(x)| \\
&\ge \sup_{j=1,\ldots,n-1} \frac{n-1}{\beta-\alpha} \int_{x_j}^{x_{j+1}} \left| f'(x) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right| \mathrm{d}x \\
&\ge \sup_{j=1,\ldots,n-1} \frac{n-1}{\beta-\alpha} \left| \int_{x_j}^{x_{j+1}} \left[f'(x) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right] \mathrm{d}x \right| \\
&= \sup_{j=1,\ldots,n-1} \frac{n-1}{\beta-\alpha} \left| f(x_{j+1}) - f(x_j) - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right| = G_n(f;[\alpha,\beta]).
\end{aligned}
$$

Thus, we have $g_-(n,[\alpha,\beta]) := 0$.

To find an upper bound on $|f|_{\mathcal{G_Y}} - G_n(f;[\alpha,\beta])$, note that

$$
|f|_{\mathcal{G_Y}} - G_n(f;[\alpha,\beta]) = |f|_{\mathcal{G_Y}} - |A_n(f,[\alpha,\beta])|_{\mathcal{G_Y}} \le |f - A_n(f,[\alpha,\beta])|_{\mathcal{G_Y}} = \left\| f' - A_{n_i}(f,[\alpha,\beta])' \right\|_\infty,
$$

since $(f - A_n(f,[\alpha,\beta]))(x)$ vanishes for $x = \alpha, \beta$. Using (4) it then follows that

$$
\begin{aligned}
|f|_{\mathcal{G_Y}} - G_n(f;[\alpha,\beta]) &\le \left\| f' - A_n(f,[\alpha,\beta])' \right\|_\infty \\
&= \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} \left| f'(x) - \frac{n-1}{\beta-\alpha}[f(x_{j+1}) - f(x_j)] \right| \\
&= \frac{n-1}{\beta-\alpha} \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n_i-1}} \left| \int_{x_j}^{x_{j+1}} \frac{\partial v_j}{\partial x}(t,x) f''(t)\,\mathrm{d}t \right| \\
&\le \frac{n-1}{\beta-\alpha} \|f''\|_\infty \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n_i-1}} \int_{x_j}^{x_{j+1}} \left| \frac{\partial v_j}{\partial x}(t,x) \right| \mathrm{d}t \\
&= \frac{n-1}{\beta-\alpha} \|f''\|_\infty \sup_{\substack{x_j \le x \le x_{j+1} \\ j=1,\ldots,n-1}} \left\{ \frac{(\beta-\alpha)^2}{2(n-1)^2} - (x - x_j)(x_{j+1} - x) \right\} \\
&= g_+(n,[\alpha,\beta]) \|f''\|_\infty, \qquad g_+(n,[\alpha,\beta]) := \frac{\beta-\alpha}{2(n-1)}.
\end{aligned}
$$

Thus, we obtain

$$
G_n(f;[\alpha,\beta]) \le \left\| f' - \frac{f(\beta)-f(\alpha)}{\beta-\alpha} \right\|_\infty \le \frac{G_n(f;[\alpha,\beta])}{1 - \tau(\mathrm{vol}([\alpha,\beta]))(\beta-\alpha)/(2n-2)}, \quad \forall f \in \mathcal{C}_\tau,
$$

$$
\tau(\mathrm{vol}([\alpha,\beta]))G_n(f;[\alpha,\beta]) \le \|f''\|_\infty \le \frac{\tau(\mathrm{vol}([\alpha,\beta]))G_n(f;[\alpha,\beta])}{1 - \tau(\mathrm{vol}([\alpha,\beta]))(\beta-\alpha)/(2n-2)}, \quad \forall f \in \mathcal{C}_\tau.
$$

Therefore,

$$
\begin{aligned}
\|f - A_n(f,[\alpha,\beta])\|_\infty &\le \frac{(\beta-\alpha)^2}{8(n-1)^2}\,\|f''\|_\infty\,,\\
&\le \frac{(\beta-\alpha)^2}{8(n-1)^2}\cdot\frac{\tau(\mathrm{vol}([\alpha,\beta]))G_n(f;[\alpha,\beta])}{1-\tau(\mathrm{vol}([\alpha,\beta]))(\beta-\alpha)/(2n-2)}\\
&\le \frac{\tau(\mathrm{vol}([\alpha,\beta]))(\beta-\alpha)^2 G_n(f;[\alpha,\beta])}{4(n-1)(2n-2-\tau(\mathrm{vol}([\alpha,\beta]))(\beta-\alpha))} = \varepsilon_{[\alpha,\beta]}.
\end{aligned}
$$

Then we know, for a partition $\mathcal{P}$, we have

$$
\begin{aligned}
\Phi(\widetilde{\boldsymbol{S}}_{\mathcal{P}};\mathcal{P}) = \widetilde{\Phi}(\widetilde{\boldsymbol{S}}_{\mathcal{P}},\boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) &= \sum_{l=1}^{L} A_{n_l^{(L)}}(\cdot;[t_{l-1},t_l]),\\
\mathrm{err}(\widetilde{\boldsymbol{S}}_{\mathcal{P}},\boldsymbol{\varepsilon}_{\mathcal{P}};\mathcal{P}) = \|\boldsymbol{\varepsilon}_{\mathcal{P}}\|_\infty &= \max_{l=1,\dots,L}\varepsilon_{[t_{l-1},t_l]},
\end{aligned}
$$

where $L = |\mathcal{P}|$.

Algorithm 1 (Locally Adaptive Univariate Function Recovery)**.** Let the sequence of algorithms $\{A_n\}$, $\{G_n\}$, $\{F_n\}$, and n be as described above. Set $L=1,\ l=1,\dots,L$. Choose integer $\tau_{\mathrm{lo}},\tau_{\mathrm{hi}}$, where $\tau_{\mathrm{hi}}\ge\tau_{\mathrm{lo}}$ such that

$$
\tau_{[t_{l-1},t_l]} = \max\left\{\left\lceil \tau_{\mathrm{hi}}\left(\frac{\tau_{\mathrm{lo}}}{\tau_{\mathrm{hi}}}\right)^{\frac{1}{1+t_i-t_{i-1}}}\right\rceil, 3\right\}.
$$

Then let $n_i = \left\lceil \frac{\tau_{[t_{l-1},t_l]}+1}{2}\right\rceil + 1,\ \varepsilon_{[t_{l-1},t_l]} = \infty$. For any error tolerance ε and input function f, do the following:

Stage 1. Find the maximum error If $\max_{l=1,\dots,L}\varepsilon_{[t_{l-1},t_l]} < \varepsilon$, stop. Otherwise, find

$$
k = \arg\max_{l=1,\dots,L}\varepsilon_{[t_{l-1},t_l]}.
$$

Stage 2. Compute $\left\| f' - \frac{f(t_k)-f(t_{k-1})}{t_k-t_{k-1}}\right\|_\infty$ **and bound** $\|f''\|_{\infty,[t_{k-1},t_k]}$**.** Compute $G_{n_k}(f)$ in (8) and $F_{n_k}(f)$ in (7).

Stage 3. Check the necessary condition for $f\in\mathcal{C}_{\tau_{[t_{k-1},t_k]}}$**.** Compute

$$
\tau_{\min,n_k} = \frac{F_{n_k}(f)}{G_{n_k}(f)/(t_k-t_{k-1}) + F_{n_k}(f)/(2n_k-2)}.
$$

If $\tau_{[t_{k-1},t_k]} \ge \tau_{\min,n_k}$, then go to stage 4. Otherwise, set $\tau_{[t_{k-1},t_k]} = 2\tau_{[t_{k-1},t_k]}$. If $n_k \ge (\tau_{[t_{k-1},t_k]}+1)/2$, then go to stage 4. Otherwise, go to Stage 5.

Stage 4. Check for convergence. Estimate $\varepsilon_{[t_{k-1},t_k]}$.

$$\varepsilon_{[t_{l-1},t_l]} = \frac{\tau_{[t_{k-1},t_k]}(t_k - t_{k-1})G_{n_k}(f)}{4(n_k - 1)(2n_k - 2 - \tau_{[t_{k-1},t_k]})}.$$

If $\varepsilon_{[t_{l-1},t_l]} < \varepsilon$, go to Stage 1. Otherwise go to Stage 5.

Stage 5. Double the initial number of points and split the interval Let $L = L + 1$. Then

$$t_l = t_{l-1},\ n_l = n_{l-1}, \varepsilon_{[t_{l-1},t_l]} = \varepsilon_{[t_{l-2},t_{l-1}]} \text{ when } l > k.$$
$$t_k = \frac{t_{k-1} + t_k}{2},\ \varepsilon_{[t_{k-1},t_k]} = \infty,\ \varepsilon_{[t_k,t_{k+1}]} = \infty.$$

Go to Stage 1.