Worksheet 2: Inclusion-Exclusion principle

Answers to Exercise 1:

We divide the set of 100 integers into 50 pairs of consecutive integers, treating each pair as a pigeonhole

$$\{(1, 2), (3, 4), (5, 6), \ldots, (99, 100)\}.$$

Let the 51 chosen integers $\{i_1, i_2, \ldots, i_{50}\}$ represent the pigeons.

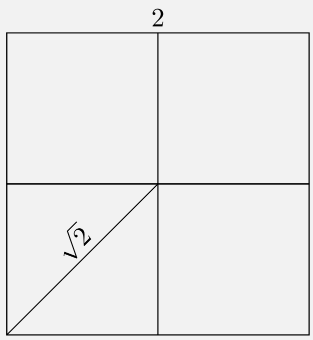
According to the Pigeonhole Principle (PP),

- Pigeons: The 51 chosen numbers from the integers 1 to 100.
- Pigeonholes: The pairs of consecutive integers $(1, 2), (3, 4), \ldots, (99, 100)$.

when we select 51 integers, at least one pigeonhole (corresponding to a pair of integers) must contain 2 pigeons (chosen integers). Consequently, there exist 2 consecutive integers within the set of 51 chosen integers.

Answers to Exercise 2:

We partition the square with a side length of 2 into four smaller squares, each with a side length of 1, achieved by bisecting its sides.



Each of the five points must fall within one of these four small squares.

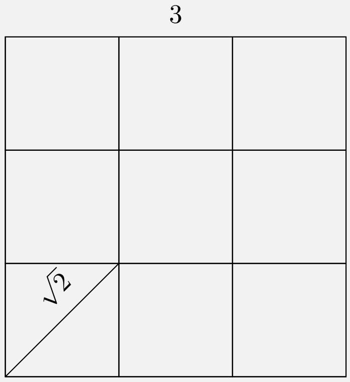
According to the Pigeonhole Principle:

- Pigeons: The 5 points in the square.
- Pigeonholes: 4 small squares of side 1.

there must exist a small square containing at least two of the five points. The diameter of a small square, representing the largest possible distance between two points within it, is the length of its diagonal, which is $\sqrt{2}$. Consequently, the distance between the two points within the same small square is at most $\sqrt{2}$.

Answers to Exercise 3:

We partition the square with a side length of 3 into nine smaller squares, each with a side length of 1, achieved by trisecting its sides.



Each of the ten points must fall within one of these nine small squares.

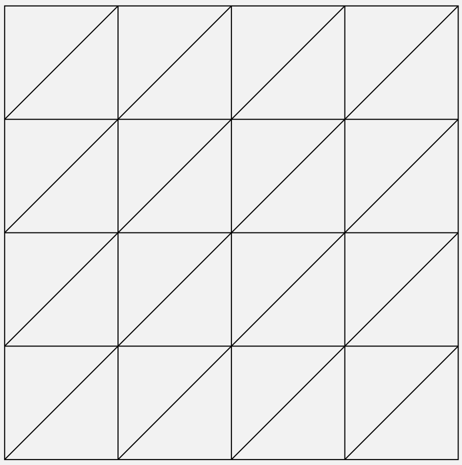
According to the Pigeonhole Principle:

- Pigeons: The 10 points in the square.
- Pigeonholes: 9 small squares of side 1.

There must exist a small square containing at least two of the ten points. The diameter of a small square, representing the largest possible distance between two points within it, is the length of its diagonal, which is $\sqrt{2}$. Consequently, the distance between the two points within the same small square is at most $\sqrt{2}$.

Answers to Exercise 4:

Let us cut the square of side 1 into 32 congruent parts. This yields 32 parts of area $\frac{1}{32}$ each.



According to the generalized version of the Pigeonhole Principle,

- Pigeons: The 65 points inside the square.
- Pigeonholes: The 32 smaller parts obtained by dividing the larger square.

at least one of these 32 parts must contain at least three of our points.

Consequently, the triangle formed by the 3 points has an area less than $\frac{1}{32}$ (area of each small part).

Answers to Exercise 5:

We partition the set of 20 integers into 10 pairs, each with a sum of 21, treating each pair as a pigeonhole:

$$\{(1,20),(2,19),(3,18),\dots,(10,11)\}$$

Let $\{i_1, i_2, \dots, i_{11}\}$ represent the 11 chosen integers as pigeons.

According to the Pigeonhole Principle (PP),

- Pigeons: The 11 chosen numbers from the integers 1 to 20.
- Pigeonholes: The pairs of integers $\{(1,20),(2,19),(3,18),\dots,(10,11)\}$ with a sum of 21.

When we select 11 integers, PP guarantees that at least one pigeonhole (corresponding to a pair of integers) must contain 2 pigeons (chosen integers). Therefore, there exist 2 integers with a sum of 21.