Chapter 2: Advanced Counting Principles

In the first chapter, we covered the foundational concepts and rules of counting. Now, we will explore more advanced counting principles that allow us to tackle more complex problems.

1. Cardinality of Sets

Let S be a finite set, and let A and B be subsets of S.

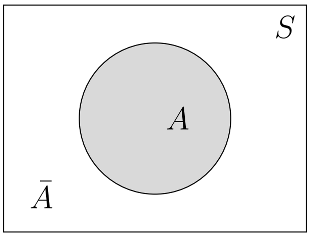
Definition 1.1. The *cardinality* of a set S, denoted $|S|$, is the number of elements in S. Specifically, if $S = \{e_1, e_2, \ldots, e_n\}$, then $|S| = n$.

Definition 1.2 (Complement of A)**.** The complement of a subset A of S, denoted $\bar{A}$, consists of all elements of S that are not in A:

$$\bar{A} = \{x \in S \mid x \notin A\}.$$

Cardinality: The cardinality of the complement $\bar{A}$ is given by

$$|\bar{A}| = |S| - |A|.$$

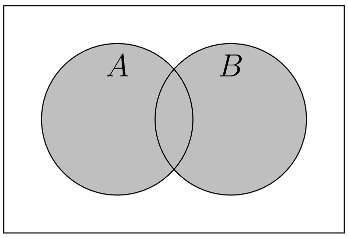


Definition 1.3 (Union of Two Sets)**.** The *union* of two subsets A and B of S, denoted by $A \cup B$, is the set of all elements that are in A, in B, or in both:

$$A \cup B = \{x \in S \mid x \in A \text{ or } x \in B\}.$$

Cardinality: The cardinality of $A \cup B$ is calculated by

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

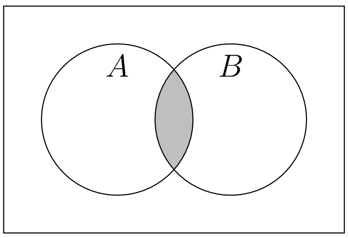


Definition 1.4 (Intersection of Two Sets)**.** The *intersection* of two subsets A and B of S, denoted by $A \cap B$, is the set of all elements that are in both A and B:

$$A \cap B = \{x \in S \mid x \in A \text{ and } x \in B\}.$$

Cardinality: The cardinality of $A \cap B$ is simply

$$|A \cap B|.$$

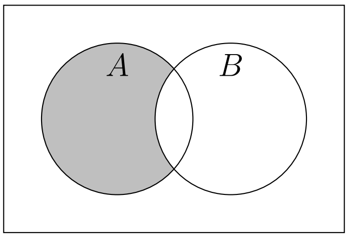


Definition 1.5 (Difference of Two Sets). The *difference* of two sets A and B, denoted $A \setminus B$, is the set of all elements that are in A but not in B:

$$A \setminus B = \{x \in S \mid x \in A \text{ and } x \notin B\}.$$

Cardinality: The cardinality of $A \setminus B$ is

$$|A \setminus B| = |A| - |A \cap B|.$$



1.1 Finite Unions and Intersections

Now, let $A_1, A_2, \ldots, A_n$ be subsets of S.

Definition 1.6 (Finite Unions). The *finite union* of $A_1, A_2, \ldots, A_n$, subsets of S, is defined as:

$$\bigcup_{i=1}^{n} A_i = \{x \in S \mid x \in A_i \text{ for at least one } i\}.$$

Cardinality: The cardinality of $\bigcup_{i=1}^{n} A_i$ is determined by the inclusion-exclusion principle, which leads to a more complex formula.

Definition 1.7 (Finite Unions of Disjoint Subsets). If $A_1, A_2, \ldots, A_n$ are subsets of S such that each pairwise intersection is empty, i.e.,

$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j,$$

then their union is denoted by $\bigoplus_{i=1}^{n} A_i$.

Cardinality: The cardinality of $\bigoplus_{i=1}^{n} A_i$ is simply the sum of the cardinalities of the individual sets:

$$\left| \bigoplus_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|.$$

Definition 1.8 (Finite Intersections). The *finite intersection* of $A_1, A_2, \ldots, A_n$, subsets of S, is defined as:

$$\bigcap_{i=1}^{n} A_i = \{x \in S \mid x \in A_i \text{ for all } i\}.$$

Cardinality: The cardinality of $\bigcap_{i=1}^{n} A_i$ is simply

$$\left| \bigcap_{i=1}^{n} A_i \right|.$$

Definition 1.9 (Power Set). The *power set* of a set S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Proposition 1.10 (Cardinality of the Power Set). If $|S| = n$, the cardinality of the power set $\mathcal{P}(S)$ is given by:

$$|\mathcal{P}(S)| = 2^n.$$

1.2 Floor and Ceiling Functions

Definition 1.11 (Floor and Ceiling Functions)**.** For any real number x, we know that x lies between two integers n and $n+1$, where $n \leq x < n+1$. In this case:

- The integer n is denoted by $\lfloor x \rfloor$, called the *floor function*, which represents the greatest integer less than or equal to x.
- The integer $n+1$ is denoted by $\lceil x \rceil$, called the *ceiling function*, which represents the smallest integer greater than or equal to x.

2. Pigeonhole Principle (Dirichlet box principle)

Theorem 2.1 (Pigeonhole Principle - Version 1)**.** If $n+1$ or more pigeons are placed into n pigeonholes, then at least one pigeonhole must contain more than one pigeon.

Examples.

- In a group of 8 students, at least two of them must have the same day of the week as their birthday.
  To solve this using the pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (8 students in total).
  - **Pigeonholes (boxes):** The days of the week (7 days: Monday through Sunday).

  By the pigeonhole principle (PHP), with 8 pigeons and only 7 pigeonholes, at least one day must be shared by two students. Thus, at least two students have the same birthday day of the week.
- In a group of 13 students, at least two of them must have the same birth month.
  To solve this using the pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (13 students).
  - **Pigeonholes (boxes):** The months of the year (12 months).

  By the pigeonhole principle (PHP), with 13 pigeons and only 12 pigeonholes, at least one month must be shared by two students. Thus, at least two students have the same birth month.

Theorem 2.2 (Pigeonhole Principle - Version 2)**.** If $kn+1$ pigeons are placed into n pigeonholes, where k is a positive integer, then at least one pigeonhole must contain at least $k+1$ pigeons.

Examples.

- In a group of 22 students, at least 3 of them must have the same day of the week as their birthday.
  To solve this using the pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (22 students in total).
  - **Pigeonholes (boxes):** The days of the week (7 days: Monday through Sunday).

  By the pigeonhole principle (PHP), with 15 pigeons and only 7 pigeonholes, at least one day must be shared by at least two students. Thus, at least two students have the same birthday day of the week.
- In a group of 37 people, at least 4 must have been born in the same month.
  To solve this using the pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:

- **Pigeons:** Each person in the group (37 people in total).
- **Pigeonholes (boxes):** The months of the year (12 months).

By the pigeonhole principle (PHP), with 37 pigeons and only 12 pigeonholes, at least one month must be shared by at least 4 people. Thus, at least 4 people must have been born in the same month.

Theorem 2.3 (Generalized Pigeonhole Principle - Version 3)**.** If m pigeons are placed into n pigeonholes, then at least one pigeonhole contains at least $\lceil \frac{m}{n} \rceil$ pigeons.

Example.
Suppose there are 316 students in the first year of NHSM, and we want to distribute them into 12 groups, at least one group must contain at least 27 students.
To solve this using the generalized pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:

- **Pigeons:** The 316 students in total.
- **Pigeonholes (boxes):** The 12 groups.

By the generalized pigeonhole principle (PHP), with 316 students and only 12 groups, at least one group must contain at least $\lceil \frac{316}{12} \rceil = 27$ students. Thus, at least one group must have at least 27 students.

Exercise. Suppose we have gloves of 4 colors: black, red, green, and blue. How many gloves should we grab to ensure we get a pair of the same color?

Solution. To guarantee a pair, we define the pigeons as the gloves and the pigeonholes as the 4 colors. By the pigeonhole principle, grabbing 5 gloves ensures at least two gloves of the same color, since there are only 4 colors.
Thus, we need to grab at least 5 gloves.

Exercise. Suppose we have gloves of 4 colors: black, red, green, and blue. How many gloves should an octopus (with 8 hands) grab to ensure it gets a pair of the same color?

Solution. To guarantee a pair, we define the pigeons as the gloves and the pigeonholes as the 4 colors. By the pigeonhole principle, even if the octopus grabs $7 \times 4 + 1$ gloves, it ensures that at least 8 gloves of the same color are taken, since there are only 4 colors.
Thus, the octopus needs to grab at least 29 gloves.

Exercise. Show that in a set $S = \{a_1, a_2, \dots, a_{n+1}\}$ of $n + 1$ integers, there are at least two integers whose difference is divisible by n.

3. Inclusion-Exclusion Principle

Let S be a finite set, and $A_1, A_2 \subseteq S$. To compute $|A_1 \cup A_2|$, we sum the cardinalities of the individual sets:

$$|A_1| + |A_2|.$$

However, elements in $A_1 \cap A_2$ are counted twice, so we subtract the intersection:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

For the complement of the union, the size of the set $\overline{A_1} \cap \overline{A_2}$ (the elements not in $A_1 \cup A_2$) is:

$$|\overline{A_1} \cap \overline{A_2}| = |S \setminus (A_1 \cup A_2)| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

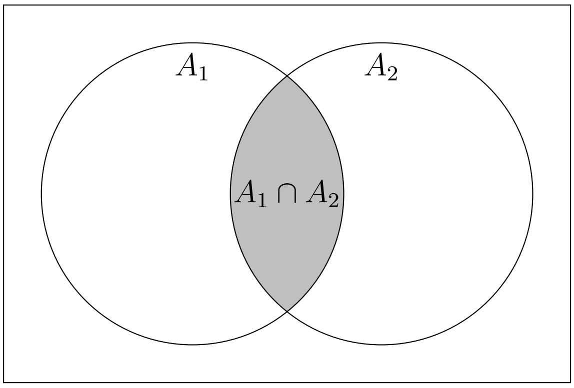


Figure 2: Venn diagram of two sets

Let S now be a finite set, and $A_1, A_2, A_3 \subseteq S$. To compute $|A_1 \cup A_2 \cup A_3|$, we begin by summing the cardinalities of the individual sets:

$$|A_1| + |A_2| + |A_3|.$$

However, elements in the pairwise intersections $A_1 \cap A_2$, $A_1 \cap A_3$, and $A_2 \cap A_3$ are counted twice, so we subtract the sizes of these intersections:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|.$$

But now, the elements in the triple intersection $A_1 \cap A_2 \cap A_3$ have been subtracted three times, so we add back the size of this triple intersection:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

For the complement of the union, the size of the set $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ (the elements not in $A_1 \cup A_2 \cup A_3$) is given by:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S \setminus (A_1 \cup A_2 \cup A_3)| = |S| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|.$$

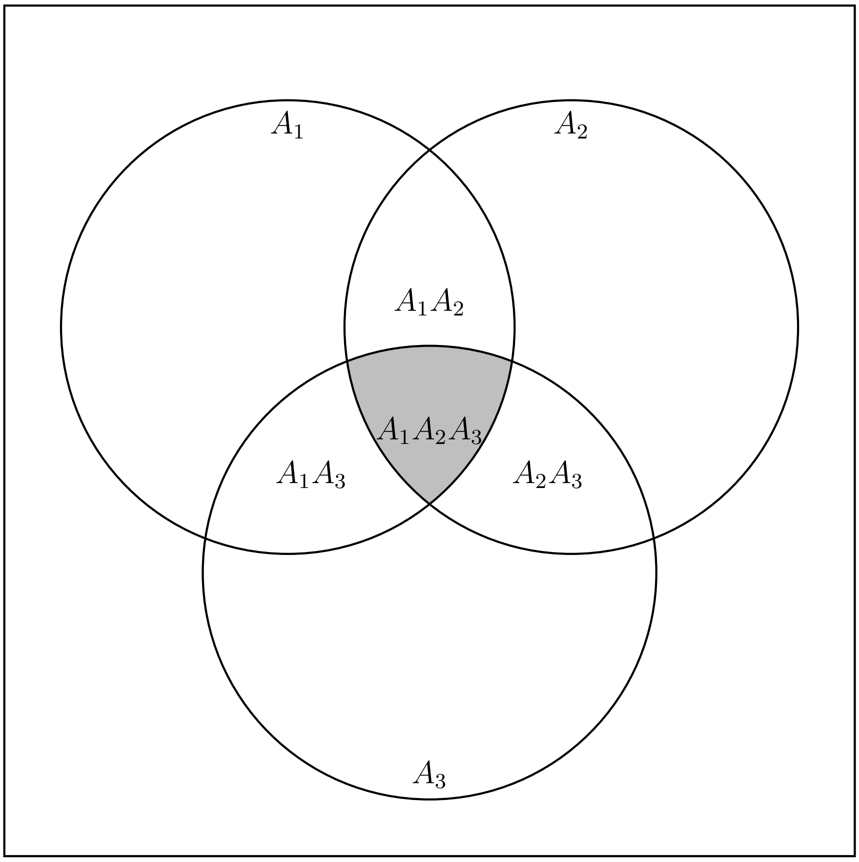


Figure 3: Venn diagram of three sets