Chapter 3: Introduction to Graph Theory

The Königsberg Bridge Problem

The city of Königsberg (now Kaliningrad, Russia) is divided by a river, creating two large islands and two mainland portions, all connected by seven bridges as shown in the next figure.

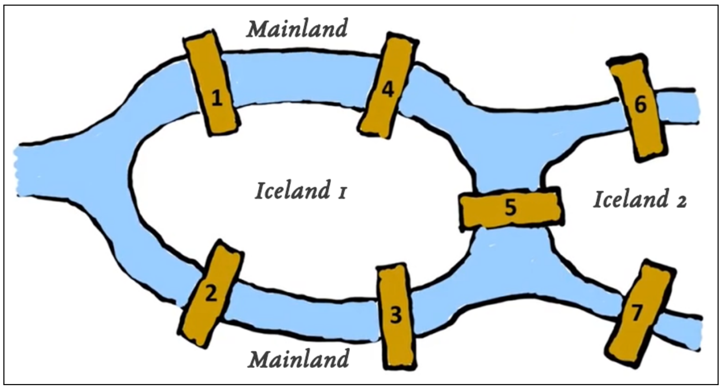


Figure 4: Seven Bridges of Königsberg

The problem asks if it's possible to walk through the city and cross each bridge exactly once, starting and ending at the same point. We can represent the problem using the following structure,

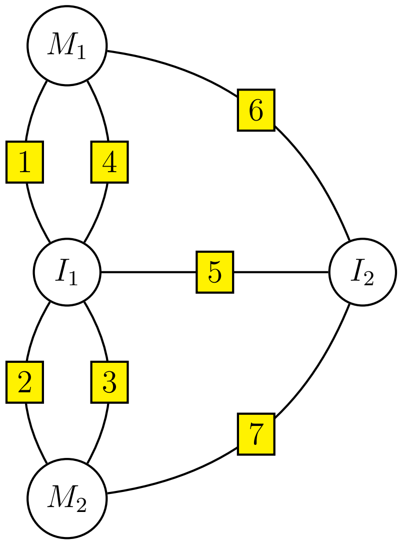


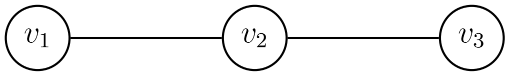
Figure 5: Graph representation of the Königsberg bridges

where the land areas are represented as points and the bridges as lines. This structure is known as **a graph**.

1. What's a graph?

Definition 1.1.
A graph $G = (V, E)$ is defined by two finite sets: V and E, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices (with each v_i representing a vertex), and $E = \{e_1, e_2, \ldots, e_m\}$ is the set of edges, which are the connections between vertices. The vertices are typically represented as points, and the edges as lines connecting the corresponding vertices.

Example.
Consider a graph with vertices $V = \{v_1, v_2, v_3\}$ and edges $E = \{(v_1, v_2), (v_2, v_3)\}$. The corresponding graph is:



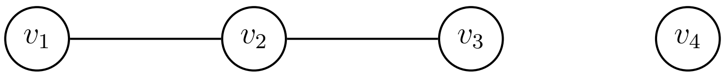
Definition 1.2 (Adjacent, Incident, and Isolated).

- Two vertices v_i and v_j in a graph are said to be ***adjacent*** if there exists an edge that connects v_i and v_j.
- Similarly, two edges e_i and e_j are said to be ***adjacent*** if they share a common vertex.
- An edge is said to be ***incident*** to a vertex if the vertex is one of the endpoints of the edge.
- A vertex v_i is said to be ***isolated*** if it is not incident to any edge in the graph.

Example.
Consider the following graph $G = (V, E)$, where the set of vertices is $V = \{v_1, v_2, v_3, v_4\}$ and the set of edges is $E = \{(v_1, v_2), (v_2, v_3)\}$.

- The vertices v_1 and v_2 are ***adjacent*** because there is an edge (v_1, v_2) connecting them.
- The edges (v_1, v_2) and (v_2, v_3) are ***adjacent*** because they share the common vertex v_2.
- The edge (v_1, v_2) is ***incident*** to the vertices v_1 and v_2.
- The vertex v_4 is ***isolated*** because it is not incident to any edge.

The graph is illustrated below:



Definition 1.3 (Order and Size of a Graph).
The **order** of a graph $G = (V, E)$, denoted $\mathrm{ord}(G)$, is the number of vertices in the graph. That is,
$$\mathrm{ord}(G) = |V|.$$
The **size** of a graph $G = (V, E)$, denoted $e(G)$, is the number of edges in the graph. That is,
$$e(G) = |E|.$$

Example.
In the previous example, $ord(G) = 4$ and $e(G) = 2$.

Definition 1.4 (Degree of a Vertex).
Let $G = (V, E)$ be a graph and $v \in V$ be a vertex.
The ***degree*** of a vertex v, denoted $\deg(v)$, is the number of edges incident to v. Formally, the degree of v is given by
$$\deg(v) = |\{e \in E : v \text{ is an endpoint of } e\}|.$$

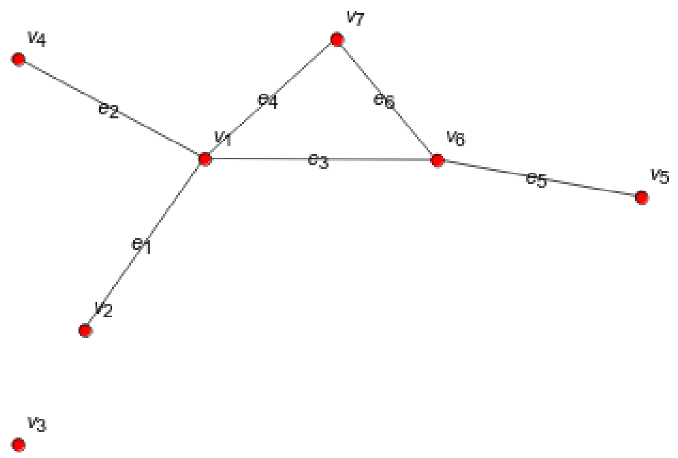
- The ***maximum degree*** of the graph, denoted $\Delta(G)$, is the highest degree of any vertex in the graph:
$$\Delta(G) = \max_{v \in V} \deg(v).$$
- The ***minimum degree*** of the graph, denoted $\delta(G)$, is the lowest degree of any vertex in the graph:
$$\delta(G) = \min_{v \in V} \deg(v).$$

Example.
Consider the graph $G = (V, E)$, where
$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

and

$$E = \{e_1 = (v_1, v_2), e_2 = (v_1, v_4), e_3 = (v_1, v_6), e_4 = (v_1, v_7), e_5 = (v_5, v_6), e_6 = (v_6, v_7)\}.$$

The corresponding graph is illustrated below:



The degrees of the vertices are:

- $\deg(v_1) = 4$
- $\deg(v_2) = 1$
- $\deg(v_3) = 0$
- $\deg(v_7) = 2$

Thus, the maximum degree is $\Delta(G) = 4$ and the minimum degree is $\delta(G) = 0$.

Definition 1.5 (Loop).
A ***loop*** in a graph is an edge that connects a vertex to itself.

Definition 1.6 (Simple Graph).
A graph is called a ***simple graph*** if it does not contain any loops or multiple edges between the same pair of vertices.

Definition 1.7 (Empty Graph).
An ***empty graph*** is a graph with no edges. It can have any number of vertices, but there are no edges connecting them. In other words, $E(G) = \emptyset$.

Definition 1.8 (Complete Graph).
A ***complete graph*** is a simple graph in which every pair of distinct vertices is connected by a unique edge. A complete graph on n vertices is denoted by K_n.

2. Directed and Undirected Graphs

Definition 2.1 (Undirected Graph).
An ***undirected graph*** is a graph in which the edges have no direction. That is, if there is an edge between vertices v_i and v_j, it can be traversed in both directions, and we denote it by (v_i, v_j) or (v_j, v_i).

Definition 2.2 (Directed Graph).
A ***directed graph*** (or ***digraph***) is a graph in which the edges have a direction. Each edge is represented as an ordered pair of vertices, indicating a directed edge from one vertex to another. If there is an edge from v_i to v_j, we write it as (v_i, v_j).

3. Paths, Chains, Cycles, Circuits, Trees, Connectivity, Planarity