Discrete Mathematics 1

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2024/2025

Contents

Chapter 1: Counting

1. Introduction

1.1 What is Discrete Mathematics?

Discrete Mathematics is a field of mathematics that deals with distinct and countable objects, rather than quantities that change continuously. It focuses on the study of structures that are fundamentally discrete, such as numbers, logical statements, and algebraic structures. The topics covered in this field form the backbone of areas like computer science, cryptography, and various branches of pure mathematics.

1.1.1 Key Concepts in Discrete Mathematics:

- **Sets:** Collections of distinct objects or elements.
- **Logic:** Principles of reasoning and constructing valid arguments.
- **Algebraic Structures:** The study of mathematical structures such as groups, rings, and fields, which are essential in various areas of mathematics and computer science.
- **Number Theory:** The study of properties and relationships of integers.
- **Combinatorics:** The art of counting, organizing, and selecting objects. In our course, we will focus on **enumerative combinatorics**, which is concerned with counting discrete structures and analyzing how they can be organized or constructed. Often in combinatorics, we are faced with answering key questions such as:
  - Is the configuration possible?
  - In how many ways can the configuration be made?
  - How do we go about finding such a configuration?

Examples.
Here are some example questions in combinatorics.

- How many distinct ways can the letters in the word "NHSM" be arranged?
- A class has 12 students, and 3 different prizes (gold, silver, bronze) are to be awarded. In how many ways can the prizes be distributed?
- How many ways can you arrange 6 balls into 4 boxes?
- How many ways can you assign 5 distinct colors to 5 identical chairs in a row?
- How many passwords of length n are possible using 26 letters?
- A company has 7 tasks to assign to 3 employees. In how many ways can the tasks be assigned if each employee must get at least one task?

Remark.
It is very important when doing the enumeration (counting the number of possible ways) to understand the distinction between listing all the configurations and determining their number.

1.2 Why Study Combinatorics?

Combinatorics is a fundamental area of Discrete Mathematics that provides tools to solve problems involving counting, organization, and construction of discrete structures. It plays a crucial role in many applications, especially in areas requiring detailed analysis of finite systems.

1.2.1 Applications of Combinatorics:

- **Computer Science:** Combinatorics is essential for designing efficient algorithms, data structures, and cryptographic systems, as well as in programming and network theory.
- **Engineering:** Combinatorics helps optimize system designs, network configurations, and resource allocations.

- **Mathematics:** It supports advancements in areas like number theory, algebraic structures, and discrete optimization.
- **Statistics:** Combinatorial methods lay the foundation for probability theory and statistical models.
- **Physics:** Applied in fields such as quantum mechanics and statistical physics, where discrete models and structures are essential.

Studying combinatorics equips you with powerful techniques for logical reasoning, problem-solving, and analyzing complex systems. These skills are highly valuable in technology, mathematics, engineering, and beyond.

Exercise.
How many different 2-letter words, including nonsense words can be produced by arranging the letters H,I,M?

Solution.
The set of all possible 2-letter words, including nonsense words is HI,HM,IM,IH,MH,MI,II,HH,MM, there is 9 such words.

Let change the question a little

Exercise.
How many different 26-letter words, including nonsense words can be produced by arranging the letters of the alphabet?

Solution.
There are 26^{26} such words, even using a computer that prints 1 billion words per second it will take 1.27×10^{17} centuries to print all the possible words.

2. Fundamental Principles of Counting: Product or Sum?

In this section, we introduce two fundamental counting principles: the *addition principle* and the *multiplication principle*, both of which are essential for solving combinatorial problems.

First, let's define the *Cardinality of a Set* which is the number of elements in a finite set S. The cardinality of S is denoted by $|S|$. For example, if $S = \{a, b, c, d\}$, then:

$$|S| = 4$$

Similarly, if $T = \{1, 2, 3, 4, 5, 6, 7\}$, then:

$$|T| = 7$$

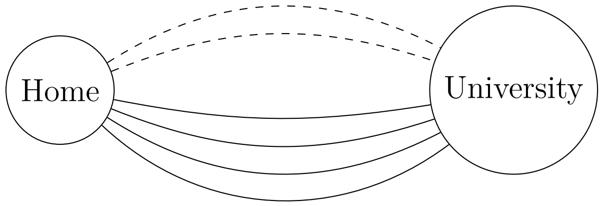
2.1 Addition principle

Theorem 2.1 (Addition Principle).
If A and B are two disjoint sets ($A \cap B = \emptyset$), then:

$$|A \cup B| = |A| + |B|$$

Examples.

1. Suppose you have 2 different train routes and 4 different bus routes from your home to the university. Let A be the set of train routes and B the set of bus routes. According to the Addition Principle, you have a total of $|A| + |B| = 6$ possible routes to reach the university.



2. In a class, we have 8 boys and 5 girls. We want to choose a representative of the group. By the Addition Principle, we have $|Boys| + |Girls| = 13$ possible choices .

Corollary 2.2 (Generalised Addition Principle).
If $A_1, A_2, \ldots, A_n$ are n pairwise disjoint sets ($A_i \cap A_j = \emptyset$ for all $i \neq j$), then:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

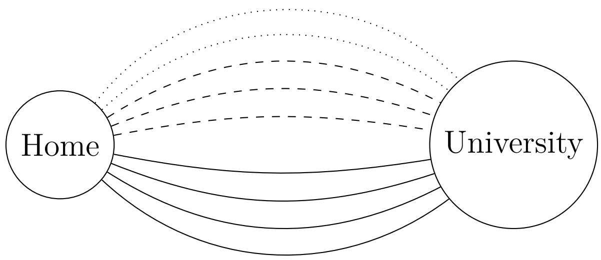
Example.
Suppose you have three different modes of transportation from your home to the university:

- 3 different train routes (A_1),
- 4 different bus routes (A_2),
- 2 different tram routes (A_3).

Since these sets of routes are pairwise disjoint (you can't take more than one mode of transportation at the same time), So by generalised adddition principle the total number of routes available is the sum of the routes for each mode of transportation:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| = 3 + 4 + 2 = 9$$

Thus, you have 9 total route options to get to the university.



Theorem 2.3 (Addition Principle: Alternate Form).
If a problem consists of solving one of k tasks, and each task i (where $1 \leq i \leq k$) can be done in n_i ways, then the total number of solutions to the problem is:

$$\sum_{i=1}^{k} n_i = n_1 + n_2 + \cdots + n_k.$$

2.2 Multiplication principle

Let us consider the set $A \times B$, which contains all ordered pairs (a, b) where $a \in A$ and $b \in B$. We call $A \times B$ the *Cartesian product* of A and B.

Theorem 2.4 (Multiplication Principle).
If A and B are two sets, then:

$$|A \times B| = |A| \cdot |B|$$

Examples. 1. Suppose you need to travel from your home to the university. You can:

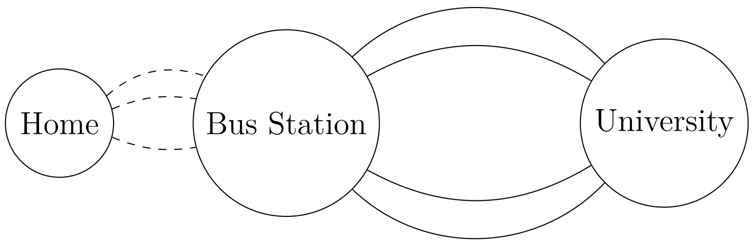
(a) Choose one of 3 train routes from home to the bus station.

(b) Choose one of 4 bus routes from the bus station to the university.

By the Multiplication Principle, the total number of ways to complete the journey is:

$$\text{Total routes} = 3\,(\text{train routes}) \times 4\,(\text{bus routes}) = 12\,\text{routes}.$$

Thus, there are 12 possible ways to travel from home to the university.



2. Suppose a class has 8 boys and 5 girls, and we need to form a committee consisting of one boy and one girl. The number of ways to choose a boy and a girl for the committee is determined by the Multiplication Principle:

(a) First, you choose one of the 8 boys.

(b) Then, you choose one of the 5 girls.

By the Multiplication Principle, the total number of ways to form the committee is:

$$\text{Total ways} = 8\,(\text{boys}) \times 5\,(\text{girls}) = 40\,\text{ways}.$$

Thus, there are 40 possible ways to form the committee.

Corollary 2.5 (Generalized Multiplication Principle).
If $A_1, A_2, \ldots, A_n$ are n sets, then:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|$$

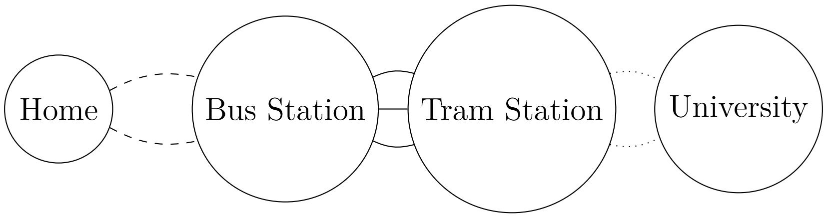
Example.
Suppose you want to travel from home to the university. The journey consists of three stages:

1. Choose 2 different train routes to go from home to the bus station,
2. Choose 3 different bus routes to go from the bus station to the tram station,
3. Choose 2 different tram routes to reach the university.

By the **Multiplication Principle**, the total number of ways to complete the journey is:

$$|A_1 \times A_2 \times A_3| = 2\,(\text{train routes}) \times 3\,(\text{bus routes}) \times 2\,(\text{tram routes}) = 12\,\text{ways}.$$

Thus, there are 12 different ways to travel from home to the university.



Theorem 2.6 (Multiplication Principle: Alternate Form).
If a problem consists of solving k tasks sequentially, and each task i (where $1 \leq i \leq k$) can be performed in n_i ways, then the total number of solutions to the problem is:

$$\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k.$$

3. Distinguishability, Ordering, and Repetition

In combinatorial problems, we often need to select a subset A from the elements of a finite set S, based on certain criteria. To solve these problems, we must consider the following three key properties:

Definition 3.1 (Distinguishability).
Two elements of the set S are considered indistinguishable if they are identical with respect to the property in question. Otherwise, they are distinguishable if they differ with respect to this property.

Example.
In the following example, depending on the property we consider, the elements are distinguishable or indistinguishable:

- **By color**: We have 3 distinguishable groups, each containing 4 indistinguishable squares.
- **By number**: We have 4 distinguishable groups, each containing 3 indistinguishable squares.
- **By both color and number**: We have 12 pairwise distinguishable squares.

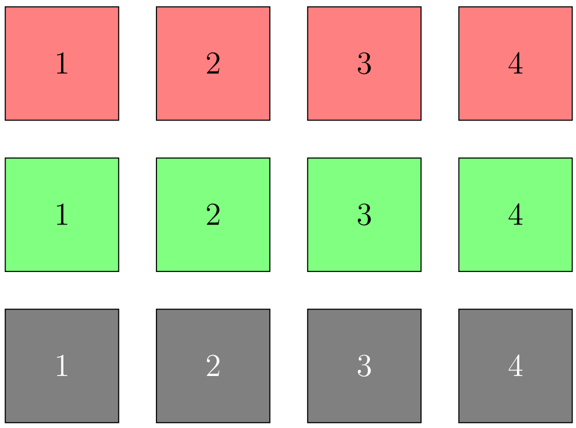


Figure 1: Distinguishability based on different properties

Definition 3.2 (Ordering).
If the elements in the selected subset A are treated equally and play the same role, the selection is unordered. If the elements can occupy different roles (i.e., their positions matter), the selection is ordered (a list).

Example.
Consider the 100-meter final at the Olympics with 8 participants:

- If we want to randomly select 3 runners, the selection is unordered.
- If we want to select 3 runners for gold, silver, and bronze medals, the selection is ordered since the positions matter.

Definition 3.3 (Repetition).
If an element from the set S can appear more than once in the selection, repetition is allowed. If each element can only appear once, repetition is not allowed.

Example.
Suppose we need to select a leader from a group of students each day. Repetition is possible because a student can be chosen more than once as the leader. Does the order matter in this case?

Now, let's consider the possible cases of selecting a part of k elements under the conditions of ordering and repetition.

3.1 Arrangements (k-Permutations)

Definition 3.4.
Let S be a set of n elements $\{e_1, e_2, \ldots, e_n\}$. An **arrangement** (or k-**permutation**) of k elements from S refers to any ordered sequence of k elements $(a_1, a_2, \ldots, a_k)$ selected from S. In other words, an arrangement is a selection **with order** of k elements from a set S of n elements.

Examples.

- Let $S := \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The numbers $123, 112, 344, 789, 258$ are arrangements of 3 elements from S.
- Let $S := \{A, B, C, D, E, F\}$. The words ABC, DEF, AAB, CDC, CDA are arrangements of 3 elements from S.

Remark.
An arrangement can simply be described as a listing of elements.

3.1.1 k-permutations without repetition

Definition 3.5.
A k-**permutation without repetition** is an arrangement of k distinct elements selected from a set S, where no element can appear more than once. In other words, it is an ordered sequence of k different elements chosen from S.

Example.
From the previous examples of arrangements, $123, 789$ and 258 are 3-permutations with repetition from the first set, while ABC, DEF and CDA are 3-permutations with repetition from the second set.

Theorem 3.6 (Number of k-permutations without repetition)**.**
The number of all k-permutations without repetition from a set S of n elements, denoted by $(n)_k$, is given by:

$$(n)_k = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$$

Exercise.

- Let $S = \{A, B, C\}$. How many 2-permutations without repetition can be formed from the set S? List all the possible 2-permutations.
- Let $S = \{A, B, C, D, E, F\}$. How many 4-permutations without repetition can be formed from the set S?

Solution.

- The number of 2-permutations from $S = \{A, B, C\}$ is:
$$(3)_2 = 3 \times 2 = 6$$
The possible 2-permutations are: AB, AC, BA, BC, CA, CB.
- The number of 4-permutations from $S = \{A, B, C, D, E, F\}$ is:
$$(6)_4 = 6 \times 5 \times 4 \times 3 = 360$$
Thus, there are 360 possible 4-permutations.

3.1.2 k-permutations with repetition

Definition 3.7.
A k-**permutation with repetition** is an arrangement of k elements selected from a set S, where the repetition of elements is allowed. In other words, it is an ordered sequence of k elements, where each element is chosen from S and can appear more than once.

Example.
From the previous examples, 112 and 344 are 3-permutations with repetition from the first set, while AAB and CDC are 3-permutations with repetition from the second set.

Theorem 3.8 (Number of k-permutations with repetition)**.**
The number of all k-permutations with repetition from a set S of n elements, denoted by $\mathcal{P}(n, k)$, is given by:

$$\mathcal{P}(n, k) = n^k$$

Remark.
The repetition of elements here is allowed and unlimited.

Exercise.
Let $S = \{1, 2, 3\}$. How many 2-permutations with repetition can be formed from the set S? List all the possible 2-permutations.

Solution.
The number of 2-permutations with repetition from the set of $n = 3$ elements is given by:

$$\mathcal{P}(n, k) = n^k$$

Here, $n = 3$ and $k = 2$, so the total number of 2-permutations is:

$$\mathcal{P}(3, 2) = 3^2 = 9$$

Now, let's list all the possible 2-permutations with repetition:

$$11, 12, 13, 21, 22, 23, 31, 32, 33$$

Thus, there are 9 possible 2-permutations with repetition.

3.1.3 Permutations without Repetition

Before defining a permutation, let $n!$ represent the factorial of a non-negative integer n, which is defined as:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1,$$

with the special case $0! = 1$.

Definition 3.9.
A permutation of a set $S = \{e_1, e_2, \dots, e_n\}$ is an arrangement of all the elements of S without repetition (i.e., an n-permutation without repetition).

Example.
Let $S = \{A, B, C\}$. The set S has 3 elements, and the possible permutations (arrangements without repetition) are:

$$ABC, ACB, BAC, BCA, CAB, CBA$$

Thus, there are 6 permutations of S.

Theorem 3.10 (Number of Permutations)**.**
The number of permutations of a set S with n elements, denoted by P_n, is given by:

$$P_n = n!$$

Exercise.
How many nonsense words of length 26 can be formed using all the letters of the English alphabet without repetition?

Solution.
Since the word must be 26 letters long and use all the letters of the English alphabet without repetition, this is a permutation of all 26 letters. The number of such permutations is denoted by

P_{26}, where:

$$P_{26} = 26!$$

Therefore, the total number of possible nonsense words is:

$$P_{26} = 26! = 26 \times 25 \times 24 \times \cdots \times 2 \times 1.$$

Proposition 3.11.

- For all $n \geq 0$,
$$P_n = (n)_n = n!$$
- More generally, for $k \leq n$, the number of k-permutations of a set with n elements is given by:
$$(n)_k = \frac{n!}{(n-k)!}$$
- Additionally, the relationship between permutations can be expressed as:
$$P_n = (n)_k P_{n-k}$$
where $(n)_k = n \times (n-1) \times \cdots \times (n-k+1)$ represents the number of ways to arrange k elements from n elements.

Exercise.

Prove the proposition.

3.2 Combinations, Binomial Theorem and Pascal's Triangle

Definition 3.12.

A combination is a selection of a subset from a finite set, where the order of selection does not matter. Simply put, a combination of a set is a subset of that set.

Examples.

- $\{1, 2, 4\}$ and $\{5, 4, 1\}$ are both valid combinations of size 3 from the set $\{1, 2, 3, 4, 5\}$.
- $\{A, B, C\}$ and $\{B, A, C\}$ represent the same combination from the set $\{A, B, C, D\}$, because the order does not matter in a combination.

3.2.1 k-combination without repetition

Definition 3.13.

A k-combination without repetition from a set S of n elements is a selection of k elements from S, where order does not matter. Simply put, it is a subset of k elements from S.

Examples.

- The 2-combinations of the set $\{A, B, C, D\}$ are as follows:
$$\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}$$
Thus, there are 6 possible 2-combinations.
- The combinations of the set $S = \{1, 2, 3\}$ are as follows:
  - The 0-combination: $\{\}$ (the empty set)
  - The 1-combinations: $\{1\}, \{2\}, \{3\}$
  - The 2-combinations: $\{1, 2\}, \{1, 3\}, \{2, 3\}$
  - The 3-combination: $\{1, 2, 3\}$

  Thus, there are 8 possible combinations in total, which are all the subsets of S, including the empty set.

Theorem 3.14 (Number of k-combinations without repetition).
The number of all k-combinations without repetition from a set S of n elements, denoted by $\binom{n}{k}$, is given by:

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$

The quantities $\binom{n}{k}$ are also called binomial coefficients.

Exercise.

- How many ways can we select a team of 3 people from a group of 3 persons $S = \{P1, P2, P3, P4, P5\}$? List all the possible teams.
- From a group of 10 people (6 boys and 4 girls), how many ways can we select a team containing:
  - 4 people?
  - 2 boys and 2 girls?

Solution.
- To select a team of three people from a group of five persons $S = \{P1, P2, P3, P4, P5\}$, we calculate the number of 3-combinations without repetition from 5 elements:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3!}{3! \times 2!} = \frac{5 \times 4}{2!} = 10.$$

  The possible teams are:

$$\{P1, P2, P3\}, \{P1, P2, P4\}, \{P1, P2, P5\}, \{P1, P3, P4\}, \{P1, P3, P5\},$$
$$\{P1, P4, P5\}, \{P2, P3, P4\}, \{P2, P3, P5\}, \{P2, P4, P5\}, \{P3, P4, P5\}.$$

- From a group of 10 people (6 boys and 4 girls):
  - To select 4 people, we calculate the number of 4-combinations from 10:

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$$

    Therefore, there are 210 possible teams of 4 people.
  - To select 2 boys from 6 and 2 girls from 4, we calculate:

$$\binom{6}{2} \times \binom{4}{2} = \frac{6 \times 5}{2!} \times \frac{4 \times 3}{2!} = 15 \times 6 = 90.$$

    Therefore, there are 90 possible teams with 2 boys and 2 girls.

3.2.2 Binomial coefficients triangle (Pascal's triangle)

If we arrange the binomial coefficients $\binom{n}{k}$ in a triangle, where n represents the row number and k represents the position in the row (column number), we obtain what is called the Binomial Coefficient Triangle (or Pascal's Triangle). The following figure illustrates the triangle:

n/k	0	1	2	3	4	5	6	7	8	9
0	$\binom{0}{0}=1$									
1	$\binom{1}{0}=1$	$\binom{1}{1}=1$								
2	$\binom{2}{0}=1$	$\binom{2}{1}=2$	$\binom{2}{2}=1$							
3	$\binom{3}{0}=1$	$\binom{3}{1}=3$	$\binom{3}{2}=3$	$\binom{3}{3}=1$						
4	$\binom{4}{0}=1$	$\binom{4}{1}=4$	$\binom{4}{2}=6$	$\binom{4}{3}=4$	$\binom{4}{4}=1$					
5	$\binom{5}{0}=1$	$\binom{5}{1}=5$	$\binom{5}{2}=10$	$\binom{5}{3}=10$	$\binom{5}{4}=5$	$\binom{5}{5}=1$				
6	$\binom{6}{0}=1$	$\binom{6}{1}=6$	$\binom{6}{2}=15$	$\binom{6}{3}=20$	$\binom{6}{4}=15$	$\binom{6}{5}=6$	$\binom{6}{6}=1$			
7	$\binom{7}{0}=1$	$\binom{7}{1}=7$	$\binom{7}{2}=21$	$\binom{7}{3}=35$	$\binom{7}{4}=35$	$\binom{7}{5}=21$	$\binom{7}{6}=7$	$\binom{7}{7}=1$		
8	$\binom{8}{0}=1$	$\binom{8}{1}=8$	$\binom{8}{2}=28$	$\binom{8}{3}=56$	$\binom{8}{4}=70$	$\binom{8}{5}=56$	$\binom{8}{6}=28$	$\binom{8}{7}=8$	$\binom{8}{8}=1$	
9	$\binom{9}{0}=1$	$\binom{9}{1}=9$	$\binom{9}{2}=36$	$\binom{9}{3}=84$	$\binom{9}{4}=126$	$\binom{9}{5}=126$	$\binom{9}{6}=84$	$\binom{9}{7}=36$	$\binom{9}{8}=9$	$\binom{9}{9}=1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Binomial coefficients triangle

The first property to note is that

$$\binom{n}{0} = 1 \quad \text{and} \quad \binom{n}{n} = 1.$$

Combinatorially, this can be interpreted as follows: the number of ways to select 0 elements from a set of n elements is 1 (the empty set), and the number of ways to select all n elements from the set is also 1 (the entire set).

Theorem 3.15 (Symmetry Property)**.**
For any integers n and k such that $0 \leq k \leq n$, the following holds:

$$\binom{n}{k} = \binom{n}{n-k}$$

Theorem 3.16 (Recurrence Relation)**.**
For any integers n and k, $(0 \leq k \leq n)$, the binomial coefficient satisfies the following recurrence relation:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Remark.
For now, we assume that if $k > n$, $k < 0$, or $n < 0$, the binomial coefficient $\binom{n}{k} = 0$.

Theorem 3.17 (Vandermonde's Identity)**.**
For any non-negative integers m, n and k, Vandermonde's identity is given by:

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r}$$

Exercise.
Prove the following properties of binomial coefficients using both algebraic and combinatorial argu-

ments:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \quad \text{for } k \geq 1,$$
$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}.$$

3.2.3 The Binomial Theorem

One of the most widely used theorems in mathematics is the expansion of $(a+b)^n$, known as the Binomial Theorem.

Theorem 3.18 (The Binomial Theorem).
For any variables a and b, and for each positive integer n, the expansion of $(a+b)^n$ is:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}a^0 b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.$$

Exercise.
Determine the coefficient of $a^{17}b^9$ in the expansion of $(a+b)^{26}$.

Solution.
The general term in the expansion of $(a+b)^{26}$ is given by $\binom{26}{k}a^{26-k}b^k$.
To find the term $a^{17}b^9$, we need $26-k=17$, which gives $k=9$.
Therefore, the coefficient is:

$$\binom{26}{9}.$$

3.3 Permutations with (limited) repetitions, Multinomial Coefficients, and the Multinomial Theorem

3.3.1 The MISSISSIPPI Problem

Consider the following problem: How many distinct words can be formed by rearranging the letters of the word "MISSISSIPPI"?

At first glance, one might assume there are 11! possible arrangements since the word consists of 11 letters. However, this is incorrect because permuting identical letters yields the same word multiple times. To solve this problem, we must account for the repeated letters by selecting the positions for each without permuting them internally. Here's how it works:

- $\binom{11}{1}$ possible positions for the 'M',
- $\binom{11-1}{4}$ positions for the 'I',
- $\binom{11-1-4}{4}$ positions for the 'S',
- $\binom{11-1-4-4}{2}$ positions for the 'P'.

Using the multiplication principle, the total number of distinct permutations is given by:

$$\binom{11}{1}\binom{11-1}{4}\binom{11-1-4}{4}\binom{11-1-4-4}{2} = \frac{11!}{1!10!}\frac{10!}{4!6!}\frac{6!}{4!2!}\frac{2!}{2!} = \frac{11!}{1!4!4!2!}$$

This formula represents the total number of ways to permute all 11 letters, while dividing by the factorial of repeated letters to exclude identical permutations.

3.3.2 Permutations with (limited) repetitions

More generally, we can express this concept in the form of the following theorem:

Theorem 3.19 (Permutation with Repetition).
Let S be a collection of k distinct elements $e_1, e_2, \ldots, e_k$, where each element e_i is repeated n_i times

$(1 \le i \le k)$, and the total number of elements with repetitions is n (i.e., $n_1 + n_2 + \cdots + n_k = n$). The number of permutations of all the elements in S, denoted by $\binom{n}{n_1, n_2, \ldots, n_k}$, is given by:

$$\binom{n}{n_1, n_2, \ldots, n_k} = \binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-\sum_{i=1}^{k-1} n_i}{n_k} = \frac{n!}{n_1!n_2!n_3!\cdots n_k!}$$

where the coefficients $\binom{n}{n_1, n_2, \ldots, n_k}$ are called multinomial coefficients.

Remarks.

- The collection S can be denoted as:
$$\{n_1 \cdot e_1, n_2 \cdot e_2, \ldots, n_k \cdot e_k\},$$
which is called a *multiset*. Unlike sets, multisets can contain the same element multiple times.
- Infinite repetition is represented by ∞. For example, the multiset:
$$\{\infty \cdot a, 5 \cdot b, 3 \cdot c\}$$
means that element a appears infinitely often, b appears 5 times, and c appears 3 times.
- Problems involving infinite repetition are generally simpler than those with finite repetition. Given a multiset:
$$A = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\},$$
the task is to determine how many permutation of n elements can be formed. Since each element can be used repeatedly without limit, the number of such arrangements is:
$$n^n.$$

3.3.3 Properties of Multinomial Coefficients

Multinomial coefficients generalize binomial coefficients by representing the number of ways to divide n distinct objects into k groups of sizes $n_1, n_2, \ldots, n_k$, where the total size of all groups equals n.

Theorem 3.20 (Recursive Formula for Multinomial Coefficients)**.**
Let n be a positive integer, and let $n_1, n_2, \ldots, n_k$ be non-negative integers such that $n = n_1 + n_2 + \cdots + n_k$ with $k \ge 2$. The multinomial coefficient can be expressed recursively as follows:

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_k} &= \sum_{i=1}^{k} \binom{n-1}{n_1, \ldots, n_i - 1, \ldots, n_k} \\ &= \binom{n-1}{n_1 - 1, n_2, \ldots, n_k} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_k} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_k - 1}.\end{aligned}$$

Exercise.
Prove the theorem both algebraically and combinatorially.

3.3.4 Multinomial Theorem

The multinomial theorem extends the binomial theorem.

Theorem 3.21 (Multinomial Theorem)**.**
For any positive integer n and any k-tuple of non-negative integers $x_1, x_2, \ldots, x_k$, the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ is given by:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_1 \ge 0, \ldots, n_k \ge 0}} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$

where $\binom{n}{n_1, n_2, \ldots, n_k}$ are the multinomial coefficients.

Exercise.

- Provide a combinatorial proof using similar arguments to those used in the binomial theorem.
- Prove the multinomial theorem using induction:
  - On n.
  - On k.

Example.
Expand $(x_1 + x_2 + x_3)^4$:

$$\begin{aligned}(x_1 + x_2 + x_3)^3 &= \sum_{\substack{n_1+n_2+n_3=3\\ n_1,n_2,n_3\geq 0}} \binom{3}{n_1, n_2, n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}, \\ &= \binom{3}{0,0,3} x_3^3 + \binom{3}{0,3,0} x_2^3 + \binom{3}{0,1,2} x_2 x_3^2 + \binom{3}{0,2,1} x_2^2 x_3^1 + \binom{3}{1,1,1} x_1 x_2 x_3 \\ &\quad + \binom{3}{1,2,0} x_1 x_2^2 + \binom{3}{1,0,2} x_1 x_3^2 + \binom{3}{2,0,1} x_1^2 x_3 + \binom{3}{2,1,0} x_1^2 x_2 + \binom{3}{3,0,0} x_1^3.\end{aligned}$$

Exercise.
Expand $(a + b + 2c)^5$ and find the coefficients of:

- a^2b^2c
- a^3bc
- b^5

3.4 Combinations with repetitions

We have already seen the k-combinations without repetition, which are counted by $\binom{n}{k}$. Now, let us consider the following problem: let $S = \{1, 2, 3\}$. What are the 2-combinations of S without repetition? Clearly, they are

$$\{1, 2\}, \{1, 3\}, \{2, 3\}.$$

Now, let us consider the same problem, but with repetition allowed. The solutions are:

$$\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 2\}, \{2, 3\}, \{3, 3\}$$

. Thus, we have 6 solutions.

Can you now answer the question: if the set is $S = \{1, 2, 3, \ldots, n\}$ and we want to form a k-combination with repetition allowed, how many such combinations are there?

Definition 3.22.
A *k-combination with repetition* from a set $S = \{1, 2, 3, \ldots, n\}$ is an unordered selection of k elements, where repetition of elements is allowed.

Remarks.

- It is evident that the repetitions of elements cannot exceed k. If $x_1, x_2, \ldots, x_n$ represent the repetitions of the elements $e_1, e_2, \ldots, e_n$, then we must have the equation $x_1 + x_2 + \cdots + x_n = k$ with $0 \leq x_i \leq k$ for each $1 \leq i \leq n$.
- The problem of finding a k-combination with repetition is equivalent to forming a multiset of k elements, denoted as $\{x_1 \cdot e_1, x_2 \cdot e_2, \ldots, x_n \cdot e_n\}$, where the sum $x_1 + x_2 + \cdots + x_n = k$ holds.

Theorem 3.23 (Number of k-combinations with repetition)**.**
The number of k-combinations with repetition allowed from a set $S = \{1, 2, 3, \ldots, n\}$, denoted $\left(\!\binom{n}{k}\!\right)$,

is given by:

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

3.4.1 Application: Number of Integer Solutions of Simple Linear Equations

An important application of k-combinations with repetition is counting the number of nonnegative integer solutions to linear equations. Consider the following general problem:

Question: How many nonnegative integer solutions exist for the equation:

$$x_1 + x_2 + \cdots + x_n = k?$$

Theorem 3.24.
The number of nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_n = k$ is given by:

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{k+n-1}{k}.$$

Exercise.
Solve the following problems:

- How many nonnegative integer solutions are there for the equation:

$$x_1 + x_2 + x_3 = 6?$$

- How many solutions exist for the equation:

$$x_1 + x_2 + x_3 + x_4 = 18,$$

where $x_1 \geq 3$, $x_2 \geq 5$, $x_3 \geq 1$, and $x_4 \geq 0$?

Exercise.
How many nonnegative integer solutions exist for the inequality:

$$x_1 + x_2 + \cdots + x_n \leq k?$$

Chapter 2: Advanced Counting Principles

In the first chapter, we covered the foundational concepts and rules of counting. Now, we will explore more advanced counting principles that allow us to tackle more complex problems.

1. Cardinality of Sets

Let S be a finite set, and let A and B be subsets of S.

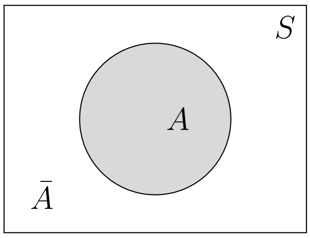
Definition 1.1.
The *cardinality* of a set S, denoted $|S|$, is the number of elements in S. Specifically, if $S = \{e_1, e_2, \ldots, e_n\}$, then $|S| = n$.

Definition 1.2 (Complement of A)**.**
The complement of a subset A of S, denoted $\bar{A}$, consists of all elements of S that are not in A:

$$\bar{A} = \{x \in S \mid x \notin A\}.$$

Cardinality: The cardinality of the complement $\bar{A}$ is given by

$$|\bar{A}| = |S| - |A|.$$

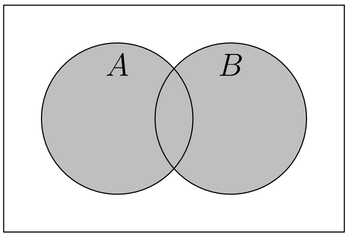


Definition 1.3 (Union of Two Sets)**.**
The *union* of two subsets A and B of S, denoted by $A \cup B$, is the set of all elements that are in A, in B, or in both:

$$A \cup B = \{x \in S \mid x \in A \text{ or } x \in B\}.$$

Cardinality: The cardinality of $A \cup B$ is calculated by

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

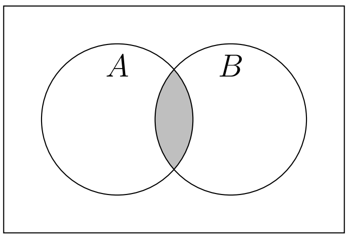


Definition 1.4 (Intersection of Two Sets)**.**
The *intersection* of two subsets A and B of S, denoted by $A \cap B$, is the set of all elements that are in both A and B:

$$A \cap B = \{x \in S \mid x \in A \text{ and } x \in B\}.$$

Cardinality: The cardinality of $A \cap B$ is simply

$$|A \cap B|.$$

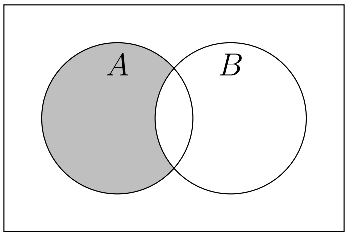


Definition 1.5 (Difference of Two Sets).
The *difference* of two sets A and B, denoted $A \setminus B$, is the set of all elements that are in A but not in B:

$$A \setminus B = \{x \in S \mid x \in A \text{ and } x \notin B\}.$$

Cardinality: The cardinality of $A \setminus B$ is

$$|A \setminus B| = |A| - |A \cap B|.$$



1.1 Finite Unions and Intersections

Now, let $A_1, A_2, \dots, A_n$ be subsets of S.

Definition 1.6 (Finite Unions).
The *finite union* of $A_1, A_2, \dots, A_n$, subsets of S, is defined as:

$$\bigcup_{i=1}^{n} A_i = \{x \in S \mid x \in A_i \text{ for at least one } i\}.$$

Cardinality: The cardinality of $\bigcup_{i=1}^{n} A_i$ is determined by the inclusion-exclusion principle, which leads to a more complex formula.

Definition 1.7 (Finite Unions of Disjoint Subsets).
If $A_1, A_2, \dots, A_n$ are subsets of S such that each pairwise intersection is empty, i.e.,

$$A_i \cap A_j = \emptyset \quad \text{for all } i \neq j,$$

then their union is denoted by $\bigoplus_{i=1}^{n} A_i$.
Cardinality: The cardinality of $\bigoplus_{i=1}^{n} A_i$ is simply the sum of the cardinalities of the individual sets:

$$\left| \bigoplus_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i|.$$

Definition 1.8 (Finite Intersections).
The *finite intersection* of $A_1, A_2, \dots, A_n$, subsets of S, is defined as:

$$\bigcap_{i=1}^{n} A_i = \{x \in S \mid x \in A_i \text{ for all } i\}.$$

Cardinality: The cardinality of $\bigcap_{i=1}^{n} A_i$ is simply

$$\left| \bigcap_{i=1}^{n} A_i \right|.$$

Definition 1.9 (Power Set).
The *power set* of a set S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Proposition 1.10 (Cardinality of the Power Set).
If $|S| = n$, the cardinality of the power set $\mathcal{P}(S)$ is given by:

$$|\mathcal{P}(S)| = 2^n.$$

1.2 Floor and Ceiling Functions

Definition 1.11 (Floor and Ceiling Functions).
For any real number x, we know that x lies between two integers n and $n+1$, where $n \leq x < n+1$. In this case:

- The integer n is denoted by $\lfloor x \rfloor$, called the *floor function*, which represents the greatest integer less than or equal to x.
- The integer $n+1$ is denoted by $\lceil x \rceil$, called the *ceiling function*, which represents the smallest integer greater than or equal to x.

2. Pigeonhole Principle (Dirichlet box principle)

Theorem 2.1 (Pigeonhole Principle - Version 1).
If $n+1$ or more pigeons are placed into n pigeonholes, then at least one pigeonhole must contain more than one pigeon.

Examples.

- In a group of 8 students, at least two of them must have the same day of the week as their birthday.
  To solve this using the pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (8 students in total).
  - **Pigeonholes (boxes):** The days of the week (7 days: Monday through Sunday).

  By the pigeonhole principle (PHP), with 8 pigeons and only 7 pigeonholes, at least one day must be shared by two students. Thus, at least two students have the same birthday day of the week.
- In a group of 13 students, at least two of them must have the same birth month.
  To solve this using the pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (13 students).
  - **Pigeonholes (boxes):** The months of the year (12 months).

  By the pigeonhole principle (PHP), with 13 pigeons and only 12 pigeonholes, at least one month must be shared by two students. Thus, at least two students have the same birth month.

Theorem 2.2 (Pigeonhole Principle - Version 2).
If $kn+1$ pigeons are placed into n pigeonholes, where k is a positive integer, then at least one pigeonhole must contain at least $k+1$ pigeons.

Examples.

- In a group of 22 students, at least 3 of them must have the same day of the week as their birthday.
  To solve this using the pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each student in the group (22 students in total).
  - **Pigeonholes (boxes):** The days of the week (7 days: Monday through Sunday).

  By the pigeonhole principle (PHP), with 15 pigeons and only 7 pigeonholes, at least one day must be shared by at least two students. Thus, at least two students have the same birthday day of the week.

- In a group of 37 people, at least 4 must have been born in the same month.
  To solve this using the pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:
  - **Pigeons:** Each person in the group (37 people in total).
  - **Pigeonholes (boxes):** The months of the year (12 months).

  By the pigeonhole principle (PHP), with 37 pigeons and only 12 pigeonholes, at least one month must be shared by at least 4 people. Thus, at least 4 people must have been born in the same month.

Theorem 2.3 (Generalized Pigeonhole Principle - Version 3)**.**
If m pigeons are placed into n pigeonholes, then at least one pigeonhole contains at least $\lceil \frac{m}{n} \rceil$ pigeons.

Example.

Suppose there are 316 students in the first year of NHSM, and we want to distribute them into 12 groups, at least one group must contain at least 27 students.
To solve this using the generalized pigeonhole principle, we define the "pigeons" and "pigeonholes" as follows:

- **Pigeons:** The 316 students in total.
- **Pigeonholes (boxes):** The 12 groups.

By the generalized pigeonhole principle (PHP), with 316 students and only 12 groups, at least one group must contain at least $\lceil \frac{316}{12} \rceil = 27$ students. Thus, at least one group must have at least 27 students.

Exercise.
Suppose we have gloves of 4 colors: black, red, green, and blue. How many gloves should we grab to ensure we get a pair of the same color?

Solution.
To guarantee a pair, we define the pigeons as the gloves and the pigeonholes as the 4 colors. By the pigeonhole principle, grabbing 5 gloves ensures at least two gloves of the same color, since there are only 4 colors.
Thus, we need to grab at least 5 gloves.

Exercise.
Suppose we have gloves of 4 colors: black, red, green, and blue. How many gloves should an octopus (with 8 hands) grab to ensure it gets a pair of the same color?

Solution.
To guarantee a pair, we define the pigeons as the gloves and the pigeonholes as the 4 colors. By the pigeonhole principle, even if the octopus grabs $7 \times 4 + 1$ gloves, it ensures that at least 8 gloves of the same color are taken, since there are only 4 colors.
Thus, the octopus needs to grab at least 29 gloves.

Remarks.
- The Pigeonhole Principle can be used to prove the existence of certain properties within a set of objects.
- Applying the Pigeonhole Principle is not always straightforward and may require thoughtful construction of "pigeons" and "holes."
- A well-constructed approach can lead to a concise and elegant proof.

- The Pigeonhole Principle guarantees that there is a *certain* box that contains at least two objects. However, it does not tell us *which* box it is or *which* objects it contains.

Exercise.
Show that in a set $S = \{a_1, a_2, \dots, a_{n+1}\}$ of $n+1$ integers, there are at least two integers whose difference is divisible by n.

3. Inclusion-Exclusion Principle

Let S be a finite set, and $A_1, A_2 \subseteq S$. To compute $|A_1 \cup A_2|$, we sum the cardinalities of the individual sets:

$$|A_1| + |A_2|.$$

However, elements in $A_1 \cap A_2$ are counted twice, so we subtract the intersection:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

For the complement of the union, the size of the set $\overline{A_1} \cap \overline{A_2}$ (the elements not in $A_1 \cup A_2$) is:

$$|\overline{A_1} \cap \overline{A_2}| = |S \setminus (A_1 \cup A_2)| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|.$$

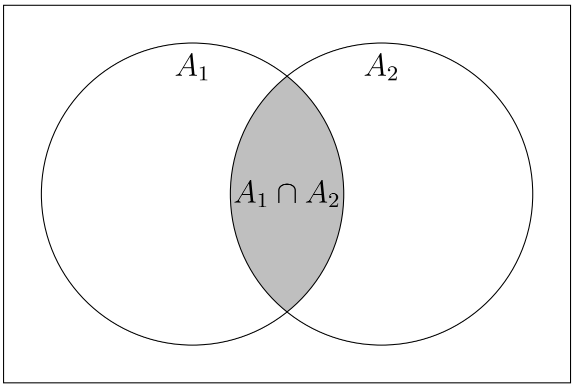


Figure 2: Venn diagram of two sets

3.1 Case of the three subsets

Let S now be a finite set, and $A_1, A_2, A_3 \subseteq S$. To compute $|A_1 \cup A_2 \cup A_3|$, we begin by summing the cardinalities of the individual sets:

$$|A_1| + |A_2| + |A_3|.$$

However, elements in the pairwise intersections $A_1 \cap A_2$, $A_1 \cap A_3$, and $A_2 \cap A_3$ are counted twice, so we subtract the sizes of these intersections:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|.$$

But now, the elements in the triple intersection $A_1 \cap A_2 \cap A_3$ have been subtracted three times, so we add back the size of this triple intersection:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

For the complement of the union, the size of the set $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$ (the elements not in $A_1 \cup A_2 \cup A_3$) is given by:

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S \setminus (A_1 \cup A_2 \cup A_3)| = |S| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|.$$

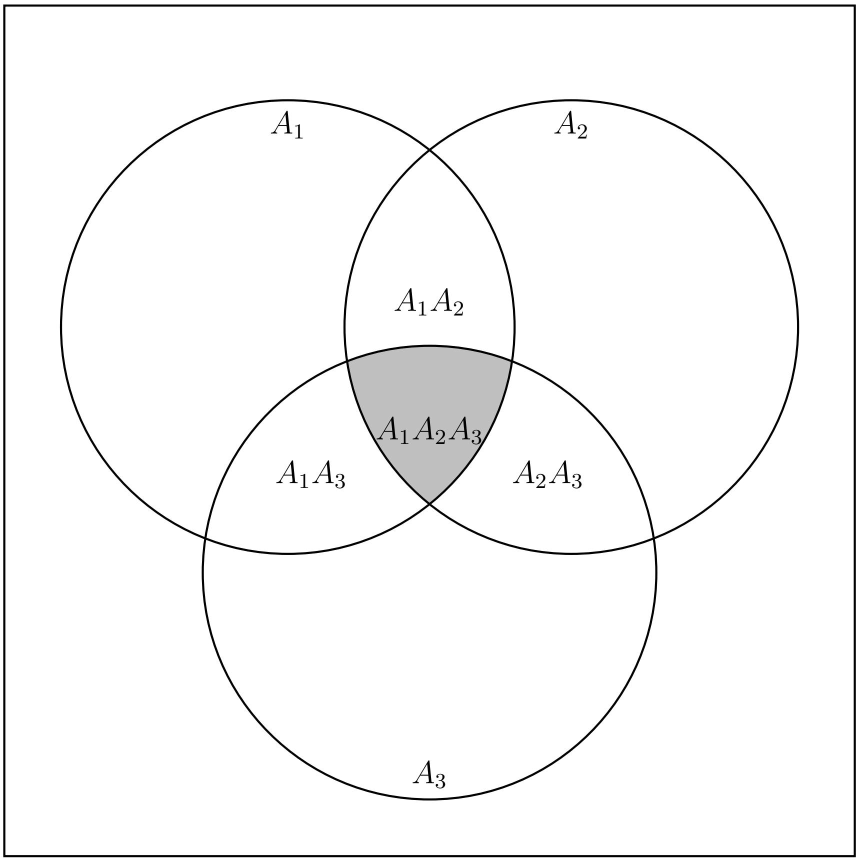


Figure 3: Venn diagram of three sets

3.2 General case

As we observe in the cases of two sets and three sets, the computation of the union always starts with the **inclusion** of the cardinalities of the individual sets. Next, we **exclude** the intersections of pairs of sets because these elements are counted multiple times. This process of alternating **inclusion-exclusion** continues, giving rise to the concept known as the **Inclusion-Exclusion Principle (IEP)**.

In the following theorem, we will generalize this principle to compute the cardinality of the union of n subsets of a finite (universal) set S.

Before proceeding, let us consider subsets $A_1, A_2, \ldots, A_n \subseteq S$, where each A_i represents the set of elements satisfying property i (for $i = 1, 2, \ldots, n$). To simplify notation, we denote the intersection $A_i \cap A_j$ directly as A_iA_j. For example, $A_1A_2A_3$ represents $A_1 \cap A_2 \cap A_3$.

Theorem 3.1 (General Inclusion-Exclusion Principle)**.**
Let $A_1, A_2, \ldots, A_n$ be subsets of a finite universal set S. The cardinality of the union $|A_1 \cup A_2 \cup \cdots \cup A_n|$ is given by the following formula:

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= \sum_{1 \le i_1 \le n} |A_{i_1}| - \sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \le i_1 < i_2 < i_3 \le n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \\ &\quad - \cdots + (-1)^{n-1} \sum_{1 \le i_1 < i_2 < \cdots < i_n \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_n}| \\ &= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|. \end{aligned}$$

Corollary 3.2 (Complementary form)**.**
The cardinality of the complement of the union of subsets $A_1, A_2, \ldots, A_n$ is:

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right| = |S| - \sum_{k=1}^{n}(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left|A_{i_1} \cap \cdots \cap A_{i_k}\right|.$$

When to Use IEP

- **Union of Sets:** Apply IEP to calculate the size of a union when overlaps cause over-counting.
- **Complement of Union:** To compute the complement of a union, subtract the union's size from the size of the universal set, and this is the common application of IEP.
- The universal set S includes all the elements under consideration for the problem, without considering any specific conditions or restrictions imposed by subsets.

Steps for Applying IEP

1. **Define the Sets:** Specify the subsets $A_1, A_2, \ldots, A_n$, where each represents a specific condition or property.
2. **Compute Intersections:** Calculate the sizes of all intersections, such as:
$$\left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right|.$$
3. **Apply the IEP Formula:** Compute the union's size using:
$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left|A_{i_1} \cap \cdots \cap A_{i_k}\right|.$$

4. Classical examples of using IEP

4.1 Derangements: Permutations Without Fixed Points

Definition 4.1 (Derangement).
A **derangement** is a permutation of n elements $\{e_1, e_2, \ldots, e_n\}$ such that no element e_i is in its original position i.

Theorem 4.2 (Number of derangements).
The number of derangements of a set of n elements, denoted by D_n, is given by:
$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

From the definition, a derangement of $\{e_1, \ldots, e_n\}$ is a permutation where e_1 is not in position 1, e_2 is not in position 2, and so on, with e_n not in position n. To calculate the number of such permutations, we count the number of elements in the set of permutations that satisfy $e_1 \notin$ position $1, e_2 \notin$ position $2, \ldots, e_n \notin$ position n. This requires counting the intersections of sets, which leads us to the complementary form of Inclusion-Exclusion Principle (IEP):

$$\left|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}\right| = |S| - \sum_{k=1}^{n}(-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left|A_{i_1} \cap \cdots \cap A_{i_k}\right|.$$

- **Define the Sets:**
  - $\overline{A_i}$: The set of permutations where e_i is not in position i.
  - A_i: The set of permutations where e_i is in position i.
  - S: The universal set, representing all permutations of $\{e_1, \ldots, e_n\}$, without any restrictions.
- **Compute Intersections:** Using the definition of intersections:
  - Cardinality of S:
$$|S| = n!.$$

- Single Sets: Each $|A_{i_1}| = (n-1)!$, so:
$$\sum_{1 \le i_1 \le n} |A_{i_1}| = \binom{n}{1}(n-1)!.$$
- Intersections of Two Sets: The intersection $|A_{i_1} \cap A_{i_2}| = (n-2)!$, so:
$$\sum_{1 \le i_1 < i_2 \le n} |A_{i_1} \cap A_{i_2}| = \binom{n}{2}(n-2)!.$$
- General Case for k Sets: The intersection of k subsets $|A_{i_1} \cap \cdots \cap A_{i_k}| = (n-k)!$. Thus:
$$\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| = \binom{n}{k}(n-k)!.$$

- **Apply the IEP Formula:** Substituting these results into the Inclusion-Exclusion formula:
$$\begin{aligned}
|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| &= |S| - \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|, \\
&= n! - \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k}(n-k)!, \\
&= n! + \sum_{k=1}^{n} (-1)^{k} \frac{n!}{k!}, \\
&= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\end{aligned}$$
This proves the formula for the number of derangements $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

4.2 Number of Integer Solutions to a Linear Equation with Constraints

We aim to count the number of solutions to the equation (S), defined as:
$$(S) \quad \begin{cases} x_1 + x_2 + \cdots + x_n = k \\ r_i \le x_i \le s_i \quad \forall i \in \{1, 2, \cdots, n\} \end{cases}$$
First, let us define new variables $y_i = x_i - r_i$ for each $i \in \{1, 2, \cdots, n\}$. Substituting these into the equation, we obtain the equivalent system:
$$(S) \quad \begin{cases} y_1 + y_2 + \cdots + y_n = k - \underbrace{\sum_{i=1}^{n} r_i}_{r} \\ 0 \le y_i \le \underbrace{s_i - r_i}_{l_i} \quad \forall i \in \{1, 2, \cdots, n\} \end{cases} \iff \begin{cases} y_1 + y_2 + \cdots + y_n = k - r \\ 0 \le y_i \le l_i \quad \forall i \in \{1, 2, \cdots, n\} \end{cases}$$
Next, consider the equation (Ω) obtained by relaxing the upper bounds on y_i, i.e., without the constraints on y_i:
$$(\Omega) \quad \begin{cases} y_1 + y_2 + \cdots + y_n = k - r \\ y_i \ge 0 \quad \forall i \in \{1, 2, \cdots, n\} \end{cases}$$
Our task is to find the number of solutions to (Ω) under the constraints $0 \le y_1 \le l_1$, $0 \le y_2 \le l_2$, and so on for each y_i. This is equivalent to counting the size of the intersection of sets:
$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$$
where
$$\overline{A_i} = \{(y_1, y_2, \cdots, y_n) \in \Omega : 0 \le y_i \le l_i\} \quad \forall i \in \{1, 2, \ldots, n\}.$$
Using the Inclusion-Exclusion Principle (IEP), we have:
$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |\Omega| - \sum_{m=1}^{n} (-1)^{m-1} \sum_{1 \le i_1 < \cdots < i_m \le n} |A_{i_1} \cap \cdots \cap A_{i_m}|,$$

where
$$A_i = \{(y_1, y_2, \cdots, y_n) \in \Omega : y_i \geq l_i + 1\} \quad \forall i \in \{1, 2, \ldots, n\}.$$
Now, to find the number of solutions for each A_i, we make a substitution: let $z_i = y_i - l_i - 1$ for each i, so that $y_i \geq l_i + 1$ corresponds to $z_i \geq 0$. The number of solutions to A_i is then:
$$|A_i| = \binom{n}{k - r - l_i - 1}.$$
For the intersection of two sets A_i and A_j, the number of solutions is:
$$|A_i \cap A_j| = \binom{n}{k - r - l_i - l_j - 2}.$$
In the general case, the number of solutions for the intersection of m sets $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$ is:
$$|A_{i_1} \cap \cdots \cap A_{i_m}| = \binom{n}{k - r - l_{i_1} - l_{i_2} - \cdots - l_{i_m} - m}.$$
Finally, the cardinality of Ω is:
$$|\Omega| = \binom{n}{k - r}.$$
Substituting these values into the Inclusion-Exclusion formula, we obtain the number of solutions to the original equation under the given constraints:
$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = \binom{n}{k - r} - \sum_{m=1}^{n} (-1)^{m-1} \sum_{1 \leq i_1 < \cdots < i_m \leq n} \binom{n}{k - r - l_{i_1} - l_{i_2} - \cdots - l_{i_m} - m}.$$

4.3 IEP and Euler's Totient Function

Definition 4.3 (Euler's Totient Function)**.**
The Euler's Totient function, denoted as $\varphi(n)$, counts the number of positive integers x such that $1 \leq x \leq n$ and $\gcd(x, n) = 1$. That is,
$$\varphi(n) = |\{x \mid 1 \leq x \leq n,\ \gcd(x, n) = 1\}|$$
where $\gcd(x, n)$ denotes the greatest common divisor of x and n, and the set $\{x \mid 1 \leq x \leq n,\ \gcd(x, n) = 1\}$ contains all integers from 1 to n that are coprime with n.

Counting directly the integers that are coprime with n is not an efficient method for computing $\varphi(n)$. Instead, we use the **inclusion-exclusion principle** to derive a more efficient formula for $\varphi(n)$.

Any positive integer n has a unique prime factorization of the form:
$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$$
where $p_1, p_2, \ldots, p_m$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_m$ are their respective multiplicities.

To use the inclusion-exclusion principle, we will use the fact that any number that is not coprime with n has at least one factor other than 1 in common with n. So, consider the set of integers less than or equal to n that have the factor p_i in common with n, denoted by A_i. Therefore, the number of integers that are coprime with n is given by:
$$\varphi(n) = n - |A_1 \cup A_2 \cup \cdots \cup A_m|$$
Now, using the inclusion-exclusion principle, we can express the size of the union of these sets as:
$$|A_1 \cup A_2 \cup \cdots \cup A_m| = \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$
where $|A_i|$ is the number of integers divisible by p_i, and the other terms account for intersections of the sets.

Counting the Intersections:

- **Single Sets $|A_{i_1}|$:** The set A_{i_1} consists of all integers divisible by p_{i_1}. The number of such integers is:

$$|A_{i_1}| = \left\lfloor \frac{n}{p_{i_1}} \right\rfloor = \frac{n}{p_{i_1}}$$

where $\lfloor\rfloor$ is the floor function, which counts the number of multiples of p_{i_1} less than or equal to n.

- **Intersections of Two Sets $|A_{i_1} \cap A_{i_2}|$:** The intersection $A_{i_1} \cap A_{i_2}$ consists of integers divisible by both p_{i_1} and p_{i_2}. Since p_{i_1} and p_{i_2} are distinct primes, we know that:

$$\mathrm{lcm}(p_{i_1}, p_{i_2}) = p_{i_1} p_{i_2}$$

Therefore, the number of elements in this intersection is:

$$|A_{i_1} \cap A_{i_2}| = \left\lfloor \frac{n}{p_{i_1} p_{i_2}} \right\rfloor = \frac{n}{p_{i_1} p_{i_2}}$$

- **Intersections of Three Sets $|A_{i_1} \cap A_{i_2} \cap A_{i_3}|$:** The intersection $A_{i_1} \cap A_{i_2} \cap A_{i_3}$ consists of integers divisible by the least common multiple of $p_{i_1}, p_{i_2}, p_{i_3}$. Since these are distinct primes, we have:

$$\mathrm{lcm}(p_{i_1}, p_{i_2}, p_{i_3}) = p_{i_1} p_{i_2} p_{i_3}$$

The number of integers in this intersection is:

$$|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = \left\lfloor \frac{n}{p_{i_1} p_{i_2} p_{i_3}} \right\rfloor = \frac{n}{p_{i_1} p_{i_2} p_{i_3}}$$

- **General Case $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$:** For any k-tuple $(i_1, i_2, \ldots, i_k)$, the intersection $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ consists of integers divisible by the least common multiple of the primes $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$. Since all $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ are distinct primes, we have:

$$\mathrm{lcm}(p_{i_1}, p_{i_2}, \ldots, p_{i_k}) = p_{i_1} p_{i_2} \ldots p_{i_k}$$

The number of such integers is:

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = \left\lfloor \frac{n}{p_{i_1} p_{i_2} \ldots p_{i_k}} \right\rfloor = \frac{n}{p_{i_1} p_{i_2} \ldots p_{i_k}}$$

Using the inclusion-exclusion principle, the final formula for $\varphi(n)$ becomes:

$$\begin{aligned}
\varphi(n) &= n - \sum_{i_1=1}^{m} \frac{n}{p_{i_1}} + \sum_{1 \le i_1 < i_2 \le m} \frac{n}{p_{i_1} p_{i_2}} - \sum_{1 \le i_1 < i_2 < i_3 \le m} \frac{n}{p_{i_1} p_{i_2} p_{i_3}} + \cdots + (-1)^{m-1} \frac{n}{p_1 p_2 \ldots p_m} \\
&= n \left(1 - \sum_{i_1=1}^{m} \frac{1}{p_{i_1}} + \sum_{1 \le i_1 < i_2 \le m} \frac{1}{p_{i_1} p_{i_2}} - \sum_{1 \le i_1 < i_2 < i_3 \le m} \frac{1}{p_{i_1} p_{i_2} p_{i_3}} + \cdots + (-1)^{m-1} \frac{1}{p_1 p_2 \ldots p_m} \right) \\
&= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right) = n \prod_{k=1}^{m} \left(1 - \frac{1}{p_k}\right).
\end{aligned}$$

Example (Numerical example)**.**
We begin by finding the prime factorization of 2024:

$$2024 = 2^3 \times 11 \times 23$$

Now, we apply the formula for Euler's Totient Function:

$$\varphi(2024) = 2024 \times \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{11}\right) \times \left(1 - \frac{1}{23}\right)$$

We simplify each term:

$$1 - \frac{1}{2} = \frac{1}{2}, \quad 1 - \frac{1}{11} = \frac{10}{11}, \quad 1 - \frac{1}{23} = \frac{22}{23}$$

Now, we compute:

$$\varphi(2024) = 2024 \times \frac{1}{2} \times \frac{10}{11} \times \frac{22}{23}$$

First, simplify step by step:

$$2024 \times \frac{1}{2} = 1012$$

$$1012 \times \frac{10}{11} = 920$$

$$920 \times \frac{22}{23} = 880$$

Thus, the value of Euler's Totient Function for 2024 is:

$$\varphi(2024) = 880$$

4.4 The Ménage Problem

The Ménage problem is a classical combinatorics problem that asks:

Given n married couples, how many ways can one arrange these n couples around a circular table such that men and women alternate in seating, and no woman sits next to her husband?

This problem was formulated in 1891 by Édouard Lucas. The first explicit formula for the problem was published by Touchard in 1934, although it lacked a proof. In 1943, Kaplansky provided a proof of Touchard's formula. The formula for the number of valid arrangements, M_n, is:

$$M_n = 2n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!.$$

From the problem statement, we seek the number of alternating arrangements such that no couple is seated together. This translates to finding the cardinality of the set $|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$, where $\overline{A_i}$ is the set of configurations in which couple i is not seated together. Using the Inclusion-Exclusion Principle (IEP), we have:

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = |S| - \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \left| A_{i_1} \cap \cdots \cap A_{i_k} \right|,$$

where A_i is the set of configurations in which couple i is seated together, and S is the set of all alternating seating arrangements.

It is straightforward to show that $|S| = 2(n!)^2$. This follows from the fact that women can occupy either odd or even positions, resulting in 2 configurations. For each configuration, the women and men can be independently arranged in $n!$ ways, yielding $2(n!)^2$ total alternating arrangements.

Next, we determine the formula for:

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \left| A_{i_1} \cap \cdots \cap A_{i_k} \right|.$$

This requires analyzing the placement of k dominoes on a circular arrangement.

4.4.1 Placing k Identical, Non-Overlapping Dominoes on m Labeled Circular Positions

To analyze the circular case effectively, we first examine the simpler linear case. Consider k identical dominoes:

,

which we aim to place on a linear grid of m labeled positions such that no two dominoes overlap. Each domino occupies exactly two consecutive positions, and the goal is to compute the total number of distinct arrangements.

Linear Grid Case For a linear grid of m labeled positions, we aim to compute A_m^k, the number of ways to arrange k dominoes. Each domino occupies two positions, leaving $m - 2k$ empty spaces. These spaces are divided into $k + 1$ groups (before, between, and after the dominoes). The number of arrangements corresponds to the non-negative integer solutions of:

$$x_1 + x_2 + \cdots + x_{k+1} = m - 2k,$$

where $x_i \ge 0$ represents the empty positions in each segment. The solution is given by:

$$A_m^k = \binom{m-k}{k}.$$

Circular Grid Case For a circular arrangement, the positions form a closed loop. Each position can be:

- Occupied by a domino paired with the previous position.
- Occupied by a domino paired with the next position.

- Unoccupied.

This introduces an additional challenge since the circular nature prevents a direct division of the empty spaces into groups. To handle this, we fix one domino's position (breaking the symmetry) and analyze the remaining $m-2$ positions linearly.

The number of ways to place k dominoes on a circular grid of m positions is:

$$W_m^k = 2A_{m-2}^{k-1} + A_{m-1}^k,$$

where A_{m-2}^{k-1} accounts for configurations where the fixed domino is paired with the next position, and A_{m-1}^k accounts for configurations where the fixed domino is paired with the previous position.

Substituting $A_m^k = \binom{m-k}{k}$, we have:

$$W_m^k = 2\binom{m-1-k}{k-1} + \binom{m-1-k}{k}.$$

Simplifying further:

$$W_m^k = \binom{m-1-k}{k-1} + \binom{m-k}{k}.$$

Using the combinatorial identity:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$

we rewrite:

$$W_m^k = \left(\frac{k}{m-k} + 1\right)\binom{m-k}{k}.$$

Finally:

$$W_m^k = \frac{m}{m-k}\binom{m-k}{k}.$$

Thus, the number of ways to place k identical, non-overlapping dominoes on m circular positions is:

$$W_m^k = \frac{m}{m-k}\binom{m-k}{k}.$$

4.4.2 Solution of the Ménage Problem

Returning to the original formula:

$$\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$

To arrange k couples around a circular table of size $2n$:

- Choose k couples from n, which can be done in $\binom{n}{k}$ ways.
- Arrange k couples as blocks (dominoes), multiplied by $k!$.
- Alternate starting with either a man or a woman (2 choices).
- Place the remaining $(n-k)$ men and $(n-k)$ women arbitrarily, yielding $((n-k)!)^2$ arrangements.

Thus:

$$\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| = 2\binom{n}{k} k!((n-k)!)^2 W_{2n}^k.$$

Simplifying:

$$\begin{aligned}\sum_{1 \le i_1 < \cdots < i_k \le n} |A_{i_1} \cap \cdots \cap A_{i_k}| &= 2n!(n-k)!W_{2n}^k, \\ &= 2n!(n-k)!\frac{2n}{2n-k}\binom{2n-k}{k}.\end{aligned}$$

Finally, the solution to the Ménage problem using IEP is:

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| &= 2(n!)^2 - \sum_{k=1}^{n}(-1)^{k-1}2n!(n-k)!\frac{2n}{2n-k}\binom{2n-k}{k}, \\ &= 2n!\sum_{k=0}^{n}(-1)^k(n-k)!\frac{2n}{2n-k}\binom{2n-k}{k}.\end{aligned}$$

5. Bonferroni Inequalities and Inclusion-Exclusion Principle

Theorem 5.1 (Bonferroni Inequalities)**.**
Let $A_1, A_2, \ldots, A_n$ be n sets. The Bonferroni inequalities are:

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \geq \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \quad \text{(if } m \text{ is even)}$$
$$|A_1 \cup A_2 \cup \cdots \cup A_n| \leq \sum_{k=1}^{m} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \quad \text{(if } m \text{ is odd.)}$$

Chapter 3: Introduction to Graph Theory

The Königsberg Bridge Problem

The city of Königsberg (now Kaliningrad, Russia) is divided by a river, creating two large islands and two mainland portions, all connected by seven bridges as shown in the next figure.

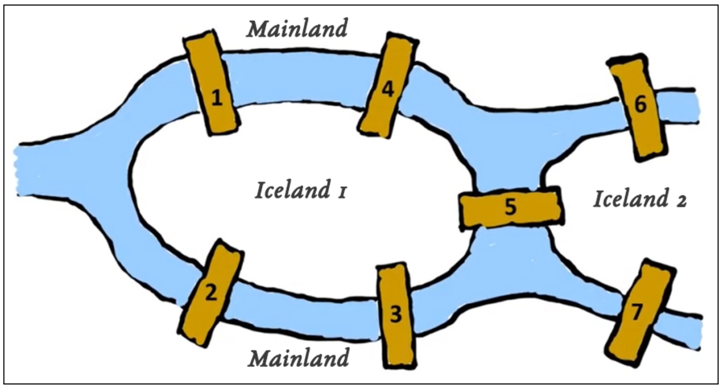


Figure 4: Seven Bridges of Königsberg

The problem asks if it's possible to walk through the city and cross each bridge exactly once, starting and ending at the same point. We can represent the problem using the following structure,

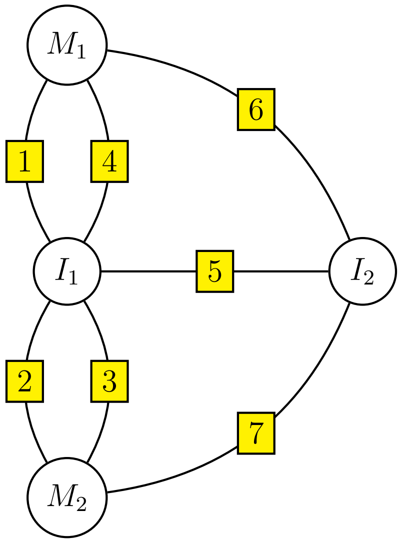


Figure 5: Graph representation of the Königsberg bridges

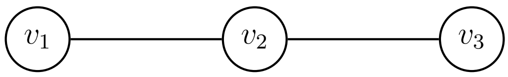
where the land areas are represented as points and the bridges as lines. This structure is known as **a graph**.

1. What's a graph?

Definition 1.1.
A graph $G = (V, E)$ is defined by two finite sets: V and E, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices (with each v_i representing a vertex), and $E = \{e_1, e_2, \ldots, e_m\}$ is the set of edges, which are the connections between vertices. The vertices are typically represented as points, and the edges as lines connecting the corresponding vertices.

Remark.
Algebraically, a graph represents a binary relation E between the elements of a finite set V of vertices. The relation $R \subseteq V \times V$ consists of ordered pairs (v_i, v_j), indicating an edge between vertices v_i and v_j.

Example.
Consider a graph with vertices $V = \{v_1, v_2, v_3\}$ and edges $E = \{(v_1, v_2), (v_2, v_3)\}$. The corresponding graph is:



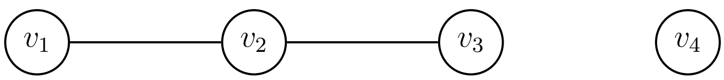
Definition 1.2 (Adjacent, Incident, and Isolated)**.**

- Two vertices v_i and v_j in a graph are said to be ***adjacent*** if there exists an edge that connects v_i and v_j.
- Similarly, two edges e_i and e_j are said to be ***adjacent*** if they share a common vertex.
- An edge is said to be ***incident*** to a vertex if the vertex is one of the endpoints of the edge.
- A vertex v_i is said to be ***isolated*** if it is not incident to any edge in the graph.

Example.
Consider the following graph $G = (V, E)$, where the set of vertices is $V = \{v_1, v_2, v_3, v_4\}$ and the set of edges is $E = \{(v_1, v_2), (v_2, v_3)\}$.

- The vertices v_1 and v_2 are ***adjacent*** because there is an edge (v_1, v_2) connecting them.
- The edges (v_1, v_2) and (v_2, v_3) are ***adjacent*** because they share the common vertex v_2.
- The edge (v_1, v_2) is ***incident*** to the vertices v_1 and v_2.
- The vertex v_4 is ***isolated*** because it is not incident to any edge.

The graph is illustrated below:



Definition 1.3 (Order and Size of a Graph)**.**
The **order** of a graph $G = (V, E)$, denoted $\operatorname{ord}(G)$, is the number of vertices in the graph. That is,

$$\operatorname{ord}(G) = |V|.$$

The **size** of a graph $G = (V, E)$, denoted $e(G)$, is the number of edges in the graph. That is,

$$e(G) = |E|.$$

Example.
In the previous example, $ord(G) = 4$ and $e(G) = 2$.

Definition 1.4 (Degree of a Vertex)**.**
Let $G = (V, E)$ be a graph and $v \in V$ be a vertex.
The ***degree*** of a vertex v, denoted $\deg(v)$, is the number of edges incident to v. Formally, the degree of v is given by

$$\deg(v) = |\{e \in E : v \text{ is an endpoint of } e\}|.$$

- The ***maximum degree*** of the graph, denoted $\Delta(G)$, is the highest degree of any vertex in the graph:
$$\Delta(G) = \max_{v \in V} \deg(v).$$
- The ***minimum degree*** of the graph, denoted $\delta(G)$, is the lowest degree of any vertex in the

graph:

$$\delta(G) = \min_{v \in V} \deg(v).$$

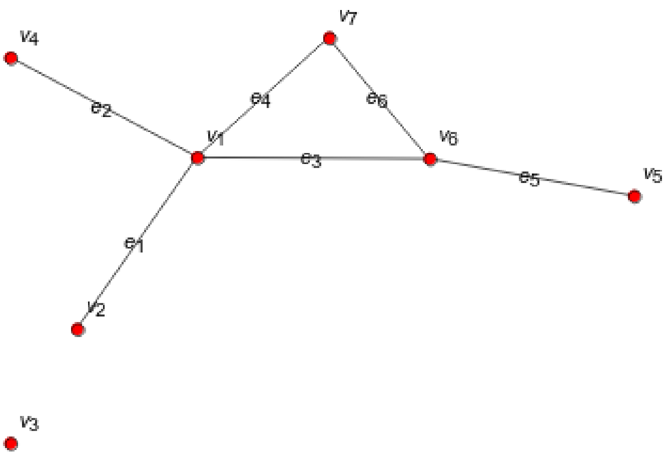
Example.
Consider the graph $G = (V, E)$, where

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

and

$$E = \{e_1 = (v_1, v_2), e_2 = (v_1, v_4), e_3 = (v_1, v_6), e_4 = (v_1, v_7), e_5 = (v_5, v_6), e_6 = (v_6, v_7)\}.$$

The corresponding graph is illustrated below:



The degrees of the vertices are:

- $\deg(v_1) = 4$
- $\deg(v_2) = 1$
- $\deg(v_3) = 0$
- $\deg(v_7) = 2$

Thus, the maximum degree is $\Delta(G) = 4$ and the minimum degree is $\delta(G) = 0$.

Definition 1.5 (Loop).
A ***loop*** in a graph is an edge that connects a vertex to itself.

Definition 1.6 (Simple Graph).
A graph is called a ***simple graph*** if it does not contain any loops or multiple edges between the same pair of vertices.

Definition 1.7 (Empty Graph).
An ***empty graph*** is a graph with no edges. It can have any number of vertices, but there are no edges connecting them. In other words, $E(G) = \emptyset$.

Definition 1.8 (Complete Graph).
A ***complete graph*** is a simple graph in which every pair of distinct vertices is connected by a unique edge. A complete graph on n vertices is denoted by K_n.

2. Directed and Undirected Graphs

Definition 2.1 (Undirected Graph).
An ***undirected graph*** is a graph in which the edges have no direction. That is, if there is an edge between vertices v_i and v_j, it can be traversed in both directions, and we denote it by (v_i, v_j) or

(v_j, v_i).

Definition 2.2 (Directed Graph)**.**
A ***directed graph*** (or ***digraph***) is a graph in which the edges have a direction. Each edge is represented as an ordered pair of vertices, indicating a directed edge from one vertex to another. If there is an edge from v_i to v_j, we write it as (v_i, v_j).

3. Paths, Chains, Cycles, Circuits, Trees, Connectivity, Planarity