Chapter 3: Introduction to Graph Theory

The Königsberg Bridge Problem

The city of Königsberg (now Kaliningrad, Russia) is divided by a river, creating two large islands and two mainland portions, all connected by seven bridges as shown in the next figure.

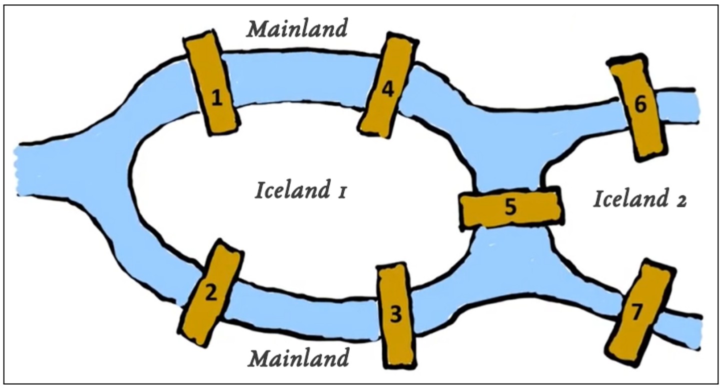


Figure 4: Seven Bridges of Königsberg

The problem asks if it's possible to walk through the city and cross each bridge exactly once, starting and ending at the same point. We can represent the problem using the following structure,

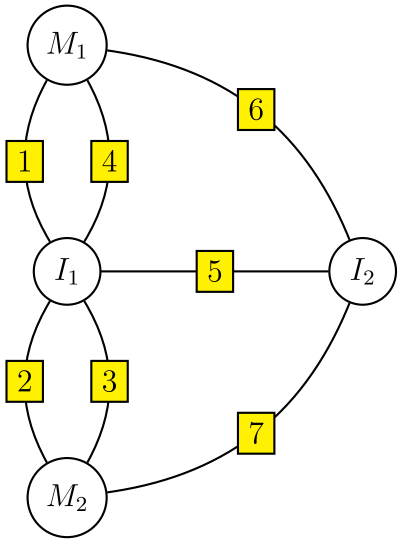


Figure 5: Graph representation of the Königsberg bridges

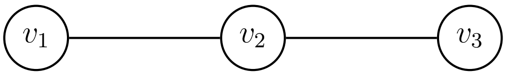
where the land areas are represented as points and the bridges as lines. This structure is known as **a graph**.

1. What's a graph?

Definition 1.1.
A graph $G = (V, E)$ is defined by two finite sets: V and E, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices (with each v_i representing a vertex), and $E = \{e_1, e_2, \ldots, e_m\}$ is the set of edges, which are the connections between vertices. The vertices are typically represented as points, and the edges as lines connecting the corresponding vertices.

Remark.
Algebraically, a graph represents a binary relation E between the elements of a finite set V of vertices. The relation $R \subseteq V \times V$ consists of ordered pairs (v_i, v_j), indicating an edge between vertices v_i and v_j.

Example.
Consider a graph with vertices $V = \{v_1, v_2, v_3\}$ and edges $E = \{(v_1, v_2), (v_2, v_3)\}$. The corresponding graph is:



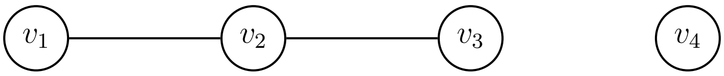
Definition 1.2 (Adjacent, Incident, and Isolated)**.**

- Two vertices v_i and v_j in a graph are said to be ***adjacent*** if there exists an edge that connects v_i and v_j.
- Similarly, two edges e_i and e_j are said to be ***adjacent*** if they share a common vertex.
- An edge is said to be ***incident*** to a vertex if the vertex is one of the endpoints of the edge.
- A vertex v_i is said to be ***isolated*** if it is not incident to any edge in the graph.

Example.
Consider the following graph $G = (V, E)$, where the set of vertices is $V = \{v_1, v_2, v_3, v_4\}$ and the set of edges is $E = \{(v_1, v_2), (v_2, v_3)\}$.

- The vertices v_1 and v_2 are ***adjacent*** because there is an edge (v_1, v_2) connecting them.
- The edges (v_1, v_2) and (v_2, v_3) are ***adjacent*** because they share the common vertex v_2.
- The edge (v_1, v_2) is ***incident*** to the vertices v_1 and v_2.
- The vertex v_4 is ***isolated*** because it is not incident to any edge.

The graph is illustrated below:



Definition 1.3 (Order and Size of a Graph)**.**
The **order** of a graph $G = (V, E)$, denoted $\mathrm{ord}(G)$, is the number of vertices in the graph. That is,

$$\mathrm{ord}(G) = |V|.$$

The **size** of a graph $G = (V, E)$, denoted $e(G)$, is the number of edges in the graph. That is,

$$e(G) = |E|.$$

Example.
In the previous example, $ord(G) = 4$ and $e(G) = 2$.

Definition 1.4 (Degree of a Vertex)**.**
Let $G = (V, E)$ be a graph and $v \in V$ be a vertex.
The ***degree*** of a vertex v, denoted $\deg(v)$, is the number of edges incident to v. Formally, the degree of v is given by

$$\deg(v) = |\{e \in E : v \text{ is an endpoint of } e\}|.$$

- The ***maximum degree*** of the graph, denoted $\Delta(G)$, is the highest degree of any vertex in the graph:

$$\Delta(G) = \max_{v \in V} \deg(v).$$

- The ***minimum degree*** of the graph, denoted $\delta(G)$, is the lowest degree of any vertex in the graph:

$$\delta(G) = \min_{v \in V} \deg(v).$$

Definition 1.5 (Neighbor and Neighborhood)**.**

- A vertex u is said to be a **neighbor** of vertex v if they are adjacent, i.e., there exists an edge $e = \{u, v\} \in E$.
- The **neighborhood** of v, denoted $N(v)$, is the set of vertices adjacent to v. Formally:

$$N(v) = \{u \in V : \exists e \in E \ (e = \{u, v\})\}$$

  - $N(v)$ is called the **open neighborhood** of v.
  - $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of v.

Example.

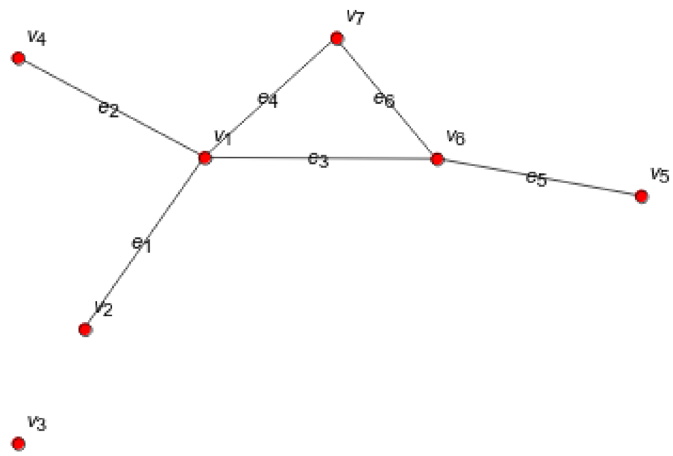
Consider the graph $G = (V, E)$, where

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$$

and

$$E = \{e_1 = (v_1, v_2), e_2 = (v_1, v_4), e_3 = (v_1, v_6), e_4 = (v_1, v_7), e_5 = (v_5, v_6), e_6 = (v_6, v_7)\}.$$

The corresponding graph is illustrated below:



The degrees of the vertices are:

- $\deg(v_1) = 4$
- $\deg(v_2) = 1$
- $\deg(v_3) = 0$
- $\deg(v_7) = 2$

Thus, the maximum degree is $\Delta(G) = 4$ and the minimum degree is $\delta(G) = 0$.

Definition 1.6 (Loop)**.**

A ***loop*** in a graph is an edge that connects a vertex to itself.

Definition 1.7 (Simple Graph)**.**

A graph is called a ***simple graph*** if it does not contain any loops or multiple edges between the same pair of vertices.

Definition 1.8 (Empty Graph)**.**

An ***empty graph*** is a graph with no edges. It can have any number of vertices, but there are no edges connecting them. In other words, $E(G) = \emptyset$.

Definition 1.9 (Complete Graph).
A ***complete graph*** is a simple graph in which every pair of distinct vertices is connected by a unique edge. A complete graph on n vertices is denoted by K_n.

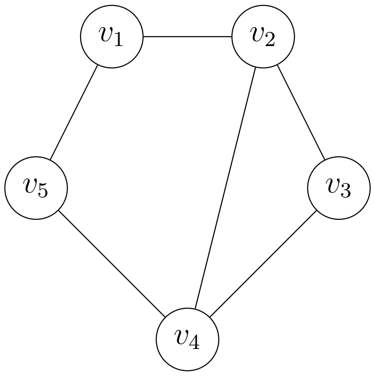
2. Directed and Undirected Graphs

Definition 2.1 (Undirected Graph).
An ***undirected graph*** is a graph in which the edges have no direction. That is, if there is an edge between vertices v_i and v_j, it can be traversed in both directions, and we denote it by $\{v_i, v_j\}$.

Example.
Consider an undirected graph $G = (V, E)$, where:

$$V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_1\}\}.$$

Here, the edges connect the vertices, and each edge can be traversed in both directions.



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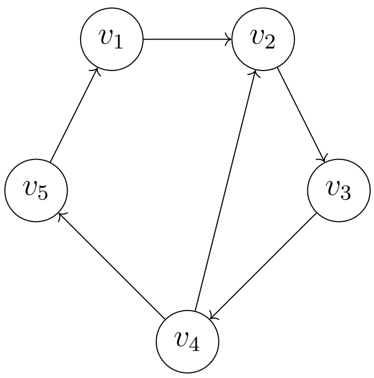
Definition 2.2 (Directed Graph).
A ***directed graph*** (or ***digraph***) is a graph in which the edges have a direction. Each edge is represented as an ordered pair of vertices, indicating a directed edge from one vertex to another. If there is an edge from v_i to v_j, we write it as (v_i, v_j).

Example.
Consider a directed graph $G = (V, E)$, where:

$$V = \{v_1, v_2, v_3, v_4, v_5\}, \quad E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_2), (v_4, v_5), (v_5, v_1)\}.$$

Here, each edge can only be traversed in the specified direction.



Note: In this chapter, we will focus only on *undirected graphs.*

Lemma 2.3 (Handshaking Lemma).

For any graph $G = (V, E)$, the sum of the degrees of all vertices is twice the number of edges, i. e.,

$$\sum_{v \in V} deg(v) = 2|E|.$$

Proof. Each edge contributes 2 to the sum of degrees (1 for each of its two endpoints). Therefore, the sum of degrees counts each edge exactly twice. Hence

$$\sum_{v \in V} \deg(v) = 2|E|.$$

□

Theorem 2.4.
In any graph, the number of vertices with odd degree is even.

Proof. By the Handshaking Lemma, $\sum_{v \in V} \deg(v) = 2|E|$, which is even. Split the sum into vertices with odd degrees (V_{odd}) and even degrees (V_{even}):

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_{\text{odd}}} \deg(v) + \sum_{v \in V_{\text{even}}} \deg(v).$$

Since $\sum_{v \in V_{\text{even}}} \deg(v)$ is even, $\sum_{v \in V_{\text{odd}}} \deg(v)$ must also be even. The sum of an odd number of odd integers is odd, so $|V_{\text{odd}}|$ must be even. □

3. Walk, Trail, Path, Circuit and Cycle

Definition 3.1 (Walk, Trail, Path, Circuit, Cycle)**.**
Let $G = (V, E)$ be a graph.

- A ***walk*** is a sequence of vertices $v_0, v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for all $0 \leq i < k$. The length of the walk is the number of edges, k.
- A ***trail*** is a walk in which no edge is repeated.
- A ***path*** is a walk in which no vertex is repeated (and thus no edge is repeated).
- A ***circuit*** is a closed walk (i.e., $v_0 = v_k$) in which no edge is repeated.
- A ***cycle*** is a closed walk (i.e., $v_0 = v_k$) in which no vertex is repeated except for the starting and ending vertex, and no edge is repeated.

Definition 3.2 (Length of a Walk, Path, or Trail)**.**
The length of a walk, path, or trail is defined as the total number of edges it contains, with repeated edges counted each time they appear in the sequence.

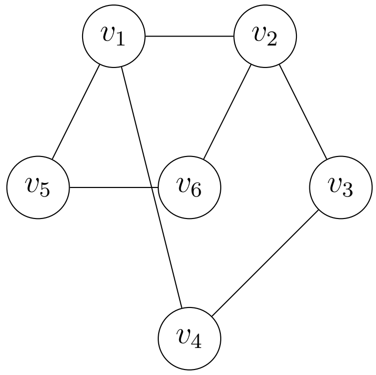
3.1 Key Differences

Term	**Repeated Vertices**	**Repeated Edges**	**Closed** ($v_0 = v_k$)
Walk	Allowed	Allowed	Not Required
Trail	Allowed	Not Allowed	Not Required
Path	Not Allowed	Not Allowed	Not Required
Circuit	Allowed	Not Allowed	Required
Cycle	Not Allowed (except start/end)	Not Allowed	Required

Table 2: Comparison of Walk, Trail, Path, Circuit, and Cycle

Example.
Consider the graph $G = (V, E)$, where:

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}, \quad E = \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}, \{v_1, v_5\}, \{v_5, v_6\}, \{v_6, v_2\}\}.$$



- **Walk**: $v_1, v_2, v_3, v_4, v_1, v_5, v_6, v_2$ (vertices and edges repeat).
- **Trail**: $v_1, v_2, v_3, v_4, v_1, v_5, v_6$ (no repeated edges).
- **Path**: $v_1, v_5, v_6, v_2, v_3, v_4$ (no repeated vertices or edges).
- **Circuit**: v_1, v_2, v_6, v_5, v_1 (closed trail).
- **Cycle**: v_1, v_2, v_3, v_4, v_1 (closed path).

Remarks.

- The following inclusions hold:
$$\text{Paths} \subset \text{Trails} \subset \text{Walks}.$$
That is, all paths are trails, and all trails are walks.
- The length of a cycle must be at least 3. That is, a cycle must contain 3 or more edges.

Exercise.
Show that in any simple graph with n vertices, there are at most $\frac{n(n-1)}{2}$ edges.

Solution (Combinatorial Proof)**.**
Let $G = (V, E)$ be a simple graph with n vertices. Since G is simple, it has no loops or multiple edges. The graph has the most edges when every pair of vertices is connected by exactly one edge. This is equivalent to the number of ways to choose 2 distinct vertices from n, which is:
$$|E| \leq \binom{n}{2} = \frac{n(n-1)}{2}.$$
This result can also be proved by induction on n.

Exercise.
Determine the number of distinct simple graphs that can be formed with n vertices.

Solution.
A simple graph with n vertices is determined by its set of edges. The maximum number of edges in a simple graph with n vertices is $\binom{n}{2}$, since each edge is uniquely determined by a pair of distinct vertices.
For each pair of vertices, there are two choices:

- Include the edge between them.
- Exclude the edge between them.

Since there are $\binom{n}{2}$ pairs of vertices, the total number of distinct simple graphs with n vertices is:
$$2^{\binom{n}{2}}.$$

Lemma 3.3.
In any graph with at least two vertices, there are at least two vertices with the same degree.

Proof. In a graph with $n \geq 2$ vertices, the possible degrees range from 0 to $n-1$. However, a graph cannot simultaneously have a vertex of degree 0 (isolated) and a vertex of degree $n-1$ (connected to all others). Therefore, there are at most $n - 1$ distinct possible degrees for n vertices. By the **Pigeonhole Principle**, at least two vertices must have the same degree. (See Worksheet 2, Exercise 6.) □

Theorem 3.4.
Let $G = (V, E)$ be a graph, and let $u, v \in V$ be two distinct vertices. Any u-v walk in G contains a u-v path in G.

Proof. Let $W = u, v_1, \ldots, v_{k-1}, v$ be a u-v walk. If W has no repeated vertices, it is already a u-v path. Otherwise:

- Let $v_i = v_j$ be the first repeated vertices in W with $i < j$.
- Remove the subsequence $v_{i+1}, \ldots, v_j$ to obtain a shorter walk W'.
- Repeat this process until no repeated vertices remain. The result is a u-v path.

Thus, any u-v walk contains a u-v path. □

4. Connectivity and Planarity