Exercise 1 (Topological Spaces)

Let $X = \{a, b, c\}$.

a) List all possible topologies on X that contain the singleton set $\{a\}$.

b) For one of the topologies found in part (a) that is not the discrete topology, determine the closure of the set $\{b\}$.

Exercise 2 (Basis)

Consider the real line $\mathbb{R}$ with the standard topology.

a) Define what it means for a collection $\mathcal{B}$ of subsets of X to be a basis for a topology on X.

b) Show that the collection of open intervals (q_1, q_2) where $q_1, q_2 \in \mathbb{Q}$ is a basis for the standard topology on $\mathbb{R}$.

Exercise 3 (Continuity)

Let X and Y be topological spaces and $f : X \to Y$ be a function.

a) State the definition of continuity in terms of open sets.

b) Prove that f is continuous if and only if for every closed set $C \subseteq Y$, the preimage $f^{-1}(C)$ is closed in X.

Exercise 4 (Compactness)

a) Define what it means for a topological space X to be compact.

b) Prove that a closed subset of a compact space is compact.

Exercise 5 (Connectedness)

Let X be a topological space.

a) Define what it means for X to be connected.

b) Let A be a connected subset of X. Prove that if $A \subseteq B \subseteq \overline{A}$, then B is connected.