Worksheet 2: Inclusion-Exclusion principle

Answers to Exercise 1:

We divide the set of 100 integers into 50 pairs of consecutive integers, treating each pair as a pigeonhole

$$\{(1, 2), (3, 4), (5, 6), \ldots, (99, 100)\}.$$

Let the 51 chosen integers $\{i_1, i_2, \ldots, i_{50}\}$ represent the pigeons.

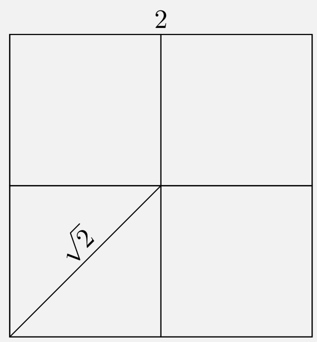
According to the Pigeonhole Principle (PP),

- Pigeons: The 51 chosen numbers from the integers 1 to 100.
- Pigeonholes: The pairs of consecutive integers $(1, 2), (3, 4), \ldots, (99, 100)$.

when we select 51 integers, at least one pigeonhole (corresponding to a pair of integers) must contain 2 pigeons (chosen integers). Consequently, there exist 2 consecutive integers within the set of 51 chosen integers.

Answers to Exercise 2:

We partition the square with a side length of 2 into four smaller squares, each with a side length of 1, achieved by bisecting its sides.



Each of the five points must fall within one of these four small squares.

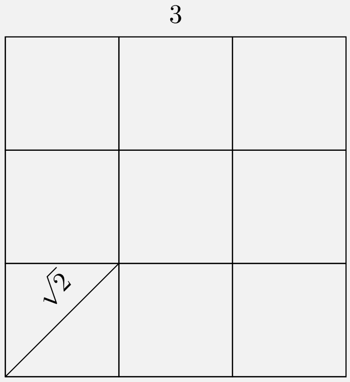
According to the Pigeonhole Principle:

- Pigeons: The 5 points in the square.
- Pigeonholes: 4 small squares of side 1.

there must exist a small square containing at least two of the five points. The diameter of a small square, representing the largest possible distance between two points within it, is the length of its diagonal, which is $\sqrt{2}$. Consequently, the distance between the two points within the same small square is at most $\sqrt{2}$.

Answers to Exercise 3:

We partition the square with a side length of 3 into nine smaller squares, each with a side length of 1, achieved by trisecting its sides.



Each of the ten points must fall within one of these nine small squares.

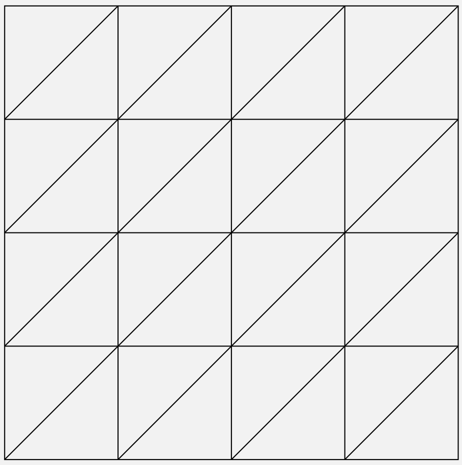
According to the Pigeonhole Principle:

- Pigeons: The 10 points in the square.
- Pigeonholes: 9 small squares of side 1.

There must exist a small square containing at least two of the ten points. The diameter of a small square, representing the largest possible distance between two points within it, is the length of its diagonal, which is $\sqrt{2}$. Consequently, the distance between the two points within the same small square is at most $\sqrt{2}$.

Answers to Exercise 4:

Let us cut the square of side 1 into 32 congruent parts. This yields 32 parts of area $\frac{1}{32}$ each.



According to the generalized version of the Pigeonhole Principle,

- Pigeons: The 65 points inside the square.
- Pigeonholes: The 32 smaller parts obtained by dividing the larger square.

at least one of these 32 parts must contain at least three of our points.

Consequently, the triangle formed by the 3 points has an area less than $\frac{1}{32}$ (area of each small part).

Answers to Exercise 5:

We partition the set of 20 integers into 10 pairs, each with a sum of 21, treating each pair as a pigeonhole:

$$\{(1,20),(2,19),(3,18),\ldots,(10,11)\}$$

Let $\{i_1, i_2, \ldots, i_{11}\}$ represent the 11 chosen integers as pigeons.

According to the Pigeonhole Principle (PP),

- Pigeons: The 11 chosen numbers from the integers 1 to 20.
- Pigeonholes: The pairs of integers $\{(1,20),(2,19),(3,18),\ldots,(10,11)\}$ with a sum of 21.

When we select 11 integers, PP guarantees that at least one pigeonhole (corresponding to a pair of integers) must contain 2 pigeons (chosen integers). Therefore, there exist 2 integers with a sum of 21.

Answers to Exercise 6:

- **Case 1:** Suppose every member of the group has at least one friend.

  In this scenario, each of the n members will have between 1 and $n-1$ friends. We can conceptualize these friend counts as holes, labeled with numbers from 1 to $n-1$, and the n members as pigeons.

  With $n-1$ holes and n pigeons, by the Pigeonhole Principle, there must be at least one hole that contains more than one pigeon. This implies that there exists a number from 1 to $n-1$ representing the count of friends, which is shared by more than one member.

  Thus, in a group of n members, there must be at least two individuals with an equal number of friends.

- **Case 2:** Assume there is a person in the group with no friends.

  In this case, excluding the person with no friends, consider the remaining $n-1$ members. If any one of them has zero friends, we immediately find two individuals with an identical number of friends.

  If none of the remaining $n-1$ members has zero friends, excluding the person with no friends, the remaining $n-1$ individuals will have between 1 and $n-2$ friends. This situation aligns with Case 1, and thus, even in this scenario, we would have two individuals with an identical number of friends.

Consequently, in a group of n people, there must always exist two persons with the same number of friends.

Answers to Exercise 7:

Consider the residues of integers when divided by 10.

There are 10 possible residues when an integer is divided by 10: $\{0, 1, 2, \ldots, 9\}$.

Now, let's divide the set of residues into pairs with a sum of 10 as follows:

$$(0),(1,9),(2,8),(3,7),(4,6),(5).$$

By the Pigeonhole Principle (PP), if we distribute 7 numbers among the 6 subsets, there must be at least two integers x and y in the same set, so either $r_x = r_y$ or $r_x + r_y = 10$.

- If $r_x = r_y$, then $x - y$ is divisible by 10.
- If $r_x + r_y = 10$, then $x + y$ is divisible by 10.

Therefore, in any set of 7 distinct integers, there must be two integers x and y such that either $x + y$ or $x - y$ is divisible by 10.

Answers to Exercise 8:

Let $x_1, x_2, \dots, x_{15}$ represent the number of games played on each of the 15 days. Given that the team plays at least one game daily, we have $x_i \geq 1$ for all i.

The total number of games played in the 15-day season is expressed as the sum:

$$x_1 + x_2 + \dots + x_{15} = 20$$

Now, let's define the cumulative sums:

$$\begin{aligned} S_1 &= x_1 \\ S_2 &= x_1 + x_2 \\ &\dots \\ S_{15} &= x_1 + x_2 + \dots + x_{15} \end{aligned}$$

The objective is to find indices i and j such that $S_j - S_i = x_{i+1} + x_{i+2} + \dots + x_j = 9$, and $j > i$. This ensures that during the consecutive days from the $(i+1)$-th day to the j-th day, the team played exactly 9 games.

Alternatively, we can find S_j such that $S_j = S_i + 9$. Consider the possible sums $T_i = S_i + 9$:

$$\begin{aligned} T_1 &= S_1 + 9 \\ T_2 &= S_2 + 9 \\ &\dots \\ T_{15} &= S_{15} + 9 \end{aligned}$$

There are 15 different values for S_i ranging from 1 to 20, and 15 different values for T_i ranging from 10 to 29. By the Pigeonhole Principle, since we have 30 sums (S_j and T_i) (pigeons) distributed over the range $\{1, 2, \dots, 29\}$ (pigeonholes), there must be two sums that are equal.

Hence, there exist i and j such that $S_j = T_i = S_i + 9$, and consequently, $S_j - S_i = x_{i+1} + x_{i+2} + \dots + x_j = 9$.

This resolves the question.

Answers to Exercise 9:

- We start by dividing the set of $2n$ integers into n pairs, each with a sum of $2n+1$, treating each pair as a pigeonhole:
$$\{(1, 2n), (2, 2n-1), (3, 2n-2), \dots, (n, n+1)\}$$
Let $\{i_1, i_2, \dots, i_{n+1}\}$ denote the $n+1$ chosen integers as pigeons. Applying the Pigeonhole Principle (PP):
  - **Pigeons:** The $n+1$ chosen numbers from the integers 1 to $2n$.
  - **Pigeonholes:** The pairs of integers $\{(1, 2n), (2, 2n-1), (3, 2n-2), \dots, (n, n+1)\}$ with a sum of $2n+1$.

  When $n+1$ integers are selected, PP ensures that at least one pigeonhole (representing a pair of integers) must contain 2 pigeons (chosen integers). Consequently, there exist 2 integers with a sum of $2n+1$.
- Similarly, if we partition the set into n pairs, each with a difference of n, treating each as a pigeonhole:
$$\{(2n, n), (2n-1, n-1), (2n-2, n-2), \dots, (n+1, 1)\}$$
According to the Pigeonhole Principle (PP):
  - **Pigeons:** The $n+1$ chosen numbers from the integers 1 to $2n$.
  - **Pigeonholes:** The pairs of integers $\{(2n, n), (2n-1, n-1), (2n-2, n-2), \dots, (n+1, 1)\}$ with a difference of n.

  When $n+1$ integers are selected, PP guarantees that at least one pigeonhole (corresponding to a pair of integers) must contain 2 pigeons (chosen integers). Hence, there exist 2 integers with a difference of n.

Answers to Exercise 10:

Let the set of n integers be $\{a_1, a_2, \ldots, a_n\}$. Consider the cumulative sums of subsets from the set of n integers:

$$\{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, a_1 + a_2 + a_3 + \ldots + a_n\}$$

There are n distinct values for these cumulative sums.

Now, in two cases:

Case 1: If any cumulative sum is divisible by n, we've found a subset with a sum divisible by n, solving the problem.

Case 2: If none of the cumulative sums is divisible by n, each sum has a remainder from 1 to $n-1$ when divided by n. With n cumulative sums and $n-1$ possible remainders, the Pigeonhole Principle implies at least two sums share the same remainder. Let $a_1 + a_2 + \ldots + a_i \equiv a_1 + a_2 + \ldots + a_j \pmod{n}$, where $i < j$.

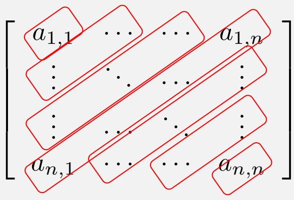
Then,

$$(a_1 + a_2 + \ldots + a_j) - (a_1 + a_2 + \ldots + a_i) = a_{i+1} + a_{i+2} + \ldots + a_j \equiv 0 \pmod{n}$$

This demonstrates that we've found a subset of consecutive integers with a sum divisible by n.

Answers to Exercise 11:

We examine the set of entries in the matrix A that form anti-diagonals, defined as the intersection of A with a line of slope 1. There are $2n-1$ such anti-diagonals in A, and each entry belongs to exactly one of them.



If two '1's are in the same anti-diagonal, they satisfy the property of being "one strictly above and strictly to the right of the other."

Since there are at least $2n$ entries equal to 1, and there are only $2n-1$ anti-diagonals in a matrix of size $n \times n$, by the Pigeonhole Principle, at least one anti-diagonal must contain at least two entries equal to 1. This is because we have more "pigeons" (entries equal to 1) than "pigeonholes" (anti-diagonals) to place them into.

Thus, there exist two entries in A with the property that one is strictly above and strictly to the right of the other. The statement is, therefore, proven.

Answers to Exercise 12:

To prove that the product is even, we just need to show that there is at least one factor that is even. Let's call this factor $(a_{i_k} - a_k)$.

Now, a number like $(a_{i_k} - a_k)$ is even only if both a_{i_k} and a_k are either both even or both odd. We say they have the same "parity" in this case.

So, the key is to consider the parity (even or odd) of the 5 numbers in the set A. To do this, let's separate them into two groups: one for even numbers and one for odd numbers. Since there are 5 numbers in total, by the Pigeonhole Principle (PP), at least 3 of them must be in the same group. Let's call these three elements a_1, a_2, and a_3.

Now, notice that the three elements we selected, $\{a_{i_1}, a_{i_2}, a_{i_3}\}$, are distinct from the original set $\{a_1, a_2, a_3\}$. So, without loss of generality, we can assume that $a_1 = a_{i_3}$.

This means that $a_{i_3} - a_3 = a_1 - a_3$. The difference on the right side is even because a_1 and a_3 have the same parity. Therefore, the factor $(a_{i_3} - a_3)$ is even, and this completes the proof.

Answers to Exercise 13:

Consider pairs of subjects (A, B) and treat each pair as a pigeonhole. If a student studies both subjects in a pair, we place a pigeon into that specific pigeonhole. Since a student chooses 4 subjects, the number of ways to choose 2 out of those 4 subjects is given by $\binom{4}{2}$, which is 6.

With 15 subjects, we have $\binom{15}{2} = 105$ pigeonholes (pairs of subjects). Therefore, each student contributes 6 pigeons, and considering 18 students, the total number of pigeons is $18 \times 6 = 108$.

According to the Pigeonhole Principle (PP), when the number of pigeons exceeds the number of pigeonholes, there must be at least two pigeons in the same pigeonhole.

Hence, by the Pigeonhole Principle, we can conclude that there are at least two students who have chosen the same pair of subjects.

Answers to Exercise 14:

For any integer $n \geq 1$, it can be uniquely expressed in the form $n = 2^a b$, where b denotes its odd factor. Applying this concept, consider the set $\{1, 2, 3, \ldots, 2n\}$ and replace each element with its odd factor. As there are only n odd numbers less than $2n$, there are precisely n choices for this factor. Consequently, according to the pigeonhole principle, when $n + 1$ numbers are selected from the set, at least two of them must share the same odd factor. These two numbers can be represented as $2^a b$ and $2^c b$, where it is evident that the smaller one divides the larger one.

Answers to Exercise 15:

Analyze the extended diagonals identified from 1 to 10 in the 10x10 chessboard arrangement presented below:

1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	1
3	4	5	6	7	8	9	10	1	2
4	5	6	7	8	9	10	1	2	3
5	6	7	8	9	10	1	2	3	4
6	7	8	9	10	1	2	3	4	5
7	8	9	10	1	2	3	4	5	6
8	9	10	1	2	3	4	5	6	7
9	10	1	2	3	4	5	6	7	8
10	1	2	3	4	5	6	7	8	9

There are a total of 10 such diagonals, each comprising 10 squares. According to the pigeonhole principle, if 41 rooks are distributed on the chessboard, at least one diagonal must contain a minimum of five rooks, indicating the presence of 5 non-attacking rooks. We can see the problem as follow also.

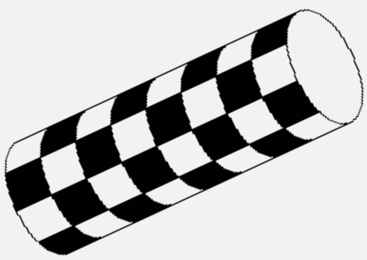


Figure 1: cylindrical chessboard