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# On an accelerated Krasnosel'skiĭ-Mann iteration

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**Abstract** In this note, we present an accelerated Krasnosel'skiĭ-Mann iteration in real Hilbert spaces. Through the utilization of new, self-contained, and simplified techniques, we prove its weak convergence. Notably, our upper bounds on acceleration factors surpass existing ones, assuming that the involved factors remain constant.

**Keywords** Non-expansive operator, Krasnosel'skiĭ-Mann iteration, Acceleration factors, Weak convergence

## 1 Introduction

In many branches of mathematics, fixed point theory and algorithms play a crucial role as fundamental tools.

The Krasnosel'skiĭ-Mann (KM) iteration [1, 2] serves as a fundamental iterative scheme for locating fixed points of non-expansive operators in real Hilbert spaces. The distinctive characteristic of the KM iteration is that the new iterate is obtained as a convex combination of the current iterate and its operator evaluation.

Recently, [3, 4] investigated an accelerated version of the KM iteration, incorporating an additional term that involves a nonnegative factor and the difference between the two most recent iterates. Notably, the assumptions [4]

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regarding this factor are less restrictive compared to those [3], and they are independent of the iterates themselves.

The objective of this note is to introduce significantly different assumptions regarding the aforementioned factor. As illustrated below, these assumptions are weaker than those [4] and remarkably enable a new, concise, self-contained, and simplified proof of the weak convergence of the accelerated KM iteration. Impressively, unlike the proof of [4, Theorem 1], this proof no longer relies on the seminal result [5, Lemma 2.3].

## 2 Preliminaries

In this section, we first give some basic definitions and then provide some auxiliary results for later use.

Let $\mathcal{H}$ be an infinite-dimensional real Hilbert space equipped with the standard inner product $\langle x, y\rangle$ and the induced norm $|x| = \sqrt{\langle x, x\rangle}$ for $x, y \in \mathcal{H}$.

An operator $T : \mathcal{H} \to \mathcal{H}$ is said to be non-expansive if it satisfies the inequality:

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all $x, y$ in $\mathcal{H}$.

The Krasnosel'skiĭ-Mann (KM) iteration is defined as follows:

$$x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T(x_k), \quad k = 0, 1, \ldots,$$

where the coefficient $\alpha_k \in [0, 1]$ and the series $\sum \alpha_k(1 - \alpha_k)$ diverges. Refer to [6–8] for relevant discussions and the cited references.

To accelerate the KM iteration, [3] proposed the following modification:

$$x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T(x_k) + (1 - \alpha_k)\delta_k(x_k - x_{k-1}),$$

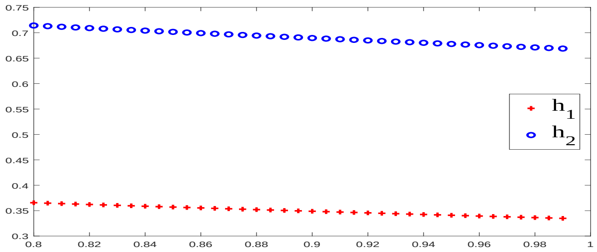
where $\delta_k \geq 0$ and $x_{-1} := x_0$.

For weak convergence, the following assumptions were introduced in [4]:

$$\begin{aligned}
&\varepsilon \leq \alpha_k < 1, \quad 0 \leq \delta_k \leq 1, \quad (1 - \alpha_{k-1})\delta_{k-1} \leq (1 - \alpha_k)\delta_k, \\
&\left(\frac{1}{\alpha_{k-1}} - 1\right)(1 - \delta_{k-1}) - \left(2 - \frac{1}{\alpha_k} - \alpha_k\right)\delta_k^2 - \left(\frac{1}{\alpha_k} - \alpha_k\right)\delta_k \geq \varepsilon,
\end{aligned} \tag{1}$$

where $\varepsilon$ is a given sufficiently small positive number.

Fig. 1: The graphs of $h_1$ and $h_2$




## 3 Results

In this section, we propose accelerated KM iteration, and we suggest new and weaker assumptions for analyzing weak convergence.

First of all, we would like to point out that, by our numerical experiments [4], $\alpha_k$ in the KM iteration shall be close to 1 for numerical efficiency in practice and weak convergence in theory. Thus, we give a practical, accelerated KM iteration — Algorithm 3.1.

**Algorithm 3.1** a practical, accelerated KM iteration

1: Choose $x_{-1} = x_0 \in \mathcal{H}$. Choose $\varepsilon = 10^{-9}$ and $\alpha \in [0.80, 0.99]$. Compute $h_1(\alpha)$ via (2), and denote by $\delta^+$. Choose $\delta_{-1} = 0$. Set $k := 0$.

2: Choose $\delta_k \in [\delta_{k-1}, \delta^+]$. Compute

$$x_{k+1} = (1-\alpha)x_k + \alpha T(x_k) + (1-\alpha)\delta_k(x_k - x_{k-1}).$$

Set $k := k+1$.

To provide a better understanding of the practical, accelerated KM iteration, we define $h_1(\alpha)$ and $h_2(\alpha)$ as follows.

$$h_1(\alpha) := 0.5\frac{-1+\sqrt{1+4(\frac{2}{\alpha}+1)(\frac{2}{3\alpha}-\frac{\varepsilon}{1-\alpha})}}{\frac{2}{\alpha}+1}, \tag{2}$$
$$h_2(\alpha) := \frac{1}{1-\alpha}\left(1-\varepsilon-\frac{1}{1+\frac{2}{3}\frac{1-\alpha}{\alpha}}\right),$$

where $\varepsilon = 10^{-9}$, and the graphs of $h_1(\alpha)$ and $h_2(\alpha)$ are plotted in Fig. 1 using MATLAB. Also, it is direct to check that

$$\frac{1}{3} < h_1(\alpha) < \frac{1}{2} < h_2(\alpha), \quad \alpha \in [0.80, 0.99]. \tag{3}$$

Below, we describe the accelerated KM iteration in a general case of $\alpha_k \in [\varepsilon, 1-\varepsilon]$, for a given sufficiently small positive number $\varepsilon$.

For the following accelerated KM iteration

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T(x_k) + (1-\alpha_k)\delta_k(x_k - x_{k-1}), \quad k = 0, 1, ..., \tag{4}$$

with $\alpha_{-1} = \alpha_0$ in $[\varepsilon, 1-\varepsilon]$ and $\delta_{-1} = 0$, we assume that, (i) the sequences $\{\alpha_k\}$ and $\{\delta_k\}$ satisfy

$$\alpha_k \in [\varepsilon, 1-\varepsilon], \quad \delta_k \geq \delta_{k-1}(1-\alpha_{k-1})/(1-\alpha_k); \tag{5}$$

(ii)

$$\begin{aligned} \delta_k^+ &:= 0.5\,\frac{-1+\sqrt{1+4((\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1)((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k}-\frac{\varepsilon}{1-\alpha_k})}}{(\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1}, \\ \delta_k &\leq \min\left\{\delta_k^+, \frac{1}{1-\alpha_k}\left(1-\varepsilon-\frac{1}{1+(1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1}}\right)\right\}, \end{aligned} \tag{6}$$

where $\sigma$ is chosen in $(0,1)$ in advance.

*Unlike (1), the corresponding assumption (6) no longer includes $\delta_{k-1}$. This fully shows that they are widely different.*

Obviously, for this accelerated KM iteration described by (4)-(6), it reduces to Algorithm 3.1 provided that $\alpha_k \equiv \alpha$ and $\sigma = 1/3$.

Notice that, the extra variable $\sigma$ in (6) shall be chosen to maximize the min function defined in the interval $(0,1)$. See Remark 3.2 and Fig. 2 below for more details.

In the analysis of weak convergence for the accelerated KM iteration given by (4)-(6), we make use of the following lemmas to simplify the analysis.

**Lemma 3.1** *Assume that $\alpha > 0$. If $4\alpha\beta \geq \gamma^2$, then*

$$\alpha\|a\|^2 + \beta\|b\|^2 + \gamma\langle a, b\rangle \geq 0, \quad \forall a, b \in \mathcal{H}.$$

**Lemma 3.2** *([9, Sect. 3]) Let $\alpha > 0$, $t \in \mathcal{R}$. If $4\alpha > t^2\beta$, then the following*

$$\langle x, \alpha x\rangle + \langle u, \beta u\rangle - t\langle x, \beta u\rangle \geq \frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2+t^2\beta^2}}{2}\left(\|x\|^2+\|u\|^2\right)$$

*holds for all $x$, $u \in \mathcal{H}$.*

Using these lemmas, we can establish the weak convergence of the accelerated KM iteration.

**Theorem 3.1** *The sequence $\{x_k\}$ generated by Algorithm 3.1 converges weakly to a fixed point of $T$.*

For Theorem 3.1, we decide to omit its proof details here. This is because that (i) it can be viewed as an instance $\sigma = 1/3$ of that of the next theorem and (ii) the statement "Choose $\delta_k \in [\delta_{k-1}, \delta^+]$" in Algorithm 3.1 corresponds to (5) and (6); see Fig. 1 and the desired inequality (3).

**Theorem 3.2** *If the assumptions (5)-(6) hold, then the sequence $\{x_k\}$ generated by (4) converges weakly to a fixed point of $T$.*

*Proof* In view of (4), we have

$$\hat{x}_k = x_k + \delta_k (x_k - x_{k-1}), \tag{7}$$

$$x_{k+1} = (1 - \alpha_k)\hat{x}_k + \alpha_k T(x_k). \tag{8}$$

For any given fixed point $z$ of $T$, i.e., $T(z) = z$, it follows from (8) that

$$x_{k+1} - z = (1 - \alpha_k)(\hat{x}_k - z) + \alpha_k (Tx_k - Tz).$$

Since $T$ is non-expansive, we have

$$\begin{aligned} \|x_{k+1} - z\|^2 &= \alpha_k \|Tx_k - Tz\|^2 + (1 - \alpha_k)\|\hat{x}_k - z\|^2 - \alpha_k(1 - \alpha_k)\|Tx_k - \hat{x}_k\|^2 \\ &\le \alpha_k \|x_k - z\|^2 + (1 - \alpha_k)\|\hat{x}_k - z\|^2 - \alpha_k(1 - \alpha_k)\|Tx_k - \hat{x}_k\|^2. \end{aligned}$$

From (8) and (7), we have

$$\alpha_k (Tx_k - \hat{x}_k) = x_{k+1} - \hat{x}_k = x_{k+1} - x_k - \delta_k(x_k - x_{k-1}),$$

so, we get

$$\begin{aligned} &\alpha_k^2 \|Tx_k - \hat{x}_k\|^2 \\ &= \|x_{k+1} - x_k\|^2 + \delta_k^2\|x_k - x_{k-1}\|^2 - 2\delta_k \langle x_{k+1} - x_k, x_k - x_{k-1}\rangle. \end{aligned}$$

Observe that

$$\begin{aligned} \|\hat{x}_k - z\|^2 &= \|(1 + \delta_k)(x_k - z) - \delta_k(x_{k-1} - z)\|^2 \\ &= (1 + \delta_k)\|x_k - z\|^2 - \delta_k\|x_{k-1} - z\|^2 + \delta_k(1 + \delta_k)\|x_k - x_{k-1}\|^2. \end{aligned}$$

Thus, we further get

$$\begin{aligned} \|x_{k+1} - z\|^2 \le{}& (1 + (1 - \alpha_k)\delta_k)\|x_k - z\|^2 - (1 - \alpha_k)\delta_k\|x_{k-1} - z\|^2 \\ &- \frac{1 - \alpha_k}{\alpha_k}\|x_{k+1} - x_k\|^2 + 2\frac{1 - \alpha_k}{\alpha_k}\delta_k\langle x_{k+1} - x_k, x_k - x_{k-1}\rangle \\ &+ \left((1 - \alpha_k)\delta_k(1 + \delta_k) - \frac{1 - \alpha_k}{\alpha_k}\delta_k^2\right)\|x_k - x_{k-1}\|^2. \end{aligned}$$

From the assumption (5), we have

$$\begin{aligned}&\|x_{k+1}-z\|^2-(1-\alpha_{k+1})\,\delta_{k+1}\|x_k-z\|^2+(1-\sigma)\frac{1-\alpha_k}{\alpha_k}\|x_{k+1}-x_k\|^2\\ \le\ &\|x_k-z\|^2-(1-\alpha_k)\,\delta_k\|x_{k-1}-z\|^2+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k-x_{k-1}\|^2-\Delta_k,\end{aligned} \tag{9}$$

where $\sigma\in(0,1)$ and $\Delta_k$ is given by

$$\begin{aligned}\Delta_k:=\ &\sigma\frac{1-\alpha_k}{\alpha_k}\|x_{k+1}-x_k\|^2-2\frac{1-\alpha_k}{\alpha_k}\delta_k\langle x_{k+1}-x_k,x_k-x_{k-1}\rangle\\ &+\left((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}-(1-\alpha_k)\delta_k(1+\delta_k)+\frac{1-\alpha_k}{\alpha_k}\delta_k^2\right)\|x_k-x_{k-1}\|^2.\end{aligned}$$

Set

$$\varphi_k:=\|x_k-z\|^2-(1-\alpha_k)\,\delta_k\|x_{k-1}-z\|^2+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k-x_{k-1}\|^2.$$

Then

$$\varphi_{k+1}\le\varphi_k-\Delta_k. \tag{10}$$

Consider

$$\begin{aligned}\varphi_k:=\ &\|x_k-z\|^2-(1-\alpha_k)\,\delta_k\|x_{k-1}-z\|^2+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k-x_{k-1}\|^2\\ =\ &\|x_{k-1}-z\|^2+2\langle x_{k-1}-z,x_k-x_{k-1}\rangle+\|x_k-x_{k-1}\|^2\\ &-(1-\alpha_k)\,\delta_k\|x_{k-1}-z\|^2+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\|x_k-x_{k-1}\|^2\\ =\ &(1-(1-\alpha_k)\,\delta_k)\,\|x_{k-1}-z\|^2+2\langle x_{k-1}-z,x_k-x_{k-1}\rangle\\ &+\left(1+(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\right)\|x_k-x_{k-1}\|^2.\end{aligned}$$

Combining this with Lemma 3.1 and the assumption (6)

$$\begin{aligned}&\delta_k\le\frac{1}{1-\alpha_k}\left(1-\varepsilon-\frac{1}{1+(1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1}}\right),\\ \Leftrightarrow\quad&(1-\varepsilon-(1-\alpha_k)\,\delta_k)\,(1+(1-\sigma)(1-\alpha_{k-1})/\alpha_{k-1})\ge 1\end{aligned}$$

yields

$$\varphi_k\ge\varepsilon\,\|x_{k-1}-z\|^2.$$

Similarly, by Lemma 3.1 and the assumption (6)

$$\begin{aligned}
&\delta_k \le 0.5\,\frac{-1+\sqrt{1+4((\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1)((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k}-\frac{\varepsilon}{1-\alpha_k})}}{(\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1}\\
\Leftrightarrow\quad &\left((\frac{1}{\sigma}-1)\frac{1}{\alpha_k}+1\right)\delta_k^2+\delta_k-(1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}\frac{1}{1-\alpha_k}+\frac{\varepsilon}{1-\alpha_k}\le 0\\
\Leftrightarrow\quad &\sigma\frac{1-\alpha_k}{\alpha_k}\left((1-\sigma)\frac{1-\alpha_{k-1}}{\alpha_{k-1}}-(1-\alpha_k)\delta_k(1+\delta_k)+\frac{1-\alpha_k}{\alpha_k}\delta_k^2-\varepsilon\right)\\
&\ge \frac{(1-\alpha_k)^2}{\alpha_k^2}\delta_k^2,
\end{aligned}$$

we can get

$$\Delta_k \ge \varepsilon\,\|x_k-x_{k-1}\|^2.$$

Obviously, from these two relations and (10), we conclude that

$$\begin{aligned}
&\lim \varphi_k \text{ exists} &&\Rightarrow\quad \|x_{k-1}-z\|\ (\text{thus } \|x_k-z\|) \text{ is bounded in norm;}\\
&\lim \Delta_k = 0 &&\Rightarrow\quad \lim \|x_k-x_{k-1}\| = 0.
\end{aligned}$$

From (5) and

$$(I-T)(x_k)=\frac{(1-\alpha_k)\delta_k(x_k-x_{k-1})-(x_{k+1}-x_k)}{\alpha_k},$$

it is not difficult to follow [10, Theorem 3.1] to complete the proof. □

*Remark 3.1* Next, we numerically demonstrate the assumption (6) to some extent. For brevity, we simply set $\alpha_k \equiv \alpha$, $\delta_k \equiv \delta$. Then the assumption (6) above reduces to

$$\begin{aligned}
&\delta^+ := 0.5\,\frac{-1+\sqrt{1+4((\frac{1}{\sigma}-1)\frac{1}{\alpha}+1)(1-\sigma)\frac{1}{\alpha}}}{(\frac{1}{\sigma}-1)\frac{1}{\alpha}+1},\\
&\delta < \min\left\{\delta^+,\ \frac{1}{1-\alpha}\left(1-\frac{1}{1+(1-\sigma)(1-\alpha)/\alpha}\right)\right\} := f(\sigma). \qquad (11)
\end{aligned}$$

Be aware that, in contrast to (6), we no longer introduce the extra $\varepsilon$ above because we turn to resort to Lemma 3.2. In addition, we have replaced $\le$ there by $<$ here.

Numerical demonstration of (11) is given in Table 1, where $\delta_{\text{new}}$ stands for a slightly lower approximation of the maximum of $f$ in (11) with respect to $\sigma$. We also provide the values from [4, Table 1] for comparison.

From Tables 1 and 2, we can observe that our computed values of $\delta_{\text{new}}$ are consistently larger than the corresponding values from [4, Table 1] for each sampling point.

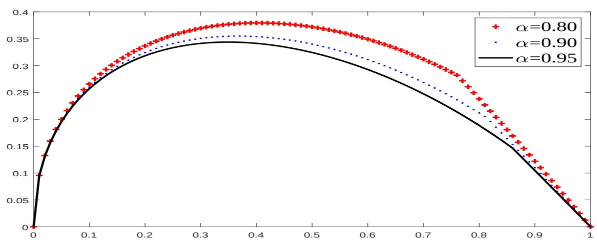
Table 1: Numerical demonstration of (11) with respect to $\sigma$

| $\alpha$ | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|
| $\delta_{\text{new}}$ | 0.4397 | 0.4230 | 0.4075 | 0.3930 | 0.3795 | 0.3668 | 0.3549 | 0.3437 | 0.3353 |
| $\sigma$ | 0.49 | 0.46 | 0.45 | 0.42 | 0.40 | 0.38 | 0.36 | 0.34 | 0.33 |
| $\alpha$ | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 | 0.55 |
| $\delta_{\text{new}}$ | 0.6394 | 0.6389 | 0.6038 | 0.5730 | 0.5455 | 0.5206 | 0.4978 | 0.4769 | 0.4575 |
| $\sigma$ | 0.75 | 0.70 | 0.66 | 0.63 | 0.61 | 0.58 | 0.56 | 0.54 | 0.50 |

Table 2: Numerical demonstration of [4, Table 1]

| $\alpha$ | 0.60 | 0.65 | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
|---|---|---|---|---|---|---|---|---|---|
| $\delta_{-}$ | 0.4105 | 0.3983 | 0.3870 | 0.3765 | 0.3668 | 0.3576 | 0.3490 | 0.3410 | 0.3348 |
| $\alpha$ | 0.15 | 0.20 | 0.25 | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 | 0.55 |
| $\delta_{-}$ | 0.6143 | 0.5746 | 0.5426 | 0.5157 | 0.4927 | 0.4725 | 0.4545 | 0.4384 | 0.4239 |

Fig. 2: The graph of f($\sigma$) for different $\alpha$




*Remark 3.2* For the KM iteration, choosing $\alpha$ close to 1 in its accelerated and inertial versions [4, 10] is generally a good strategy. In this case, it is noted that selecting $\sigma$ to be equal to or close to $1/3$ has been found to be a favorable choice; see Table 1 and Fig. 2.

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**Declarations**

**Conflict of interest** Authors declare that they have no conflict of interest.

## Data Availability

Has no associated data.

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