Masterpiece

Dong' s criteria

Dong' s criteria is a step length criteria that contains only gradient information

$$c_2\langle\nabla f(x), d\rangle \le \langle\nabla f(x+\alpha d), d\rangle \le c_1\langle\nabla f(x), d\rangle,$$

where $0 < c_1 < c_2 < 1$.

In contrast to the Wolfe conditions, Dong' s criteria rely solely on gradient evaluations. This is beneficial, as evaluating f requires either the primal or dual condensed incremental potential, cf. Table 1, which is generally not available in FFT-based homogenization.

New properties of forward–backward splitting and a practical proximal-descent algorithm

Lemma Let $F : \mathcal{H} \to 2^{\mathcal{H}}$ be continuous monotone in $\mathcal{H}$, and $B : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal monotone on dom$B$. Then for any given $x \in$ dom$B$ and $\alpha \geqslant \alpha' > 0$, the following hold

$$\|(I+\alpha B)^{-1}(I-\alpha F)(x) - x\| \geqslant \left\|(I+\alpha' B)^{-1}(I-\alpha' F)(x) - x\right\|,$$

$$\frac{\|(I+\alpha B)^{-1}(I-\alpha F)(x) - x\|}{\alpha} \leqslant \frac{\|(I+\alpha' B)^{-1}(I-\alpha' F)(x) - x\|}{\alpha'},$$

$$\liminf_{\alpha\to 0} \frac{\|(I+\alpha B)^{-1}(I-\alpha F)(x) - x\|}{\alpha} = \min\{\|w\| : w \in B(x) + F(x)\}.$$

Moreover, if $B + F$ is maximal monotone on dom$B$, then the minimum on the right-hand side must be uniquely attainable.

Algorithm 1 A Proximal-Descent Algorithm

Step 0. Choose $x^0 \in \mathcal{X}, \alpha_{-1} = 1, \beta, \rho \in (0,1), \theta \in (0,2)$. Set $k := 0$.

Step 1. (*step length*) Choose $\tilde{\alpha}_{k-1}$ by either of the following two ways:

$$\tilde{\alpha}_{k-1} = \alpha_{k-1}, \quad \text{if } F \text{ is strongly monotone; otherwise,}$$

$$\tilde{\alpha}_{k-1} = \alpha_{k-1}, \quad k \leq K, \quad \tilde{\alpha}_{k-1} = 2\alpha_{K-1}, k > K,$$

where K is a prescribed natural number, and determine the step length α_k to be the largest

$$\alpha \in \left\{ \tilde{\alpha}_{k-1}\beta^{-1}, \alpha_{k-1}\beta^i, i = 0, 1, \ldots \right\}, \quad x^k(\alpha) = (I + \alpha B)^{-1}(I - \alpha F)(x^k),$$

such that

$$\alpha \langle x^k - x^k(\alpha), F(x^k) - F(x^k(\alpha)) \rangle \leq (1 - \rho) \| x^k - x^k(\alpha) \|^2.$$

Step 2. (*Proximal step*) Use the step length α_k just determined in Step 1, and compute

$$x^k(\alpha_k) = (I + \alpha_k B)^{-1}(I - \alpha_k F)(x^k).$$

If $x^k(\alpha_k) = x^k$, then stop. Otherwise go to Step 3.

Step 3. (*Descent step*) First compute the relaxation factor γ_k through the following formula

$$\gamma_k = \frac{\theta \rho \| x^k - x^k(\alpha_k) \|^2}{\| x^k - x^k(\alpha_k) - \alpha_k F(x^k) + \alpha_k F(x^k(\alpha_k)) \|^2}.$$

Then the new iterate is given by

$$x^{k+1} = P_{\mathcal{X}} \left[x^k - \gamma_k (x^k - x^k(\alpha_k) - \alpha_k F(x^k) + \alpha_k F(x^k(\alpha_k)) \right].$$

Set $k := k + 1$, and go to Step 1.

The Proximal Point Algorithm Revisited

Lemma Let $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ be sequences of positive numbers. Assume that they satisfy

$$\alpha_{k+1}^2 \leq \alpha_k^2 - \beta_k \gamma_k, \quad k = 0, 1, \ldots,$$

the sequence $\{\beta_k\}$ is nonsummable and the sequence $\{\gamma_k\}$ is decreasing. Then there exists ε_k such that

$$\gamma_k \sum_{i=0}^{k} \beta_i \leq 2\alpha_0 \varepsilon_k,$$

$$\alpha_k \le \varepsilon_k \le \alpha_0, \quad \lim_{k\to+\infty} \varepsilon_k = \lim_{k\to+\infty} \alpha_k.$$

Theorem Let $\{(x^k, \lambda_k)\}$ be the corresponding iterate-parameter sequence in the proximal point algorithm. Assume that $\lim_{k\to+\infty} \sum_{i=0}^{k-1} \lambda_i^2 = +\infty$. Then

$$\|\lambda_k^{-1}(x^k - x^{k+1})|^2 \sum_{i=0}^{k} \lambda_i^2 \le 2|x^0 - A^{-1}(0)|\varepsilon_k,$$

$$|x^k - A^{-1}(0)| \le \varepsilon_k \le |x^0 - A^{-1}(0)|, \quad \lim_{k\to+\infty} \varepsilon_k = \lim_{k\to+\infty} |x^k - A^{-1}(0)|.$$

As a consequence, the following relation:

$$\lim_{k\to+\infty} \sup |A(x^k) - 0|^2 \sum_{i=0}^{k-1} \lambda_i^2 \le 2|x^0 - A^{-1}(0)| \lim_{k\to+\infty} |x^k - A^{-1}(0)|$$

holds as well. In particular, if the sequence of the regularization parameters has a positive lower bound, then the proximal point algorithm has the following estimate of convergence rate in the context of convex minimization:

$$f(x^k) - \min f(x) = o\left(\frac{1}{k}\right).$$

And in the context of the monotone inclusion, we have

$$\left|A\left(x^k\right) - 0\right|^2 = o\left(\frac{1}{k}\right).$$

Anextension of Luques growth condition

Theorem Consider any maximal monotone mappings $B, F : H \to H$ such that F is Lipschitz continuous. Let $T := B + F$. Then the following statements are equivalent.

(a) There exist $\epsilon' > 0, \tau' > 0$ such that

$$\forall y \in \bar{B}(0, \epsilon'), \forall x \in T^{-1}y \quad |x - Z| \le \tau' \|y\|.$$

(b) There exist $\epsilon > 0, \tau > 0$ such that

$$\forall x \in Z_\alpha(\epsilon) \quad |x - Z| \le \tau \|\pi_\alpha(x)\|$$

where $Z_\alpha(\epsilon) := \{x : \|\pi_\alpha(x)\| \le \epsilon\}$.

An LS-free splitting method for composite mappings

This splitting method is an extension of projection type methods for monotone variational inequalities. For any given starting point $x^0 \in \mathcal{H}$, its recursive formulae are given by

$$\begin{aligned} (I+B)(y^k) &\ni (I-F)(x^k), \\ x^{k+1} &= x^k - \gamma_k (I+F^*)(x^k - y^k), \end{aligned}$$

where $\gamma_k := \frac{\theta \|x^k - y^k\|^2}{\|(I+F^*)(x^k-y^k)\|^2}$ and $\theta \in (0,2)$.