董云达的代表作系列一

董氏引理 设 $\{\alpha_k\},\{\beta_k\},\{\gamma_k\}$ 是三个正序列。假设它们满足：

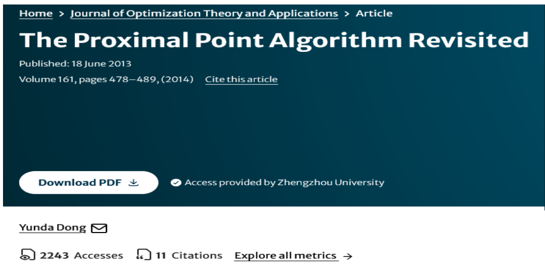
$$\alpha_{k+1}^2 \leq \alpha_k^2 - \beta_k\gamma_k, \quad k=0,1,...,$$

且序列 $\{\beta_k\}$ 是不可和的，序列 $\{\gamma_k\}$ 单调递减。那么，存在序列 $\{\varepsilon_k\}$ 使得下述关系成立：

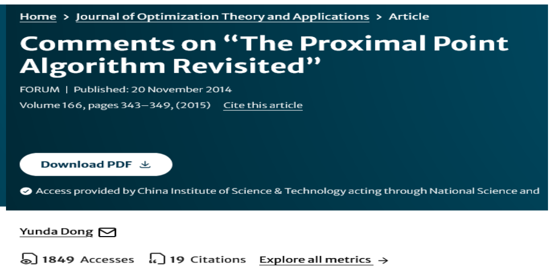
$$\gamma_k \textstyle\sum_{i=0}^k \beta_i \leq 2\alpha_0\varepsilon_k,$$

$$\alpha_k \leq \varepsilon_k \leq \alpha_0, \qquad \lim_{k\to+\infty} \varepsilon_k = \lim_{k\to+\infty} \alpha_k.$$

Brezis 与菲尔兹奖获得者 Lions 证明邻近点方法收敛率与 $1/k$ 的同阶。董氏引理进一步将其提升至高阶无穷小。截至2024年末，该成果已被下载超4000次!



(a)



(b)

值得指出的是，利用董氏引理可以证明 Abel-Dini 定理。

董云达的代表作系列二

假设 f 在欧氏空间中是二次连续可微的凸函数。考虑无约束极小化问题 min $f(x)$. 相应的拟 Newton 法为

$$x^{k+1} = x^k - \alpha_k W_k \nabla f(x^k),$$

其中 W_k 由某个对称秩二校正公式自动生成，而 α_k 作为步长由下面 Wolfe 条件确定

$$f(x+\alpha d) \le f(x) + c_1 \alpha \nabla f(x)^T d,$$
$$\nabla f(x+\alpha d)^T d \ge c_2 \nabla f(x)^T d,$$

其中 $0 < c_1 < c_2 < 1$.

在 2007-2010 年间，董云达提出了董氏条件

$$c_2 \nabla f(x)^T d \le \nabla f(x+\alpha d)^T d \le c_1 \nabla f(x)^T d, \quad 0 < c_1 < c_2 < 1.$$

对于董氏条件，2020年德国卡尔斯鲁厄理工学院（德国版MIT）的学者在计算力学顶刊 Comput Method Appl M 上予以好评：“与 Wolfe 条件形成鲜明对比的是，董氏条件仅仅依赖梯度的估值...”

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iteration reads

$$x^{k+1} = x^k - a^k \mathcal{H}^{-1}(x^k) \nabla f(x^k)$$

with the Hessian $\mathcal{H}$ of f and a damping factor $a^k \in (0, 1]$, cf. Chapter 9 in Boyd's book [60]. Instead of inverting $\mathcal{H}(x^k)$, the update can be performed by

$$x^{k+1} = x^k + a^k \Delta x$$

where Δx is an approximate solution of

$$\mathcal{H}(x^k)\Delta x = -\nabla f(x^k). \tag{5.10}$$

The damping factor a^k is determined by a back-tracking procedure. In this paper, we use the stopping criteria of Dong [63]

$$c_2 \langle \nabla f(x^k), \Delta x \rangle_V \le \langle \nabla f(x^k + a^k \Delta x), \Delta x \rangle_V \le c_1 \langle \nabla f(x^k), \Delta x \rangle_V$$

with $0 < c_1 < c_2 < 1$. In contrast to the Wolfe conditions [64], Dong's criteria rely solely on gradient evaluations. This is beneficial, as evaluating f requires either the primal or dual condensed incremental potential, cf. Table 1, which is generally not available in FFT-based homogenization. Both w and w^* carry no physical meaning as they

董云达的代表作系列三

在 2003 年博士论文的第四章，董云达在Hilbert空间中考虑了下面的问题

$$0 \in Mx + q + B(x),$$

其中 M 是有界、线性和单调的，q 是一个向量，B 是极大单调的，并且提出了下面的方法

CHAPTER 4. NEW SPLITTING METHODS 29

Algorithm 4.2.1. *Let $F(x) = Mx + q$. Choose any starting point $x^0 \in \mathcal{H}$. For any given $k \geq 0$, computing x^{k+1} from x^k according to*

$$x^{k+1} = x^k - \gamma_k (I + M^T) r(x^k) \quad \forall k \geq 0. \tag{4.9}$$

where

$$r(x) = x - (I+B)^{-1}(x - Mx - q),$$
$$\gamma_k = \|(I + M^T) r(x^k)\|^{-2} \|r(x^k)\|^2. \tag{4.10}$$

更多细节参考 https://ydong2024.github.io/downloads/dongthesis.pdf

该结果发表在 Appl Math Letters (2005)，2024年获排名第一的数学项刊Acta Numerica的好评。

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sequence in $[\varepsilon, 2-\varepsilon]$, let $x_0 \in \mathcal{H}$, and let $y_0^* \in \mathcal{G}$. Iterate

$$\begin{array}{l}
\text{for } n = 0, 1, \ldots \\
\left\lfloor \begin{array}{l}
a_n = J_{\gamma_n A}(x_n - \gamma_n L^* y_n^*) \\
l_n = L x_n \\
b_n = J_{\sigma_n B}(l_n + \sigma_n y_n^*) \\
t_n = b_n - L a_n \\
t_n^* = \gamma_n^{-1}(x_n - a_n) + \sigma_n^{-1} L^*(l_n - b_n) \\
\tau_n = \|t_n\|^2 + \|t_n^*\|^2 \\
\text{if } \tau_n > 0 \\
\lfloor\ \theta_n = \lambda_n(\gamma_n^{-1}\|x_n - a_n\|^2 + \sigma_n^{-1}\|l_n - b_n\|^2)/\tau_n \\
\text{else } \theta_n = 0 \\
x_{n+1} = x_n - \theta_n t_n^* \\
y_{n+1}^* = y_n^* - \theta_n t_n.
\end{array}\right.
\end{array} \tag{9.11}$$

Then $(x_n)_{n\in\mathbb{N}}$ converges weakly to a point $x \in Z$ and $(y_n^*)_{n\in\mathbb{N}}$ converges weakly to a point $y^* \in Z^*$.

Remark 9.2. Here are notable instantiations of Proposition 9.1.

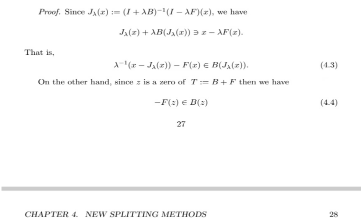
(i) The first instance of (9.11) in the literature seems to be that of Dong (2005), where $\mathcal{H}$ and $\mathcal{G}$ are Euclidean spaces, $A = 0$, and $(\forall n \in \mathbb{N})\ \gamma_n = \sigma_n = 1$ and $\lambda_n = \lambda \in]0, 2[$. Convergence of the primal sequence $(x_n)_{n\in\mathbb{N}}$ was established by different means.

董云达的代表作系列四

在 2003 年博士论文的第四章，董云达在Hilbert空间中考虑了下面的问题

$$0 \in F(x) + B(x),$$

其中 F 是连续单调的，B 是极大单调的。利用图(a)中的记号 $J_\lambda(x)$, 提出了下面的方法



Proof. Since $J_\lambda(x) := (I+\lambda B)^{-1}(I-\lambda F)(x)$, we have

$$J_\lambda(x) + \lambda B(J_\lambda(x)) \ni x - \lambda F(x).$$

That is,

$$\lambda^{-1}(x - J_\lambda(x)) - F(x) \in B(J_\lambda(x)). \tag{4.3}$$

On the other hand, since z is a zero of $T := B + F$ then we have

$$-F(z) \in B(z) \tag{4.4}$$

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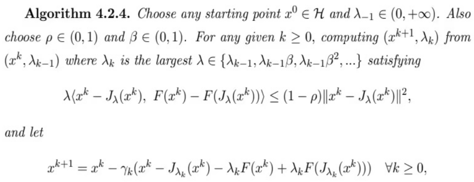
(a)



Algorithm 4.2.4. *Choose any starting point $x^0 \in \mathcal{H}$ and $\lambda_{-1} \in (0,+\infty)$. Also choose $\rho \in (0,1)$ and $\beta \in (0,1)$. For any given $k \geq 0$, computing (x^{k+1}, λ_k) from (x^k, λ_{k-1}) where λ_k is the largest $\lambda \in \{\lambda_{k-1}, \lambda_{k-1}\beta, \lambda_{k-1}\beta^2, ...\}$ satisfying*

$$\lambda\langle x^k - J_\lambda(x^k),\ F(x^k) - F(J_\lambda(x^k))\rangle \leq (1-\rho)\|x^k - J_\lambda(x^k)\|^2,$$

and let

$$x^{k+1} = x^k - \gamma_k(x^k - J_{\lambda_k}(x^k) - \lambda_k F(x^k) + \lambda_k F(J_{\lambda_k}(x^k))) \quad \forall k \geq 0,$$

(b)

更多细节参考 https://ydong2024.github.io/downloads/dongthesis.pdf

利用董云达论文 [Math Comput Simulat, 223, 86-107, 2024] 中的记号，我们将其重新叙述/摘录如下：

Choose $x^0 \in \mathrm{dom}F \cap \mathrm{dom}A$. Choose $\rho \in (0,1)$ and $\alpha_{-1} > 0$. At k-th iteration, find the smallest j_k in $\alpha = \alpha_{k-1}\, 0.5^j$, $j = 0, 1, ...$, such that $x^k(\alpha) = (I + \alpha A)^{-1}(x^k - \alpha F(x^k))$ satisfies

$$\alpha\langle x^k - x^k(\alpha),\, F(x^k) - F(x^k(\alpha))\rangle \leq (1-\rho)\|x^k - x^k(\alpha)\|^2.$$

Take $\alpha_k = \alpha_{k-1}\, 0.5^{j_k}$, and compute $\bar{x}^k = x^k(\alpha_k)$. Compute in order

$$d^k = x^k - \bar{x}^k - \alpha_k(F(x^k) - F(\bar{x}^k)), \quad \gamma_k = \langle x^k - \bar{x}^k,\, d^k\rangle / \|d^k\|^2.$$

Then choose $\theta_k \in (0, 2]$ and compute

$$x^{k+1} = x^k - \theta_k \gamma_k d^k.$$

由于不需要估计 Lipschitz 常数以及克服了 Tseng 方法的小步长现象，从而成为求解上述问题的最实用的基本方法。2014年董云达与学生合作，通过具体例子，证实了它在数值方面的明显优势。2024年这篇论文，还讨论了 F 仅为一致连续的情形，并给出方法弱收敛性的一个严格证明。

至于凸极小化问题 min $f(x) + g(x)$，我们考虑考虑其最优性条件

$$0 \in \nabla f(x) + \partial g(x),$$

并且取 $F := \nabla f$ 以及 $B := \partial g$ 即可。

董云达的代表作系列五

董云达在论文 [Math Comput Simulat, 223, 86-107, 2024] 中考虑了下面的问题

$$0 \in F(x) + A(x) + Q^* B(Qx - q),$$

其中 F 是连续单调的，A, B 是极大单调的，Q 是线性的而 Q^* 是它的伴随，q 是一个向量，并且提出了一个新的实用算法

In the case of F_i being Lipschitz continuous, we give

Algorithm 2.1. Our proposed splitting algorithm in Lipschitz continuity case

Step 0. For $i = 1, \dots, n$, choose $x_i^0 \in \mathcal{H}_i$, $u^0 \in \mathcal{G}$. Choose $\alpha_{i,-1} > 0$, $\rho \in (0,1)$. Set $k := 0$.

Step 1. For $i = 1, \dots, l$, choose appropriate $\alpha_{i,k} > 0$. For $i = l+1, \dots, n$, find the smallest j_k in (19) such that

$$x_i^k(\alpha_i) = (I + \alpha_i A_i)^{-1}(x_i^k - \alpha_i(F_i(x_i^k) + Q_i^* u^k))$$

satisfies

$$\alpha_i \langle x_i^k - x_i^k(\alpha_i), F_i(x_i^k) - F_i(x_i^k(\alpha_i))\rangle \le (1-\rho)\|x_i^k - x_i^k(\alpha_i)\|^2. \tag{20}$$

Take $\alpha_{i,k} = \alpha_{i,k-1} t^{j_k}$, and compute

$$\bar{x}_i^k = x_i^k(\alpha_{i,k}). \tag{21}$$

Choose $\beta_k > 0$ via (29) below and find $\bar{u}^k$ such that

$$\left(\beta_k I + B^{-1}\right)(\bar{u}^k) \ni \beta_k u^k + \sum_{i=1}^n Q_i \bar{x}_i^k - q. \tag{22}$$

If $\bar{x}_i^k = x_i^k$, $i = 1, \dots, n$, and $\bar{u}^k = u^k$, then stop. Otherwise go to Step 2.

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Step 2. For $i = 1, \dots, n$, compute

$$d_{x_i}^k = \alpha_{i,k}^{-1}(x_i^k - \bar{x}_i^k) - (F_i(x_i^k) - F_i(\bar{x}_i^k)) - Q_i^*(u^k - \bar{u}^k), \tag{23}$$

$$d_u^k = \beta_k(u^k - \bar{u}^k), \tag{24}$$

$$\gamma_k = \frac{\sum_{i=1}^n \langle x_i^k - \bar{x}_i^k, d_{x_i}^k\rangle + \langle u^k - \bar{u}^k, d_u^k\rangle}{\sum_{i=1}^n \|d_{x_i}^k\|^2 + \|d_u^k\|^2}. \tag{25}$$

Then choose $\theta_k \in (0,2]$ and the new iterates are given by

$$x_i^{k+1} = x_i^k - \theta_k \gamma_k d_{x_i}^k, \ i = 1, \dots, n, \tag{26}$$

$$u^{k+1} = u^k - \theta_k \gamma_k d_u^k. \tag{27}$$

Set $k := k+1$, and go to Step 1.

它是“董云达的代表作系列四”中方法的一个拓展，尤其适用于高光谱解混、深度学习中的特征选取等前沿问题。更多细节参考

https://ydong2024.github.io/downloads/journal2024.pdf