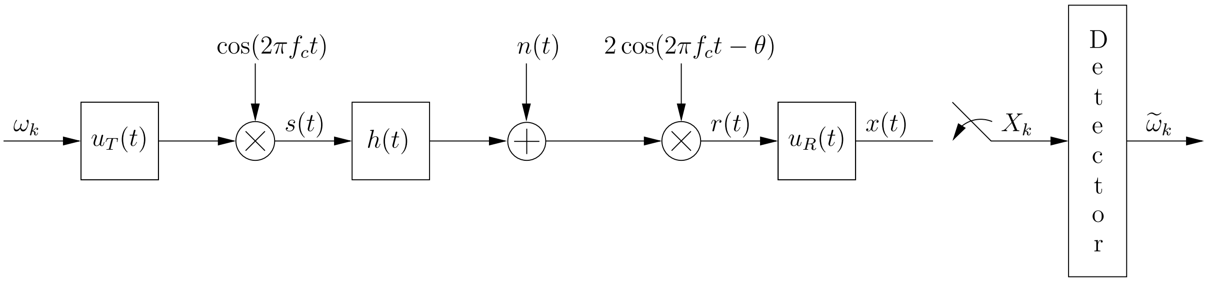

Figure 1: Narrowband band-pass PAM system model.

1 System model

A general narrowband band-pass PAM system model is shown in Figure 1. The information source generates statistically independent source symbols, ω_k, at a rate of $R = 1/T$ symbols/s. Considering antipodal signalling formats with equidistant adjacent source symbols, ω_k belong to the set

$$\omega_k \in \{2i - 1 - L \;:\; i \leq L,\; i \in \mathbb{Z}^+\}, \tag{1}$$

where L denotes the number of possible symbol amplitudes and $\mathbb{Z}^+$ denotes the set of all positive integers. The source symbols are passed through the pulse-shaping filter with impulse response $u_T(t)$ (and frequency response $U_T(\omega)$) and modulated on a carrier frequency, f_c, much larger than the signal bandwidth. The resulting narrowband band-pass transmitted signal, $s(t)$, is given by

$$s(t) = \sum_{k=-\infty}^{\infty} s_k(t - kT), \tag{2}$$

where

$$s_k(t) = \omega_k u_T(t) \cos(2\pi f_c t). \tag{3}$$

The signal is transmitted through a narrowband band-pass channel having impulse response

$$h(t) = 2h_l(t) \cos(2\pi f_c t), \tag{4}$$

where $h_l(t)$ is the impulse response of the equivalent low-pass channel, and subjected to zero-mean AWGN, $n(t)$, of double-sided PSD $\frac{1}{2}N_0$ W/Hz, where

$$n(t) = n_l(t) \cos(2\pi f_c t) \tag{5}$$

and $n_l(t)$ is the envelope of the AWGN. The average SNR per information bit, E_b/N_0, is given by

$$\frac{E_b}{N_0} = \frac{\mathrm{E}[\,\omega_k^2\,]\varepsilon_T}{2k_b N_0}, \tag{6}$$

where $\mathrm{E}[\,\bullet\,]$ denotes the expected value operator, $k_b = \log_2(L)$ is the number of binary digits required to represent the source symbol ω_k and ε_T is the energy in the transmitter filter's impulse response signal, which, assuming a 1 Ω reference resistor, is given by

$$\varepsilon_T = \frac{1}{2\pi} \int_{-\infty}^{\infty} |U_T(\omega)|^2 \, d\omega. \tag{7}$$

For equiprobable antipodal signalling formats with equidistant adjacent source symbols, the source symbol mean square magnitude, $\mathrm{E}[\,\omega_k^2\,]$, is given by

$$\begin{aligned} \mathrm{E}[\,\omega_k^2\,] &= \frac{2}{L} \sum_{i=1}^{L/2} (2i - 1)^2 \\ &= \frac{1}{3}(L^2 - 1). \end{aligned} \tag{8}$$

The received signal is demodulated subject to a carrier phase offset, θ, between the receiver and transmitter. The received signal after demodulation, $r(t)$, is given by

$$r(t) = \sum_{k=-\infty}^{\infty} r_k(t - kT) + n_l(t)\Big(\cos(\theta) + \cos(4\pi f_c t - \theta)\Big), \tag{9}$$

where

$$r_k(t) = \Big(\omega_k\big((u_T * h_l)(t)\big)\Big)\Big(\cos(\theta) + \cos(4\pi f_c t - \theta)\Big). \tag{10}$$

The bandwidth of the receiver filter with impulse response $u_R(t)$ (and frequency response $U_R(\omega)$) is much smaller than the carrier frequency, f_c. Therefore, the high frequency components present in the received demodulated signal are sufficiently attenuated by the receiver filter so as to render them negligible. Hence, the output of the receiver filter, $x(t)$, may be written as

$$x(t) = \sum_{k=-\infty}^{\infty} \omega_k g(t - kT) + \nu(t), \tag{11}$$

where

$$g(t) = \big((u_T * h_l * u_R)(t)\big)\cos(\theta) \tag{12}$$

and

$$\nu(t) = \big((n_l * u_R)(t)\big)\cos(\theta). \tag{13}$$

The receiver filter output, $x(t)$, is sampled every T seconds, at times $t_k = (\Delta + k)T$, where Δ denotes the normalised timing offset between the receiver and transmitter. Without loss of generality, we shall consider the detection of symbol ω_0. Assuming ideal sampling (i.e. no quantisation effects), the decision variate at the detector at time t_0, X, is given by[1]

$$X = \omega_0 g_0 + \sum_{k=1}^{\infty}\big(\omega_{-k} g_{-k} + \omega_k g_k\big) + \nu, \tag{14}$$

where $g_k = g((\Delta - k)T)$ and ν is a zero-mean Gaussian random variable with variance $\sigma_\nu^2 = N_0 \varepsilon_R$, where ε_R is the energy in the receiver filter's impulse response signal, which, in turn, is given by

$$\varepsilon_R = \frac{1}{2\pi}\int_{-\infty}^{\infty} |U_R(\omega)|^2 \, d\omega. \tag{15}$$

The first term in equation (14), $\omega_0 g_0$, represents the desired signal (subject to timing and carrier phase offsets and transmission channel effects), the second term,

$$\sum_{k=1}^{\infty}\big(\omega_{-k} g_{-k} + \omega_k g_k\big),$$

represents the ISI (both pre and post-cursor i.e. ISI arising from symbols transmitted before and after ω_0, respectively) and the third term, ν, represents the AWGN component present in the decision statistic. A symbol is correctly detected at the receiver if the decision variate, X, lies within the symbol's corresponding decision region in the receiver signal constellation. Without loss of generality, we take the receiver signal constellation to be determined in the absence of any information regarding the transmission channel or receiver synchronisation errors. Therefore, the decision region boundaries at the detector are given by

$$-(L-2)\zeta_0, \ldots, -2\zeta_0, 0, 2\zeta_0, \ldots, (L-2)\zeta_0 \ ,$$

[1] In order to simplify the notation, subscripts are omitted from the decision variate and AWGN components, i.e. $X \equiv X_0 = x(\Delta T)$ and $\nu \equiv \nu_0 = \nu(\Delta T)$.

where ζ_0 is the overall system impulse response, $g(t)$, under ideal transmission conditions (i.e. $\theta = \Delta = 0$ and $h(t) = \delta(t)$, where $\delta(t)$ denotes the Dirac delta function), evaluated at $t = 0$, i.e.

$$\begin{aligned} \zeta_0 &= (u_T * u_R)(0) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} U_T(\omega) U_R(\omega)\, d\omega. \end{aligned} \tag{16}$$

2 Gram-Charlier series approximation for $f_X(y)$

The decision variate pdf may be approximated via a Gram-Charlier series according to

$$f_X(y) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\Big(-\frac{(y-\omega_0 g_0)^2}{2\sigma_X^2}\Big)\Big(1 + \sum_{m=2}^{\infty} \frac{\alpha_{2m}}{(2m)!\sigma_X^{2m}} H_{2m}\Big(\frac{y-\omega_0 g_0}{\sigma_X}\Big)\Big), \tag{17}$$

where $H_m(y)$ denotes a Hermite polynomial of order m, σ_X^2 is the variance of the decision variate X,

$$\sigma_X^2 = \sigma_\nu^2 + \frac{1}{3}(L^2 - 1) \sum_{k=1}^{\infty} \big(g_{-k}^2 + g_k^2\big), \tag{18}$$

the α_m coefficients may be evaluated recursively via the cumulants, κ_m, according to

$$\alpha_{2m} = \kappa_{2m} + \sum_{i=2}^{m-2} \binom{2m-1}{2i} \kappa_{2(m-i)} \alpha_{2i}, \quad m \geq 2 \tag{19}$$

and the cumulants are given by

$$\kappa_m = -\jmath^m T_{m-1} \frac{L^m - 1}{2^m - 1} \sum_{k=1}^{\infty} \big(g_{-k}^m + g_k^m\big), \quad m > 2, \tag{20}$$

where T_m denote the tangent numbers,

$$T_m = \frac{d^m}{dt^m} \tan(t)\Big|_{t=0}. \tag{21}$$

Note that $\kappa_{2m+1} = 0$.

The definition for the Hermite polynomials employed here is the probabilists' version,

$$\begin{aligned} H_m(y) &= (-1)^m e^{y^2/2} \frac{d^m}{dy^m} e^{-y^2/2} \\ &= \sum_{i=0}^{\lfloor m/2 \rfloor} (-1)^i \frac{m!}{(m-2i)!} \frac{1}{2^i i!} y^{m-2i}, \end{aligned} \tag{22}$$

where $\lfloor x \rfloor$ represents the maximum integer less than or equal to x. ***Mathematica*, however, uses the physicists' definition so it is important to account for this when using the *HermiteH* command. The two definitions are related via**

$$H_m^{\text{phys}}(y) = 2^{m/2} H_m^{\text{prob}}(\sqrt{2}y). \tag{23}$$

3 Initial project goal

Investigate the kernel density estimation technique for approximating the decision variate pdf.

System parameters:

- BPSK transmission: $\omega_k = \pm 1$.

- $\omega_0 = 1$.
- Root-raised cosine transmit and receive filters with the roll-off factor of 0.5.
- Ideal transmission channel ($g(t)$ is, therefore, given by a raised cosine impulse response).
- Timing offset: $\Delta = 0.1$.
- SNR: $E_b/N_0 = 8$dB (i.e. $10\log_{10}(E_b/N_0) = 8$).
- Consider 80 ISI terms, i.e.

$$X = \omega_0 g_0 + \sum_{k=1}^{40} \big(\omega_{-k} g_{-k} + \omega_k g_k\big) + \nu. \tag{24}$$

Randomly generate decision statistics, X, (ω_k randomly chosen as ± 1 for all $k \neq 0$ and ν drawn from a zero mean Gaussian distribution with variance σ_ν^2 (determined via the SNR)) and apply a kernel density estimation to the resulting sequence. Compare the resulting pdf graphically with an order-40 Gram-Charlier series approximation,

$$f_X(y) = \frac{1}{\sqrt{2\pi}\sigma_X} \exp\Big(-\frac{(y-\omega_0 g_0)^2}{2\sigma_X^2}\Big)\Big(1 + \sum_{m=2}^{20} \frac{\alpha_{2m}}{(2m)!\sigma_X^{2m}} H_{2m}\Big(\frac{y-\omega_0 g_0}{\sigma_X}\Big)\Big). \tag{25}$$

How many decision statistics are required to achieve an accurate approximation using a kernel density estimation? Is this prohibitive? Change the timing offset to $\Delta = 0.05$ and $\Delta = 0.15$ and regenerate/re-evaluate. How does this affect the performance of the kernel density estimation technique, if at all?