A Simple Evaluation of DPSK Error Probability Performance in the Presence of Bit Timing Error

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ABSTRACT

This paper presents a simple closed form expression for the conditional (on the timing error) bit error probability of an otherwise ideal DPSK receiver. Numerical results are given for average bit error probability obtained by averaging the conditional error probability over a Tikhonov pdf which is a typical characterization of the timing error in practical bit synchronizers.

The effect of bit timing error on the bit error probability of an otherwise ideal DPSK (differential detection of binary PSK) receiver was considered in [1]. The expression obtained for the conditional (on the timing error) bit error probability was in the form of a double integral which had to be evaluated numerically. Furthermore, to compute the average [over the bit timing error probability density function (pdf)] bit error probability performance required yet a third integration.

Here we present a simple closed form (in terms of known and tabulated functions) expression for the above conditional bit error probability. The result is obtained based on the approach taken by Pawula et al [2] for finding the probability distribution of the angle between two random vectors. Following this, numerical results are given for average bit error probability obtained by averaging the conditional error probability over a Tikhonov pdf which is a typical characterization of the timing error in practical bit synchronizers [3].

ANALYSIS

Consider the DPSK receiver illustrated in Fig. 1 which is optimum for the case of perfect bit timing. The transmitted signal in the ith bit interval $iT_b \leq t \leq (i+1)T_b$ is given by $s^{(i)}(t) = \sqrt{2P}\cos\left(\omega_0 t + \theta^{(i)}\right)$ where ω_0 is the carrier radian frequency, P is the average power, and $\theta^{(i)}$ is the transmitted phase. For differentially encoded BPSK, $\theta^{(i)}$ is related to the information phase $\Delta\phi^{(i)}$ by $\theta^{(i)} = \Delta\phi^{(i)} + \theta^{(i-1)}$ and $\Delta\phi^{(i)}$ takes on values $\theta_0 = 0$ and $\theta_1 = \pi$ with equal probability. Assuming an additive white Gaussian noise channel, the input to this receiver is expressed by

$$r(t) = s^{(i)}(t;\phi_a) + n(t) = \sqrt{2P}\cos\left(\omega_0 t + \theta^{(i)} + \phi_a\right) + n(t) \tag{1}$$

where in addition to the noise, $n(t)$ (single-sided power spectral density N_0 watts/Hz), the channel introduces an arbitrary phase ϕ_a which is assumed to be constant for a duration of at least $2T_b$ seconds.

Assume now that the bit timing is unknown and denoted by $\varepsilon = \lambda T_b$ where λ represents the normalized (with respect to the bit time, T_b) error. The appropriate modification of Fig. 1 then is to replace the lower and upper limits of the integrate-and-dump circuits (I&D's) by $(i-\lambda)T_b$ and $(i+1-\lambda)T_b$, respectively. Having done this, the outputs of the these circuits become

$$\begin{aligned} V_{c,i} &= \sqrt{P}T_b\left[(1-\lambda)\cos\left(\theta^{(i)} + \phi_a\right) + \lambda\cos\left(\theta^{(i-1)} + \phi_a\right)\right] + N_{c,i} \\ V_{s,i} &= \sqrt{P}T_b\left[(1-\lambda)\sin\left(\theta^{(i)} + \phi_a\right) + \lambda\sin\left(\theta^{(i-1)} + \phi_a\right)\right] + N_{s,i} \end{aligned} \tag{2}$$

where

$$\begin{aligned} N_{c,i} &\overset{\Delta}{=} \int_{(i-\lambda)T_b}^{(i+1-\lambda)T_b} n(t)\left[\sqrt{2}\cos\omega_0 t\right]dt, \\ N_{s,i} &\overset{\Delta}{=} \int_{(i-\lambda)T_b}^{(i+1-\lambda)T_b} n(t)\left[-\sqrt{2}\sin\omega_0 t\right]dt \end{aligned} \tag{3}$$

are independent zero mean Gaussian random variables with variance $\sigma^2 = N_0 T_b / 2$. The equivalent phase, η_i for the ith transmission interval is found from the arctangent of the ratio of $V_{s,i}$ to $V_{c,i}$ and using (2) can be expressed in the form

$$\eta_i = \tan^{-1}\left\{\frac{\sqrt{P}T_b\Gamma_i\sin\left(\xi_i + \phi_a\right) + N_{s,i}}{\sqrt{P}T_b\Gamma_i\cos\left(\xi_i + \phi_a\right) + N_{c,i}}\right\} \tag{4}$$

where

$$\begin{aligned} \xi_i &= \tan^{-1}\left\{\frac{(1-\lambda)\sin\theta^{(i)} + \lambda\sin\theta^{(i-1)}}{(1-\lambda)\cos\theta^{(i)} + \lambda\cos\theta^{(i-1)}}\right\} \\ \Gamma_i &= \sqrt{(1-\lambda)^2 + \lambda^2 + 2\lambda(1-\lambda)\cos\left(\theta^{(i)} - \theta^{(i-1)}\right)} \\ &= \sqrt{(1-\lambda)^2 + \lambda^2 + 2\lambda(1-\lambda)\cos\Delta\phi^{(i)}} \end{aligned} \tag{5}$$

A similar expression can be written for the equivalent phase, η_{i-1}, in the $(i$-1)st transmission interval.

If $\theta^{(i)} - \theta^{(i-1)} = \Delta\phi^{(i)} = \theta_k$ is the value of the phase communicated in the ith transmission interval, then for the decision rule illustrated in Fig. 1, a correct decision is made when $\psi_i = \eta_i - \eta_{i-1}$ modulo 2π falls in the half plane defined by the

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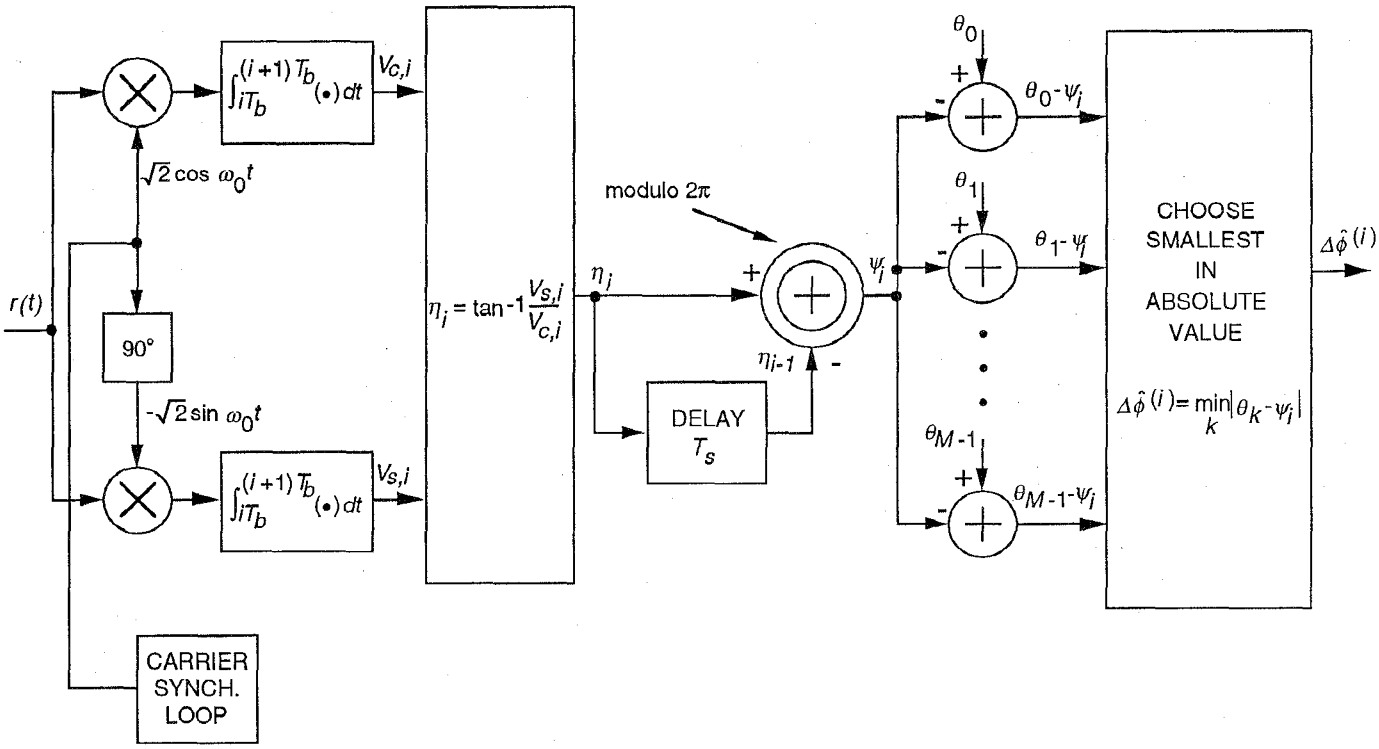

Fig. 1. Optimum receiver for classical M-DPSK.

angular interval $(\theta_k - \pi/2, \theta_k + \pi/2)$. In mathematical terms, the conditional probability of a correct symbol decision given that $\Delta\phi^{(i)} = \theta_k$ is

$$\begin{aligned} P_b(C|k;\lambda) &= \Pr\left\{\left|(\eta_i - \eta_{i-1})_{\bmod 2\pi} - \underbrace{\left(\theta^{(i)} - \theta^{(i-1)}\right)}_{\theta_k}\right| \le \frac{\pi}{2}\right\} \\ &= \Pr\left\{|\psi_i - \theta_k| \le \frac{\pi}{2}\right\} \end{aligned} \tag{6}$$

Then the conditional bit error probability is given by

$$\begin{aligned} P_b(E|\lambda) &= 1 - P_b(C|\lambda) = 1 - \frac{1}{2}[P_b(C|0;\lambda) + P_b(C|1;\lambda)] \\ &= \frac{1}{2}[P_b(E|0;\lambda) + P_b(E|1;\lambda)] \\ &= \frac{1}{2}\left(\int_{-3\pi/2}^{-\pi/2} f_\Psi(\psi)d\psi\right) + \frac{1}{2}\left(\int_{-\pi/2}^{\pi/2} f_\Psi(\psi)d\psi\right) \end{aligned} \tag{7}$$

where $f_\Psi(\psi)$ is the pdf of ψ where ψ is the angle between the two vectors $\Gamma_i e^{j\xi_i} + N_{c,i} + jN_{s,i}$ and $\Gamma_{i-1}e^{j\xi_{i-1}} + N_{c,i-1} + jN_{s,i-1}$.

Evaluation of the integrals in (7) requires determining the probability distribution of the angle ψ between two signal vectors each perturbed by complex random Gaussian noise (see the geometry of Fig. 2). This is the generic problem considered in [2] and our interest here is in the case where two signal vectors $A_1e^{j\phi_1} + x_1 + jy_1$ and $A_2e^{j\phi_2} + x_2 + jy_2$ have unequal length and the noises are uncorrelated. Furthermore, the limits of the integrations (see Eq. (7)) are separated by π radians. For this special case, Pawula and Roberts [4] were able to evaluate the integrals in closed form. In particular,

$$\Pr\{\psi_0 - \pi \le \psi \le \psi_0\} = \int_{\psi_0-\pi}^{\psi_0} f_\Psi(\psi)d\psi \tag{8}$$

$$= \begin{cases} \frac{1}{2}\left[1 - \sqrt{1 - \frac{\beta^2}{\alpha^2}}\, Ie\left(\frac{\beta}{\alpha}, \alpha\right)\right]; \\ \qquad \Delta\Phi - \pi \le \psi_0 \le \Delta\Phi \\ \frac{1}{2}\left[1 + \sqrt{1 - \frac{\beta^2}{\alpha^2}}\, Ie\left(\frac{\beta}{\alpha}, \alpha\right)\right]; \\ \qquad \text{elsewhere} \end{cases}$$

where $Ie(k,x)$ is the Rice Ie-function [5] defined by

$$Ie(k,x) \stackrel{\Delta}{=} \int_0^x \exp(-t)I_0(kt)dt \tag{9}$$

and tabulated in [5, 6], and

$$\begin{aligned} \alpha &= U \\ \beta &= \sqrt{U^2\cos^2(\Delta\Phi - \psi_0) + V^2\sin^2(\Delta\Phi - \psi_0)} \end{aligned} \tag{10}$$

where[1]

$$\Delta\Phi = (\phi_2 - \phi_1) \text{ modulo } 2\pi$$
$$U = \frac{1}{2}\left(\frac{A_1^2}{2\sigma_1^2} + \frac{A_2^2}{2\sigma_2^2}\right); \quad V = \frac{1}{2}\left(\frac{A_2^2}{2\sigma_2^2} - \frac{A_1^2}{2\sigma_1^2}\right) \tag{11}$$

and $\operatorname{var} x_1 = \operatorname{var} y_1 = \sigma_1^2$, $\operatorname{var} x_2 = \operatorname{var} y_2 = \sigma_2^2$.

The appropriate associations with the geometry in Fig. 2 are

$$\begin{aligned} &A_1 = \sqrt{P}T_b\Gamma_{i-1}; \quad A_2 = \sqrt{P}T_b\Gamma_i \\ &\phi_1 = \xi_{i-1} + \phi_a; \quad \phi_2 = \xi_i + \phi_a \\ &x_1 = N_{c,i-1}; \quad y_1 = N_{s,i-1}; \quad x_2 = N_{c,i}; \quad y_2 = N_{s,i} \\ &\sigma_1^2 = \sigma_2^2 = \sigma^2 \end{aligned} \tag{12}$$

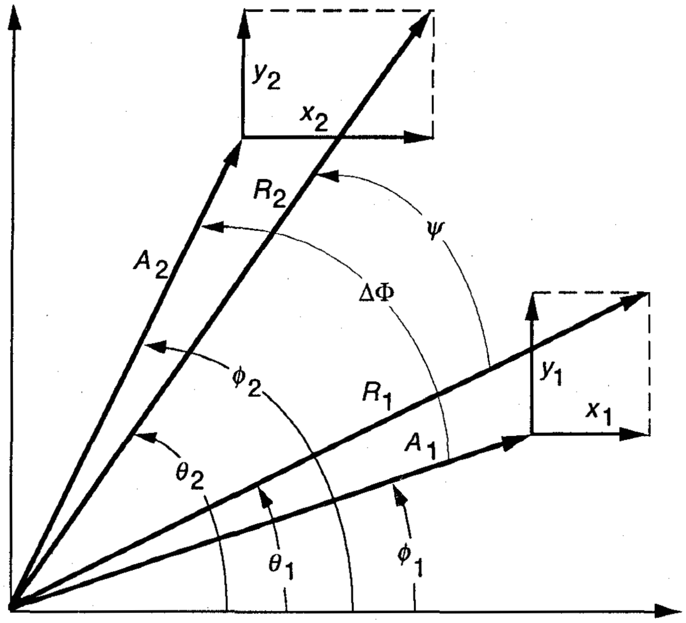

Fig. 2. Geometry for angle between vectors perturbed by Gaussian noise.

Substituting (12) in (11) gives

$$\begin{aligned} U &= \frac{E_b}{N_0}\left[\frac{(\Gamma_{i-1})^2 + (\Gamma_i)^2}{2}\right] = \frac{E_b}{N_0}\left[(1-\lambda)^2 + \lambda^2 + \lambda(1-\lambda)\left(\cos\Delta\phi^{(i)} + \cos\Delta\phi^{(i-1)}\right)\right] \\ V &= \frac{E_b}{N_0}\left[\frac{(\Gamma_i)^2 - (\Gamma_{i-1})^2}{2}\right] = \frac{E_b}{N_0}\left[\lambda(1-\lambda)\left(\cos\Delta\phi^{(i)} - \cos\Delta\phi^{(i-1)}\right)\right] \\ \Delta\Phi &= (\xi_i - \xi_{i-1}) \text{modulo } 2\pi \\ &= \left(\tan^{-1}\left\{\frac{(1-\lambda)^2\sin\Delta\phi^{(i)} + \lambda^2\sin\Delta\phi^{(i-1)} + \lambda(1-\lambda)\sin\left(\Delta\phi^{(i)} + \Delta\phi^{(i-1)}\right)}{(1-\lambda)^2\cos\Delta\phi^{(i)} + \lambda^2\cos\Delta\phi^{(i-1)} + \lambda(1-\lambda)\left[1 + \cos\left(\Delta\phi^{(i)} + \Delta\phi^{(i-1)}\right)\right]}\right\}\right) \text{modulo } 2\pi \end{aligned} \tag{13}$$

For DPSK, the information phase, $\Delta\phi^{(i)}$, to be communicated in the ith transmission interval takes on a value of 0 or π. For each of these values we must compute the conditional bit error probability when $\Delta\phi^{(i-1)}$ takes on these very same values (with equal probability). To do this, we apply the results of Case 2 in [2], in particular, the special case of these results given in (8) - (11). To begin, we observe from (13) that for $\Delta\phi^{(i-1)}$ equal to either 0 or π, $\Delta\Phi = \Delta\phi^{(i)}$. Furthermore, for $\Delta\Phi = 0$, we have $\psi_0 = -\pi/2$ while for $\Delta\Phi = \pi$, we have $\psi_0 = \pi/2$. Thus, the conditional bit error probabilities in (7) are evaluated from (8) as

$$\begin{aligned} P(E|0,\lambda) &= \Pr\left\{-\frac{3\pi}{2} \le \psi \le -\frac{\pi}{2}\right\} \\ &= \frac{1}{2}\left[1 - \sqrt{1 - \frac{\beta_0^2}{\alpha_0^2}}\, Ie\left(\frac{\beta_0}{\alpha_0}, \alpha_0\right)\right] \\ P(E|\pi,\lambda) &= \Pr\left\{-\frac{\pi}{2} \le \psi \le \frac{\pi}{2}\right\} \\ &= \frac{1}{2}\left[1 - \sqrt{1 - \frac{\beta_\pi^2}{\alpha_\pi^2}}\, Ie\left(\frac{\beta_\pi}{\alpha_\pi}, \alpha_\pi\right)\right] \end{aligned} \tag{14}$$

where from (10) and (13)

$$\begin{aligned} \alpha_0 &= \frac{E_b}{N_0} \times \begin{cases} 1; & \Delta\phi^{(i-1)} = 0 \\ 1 - 2\lambda(1-\lambda); & \Delta\phi^{(i-1)} = \pi \end{cases} \\ \beta_0 &= \frac{E_b}{N_0} \times \begin{cases} 0; & \Delta\phi^{(i-1)} = 0 \\ 2\lambda(1-\lambda); & \Delta\phi^{(i-1)} = \pi \end{cases} \\ \alpha_\pi &= \frac{E_b}{N_0} \times \begin{cases} 1 - 2\lambda(1-\lambda); & \Delta\phi^{(i-1)} = 0 \\ (1-2\lambda)^2; & \Delta\phi^{(i-1)} = \pi \end{cases} \\ \beta_\pi &= \frac{E_b}{N_0} \times \begin{cases} -2\lambda(1-\lambda); & \Delta\phi^{(i-1)} = 0 \\ 0; & \Delta\phi^{(i-1)} = \pi \end{cases} \end{aligned} \tag{15}$$

[1]Note that $\Delta\Phi$ ordinarily includes the carrier phase shift. However, this term is frequently eliminated by receiver processing and thus we do not include it in Fig. 2.

The average error probability conditioned on a given timing offset λ is then obtained by averaging (14) over equiprobable events $\Delta\phi^{(i-1)} = 0$ and $\Delta\phi^{(i-1)} = \pi$, namely,

$$\begin{aligned} P(E|\lambda) &= \frac{1}{4}P(E|0,\lambda)\Big|_{\Delta\phi^{(i-1)}=0} + \frac{1}{4}P(E|0,\lambda)\Big|_{\Delta\phi^{(i-1)}=\pi} \\ &\quad + \frac{1}{4}P(E|\pi,\lambda)\Big|_{\Delta\phi^{(i-1)}=0} + \frac{1}{4}P(E|\pi,\lambda)\Big|_{\Delta\phi^{(i-1)}=\pi} \\ &= \frac{1}{8}\left[1 - Ie\left(0,\frac{E_b}{N_0}\right)\right] + \frac{1}{8}\left[1 - \sqrt{1 - \frac{(2\lambda(1-\lambda))^2}{(1-2\lambda(1-\lambda))^2}}\right. \\ &\quad \left.\times Ie\left(\frac{2\lambda(1-\lambda)}{1-2\lambda(1-\lambda)}, (1-2\lambda(1-\lambda))\frac{E_b}{N_0}\right)\right] \\ &\quad + \frac{1}{8}\left[1 - \sqrt{1 - \frac{(-2\lambda(1-\lambda))^2}{(1-2\lambda(1-\lambda))^2}}\right. \\ &\quad \left.\times Ie\left(\frac{-2\lambda(1-\lambda)}{1-2\lambda(1-\lambda)}, (1-2\lambda(1-\lambda))\frac{E_b}{N_0}\right)\right] \\ &\quad + \frac{1}{8}\left[1 - Ie\left(0,(1-2\lambda)^2\frac{E_b}{N_0}\right)\right] \end{aligned} \tag{16}$$

Using the identity [4, Eq. (A3-5)]

$$\frac{1}{2}\left[1 - \sqrt{1 - \frac{\beta^2}{\alpha^2}}\, Ie\left(\frac{\beta}{\alpha}, \alpha\right)\right] = \frac{\sqrt{\alpha^2 - \beta^2}}{2\pi} \times \int_0^{\pi} \frac{\exp\{-[\alpha - \beta\cos\theta]\}}{\alpha - \beta\cos\theta}\, d\theta \tag{17}$$

we can rewrite (16) as

$$P(E|\lambda) = \left[\frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)\right]\mathcal{D}; \tag{18}$$

$$\begin{aligned} \mathcal{D} &= \frac{1}{4} + \frac{(1-2\lambda)}{4\pi}\int_0^{\pi} \frac{\exp\left\{\frac{E_b}{N_0}\left[4\lambda(1-\lambda)\cos^2\frac{\theta}{2}\right]\right\}}{1 - 4\lambda(1-\lambda)\cos^2\frac{\theta}{2}}\, d\theta \\ &\quad + \frac{(1-2\lambda)}{4\pi}\int_0^{\pi} \frac{\exp\left\{\frac{E_b}{N_0}\left[4\lambda(1-\lambda)\sin^2\frac{\theta}{2}\right]\right\}}{1 - 4\lambda(1-\lambda)\sin^2\frac{\theta}{2}}\, d\theta \\ &\quad + \frac{1}{4}\exp\left\{\frac{E_b}{N_0}[4\lambda(1-\lambda)]\right\} \end{aligned}$$

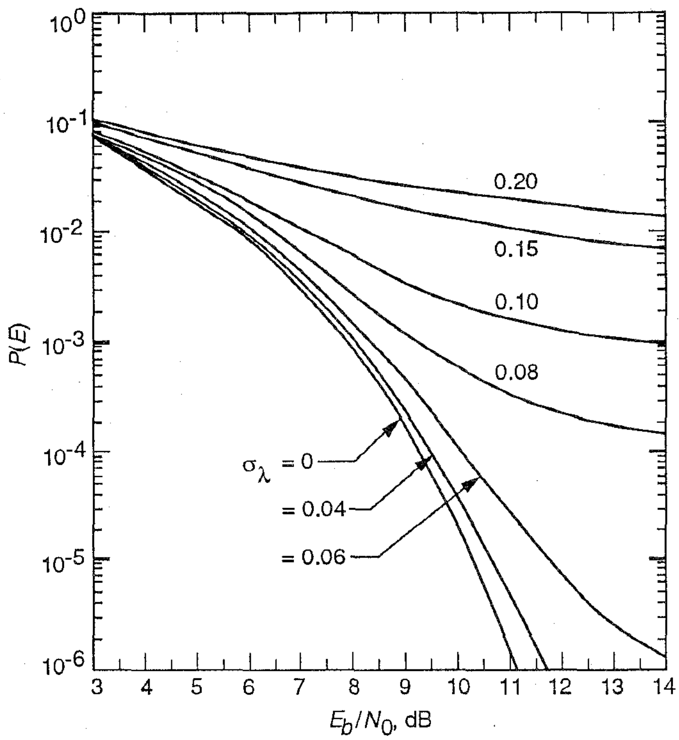

Fig. 3. Average error probability performance of DPSK in the presence of AWGN and bit timing error.

which clearly places in evidence the degradation in performance due to bit timing error relative to the ideal (perfectly timed) performance corresponding to $\mathcal{D} = 1$. Assuming a Tikhonov pdf for the normalized timing error [3, Chap. 9], namely,

$$f(\lambda) = \frac{\exp\left[\cos(2\pi\lambda)/(2\pi\sigma_\lambda)^2\right]}{I_0\left(1/(2\pi\sigma_\lambda)^2\right)}; \quad |\lambda| \le \frac{1}{2} \tag{19}$$

then the average bit error probability performance $P(E)$ of DPSK in the presence of timing error is obtained by averaging (18) over this pdf and is illustrated in Fig. 3 for various values of rms normalized timing error σ_λ. Comparing these curves with the comparable performance results for PSK (see [3, Fig. 9-37]), we see that both have virtually identical variation with σ_λ. On an absolute scale, the curves for DPSK will be slightly higher than those for PSK since the former has inferior performance.

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