Goal: *For a 4-PAM transmission scheme with non-deterministic timing offset, are the optimum decision region boundaries given by* $-2\zeta, 0, 2\zeta$*, where*

$$\zeta = \int_{-1/2}^{1/2} g(y) f_{\Delta}(y)\,\mathrm{d}y, \tag{1}$$

$g(y)$ *is the overall system impulse response (raised-cosine in our case) and* $f_{\Delta}(y)$ *is the timing offset pdf?*

Background material:

Tikhonov pdf:

$$f_{\Delta}(y) = \frac{\exp\Big(\dfrac{\cos(2\pi y)}{(2\pi\sigma_{\Delta})^2}\Big)}{I_0\Big(\dfrac{1}{(2\pi\sigma_{\Delta})^2}\Big)} \qquad -\frac{1}{2} \leq y \leq \frac{1}{2}, \tag{2}$$

where $I_0(x)$ denotes the zero-order modified Bessel function of the first kind and σ_{Δ} denotes the rms normalised timing error.

Decision variate:

$$X = \omega_0 g_0 + \sum_{k=1}^{\infty}\big(\omega_{-k}g_{-k} + \omega_k g_k\big) + \nu. \tag{3}$$

Gram-Charlier series pdf:

$$f_X(y) = \frac{1}{\sqrt{2\pi}\sigma_X}\exp\Big(-\frac{(y-\omega_0 g_0)^2}{2\sigma_X^2}\Big)\Bigg(1 + \sum_{m=2}^{\infty}\frac{\alpha_{2m}}{(2m)!\sigma_X^{2m}}H_{2m}\Big(\frac{y-\omega_0 g_0}{\sigma_X}\Big)\Bigg), \tag{4}$$

$$\sigma_X^2 = \sigma_\nu^2 + \frac{1}{3}(L^2-1)\sum_{k=1}^{\infty}\big(g_{-k}^2 + g_k^2\big), \tag{5}$$

$$\alpha_{2m} = \kappa_{2m} + \sum_{i=2}^{m-2}\binom{2m-1}{2i}\kappa_{2(m-i)}\alpha_{2i}, \quad m \geq 2, \tag{6}$$

$$\kappa_m = -\hat{\jmath}^m T_{m-1}\frac{L^m-1}{2^m-1}\sum_{k=1}^{\infty}\big(g_{-k}^m + g_k^m\big), \quad m > 2. \tag{7}$$

But, now g_k terms are functions of the random variable Δ (unlike previously where g_k terms were deterministic/fixed).

For now anyway,

- discretise $f_{\Delta}(y)$
- evaluate Gram-Charlier $f_{X|\Delta}(y)$, i.e. pdf of X conditioned on a given Δ, for each Δ
- average over the discretised $f_{\Delta}(y)$

Note that $f_{\Delta}(y)$ should be discretised in such a manner that

$$\sum_{i=1}^{n}\widetilde{f}_{\Delta}(y_i) = 1, \tag{8}$$

where $\widetilde{f}_\Delta(y_i)$ denotes the discretised pdf.

Also compare to simulation results where Δ is randomly generated from $f_\Delta(y)$ and the simulations are run as before.