Analysis of Equal-Gain Diversity with Partially Coherent Fading Signals

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***Abstract*—An analytical technique based on Gram–Charlier series expansion is presented for the computation of the error probability of equal-gain combiner (EGC) with partially coherent fading signals. Imperfect carrier recovery is attributed to the random noise present in the carrier recovery loops. The resulting noisy phase references are assumed to satisfy Tikhonov distribution. The fades on the diversity branches are assumed to be slowly varying and statistically independent with Rayleigh-distributed envelopes. The error-rate performance of coherent and differentially coherent phase-shift keying (PSK) systems are compared and the phase precision requirement for a reliable coherent detection is computed. Detection loss caused by carrier phase errors is computed for several signal-to-noise ratio (SNR) reliability and bit error probability levels. It is demonstrated that the effect of carrier phase errors on the mean SNR is negligible compared to their effect on deep fades or small bit error probabilities. It is also shown that the carrier phase precision requirement can be reduced through signal combination.**

***Index Terms*—Carrier synchronization, equal-gain combiner, Gram–Charlier series, partially coherent detection, phase-locked loops, Tikhonov distribution.**

I. Introduction

IT HAS been well recognized that diversity techniques are effective methods for combating the deleterious effect of channel fading. In equal-gain combiner (EGC), signal amplitudes of different branches are cophased, equally weighted, and added to form the decision variable, while noise and unwanted signals are added noncoherently. Cophasing is the demodulator attempt of undoing the random phase shifts introduced on the diversity channels. This is accomplished by multiplying the outputs of the matched filters by the complex conjugate of the phase shift estimates assumed to be derived either from an unmodulated carrier or the information-bearing signal. The Doppler spread in the carrier frequency and the presence of noise and interference in the carrier recovery loops do not allow for perfect estimation for the carrier phase, which may lead to serious degradation to system performance. In this study, we assume that the channel fading is sufficiently slow to allow the implementation of coherent detection. We assume further that the phase on each diversity channel is estimated using a first-order phase-locked loop (PLL) and that the phase errors follow Tikhonov distribution.

When compared with the optimum maximum ratio combiner (MRC), EGC achieves a comparable signal-to-noise ratio (SNR) performance [11], [12] with a simplified receiver structure. For this reason, EGC has received a considerable interest in the literature [11]–[19]. Unfortunately, exact analysis of the combiner's SNR and error probability becomes intractable if the diversity order exceeds two. Moreover, synchronization phase errors cause random rotations in the received phasors. As a result, the real and imaginary parts of the decision variable appear with random multiplicative functions [4]. For phase errors with Tikhonov distributions, a closed-form expression for the pdf of SNR becomes inexpressible in any useful closed form even for the simple case of no diversity. A number of studies have been devoted to the evaluation of SNR of EGC. Brennan [11] has constructed curves by numerical techniques for the SNR distribution for diversity orders of two–eight. Schwartz *et al.* [14] have related the SNR distribution of EGC to that of MRC by a reduction factor that depends only on the diversity order. This approximation gives good results for only small SNR values [15]. Beaulieu and Abu-Dayya [19] have recently presented an infinite series based on the confluent hypergeometric functions for the computation of the SNR distribution of EGC in Nakagami fading. However, their work assumed perfect carrier synchronization. Brennan [11] has derived a simple estimate for the degradation in the average output SNR of linear combiners with imperfectly coherent signals. The estimate is conservative and assumes fixed phase errors between the demodulated phasors. This paper provides exact analysis of the impact of imperfect carrier recovery on the SNR and error-rate performance of binary and quaternary coherent phase-shift keying (PSK) systems.

The remaining of this paper is organized as follows. The system model is described in Section II. In Section III, the impact of imperfect carrier phase recovery on the SNR performance is analyzed. In Section IV, we present a new technique for the computation of the average bit error probability for BPSK and QPSK when the channel fades and the carrier phase errors satisfy Rayleigh and Tikhonov distributions, respectively. A discussion of our results is given in Section V, and the main results are summarized in Section VI.

II. System Model

The PSK receiver under consideration is depicted in Fig. 1. We assume that all diversity channels are subject to multipath fading and corrupted by additive white Gaussian noise

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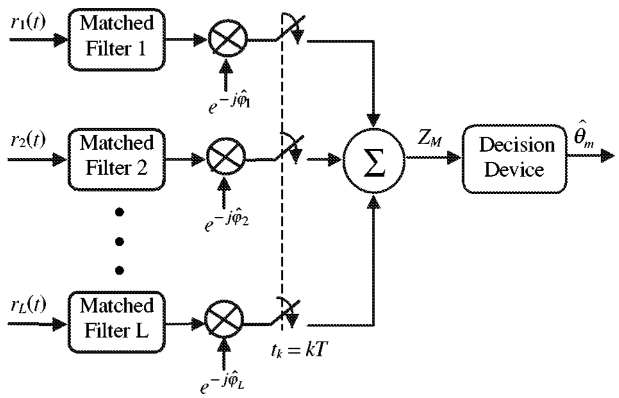

Fig. 1. Coherent PSK receiver with equal-gain diversity.

(AWGN). The low-pass representation of the mth-received signal on the ℓth channel takes the form

$$r_{m\ell}(t) = \alpha_\ell e^{j\varphi_\ell} u_M(t) e^{j\theta_m} + n_\ell(t), \qquad \begin{array}{c} \ell = 1, 2, \cdots, L \\ m = 1, 2, \cdots, M \end{array} \tag{1}$$

where φ_ℓ is the random phase shift introduced by the ℓth channel assumed to be uniformly distributed in the interval $[-\pi, \pi)$. θ_m is the modulation phase of the mth signal, which takes the discrete values 0 and π radians in the BPSK case and $-3\pi/4, -\pi/4, \pi/4, 3\pi/4$ radians in the QPSK case. $u_M(t)$ is a real-valued signaling pulse assumed to be constant over the signaling interval T, i.e.,

$$u_M(t) = \begin{cases} \sqrt{\dfrac{2E_s}{T}}, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

where $E_s = E_b \log_2 M$ is the symbol energy.

The noise terms $\{n_\ell(t)\}$ in (1) denote zero mean complex-valued white Gaussian processes. All noise terms are assumed to be mutually statistically independent with identical single-sided power spectral density of $2N_0$ W/Hz. The signal fading α_ℓ is due to the time-variant multipath characteristics on the ℓth channel. The fades on all channels are assumed to be uncorrelated, frequency-nonselective, and slowly varying with Rayleigh-distributed envelopes

$$f_{\alpha_\ell}(\alpha) = 2\alpha e^{-\alpha^2}, \qquad \alpha \ge 0, \ell = 1, 2, \cdots, L \tag{3}$$

where, without loss of generality, the mean square value of α_ℓ is normalized to one.

The receiver removes the random phase shifts introduced on all diversity channels by multiplying the received signals on the respective channels by the complex conjugate of the phase shift estimates. Thus, for the signals detected using matched filter demodulators with perfect bit synchronization, the decision variable at the output of the combiner can be expressed as

$$\begin{aligned} Z_M(\theta_m) &= \sum_{\ell=1}^{L} e^{-j\hat{\varphi}_\ell} \int_0^T r_{m\ell}(t) u_M(t)\,dt \\ &= 2E_s \sum_{\ell=1}^{L} \alpha_\ell e^{j(\varepsilon_\ell + \theta_m)} + \upsilon_M, \\ &\quad m = 1, 2, \cdots, M \end{aligned} \tag{4}$$

where $\varepsilon_\ell \equiv \varphi_\ell - \hat{\varphi}_\ell$ represents the carrier phase error in the ℓth channel. It is assumed that the phase processes $\{\varphi_\ell\}$ are slowly varying so that they are considered constant over one signaling interval T.

The noise term at the output of the combiner is given by

$$\begin{aligned} \upsilon_M &= \sum_{\ell=1}^{L} e^{-j\hat{\varphi}_\ell} \int_0^T n_\ell(t) u_M(t)\,dt \\ &= \upsilon_{M,I} + j\upsilon_{M,Q}. \end{aligned} \tag{5}$$

It follows that $u_{M,I}$ and $u_{M,Q}$ are Gaussian-distributed random variables with zero mean and variance

$$\sigma^2_{\upsilon_{M,I}} = \sigma^2_{\upsilon_{M,Q}} = 2E_s N_0. \tag{6}$$

We assume that the carrier synchronization is derived from an unmodulated carrier[1] using a first-order PLL. The steady-state probability density function of the phase error ε when Gaussian noise is present in the recovery loop is given by [8]

$$f_{\varepsilon_\ell}(\varepsilon) = \frac{\exp(\gamma_\ell \cos \varepsilon)}{2\pi I_0(\gamma_\ell)}, \qquad |\varepsilon| \le \pi, \ell = 1, 2, \cdots, L \tag{7}$$

where $I_n(.)$ is the nth-order modified Bessel function of the first kind and γ_ℓ is the SNR in the ℓth loop. For a recovery circuit with loop bandwidth of B_L and single-sided noise spectral density of N_0, the loop SNR is given by

$$\gamma = \frac{2P_s}{N_0 B_L} \tag{8}$$

where P_s is the power in the synchronization signal. We note that the Tikhonov distribution is also a good approximation for the phase error if a second-order PLL acting on unmodulated carrier is used for phase recovery, provided that the loop SNR γ is sufficiently large [8]. In addition, it resembles Gaussian distribution for large γ.

III. SNR Analysis

The SNR at the output of the L-branch combiner can be written as

$$\rho_L = \frac{\rho}{L} \left[\sum_{\ell=1}^{L} \alpha_\ell f_M(\varepsilon_\ell) \right]^2 \tag{9}$$

where $\rho = E_b/N_0$ is the average SNR per bit per branch and $f_M(\varepsilon_\ell)$ represents the correlation loss function [10] caused by the synchronization phase error ε_ℓ and is given by

$$f_2(\varepsilon) = \cos \varepsilon \tag{10}$$

for the BPSK case and by

$$f_4(\varepsilon) = \cos \varepsilon \pm \sin \varepsilon = \sqrt{2} \cos\left(\varepsilon \mp \frac{\pi}{4}\right) \tag{11}$$

[1]If carrier synchronization is derived from the information-bearing signal, then a PLL in conjuction with a nonlinear device or a decision-feedback must be used to remove the modulation and provide the phase [7]. The same analysis can be carried out with two or more considerations. First, the reference phase has a phase ambiguity of $2\pi/M$ radian which needs to be removed by differential encoding and decoding. Second, the nonlinear operation enhances noise in the loop, which leads to additional loss in the SNR of the reference phase [9].

for the QPSK case, where the positive sign is associated with one demodulated carriers and the negative sign is associated with the other carrier.

Since the sets of the random variables $\{\alpha_\ell\}$ and $\{\varepsilon_\ell\}$ are mutually statistically independent, one can show using (53) in Appendix A that if the phase estimates in all diversity branches are identical, then the decrease in the combiner's mean SNR is given by

$$\frac{\langle f_M^2\rangle + \frac{\pi}{4}(L-1)(\langle f_M\rangle)^2}{1+\frac{\pi}{4}(L-1)} \tag{12}$$

where $\langle . \rangle$ denotes the mathematical expectation. Using (56), the first two moments of the correlation loss functions can be expressed as

$$\begin{aligned}\langle f_2\rangle &= \langle f_4\rangle = \frac{I_1(\gamma)}{I_0(\gamma)}\\ \langle f_2^2\rangle &= \frac{1}{2}\left[1+\frac{I_2(\gamma)}{I_0(\gamma)}\right]\\ \langle f_4^2\rangle &= 1.\end{aligned} \tag{13}$$

Equation (12) gives the degradation in mean SNR of the combiner due to the noisy phase references. Most practical systems are usually designed to satisfy a certain SNR reliability level (i.e., the probability or percentage of time that the SNR exceeds a specified threshold [11]). That is

$$\begin{aligned}\Re &= \Pr[\rho_L > \rho_{th}]\\ &= \Pr\left[\left(\sum_{\ell=1}^{L}\alpha_\ell f_M(\varepsilon_\ell)\right)^2 > L\,\frac{\rho_{th}}{\rho}\right]\\ &= 1 - F\left(\sqrt{L\,\frac{\rho_{th}}{\rho}}\right) + F\left(-\sqrt{L\,\frac{\rho_{th}}{\rho}}\right)\end{aligned} \tag{14}$$

where $F(.)$ is the cumulative distribution function of the random variable $S=\sum_{\ell=1}^{L}\alpha_\ell f_M(\varepsilon_\ell)$. The evaluation of SNR reliability thus requires exact knowledge of the probability distribution of S. The problem of finding this distribution is solved using the series technique to be presented in the next section.

IV. Computation of the Error Probability

In this section, we derive a series technique for the computation of the average bit error probability for BPSK and QPSK systems from the knowledge of the generalized moments of the decision random variables. The series, known as the Gram–Charlier power series of type A, is based on expanding the pdf of the decision random variable Z_M in orthogonal polynomials derived from the Gaussian distribution [6]. This technique was used in [5] to compute the error probability for BPSK due to AWGN and intersymbol interference (ISI), where it was shown that the series is absolutely convergent.

A. Probability of Error for BPSK

For equally probable symbols and assuming that phase 0 was transmitted, the average probability of error for BPSK in the presence of AWGN, channel fades $\{\alpha_\ell\}$, and carrier phase errors $\{\varepsilon_\ell\}$ is the probability that the real part of the decision variable $Z_2(0)$ is less than zero. Using (4)–(6), we obtain

$$P_2 = \Pr\left[\sqrt{\frac{2\rho}{L}}\,S + n < 0\right] \tag{15}$$

where n is normal random variable with zero mean and unity variance and

$$S = \sum_{\ell=1}^{L}\alpha_\ell\cos\varepsilon_\ell \tag{16}$$

is the signal term at the output of the combiner.

Instead of expanding the pdf of S, it is more convenient to expand the random variable

$$U = \sqrt{\frac{2\rho}{L}}\,S + n. \tag{17}$$

Since S and n are mutually statistically independent, then

$$\mu_U = \sqrt{\frac{2\rho}{L}}\,\mu_S \tag{18}$$

and

$$\sigma_U^2 = 1 + \frac{2\rho}{L}\,\sigma_S^2 \tag{19}$$

where μ_S and σ_S are the mean and the standard deviation of S. Using (61) in Appendix A, we can show that

$$\mu_S = \frac{\sqrt{\pi}}{2}\sum_{\ell=1}^{L}\frac{I_1(\gamma_\ell)}{I_0(\gamma_\ell)} \tag{20}$$

and

$$\sigma_S^2 = \frac{L}{2} + \frac{1}{2}\sum_{\ell=1}^{L}\frac{I_2(\gamma_\ell)}{I_0(\gamma_\ell)} - \frac{\pi}{4}\sum_{\ell=1}^{L}\left[\frac{I_1(\gamma_\ell)}{I_0(\gamma_\ell)}\right]^2. \tag{21}$$

Consider the standardized random variable

$$u = \frac{U-\mu_U}{\sigma_U}. \tag{22}$$

The pdf of u can be obtained using the Gram–Charlier expansion as a weighted linear sum of Hermite polynomials [6]

$$f_u(\omega) = f_G(\omega)\sum_{k=0}^{\infty}\frac{a_k}{k!}\,H_k(\omega) \tag{23}$$

where f_G is the pdf of Gaussian random variable with zero mean and unity variance. $H_k(\omega)$ is the Hermite polynomial of degree k satisfying the recursive relation

$$H_{k+1}(\omega) = \omega H_k(\omega) - kH_{k-1}(\omega), \qquad k \ge 1 \tag{24}$$

with $H_0(\omega)=1$ and $H_1(\omega)=\omega$. The Hermite polynomial can also be expressed in the closed form

$$H_k(\omega) = \sum_{n=0}^{\lfloor k/2\rfloor} h_{kn}\,\omega^{k-2n}, \qquad k = 0, 1, 2, \cdots \tag{25}$$

where $\lfloor k/2 \rfloor$ is the largest integer less than or equal to $k/2$ and

$$h_{kn} = \frac{(-1)^n k!}{2^n n!(k-2n)!}, \qquad \begin{array}{l} k = 1, 2, \cdots \\ n = 1, 2, \cdots, \lfloor k/2 \rfloor \end{array} \tag{26}$$

with $h_{k0} = 1$, $k \geq 0$.

The coefficients $\{a_k\}$ are determined by utilizing the orthogonality relation of $\{H_k(\omega)\}$ with respect to $f_G(\omega)$ [1]. Thus, multiplying (23) by $H_k(\omega)$ and integrating termwise gives

$$a_k = \int_{-\infty}^{\infty} H_k(\omega) f_u(\omega)\, d\omega, \qquad k = 0, 1, 2, \cdots \tag{27}$$

where $a_0 = 1$ and $a_1 = a_2 = 0$. Since u is a standardized random variable, then $\{a_k\}$ become zero for all $k > 0$ as u approaches Gaussian. Detailed computation of $\{a_k\}$ is given in Appendix B.

Using (15), (17), and (22), the average probability of error can be expressed in terms of the pdf of u as

$$P_2 = \Pr\left[u < -\frac{\mu_U}{\sigma_U}\right] = \int_{-\infty}^{-\mu_U/\sigma_U} f_u(\omega)\, d\omega. \tag{28}$$

Inserting (23) in (28) and interchanging the order of integration and summation and rearranging, one can show that

$$P_2 = \frac{1}{2}\,\mathrm{erfc}\left(\mu_U/\sqrt{2}\sigma_U\right) + \sum_{k=3}^{\infty} (-1)^k \frac{a_k}{k!}\, d_k(\mu_U/\sigma_U) \tag{29}$$

where

$$d_k(x) = \int_x^{\infty} H_k(\omega) f_G(\omega)\, d\omega, \qquad k = 0, 1, 2, \cdots. \tag{30}$$

Using (25) and interchanging the order of integration and summation, we have

$$d_k(x) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\lfloor k/2 \rfloor} 2^{(k-2n-2/2)} h_{kn} \beta_{kn}(x), \qquad k = 0, 1, 2, \cdots \tag{31}$$

where

$$\beta_{kn}(x) = \begin{cases} \Gamma\left(\dfrac{k-2n+1}{2}, \dfrac{x^2}{2}\right), \\ \quad x \geq 0, \text{ or } x < 0,\ k \text{ odd} \\ 2\Gamma\left(\dfrac{k-2n+1}{2}, 0\right) - \Gamma\left(\dfrac{k-2n+1}{2}, \dfrac{x^2}{2}\right), \\ \quad x < 0,\ k \text{ even} \end{cases} \tag{32}$$

and $\Gamma(a, x)$ is the complement of the incomplete gamma function

$$\Gamma(a, x) = \int_x^{\infty} e^{-t} t^{a-1}\, dt. \tag{33}$$

B. Probability of Error for QPSK

We assume that the information bits are mapped to the four possible phases using the Gray-encoding scheme. Since the bits assigned to adjacent bits differ only in one position, the average bit error probability for equally probable symbols can be approximated by [3]

$$P_4 = \frac{1}{2}\Pr\left[\mathrm{Re}\left\{Z_4\left(\frac{\pi}{4}\right)\right\} < 0\right] + \frac{1}{2}\Pr\left[\mathrm{Im}\left\{Z_4\left(\frac{\pi}{4}\right)\right\} < 0\right] \tag{34}$$

where $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are the real and imaginary parts of x. Using (4)–(6), one can show

$$P_4 = \frac{1}{2}\Pr\left[\sqrt{\frac{2\rho}{L}}\,(S - C) + n_I < 0\right] + \frac{1}{2}\Pr\left[\sqrt{\frac{2\rho}{L}}\,(S + C) + n_Q < 0\right] \tag{35}$$

where n_I and n_Q are identical and statistically independent Gaussian random variables with zero mean and unity variance, S was defined by (16) and C is given by

$$C = \sum_{\ell=1}^{L} \alpha_\ell \sin \varepsilon_\ell. \tag{36}$$

Equation (35) represents the arithmetic mean of the bit error probabilities of two cross-coupled binary systems, one with constructive coupling and the other with destructive coupling. However, since the probability distributions of the phase errors are even and the noise components on the two carriers are identical and statistically independent, the average bit error probabilities of both binary systems are the same. Thus, for calculating P_4 in (35), we shall be concerned with calculating

$$P_4 = \Pr\left[\sqrt{\frac{2\rho}{L}}\,(S + C) + n < 0\right] \tag{37}$$

where $n \equiv n_I \equiv n_Q \equiv N(0, 1)$. Let

$$V = \sqrt{\frac{2\rho}{L}}\,(S + C) + n. \tag{38}$$

Since $E[C] = 0$, we have

$$\mu_V = \sqrt{\frac{2\rho}{L}}\,\mu_S \tag{39}$$

and

$$\sigma_V^2 = 1 + \frac{2\rho}{L}\left(\sigma_S^2 + \sigma_C^2\right) \tag{40}$$

where σ_S^2 is given by (21) and

$$\sigma_C^2 = E[C^2] = \frac{L}{2} - \frac{1}{2}\sum_{\ell=1}^{L} \frac{I_2(\gamma_\ell)}{I_0(\gamma_\ell)}. \tag{41}$$

Since σ_C is not necessarily zero for all degrees of phase coherence, the presence of the cross-talk interference between the demodulated quadrature components in QPSK has caused the variance of the decision variable to increase by $(2\rho/L)\sigma_C^2$ over the binary case. Obviously, this additional deterioration becomes severe at large E_b/N_0 and small-loop SNR values.

Let

$$v = \frac{V - \mu_V}{\sigma_V}. \tag{42}$$

By expanding the pdf of v using the Gram–Charlier series expansion and following the same development steps as in the BPSK case, we can show that the average bit error probability for QPSK can be expressed as

$$P_4 = \frac{1}{2}\operatorname{erfc}\left(\mu_V/\sqrt{2}\,\sigma_V\right) + \sum_{k=3}^{\infty}(-1)^k \frac{b_k}{k!}\, d_k(\mu_V/\sigma_V) \tag{43}$$

where

$$b_k = \int_{-\infty}^{\infty} H_k(\omega) f_v(\omega)\, d\omega, \qquad k = 0, 1, 2 \cdots \tag{44}$$

with $b_0 = 1$ and $b_1 = b_2 = 0$, where f_v is the pdf of the random variable v. Note also that $\{b_k\}$ become zero for all $k > 0$ as v approaches Gaussian. Detailed computation of $\{b_k\}$ is given in Appendix B.

C. Irreducible Error Probabilities

When the performance of the receiver is dominated by the carrier phase errors [i.e., when the noise term in (4) is zero], the probability of error reduces to

$$P_{2,\,\mathrm{irred}} = \Pr[S < 0] \tag{45}$$

in the binary case and to

$$P_{4,\,\mathrm{irred}} = \Pr[(S + C) < 0] \tag{46}$$

in the quaternary case. Since S and C can take negative values, imperfect coherent combination of signals in EGC can introduce irreducible error floor, which cannot be corrected merely by increasing the transmitted power.

The average error floor for BPSK can still be evaluated using (29) with μ_S and σ_S replacing μ_U and σ_U, respectively. In the absence of noise, μ_u^i in (65) reduces to

$$\mu_u^i = \sigma_S^{-i} \mu_{S_0}^i \tag{47}$$

where $\mu_{S_0}^i$ is the ith-central moment of S. Similarly, for QPSK, the average error floor can be calculated using (43) with μ_S and $\sqrt{\sigma_S^2 + \sigma_C^2}$ replacing μ_V and σ_V, respectively. Also, in the absence of noise, (68) reduces to

$$\mu_v^i = (\sigma_S^2 + \sigma_C^2)^{-i/2} \mu_{X_0}^i \tag{48}$$

where $\mu_{X_0}^i$ is the ith-central moment of $(S + C)$.

V. Discussion of Results

The first term in (29) represents the average bit error probability for the binary system if the random variable U is modeled as Gaussian with variance $\sigma_U^2 = 1 + (2\rho/L)\sigma_S^2$. Similarly, the first term in (43) represents the average bit error probability for the quaternary system if the random variable V is modeled as Gaussian with variance $\sigma_V^2 = 1 + (2\rho/L)(\sigma_S^2 + \sigma_C^2)$. The summation terms give the remainder of the bit error probability if the distributions of the above random variables deviate from Gaussian. Since the random variables S and C are the sum of L independent random variables, their distributions approximately equal Gaussian for large L, by the central limit theorem. In addition, since each term in S and C is the product of a Rayleigh variate and the cosine or sine of a Tikhonov one, each term becomes Gaussian in the limit when the reference phases are uniformly distributed. However, since L is not in practice quite large nor the phase references for coherent detection are desired to be uniformly distributed, the Gaussian approximation is fully justified. Nonetheless, the above technique still suits the problem of finding the average probability of error in (15) and (37) for moderate values of L, γ, and ρ.

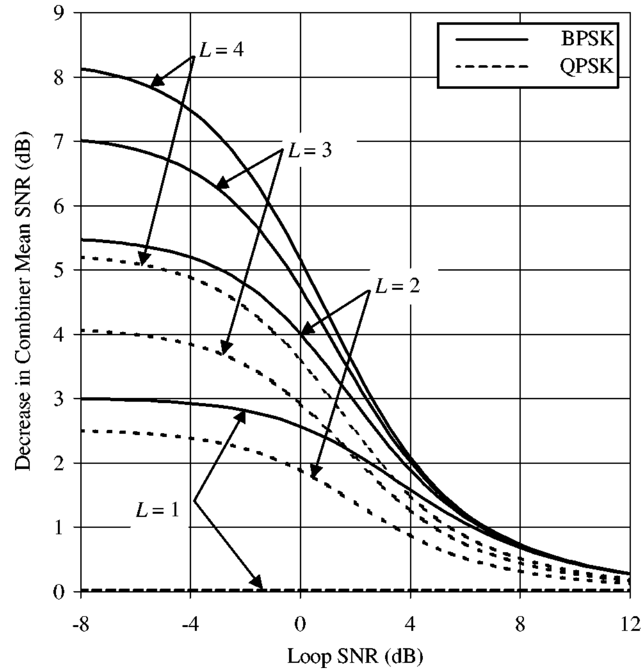

Fig. 2. Degradation in mean SNR of EGC due to imperfect carrier recovery.

The reduction in the combiner mean SNR due to carrier phase errors is plotted in Fig. 2 for diversity orders of one–four. The results indicate that the phase precision requirement for QPSK is less stringent than that for BPSK if the average SNR is used as a basis for performance comparison. This is because, the average SNR is dependent only on the first two moments of the correlation functions induced by the carrier phase errors.

The detection loss at 90% and 99% SNR reliability levels is shown in Table I[2] for BPSK and in Table II for QPSK. The SNR thresholds that achieve 90% reliability with perfect carrier recovery are −9.61, −3.33, −0.36, and 1.54 dB corresponding to $L = 1, 2, 3,$ and 4, respectively. For 99% reliability, the SNR thresholds are −23.47, −8.94, −4.45, and −1.77 dB.

The average bit error probability versus average SNR (i.e., $\rho E[\alpha^2]$) is plotted in Fig. 3 for BPSK for an rms phase error of 20° and in Fig. 4 for QPSK for an rms phase error of 12°. The average bit error probabilities for BPSK and QPSK are also plotted in Fig. 5 versus loop SNR for E_b/N_0 values that achieve an average bit error probability of 10^{-3} under perfect detection conditions. The detection loss for BPSK and QPSK at bit error probability of 10^{-3} is given in Table III for several rms phase error values. Note that the SNR values that achieve an average bit error probability of 10^{-3} under perfect detection

[2]Detection loss at a specific SNR reliability (or error probability) is defined as the increase in EGC mean SNR needed to maintain the same SNR reliability (or error probability) as that of ECG with perfect carrier recovery.

TABLE I
DETECTION LOSS IN DECIBELS IN BPSK EGC AT 90% AND 99% SNR RELIABILITY LEVELS

σ_ε	$L=1$		$L=2$		$L=3$		$L=4$	
	90%	99%	90%	99%	90%	99%	90%	99%
8^0	0.09	0.09	0.08	0.08	0.08	0.08	0.08	0.08
10^0	0.14	0.14	0.13	0.14	0.13	0.13	0.13	0.13
12^0	0.20	0.21	0.19	0.21	0.19	0.20	0.19	0.20
14^0	0.28	0.29	0.27	0.29	0.27	0.28	0.27	0.28
16^0	0.39	0.40	0.37	0.39	0.37	0.39	0.37	0.38
18^0	0.51	0.54	0.49	0.52	0.48	0.50	0.47	0.49
20^0	0.68	0.75	0.63	0.70	0.60	0.67	0.57	0.65

TABLE II
DETECTION LOSS IN DECIBELS IN QPSK EGC AT 90% AND 99% SNR RELIABILITY LEVELS

σ_ε	$L=1$		$L=2$		$L=3$		$L=4$	
	90%	99%	90%	99%	90%	99%	90%	99%
8^0	0.36	0.39	0.33	0.37	0.26	0.37	0.22	0.33
10^0	0.61	0.70	0.56	0.60	0.43	0.59	0.36	0.54
12^0	0.97	1.29	0.88	1.01	0.65	0.92	0.53	0.82
14^0	1.45	2.41	1.22	1.92	0.89	1.33	0.73	1.21
16^0	2.06	4.22	1.61	2.60	1.19	1.84	0.99	1.70
18^0	2.73	6.40	2.00	3.98	1.51	2.54	1.26	2.31
20^0	3.47	8.71	2.47	5.49	1.89	3.68	1.59	3.15

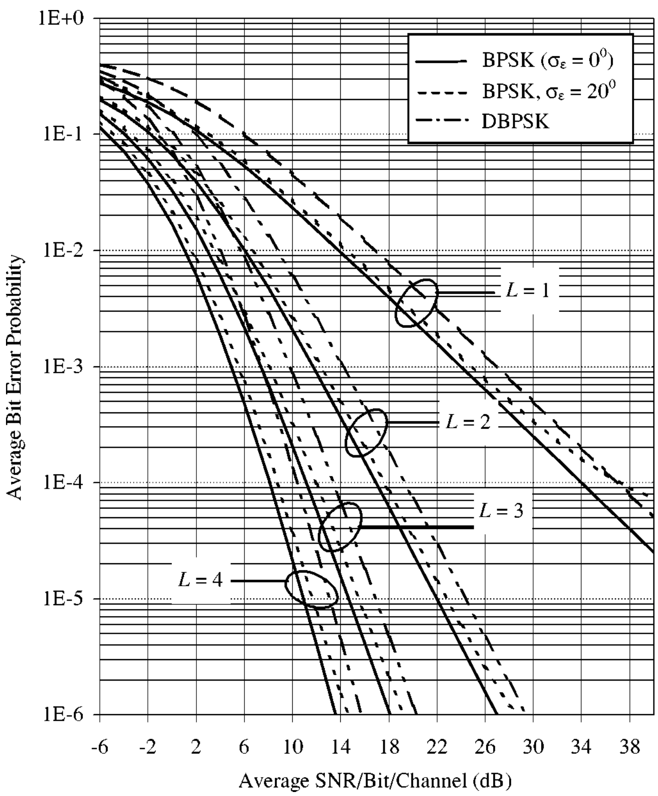

Fig. 3. Error-rate performance for BPSK EGC in the presence of AWGN, Rayleigh fading, and carrier phase errors.

conditions are 23.97, 11.71, 7.37, and 4.94 dB corresponding to $L = 1, 2, 3$, and 4. To set a limit on the degradation in the coherent systems due to imperfect carrier recovery, the bit error probabilities of the two- and four-phase differentially coherent PSK (DBPSK and DQPSK) systems are also shown in these plots. It is assumed here that the channel phase variations are constant over at least two consecutive signaling intervals. In the DQPSK case, it is assumed that the modulator implements Gray

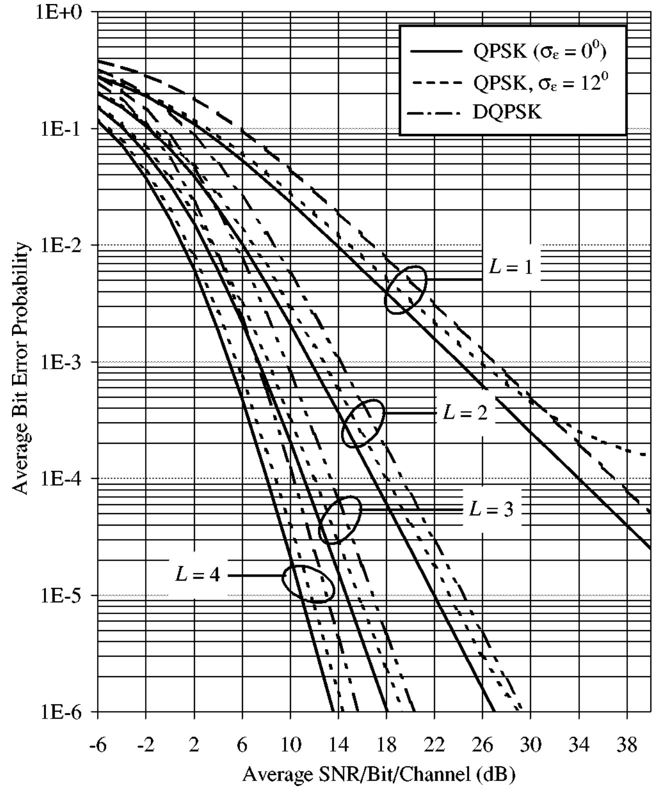

Fig. 4. Error-rate performance for QPSK EGC in the presence of AWGN, Rayleigh fading, and carrier phase errors.

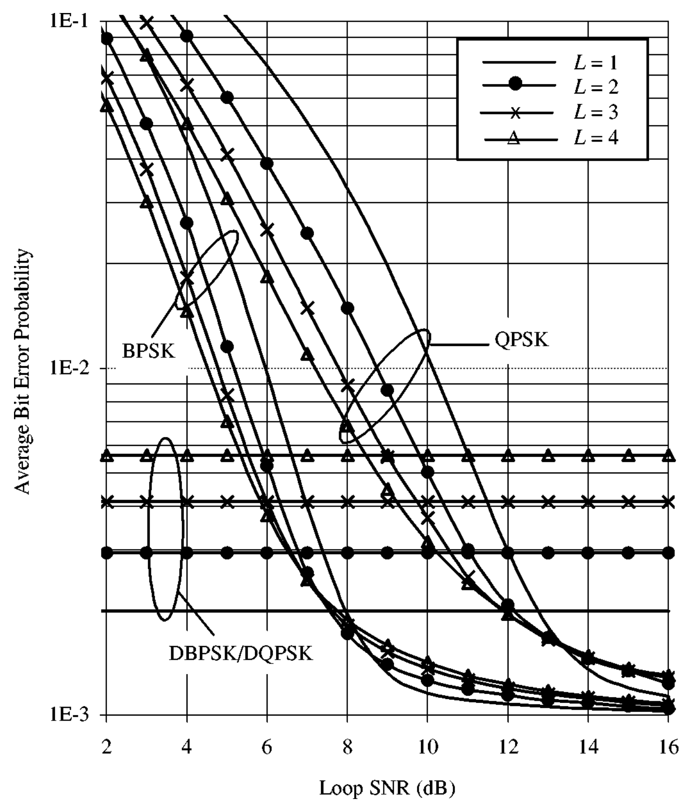

Fig. 5. Average bit error probability for EGC as a function of loop SNR.

TABLE III
DETECTION LOSS IN DECIBELS IN BPSK AND QPSK EGC'S AT BIT ERROR PROBABILITY OF 10^{-3}

σ_ε	$L = 1$		$L = 2$		$L = 3$		$L = 4$	
	BPSK	QPSK	BPSK	QPSK	BPSK	QPSK	BPSK	QPSK
8^0	0.1	0.4	0	0.3	0	0.3	0	0.2
10^0	0.1	0.7	0.1	0.6	0	0.5	0	0.4
12^0	0.2	1.8	0.1	1.0	0.1	0.7	0.1	0.6
14^0	0.3	9.3	0.2	1.6	0.2	1.0	0.2	0.9
16^0	0.4	—	0.3	4.4	0.3	1.8	0.3	1.3
18^0	0.6	—	0.4	11.3	0.4	2.7	0.4	1.8
20^0	0.9	—	0.7	—	0.6	5.7	0.6	2.8

TABLE IV
PHASE PRECISION REQUIREMENTS FOR COHERENT PSK SYSTEMS TO ACHIEVE A BIT ERROR PROBABILITY OF 10^{-3} AS THAT OF THEIR DIFFERENTIAL VERSIONS

DIVERSITY ORDER, L	BPSK	QPSK
1	22.9^0	12.5^0
2	26.5^0	15.4^0
3	28.7^0	17.5^0
4	30.1^0	18.7^0

encoding. Table IV gives the phase precision requirement for PSK systems to achieve a bit error probability of 10^{-3} as that of their differentially coherent versions. This example indicates that the phase precision requirement for QPSK is quite stringent if a reliable coherent detection is to be achieved.

The results in Tables I and II and in Figs. 3 and 4 indicate that the detection loss is dependent on modulation scheme, diversity order and reliability (or error probability) level. The values in these tables are much larger than those in Fig. 2 and can be made even larger by calculating the detection losses for higher SNR reliability values. For example, for a third-order diversity system with an rms phase error of 20°, the loss in the mean SNR is 0.52 dB for BPSK and 0.32 for QPSK. On the other hand, the detection loss at 99% reliability becomes 0.67 dB for BPSK and 3.68 dB for QPSK. For a bit error probability of 10^{-3}, the detection loss is 0.64 dB for BPSK and 5.7 dB for QPSK. We conclude that imperfect carrier recovery affects the mean SNR negligibly compared to its effect on deep fades or small bit error probabilities.

The irreducible error floors for several rms phase error values are given in Table V for BPSK and in Table VI for QPSK. We note that the effect of the error floor may not be ignored, particularly for QPSK with low diversity orders.

Thus far, the emphasis of this paper was to determine the degradation in the EGC due to imperfect carrier recovery. Thus, it was possible to calculate the decrease in the mean SNR or the detection loss for any given level of SNR reliability or bit error probability. These results are useful in the design of coherent systems, where the detection loss can be interpreted as an additional margin in the power link budget to compensate for the power loss due to imperfect carrier recovery. Alternatively, the analysis presented in this paper entitles one to calculate the diversity gain in the presence of carrier phase errors for any desired SNR reliability or bit error probability level. For example,

TABLE V
IRREDUCIBLE ERROR FLOOR FOR BPSK EGC DUE TO CARRIER PHASE ERRORS

σ_ε	DIVERSITY ORDER, L			
	1	2	3	4
18^0	5.49E–6	5.11E–7	4.97E–8	4.11E–9
20^0	4.27E–5	5.73E–6	6.32E–7	6.54E–8
25^0	9.96E–4	1.34E–4	2.36E–5	3.20E–6
30^0	5.54E–3	1.35E–3	2.46E–4	7.25E–5

TABLE VI
IRREDUCIBLE ERROR FLOOR FOR QPSK EGC DUE TO CARRIER PHASE ERRORS

σ_ε	DIVERSITY ORDER, L			
	1	2	3	4
12^0	1.20E–4	4.59E–6	3.64E–7	3.04E–8
15^0	1.61E–3	1.12E–4	9.06E–6	9.43E–7
20^0	1.30E–2	2.34E–3	4.79E–4	8.12E–5
25^0	3.58E–2	1.10E–2	3.22E–3	8.34E–4

at bit error probability of 10^{-3} with perfect coherent detection, the EGC offers diversity gains of 12.3, 16.6, and 19.0 dB, corresponding to $L = 2, 3,$ and 4. With rms phase error of 16° in every diversity branch and to maintain the same bit error probability, the diversity gains drop to 12.0, 16.3, and 18.7 dB in the binary case and to 7.9, 14.8, and 17.7 dB in the quaternary case.

VI. CONCLUSIONS

An efficient technique based on Gram–Charlier series expansion is presented for the computation of the bit error probability of EGC with imperfectly coherent Rayleigh signals. This technique utilizes the tendency of the distribution of the combiner decision variable to normal by expanding its pdf in orthogonal polynomials derived from the Gaussian distribution. In most cases, the series can be calculated with a considerable accuracy by using only a few terms of the expansion. This technique is found to be computationally much stable when compared to the Taylor series expansion method.

While the impact of imperfect carrier recovery on the combiner mean SNR is negligible, it is shown that it may not be neglected for deep fades or small bit error probabilities. Due to the cross-talk interference between the demodulated carriers, the QPSK requires much higher degrees of phase precision as compared to the BPSK case. Consequently, diversity improvements for BPSK and QPSK are no longer the same under imperfect coherent detection condition. Also, in this paper, we have compared the error probabilities of BPSK and QPSK systems to their differential versions and computed the phase precision requirement to maintain equal performances. Finally, we have provided tables for the detection loss as a function of the rms phase error at several SNR reliability and bit error probability levels. These results can be utilized in the design of practical coherent systems. One general conclusion that can be drawn from the previous analysis is that systems with diversity are less susceptible to the effect of phase errors than those without diversity.

APPENDIX A
MOMENTS OF THE COMBINED SIGNAL AT THE OUTPUT OF THE EGC

The EGC forms the decision variable that consists of the signal term $S = \sum_{\ell=1}^{L} \alpha_\ell \cos \varepsilon_\ell$ for BPSK and $(S \pm C) = \sqrt{2} \sum_{\ell=1}^{L} \alpha_\ell \cos(\varepsilon_\ell \mp (\pi/4))$ for QPSK. In this Appendix, we derive expressions for the kth moment of S and $(S \pm C)$.

Consider the general case

$$A(\lambda) = \sum_{\ell=1}^{L} A_\ell(\lambda) = \sum_{\ell=1}^{L} \alpha_\ell \cos(\varepsilon_\ell + \lambda) \tag{49}$$

from which we can express the desired random variables as

$$S = A(0) \tag{50}$$

and

$$(S \pm C) = \sqrt{2}\, A\left(\mp\frac{\pi}{4}\right). \tag{51}$$

Let

$$\mu_{\alpha_\ell}^k = \langle \alpha_\ell^k \rangle = \int_0^\infty 2\alpha^{k+1} e^{-\alpha^2}\, d\alpha. \tag{52}$$

This integral can be expressed in terms of the Gamma function as [1]

$$\mu_{\alpha_\ell}^k = \Gamma\left(1 + \frac{k}{2}\right), \qquad \ell = 1, 2, \cdots, L. \tag{53}$$

Denote by $\mu_{\varepsilon_\ell}^k(\lambda)$ the kth-cosine moment (shifted by λ radian) of the phase error ε_ℓ, i.e.,

$$\begin{aligned}\mu_{\varepsilon_\ell}^k(\lambda) &= \langle \cos^k(\varepsilon_\ell + \lambda) \rangle \\ &= 2^{-k} \langle (\exp[j(\varepsilon_\ell + \lambda)] + \exp[-j(\varepsilon_\ell + \lambda)])^k \rangle.\end{aligned} \tag{54}$$

Using the binomial expansion, we have

$$\begin{aligned}\mu_{\varepsilon_\ell}^k(\lambda) = 2^{-k} &\sum_{n=0}^{k} C_n^k \exp[j(2n-k)\lambda] \\ &\cdot \{\langle \cos(2n-k)\varepsilon_\ell \rangle + j\langle \sin(2n-k)\varepsilon_\ell \rangle\}\end{aligned} \tag{55}$$

where $C_n^k = (k!/n!(k-n)!)$. Using (7), the above expression can be written in terms of the modified Bessel functions as

$$\begin{aligned}\mu_{\varepsilon_\ell}^k(\lambda) = 2^{-k} &\sum_{n=0}^{k} C_n^k \frac{I_{|2n-k|}(\gamma_\ell)}{I_0(\gamma_\ell)} \cos(2n-k)\lambda, \\ &(\ell = 1, 2, \cdots, L)\end{aligned} \tag{56}$$

where γ_ℓ is the SNR in the ℓth loop.

Denote by $\mu_{A_\ell}^k(\lambda)$ the kth moment of the ℓth term in (49). Since $\{\varepsilon_\ell\}$ and $\{\alpha_\ell\}$ are statistically independent, we have

$$\mu_{A_\ell}^k(\lambda) = \langle A_\ell^k(\lambda) \rangle = \mu_{\alpha_\ell}^k \mu_{\varepsilon_\ell}^k(\lambda), \qquad \ell = 1, 2, \cdots, L. \tag{57}$$

Denote by T_m the partial sum $\sum_{\ell=1}^{m} A_\ell(\lambda)$. Let

$$\mu_{T_m}^k(\lambda) = \langle T_m^k(\lambda) \rangle, \qquad 1 \le m \le L. \tag{58}$$

A recursive formula for computing the moments of T_m can be developed by writing $\mu_{T_m}^k$ as

$$\mu_{T_m}^k(\lambda) = \langle (T_{m-1}(\lambda) + A_m(\lambda))^k \rangle, \qquad 1 < m \le L. \tag{59}$$

Using the binomial expansion and interchanging the orders of summation and expectation, we can express $\mu_{T_m}^k$ recursively in terms of the moments of the sum of the first $(m-1)$ terms of the random variables and the moments of the mth term as [2]

$$\mu_{T_m}^k(\lambda) = \sum_{n=0}^{k} C_n^k \mu_{T_{m-1}}^n(\lambda) \mu_{A_m}^{k-n}(\lambda), \qquad 1 < m \le L \tag{60}$$

with $\mu_{T_m}^0(\lambda) = 1$, $1 \le m \le L$, and $\mu_{T_1}^k(\lambda) = \mu_{A_1}^k(\lambda)$, $k \ge 0$. Finally, we obtain

$$\langle S^k \rangle = \langle A^k(0) \rangle = \mu_{T_L}^k(0) \tag{61}$$

and

$$\langle (S \pm C)^k \rangle = 2^{k/2} \left\langle A^k\left(\frac{\pi}{4}\right) \right\rangle = 2^{k/2} \mu_{T_L}^k\left(\frac{\pi}{4}\right). \tag{62}$$

APPENDIX B
COMPUTATION OF THE GENERALIZED MOMENTS $\{a_k\}$ AND $\{b_k\}$

For the BPSK case, we have from (27)

$$a_k = \int_{-\infty}^{\infty} H_k(\omega) f_u(\omega)\, d\omega, \qquad k = 0, 1, 2, \cdots. \tag{63}$$

Inserting (25) in (63) and interchanging the operations of integration and summation, we obtain

$$a_k = \sum_{n=0}^{\lfloor k/2 \rfloor} h_{kn} \mu_u^{k-2n}, \qquad k = 0, 1, 2, \cdots \tag{64}$$

where μ_u^i, $i = 0, 1, 2, \cdots, k$ are the moments of the standardized random variable u defined in (22). Denoting by $\mu_{S_0}^i$ the ith-central moment of the random variable S and using the binomial expansion gives

$$\mu_u^i = \left[1 + \left(\frac{2\rho}{L}\right)\sigma_S^2\right]^{-i/2} \sum_{n=0}^{i} C_n^i \left(\frac{2\rho}{L}\right)^{n/2} \mu_{S_0}^n \mu_n^{i-n} \tag{65}$$

where

$$\mu_n^i = \int_{-\infty}^{\infty} t^i \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt = \begin{cases} \dfrac{2^{(i+1)/2}}{\sqrt{2\pi}} \Gamma\left(\dfrac{i+1}{2}\right), & i \text{ even} \\ 0, & i \text{ odd} \end{cases} \tag{66}$$

is the ith moment of the standardized Gaussian random variable.

Similarly, for the QPSK case, we have from (44)

$$\begin{aligned}b_k &= \int_{-\infty}^{\infty} H_k(\omega) f_v(\omega)\, d\omega \\ &= \sum_{n=0}^{\lfloor k/2 \rfloor} h_{kn} \mu_v^{k-2n}, \qquad k = 0, 1, 2, \cdots\end{aligned} \tag{67}$$

where μ_v^i is the ith moment of the standardized random variable v defined in (42). Denote by $\mu_{X_0}^i$ the ith-central moment of the random variable $(S + C)$. Using the binomial expansion for v, we have

$$\mu_v^i = \left[1 + \frac{2\rho}{L}(\sigma_S^2 + \sigma_C^2)\right]^{-i/2} \sum_{n=0}^{i} C_n^i \left(\frac{2\rho}{L}\right)^{n/2} \mu_{X_0}^n \mu_n^{i-n}. \tag{68}$$

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