# Appendix: Proofs

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**Proposition 1**

Before proving this proposition, we need the following lemma:

**Lemma 1** *Let $\mathscr{M} = (S, R, \mathscr{V})$ be an FVEL model, $s \in S$, and $\mathscr{M}!$ be a cluster consolidation of $\mathscr{M}$ implementing $h \in H$ for agent i. If $u, v \in U_s$, then $\mathscr{M}!, s_v \models B_i\varphi$ iff $\mathscr{M}!, s_u \models B_i\varphi$, for any formula $\varphi$.*

**Proof of Lemma 1:** We have $s_v R'_i t_w$ iff ($sR_it$ and $w \in \mathcal{V}_i(t)$) iff $s_u R'_i t_w$. So $\{s' \in S' \mid s_v R'_i s'\} = \{s' \in S' \mid s_u R'_i s'\}$, and as such $\mathscr{M}!, s_v \models B_i\varphi$ iff $\mathscr{M}!, s_u \models B_i\varphi$. ■

**Proof of Prop. 1:** The proof will be by induction on $n$, but we will first prove separately the case when $n = 0$.

We want to show that if $\mathscr{M}, s \models \Box_i(p^{x_1} \vee ... \vee p^{x_m})$ and:

i. $\{x_1, ..., x_m\} \subseteq h^{-1}(1)$, then $V(p, t_u) = 1$ in all states $t_u$ such that $f(s)R'_i t_u$;

ii. $\{x_1, ..., x_m\} \cap h^{-1}(1) = \emptyset$, then $\exists t_u \in S'$ s.t. $f(s)R'_i t_u$, where $V(p, t_u) = 0$;

iii. $\{x_1, ..., x_m\} \subseteq h^{-1}(0)$, then $V(p, t_u) = 0$ in all states $t_u$ such that $f(s)R'_i t_u$;

iv. $\{x_1, ..., x_m\} \cap h^{-1}(0) = \emptyset$, then $\exists t_u \in S'$ s.t. $f(s)R'_i t_u$, where $V(p, t_u) = 1$.

This entails the proposition. We will analyse each case:

$\{x_1, ..., x_m\} \subseteq h^{-1}(1)$: Let $t_u \in S'$ be such that $f(s)R'_i t_u$. Since $\mathscr{M}, s \models \Box_i(p^{x_1} \vee ... \vee p^{x_m})$, we have that $\mathscr{M}, t \models p^{x_1} \vee ... \vee p^{x_m}$ for all $t$ such that $sR_it$. But this is true for $t$ iff $M, t \models p^{x_1}$ or ... or $\mathscr{M}, t \models p^{x_m}$. Since $f(s)R'_i t_u$, it holds that $sR_it$ and $u \in \mathcal{V}_i(t)$. So $u$ is $h$-compatible (with $\mathscr{V}$ at $t$), and since $h(x_1) = ... = h(x_m) = 1$, we have $V(p, t_u) = 1$. The case for $\{x_1, ..., x_m\} \subseteq h^{-1}(0)$ is analogous.

$\{x_1, ..., x_m\} \cap h^{-1}(1) = \emptyset$: Similar to the previous case, but now we have that $h(x_1) \neq 1, ..., h(x_m) \neq 1$, so since $\mathcal{V}_i(t)$ is the set of $h$-compatible valuations (with $\mathscr{V}$ at $t$), for any $u \in \mathcal{V}_i(t)$ either $u(p) = 0$ or $u^{-p} \in \mathcal{V}_i(t)$. In either case (ii) is satisfied. Case (iv) is analogous.

We now show that the proposition hold for the base case, where $n = 1$, and then we extend the result to all $n \geq 1$ by induction.

Assume $n = 1$ and $\mathscr{M}, s \models \Box_{i_1}\Box_{i_0}(p^{x_1} \vee ... \vee p^{x_m})$. Let us prove by cases.

$\{x_1, ..., x_m\} \subseteq h^{-1}(1)$: By $\mathscr{M}, s \models \Box_{i_1}\Box_{i_0}(p^{x_1} \vee ... \vee p^{x_m})$ and the semantics of FVEL we conclude that (1) for all $t, r$ such that $sR_{i_1}tR_{i_0}r$ we have $\mathscr{M}, s \models p^{x_1} \vee$

$... \vee p^{x_m}$. From Def. 7 we have that $f(s)R'_{i_1}t_uR'_{i_0}r_v$ iff $sR_{i_1}tR_{i_0}r$ and $u \in \mathcal{V}_{i_1}(t)$ and $v \in \mathcal{V}_{i_0}(r)$. Fact (1) implies $\mathscr{M}, r \models p^{x_1}$ or ... or $\mathscr{M}, r \models p^{x_m}$. Formulas of type $\varphi^y$ are satisfied in a state $s$ iff $\overline{\mathscr{V}}(\varphi, s) = y$. This means that (1) implies $\overline{\mathscr{V}}(p, r) \in \{x_1, ..., x_m\}$. But since $\{x_1, ..., x_m\} \subseteq h^{-1}(1)$ and $\mathcal{V}_{i_0}$ is $h$-compatible with $\mathscr{V}$ at $r$, we have that $\mathscr{M}!, r_v \models p$ for all $t_u$ and $r_v$ such that $f(s)R'_{i_1}t_uR'_{i_0}r_v$. This concludes this case. The case for $\{x_1, ..., x_m\} \subseteq h^{-1}(0)$ is analogous.

$\{x_1, ..., x_m\} \cap h^{-1}(1) = \emptyset$: Similar to the previous case but now $\mathcal{V}_{i_0}$ being $h$-compatible with $\mathscr{V}$ at $r$ implies that for all $t_u$ there is some $r_v$ s.t. $f(s)R'_{i_1}t_uR'_{i_0}r_v$ with $\mathscr{M}!, r_v \models \neg p$. This concludes this case. The case for $\{x_1, ..., x_m\} \cap h^{-1}(0) = \emptyset$ is analogous.

Now we can use induction to finish the proof. As Induction Hypothesis (I.H.) we assume the proposition is valid for $n = k - 1$, and from this we prove that it is valid for $n = k$. Suppose that $\mathscr{M}, s \models \Box_{i_k} ... \Box_{i_0}(p^{x_1} \vee ... \vee p^{x_m})$. Again, let us go by cases.

$\{x_1, ..., x_m\} \subseteq h^{-1}(1)$: we have to show that $\mathscr{M}!, f(s) \models B_{i_k} ... B_{i_0}p$. By the semantics of FVEL we have that for all $t$ s.t. $sR_{i_k}t$ we have $\mathscr{M}, t \models \Box_{i_{k-1}} ... \Box_{i_0}(p^{x_1} \vee ... \vee p^{x_m})$, but the I.H. this implies $\mathscr{M}!, f(t) \models B_{i_{k-1}} ... B_{i_0}p$ But by Def. 7 we have that $s_vR'_{i_k}t_u$ iff $sR_{i_k}t$ and $u \in \mathcal{V}_{i_k}(t)$. Using Lemma 1 we have that for any such $t_u$ it holds that $\mathscr{M}!, t_u \models B_{i_{k-1}} ... B_{i_0}p$. This, of course, implies $\mathscr{M}!, f(s) \models B_{i_k} ... B_{i_0}p$, which concludes this case. The other cases are identical, since the case condition is only relevant for the application of the I.H. ■

**Proposition 2**

**Proof:** This proposition can be proven by a simple induction on the structure of $\varphi$. The base case is the case for atoms, and the Induction Hypothesis is that the proposition holds for proper subformulas of $\varphi$. ■

We omit the proofs of propositions 3, 4 and 5, which are rather simple.

**Proposition 6**

**Proof:** By the construction of $\mathsf{FV}(M)$ we know that $\mathsf{FV}(M), s \models p$ iff $M, s \models \Box p$ and the same for $\neg p$. But since $R$ consists exactly of all reflexive arrows, $\mathsf{FV}(M), s \models \Box p$ iff $\mathsf{FV}(M), s \models p$ (again, the same for $\neg p$). ■

**Proposition 7**

**Proof:** Let us first show that $\mathscr{M} \models \Box p$ entails $\mathsf{BP}(\mathscr{M}), s_v \models \Box p$. Let us assume $\mathscr{M} \models \Box p$. We need to show that (i) $\exists X \in E(s_v)$ such that $\forall t \in X$ it holds that $\mathsf{BP}(\mathscr{M}), t \models p$.

$S'$ is not necessarily a piece of evidence matching the $X$ of condition (i), so we have to check whether there is some $X_p$ according to Def. 22 respecting those conditions. But $X_p$ can only fail the condition if $\exists t_u \in X_p$ s.t. $\mathsf{BP}(\mathscr{M}), t_u \models \neg p$, which means that $t_u \notin V(p)$ and thus $u(p) = 0$. If $X_p$ is built according to Def. 22 this is not possible. So, if we can prove that a non-empty $X_p$ according to Def. 22 exists, we are done. $C(s)$ is empty iff $\nexists t$ s.t. $sRt$, but since the model is serial this is not possible. So $C(s)$ is non-empty and $\mathscr{M}, s \models \Box p$ is assumed, so we just need to guarantee that there is one $t_u \in C(s)$ s.t. $u(p) = 1$. But since $\mathscr{M}, s \models \Box p$, for all $t$ s.t. $sRt$ we have $\mathscr{M}, t \models p$, which by the definition of $S'$, $V$ and $C(s)$ will guarantee that for all such $t$ there is at least one $u$ s.t. $t_u \in C(s)$ and $u(p) = 1$. This concludes this direction.

For the other direction, we will prove that $\mathscr{M} \not\models \Box p$ entails $\mathsf{BP}(\mathscr{M}), s_v \not\models \Box p$, which gives us the desired result by *modus tollens*. We assume the former, which entails

$\exists t$ s.t. $sRt$ and $\mathscr{M}, t \not\models p$. Now for $\mathsf{BP}(\mathscr{M}), s_v \not\models \Box p$ we just have to show that $\nexists X \in E(s_v)$ s.t. $\forall t_u \in X$, $\mathsf{BP}(\mathscr{M}), t_u \models p$. We will show that this condition is indeed not satisfied by any $X \in E(s_v)$, for each case of Def. 22.

$X = S'$. If $\forall t_u \in S'$ it holds that $\mathsf{BP}(\mathscr{M}), t_u \models p$, then there is no $t_u \in S'$ s.t. $u(p) = 0$. By the definition of $S'$, this means that in all states $w$, $\overline{\mathscr{V}}(p, w) = t$. But this contradicts our assumption that $\exists t$ s.t. $sRt$ and $\mathscr{M}, t \not\models p$.

$X = X_p$, where $X_p \subseteq C(s)$ and $t_u \in X_p$ iff $\mathscr{M}, s \models \Box p$ and $u(p) = 1$. Since we are assuming $\mathscr{M}, s \not\models \Box p$, there is no non-empty $X_p$ satisfying these conditions.

$X = X_q$, where $q \neq p$, $X_q \subseteq C(s)$ and $t_u \in X_q$ iff $\mathscr{M}, s \models \Box q$ and $u(q) = 1$. Since $\exists t$ s.t. $sRt$ and $\mathscr{M}, t \not\models p$, then by the definitions of $S'$, $V$ and $C(s)$ there is a $t_u \in C(s)$ s.t. $u(p) = 0$. Moreover, for any $u \in \mathit{Val}_t^{h_1}$ s.t. $u(q) = 1$ (as required by any $t_u \in X_q$) there is a $u' \in \mathit{Val}_t^{h_1}$ s.t. $u'(r) = u(r)$ for all $r \neq p$ and $u'(p) = 0$ – by the combinatorial nature of $\mathit{Val}_t^{h_1}$. So $\exists t_u \in X_q$ s.t. $\mathsf{BP}(\mathscr{M}), t_u \not\models p$.

$X = X_{\neg p}$, where $X_{\neg p} \subseteq C(s)$ and $t_u \in X_{\neg p}$ iff $\mathscr{M}, s \models \Box\neg p$ and $u(p) = 0$. If $X_{\neg p}$ is non-empty, then by definition $\forall t_u \in X_{\neg p}$ has $\mathsf{BP}(\mathscr{M}), t_u \not\models p$.

$X = X_{\neg q}$, where $q \neq p$, $X_{\neg q} \subseteq C(s)$ and $t_u \in X_{\neg q}$ iff $\mathscr{M}, s \models \Box\neg q$ and $u(q) = 0$. The argument is identical to the $X = X_q$ case.

The cases for $\neg p$ are completely analogous. ■

**Proposition 9**

**Proof:** SIMP $\Rightarrow$ GOOD: easily verifiable.

SIMP $\Rightarrow$ COMP: easily verifiable (take $\{s\}$ as $X$).

SIMP $\Rightarrow$ CONS: easily verifiable (if all states in $S$ support $p$, then $s$ supports $p$; if $s$ supports $p$, then there is a state in $S$ which supports $p$: $s$ itself).

For the following proofs we assume a B&P model $M = (S, E, V)$, $\mathsf{FV}(M) = (S, R, \mathscr{V})$ and $\mathsf{BP}(\mathsf{FV}(M)) = (S', E', V')$. Preservation of $S$, more precisely, means $|S| = |S'|$. Preservation of $V$ means that there is a bijection $f$ from $S$ to $S'$ such that for all $s \in S$ and all $p \in At$: $s \in V(p)$ iff $f(s) \in V'(p)$. Preservation of $E$ means that there is a bijection $f$ from $S$ to $S'$ such that $\forall X \subseteq S$: $X \in E(s)$ iff $\{f(w) \mid w \in X\} \in E'(f(s))$.

**Preservation of** $S$: First, let us show that CONS and COMP imply $|S| = |S'|$, then the converse. By the definition of $\mathsf{FV}$ (Def. 19), we know that a proposition $p$ in some state $s$ of $\mathsf{FV}(M)$ cannot have value *both* unless $M, s \models \Box p$ and $M, s \models \Box\neg p$. CONS prevents this. For *none*, $M, s \not\models \Box p$ and $M, s \not\models \Box\neg p$ are needed. COMP prevents this. So CONS and COMP together imply that $\mathsf{FV}(M)$ does not have any atom in any state with value $b$ or $n$. So by the definitions of $S'$ (in Def. 22) and of $\mathit{Val}_s^{h_1}$ we know that each state will have only one accepted valuation, and therefore $|S| = |S'|$.

Now let us show that $|S| = |S'|$ implies CONS and COMP. If CONS is violated, then for some $p, s$ we have $M, s \models \Box p$ and $M, s \models \Box\neg p$. If COMP is violated, then for some $p, s$ we have $M, s \not\models \Box p$ and $M, s \not\models \Box\neg p$. In either case, $\mathsf{FV}(M)$ will have some proposition with value $b$ or $n$, which again by Def. 22 will imply that $|S'| > |S|$.

**Preservation of** $V$: First, let us show that CONS, COMP and GOOD imply that $V$ is preserved. We just showed that CONS and COMP imply $|S| = |S'|$. For all $s \in S$, let $f(s) = s_v$, where $s_v \in S'$. We have to show that for all $s, p$: $s \in V(p) \Rightarrow s_v \in V'(p)$ and $s_v \in V'(p) \Rightarrow s \in V(p)$.

By GOOD, $s \in V(p)$ implies that $M, s \models \Box p$. This implies that $1 \in \mathscr{V}(p, s)$. CONS implies $M, s \not\models \Box\neg p$, which makes $\mathscr{V}(p, s) = t$. Now there is only one $v$ s.t. $s_v \in S'$, and by the definition of $S'$ we have that $v(p) = 1$, and therefore $s_v \in V'(p)$.

For the other direction, we assume $s_v \in V'(p)$. This implies $v(p) = 1$, but by CONS and COMP we know this $v$ is unique, which means that $\mathscr{V}(p, s) = t$, which is only the case if $M, s \models \Box p$ and $M, s \not\models \Box\neg p$. By GOOD, we derive that $s \in V(p)$.

Now we give a counterexample for why preservation of $V$ does not imply GOOD. Let $S = \{s, t\}$, with $s \in V(p)$ and $t \notin V(p)$, $E(s) = \{\{t\}, S\}$ and $E(t) = \{\{s\}, S\}$ (notice that this violates GOOD). Now, $\mathsf{BP}(\mathsf{FV}(M))$ will have $S' = \{s_v, t_u\}$ for some $v, u$. If we make $f(s) = t_u$ and $f(t) = s_v$, $V$ is preserved, but GOOD does not hold.

**Preservation of** $E$: First, let us show that SIMP implies the preservation of $E$. Since SIMP entails the other conditions, we know that it also preserves $S$ and $V$. Let $f(s) = s_v$, where $s_v \in S'$ (this bijection was just shown to preserve $V$). Given this and $E(s) = \{\{s\}, S\}$ for all $s$ (SIMP), we just need to show that $E'(s_v) = \{\{s_v\}, S'\}$. By CONS and COMP and the definition of $S'$ we have that there is only one valuation compatible with each $s \in S$, and therefore (by the definition of $C(s)$) $E'(s_v) \subseteq \{\{s_v\}, S'\}$. Now $S' \in E'(s, v)$, so we only have to show that $\{s_v\} \in E'(s_v)$. First, note that if $v(p) = 1$ then $\mathscr{M}, s \models \Box p$ (and if $v(p) = 0$ then $\mathscr{M}, s \models \Box\neg p$), by the definition of $\mathit{Val}_s^{h_1}$ and $R$. So $\{s_v\}$ will be added either as $X_p$ or $X_{\neg p}$ (definition of $E'(s_v)$ in Def. 22).

Now we show that the preservation of $E$ entails SIMP. The preservation of $E$ entails the existence of a bijection between $S$ and $S'$, which in turn entails CONS and COMP. The exact same reasoning as in the previous proof can be used to show that $E'(s_v) = \{\{(s_v)\}, S'\}$. Now assume $f(t) = (s_v)$, for some $t$ (this $t$ has to exist as $f$ is a bijection). Then $E'(f(t)) = \{\{f(t)\}, S'\}$. But since $E$ is preserved, $E(t) = \{\{t\}, S\}$, and since $t$ is arbitrary, this just proves SIMP. ■

**Corollary 3**

**Proof:** Prop. 27 just showed that SIMP implies CONS, COMP and GOOD, which in turn imply the preservation of $V$ and $S$. So SIMP implies the preservation of $S$, $V$ and $E$. Moreover, in the proof of Prop. 27 we saw that there is a bijection that preserves simultaneously $V$ and $E$. This guarantees that $M \cong \mathsf{BP}(\mathsf{FV}(M))$. If $M \cong \mathsf{BP}(\mathsf{FV}(M))$ holds, then obviously $E$ is preserved, which in turn implies, by Prop. 27, that SIMP holds. (Notice that the importance of this corollary is not just to show that the satisfaction of SIMP is equivalent to the preservation of $S$, $R$ and $V$, but to show that this preservation occurs under one and the same bijection.) ■

**Proposition 10**

**Proof:** Let $\mathscr{M} = (S, R, \mathscr{V})$, $\mathsf{BP}(\mathscr{M}) = (S', E, V)$ and $\mathsf{FV}(\mathsf{BP}(\mathscr{M})) = (S', R', \mathscr{V}')$. That REFL entails KNOW is easy to check.

**Preservation of** $S$: If CLAS holds, for each $s \in S$ there will be only one $v \in \mathit{Val}_s^{h_1}$, so $|S| = |S'|$. Now for the other direction we will assume that CLAS does not hold. Then there is some $p \in \mathit{At}$ and $s \in S$ s.t. $\mathscr{V}(p, s) \in \{b, n\}$. In either case, by the definition of $S'$ (Def. 22), there will be more than one $v \in \mathit{Val}_s^{h_1}$, and since for any $t$ there is at least one $u \in \mathit{Val}_t^{h_1}$, we have $|S'| > |S|$.

**Preservation of** $\mathscr{V}$: By this we mean that there is a bijection $f$ from $S$ to $S'$ s.t. for all $p \in \mathit{At}$ and all $s \in S$: $\mathscr{V}(p, s) = \mathscr{V}'(p, f(s))$.

First we show that CLAS and KNOW imply the preservation of $\mathscr{V}$. By CLAS we have $|S| = |S'|$, and by the def. of $S'$ (Def. 22) there is a unique $v \in Val_s^{h_1}$, so take $f(s) = s_v$ where $v$ is s.t. $s_v \in S'$. By Prop. 23 we know that $\mathscr{M}, s \models \Box p$ iff $\mathsf{BP}(\mathscr{M}), s_v \models \Box p$. By Def. 19 we know that $1 \in \mathscr{V}'(p, s_v)$ iff $\mathsf{BP}(\mathscr{M}), s_v \models \Box p$. Thus, $1 \in \mathscr{V}'(p, s_v)$ iff $\mathscr{M}, s \models \Box p$ and, by KNOW, $\mathscr{M}, s \models \Box p$ iff $\mathscr{M}, s \models p$, which boils down to $1 \in \mathscr{V}'(p, s_v)$ iff $1 \in \mathscr{V}(p, s)$. The reasoning for 0 and $\neg p$ is analogous. Since $f(s) = s_v$, $\mathscr{V}$ is preserved.

Now we show that the preservation of $\mathscr{V}$ implies CLAS, but does not imply KNOW. If $\mathscr{V}$ is preserved then there is a bijection between $S$ and $S'$, therefore $S$ is preserved, which implies CLAS (as we shown above). Now a counterexample of $\mathscr{M}$ where $\mathscr{V}$ is preserved but KNOW does not hold. Let $S = \{s, t\}$, $R = \{(s, t), (t, s)\}$ and $\mathscr{V}(p, s) = t$ and $\mathscr{V}(p, t) = f$. Let $S' = \{s_v, t_u\}$. If we make $f(s) = t_u$ and $f(t) = s_v$, $\mathscr{V}$ is preserved, but KNOW does not hold.

**Preservation of** $R$: By this we mean that there is a bijection $f$ from $S$ to $S'$ s.t. $sRt$ iff $f(s)R'f(t)$, for all $s, t \in S$.

First let us show that CLAS and REFL together imply the preservation of $R$. By CLAS we have $|S| = |S'|$, and by Def. 19 we have $R' = \{(s_v, s_v) \mid s_v \in S'\}$. Since REFL means $R = \{(s, s) \mid s \in S\}$ for all $s \in S$, just take $f(s) = s_v$, with $v$ s.t. $s_v \in S'$, for all $s \in S$.

The other direction: since $R' = \{(s', s') \mid s' \in S'\}$, and we have a bijection $f$ between $S$ and $S'$, we conclude that $s'R't'$ iff $f^{-1}(s')Rf^{-1}(t')$, and therefore $R = \{(s, s) \mid s \in S\}$. ■

**Corollary 4**
**Proof:** The only thing worth noting here is that CLAS and REFL imply KNOW, and by CLAS and KNOW we have that $\mathscr{V}$ is preserved with the bijection $f(s) = (s, v)$ for $v$ s.t. $(s, v) \in S'$. The same bijection, as shown before, under CLAS and REFL, preserves $R$. Again, this is to guarantee that these properties not only preserve $S$, $R$ and $\mathscr{V}$, but also do so under one and the same bijection. ■

**Proposition 11**
**Proof:** The proof will be by induction on the structure of $\varphi$. Base: $\varphi$ atomic; the proposition holds because $V$ is the same for $M$ and $M!$. I.H.: $M \models \varphi'$ iff $M! \models \varphi'$ for $\varphi'$ subformula of $\varphi$. Step: $M, s \models \neg\varphi$ iff $M', s \not\models \varphi$ iff (by I.H.) $M', s \not\models \varphi$ iff $M', s \models \neg\varphi$. $M, s \models \varphi \wedge \psi$ iff ($M, s \models \varphi$ and $M, s \models \psi$) iff (by I.H.) ($M', s \models \varphi$ and $M', s \models \psi$) iff $M', s \models \varphi \wedge \psi$. $M, s \models B\varphi$ iff (for all $s$-scenarios $\chi$, $\forall t \in \bigcap \chi$: $M, t \models \varphi$) iff (by I.H.) (for all $s$-scenarios $\chi$, $\forall t \in \bigcap \chi$: $M', t \models \varphi$) iff ($\forall t$ s.t. $sB_Et$: $M', t \models \varphi$) iff $M', s \models B\varphi$. ■

**Proposition 12**
**Proof:** Let $M = (S, E, V)$, $\mathscr{M} = (S', R, \mathscr{V})$, $M! = (S, B_E, V)$ and $\mathscr{M}! = (S'', R', V')$.

$\Leftarrow$: Since $V$ and $\mathscr{V}$ match, $\mathscr{V}$ is classical (that is, it only assigns values $t$ and $f$), which means that there will be a one-to-one correspondence between states of $\mathscr{M}$ and $\mathscr{M}!$. $M$ and $M!$ already have the same states, so through $M \doteq \mathscr{M}$ we have a correspondence between states of $M!$ and $\mathscr{M}!$. They will also have the same valuation, because the valuations of $M$ and $\mathscr{M}$ match, $V$ is the same for $M$ and $M!$, and by the definition of cautious consolidation $\mathscr{M}!$ will also have the same valuation as $M!$. Now, by assumption, $M!$ and $\mathscr{M}$ have matching valuations, and since $\mathscr{V}$ is classical, by the definition of cautious consolidation we have that $R$ will be identical to $R'$ under the bijection specified earlier, and by assumption $R$ is isomorphic to $B_E$.

$\Rightarrow$: Since $M \stackrel{\circ}{=} \mathscr{M}$, these models have the same number of states. The same goes for $M!$, and since $M! \cong \mathscr{M}!$, $\mathscr{M}!$ also has the same number of states. If $\mathscr{V}$ were not classical, $\mathscr{M}!$ would have more states than $\mathscr{M}$, therefore $\mathscr{V}$ is classical.

Since $S$ is the same for $M$ and $M!$, we can use $f$ to map states of $M!$ into $\mathscr{M}$. By the definition of cluster consolidation and the fact that $\mathscr{V}$ is classical we conclude that $R$ and $R'$ will be isomorphic, but since $M! \cong \mathscr{M}!$, this implies that $R$ is isomorphic to $B_E$ (in other words: the last condition of this proposition holds).

For each state of $\mathscr{M}$ there is only one accepted valuation, and this valuation is compatible with $\mathscr{V}$. Since $\mathscr{V}$ is classical, we will have that $V'$ will match it. Now $V$ and $V'$ are isomorphic by assumption, so $V$ and $\mathscr{V}$ will match. ■