Appendix: Proofs

Proof of Prop. 1: ($\mathscr{L}_0$) This can be proved easily by induction, consulting the truth tables in [1]. Base case: atoms. Clearly all atoms have truth range $\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. I.H.: for any φ' proper subformula of φ, the proposition holds. Step: $\varphi = {\sim}\psi$. Clearly any formula of this format can only have truth values $\{0\}$ or $\{1\}$, and therefore satisfy the proposition. $\varphi = \psi \wedge \chi$. This case is tedious but easy. We just have to check what are the possible truth values for $\psi \wedge \chi$ given each truth range for ψ and χ. By the I.H., ψ and χ have one of the truth ranges listed in the proposition. As an example, let us check the case for when ψ and χ have truth ranges $\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ and $\{\{0\}, \{1\}\}$, respectively. Well, in that case the possible truth values for $\psi \wedge \chi$ are the values in the truth table when we restrict one of the parameters to $\{0\}$ and $\{1\}$, which gives us the truth range $\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. If we do the same considering each of the other truth ranges listed in the proposition for ψ and χ, we conclude that all possible truth ranges for $\psi \wedge \chi$ are within the ones listed in the proposition statement.

($\mathscr{L}_1$) If a valuation assigns no value $\emptyset$ or $\{0,1\}$ to any atom, then all formulas have "classical" truth values ($\{0\}$ or $\{1\}$), so it is not possible to have formulas with the truth ranges mentioned in the statement of the proposition. To show that the other truth ranges are possible, we give examples, followed by their truth ranges: ${\sim}(p \wedge {\sim}p)$: $\{\{1\}\}$. ${\sim}{\sim}(p \wedge {\sim}p)$: $\{\{0\}\}$. ${\sim}p$: $\{\{0\}, \{1\}\}$. p: $\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. $p \wedge {\sim}p$: $\{\emptyset, \{0\}\}$. $\neg(p \wedge {\sim}p)$: $\{\emptyset, \{1\}\}$. $p \wedge \neg p$: $\{\emptyset, \{0\}, \{0,1\}\}$. $\neg(p \wedge \neg p)$: $\{\emptyset, \{1\}, \{0,1\}\}$. $p \wedge \neg p \wedge {\sim}(p^n)$: $\{\{0\}, \{0,1\}\}$. $\neg(p \wedge \neg p \wedge {\sim}(p^n))$: $\{\{1\}, \{0,1\}\}$. $p \wedge {\sim}(p^n)$: $\{\{0\}, \{1\}, \{0,1\}\}$. $p \wedge {\sim}(p^b)$: $\{\emptyset, \{0\}, \{1\}\}$. ■

Proof of Prop. 2: Take a contingent $\varphi \in \mathscr{L}_0$ and two agents $s, t \in S$, $s \neq t$, and a model $M = (S, R, V)$. If $\mathrm{Att}(\varphi, s) \neq \mathrm{Att}(\varphi, t)$, then we are done, do let us assume that $\mathrm{Att}(\varphi, s) = \mathrm{Att}(\varphi, t)$. Also, assume that neither sRt nor tRs hold. By Modesty, there is a $M' = (S', R', V')$ with $S \subseteq S'$ such that $\mathrm{Att}'(\varphi, s) \neq \mathrm{Att}(\varphi, s)$. Now notice that we can build a M'' by adding to M' an isomorphic copy of M (with fresh agent labels, say from s to $s*$). Now we can exchange the agent label of t (which was already in M') with the relabelled $t*$ (that came from the copy of M). In this way, $(M'', s) \leftrightarrows (M', s)$ and $(M'', t) \leftrightarrows (M, t)$. By Prop. 3, $\mathrm{Att}''(\varphi, s) = \mathrm{Att}'(\varphi, s)$ and $\mathrm{Att}''(\varphi, t) = \mathrm{Att}(\varphi, t)$, and therefore $\mathrm{Att}''(\varphi, s) \neq \mathrm{Att}(\varphi, t)$. ■

Proof of Obs. 1: Naive consolidation satisfies Atom Independence but violates Doxastic Freedom. For the other direction, consider a set of atoms $At = \{p_1, p_2, ...\}$, and a consolidation similar to Policy V, but which instead of deciding Bp_i based on p_i, does the following: if At is infinite, decides p_i based on p_{i+1} for odd i, and based on p_{i-1} for even i; if $At = \{p_1, ..., p_n\}$ is finite, decides belief in p_i based on p_{i+1}, except for p_n, which is decided based on p_1. Policy V and this modification satisfy Doxastic Freedom, but this modification does not satisfy Atom Independence (and therefore is not a $\mathscr{C}$-consolidation). ■

Proof of Obs. 2: One just has to see that $M, s \models B\varphi$ implies $M, s \not\models B{\sim}\varphi$ for a consolidations satisfying Consistency (and similarly for the $B{\sim}\varphi$ case). ■

Proof of Prop. 3: The first direction is easy to prove by induction on the structure of φ. Base: it is immediately evident (by Def. 11) that if $(M, s) \leftrightharpoons (M', s')$ then $M, s \models \varphi$ iff $M', s' \models \varphi$, for all $\varphi \in \mathscr{L}_1$. I.H.: For all φ' proper subformula of φ, $(M, s) \leftrightharpoons (M', s')$ implies $M, s \models \varphi'$ iff $M', s' \models \varphi'$. Step: $\varphi = {\sim}\psi$. By I.H. $M, s \models \psi$ iff $M', s' \models \psi$, but then $M, s \not\models \psi$ iff $M', s' \not\models \psi$. $\varphi = \psi \wedge \chi$. By I.H., $M, s \models \psi$ iff $M', s' \models \psi$ and $M, s \models \chi$ iff $M', s' \models \chi$. Then, $M, s \models \psi \wedge \chi$ iff $M', s' \models \psi \wedge \chi$. $\varphi = \Box\psi$, where ψ has no $\Box$ nor B. Since $(M, s) \leftrightharpoons (M', s')$, for all t such that sRt there is a t' such that $s'R't'$ with $V(p, t) = V'(p, t')$ for all $p \in At$. Then, since ψ has no $\Box$ nor B, $\psi \in \mathscr{L}_1$, and for any $\psi' \in \mathscr{L}_1$ and t such that sRt, $M, t \models \psi'$ implies that there exists a t' such that $s'R't'$ and $M', t' \models \psi'$. The other direction follows by **back**.

Now for the other direction (the second part of the proposition). First, from $M, s \models \varphi$ iff $M', s' \models \varphi$ for $\varphi \in \mathscr{L}$ not containing B nor nested $\Box$, we can easily see that **atoms** holds. The argument for **back** and **forth** are analogous, so we just show **forth** here. Consider a t such that sRt, and consider the set $\Sigma = \{p^x \mid p \in At \text{ and } M, t \models p^x, \text{ where } x \in \mathscr{P}(\{0, 1\})\}$. We want to show that there is a t' such that $s'R't'$ and $V(p, t) = V'(p, t')$ for all $p \in At$. For any finite conjunction γ of elements of Σ, we have $M, t \models \gamma$ and therefore $M, s \models \Diamond\gamma$. But then $M', s' \models \Diamond\gamma$, as $\gamma \in \mathscr{L}_1$. This implies that every finite conjunction γ of elements of Σ are satisfied in some successor of s'. Assume, then, that no successor of s' satisfies all elements of Σ. Then, for each such successor t'_i there is a $p_i^{x_i} \in \Sigma$ such that $M', t' \not\models p_i^{x_i}$. But then, the finite conjunction $p_1^{x_1} \wedge p_2^{x_2} \wedge ... \wedge p_n^{x_n}$, where $s'R't'_1, ..., s'R't'_n$, is not satisfied in any successor of s'. Contradiction. So Σ is satisfied in some successor of s' and therefore **forth** holds. ■

Proof of Prop. 4: That reflexivity and symmetry are satisfied is trivial. One just have to check whether $\leftrightharpoons$ is also transitive, which can be done straightforwardly by checking Def. 11in the case where $(M, s) \leftrightharpoons (M', s')$ and $(M', s') \leftrightharpoons (M'', s'')$, to derive $(M, s) \leftrightharpoons (M'', s'')$. ■

Proof of Prop. 5: The logic of $\mathscr{L}_0$ is basically classical propositional logic (as mentioned in [1]), and is, therefore, compact. So for any $\Sigma \models \varphi$ with $\varphi \in \mathscr{L}_0$, there is a finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \varphi$. The case where $\varphi = p \wedge {\sim}p$ is a particular case of this. So all inconsistent subsets of $\mathscr{L}_0$ have a finite inconsistent subset. ■

Proof of Prop. 6: The reasoning is similar to the case for Prop. 5. ■

Proof of Prop. 7: ($\Rightarrow$) Suppose ${\sim}((\varphi^t \wedge \Box\varphi^t) \wedge B{\sim}\varphi)$ is not valid. Then there is a model M and state s such that $M, s \models (\varphi^t \wedge \Box\varphi^t) \wedge B{\sim}\varphi$. By semantics, we find that this is the case iff $\overline{V}(\varphi, s) = \{1\}$ and for all t such that sRt, $M, t \models \varphi^t$ and $\mathrm{Att}(\varphi, s) = -1$ (recall that Consistency is assumed). Therefore Consensus is violated. The case for **C2** is analogous.

($\Leftarrow$) Take an arbitrary M and s. Since **C1** is valid, $M, s \models {\sim}(\varphi^t \wedge \Box\varphi^t) \wedge B{\sim}\varphi$. By semantics, this corresponds to $\overline{V}(\varphi, s) = \{1\}$ and for all t such that sRt, $\overline{V}(\varphi, t) = \{1\}$ implies $M, s \not\models B{\sim}\varphi$, therefore $\mathrm{Att}(\varphi, s) \neq -1$. With similar reasoning starting from **C2**, we get the other condition for Consensus, and therefore this postulate is satisfied. ■

Proof of Prop. 10: **Consistency.** By Def. 14, the agents can only belief a consistent set of atoms, and from that, given the "classical" nature of the rules to form beliefs in complex formulas, only classical consequences of this consistent set of atoms can be derived, resulting in a consistent belief state. **Atom Independence.** Given that belief in an atom is only determined by $\mathscr{C}$, and that if V does not change for an atom p none of the parameters for $\mathscr{C}$ will change, we conclude that $\mathrm{Att}(p, s)$ will not change for any s. ■

Proof of Prop. 11: This is a straightforward proof by structural induction on φ. The only thing to pay attention here is that, on the step where $\varphi = {\sim}(\psi \wedge \chi)$, if we assume $\Sigma \models {\sim}(\psi \wedge \chi)$, we can only conclude that $\Sigma \models {\sim}\psi$ or $\Sigma \models {\sim}\chi$ (and then use the I.H.) because Σ is maximal, and therefore for any contingent formula ζ, either $\Sigma \models \zeta$ or $\Sigma \models {\sim}\zeta$. ■

Proof of Corollary 1: First, recall that $\mathscr{L}_0$ is equivalent to classical logic in the sense that if $\Sigma \models \varphi$ in classical logic, then $\Sigma \models \varphi$ in $\mathscr{L}_0$ (see [1]). Also, notice that any contingent $\varphi \in \mathscr{L}_0$ is a consequence of some consistent set of literals (of form p or ${\sim}p$). To see this just think about truth tables. Now with Prop. 11 we get that, for any $\mathscr{C}$-consolidation, if a maximally consistent set of literals is believed, its consequences are also believed. From this it follows that for any $\mathscr{C}$-consolidation satisfying Doxastic Freedom, any set of agents S with $s, t \in S$ and any contingent $\varphi \in \mathscr{L}_0$ there will be a model where $\mathrm{Att}(\varphi, s) \neq \mathrm{Att}(\varphi, t)$, which implies No Gurus. ■

Proof of Prop. 12: Suppose $M, s \models B\varphi$ and $M, s \models B{\sim}(\varphi \wedge {\sim}\psi)$. By semantics, we know that $M, s \models B{\sim}(\varphi \wedge {\sim}\psi)$ iff $M, s \models B{\sim}\varphi$ or $M, s \models B\psi$. Since $\mathscr{C}$-consolidations satisfy Consistency, $M, s \models B\varphi$ implies $M, s \not\models B{\sim}\varphi$, therefore $M, s \models {\sim}{\sim}\psi$, and by semantics $M, s \models B\psi$. ■

Proof of Prop. 13: **Doxastic Freedom.** Let $f : At \times S \to \{1, 0, -1\}$ be arbitrary. Take a model $M = (S, R, V)$ where $R = \emptyset$ and make, for all $p \in At$ and $s \in S$, $V(p, s) = \{1\}$ iff $f(p, s) = 1$, $V(p, s) = \{0\}$ iff $f(p, s) = -1$ and $V(p, s) = \emptyset$ otherwise.

Monotonicity. We have to check each case of variation in V.

$V(p, s) = \emptyset$ and $V'(p, s) = \{1\}$. In this case, by the definition of $\mathscr{C}$, $\mathrm{Att}'(p, s) = 1$. So s does not violate Monotonicity. Now take an arbitrary agent $t \neq s$. $V(p, t) = V'(p, t)$, so V^t_p and $V^t_{\neg p}$ do not change. If not tRs, then the other values also do not change, and then Monotonicity is not violated. Even if tRs, $V^t_{\Diamond\neg p}$ and $V^t_{\Box\neg p}$ do not change. Values $V^t_{\Box p}$ and $V^t_{\Diamond p}$ may change from 0 to 1. By looking at the decision trees for Policy I and II we see that These possible changes in parameters can cause the following changes from $\mathrm{Att}(p,t)$ to $\mathrm{Att}'(p,t) =$: 0 to 1, -1 to 0 and -1 to 1. This last step, of determining what are the changes in the output of $\mathscr{C}$ given the possible changes in parameters, is more reliably done computationally by a simple algorithm on the decision tree of the policy. We will not go through all the cases here, but the reasoning is similar and the last step was always checked via an algorithm.

Consensus. We will prove a stronger version of Consensus, which imply the actual postulate. Consensus': If for some agent $s \in S$ and some $\varphi \in \mathscr{L}_0$ we have that $1 \in \overline{V}(\varphi, s)$ (or $1 \notin \overline{V}(\varphi, s)$), and for all $t \in S$ such that sRt: $1 \in \overline{V}(\varphi, t)$ (or $1 \notin \overline{V}(\varphi, t)$), then $\mathrm{Att}(\varphi, s) \neq -1$ (or 1).

We prove by structural induction on φ. Base: $\varphi = p$. If $1 \in V(p, s)$ and for all t with sRt also $1 \in V(p, t)$, then (by looking at the decision trees of the policies) $M, s \not\models B{\sim}p$. Similarly for negative case where $1 \notin V(p, s)$ and $1 \notin V(p, t)$ for all t such that sRt.

Step: $\varphi = {\sim}\psi$. Suppose $1 \in \overline{V}(\varphi, s)$ (which in this case means just $\overline{V}(\varphi, s) = \{1\}$) and $1 \in \overline{V}(\varphi, t)$ for all t with sRt. But then $1 \notin \overline{V}(\psi, s)$ and $1 \notin \overline{V}(\psi, t)$ for all t with sRt. By I.H. $M, s \not\models B\psi$, but by our semantics the only way to obtain $M, s \models B{\sim}\varphi (= {\sim}{\sim}\psi)$ is if we have $M, s \models B\psi$. The negative case $(1 \notin \overline{V}(\varphi, s)...)$ is very similar.

$\varphi = \psi \wedge \chi$. Suppose $1 \in \overline{V}(\varphi, s)$ and $1 \in \overline{V}(\varphi, t)$ for all t with sRt. This implies that the valuations of ψ and χ contain 1 for s and her peers. By the I.H. $M, s \not\models B{\sim}\psi$ and $M, s \not\models B{\sim}\chi$. But by our semantics $M, s \models {\sim}(\psi \wedge \chi)$ only happens if $M, s \models B{\sim}\psi$ or $M, s \models B{\sim}\chi$. The negative case follow similar reasoning.

Since Consensus' implies Consensus, Consensus is satisfied. What this proof shows is actually that: *If a $\mathscr{C}$-consolidation satisfies a version of Consensus' for atoms, it satisfies Consensus' (and therefore Consensus).*

Modesty. Take any atom p. If $V(p, s) = \{1\}$, then $\mathrm{Att}(p, s) = 1$ and no changes in R or V can change that.

Equal Weight. Take a model $M = (S, R, V)$ where $V(p, s)\{1\}$, $V(p, t) = \{0\}$ and sRt for some $p \in At$. If we swap the values in V between s and t for p, we have $1 = \mathrm{Att}(p, s) \neq \mathrm{Att}'(p, s) = -1$. ■

Proof of Prop. 14: Modesty. First, we show that for any valuation of an atom, at least two distinct belief attitudes are possible for such atom. For this we need also to use the fact that this consolidation satisfies Doxastic Freedom. By cases:

$V(p, s) = \emptyset$. If s has no peers, $\mathrm{Att}(p, s) = 0$. If additionally there is an arrow to t with $V(p, t) = \{0\}$, then $\mathrm{Att}(p, s) = -1$. Or if $V(p, t) = \{1\}$, then $\mathrm{Att}(p, s) = 1$.

$V(p, s) = \{0\}$. If s has no peers, $\mathrm{Att}(p, s) = -1$. If there is an arrow to t with $V(p, t) = \{1\}$, then $\mathrm{Att}(p, s) = 0$. It is not possible to obtain $\mathrm{Att}(p, s) = 1$.

$V(p,s) = \{1\}$. If s has no peers, $\mathrm{Att}(p,s) = 1$. If there is an arrow to t with $V(p,t) = \{0\}$, then $\mathrm{Att}(p,s) = 0$. It is not possible to obtain $\mathrm{Att}(p,s) = -1$.

$V(p,s) = \{0,1\}$. If s has no peers, $\mathrm{Att}(p,s) = 0$. If there is an arrow to t, then if $V(p,t) = \{0\}$ we have $\mathrm{Att}(p,s) = -1$, if $V(p,t) = \{1\}$ we have $\mathrm{Att}(p,s) = 1$.

In summary, for any atom, it is possible at to abstain about it, or at least have one attitude among belief/disbelief. As is easy to see, if we make our agent abstain w.r.t. all atoms, $\mathrm{Att}(\varphi,s) = 0$ for any $\varphi \in \mathscr{L}_0$. But if our agent do not abstain for any atom, she will believe a maximal set of literals, and therefore (by Prop. 11) she will either believe φ or ${\sim}\varphi$, for any $\varphi \in \mathscr{L}_0$. Since this was done with the valuation for s fixed, Modesty follows.

Equal Weight. It is easy to see by Fig. 2 (right) that for any atom p, if we exchange the valuation of s with that of t, all parameters will be kept the same, and therefore the attitude towards all atoms (and therefore all formulas) will be kept the same.

Monotonicity. This can be proved using the same procedure that was used in Prop. 13.

Doxastic Freedom. Same as Prop. 13.

Consensus. Same as Prop. 13. ■

Proof of Prop. 15: ($\Leftarrow$) If ${\sim}(Bp \wedge \langle +p \rangle {\sim} Bp)$ is valid, then for any M, s, it holds that $M, s \not\models Bp$ or $M, s \not\models \langle +p \rangle {\sim} Bp$, which implies that $M, s \models Bp$ implies $M, s \not\models \langle +p \rangle {\sim} Bp$. This implies that if $M, s \models Bp$, then there is no M_p^+ such that $M_p^+, s \not\models Bp$. This covers one of the cases of Monotonicity. By analogous reasoning with the other axioms, we get all the other cases.

($\Rightarrow$) The axiom ${\sim}(Bp \wedge \langle +p \rangle {\sim} Bp)$ is valid if, for arbitrary M and s, $M, s \models Bp$ implies there is no M_p^+ such that $M_p^+, s \not\models Bp$. Indeed a model M_p^+ satisfies the condition $V(p,t) \preceq V'(p,t)$ for some t (by Def. 15). In this case Monotonicity implies that $\mathrm{Att}'(p,s) \geq \mathrm{Att}(p,s)$. So indeed, if $M, s \models Bp$, which by Consistency means that $\mathrm{Att}(p,s) = 1$, we can only have $\mathrm{Att}'(p,s) = 1$, so $M_p^+, s \models Bp$. So the semantic conditions for **M1** are satisfied. Notice that the case for **M2** is similar, for a model $M_{\neg p}^-$ also satisfies $V(p,t) \preceq V'(p,t)$ for some t. The case for the other axioms are similar. ■

Proof of Prop. 16: ($\Leftarrow$) If our models are image-finite and At is finite, then for any two models M and M' such that $V(p,s) = V'(p,s)$ for all $s \in S$ and some $p \in At$, there is a finite sequence: $M, M_{l_1}^{\circ_1}, (M_{l_1}^{\circ_1})_{l_2}^{\circ_2}, ..., M'$. If $\mathrm{Att}(p,s) \neq \mathrm{Att}'(p,s)$ (for M and M', respectively), then there is one M_i in this sequence such that $\mathrm{Att}_i(p,s) \neq \mathrm{Att}_{i+1}(p,s)$. But if **AI1-AI4** are valid, this is not possible.

($\Rightarrow$) Assume that Atom Independence is satisfied, and $Bp \wedge \langle \circ l \rangle {\sim} Bp$ are satisfiable. Then there is a M_l° such that $M_l^\circ, s \not\models Bp$, while $M, s \models Bp$. But then $V_l^\circ(p,s) = V(p,s)$ for all s, but $\mathrm{Att}(p,s) \neq \mathrm{Att}_l^\circ(p,s)$, and therefore Atom Independence do not hold. Contradiction. Therefore **AI1** is valid. The other cases are similar.

Now we show a consolidation which satisfies **AI1-AI4** but violates Atom Independence (in a setting with infinite At). First, we will need to define some preliminary notions. Let M, s have a *p-canonical* valuation iff $V(p, s) = \{1\}$ and $V(p, t) = \{1\}$ for all t with sRt and $V(q, s) = \{0\}$ and $V(q, t) = \{0\}$ for all t with sRt. The *p-canonical* model of M is M with its p-canonical valuation. For two pointed models M, s and M', s which differ only in V, define the *distance* between them to be the size of the sequence (similar to the one built in the first part of this proof) needed to go from M to M'. If no such sequence exists, the distance is infinite. We can easily show that (*) if $M, s \underline{\leftrightarrow} M', s'$, then M, s is at a finite distance from its p-canonical model iff M', s' is at a finite distance from its p-canonical model. Now define a consolidation $\mathbb{C}$ as follows: $M, s \models Bp$ iff M, s is at a finite distance from its p-canonical model, and $M, s \not\models B\varphi$ for all non-atomic φ. This consolidation respects Def. 12, due to (*). Moreover, this definition violates Atom Independence, for if we take a p-canonical M, s (with $\mathrm{Att}(p, s) = 1$) and change the valuation of infinitely many atoms (without changing p) to obtain M^*, s, this new pointed model is not at a finite distance from its canonical model M, s, and therefore $\mathrm{Att}^*(p, s) \neq 1$. This violates Atom Independence. Axioms **AI1** to **AI4**, however, are valid. Suppose $M, s \models Bp$. Then M, s is at a finite distance from its p-canonical model. For $M, s \models \langle \circ l \rangle {\sim} Bp$ to be satisfied, there need to be a M_l°, s such that $M_l^\circ, s \models {\sim} Bp$. But that would mean that M_l° is at an infinite distance from its p-canonical model. This is impossible, for M, s is p-canonical and M_l° only differs from it in one atom for one agent. The cases for the other axioms are similar. ■

References

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