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Exercises, Algebraic Geometry I – Week 14

Except in the bonus exercise, we assume that k is a perfect field and every curve is geometrically integral, smooth, and complete over k.

Exercise 77. *Curves of genus* 0
Let X be a curve over k.

(i) Show that $X \cong \mathbb{P}^1_k$ if and only if there exists a divisor D on X with $\deg(D) = 1$ and $h^0(X, \mathcal{O}_X(D)) \geq 2$.

(ii) Show that $X \cong \mathbb{P}^1_k$ if and only if $g(X) = 0$ and $X(k) \neq \emptyset$.

Exercise 78. *Existence of curves of given genus*
Let $Q := \mathbb{P}^1_k \times \mathbb{P}^1_k$ and let $g \geq 0$ be any non-negative integer.

(i) Find a bihomogeneous polynomial $f(x_0, x_1, y_0, y_1)$ which has degree $g+1$ in the x_i and degree 2 in the y_i such that $V_+(f) \subseteq Q$ is smooth.

(ii) Conclude that curves of genus g exist.

Exercise 79. *Genus degree formula*
Let $X = V_+(f_d) \subseteq \mathbb{P}^2_k$ be a smooth hypersurface with f_d homogeneous of degree d.

(i) Show that $g(X) = \frac{1}{2}(d-1)(d-2)$.

(ii) Conclude that curves of genus 2 never admit a closed immersion to $\mathbb{P}^2_k$.

Exercise 80. *Hyperelliptic curves*
A curve X over k is called *hyperelliptic* if $g(X) \geq 2$ and X admits a finite morphism of degree 2 to $\mathbb{P}^1_k$.

(i) Show that every curve of genus 2 is hyperelliptic.

(ii) Show that all the curves constructed in Exercise 78 are hyperelliptic.

Exercise 81. *Plane quartics*
Let X be a curve over k. Assume that $g(X) = 3$ and X admits a closed immersion to $\mathbb{P}^2_k$.

(i) Give an explicit example of such an X.

(ii) Show that the morphism $X \to \mathbb{P}^2_k$ is given by a basis of $H^0(X, \omega_X)$. In other words, the effective divisors D on X given by global sections of $H^0(X, \omega_X)$ are exactly the intersections of X with lines in $\mathbb{P}^2_k$.

This sheet will not be corrected.

(iii) Show that if D is an effective divisor of degree 2 on X, then $|D| = \{D\}$. Conclude that X is not hyperelliptic.

(Hint: Reduce to the case where k is algebraically closed and apply the criterion for very ampleness that will be discussed in Monday's lecture)

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 82. *Curves are quasi-projective*
In this exercise, k is an arbitrary field. In the lecture, we have seen that regular complete curves over a field k are projective. The goal of this exercise is to see that, more generally, all curves over a field k are quasi-projective. So, let X be a curve over k.

(i) Let $U \subseteq X$ be a non-empty open affine subset. Show that there exists a globally generated invertible sheaf $\mathcal{L}$ and a global section s of $\mathcal{L}$ such that $X_s = U$.

(ii) Conclude that there exists a globally generated invertible sheaf $\mathcal{L}$ on X such that for every $x \in X$, there exists a global section s of $\mathcal{L}$ such that $x \in X_s$.

(iii) Show that the invertible sheaf $\mathcal{L}$ in (ii) is very ample. Conclude that X is quasi-projective over k. In particular, if X is complete, then it is projective over k, and we do not need to assume that X is regular for this.