Reading Seminar in Derived Categories

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Contents

1 Lecture 1: Fourier Mukai Transforms

1.1 Grothendieck-Verdier Duality

Let $f : X \to Y$ be a morphism of smooth schemes over a field k (any char). Denote the relative canonical bundle by

$$\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$$

Then, for any $\mathcal{F}^\bullet \in D^b(X)$ and $\mathcal{E}^\bullet \in D^b(Y)$ there exists a natural isomorphism

$$Rf_* R\mathcal{H}om(\mathcal{F}^\bullet, Lf^*(\mathcal{E}^\bullet) \otimes^{\mathbb{L}} \omega_{X/Y}[\dim X - \dim Y] \simeq R\mathcal{H}om(Rf_*\mathcal{F}^\bullet.\mathcal{E}^\bullet)$$

Since for smooth maps $\omega_{X/Y}$ is locally free the tensor product is underived. Define,

$$f^! : \mathrm{D}^b(Y) \to \mathrm{D}^b(X) \qquad \mathcal{E}^\bullet \mapsto Lf^*(\mathcal{E}^\bullet) \otimes \omega_{X/Y}[\dim X - \dim Y]$$

Then $f^! \dashv Rf_*$.

1.2 Corollaries

1. Taking cohomologies we get,

   $$R\Gamma Rf_* R\mathcal{H}om(\mathcal{F}^\bullet, Lf^*(\mathcal{E}^\bullet) \otimes \omega_{X/Y}[\dim X - \dim Y] \simeq R\Gamma R\mathcal{H}om(Rf_*\mathcal{F}^\bullet.\mathcal{E}^\bullet)$$

   But, $R\Gamma \circ Rf_* = R\Gamma$ and $R\Gamma \circ R\mathcal{H}om = R\mathrm{Hom}$. Therefore in degree zero we get,

   $$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}^\bullet, Lf^*(\mathcal{E}^\bullet) \otimes \omega_{X/Y}[\dim X - \dim Y]) \simeq \mathrm{Hom}_{\mathrm{D}^b(Y)}(Rf_*\mathcal{F}^\bullet.\mathcal{E}^\bullet)$$

2. (Serre Duality) Grothendieck Duality applied to $f : X \to k$ yields classical Serre duality:

   $$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}^\bullet[i], \omega_X[\dim X]) \simeq \mathrm{Hom}_k(Rf_*\mathcal{F}^\bullet[i], k)$$

   In particular, for a sheaf we have, $\mathcal{F}^\bullet = \mathcal{F}$. This together with the facts, $R^i f_* \mathcal{F} = H^i(X, \mathcal{F})$ and $Hom_{\mathrm{D}^b(X)}(\mathcal{F}, \mathcal{G}[\dim X - i]) \simeq \mathrm{Ext}^{n-i}(\mathcal{F}, \mathcal{G})$, yield

   $$\mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X) \simeq H^i(X, \mathcal{F})^*$$

3. For $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{D}^b(X)$, the derived version of Serre duality gives

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes \omega_X[\dim X]) \\
&\qquad \simeq \mathrm{Hom}_{\mathrm{D}^b(X)}(R\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{F}^\bullet), \omega_X[\dim X]) \\
&\qquad \simeq \mathrm{Hom}_k(R\Gamma R\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{F}^\bullet), k) \\
&\qquad \simeq \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{G}^\bullet, \mathcal{F}^\bullet)^*
\end{aligned}$$

1.3 Fourier Mukai Transforms

For this section, unless otherwise mentioned the standard notation will always mean derived. For instance, we will write $\mathcal{F}$ for $\mathcal{F}^\bullet$, $\otimes$ for $\otimes^{\mathbb{L}}$ etc.

Definition 1.1 (Fourier-Mukai Transforms)**.** Let $\mathcal{P} \in \mathrm{D}^b(X \times Y)$. Let $q : X \times Y \to X$ and $p : X \times Y \to Y$ be standard projections. This Fourier-Mukai transform is a functor $\Phi_{\mathcal{P}} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ defined by

$$\mathcal{E}^\bullet \mapsto Rp_*(q^*\mathcal{E}^\bullet \otimes \mathcal{P})$$

The object $\mathcal{P}$ is called the Fourier Mukai kernel of the Fourier-Mukai transform $\Phi_{\mathcal{P}}$.

Remark.

1. Since q is smooth (hence flat) q^* is underived.
2. If $\mathcal{P}$ is a complex of locally free sheaves, $\otimes$ in the formula is also underived. This will be the case in most of the applications.
3. It is a composition of exact functors and hence exact.

Examples:

1. (The identity functor) $id : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$. Then, $id \simeq \Phi_{\mathcal{O}_\Delta}$, where $\Delta \subset X \times X$ is the diagonal.
2. We will show more generally that for a morphism $f : X \to Y$, $Rf_* \simeq \Phi_{\mathcal{O}_\Gamma}$, where $g : X \to \Gamma \subset X \times Y$ is the graph of the morphism f.
   For $\mathcal{F} \in \mathrm{D}^b(X)$,

$$\begin{aligned}
Rf_*(\mathcal{F}) &= R(q \circ g)_*(g^*p^*\mathcal{F}) \\
&= Rq_* \circ Rg_*(g^*p^*\mathcal{F}) \\
&= Rq_*(p^*\mathcal{F} \otimes Rg_*\mathcal{O}_X) \quad \text{(projection formula)} \\
&= Rq_*(p^*\mathcal{F} \otimes \mathcal{O}_\Gamma) \quad (g \text{ is an isomorphism.})
\end{aligned}$$

3. Let L be a line bundle on X then the functor $\mathcal{F}^\bullet \mapsto \mathcal{F}^\bullet \otimes L$ is isommorphic to Φ_{i_*L} where, $i : X \to \Delta \subset X \times X$ is the diagonal embedding.

4. The shift functor $T : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is given by $\Phi_{\mathcal{O}_\Delta[1]}$.

5. Let $\mathcal{P}$ be a flat coherent sheaf on $X \times Y$, then for a closed point $x \in X$,

$$\Phi_{\mathcal{P}}(k(x)) = Rp_*(q^*k(x) \otimes \mathcal{P}) = \mathcal{P}_x$$

Definition 1.2 (Theorem)**.** For any object $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ we define the following objects in $\mathrm{D}^b(X \times Y)$

$$\mathcal{P}_L = \mathcal{P}^\vee \otimes p^*\omega_Y[\dim Y] \qquad \mathcal{P}_R = \mathcal{P}^\vee \otimes q^*\omega_X[\dim X]$$

where, $\mathcal{P}^\vee = R\mathcal{H}om(\mathcal{P}, \mathcal{O}_{X\times Y})$. Then, $G = \Phi_{\mathcal{P}_L}$ is the left adjoint and $H = \Phi_{\mathcal{P}_R}$ is the right adjoint of the the fourier mukai transform with kernel $\mathcal{P}$.

Proof. $G \dashv \Phi_{\mathcal{P}}$: For $\mathcal{F}^\bullet \in \mathrm{D}^b(Y)$ and $\mathcal{E}^\bullet \in \mathrm{D}^b(X)$,

$$\begin{aligned}
\mathrm{Hom}_{D^b(X)}(G(\mathcal{F}^\bullet), \mathcal{E}^\bullet) &= \mathrm{Hom}_{D^b(X)}(Rq_*(p^*\mathcal{F}^\bullet \otimes \mathcal{P}_L), \mathcal{E}^\bullet) \\
&= \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}_L \otimes p^*\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet \otimes \omega_{X\times Y/X}[\dim Y]) \\
&\text{(GV Duality; } q^* \text{is underived since } q \text{ is flat)} \\
&= \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}_L \otimes p^*\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet \otimes p^*\omega_Y[\dim Y]) \\
&= \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}^\vee \otimes p^*\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet) \\
&= \mathrm{Hom}_{D^b(X\times Y)}(p^*\mathcal{F}^\bullet, \mathcal{P} \otimes q^*\mathcal{E}^\bullet) \\
&= \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, Rp_*(\mathcal{P} \otimes q^*\mathcal{E}^\bullet)) \quad (Lp^* \dashv Rp_*) \\
&= \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \Phi_{\mathcal{P}}) \quad (p^* \dashv p_*)
\end{aligned}$$

□

1.4 Composition of FM transforms

Proposition 1.3. *The composition*

$$D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y) \xrightarrow{\Phi_{\mathcal{Q}}} D^b(Z)$$

is isomorphic to $\Phi_{\mathcal{R}} : D^b(X) \to D^b(Z)$*, where* $\mathcal{R} = R\pi^*_{XZ}(\pi^*_{XY}\mathcal{P} \otimes \pi^*_{YZ}\mathcal{Q})$.

1.5 Orlov's theorem and applications

Theorem 1.4 (Orlov)**.** *Let X and Y be two smooth projective varieties and let $F : D^B(X) \to D^b(Y)$ be a fully faithful exact functor. If F admits right (and hence left) adjoint functors, then there exists an object $\mathcal{P} \in D^b(X \times Y)$ unique up to isomorphism such that $F \simeq \Phi_{\mathcal{P}}$.*

Proof. Future lecture(maybe) □

Corollaries:

Corollary 1.5. *Let $F : D^b(X) \to D^b(Y)$ be an equivalence between the derived categories of two smooth projective varieties. Then F is isomorphic to a FM transform $\Phi_{\mathcal{P}}$ associated to a certain object $\mathcal{P} \in D^b(X \times Y)$ unique up to isomorphism. Moreover,* $\dim X = \dim Y$ *and* $\mathcal{P} \otimes q^*\omega_Y \simeq \mathcal{P} \otimes p^*\omega_X$.

Proof. Note that, equivalence of categories ensures existence of adjoints, namely the quasi-inverse F'. Therefore we can apply Orlov's theorem to find a $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ unique up to isomorphism such that the FM transform with kernel $\mathcal{P}$ is isomorphic to F. Again by applying the uniqueness of Orlov's theorem to the quasi-inverse F' we see that, $\mathcal{P}_L \simeq \mathcal{P}_R$. In other words, $\mathcal{P}^\vee \simeq \mathcal{P}^\vee \otimes p^*\omega_X \otimes q^*\omega_Y[\dim X - \dim Y]$. Since $\mathcal{P}$ is a bounded complex that is not quasi-isomorphic to zero, $\dim X = \dim Y$. □

Corollary 1.6. *Suppose $\Phi : D^b(X) \simeq D^b(Y)$ is an equivalence such that, for any close point $x \in X$, there exists a closed point $f(x) \in Y$ such that $\Phi(k(x)) \simeq k(f(x))$. Then, $f : X \to Y$ defines an isomorphism and Φ simeq$(M \otimes _) \circ f_*$ for some line bundle $M \in Pic(Y)$.*

Definition 1.7 (Spanning Class)**.** A collection Ω os objects in a triangulated category $\mathcal{D}$ is a spanning class of $\mathcal{D}$ f for all $B \in \mathcal{D}$ the following condition hold:
If $\mathrm{Hom}(B, A[i]) = 0$ for all $A \in \Omega$ and $\forall i \in \mathbb{Z}$ then, $B \simeq 0$

Proof. By Orlov's theorem, $\Phi \simeq \Phi_{\mathcal{P}}$ for some object $\mathcal{P} \in \mathrm{D}^b(X \times Y)$. Note that, $\Phi(k(x)) = q^*(k(x)) \otimes^{\mathbb{L}} \mathcal{P} \simeq k(f(x))$. Therefore, for any closed point $x \in X$ the embedding $i : x \times Y \hookrightarrow X \times Y$, $Li^*\mathcal{P}$ is also a sheaf. Then the lemma below implies that, $\mathcal{P}$ is a coherent sheaf flat over X. Therefore, $R\Phi(k(x)) = \mathcal{P}|_{\{x\}\times Y} \simeq k(f(x))$. Hence, Supp$\mathcal{P}$ is precisely the graph of f and thus Γ_f has a reduced induced scheme structure. Γ_f is then a variety isomorphic to X by first projection and f is a composition of this isomorphism with projection to Y. Therefore, $f : X \to Y$ defines a morphism.

Now, for $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$, if $\mathrm{Hom}(\mathcal{F}^\bullet, k(x)[-i]) = \mathrm{Ext}^{-i}(\mathcal{F}^\bullet, k(x)) = 0$ for all $x \in X$ and $i \in \mathbb{Z}$, then since $\mathrm{Ext}^p(\mathcal{H}^q(\mathcal{F}^\bullet), k(x)) \Rightarrow \mathrm{Ext}^{p+q}(\mathcal{F}^\bullet, k(x))$, we get that, $\mathrm{Ext}^p(\mathcal{H}^q(\mathcal{F}^\bullet), k(x)) = 0$. But if $x \in \mathrm{Supp}(\mathcal{H}^{p+q})$, $\mathrm{Hom}(\mathcal{H}^{p+q}(\mathcal{F}^\bullet), k(x)) \neq 0$. This contradiction shows that $k(x)$ spans the category $\mathrm{D}^b(X)$. Since, Φ is a equivalence, it is easy to check that, $k(f(x))$ spans $\mathrm{D}^b(Y)$. Therefore, for $y \in Y$ there is an integer m and $x \in X$ such that, $\mathrm{Hom}(k(f(x)), k(y)[m]) \neq 0$. This implies that, $m = 0$ and $y = f(x)$.
For injectivity, pick $x_1 \neq x_2$. then $\Phi(k(x_1)) \neq \Phi(k(x_2))$. Therefore, $f(x_1) \neq f(x_2)$.

We can use similar argument on the quasi inverse to Φ to show that f has an honest inverse.
Now, $\mathcal{P}|_{\mathrm{Supp}\mathcal{P}}$ has fibre of dimension 1, therefore, $\mathcal{P}|_{\mathrm{Supp}\mathcal{P}}$ is line bundle. Since, *Supp*($\mathcal{P}$) $\simeq Y$ via the projection p, it gives rise to a line bundle $M = p_*\mathcal{P}$ on Y. From the formula for composition, it is possible to calculate and see that $\Phi_{\mathcal{P}} \simeq (\_ \otimes M) \circ f_*$. $\square$

Lemma 1.8. *Consider a morphism $S \to X$. Suppose $\mathcal{P} \in D^b(S)$ and assume that for all closed points $x \in X$ the derived pull back $Li_x^*\mathcal{P} \in D^b(S_x)$, for $i_x : S_x \hookrightarrow S$, is a complex concentrated in degree 0, i.e. a sheaf. Then $\mathcal{P}$ is isomorphic to a coherent sheaf which is flat over X.*

Proof. For a proof, see [1, pg.82 Lemma 3.31] $\square$

Corollary 1.9 (**Gabriel**)**.** *If $\Phi : \boldsymbol{Coh}(X) \to \boldsymbol{Coh}(Y)$ is an equivalence of categories then $\exists$ a morphism f such that $f : X \simeq Y$ and $\Phi \simeq (M \otimes \_) \circ f_*$, for a line bundle M on Y.*

Proof. In order to apply the previous corollary, we need to check that, $\Phi(k(x)) \simeq k(f(x))$. $\square$

2 Lecture 2: Passing to cohomology

2.1 Grothendieck ring

Let $K(X)$ be the Grothendieck group of X. Recall that the elements are coherent sheaves on X with the following equivalence relation: if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, then $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$. On a smooth projective variety any coherent sheaf admits a locally free resolution, hence any element in the Grothendieck group may be written asa linear combination of locally free sheaves.

We define a map $[\ ] : D^b(X) \to K(X)$ by $[\mathcal{F}^\bullet] = \sum(-1)^i[F^i]$. We also define a ring structure on $K(X)$ by $[\mathcal{E}_1] \cdot [\mathcal{E}_2] = [\mathcal{E}_1 \otimes \mathcal{E}_2]$.

Remark.

- $[\mathcal{F}^\bullet[k]] = (-1)^k[\mathcal{F}^\bullet]$.
- $[\mathcal{F}^\bullet{}_1 \oplus \mathcal{F}^\bullet{}_2] = [\mathcal{F}^\bullet{}_1] + [\mathcal{F}^\bullet{}_2]$.
- $[\mathcal{F}^\bullet] = \sum(-1)^i[\mathcal{H}^i(\mathcal{F}^\bullet)] \in K(X)$.
- $[\mathcal{F}^\bullet{}_1 \otimes \mathcal{F}^\bullet{}_2] = [\mathcal{F}^\bullet{}_1] \cdot [\mathcal{F}^\bullet{}_2]$.

So $[\ ]$ is a ring map.

We can define the pullback of a morphism $f : X \to Y$ at the Grothendieck ring in the usual way and it will be a ring homomorphism. Moreover, we will have:

$$\begin{tikzcd} D^b(Y) \arrow[r, "f^*"] \arrow[d, "{[\ ]}"'] & D^b(X) \arrow[d, "{[\ ]}"] \\ K(Y) \arrow[r, "f^*"'] & K(X) \end{tikzcd}$$

We also want to define a map of Grothendieck rings that commute with the pushforward. Let $f_! : K(X) \to K(Y)$ be defined by $f_![\mathcal{F}] = \sum(-1)^i[R^i f_*\mathcal{F}]$. Using the fact that for a spectral sequence $\sum(-1)^{p+q}[E_r^{p,q}] = \sum(-1)^{p+q}[E_{r+1}^{p,q}]$ is satisfied and using it for the spectral sequence $E_2^{p,q} = R^p f_*\mathcal{H}^q\mathcal{E}^\bullet \Rightarrow R^{p+q} f_*\mathcal{E}^\bullet$ we have that:

$$\begin{tikzcd} D^b(X) \arrow[r, "f_*"] \arrow[d, "{[\ ]}"'] & D^b(Y) \arrow[d, "{[\ ]}"] \\ K(X) \arrow[r, "f_!"'] & K(Y) \end{tikzcd}$$

Definition 2.1 (K-theoretic Fourier-Mukai transform)**.** Let $e \in K(X \times Y)$. The K-theoretic Fourier-Mukai transform is the map $\Phi_e^K : K(X) \to K(Y)$ defined by $\Phi_e^K(g) = p_!(e \cdot q^*(g))$.

Due to the previous remarks we have that:

$$\begin{array}{ccc} D^b(X) & \xrightarrow{\Phi_{\mathcal{P}}} & D^b(Y) \\ {\scriptstyle [\,]}\downarrow & & \downarrow{\scriptstyle [\,]} \\ K(X) & \xrightarrow[\Phi^K_{[\mathcal{P}]}]{} & K(Y) \end{array}$$

2.2 Cohomological Fourier-Mukai transform

We will assume from now that the ground field is $\mathbb{C}$.
Consider the ring $H^*(X, \mathbb{Q})$. The product of two classes $\alpha, \beta \in H^*(X, \mathbb{Q})$ will be denoted by $\alpha.\beta$ or simply $\alpha\beta$. Let $f : X \to Y$ be a morphism. The pullback is defined for cohomology rings in the usual way. We assume X and Y are projective varieties and hence Poincaré duality holds for them. The composition of the dual map of the pullback with the isomorphisms of Poincaré duality give us a map $f_* : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ and moreover we know that the map for degree k elements satisfy $f_* : H^k(X, \mathbb{Q}) \to H^{k+2\dim(Y)-2\dim(X)}(Y, \mathbb{Q})$. By definition, this map satisfies the Projection Formula, that is, $f_*(f^*\alpha.\beta) = \alpha.f_*\beta$.

Definition 2.2 (Cohomological Fourier-Mukai)**.** Let $\alpha \in H^*(X \times Y, \mathbb{Q})$. The cohomological Fourier-Mukai transform is the map $\Phi^H_\alpha : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ defined by $\Phi^H_\alpha(\beta) = p_*(\beta.q^*(\alpha))$.

We will now define a map $ch : K(X) \to H^*(X, \mathbb{Q})$ called the Chern character. Let $A^i(X)$ be the cycles of codimension i. First we define an element $c_i(\mathcal{E}) \in A^i(X)$ for a locally free sheaf $\mathcal{E}$. Recall that we also have a map $A^i(X) \to H^{2i}(X, \mathbb{Q})$. Notice that taking coefficients in $\mathbb{C}$ and using the Hodge decomposition, this map will land in $H^{i,i}(X)$. For the current purposes it is enough to give some conditions for $c_t(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \ldots$ to satisfy, as it will define the elements in a unique way. The conditions are:

1. If $\mathcal{E} \cong O_X(D)$ for a divisor D, then $c_t(\mathcal{E}) = 1 + Dt$.
2. For $f : Y \to X$, $c_i(f^*\mathcal{E}) = f^*c_i(\mathcal{E})$ for all i.
3. For an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, $c_t(\mathcal{E}) = c_t(\mathcal{E}').c_t(\mathcal{E}'')$.

It satisfies the following condition: suppose we have a filtration $\mathcal{E} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \ldots \supseteq \mathcal{E}_r = 0$ such that $\mathcal{E}_i/\mathcal{E}_{i+1} \cong \mathcal{L}_i$ an invertible sheaf. Then we have that $c_t(\mathcal{E}) = \prod c_t(\mathcal{L}_i)$ and for this we can use condition 1.

Suppose now that $c_t(\mathcal{E}) = \prod(1+a_i t)$. We define $ch(\mathcal{E}) = \sum \exp(a_i)$. Notice that for $\mathcal{L} \in \mathrm{Pic}(X)$, we have $ch(\mathcal{L}) = \sum \frac{c_1(\mathcal{L})^i}{i!}$. We also define the Todd class as $\mathrm{td}(\mathcal{E}) = \prod \frac{a_i}{1-e^{-a_i}}$. We have that for an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, $\mathrm{td}(\mathcal{E}) = \mathrm{td}(\mathcal{E}').\mathrm{td}(\mathcal{E}'')$.

Definition 2.3. Let X be a smooth variety. The Todd class of X is defined as $\mathrm{td}(X) = \mathrm{td}(\mathcal{T}_X)$.

Theorem 2.4 (Grothendieck-Riemann-Roch formula)**.** *Let $f : X \to Y$ be a projective morphism of smooth projective varieties. Then for any $e \in K(X)$*

$$ch(f_!(e)).td(Y) = f_*(ch(e).td(X))$$

For the case Y =Spec(k) we have that the pushforward may only be the nonzero map for $H^{2n}(X, \mathbb{Q})$ where $n = \dim(X)$. We denote by $\int_X$ the pushforward in this case. Notice that the map tells us the element in that degree. Also, notice that $f_![\mathcal{F}] = \sum(-1)^i[R^i f_*\mathcal{F}] = \sum(-1)^i[H^i(X, \mathcal{F})] = \chi(\mathcal{F})$. Taking this case we recover the folowing theorem:

Theorem 2.5 (Hirzebruch-Riemann-Roch)**.** *For any $e \in K(X)$ we have $\chi(e) = \int_X (ch(e).td(X))$.*

Definition 2.6. The Mukai vector of $e \in K(X)$ is $v(e) = ch(e).\sqrt{\mathrm{td}(X)}$. For $\mathcal{E}^\bullet \in D^b(X)$ we define $v(\mathcal{E}^\bullet) = v([\mathcal{E}^\bullet]) = ch(\mathcal{E}^\bullet).\sqrt{\mathrm{td}(X)}$.

Notice that it makes sense to write $\sqrt{\mathrm{td}(X)}$ as $\mathrm{td}(X) = 1 + \ldots$ then we can construct an element such that its square is $\mathrm{td}(X)$.

The following is a Corollary of 2.4.

Corollary 2.7. *Let $e \in K(X \times Y)$. Then the following diagram commute:*

$$\begin{tikzcd}
K(X) \arrow[r, "\Phi_e^K"] \arrow[d, "v"'] & K(Y) \arrow[d, "v"] \\
H^*(X, \mathbb{Q}) \arrow[r, "\Phi_{v(e)}^H"'] & H^*(Y, \mathbb{Q})
\end{tikzcd}$$

Proof. We check that the following commute:

$$\begin{tikzcd}
K(X) \arrow[r, "q^*"] \arrow[d, "v"'] & K(X \times Y) \arrow[r, "\cdot e"] \arrow[d, "v\sqrt{\mathrm{td}(Y)}^{-1}"] & K(X \times Y) \arrow[r, "p_!"] \arrow[d, "v\sqrt{\mathrm{td}(X)}"] & K(Y) \arrow[d, "v"] \\
H^*(X, \mathbb{Q}) \arrow[r, "q^*"'] & H^*(X \times Y, \mathbb{Q}) \arrow[r, ".v(e)"'] & H^*(X \times Y, \mathbb{Q}) \arrow[r, "p_*"'] & H^*(Y, \mathbb{Q})
\end{tikzcd}$$

□

Given $\mathcal{P} \in D^b(X \times Y)$ we will denote by $\Phi_{\mathcal{P}}^H$ the induced cohomological Fourier-Mukai transform $\Phi_{v(\mathcal{P})}^H$. As characteristic classes ara in even degree as they come from cycles, we have that $\Phi_{\mathcal{P}}^H$ respects the parity. Indeed, $\Phi_{\mathcal{P}}^H$ is an intersection with an even element followed by a pushforward which respects parity, so we have: $\Phi_{\mathcal{P}}^H(H^{\text{even}}(X)) \subseteq H^{\text{even}}(Y)$ and $\Phi_{\mathcal{P}}^H(H^{\text{odd}}(X)) \subseteq H^{\text{odd}}(Y)$.//
Using the same notation as for Fourier-Mukai we have the following lemma.

Lemma 2.8. *Let $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ and $\Phi_{\mathcal{Q}} : D^b(Y) \to D^b(Z)$ be two Fourier-Mukai transforms and let $\Phi_{\mathcal{R}} : D^b(X) \to D^b(Z)$ be the composition. Then $\Phi_{\mathcal{R}}^H = \Phi_{\mathcal{Q}}^H \circ \Phi_{\mathcal{P}}^H$*

Proposition 2.9. *Suppose that $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is an equivalence for some $\mathcal{P} \in D^b(X \times Y)$. Then the induced cohomological Fourier-Mukai transform $\Phi_{\mathcal{P}}^H : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ is an isomorphism of rational vector spaces.*

Proof. As $\Phi_{\mathcal{P}}$ is an equivalence then $\Phi_{\mathcal{P}_R} \circ \Phi_{\mathcal{P}} \cong \text{ id } \cong \Phi_{O_\Delta}$ and $\Phi_{\mathcal{P}} \circ \Phi_{\mathcal{P}_R} \cong \text{ id } \cong \Phi_{O_\Delta}$. Due to the previous lemma, we can conclude then that $\Phi_{\mathcal{P}_R}^H \circ \Phi_{\mathcal{P}}^H \cong \Phi_{O_\Delta}^H$ and $\Phi_{\mathcal{P}}^H \circ \Phi_{\mathcal{P}_R}^H \cong \Phi_{O_\Delta}^H$. So it is enough to show that $\Phi_{O_\Delta}^H \cong \text{ id }$.
Let $i : X \tilde{\to} \Delta \hookrightarrow X \times X$. Using Grothendieck-Riemann-Roch we have that $ch(O_\Delta).\text{td}(X \times X) = ch(i_! O_X).\text{td}(X \times X) = i_*(ch(O_X).\text{td}(X)) = i_*\text{td}(X)$. The last equality is because $ch(O_X) = 1$. So we get that $v(O_\Delta) = ch(O_\Delta).\sqrt{\text{td}(X \times X)} = i_*(\text{td}(X)).\sqrt{\text{td}(X \times X)}^{-1} =$
$= i_*(\text{td}(X).i^*\sqrt{\text{td}(X \times X)}^{-1}) = i_*(\text{td}(X).\text{td}(X)^{-1}) = i_*(1)$. We used that $i^*(\sqrt{\text{td}(X \times X)}) = \text{td}(X)$.
Finally, we get that $\Phi_{O_\Delta}^H(\beta) = p_*(q^*(\beta).v(O_\Delta)) = p_*(q^*(\beta).i_*(1)) =$
$= p_*i_*(i^*q^*(\beta)) = \beta$. □

As we are working with X smooth projective variety over $\mathbb{C}$, we have a Hodge structure $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ such that $\bar{H}^{p,q}(X) = H^{q,p}(X)$ and $H^{p,q}(X) = H^q(X, \Omega_X^p)$. As explained before all characteristic classes are of type (p,p), so $v() : K(X) \to \bigoplus H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$.

We have seen that the cohomological Fourier-Mukai does not respect the grading, only the parity. But we can improve that using the Hodge decomposition.

Proposition 2.10. *Suppose $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is an equivalence. Then the induced $\Phi_{\mathcal{P}}^H$ yields isomorphisms $\bigoplus_{p-q=i} H^{p,q}(X) \cong \bigoplus_{p-q=i} H^{p,q}(Y)$ $\forall i = -\dim(X), \ldots, \dim(X)$.*

Proof. $\Phi_{\mathcal{P}}^H$ defines an isomorphism between $H^*(x, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$ as we saw before. So we need to see that its $\mathbb{C}$-linear extension satisfies $\Phi_{\mathcal{P}}^H(H^{p,q}(X)) \subseteq \bigoplus_{p-q=r-s} H^{r,s}(Y)$. Consider the Künneth decomposition of $v(\mathcal{P}) = \sum \alpha^{p',q'} \times \beta^{r,s}$ with $\alpha^{p',q'} \in H^{p',q'}(X)$ and $\beta^{r,s} \in H^{r,s}(Y)$. We know the element is a sum of element of type (t,t) we only consider the part of the sum such that $p' + r = q' + s$.
Let $\alpha \in H^{p,q}(X)$. Then $\Phi_{\mathcal{P}}^H(\alpha) = p_*(q^*(\alpha). \sum \alpha^{p',q'} \times \beta^{r,s}) =$
$= \sum (\int_X (\alpha.\alpha^{p',q'}))\beta^{r,s} \in \bigoplus H^{r,s}$. The last equality is by definition of pushforward and how the pushforward behaves in the Künneth decomposition. Then, it is needed that $(p + p', q + q') = (\dim(X), \dim(X))$. The result follows. □

Corollary 2.11. *Let E, E' be two elliptic curves. Then $D^b(E) \cong D^b(E')$ if and only if $E \cong E'$.*

Proof. Let $\Phi_{\mathcal{P}}$ be the equivalence. As $\Phi_{\mathcal{P}}^H$ respects parity, it induces and isomorphism $H^1(E) \cong H^1(E')$ and $H^0(E) \bigoplus H^2(E) \cong H^0(E') \bigoplus H^2(E')$. By the previous proposition we know it also induces an isomorphism on $H^{0,1}$ and $H^{1,0}$. We also know that $E \cong H^{1,0}(E)^*/H_1(E, \mathbb{Z}) \cong H^{0,1}(E)/H^1(E, \mathbb{Z})$. So we need to show that $\Phi_{\mathcal{P}}^H(H^1(E, \mathbb{Z})) \subseteq H^1(E', \mathbb{Z})$. As they are elliptic curves we have that $\mathrm{td}(E \times E') = 1$ and $ch(\mathcal{P}) = r + c_1(\mathcal{P}) + \frac{1}{2}(c_1^2 - 2c_2)(\mathcal{P})$. But the last term does not contribute to H^1. The result follows. □

References

[1] Huybrechts, Daniel, *Fourier Mukai Transforms in Algebraic Geometry,*