ON THE EFFECTIVE FREENESS OF THE DIRECT IMAGES OF PLURICANONICAL BUNDLES

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Abstract. We give an effective bound on the generation of pushforwards of pluricanonical bundles twisted by ample line bundles. This gives a partial answer to a slightly weaker version of a conjecture proposed by Popa and Schnell. We prove two types of statements: first, more in the spirit of the general conjecture, we show generic global generation with Angehrn-Siu type bound. Secondly, assuming that the relative canonical bundle is relatively semi-ample, we make a very precise statement. In particular, when the map is smooth, it solves the conjecture with the same bound, for certain pluricanonical bundles.

1. Introduction. The main purpose of this paper is to give a partial answer to a version of the Fujita-type conjecture proposed by Popa and Schnell [PS14, Conjecture 1.3], on the global generation of pushforwards of pluricanonical bundles twisted by ample line bundles. All varieties considered below are over the field of complex numbers.

Notation. We fix $N = \binom{n+1}{2}$ for what follows.

Conjecture 1.1 (Popa-Schnell)**.** *Let $f : Y \to X$ be a morphism of smooth projective varieties, with $\dim X = n$, and let L be an ample line bundle on X . Then, for every $k \geqslant 1$, the sheaf*

$$f_*\omega_Y^{\otimes k} \otimes L^{\otimes l}$$

is globally generated for $l \geqslant k(n+1)$.

In [PS14], Popa and Schnell proved the conjecture in the case when L is an ample and globally generated line bundle, and observed that it holds in general when $\dim X = 1$. With the additional assumption that L is globally generated, they could use Kollár and Ambro-Fujino type vanishing along with Castelnuovo-Mumford regularity to conclude global generation. We remove the global generation assumption on L making a statement about generic global generation with weaker bound on the twist, as in the work of Angehrn and Siu [AS95], on the effective freeness of adjoint bundles.

Theorem A. *Let $f : Y \to X$ be a surjective morphism of projective varieties, with X smooth and $\dim X = n$. Let L be an ample line bundle on X. Consider a log canonical $\mathbb{Q}$-pair (Y, Δ) on Y, with Δ effective, such that $k(K_Y + \Delta)$ is Cartier for some $k \geqslant 1$. Denote $P = \mathcal{O}_Y\big(k(K_Y + \Delta)\big)$. Then* *the sheaf*

$$f_*P \otimes L^{\otimes l}$$

is generated by global sections at a general point $x \in X$, either

(a) *for all $l \geqslant k(N+1)$.*
or,
(b) *for all $l \geqslant k(n+1)$ when $n \leqslant 4$.*

In [Den17], using analytic methods and pseudo-effectiveness of $K_X + (n+1)L$, Deng showed that, when X and Y are both smooth and $\Delta = 0$, for every $k \geqslant 1$, the above generation in fact holds for $l \geqslant k(n+1) + n^2 - n$.

As a particular case of Theorem A, we have the following corollary, which is a generic version of Conjecture 1.1 with Angehrn-Siu type bound.

Corollary B. *Let $f : Y \to X$ be a surjective morphism of smooth projective varieties, with $\dim X = n$. Let L be an ample line bundle on X. Then for all $k \geqslant 1$, the sheaf*

$$f_*\omega_Y^{\otimes k} \otimes L^{\otimes l}$$

is generated by global sections at a general point $x \in X$ for all l as in Theorem A.

According to [PS14, Section 4], this could be interpreted as an effective version of Viehweg's weak-positivity for $f_*\omega_{Y/X}^{\otimes k}$ [Vie83] (also see [Kol86, Theorem 3.5(i)]).

One can in fact describe the locus on which global generation holds, but not in a very explicit fashion. This suffices however in order to deduce the next Theorem, where assuming semiampleness of the canonical bundle along the smooth fibres, we prove that the global generation holds outside of the branch locus.

Theorem C. *Let $f : Y \to X$ be a surjective morphism of smooth projective varieties, with $\dim X = n$. Suppose f is smooth outside of a subvariety $B \subset X$. Assume in addition that $\omega_Y^{\otimes k}$ is relatively free outside B for some $k \geqslant 1$, and let L be an ample line bundle on X. Then the sheaf*

$$f_*\omega_Y^{\otimes k} \otimes L^{\otimes l}$$

is generated by global sections at x, for all $x \notin B$, either

(a) *for all $l \geqslant k(N+1)$*
or,
(b) *for all $l \geqslant k(n+1)$ when $n \leqslant 4$.*

Note, for instance, that this applies when $f : Y \to X$ is a projective surjective morphism with generalised Calabi-Yau fibres (i.e. $\omega_F = \mathcal{O}_F$ for any smooth fibre F of f), or with fibres having nef and big canonical bundle (i.e. they are minimal varieties of general type). Indeed, in the second case there is an integer $s \gg 0$ such that $f^*f_*\omega_Y^{\otimes s} \to \omega_Y^{\otimes s}$ is surjective [Fuj09, Theorem 1.3].

In particular, if f is smooth, i.e. $B = \emptyset$, Theorem C solves Conjecture 1.1 for the pluricanonical bundles that are relatively globally generated, however with Angehrn-Siu type bound .

This in turn leads to an effective vanishing theorem (see Theorem 3.1), in the case of smooth morphisms, for the pushforwards of pluricanonical bundles that are relatively free. This is in the flavour of [PS14, Theorem 1.7], but with the global generation assumption on L removed.

The proof of Theorem A is, in part, inspired by arguments in [PS14, Theorem 1.4]. However, since we do not assume that L is globally generated, we need to follow a different path, avoiding Castelnuovo-Mumford regularity. To do this, we need to argue locally around each point and to appeal to the following local version of Kawamata's effective freeness result (see [Kaw02, Theorem 1.7]), another main source of inspiration for this paper.

Proposition 1.2. *Let $f : Y \to X$ be a surjective morphism of smooth projective varieties, with $\dim X = n$, such that f is smooth outside of a subvariety B in X. Fix a point $x \in X \setminus B$. Let Δ be a $\mathbb{Q}$-divisor on Y with the following properties:*

(a) *Δ is effective and has simple normal crossing support.*
(b) *$\Delta = \dfrac{l}{k}D + F$, where $0 \leqslant l < k$, the divisor D is smooth and intersects the fibre over x transversally and the divisor F is klt, i.e. $F = \sum a_i F_i$ with $0 < a_i < 1$.*
(c) *$\mathrm{Supp}(F) \subset f^{-1}(B)$.*

Further, let A be a nef and big $\mathbb{Q}$-divisor on X such that $A^n > N^n$ and $A^d V > N^d$ for any irreducible subvariety $V \subset X$ of dimension d that contains x and such that $\Delta + f^*A$ is Cartier. Then

$$f_*\mathcal{O}_Y(K_Y + \Delta + f^*A)$$

is generated by global sections at x.

For the proof please refer to Section 2.

Remark 1.3.

(1) When $\Delta = 0$ and B is a simple normal crossing divisor, a little more is true. The sheaf $f_*\mathcal{O}_Y(K_Y + f^*A)$ is in fact globally generated at every $x \in X$ around which A satisfies Angehrn-Siu type intersection properties. This is Kawamata's freeness result [Kaw02, Theorem 1.7]. We will use this result to prove Proposition 1.2. His proof relies on the existence of an effective $\mathbb{Q}$-divisor $\mathfrak{D} \sim_{\mathbb{Q}} \lambda A$ for some $0 < \lambda < 1$, such that the pair $(X, \mathfrak{D})$ has an isolated log canonical singularity at a given point $x \in X$. Existence of such divisors is known:
    (a) In any dimension: due to Angehrn and Siu [AS95], when A satisfies the intersection properties as in the hypothesis of Proposition 1.2 (see also [Kol97, Theorem 5.8]).

  (b) When $\dim X \leqslant 4$: due to Kawamata [Kaw97], for A satisfying better bounds: $A^n > n^n$ and $A^d V > n^d$ for any irreducible subvariety $V \subset X$ of dimension d that contains x.

  Therefore in the Proposition, when $n \leqslant 4$, we may replace the intersection hypothesis with that in (b).

(2) If we removed the assumption that $\mathrm{Supp}(F) \subset f^{-1}(B)$, the sheaf $f_*(K_Y + \Delta + f^*A)$ would still be generated by global sections at x for a very general point $x \in X$. This is the main point of the proof of Theorem A.

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2. Technical Background. We begin this section with the proof of the generalisation of Kawamata's freeness result, Proposition 1.2. We proceed by reducing from the case of a pair to the absolute case. We do this by using an inductive procedure of removing the coeffecients of the components of Δ, using Kawamata coverings [Laz04a, Theorem 4.1.12]. The proof also uses the relative freeness result in [Kaw02, Theorem 1.7] for the base case of this inductive procedure.

Proof of 1.2. We proceed inductively on the number of components of Δ. This process involves choosing a suitable Bloch-Gieseker covering (see [Laz04a, Theorem 4.1.10]) followed by the cyclic covering of the components of Δ, as follows:

Kawamata covering of D: It is enough to show when D is irreducible. We take a Bloch-Gieseker cover $p : Z \to Y$ of Y, such that the preimage of D, $p^*D \in |M^{\otimes k}|$ for some line bundle M on Z. Using [Laz04a, Lemma 4.1.11], we ensure that p^*D and the fibre $\mathfrak{F}'_x$, of $f \circ p$, over x are both smooth and intersect each other transversally. We can further make sure that the preimages, p^*F_i, of the components of F, are also irreducible and $p^*D + p^*F$ has simple normal crossing support. Moreover since p is flat and f is smooth over a neighbourhood around x, we can conclude that there is a (possibly smaller) open neighbourhood U around x such that $f \circ p$ is still smooth over U.

For the ease of notation set $g = f \circ p$ and denote by B, the branch locus of g in X. Further note that $x \notin B$.

Now, ω_Y is a direct summand of $p_*\omega_Z$ via the trace map. Therefore

$$f_*\mathcal{O}_Y(K_Y + \Delta + f^*A)$$

is a direct summand of

$$(f \circ p)_* \mathcal{O}_Z\big(K_Z + \frac{l}{k} p^*D + p^*F + (f \circ p)^*A\big).$$

Hence it is enough to show that the later, i.e.

$$g_* \mathcal{O}_Z(K_Z + lM + p^*F + g^*A)$$

is generated by global sections at x.

To do this we take a k^{th} cyclic cover $q : Y' \to Z$ of p^*D. Since p^*D intersects $\mathfrak{F}'_x$ transversally, by Lemma 2.1 there is an open set U around x such that p^*D intersects all the fibres over U transversally. Further by Lemma 2.2 we see that $g \circ q$ is still smooth over U, in other words x is not in the branch locus of $g \circ q$. We call this branch locus by $B \subset X$ again. For the ease of notation set $f' := g \circ q$. Note that,

$$q_* \mathcal{O}_{Y'}(K_{Y'}) \simeq \bigoplus_{i=0}^{b} \mathcal{O}_Z(K_Z + p^*D - iM) \simeq \bigoplus_{i=0}^{b} \mathcal{O}_Z(K_Z + (k-i)M).$$

Indeed, since $p^*D \sim kM$. Further, since $k > l$ the direct sum on the right hand side contains the term $\mathcal{O}_Z(K_Z + lM)$ when $i = k - l$.

Therefore it is enough to show that,

$$f'_* \mathcal{O}_{Y'}(K_{Y'} + q^*p^*F + f'^*A)$$

is generated by global sections at x.

We rename f' by f, Y' by Y and q^*p^*F by F. Note that we still have that $\mathrm{Supp}(F) \subset f^{-1}(B)$, the divisor F still has simple normal crossing support, and $x \notin B$. In the above process we ensure that the inverse images of the F_i's remain irreducible.

Induction on the components of F: We proceed by induction on the components of $F = \sum_i a_i F_i$. Suppose $a_1 = \frac{a}{b} \neq 0$. As before, we start by choosing a b^{th} Bloch-Gieseker cover [Laz04a, Thm. 4.1.10], $p : Z \to Y$ of Y, so that the preimage p^*F_1 is irreducible and is in the linear system of the b^{th} tensor power of some line bundle. Since the fibre $\mathfrak{F}_x$ over x does not intersect F_1, we can choose p ensuring that $f \circ p$ is still smooth around x. As before, we then take the cyclic cover along p^*F_1. Again since p^*F_1 does not intersect the fibre over x, the composition of $f \circ p$ with this cyclic cover remains smooth around the point x. Replacing f by the its composition with this b^{th} Kawamata covering of F_1, we are reduced to showing that

$$f_* \mathcal{O}_Y(K_Y + \sum_{i \geqslant 2} a_i F_i + f^*A))$$

is generated by global sections at x. Therefore by induction we are now left with showing that

$$f_* \mathcal{O}_Y(K_Y + f^*A))$$

is generated by global sections at x.

Denote by B the branch locus of f composed with all these Kawamata coverings. We emphasise that the point $x \notin B$.

Base case of the induction: Take a birational modification X' of X such that $\mu^{-1}(B)_{\mathrm{red}} =: \Sigma$ in X', as in the diagram below, has simple normal crossing support and $X' \setminus \Sigma \simeq X \setminus B$. In particular, μ is an isomorphism around x. Let $\tau : Y' \to Y$ be a resolution of the fibre product $X' \times_X Y$. We have the following commutative diagram:

$$\begin{array}{ccc} Y' & \xrightarrow{\tau} & Y \\ \Big\downarrow h & & \Big\downarrow f \\ X' & \xrightarrow{\mu} & X \end{array}$$

Since μ is an isomorphism over the neighbourhood U around x, $\mu^* A$ is a big and nef divisor that satisfies the intersection properties, as in the hypothesis, at the point $\mu^{-1}(x)$. Moreover since h is smooth outside of the simple normal crossing divisor Σ, we can apply Kawamata's freeness result [Kaw02, Theorem 1.7] to conclude that, $h_* \mathcal{O}_{Y'}(K_{Y'} + h^* \mu^* A)$ is generated by global sections at $\mu^{-1}(x)$. Additionally we have that,

$$\mu_* h_* \mathcal{O}_{Y'}(K_{Y'} + h^* \mu^* A) \simeq f_* \mathcal{O}_Y(K_Y + f^* A).$$

Therefore the sheaf $f_* \mathcal{O}_Y(K_Y + f^* A)$ is generated by global sections at x.

□

The following two facts were used in the proof of Proposition 1.2.

Lemma 2.1. *Let $f : Y \to X$ be a smooth and proper surjective map of smooth varieties. Let D be a smooth effective divisor on Y such that D intersects a fibre $\mathfrak{F}_x$ over a smooth point $x \in X$ transversally. Then there is an analytic open set U around x such that $f|_{D \cap f^{-1}(U)} : D \cap f^{-1}(U) \to U$ is smooth, or equivalently D intersects all the fibres over U transversally.*

Proof. For every $y \in D \cap \mathfrak{F}_x$, we choose coordinates $V'_y \subset Y$ around y and coordinates $U_y \subset X$ around x such that f is smooth over U_y and $f(V'_y) \subset U_y$. We choose V'_y such that in these local coordinates we can write $D = (y_m = 0)$. Since D intersects the fibre $\mathfrak{F}_x$ over x transversally, we have that

$$\mathrm{Jac}(f|_D) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_1} \\ \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial y_{m-1}} & \cdots & \frac{\partial f_n}{\partial y_{m-1}} \end{bmatrix}_{(y_1,\ldots,y_{m-1},0)}$$

Let us denote $S_y := Z(\text{determinant of } n \times n \text{ minors of } f)$. Since D intersects $\mathfrak{F}_x$ transversally, S_y does not intersect $\mathfrak{F}_x$. Then on the neighbourhood V'_y, the points where f is smooth is given by the open set $V_y := V'_y \setminus S_y$ containing

y. We can cover $\mathfrak{F}_x$ by finitely many such open sets V_y. Recall that f is smooth over $\bigcup_y U_y$. Pick

$$U := \bigcap_y f(V_y).$$

This is open since a smooth map is open and is non-empty since $x \in U$. Then $\mathrm{Jac}(f|_D)$ is non-degenerate on $f^{-1}(U)$. □

We next show that taking a cyclic cover along a smooth divisor that intersects the smooth fibres of a map transversally, does not introduce new branch points.

Lemma 2.2 (Non-smooth locus under cyclic cover)**.** *Let $f : Y \to X$ be a smooth map between smooth varieties. Let D be a divisor on Y that intersects the fibres transversally. Assume in addition that $D \in |L^k|$ for some line bundle L on Y and for some $k \geqslant 2$. Consider the k^{th}-cyclic cover $\nu : Y' \to Y$ branched along D. Then $f \circ \nu$ is also a smooth map.*

Proof. Pick a point $x \in X$ and a local system of coordinates $x_1, \dots, x_n$ around x. Similarly pick a point y on the fibre $\mathfrak{F}_x$ over x and consider a local system of coordinates $y_1, \dots y_m$. In this local coordinates suppose f is given by $(f_1, \dots, f_n)$. Then the Jacobian of f is given by:

$$\mathrm{Jac}(f) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_1} \\ \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_2} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial y_m} & \cdots & \frac{\partial f_n}{\partial y_m} \end{bmatrix}$$

Since f is smooth, the non-smooth locus of f in Y, given by the common zero loci of the determinant of the $n \times n$ minors of $\mathrm{Jac}(f)$, is empty. We denote this locus by $C := Z(\phi_1, \dots, \phi_l)$, where $l = \binom{m}{n}$ and ϕ_i's are the determinants of the $n \times n$ minors of $\mathrm{Jac}(f)$.

Assume further that around y, D can be written as $D = (y_m = 0)$. Then in local coordinates the k^{th}-cyclic cover of D, $\nu : Y' \to Y$ looks like

$$\nu : (y_1, \dots, y_m) \mapsto (y_1, \dots, y_m^k)$$

Then,

$$\mathrm{Jac}(f \circ \nu) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_1} \\ \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_n}{\partial y_2} \\ \vdots & \ddots & \vdots \\ k y_m^{k-1} \frac{\partial f_1}{\partial y_m} & \cdots & k y_m^{k-1} \frac{\partial f_n}{\partial y_m} \end{bmatrix}$$

As before, the non-smooth locus C' of $f \circ \nu$ in Y' is given by the common zero of the determinant of the $n \times n$ minors of $\mathrm{Jac}(f \circ \nu)$. We write these determinant equations in terms of ϕ_i's.

$$C' := Z(k y_m^{k-1} \phi_1, \dots, k y_m^{k-1} \phi_s, \phi_{s+1}, \dots, \phi_l),$$

where $\phi_1, \ldots, \phi_s$ are equations of the minors that involve the last row of $\mathrm{Jac}(f)$.

We want to show that $C' = \emptyset$. Suppose there is a point $p \in C'$, since $p \notin C$, p must lie on $y_m = 0$. Now since $D = (y_m = 0)$ intersects all the smooth fibres of f transversally, p cannot be a singular point of $f\big|_D$. In other words, the point

$$p \notin Z\big(\phi_{s+1}(y_1, \ldots, y_{m-1}, 0), \ldots, \phi_L(y_1, \ldots, y_{m-1}, 0)\big)$$

and hence $p \notin C'$.

□

3. Proof of the main theorems. Inspired by [PS14], the strategy is to turn the generation problem for pluricanonical bundles into one for canonical bundles on pairs. We will show that such pair can be carefully chosen to satisfy the properties in the hypothesis of Proposition 1.2.

Proof of Theorem A. We prove Case (a) of the theorem. Case (b) follows similarly (see Remark 1.3 (2)).

Following the proof of [PS14, Theorem 1.7], we first take a log resolution $\mu : \widetilde{Y} \to Y$ of the base ideal of the adjunction morphism $f^*f_*P \xrightarrow{\pi} P$ and the pair (Y, Δ). Write:

$$K_{\widetilde{Y}} - \mu^*(K_Y + \Delta) = Q - N$$

where $Q + N$ is an effective $\mathbb{Q}$-divisor with simple normal crossing support, N is the strict transform of Δ, and therefore has coefficients smaller than or equal to 1, and Q is supported on the exceptional locus. Define:

$$\widetilde{P} := \mu^*P \otimes \mathcal{O}_{\widetilde{Y}}(\lceil Q \rceil)$$

and

$$\widetilde{\Delta} := N + \lceil Q \rceil - Q.$$

Then by definition, the line bundle $\widetilde{P}$ is the same as $\mathcal{O}_{\widetilde{Y}}\big(k\big(K_{\widetilde{Y}} + \widetilde{\Delta}\big)\big)$. Moreover, since Q is exceptional, we have the sheaf isomorphism $\mu_*\widetilde{P} \simeq P$. We rename $\widetilde{Y}$ by Y, $\widetilde{P}$ by P and $\widetilde{\Delta}$ by Δ, so that the image of the adjunction morphism π is given by $P \otimes \mathcal{O}_Y(-E)$, for an effective divisor E and so that Y is smooth and the divisor $\Delta + E$ has simple normal crossing support.

Next, write $\Delta = \sum_i a_i \Delta_i$, where Δ_i's are the irreducible components of Δ. Let E_j's denote the irreducible components of E. Similar to the construction in the proof of Proposition 1.2, we take k^{th} Kawamata covers of Δ_i's and E_j's and denote the composition of these covers by $p : Y' \to Y$. We choose these covers so that $p^*\Delta_i = k\Delta'_i$ and $p^*E_j = kE'_j$ for irreducible divisors Δ'_i and E'_j. We further ensure that $p^*(\Delta + E)$ has simple normal crossing support.

Denote by B, the branch locus of $f \circ p$. Consider the following Cartesian diagram:

$$\begin{tikzcd} Y \setminus C \simeq V \arrow[r, hook, "i_V"] \arrow[d, "f_V"] & Y \arrow[d, "f"] \\ X \setminus B =: U \arrow[r, hook, "i"] & X \end{tikzcd}$$

where $C = f^{-1}(B)$.

Fix $x \in X \setminus B$. Now, pick a positive integer m which is smallest with the property that the sheaf $f_* P \otimes L^{\otimes m}$ is generated by global sections at each point on U. Therefore by adjunction, we have that $P(-E) \otimes f^* L^{\otimes m}$ is globally generated on V. Moreover, since p is a finite map, $p^*\big(P(-E) \otimes f^* L^{\otimes m}\big)$ is globally generated on $Y' \setminus p^{-1}(C)$ by the sublinear system $p^*\big|P(-E) \otimes f^* L^{\otimes m}\big|$. By Bertini's theorem (see Remark III.10.9.2 [Har77] and [Jou83]), we can pick $\mathfrak{D} \in \big|P(-E) \otimes f^* L^{\otimes m}\big|$ so that $\mathfrak{D}$ is smooth outside of C and such that $p^*\mathfrak{D}$ is also smooth outside $p^{-1}(C)$. We further ensure that the divisor $p^*\mathfrak{D}$ intersects the smooth fibre $\mathfrak{F}'_x$ over x transversally. To simplify notations, we denote $p^{-1}(C)$ by C again.

We can write:

$$kP + mf^*L \sim \mathfrak{D} + E$$

From this we get,

$$k\big(K_Y + \Delta\big) \sim_{\mathbb{Q}} K_Y + \Delta + \frac{k-1}{k}\mathfrak{D} + \frac{k-1}{k}E - \frac{k-1}{k}mf^*L,$$

and hence for any integer l,

$$k\big(K_Y + \Delta\big) + lf^*L \sim_{\mathbb{Q}} K_Y + \Delta + \frac{k-1}{k}\mathfrak{D} + \frac{k-1}{k}E + \Big(l - \frac{k-1}{k}m\Big)f^*L.$$

Now, since E is the relative base locus of the adjunction morphism $f^*f_*P \xrightarrow{\pi} P$, for every effective Cartier divisor E' such that $E - E'$ is effective we have

$$f_*\big(P(-E')\big) \simeq f_*P.$$

We would like to pick integral divisors, E' as above so that

$$\Delta + \frac{k-1}{k}E - E'$$

has coefficients strictly smaller than 1. We do so as follows: write:

$$E = \sum_i s_i \Delta_i + \widetilde{E}$$

and

$$\Delta = \sum_i a_i \Delta_i$$

such that $\widetilde{E}$ and Δ do not have any common component. Note that, by hypothesis, $0 < a_i \leqslant 1$ and $s_i \in \mathbb{N}$. We want to pick non-negative integers b_i, such that

$$0 \leqslant a_i + \frac{k-1}{k} s_i - b_i < 1$$

and

$$b_i \leqslant s_i.$$

Denote by

$$\gamma_i := a_i + \frac{k-1}{k} s_i$$

and note that $\gamma_i < 1 + s_i$. We pick b_i as follows: if for some integer j, such that $0 \leqslant j \leqslant s_i$, we can squeeze γ_i between $s_i - j + 1 > \gamma_i \geqslant s_i - j$, pick

$$b_i = s_i - j.$$

Now pick

$$E' := \sum_i b_i \Delta_i + \left\lfloor \frac{k-1}{k} \widetilde{E} \right\rfloor .$$

Then we can rewrite the above $\mathbb{Q}$-linear equivalence of divisors as:

$$P - E' + l f^* L \sim_{\mathbb{Q}} K_Y + \widetilde{\Delta} + \frac{k-1}{k} \mathfrak{D} + \Big(l - \frac{k-1}{k} m\Big) f^* L$$

where

$$\widetilde{\Delta} = \Delta + \frac{k-1}{k} E - E' = \sum_i \alpha_i \widetilde{\Delta}_i .$$

By construction $0 < \alpha_i < 1$ and $\widetilde{\Delta}$ has simple normal crossing support.

It is now enough to show that the pushforward of the right hand side of the above $\mathbb{Q}$-linear equivalence is globally generated at x and for all $l > \dfrac{k-1}{k} m + N$. Indeed, in that case the left hand side would satisfy similar global generation bounds and by the discussion above

$$f_* P(-E') \otimes L^{\otimes l} \simeq f_* P \otimes L^{\otimes l}.$$

Said differently, this would mean that

$$f_* P \otimes L^{\otimes l}$$

is globally generated on U for all $l > \dfrac{k-1}{k} m + N$. From our choice of m, we must have that $m \leqslant \frac{k-1}{k} m + N + 1$. This is the same as $m \leqslant k(N+1)$. As a consequence,

$$f_* P \otimes L^{\otimes l}$$

is generated by global sections on U for all $l \geqslant (k-1)(N+1) + N + 1 = k(N+1)$.

It now remains to show that

$$f_* \mathcal{O}_Y \Big(K_Y + \widetilde{\Delta} + \frac{k-1}{k} \mathfrak{D} + \Big(l - \frac{k-1}{k} m\Big) f^* L \Big)$$

is globally generated at x. To do so, first note that when $l - \dfrac{k-1}{k}m > N$, $\left(l - \frac{k-1}{k}m\right)L$ is ample and satisfies Angehrn-Siu type intersection properties as stated in Proposition 1.2. However the divisor $\widetilde{\Delta} + \frac{k-1}{k}\mathfrak{D}$ may not satisfy the hypothesis of Proposition 1.2. For instance, it may not have simple normal crossing support. Therefore we cannot apply Proposition 1.2 directly. Since we are only interested in generic global generation though, we can get around these problems. The rest of the proof is devoted to this.

We have that $k\alpha_i$ is an integer and by construction, p is a composition of k^{th} Kawamata coverings of the components $\widetilde{\Delta}_i$'s of $\widetilde{\Delta}$. Following the inductive argument as in the proof of Proposition 1.2, we see that

$$f_*\mathcal{O}_Y\left(K_Y + \widetilde{\Delta} + \frac{k-1}{k}\mathfrak{D} + \left(l - \frac{k-1}{k}m\right)f^*L\right)$$

is a direct summand of

$$\left(f \circ p\right)_*\mathcal{O}_{Y'}\left(K_{Y'} + \frac{k-1}{k}\mathfrak{D}' + (f \circ p)^*\left(l - \frac{k-1}{k}m\right)L\right)$$

where $\mathfrak{D}' = p^*\mathfrak{D}$. Therefore it is enough to show that the latter is globally generated at x.

We are now almost in the situation of Proposition 1.2: by our choice of $\mathfrak{D}'$, it intersects the fibre over x transversally, however, it may not be klt with simple normal crossing support.

To deal with this, we take a log resolution $\mu : Y'' \to Y'$ of $\mathfrak{D}'$, that is an isomorphism outside C and write

$$\mu^*\mathfrak{D}' = D + F$$

where D intersects the fibre over x transversally and F is supported on $C := \mu^{-1}(C)$. We replace, Y'' by Y', rename the divisor $\mu^*\mathfrak{D}'$ by $\mathfrak{D}'$. Therefore, we can assume that $\mathfrak{D}'$ has simple normal crossing support.

To deal with the fact that F may not be klt, consider the effective Cartier divisor $F' = \left\lfloor \dfrac{k-1}{k}F \right\rfloor$. Since, Supp$(F')$ is contained in the C and $x \notin B$, the stalks

$$\left(f \circ p\right)_*\mathcal{O}_{Y'}\left(K_{Y'} + \frac{k-1}{k}\mathfrak{D}' + \left(l - \frac{k-1}{k}m\right)(f \circ p)^*L\right)_x \simeq$$
$$\left(f \circ p\right)_*\mathcal{O}_{Y'}\left(K_{Y'} + \frac{k-1}{k}\mathfrak{D}' - F' + \left(l - \frac{k-1}{k}m\right)(f \circ p)^*L\right)_x$$

are isomorphic. Moreover the global sections of the later embed into the global sections of the former. Therefore, it is now enough to show that,

$$\left(f \circ p\right)_*\mathcal{O}_{Y'}\left(K_{Y'} + \widetilde{\Delta} + \left(l - \frac{k-1}{k}m\right)(f \circ p)^*L\right)$$

is globally generated at x for $l > \dfrac{k-1}{k}m + N$. Here

$$\widetilde{\Delta} := \frac{k-1}{k}\mathfrak{D}' - F'.$$

This is a klt divisor with simple normal crossing support and satisfies the hypothesis in Proposition 1.2. Hence the global generation follows from Proposition 1.2. □

The proof of Theorem C goes along the same lines. The main difference is that, in this case, we do not start by picking a Kawamata cover, but rather we show directly that, due to the additional relative semi-ampleness assumptions, the above argument works for all x outside of the branch locus B of f.

Proof of Theorem C. As before, we start by replacing Y by a birational modification to assume that the relative base ideal of $\omega_Y^{\otimes k}$ is $\mathcal{O}_Y(-E)$, for some effective divisor E with simple normal crossing support. Note that in this case, since $\omega_Y^{\otimes k}$ is relatively free over $X \setminus B$, the divisor E is supported on $C := f^{-1}(B)$. This process does not change the branch locus B of f.

Fix a point $x \in X \setminus B$. Consider the following Cartesian diagram:

$$\begin{tikzcd} Y \setminus C \simeq V \arrow[r, hook, "i_V"] \arrow[d, "f_V"] & Y \arrow[d, "f"] \\ X \setminus B =: U \arrow[r, hook, "i"] & X \end{tikzcd}$$

As in the proof of Theorem A, we pick a positive integer m which is smallest with the property that the sheaf $f_*\omega_Y^{\otimes k} \otimes L^{\otimes m}$ is generated by global sections at each point of U. Then $f^*f_*\big(\omega_Y^{\otimes k} \otimes f^*L^{\otimes m}\big)$ is also generated by global sections on V. Therefore by adjunction, so is $\omega_Y^{\otimes k}(-E) \otimes f^*L^{\otimes m}$ on V. As a consequence, we can pick a divisor $\mathfrak{D} \in \big|\omega_Y^{\otimes k}(-E) \otimes f^*L^{\otimes m}\big|$ such that $\mathfrak{D}$ is smooth outside of C and intersects the fibre $\mathfrak{F}_x$ over x transversally.

After replacing Y with a birational modification that is an isomorphism outside of C, we may assume that $\mathfrak{D} = D + F$, where D is smooth, intersects the fibre $\mathfrak{F}_x$ over x transversally and does not share any component with E Moreover, we assume that the divisor F is supported on C, and $E + D + F$ to have simple normal crossing support.

Write

$$kK_Y + mf^*L \sim D + F + E.$$

From this we can write,

$$kK_Y + lf^*L \sim_{\mathbb{Q}} K_Y + \frac{k-1}{k}D + \frac{k-1}{k}(F+E) + \Big(l - \frac{k-1}{k}m\Big)f^*L$$

for any integer l.

Now consider the effective divisor $\lfloor \frac{k-1}{k}(E+F) \rfloor$ and denote the fractional part by

$$\mathfrak{E} := \frac{k-1}{k}(E+F) - \left\lfloor \frac{k-1}{k}(E+F) \right\rfloor .$$

We obtain the following $\mathbb{Q}$-linear equivalence:

$$kK_Y - \left\lfloor \frac{k-1}{k}(E+F) \right\rfloor + lf^*L \sim_{\mathbb{Q}} K_Y + \frac{k-1}{k}D + \mathfrak{E} + \Big(l - \frac{k-1}{k}m\Big) f^*L.$$

Denote $\Delta := \dfrac{k-1}{k}D + \mathfrak{E}$. It is now enough to show that

$$f_*\mathcal{O}_Y\Big(K_Y + \Delta + \big(l - \frac{k-1}{k}m\big) f^*L\Big)$$

is generated by global sections at x for all $l > \frac{k-1}{k}m + N$. Indeed, this would imply that the left hand side of the equation also satisfies similar global generation bounds, i.e. $f_*\mathcal{O}_Y\Big(kK_Y - \lfloor \frac{k-1}{k}(E+F) \rfloor + lf^*L\Big)$ is globally generated at x for all $l > \frac{k-1}{k}m + N$. But note that the divisor $E+F$ is supported on C and $x \notin B$. Therefore the stalks

$$f_*\mathcal{O}_Y\Big(kK_Y - \left\lfloor \frac{k-1}{k}(E+F) \right\rfloor + lf^*L\Big)_x \simeq f_*\mathcal{O}_Y\big(kK_Y + lf^*L\big)_x$$

are isomorphic. Moreover the global sections of the former embeds into the global sections of the later. Said differently, this would in turn imply that

$$f_*\mathcal{O}_Y\big(kK_Y + lf^*L\big)$$

is globally generated on U for all $l > \dfrac{k-1}{k}m + N$. But from our choice of m and from similar arguments as in the proof of Theorem A, it follows that for all $l \geqslant k(N+1)$, the sheaf

$$f_*\omega_Y^{\otimes k} \otimes L^{\otimes l}$$

is generated by global sections on U.

It now remains to show that

$$f_*\mathcal{O}_Y\Big(K_Y + \Delta + \big(l - \frac{k-1}{k}m\big) f^*L\Big)$$

is generated by global sections at x when $l - \frac{k-1}{k}m > N$. But this follows from Proposition 1.2. Indeed, the divisor $\Delta = \frac{k-1}{k}D + \mathfrak{E}$ is klt and has simple normal crossing support, its component D was chosen to intersect the fibre $\mathfrak{F}_x$ over x transversally and the divisor $\mathfrak{E}$ is supported on C. More importantly, since the line bundle L is ample, the $\mathbb{Q}$-divisor $\big(l - \frac{k-1}{k}m\big)L$ satisfies Angehrn-Siu type intersection properties at x for all $l - \frac{k-1}{k}m > N$. $\square$

On a different note, when f is smooth, Kollár's vanishing theorem applied to the right hand side of the equivalence, leads to the following vanishing statement for pluricanonical bundles, with essentially the same proof.

Theorem 3.1 (Effective Vanishing Theorem)**.** *Let $f : Y \to X$ be a smooth surjective morphism of smooth projective varieties, with $\dim X = n$. Assume in addition that $\omega_Y^{\otimes k}$ is relatively free for some $k \geqslant 1$, and let L be an ample line bundle on X. Then,*

$$H^i\Big(X, f_*\omega_Y^{\otimes k} \otimes L^{\otimes l}\Big) = 0$$

for all $i > 0$ and $l \geqslant k(N+1) - N$.[1]

Proof. Since f is smooth, by Theorem C, we know that the sheaf $f_*\omega_Y^{\otimes k} \otimes L^{\otimes l}$ is globally generated for all $l \geqslant k(N+1)$. Therefore by adjunction $\omega_Y^{\otimes k} \otimes f^*L^{\otimes k(N+1)}$ is globally generated as well. As a consequence, we can pick a smooth divisor $D \in \left|\omega_Y^{\otimes k} \otimes f^*L^{\otimes k(N+1)}\right|$ such that D intersects the fibre $\mathfrak{F}_x$ over x transversally.

Write:

$$kK_Y + k(N+1)f^*L \sim D.$$

This is the same as

$$kK_Y + lf^*L \sim_{\mathbb{Q}} K_Y + \frac{k-1}{k}D + \big(l - (k-1)(N+1)\big)f^*L,$$

for any integer l. By applying Kollár's vanishing theorem [Kol95, Corollary 10.15] on the right hand side, we get that

$$H^i\Big(X, f_*\mathcal{O}_Y\big(K_Y + \frac{k-1}{k}D + \big(l - (k-1)(N+1)\big)f^*L\big)\Big) = 0$$

for all $i > 0$ and $l > (k-1)(N+1)$. Therefore, the left hand side satisfies similar vanishing properties

$$H^i\Big(X, f_*\omega_Y^{\otimes k} \otimes L^{\otimes l}\Big) = 0$$

for all $i > 0$ and $l \geqslant k(N+1) - N$ $\square$

References

[AS95] U. Angehrn and Y. T. Siu, *Effective freeness and point separation for adjoint bundles*, Invent. Math. **122** (1995), no. 2, 291–308, DOI 10.1007/BF01231446.

[Den17] Y. Deng, *Applications of the Ohsawa-Takegoshi Extension Theorem to Direct Image Problems*, arXiv:1703.07279 [math.AG] **Preprint.** (2017).

[DH91] S. Diaz and D. Harbater, *Strong Bertini theorems*, Trans. Amer. Math. Soc. **324** (1991), no. 1, 73–86, DOI 10.2307/2001496.

[Jou83] J.-P. Jouanolou, *Théorèmes de Bertini et applications*, Progress in Mathematics, vol. 42, Birkhäuser Boston, Inc., Boston, MA, 1983.

[Kaw02] Y. Kawamata, *On a relative version of Fujita's freeness conjecture*, Complex geometry (Göttingen, 2000), 2002, pp. 135–146.

[1] or, for $l \geqslant k(n+1) - n$ when $n \leqslant 4$

[Kaw97] ———, *On Fujita's freeness conjecture for* 3*-folds and* 4*-folds*, Math. Ann. **308** (1997), no. 3, 491–505, DOI 10.1007/s002080050085.

[Kol97] J. Kollár, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, 1997, pp. 221–287.

[Kol86] ———, *Higher direct images of dualizing sheaves. I*, Ann. of Math. (2) **123** (1986), no. 1, 11–42, DOI 10.2307/1971351.

[Kol95] J. Kollár, *Shafarevich maps and automorphic forms*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1995.

[Laz04a] R. Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Laz04b] ———, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Fuj09] O. Fujino, *Effective base point free theorem for log canonical pairs—Kollár type theorem*, Tohoku Math. J. (2) **61** (2009), no. 4, 475–481, DOI 10.2748/tmj/1264084495.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[PS14] M. Popa and C. Schnell, *On direct images of pluricanonical bundles*, Algebra Number Theory **8** (2014), no. 9, 2273–2295.

[Vie83] E. Viehweg, *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, Algebraic varieties and analytic varieties (Tokyo, 1981), 1983, pp. 329–353.

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