SEMINAR ON HODGE MODULES

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1. Introduction

1.1. Notation.

(1) For a map $f : B \to \mathbb{C}$, $B_t = f^{-1}(t)$ for $t \in \mathbb{C}$.
(2) $B_{\leq s} := \{z \in B || f(z)| \leq s\}$
(3) S^{n-1} and B^n will always denote a real sphere and a real ball respectively, no matter whether they come with a suffix or not.
(4) For the sake of consistency, unlike in the talk, I am reverting to Voisin's notation for cone over vanishing spheres. Please replace $B^n_{\leq s}$ by B^n_t in the talk.

1.2. **Goal.** The goal of the next two talks in this seminar is to understand nearby and vanishing cycles in the classical setting of Lefschetz degeneration. We will use that to show that if $f : X \to \Delta$ is a proper holomorphic map from an n-dimensional comlex variety to a disk, that is a submersion ver Δ^* and x_0 is a non-degenerate critical point over $0 \in \Delta$, then there exists a deformation retraction of X onto $X_t \cup B^n$ where B^n is an n-dimensional ball glued to X_t along the "vanishing" sphere S^{n-1}_t for $t \in \Delta^*$. Stealing the terminology from Morse theory we will refer to this phenomenon as homotopy type around critical points.

We will use this to give a proof of Lefschetz hyperplane theorem. For a smooth hyperplane section $j : Y \hookrightarrow X$ of an n-dimensional smooth projective variety, the vanishing homology is defined by:

$$H_*(Y, \mathbb{Z})_{\text{van}} := \ker(j_* : H_*(Y, \mathbb{Z}) \to H_*(X, \mathbb{Z}))$$

We will eventually show the following theorem:

Theorem 1.1. *Let Y be a smooth hyperplane section $j : Y \hookrightarrow X$ of an n-dimensional smooth projective variety. Then,*

(1) *$H_n(X, Y, \mathbb{Z})$ is generated by classes of the cones on the vanishing spheres.*
(2) *$H_n(Y, \mathbb{Z})_{van}$ is generated by the classes of the vanishing spheres.*
(3) *$j_* : H_k(Y, \mathbb{Z}) \to H_k(X, \mathbb{Z})$ is an isomorphism for $k < n - 1$.*

2. Vanishing spheres in the context of Lefschetz degeneration

Let $f : \mathbb{C}^n \to \mathbb{C}$ be defined by $f(z) = \sum z_i^2$. Note that the point 0 is a non-degenerate (i.e. $Hess_0 f$ is non singular) critical point or an ordinary double point. Let B be a ball of radius r around 0, then $f(B)$ has values in a disk of radius r^2. Such a map $f : B \to \Delta$ is called *Lefschetz degeneration.* Define,

$$S^{n-1}_t := \{z \in B | z_i = \sqrt{s} x_i e^{i\theta}, x_i \in \mathbb{R} \text{ and } \sum_{1 \leq i \leq n} x_i^2 \leq 1\} \subset B_t$$

where $t = se^{i2\theta} \in \Delta$ and

$$B_t^n := \{z \in B | z_i = \sqrt{s} x_i e^{i\theta}, x_i \in R \text{ and } \sum_{1 \le i \le n} x_i^2 \le 1\} \subset B_{\le s} = \{z \in B || f(x)| \le s\}$$

The sphere is called a vanishing sphere and the ball B_t^n is called the cone over the vanishing sphere.

Definition 2.1 (vanishing cycle)**.** The homology class $\delta \in H_{n-1}(B_t, \mathbb{Z})$ with a choice of orientation on S_t^{n-1} is called the vanishing cycle of the Lefschetz degeneration $f : B \to \Delta$.

2.1. **Retraction statement:**

Theorem 2.2. *Let $f : X \to \Delta$ be a proper holomorphic map from an n-dimensional complex variety to a disk, that is a submersion over Δ^* and x_0 is a non-degenerate critical point over $0 \in \Delta$, then there exists a deformation retraction of X onto $X_t \cup B^n$ where B^n is an n-dimensional ball glued to X_t along the "vanishing" sphere S_t^{n-1} for $t \in \Delta^*$.*

sketch: Since f is a proper submersion over Δ^*, by Ehresmann's theorem it suuffice to show the result for an arbitrary small disk centred around 0.

By holomorphic Morse lemma, we can assume that there is a ball B around x_0 and holomorphic coordinates $z_1, \dots, z_n$ such that $f(z) = \sum z_i^2$ for $f = f|_B : B \to \Delta$. Now we are in situation of a Lefschetz degeneration. The homotopy type properties around critical points of a Morse function tells us that by further shrinking Δ we have that $B_\Delta := f^{-1}(\Delta)$, the set B_Δ deformation retracts onto $B_t \cup B_{<s}^n$ where s, t and B^n are as above and $B_{<s}^n$ is glued to B_t along S_t^{n-1}.

Now let B^0 be the interior of B. Let, $B_\Delta^0 := B^0 \cap B_\Delta$. Then $f : X_\Delta \setminus B_\Delta^0 \to \Delta$. is a fibration of varieties with boundary, by Ehresmann we there is a deformation of $X_\Delta \setminus B_\Delta^0$ to X_t via the trivialisation $X_t \times \Delta$. The boundary, $S_\Delta := B_\Delta \cap \partial B$ also retracts onto S_t via the same trivialisation. These two deformation processes are compatible because the deformation of B_Δ can be chosen to preserve a specified retraction of S_Δ to S_t. Hence the result. □

2.2. **Theorem 1.1 for Lefschetz degeneration:**

Corollary 2.3. *Let $i : X_t \hookrightarrow X_\Delta$ be the inclusion. Then $i_* : H_k(X_t, \mathbb{Z}) \to H_k(X_\Delta, \mathbb{Z})$ is an isomorphism for $k < n-1$ and the surjective for $k = n-1$. Moreover, the kernel of i_* is generated by the class of 'the' vanishing sphere S_t^{n-1} of X_t for $k = n-1$.*

Proof. By the discussion in the earlier section we know that X_Δ has th ehomotopy type of X_t with B_t^n glued to it along S_t^{n-1} of X_t. Therefore, $H_*(X_\Delta, \mathbb{Z}) \simeq H_*(X_t \cup_{S_t^{n-1}} B_t^n, \mathbb{Z})$. Moreover, by excision we have

$$H_*(X_t \cup_{S_t^{n-1}} B_t^n, X_t, \mathbb{Z}) \simeq H_*(B^n, S^{n-1}, \mathbb{Z}).$$

We know, $H_k(B^n, S^{n-1}, \mathbb{Z}) = 0$ for $k \le n-1$. Hence the isomorphism. For the second part, we need to write down the relative homology sequence for (B^n, S^{n-1}) and (X_Δ, X_t) along with the maps between them. This will show that the kernel is precisely given by δ as defined in 2.1. □

3. Lefschetz Pencils:

A Pencil of hyperplane is a family of hyperplanes parametrised $\mathbb{P}^1$. Consider the setting of theorem 1.1. Suppose $\{X_t\}$ is a pencil of hyperplane passing through $Y = X_0 = (\sigma_0 = 0)$ i.e. for a hyperplane σ_∞, $X_t := (\sigma_0 + t\sigma_\infty = 0)$. Suppose also that the base locus $B := Z(\sigma_0, \sigma_\infty)$ is smooth, or in other words $(d\sigma_0, d\sigma_\infty)$ are linearly independent along B. Then the variety,

$$\tilde{X} := Bl_B X = \{(x,t) \in X \times \mathbb{P}^1 | \sigma_0(x) + t\sigma_\infty(x) = 0\}$$

is smooth and $f : \tilde{X} \to \mathbb{P}^1$ is a proper surjective morphism submersion outside the singularities of X_t. If we would like to apply the techniques of Morse theory around singularities of X_t for the map $f : X - X_\infty \to \mathbb{C}$ we would need to ensure that X_t do not have any worse than an ordinary double point singularity. Lefschetz proved that given a smooth hyperplane Y such a family always exists. Such a family is known as *Lefschetz Pencils*. More generally:

Definition 3.1 (Lefschetz Pencils:)**.** A Lefschetz pencil $(X_t)_{\mathbb{P}^1}$ is a pencil of hypersurfaces satisfying the following conditions:

(1) The base locus B is smooth of codimension 2
(2) Every hypersurface X_t has at most one ordinary double point as singularity.

Definition 3.2 (ordinary double point:)**.** For a map $f : X \to \mathbb{C}$, we will say that X_t has an ordinary double point at 0 if 0 is a non-degenerate critical point of f.

3.1. **Existence of Lefschetz pencils:** Let $X \subset \mathbb{P}^n$ be a non-degenrate (smooth??) subvariety of $\mathbb{P}^N$ i.e. the map

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(X, \mathcal{O}_X(1))$$

is injective. Then consider

$$Z := \{(x,H) \in X \times \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))) | X_H := X \cap H \text{ is singular at } x\}$$

Denote, $N = h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$. The first projection $p_1 : Z \to X$ makes Z a fibre bundle over X with fibres of dimension $\mathbb{P}^{N-n-1}$. Therefore, Z is smooth and of dimension $N-1$. It can be shown that, $\dim p_2(Z) = N-1$ iff there is a point $(x,H) \in Z$ such that $H \cap X$ has an ordinary double point x. For convenience, we denote $k := H^0((\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)))$. More formally,

Lemma 3.3. *Let $(x,H) \in Z$. Then X_H has only one ordinary double point at x iff $p_2 : Z \to K$ is an immersion at the point (x,H).*

Proof. local calculation see Voisin. □

Therefore, we have the following existence theorem for Lefschetz pencils:

Theorem 3.4 (Existence:)**.** *Let $X \subset \mathbb{P}^N$ be a smooth subvariety. Then a line $\Delta \subset K$ describes a Lefschetz pencil if and only if*

(1) *$p_2(Z) =: \mathcal{D}_X$ is a hypersurface in K and Δ meets the hypersurface $\mathcal{D}_X$ transversally in the open dense set*

$$\mathcal{D}_X^0 := \{H | X_H \text{ has at worst ordinary double points }\}$$

.

(2) $\dim D_X < N-1$ *then Δ does not intersect $\mathcal{D}_X$.*

Proof. The second statement is clear since by the lemma, for all $H \in \mathcal{D}_X$, $X \cap H$ have worse than ordinary double points and by definition of a Lefschetz pencil Δ cannot admit such a point. For the first statement similar argument says that Δ must intersect $\mathcal{D}_X$ in $\mathcal{D}_X^0$. Why transversally will require the fact that the base locus of Δ is smooth and therefore cannot contain the double point. In other words, if Δ passes through $H \in \mathcal{D}_X$, then since locally around H $\mathcal{D}_X$ looks like a hyperplace in K, Δ will be contained in $\mathcal{D}_X$. In other words, B will contain the double point. □

4. Lefschetz hyperplane theorem for Lefschetz pencil

Let $Y \subset X$ be a smooth hyperplane section of a smooth projective variety $X \subset \mathbb{P}^n$. Consider a generic Lefschetz pencil passing through Y such that the ∞ is a regular value of the pencil. In other words, X_∞ is smooth. Then we have seen in the earlier section that $\tilde{X} := BL_B X \subset X \times \mathbb{P}^1$ is a Lefschetz pencil under the map projection to the second factor. Then, $f : \tilde{X} \setminus X_\infty \to \mathbb{C}$ has ordinary double points over finitely many values $0_1, \dots, 0_k \in \mathbb{C}$. Restricting disks Δ_i around 0_i we are in the situation of theorem 2.2. Let x_i be the ordinary double point of X_{0_i}. In this case, there is $t_i \in \Delta_i$ such that, $X_{\Delta_i} := f^{-1}(\Delta_i)$ is homotopy equivalent to $X_{t_i} \cup_{S^{n-1}_{t_i}} B^n t_i$. Now since 0 is a regular value for the pencil the map $f : X_{\gamma_i} \to \mathbb{C}$ is a submersion, where $X_{\gamma_i} = f^{-1}(\gamma_i)$ for a path γ_i connecting 0 and x_i in $\mathbb{C}$ not containing any of the critical values of f. Therefore by Ehresmann's theorem, $X_{\gamma_i} \simeq X_0 \times \gamma_i$ and therefore deformation retracts on to $X_0 \times \gamma_i(1)$. Let $S_i^{n-1} := S_{t_i}^{n-1} \times 1$ be the vanishing sphere on X_0 under this deformation retraction and $B_i^n := B_{t_i}^n \cup S_{t_i}^{n-1} \times [0,1]$ is the cone over the vanishing sphere S_i^{n-1}. Then, $\tilde{X} \setminus X_\infty$ is homotopy equivalent to X_0 with B_i^n's glued to X_0 along S_i^{n-1}. We immediately have the following theorems:

Theorem 4.1. *The restriction arrow*
$$i_t'^* : H^k(\tilde{X} \setminus X_\infty, \mathbb{Z}) \to H^k(X_t, \mathbb{Z})$$
for $i'_t : X_t \to \tilde{X} \setminus X_\infty$ is an isomorphism for $k < n-1$ and is injective for $k = n-1$ where $n = dimX$

Proof. This is the dual statement for that of the homology. Notice that in case of homology we know that the ker of the restriction map for $k = n-1$ is generated by the homology classes of the vanishing spheres. □

We also have a description of the reative homology:

Theorem 4.2. *The relative homology $H_n(\tilde{X} \setminus X_\infty, X_0, \mathbb{Z})$ is generated by B_i^n.*

4.1. **Other related results.** For the rest of the notes, we will try to understand the following very related theorems. We follow the notations established so far. Additionally, we fix the following notaions $i_t : X_t \hookrightarrow X$, $i'_t : X_t \hookrightarrow \tilde{X} \setminus X_\infty$, $j_t : X_t \hookrightarrow \tilde{X}$ and $j : Y \hookrightarrow X$.

Theorem 4.3 (Lefschetz hyperplane theorem)**.** *The restriction arrow*
$$i_t^* : H^k(X, \mathbb{Z}) \to H^k(X_t, \mathbb{Z})$$
is an isomorphism for $k < n-1$ and is injective for $k = n-1$ where $n = dimX$. Therefore similar statement is true for Y.

Theorem 4.4 (Description of the vanishing cohomology)**.** *Define the vanishing cohomology of Y as follows:*

$$H^k(Y,\mathbb{Z})_{van} := \ker\{j_* : H^k(Y,\mathbb{Z}) \to H^{k+2}(X,\mathbb{Z})\}$$

where j_* obtained as follows:

$$H^k(Y) \xrightarrow{PD} H^{2n-k-2}(X)^* \xrightarrow{(j^*)^t} H^{2n-k-2}(X)^* \xrightarrow{PD} H^{k+2}(X)$$

where PD is the Poincaré duality map and j^* is the usual restriction map of cohomology.
Then vanishing cohomology of Y, $H^k(Y,\mathbb{Z})_{van}$ is 0 when $k \neq dimY$ and is generated by the classes of the vanishing spheres of a Lefschetz pencil passing through Y when $k = n-1$.

Theorem 4.5. *The relative homology $H_n(X,Y,\mathbb{Z})$ is generated by the images in X under $\tau : \tilde{X} \to X$ of the cones over the vanishing cycles.*

Define *primitive cohomology* of Y as follows:

$$H^k(Y,\mathbb{Z})_{prim} = \ker\{L : H^k(Y,\mathbb{Z}) \to H^k(X,\mathbb{Z})\}$$

where L is defined by the cup product with the $(1,1)$-form $c_1(\mathcal{O}_Y(1)) \in H^2(Y,\mathbb{Z})$ and is known as the *Lefschetz operator*. Also, note that $L = j^* \circ j_*$ in integral cohomology. We will discuss this briefly at the end.

Theorem 4.6 (Relation between primitive cohomology and vanishing cohomology)**.** .

(1) *We have an orthogonal direct sum w.r.t. the perfect pairing in the rational Poincaré duality:*

$$H^{n-1}(Y,\mathbb{Q}) = H^{n-1}(Y,\mathbb{Q})_{van} \oplus j^*H^k(X,\mathbb{Q})$$

(2) *Similarly,*

$$H^{n-1}(Y,\mathbb{Q})_{prim} = H^{n-1}(Y,\mathbb{Q})_{van} \oplus j^*H^k(X,\mathbb{Q})_{prim}$$

5. Proofs

Proof of Lefschetz hyperplane theorem 4.3 and 4.5 rely primarily on understanding the relation between cohomologies of $\tilde{X} \setminus X_\infty$ an cohomologies of X. It is known that $H^k(X) \oplus H^{k-2}(B) \xrightarrow{\tau^* + j_* \circ \tau^*} H^k(\tilde{X})$ is an isomorphism, where $\tau : \tilde{X} \to X$ and $j : E \hookrightarrow X$. For details of the proof of the Lefschetz hyperplane theorem we refer to [] section 2.3. However we outline the proof here:

(1) **surjectivity of $i_t^* : H^k(X) \to H^k(X_t)$ for $k < n-1$** We know from the discussions about Lefschetz pencils that $j_t'^* : \tilde{X} \setminus X_\infty \to H^k(X_t)$ is surjective for $k < n-1$. By chasing long exact sequence of pairs and identifying $H^k(\tilde{X}, \tilde{X} \setminus X_\infty) \xrightarrow{T} H^{k-2}(X_\infty)$ we note that the map $H^k(\tilde{X}) \to H^k(\tilde{X} \setminus X_\infty)$ is surjective. Therefore so is $H^k(\tilde{X}) \xrightarrow{j_t^*} H^k(X_t)$. Now we have the following commutative diagram:

$$\begin{tikzcd} H^{k-2}(E) \arrow[r, "j_*"] & H^k(\tilde{X}) \arrow[d, "j_t^*"] \\ H^{k-2}(B) \arrow[u, "\tau^*"] \arrow[r, "l_*"] & H^k(X_t) \end{tikzcd}$$

where $l : B \hookrightarrow X_t$ is the inclusion map. We want to show that, $im(i_t^*) = H^k(X_t)$. But $H^k(X_t) = im(j_t^*) = im(i_t^*) \cup im(l_*)$. Therefore it is enough to show that $im(i_t^*) \subset im(l_*)$. This follows from noting that $l_* \circ l^* = i_t^* \circ i_{t*}$ and that l^* is surjective for $k < n-2$ inductively by Lefschetz theorem on the smooth hyperplane B in X_t.

(2) **Injectivity of $i_t^* : H^k(X) \to H^k(X_t)$ for** $kn-1$ By Lefschetz theorem for the Lefschetz pencil we know that $j_t'^* : H^k(\tilde{X} \setminus X_\infty) \to H^k(X_t)$ is injective for $k \leq n-1$. We have the following commutative diagram:

$$\begin{tikzcd} H^k(\tilde{X} \setminus X_\infty) \arrow[r, "j'_{t*}"] & H^k(X_t) \\ H^k(\tilde{X}) \arrow[u, "\text{restr}"] & H^k(X) \arrow[l, "\tau^*"] \arrow[u, "i_t^*"'] \end{tikzcd}$$

Now if $i_t^*(a) = 0$, then injectivity of j'$_{t*} tellsusthat \tau^* a|_{\tilde{X} \setminus X_\infty} = 0$. By chasing l.e.s. of cohomology of pair and identifying $H^k(\tilde{X}, \tilde{X} \setminus X_\infty) \xrightarrow{T} H^{k-2}(X_\infty)$ we can conclude that there is a $b \in H^{k-2}(X_\infty)$ such that $(j_\infty)_*(b) = \tau^* a$. From the commutative diagram

$$\begin{tikzcd} H^{k-2}(X_\infty) \arrow[r, "(j_\infty)_*"] \arrow[d, equal] & H^k(\tilde{X}) \\ H^{k-2}(X_\infty) & H^k(X) \arrow[l, "(i_\infty)_*"] \arrow[u, "\tau^*"'] \end{tikzcd}$$

it follows that $\tau^* a = \tau^*(i_\infty)_* b$. Since τ^* is injective (since $\tau : X \to X$ is a surjective map between two smooth compact manifolds) we have $a = (i_\infty)_* b$. We note that $(j_\infty)_*|_{H^{k-2}(B)} = l^*$. By induction on (X_∞, B) we conclude that l^* is injective, then $b = 0$. Therefore $a = 0$.