Polarized Hodge Module

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1 The duality functor

To begin with let M is a left $\mathfrak{D}_X$-module on a complex manifold X. Note that $\mathcal{H}om_{\mathfrak{D}_X}(M, \mathfrak{D}_X)$ is a right D-module. By side changing operation, we get a good candidate for the dual D-module. $\mathcal{H}om_{\mathfrak{D}_X}(M, \mathfrak{D}_X) \otimes \Omega_X^{-1}$. But the functor $\mathcal{H}om$ is not right exact. Let $M^\bullet \in D^b(\mathfrak{D}_X)$, then it is natural to consider $R\mathcal{H}om_{\mathfrak{D}_X}(M^\bullet, \mathfrak{D}_X) \otimes \Omega_X^{-1}$ as a candidate for the dual complex in the derived category. Note that if $M = \mathfrak{D}/\mathfrak{D}P$ for some differential operator P, then

$$R\mathcal{H}om_{\mathfrak{D}_X}(M, \mathfrak{D}_X) \simeq [0 \to \mathcal{H}om(M, \mathfrak{D}_X) \to \mathfrak{D}_X \xrightarrow{\times P} \mathfrak{D}_X \to 0].$$

Since $\times P$ is injective, $R\mathcal{H}om_{\mathfrak{D}_X}(M^\bullet, \mathfrak{D}_X) \simeq \mathrm{Ext}^1(M, \mathfrak{D}_X)$. We define:

$$\mathbb{D}M := R\mathcal{H}om_{\mathfrak{D}_X}(M^\bullet, \mathfrak{D}_X) \otimes \Omega_X^{-1}[\mathrm{dim}X]$$

We have the following lemma:

Lemma 1.1 (Lemma 2.6.8 [Hot08]). *Let M be a left $\mathfrak{D}_X$-module, if M is holonomic then $\mathbb{D}M \simeq H^0(\mathbb{D}M)$.*

We are not going to prove this lemma. However, we will see an example.

Example 1.2. Let $M \in \mathfrak{D}_X$-modules be an holomorphic vector bundle with integrable connection i.e. M is a locally free $\mathcal{O}_X$-module with an integrable $\mathbb{C}$-linear connection $\nabla : \mathfrak{D}_X \to End_{\mathbb{C}}(M)$ satisfying $\nabla_{fP}(m) = f\nabla_P(m)$, $\nabla_P(fm) = P(f)m + f\nabla_P(m)$ and $\nabla_{[P,Q]}(m) = \nabla_P\nabla_Q m - \nabla_Q\nabla_P m$ and the left D-module structure on M is given by,

$$Pm := -\nabla_P m.$$

Now we have a locally free resolution of $\mathfrak{D}_x$-module:

$$\mathcal{O}_X \overset{\text{q.i.}}{\simeq} [0 \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} \bigwedge^n T_X \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} \bigwedge^{n-1} T_X \to \ldots \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} T_X \to \mathfrak{D}_X \to 0].$$

Now since M is locally free over $\mathcal{O}_X$, $\mathfrak{D}_X \otimes_{\mathcal{O}_X} M$ is locally free over $\mathfrak{D}_X$. Therefore we get the following resolution of M:

$$M \overset{\text{q.i.}}{\simeq} [0 \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} M \bigotimes_{\mathcal{O}_X} \bigwedge^n T_X \to \ldots \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} M \bigotimes_{\mathcal{O}_X} T_X \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} M \to 0].$$

Therefore

$$R\mathcal{H}om_{\mathfrak{D}_X}(M, \mathfrak{D}_X) \overset{\text{q.i.}}{\simeq} [0 \to \mathcal{H}om_{\mathfrak{D}_X}(\mathfrak{D}_X \otimes M, \mathfrak{D}_X) \to \ldots \mathcal{H}om_{\mathfrak{D}_X}(\mathfrak{D}_X \otimes_{\mathcal{O}_X} M \bigotimes \bigwedge^n T_X, \mathfrak{D}_X) \to 0].$$

Now note that,

$$\begin{aligned}\mathcal{H}om_{\mathfrak{D}_X}(\mathfrak{D}_X \otimes M \otimes_{\mathcal{O}_X} \wedge^k T_X, \mathfrak{D}_X) \simeq & \mathcal{H}om_{\mathcal{O}_X}(\wedge^k T_X \otimes M, \mathfrak{D}_X) \\ \simeq & \mathcal{H}om_{\mathcal{O}_X}(M, \Omega^k \otimes_{\mathcal{O}_X} \mathfrak{D}_X)\end{aligned} \tag{1}$$

On the other hand we have the following locally free resolution of Ω_X:

$$\Omega_X \overset{\text{q.i.}}{\simeq} [\mathfrak{D}_X \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} \Omega_X \to \ldots \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} \Omega_X^{n-1} \to \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} \Omega_X^n]$$

Since M is locally free over $\mathcal{O}_X$, $\mathcal{H}om_{\mathcal{O}_X}(M, \Omega_X) \simeq R\mathcal{H}om(M, \Omega_X)$. Therefore

$$\text{Coker}\left(\mathcal{H}om_{\mathcal{O}_X}(M, \mathfrak{D}_X \bigotimes_{\mathcal{O}_X} \Omega_X^{n-1}) \to \mathcal{H}om_{\mathcal{O}_X}(M, \mathfrak{D}_X \otimes \omega_X)\right) \simeq \mathcal{H}om_{\mathcal{O}_X}(M, \Omega_X).$$

Therefore, $R\mathcal{H}om_{\mathfrak{D}_X}(M, \mathfrak{D}_X) \overset{\text{q.i.}}{\simeq} \mathcal{H}om_{\mathcal{O}_X}(M, \Omega_X)$. Since M and Ω_X are locally free, we get , $R\mathcal{H}om_{\mathfrak{D}_X}(M, \mathfrak{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{-1} \overset{\text{q.i.}}{\simeq} \mathcal{H}om_{\mathcal{O}_X}(M, \mathcal{O}_X)$.

Following Saito, we define the dual of a holonomic right $\mathfrak{D}_X$-module to be

$$\mathbb{D}M := R\mathcal{H}om(M, \omega[n] \otimes_{\mathcal{O}_X} \mathfrak{D}_X)$$

. We can also deduce this definition from the one for left modules, using the side changing operation. Now given a filtered right D-module $(M, F^\bullet)$, we first need to define a filtration on $\mathbb{D}M$. We follow Saito to define this filtration. Define:

$$F^p\mathcal{H}om_{\mathfrak{D}_X}((M, F), (M', F')) := \{\varphi | \varphi(F^i M) \subset F^{i+p}(M')\}. \tag{2}$$

Let $\omega_X \overset{\text{q.i.}}{\simeq} K_X^\bullet$ be a resolution by filtered $\mathfrak{D}_X$-modules such that $K_X^i = 0$ for $i < -n$. Then,

$$R\mathcal{H}om(M, \omega[n] \otimes_{\mathcal{O}_X} \mathfrak{D}_X) \overset{\text{q.i.}}{\simeq} \mathcal{H}om_{\mathfrak{D}_X}(M, K_X^\bullet \otimes_{\mathcal{O}_X} \mathfrak{D}_X).$$

In other words, $(\mathbb{D}M)^i = \mathcal{H}om_{\mathfrak{D}_X}(M, K_X^i \otimes_{\mathcal{O}_X} \mathfrak{D}_X)$ which are in turn filtered for all i as described in Equation (2).

Turns out, just like in the case of regular holonomic left D-modules, we must have that $\mathbb{D}M \simeq R^n\mathcal{H}om_{\mathfrak{D}_X}(M, \omega \otimes_{\mathcal{O}_X} \mathfrak{D}_X)$. Further, if $(M, F^\bullet, K)$ is a Hodge module with $\mathbb{Q}$-structure K, then $\mathbb{D}M$ is in fact filtered, in the sense that the $\mathbb{D}M$ is strict and concentrated in degree 0. The following theorem quoted from [Theorem 5.1.13, [Sai88]] captures this idea:

Theorem 1.3. *Let $(M, F^\bullet, K) \in HM_Z(X, w)$, then $gr^F M$ is a Cohen-Macaulay $gr^F \mathfrak{D}_X$. Therefore $\mathbb{D}M$ is strict.*

Proof. (The proof is copied (for my own future reference) verbatim from [Sai88].)

We prove this by induction on the dimension of Z. Suppose $\dim Z = 0$ then there is nothing to prove. Suppose $\dim Z > 0$. Let $f : Z \to \mathbb{C}$ be a holomorphic function such that $Z \not\subset f^{-1}(0)$. We denote by $(M, F)_x$ the module $i_{f*}(M, F)_{(x,0)}$ since it is sufficient to prove that $gr^F i_{f*}(M, F)_{(x,0)}$ is Cohen-Macaulay over $gr^F \mathfrak{D}_{X \times \mathbb{C}(x,0)}$ for all $x \in Z$.

Let $(L^\bullet, F) \overset{\text{q.i.}}{\simeq} (M, F)_x$ be a resolution by free $(\mathfrak{D}_x, F[p])$ for various $p \in \mathbb{Z}$ (i.e. L^i's are direct sum of such modules). Note that,

$$\mathbb{D}(\mathfrak{D}_x, F[p]) \simeq ((\omega \otimes \mathfrak{D})_x, F[-p]).$$

By Lemma 3.3.3, 3.3.4 and 3.3.5 in [Sai88], we have: to check $\mathbb{D}M$ is strict, it is sufficient to verify that (1) for any the V-filtration, we should have that $gr^V\mathbb{D}(L^\bullet, F)$ is strict, (2) action of $t\partial_t - \alpha$ is nilpotent on $H^i gr^V_\alpha \mathbb{D}L^\bullet$and (3) that $H^i(F_p gr^V_\alpha \mathbb{D}L^\bullet) = 0$ for all p. Because, these three conditions imply that $(L^\bullet, F)$ is strict. We see that this implication does not depend on $L^\bullet$ chosen. This is because $gr^V\mathbb{D}L^\bullet \simeq Dgr^V L^\bullet \overset{\text{q.i.}}{\simeq} \mathbb{D}gr^V M$.

Since by induction $\mathbb{D}gr^W_i gr^V \alpha M$ is strict and since the resolution of $\mathbb{D}M$ is compatible with $\otimes$ and that

$$gr^W_i gr^V \mathbb{D}L \simeq gr^W_i gr^V_{-1}\mathbb{D}L \otimes \mathbb{C}[t] \oplus gr^W_i gr^V_0 \mathbb{D}L \otimes \mathbb{C}[\partial_t] \oplus gr^W_i gr^V_\alpha \mathbb{D}L \otimes \mathbb{C}[t, \partial_t]/(t\partial_t - \alpha),$$

it is enough to check the above three conditions for $(\mathbb{C}[t], F)$. $\square$

A consequence of Theorem 1.3 is the following:

Theorem 1.4. *If $(M, F^\bullet, K) \in HM(X, w)$, then $(\mathbb{D}M, F^\bullet, \mathbb{D}K) \in HM(X, -w)$*

We employ the inductive definition of Hodge module to understand Theorem 1.4. We would like to say, by induction, that given $(M, F^\bullet, K) \in HM(X, w)$ and a holomorphic function $f : X \to \mathbb{C}$ we must have that $gr^W_i \Psi_f \mathbb{D}M, F^\bullet, gr^W_i \Psi_f \mathbb{D}K)$ is a Hodge module and that all the relevant notations make sense. We discuss the compatibilities below:

Going back to the left D-module set-up, define

$$DR_X M := [M \to \Omega^1 \otimes_{\mathcal{O}_X} M \to \ldots \to \omega_X \otimes_{\mathcal{O}_X} M].$$

It is shown in [Hot08, p. 177], that $DR_X M \in Perv(\mathbb{C}_X)$. In the setting of the above example it is clear that

$$\mathbb{D}DR_X(\mathbb{D}M) \simeq M.$$

Note moreover that, for $K \in Perv(\mathbb{C}_X)$, we have that the nearby (and vanishing) functor commutes with Verdier duality:

$$\mathbb{D}\Psi(K) = \Psi\mathbb{D}(K).$$

This follows from the definition of nearby (and vanishing) cycles, as the verdier duality functor $\mathbb{D}$ commutes with derived push-forwards and pull backs. This fact together with the comparison theorem, we conclude that for a filtered holonomic D-module M such that it admits a V-filtration along a holomorphic function $f : X \to \mathbb{C}$, we have

$$\mathbb{D}\Psi_f M \simeq \Psi_f \mathbb{D}M.$$

2 Polarisations

Let X be a complex manifold. We have seen that

$$HM_{pt}(X, k, w) = \text{ Category of Hodge structures of weight } w.$$

Therefore we will first understand what it is meant by polarisation in this base case.

A polarisation of $\mathbb{Q}$-Hodge structure V of weight w on an n-dimensional complex manifold is a bilinear form

$$Q : V \otimes V \to \mathbb{Q}$$

satisfying the following conditions:

1. Q is alternating if w is odd, symmetric otherwise.
2. The Hodge decomposition $V \otimes \mathbb{C} = \{V^{p,q}\}$ is orthogonal with respect to $H(\alpha, \beta) := i^w Q(\alpha, \overline{\beta})$.
3. $i^{p-q-k}(-1)^{\frac{k(k-1)}{2}} H$ is positive definite on $V^{p,q}$.

Example 2.1 (Geometric). The prototye of polarised variation of Hodge structure arise from primitive cohomologies of projective manifolds. We know that $H^k(X, \mathbb{Q})$ is a HS of weight k and the Kähler form $[\omega] \in H^{1,1}(X, \mathbb{Q})$ defines an intersection form on $H^k(X, \mathbb{Q})$ given by:

$$Q(\alpha, \beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

Note that, in (2) above, due to multiplication i^k, H takes values in $\mathbb{R}$ and due to conjugation in the second entry, H restricts to a form on $H^{p,q}$. This intersection form satisfies the above properties except for (3). If we restrict our attention to the sub-Hodge structure of weight $k - 2r$ on

$$L^r H_0^{k-2r}(X, \mathbb{Q}) := \mathrm{Ker}\{L^{n-k-2r+1} : H^{k-2r}(X, \mathbb{Q}) \to H^{2n-k-2r+2}(X, \mathbb{Q})\},$$

then $(-1)^{\frac{k(k-1)}{2}+q}$ defines a positive definite bilinear form on $L^r H_0^{p,q}(X, \mathbb{Q})$ where $p + q + 2r = k$.

Theorem 2.2 (Structure Theorem; [Sai90]). *Let X be a complex manifold, and $Z \subset X$ an irreducible closed analytic subvariety. Then,*

1. *Every polarisable variation of $\mathbb{Q}$-Hodge structure of weight $w - dimZ$ on a Zariski-open subset of Z extends uniquely to an object of $HM_Z^p(X, w)$.*
2. *Every object $(M, F^\bullet, P) \in HM_Z^p(X, \mathbb{Q}, w)$ is a Hodge structure of weight $w - dimZ$ on a smooth open subset $U \subset Z$ on which $P[-dimZ]$ is a local system.*

3 Polarisation of Hodge modules

Let $(M, F^\bullet, K) \in HM^p(X, w)$. By Riemann-Hilbert correspondence the perverse sheaf $\mathbb{D}(DR(M)$ corresponds to the holonomic dual $\mathbb{D}M := R^n\mathcal{H}om(M, \omega_X \otimes_{\mathcal{O}_X} \mathfrak{D}_X)$.

Definition 3.1 (Polarisable Hodge module with strict support). For a closed subvariety $i : Z \hookrightarrow X$, we say that $(M, F^\bullet, K) \in HM_Z(X, w)$ is polarisable if the following conditions are satisfied:

1. There is a pairing $Q : K \otimes K \to \mathbb{Q}(-w + n)[2n]$ compatible with the filtration $F^\bullet$, i.e. there exists a unique morphism

$$(M, F, K) \to \mathbb{D}(M, F, K)(-w),$$

such that the map $K \to \mathbb{D}K$ corresponds to the map $K \otimes K \to \mathbb{Q}(-w + n)[2n]$ via Verdier duality.

2. If $Z = \{x\}$, $K \otimes K \to \mathbb{Q}$ is induced by the polarisation of Hodge structures in the usual sense. In other words, let (V, Q) be a polarisable Hodge structure corresponding to $(M, F^\bullet, K)$, then $i_* Q_V = Q$ as a morphism of Perverse sheaves.

3. If $\dim Z > 0$ and if for all holomorphic function $f : U \to \mathbb{C}$ satisfying $Z \nsubseteq f^{-1}(0)$, on a Zariski open subset U of X, the induced pairing

$$gr^{W\,p}\Psi_f Q_Z :_0 gr^W_{w-1+i}{}^p\Psi_f K \otimes_0 gr^W_{w-1+i}{}^p\Psi_f K \to \mathbb{Q}(-w+1-i+\dim Z)[2\dim X]$$

is a polarisation of the primitive part of ${}_0gr^W_{w-1+i}\Psi_f M$, where

$${}_0gr^W_{w-1+i}\Psi_f M = \mathrm{Ker}\{N^{i+1}\} \subset gr^W_{w-1+i}\Psi_f M.$$

Here $N = \frac{1}{2\pi i}\log T_u$ and W is the monodromy weight filtration i.e. $N(W_i) \subset W_{i-2}$ and $N^j : gr^W_j \Psi_f \to gr^W_{-j}\Psi_f$ is an isomorphism for $j \geq 0$. See 3.3 below for a definition of $gr^{W\,p}\Psi_f Q_Z$.

Let $(M, F^\bullet, K) \in HM(X, w)$ and $M \simeq \oplus M_Z$ be a decomposition of Hodge modules via strict support. We say that M is polarisable if M_Z's are polarisable for all Z via some pairing $Q_Z : K_Z \otimes K_Z \to \mathbb{Q}(-w+n)[2\dim Z]$. We call $Q = \oplus Q_Z$ a polarisation of $(M, F^\bullet, K)$. Moreover, a pairing $Q : K \otimes K \to \mathbb{Q}(-w+n)[2\dim X]$ is a polarisation of M if for all components M_Z of M, $i^*Q : K_Z \otimes K_Z \to \mathbb{Q}(-w+n)[2\dim Z]$. Note that, Q can be retrieved at $Q = \oplus Q_Z$. Indeed, M_Z is a Hodge module with strict support and therefore, $\mathrm{Hom}(K_Z, K_{Z'}) = 0$ and $\mathrm{Hom}(M_Z, M_{Z'}) = 0$for $Z \neq Z'$.

3.1 Definition of the pairing map

We will now discuss how a pairing Q induces ${}^p\Psi_f Q_Z$ and $gr^{W\,p}\Psi_f Q$. For this we need to refer back to how these vanishing cycles are defined: If we restric $f : X \to D$ to a disc, then $\Psi_f K := i^*\widetilde{p}_*\widetilde{p}^*K$. Then, given a pairing $Q : K \otimes K \to \mathbb{Q}(-w+n)[2n]$, we define $\Psi_f Q$ via

$$\widetilde{p}^*(\widetilde{p}_*\widetilde{p}^*K \otimes \widetilde{p}_*\widetilde{p}^*K) \to \widetilde{p}^*K \otimes \widetilde{p}^*K$$

which corresponds to the following under adjunction:

$$i^*\widetilde{p}_*\widetilde{p}^*K \otimes i^*\widetilde{p}_*\widetilde{p}^*K \to i^*\widetilde{p}_*(\widetilde{p}^*K \otimes \widetilde{p}^*K).$$

Then the later has an induced pairing map

$$\widetilde{p}^*K \otimes \widetilde{p}^*K \to \widetilde{p}^*\mathbb{Q}_X(-w+n)[2n].$$

Note that, $i^*\widetilde{p}_*\widetilde{p}^*\mathbb{Q}_X(-w+n)[2n] \simeq \mathbb{Q}_{X_0}(-w+n)[2n-2][2]$. Since ${}^p\Psi_f := \Psi_f[-1]$, Therefore we have a pairing

$${}^p\Psi_f \otimes^p \Psi_f \to \mathbb{Q}_{X_0}(-w+n)[2n-2]$$

To define $gr^{W\,p}\Psi_f Q$, we will need the following ingredient:

Lemma 3.2. *(*[Sai88] *Lemma 5.2.5)*

$${}^p\Psi_f Q \circ (N \otimes Id) +^p \Psi_f Q \circ (Id \otimes N) = 0.$$

We know by [Gri70] p. 255-256 (also see [Sch73] Lemma 6.4), such N's are said to be *infitesimal isometry of Q on V*. Further in that case the monodromy weight filtration associated to N:

$$W_{-w} \subset W_{-w+1} \subset \cdots \subset W_0 \subset W_1 \subset \cdots \subset W_w$$

becomes self dual under the non-degenerate bilinear form S:

$$W_l = W_{-l-1}^{\perp}.$$

This can be worked out very easily for $N^2 = 0$.

We are now ready for the definition

Definition 3.3. Let $Q : V \times V \to \mathbb{Q}$ be a non-degenerate bilinear form and let $N : V \to V$ be a nilpotent operator on such that $N^{w+1} = 0$ such that N is infitesimal symmetry of Q. Then we define:

$$gr_{-i}^{W} Q : gr_{-i}^{W} V \otimes gr_{-i}^{W} V \to \mathbb{C}$$

by $gr_i^W Q(\overline{v}, \overline{w}) = Q(v, N^i w)$ for some lift $v, w \in W_{-i}$

It is well defined since, $\overline{v} = v + W_{-i-1}$ and $w = \overline{w} + W_{-i-1}$ for $v', w' \in W_{-i-1}$ then

$$gr_{-i}^{W} Q(\overline{v}, \overline{w}) = Q(v, N^i w) + Q(v, N^i W_{-i-1}) + Q(W_{-i-1}, N^i w) + Q(W_{-i-1}, N^i W_{-i-1})$$

. But since $N^i(W_{-i-1}) \subset W_{i-1} = W_{-i}^{\perp}$ and $W_{-i-1} = W_i^{\perp}$ we get that the last three term in the above sum are zero. Therefore the definition of $gr_{-i}^{W} Q$ does not depend of the lift chosen.

3.2 Definition Polarization, A postiori

After knowing the structure theorem stated above, Saito's definition becomes the following a postiori.

Definition 3.4. ([PS08], Definition 14.35) Suppose that $M = (M; F, K)$ is a Hodge module of weight w with strict support in $Z \subset X$ which is of the form $M = V_Z^{\mathrm{Hdg}}$, the Hodge module extension of a polarized variation of Hodge structures V of weight $w - \dim Z$ on a Zariski-open subset U of Z. A polarization on M is a non-singular pairing on the rational component K such that

1. the quasi-isomorphism $Q : K \to \mathbb{D}K(-w)$ extends to an isomorphism $Q' : (M; F; K) \simeq \mathbb{D}(M; F; K)(w)$ of Hodge modules of weight w with strict support in Z;
2. Q' induces a polarization of the variation V (defined on U), in the sense of polarisation of variations of Hodge Structure.

A Hodge module admitting a polarization is called polarizable.

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