GRADUATE SEMINAR S4A1: JACOBIAN OF CURVES AND ABELIAN VARIETIES

In this seminar we will apply various tools from last year's algebraic geometry sequence to gain a deeper understanding of the geometry of complex algebraic curves. One of the preferred methods of studying algebraic varieties is to embed them in to projective spaces. For curves, one could also think of this as a generalisation of our classical notion of curves in terms of an equation in two variables. The *Brill–Noether theory* helps us understand how to classify all possible ways to embed a curve in some r-dimensional projective space $\mathbb{P}^r$. Joe Harris calls this "the representation theory for curves." This will be our first objective. The second objective of this seminar would be a reconstruction problem known as the *Torelli theorem*. We will learn how to associate to a given curve an abelian variety known as the Jacobian of the curve. The Torelli theorem states that a curve can be uniquely recovered from its Jacobian. For this purpose we will spend a few meetings discussing basics of abelian varieties.

Abiding by the philosophy of our main guiding book [ACGH85] "by curve we shall mean a complete reduced algebraic curve over $\mathbb{C}$; it may be singular or reducible." In this seminar, we will assume basic knowledge of sheaves on algebraic varieties and their cohomologies.

Apart from the standard terminology and notation, following [ACGH85] we will use the following additional notations:

(1) The canonical bundle will be denoted ω_C and a divisor class of this line bundle will be denoted by K_C. We shall write K when C is clear from the context.
(2) $l(D) = h^0(C, \mathcal{O}(D))$,
(3) $i(D) = h^0(C, K(-D))$,
(4) $r(D) = l(D) - 1$.
(5) D' is said to be *residual* to D if $D' \in |K(-D)|$

1. **The Riemann–Roch Theorem.** References: [Har77, IV.1-2] [ACGH85, §1.1-§1.2].
(a) Riemann–Roch for curves: Define degree, multiplicity, space of global sections and linear equivalence for divisors on smooth algebraic curves. Compute the degree and the dimension of the global sections of ω_C on a smooth curve C. Define *special divisors* and discuss examples. State the Riemann–Roch theorem without proof.
(b) Riemann–Hurwitz formula: Define *degree* and *ramification index* of a morphism between two smooth curves. Prove the Hurwitz theorem. Lüroth's theorem for curves.
(c) Embeddings in projective space: Recall the definitions: *ample*, *very ample* and *base point free* linear systems [Har77, II.5]. State and prove the criterion for a line bundle on a curve to be base point free or very ample ([Har77, Prop. 3.1]. Define degree of an embedded curve [Har77, Example 3.3.2] and g^r_d. Using this, show that for curves of genus $g \geq 2$, the canonical linear system has no base point and is very ample if and only if C admits no g^1_2 [Har77, IV.5.1 and 5.2]. Define the *canonical embedding*.

2. **Embeddings in $\mathbb{P}^3$.** *(Speaker: Felix Jäger)* Using the criterion for base point freeness and very ampleness established in the previous talk, show that any curve can be embedded in $\mathbb{P}^3$. Define *tangent variety* and *secant variety* for a smooth curve embedded in $\mathbb{P}^n$. Show

that their codimensions are ≥ 3 and ≥ 2 respectively. State and prove [Har77, Corollary 3.11] [ACGH85, Theorem(*), Appendix A], i.e. any curve is birationally equivalent to a plane curve with at most nodal singularities.
Example: [Har77, Example 5.2.2] Show that if C is a non-hyperelliptic curve of genus 4 and degree 6 in $\mathbb{P}^3$ then C is a complete intersection.

3. **Special Divisors I.** [ACGH85, IV §1], [Har77, §IV.5] Using Riemann–Roch compute $r(D)$ when $\deg D > 2g-2$. Using the canonical embedding, show that for a divisor of degree $d \leq 2g-2$, if D is in general position, we can calculate $r(D)$ [ACGH85, p. 12]. Define *effective special divisors*. Using this, show that $d-g \leq r(D) \leq d$. State and prove Clifford's theorem [ACGH85, p. 107] or [Har77, Theorem IV.5.4] to give an upper bound on $r(D)$.

4. **Special Divisors II.** [ACGH85, IV §2] and [Har77, §IV.5] Define *hyperelliptic curves*. Show that any complete g^r_d on a hyperelliptic curve can be written as $rg^1_2 + p_1 + \cdot + p_{d-2r}$ (see [Har77, Theorem IV.5.4]) for points $p_i \in C$. Show that the only cases when equality holds in Clifford's theorem are when $D = 0, K$ or C is hyperelliptic and $D = rg^1_2$ for some $r \in \mathbb{Z}_{>0}$. Define *exceptional special divisors* [ACGH85, p. 9].

5. **Special Divisors III.** [ACGH85, IV §3-5] [Har77, IV.6] Recall the genus formula for plane curves. State Castelnuovo's bound on the genus of curves embedded in $\mathbb{P}^r$. Using this, deduce the bound for $r = 3$. Present the proof after Halphen in this case as described in [Har77, IV.6.1]. Define curves of maximum possible genus allowed by Castelnuovo's theorem to be extremal curves. Show that any extremal curve in $\mathbb{P}^3$ of even degree $2k$ is a complete intersection of a quadric and a surface of degree k. Show a similar statement for odd degree.

6. **Jacobian of a Curve.** (*Speaker: Sjoerd de Vries*) [Ara, §2-4]. Define the group scheme $\mathrm{Pic}^0(C)$. Define the *Jacobian variety* $J(C)$ associated to C and define the *Abel-Jacobi map* $u\colon C \to J(C)$ and show that it is independent of the choice of base point p [ACGH85, p. 17]. State the Riemann bilinear relations. Define the first Chern class associated to a line bundle. Define the *Abel–Jacobi map* and show that it is holomorphic. Show that for an Elliptic curve $J(C) \simeq C$. State the Torelli theorem for curves.

7. **Complex Tori.** *(Speaker: Theodosis Alexandrou)* [Mum08, Ch. I.1] [Mil86, §I]. Define *group variety*. Define group varieties and abelian varieties, show that the group law on an abelian variety is commutative using the description of the tangent space at identity. Compute for all $p, q \geq 0$, the groups $H^q(A, \Omega^p_A)$ where Ω^p_A is the p^{th} exterior power of the cotangent bundle of an abelian variety A [Mum08, p. 3 (5)]. State and explain the Hodge decomposition [Mum08, p. 9] (without proof).

8. **Line Bundles on Complex Tori.** *(Speaker: Fernando Peña)* Formulate and state the theorem of Appel-Humbert [Mum08, Ch. I.2] to identify line bundles on an abelian varieties as Hermitian forms on the universal covering space of the abelian variety. Characterize the ample line bundles using the theorem of Lefschetz [Mum08, p. 28]. Using the Riemann bilinear relations show that $J(C)$ is a projective variety. Define *polarisations* of a complex torus.

9. **Varieties Associated to Linear Series: C^r_d.** Define the subvarieties C^r_d, the variety of effective divisors of degree d on C moving in a linear series of dimension at least r, W^r_d the subvariety of $\mathrm{Pic}^d(C)$ of line bundle of degree d and $h^0(C,L) > r$. Using the Riemann–Roch theorem show that for a divisor D of degree d the $r(D) \geq r$ if and only if rank of the linear map $H^0(C,\mathcal{O}(K-D)) \hookrightarrow H^0(C,K)$ is less than or equal to $d-r$. Describe the defining equations of C^r_d as embedded in the d-th symmetric product of C. Define the *Brill–Noether number*, the expected dimension on C^r_d. Determine the Zariski tangent space at $D \in C^r_d \setminus C^{r+1}_d$ [ACGH85, p.159]. Use [ACGH85, Lemma 1.6] to show when C^r_d has expected dimension.

10. **Varieties Associated to Linear Series: W^r_d.** [ACGH85, §IV.3] Recall the definition of W^r_d. Show that it can be seen as the image of $C^r_d \to \mathrm{Pic}^d(C)$ [ACGH85, Prop. 3.4] (you may use Abel's theorem without proof). Define the *Poincaré line bundle* on $C \times \mathrm{Pic}^d$ as in [ACGH85, p. 166]. Identify the defining equations of the embedding. Show that the equations are independent of the choice of Poincaré line bundle [ACGH85, p. 179]. Compute the expected dimension of W^r_d [ACGH85, Prop. 3.3]. Recall the Zariski tangent space of a variety [Har77, Ex. II.2.8] in terms of dual numbers. Show that the tangent space of $\mathrm{Pic}^d(C)$ at a point is given by $H^1(C,\mathcal{O})$ [ACGH85, p. 186]. Show that the tangent space of W^r_d at a point in $L \in W^r_d \setminus W^{r+1}_d$ given by $\mathrm{im}(\mu)^\perp$ where $\mu\colon H^0(C,L) \otimes H^0(C,\omega_C \otimes L^{-1}) \to H^0(C,\omega_C)$ is the multiplication map. Show that this tangent space is all of $T_L(\mathrm{Pic}^d(C))$ when $L \in W^{r+1}_d$ [ACGH85, Prop. 4.2]

11. **Theta divisors and Riemann's theorem.** [ACGH85, p. 21 – 28] Define *principally polarised abelian varieties*. Define *Riemann's theta function* and the *theta divisor*. Denoting by W_d the image of $u_d\colon C_d \to J(C)$, show that it is a translate of the theta divisor of $J(C)$ by the *Riemann constant*. Using the tools we have developed so far, show that for a curve of genus g, $\dim W_{g-1} = g-1$. Show that [ACGH85, Corollary IV.1.4 and p. 229 (1.5)] at a point $p \in W_{g-1}$, the variety is smooth if and only if p corresponds to a divisor $D = x_1 + \cdots + x_{g-1}$ satisfying $h^0(C,\mathcal{O}_C(D)) = 1$ if and only if the x_i's considered as points on the canonical curve span a $g-1$-dimensional linear subspace in $\mathbb{P}(H^0(C,\omega_C)^\vee)$.

12. **The Torelli Theorem.** [Pet14, §1.3] [Lan, §3,4] [ACGH85, p. 246] Recall the definition of the *dual curve*. Given a subvariety $W \subset A$ of an abelian variety, such that $W = W_{g-1}(C)$ and $A = J(C)$ for some curve non-hyperelliptic curve C, show that one can recover the curve C by identifying the discriminant of the incidence variety

$$\Phi \coloneqq \{(p,H) \in W \times \mathbb{P}(T_pA) | T_pW = H\}$$

to be the dual curve $C^\vee \subset \mathbb{P}(T_pA) = \mathbb{P}(H^0(C,\omega_C))$. Show a similar statement in the hyperelliptic case.

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