Contents

Chapter 1

Fourier Mukai Transforms

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1.1 Grothendieck-Verdier Duality

Let $f : X \to Y$ be a morphism of smooth schemes over a field k (any char). Denote the relative canonical bundle by

$$\omega_{X/Y} = \omega_X \otimes f^*\omega_Y^{-1}$$

Then, for any $\mathcal{F}^\bullet \in D^b(X)$ and $\mathcal{E}^\bullet \in D^b(Y)$ there exists a natural isomorphism

$$Rf_*R\mathcal{H}om(\mathcal{F}^\bullet, Lf^*(\mathcal{E}^\bullet) \otimes^{\mathbb{L}} \omega_{X/Y}[\dim X - \dim Y] \simeq R\mathcal{H}om(Rf_*\mathcal{F}^\bullet.\mathcal{E}^\bullet)$$

Since for smooth maps $\omega_{X/Y}$ is locally free the tensor product is underived. Define,

$$f^! : \mathrm{D}^b(Y) \to \mathrm{D}^b(X) \qquad \mathcal{E}^\bullet \mapsto Lf^*(\mathcal{E}^\bullet) \otimes \omega_{X/Y}[\dim X - \dim Y]$$

Then $f^! \dashv Rf_*$.

1.1.1 Corollaries

1. Taking cohomologies we get,

$$R\Gamma Rf_*R\mathcal{H}om(\mathcal{F}^\bullet, Lf^*(\mathcal{E}^\bullet) \otimes \omega_{X/Y}[\dim X - \dim Y] \simeq R\Gamma R\mathcal{H}om(Rf_*\mathcal{F}^\bullet.\mathcal{E}^\bullet)$$

But, $R\Gamma \circ Rf_* = R\Gamma$ and $R\Gamma \circ R\mathcal{H}om = R\mathrm{Hom}$. Therefore in degree zero we get,

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}^\bullet, Lf^*(\mathcal{E}^\bullet) \otimes \omega_{X/Y}[\dim X - \dim Y]) \simeq \mathrm{Hom}_{\mathrm{D}^b(Y)}(Rf_*\mathcal{F}^\bullet . \mathcal{E}^\bullet)$$

2. (Serre Duality) Grothendieck Duality applied to $f : X \to k$ yields classical Serre duality:

$$\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}^\bullet[i], \omega_X[\dim X]) \simeq \mathrm{Hom}_k(Rf_*\mathcal{F}^\bullet[i], k)$$

In particular, for a sheaf we have, $\mathcal{F}^\bullet = \mathcal{F}$. This together with the facts, $R^i f_*\mathcal{F} = H^i(X, \mathcal{F})$ and $Hom_{\mathrm{D}^b(X)}(\mathcal{F}, \mathcal{G}[\dim X - i]) \simeq \mathrm{Ext}^{n-i}(\mathcal{F}, \mathcal{G})$, yield

$$\mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X) \simeq H^i(X, \mathcal{F})^*$$

3. For $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{D}^b(X)$, the derived version of Serre duality gives

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes \omega_X[\dim X]) \\
&\qquad \simeq \mathrm{Hom}_{\mathrm{D}^b(X)}(R\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{F}^\bullet), \omega_X[\dim X]) \\
&\qquad \simeq \mathrm{Hom}_k(R\Gamma R\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{F}^\bullet), k) \\
&\qquad \simeq \mathrm{Hom}_{\mathrm{D}^b(X)}(\mathcal{G}^\bullet, \mathcal{F}^\bullet)^*
\end{aligned}$$

1.2 Fourier Mukai Transforms

For this section, unless otherwise mentioned the standard notation will always mean derived. For instance, we will write $\mathcal{F}$ for $\mathcal{F}^\bullet$, $\otimes$ for $\otimes^{\mathbb{L}}$ etc.

Definition 1.2.1 (Fourier-Mukai Transforms)**.** Let $\mathcal{P} \in \mathrm{D}^b(X \times Y)$. Let $q : X \times Y \to X$ and $p : X \times Y \to Y$ be standard projections. This Fourier-Mukai transform is a functor $\Phi_{\mathcal{P}} : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ defined by

$$\mathcal{E}^\bullet \mapsto Rp_*(q^*\mathcal{E}^\bullet \otimes \mathcal{P})$$

The object $\mathcal{P}$ is called the Fourier Mukai kernel of the Fourier-Mukai transform $\Phi_{\mathcal{P}}$.

Remark. 1. Since q is smooth (hence flat) q^* is underived.

2. If $\mathcal{P}$ is a complex of locally free sheaves, $\otimes$ in the formula is also underived. This will be the case in most of the applications.

3. It is a composition of exact functors and hence exact.

Examples:

1. (The identity functor) $id : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$. Then, $id \simeq \Phi_{\mathcal{O}_\Delta}$, where $\Delta \subset X \times X$ is the diagonal.

2. We will show more generally that for a morphism $f : X \to Y$, $Rf_* \simeq \Phi_{\mathcal{O}_\Gamma}$, where $g : X \to \Gamma \subset X \times Y$ is the graph of the morphism f.
For $\mathcal{F} \in \mathrm{D}^b(X)$,

$$\begin{aligned} Rf_*(\mathcal{F}) &= R(q \circ g)_*(g^* p^* \mathcal{F}) \\ &= Rq_* \circ Rg_*(g^* p^* \mathcal{F}) \\ &= Rq_*(p^*\mathcal{F} \otimes Rg_*\mathcal{O}_X) \quad \text{(projection formula)} \\ &= Rq_*(p^*\mathcal{F} \otimes \mathcal{O}_\Gamma) \quad (g \text{ is an isomorphism.}) \end{aligned}$$

3. Let L be a line bundle on X then the functor $\mathcal{F}^\bullet \mapsto \mathcal{F}^\bullet \otimes L$ is isommorphic to Φ_{i_*L} where, $i : X \to \Delta \subset X \times X$ is the diagonal embedding.

4. The shift functor $T : \mathrm{D}^b(X) \to \mathrm{D}^b(X)$ is given by $\Phi_{\mathcal{O}_\Delta[1]}$.

5. Let $\mathcal{P}$ be a flat coherent sheaf on $X \times Y$, then for a closed point $x \in X$,

$$\Phi_{\mathcal{P}}(k(x)) = Rp_*(q^*k(x) \otimes \mathcal{P}) = \mathcal{P}_x$$

Definition 1.2.2 (Theorem)**.** For any object $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ we define the following objects in $\mathrm{D}^b(X \times Y)$

$$\mathcal{P}_L = \mathcal{P}^\vee \otimes p^*\omega_Y[\dim Y] \qquad \mathcal{P}_R = \mathcal{P}^\vee \otimes q^*\omega_X[\dim X]$$

where, $\mathcal{P}^\vee = R\mathcal{H}om(\mathcal{P}, \mathcal{O}_{X\times Y})$. Then, $G = \Phi_{\mathcal{P}_L}$ is the left adjoint and $H = \Phi_{\mathcal{P}_R}$ is the right adjoint of the the fourier mukai transform with kernel $\mathcal{P}$.

Proof. $G \dashv \Phi_{\mathcal{P}}$: For $\mathcal{F}^\bullet \in \mathrm{D}^b(Y)$ and $\mathcal{E}^\bullet \in \mathrm{D}^b(X)$,

$$\begin{aligned} \mathrm{Hom}_{D^b(X)}(G(\mathcal{F}^\bullet), \mathcal{E}^\bullet) &= \mathrm{Hom}_{D^b(X)}(Rq_*(p^*\mathcal{F}^\bullet \otimes \mathcal{P}_L), \mathcal{E}^\bullet) \\ &= \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}_L \otimes p^*\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet \otimes \omega_{X\times Y/X}[\dim Y]) \\ &\text{(GV Duality; } q^* \text{is underived since } q \text{ is flat)} \\ &= \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}_L \otimes p^*\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet \otimes p^*\omega_Y[\dim Y]) \\ &= \mathrm{Hom}_{D^b(X\times Y)}(\mathcal{P}^\vee \otimes p^*\mathcal{F}^\bullet, q^*\mathcal{E}^\bullet) \\ &= \mathrm{Hom}_{D^b(X\times Y)}(p^*\mathcal{F}^\bullet, \mathcal{P} \otimes q^*\mathcal{E}^\bullet) \\ &= \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, Rp_*(\mathcal{P} \otimes q^*\mathcal{E}^\bullet)) \quad (Lp^* \dashv Rp_*) \\ &= \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \Phi_{\mathcal{P}}) \quad (p^* \dashv p_*) \end{aligned}$$

□

1.2.1 Composition of FM transforms

Proposition 1.2.3. *The composition*

$$D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y) \xrightarrow{\Phi_{\mathcal{Q}}} D^b(Z)$$

is isomorphic to $\Phi_{\mathcal{R}} : D^b(X) \to D^b(Z)$*, where* $\mathcal{R} = R\pi^*_{XZ}(\pi^*_{XY}\mathcal{P} \otimes \pi^*_{YZ}\mathcal{Q})$.

1.3 Orlov's theorem and applications

Theorem 1.3.1 (Orlov)**.** *Let* X *and* Y *be two smooth projective varieties and let* $F : D^B(X) \to D^b(Y)$ *be a fully faithful exact functor. If* F *admits right (and hence left) adjoint functors, then there exists an object* $\mathcal{P} \in D^b(X \times Y)$ *unique up to isomorphism such that* $F \simeq \Phi_{\mathcal{P}}$.

Proof. Future lecture(maybe) □

Corollaries:

Corollary 1.3.2. *Let* $F : D^b(X) \to D^b(Y)$ *be an equivalence between the derived categories of two smooth projective varieties. Then* F *is isomorphic to a FM transform* $\Phi_{\mathcal{P}}$ *associated to a certain object* $\mathcal{P} \in D^b(X \times Y)$ *unique up to isomorphism. Moreover,* $\dim X = \dim Y$ *and* $\mathcal{P} \otimes q^*\omega_Y \simeq \mathcal{P} \otimes p^*\omega_X$.

Proof. Note that, equivalence of categories ensures existence of adjoints, namely the quasi-inverse F'. Therefore we can apply Orlov's theorem to find a $\mathcal{P} \in \mathrm{D}^b(X \times Y)$ unique up to isomorphism such that the FM transform with kernel $\mathcal{P}$ is isomorphic to F. Again by applying the uniqueness of Orlov's theorem to the quasi-inverse F' we see that, $\mathcal{P}_L \simeq \mathcal{P}_R$. In other words, $\mathcal{P}^\vee \simeq \mathcal{P}^\vee \otimes p^*\omega_X \otimes q^*\omega_Y[\dim X - \dim Y]$. Since $\mathcal{P}$ is a bounded complex that is not quasi-isomorphic to zero, $\dim X = \dim Y$. □

Corollary 1.3.3. *Suppose* $\Phi : D^b(X) \simeq D^b(Y)$ *is an equivalence such that, for any close point* $x \in X$*, there exists a closed point* $f(x) \in Y$ *such that* $\Phi(k(x)) \simeq k(f(x))$*. Then,* $f : X \to Y$ *defines an isomorphism and* Φ *simeq*$(M \otimes \_) \circ f_*$ *for some line bundle* $M \in Pic(Y)$.

Definition 1.3.4 (Spanning Class)**.** A collection Ω os objects in a triangulated category $\mathcal{D}$ is a spanning class of $\mathcal{D}$ f for all $B \in \mathcal{D}$ the following condition hold:
If $\mathrm{Hom}(B, A[i]) = 0$ for all $A \in \Omega$ and $\forall i \in \mathbb{Z}$ then, $B \simeq 0$

Proof. By Orlov's theorem, $\Phi \simeq \Phi_{\mathcal{P}}$ for some object $\mathcal{P} \in \mathrm{D}^b(X \times Y)$. Note that, $\Phi(k(x)) = q^*(k(x)) \otimes^{\mathbb{L}} \mathcal{P} \simeq k(f(x))$. Therefore, for any closed point $x \in X$ the embedding $i : x \times Y \hookrightarrow X \times Y$, $Li^*\mathcal{P}$ is also a sheaf. Then the lemma below implies that, $\mathcal{P}$ is a coherent sheaf flat over X. Therefore, $R\Phi(k(x)) = \mathcal{P}|_{\{x\}\times Y} \simeq k(f(x))$. Hence, $\mathrm{Supp}\mathcal{P}$ is precisely the graph of f and thus Γ_f has a reduced induced scheme structure. Γ_f is then a variety isomorphic to X by first projection and f is a composition of this isomorphism with projection to Y. Therefore, $f : X \to Y$ defines a morphism.

Now, we want to show that $k(x)$ spans the category $\mathrm{D}^b(X)$. To do so, we need to show for $\mathcal{F}^\bullet \in \mathrm{D}^b(X)$, $\mathrm{Hom}(\mathcal{F}^\bullet, k(x)[-i]) \neq 0$. We use the spectral sequence, $E_2^{p,q} = \mathrm{Hom}(\mathcal{H}^q\mathcal{F}^\bullet, k(x)[p]) \Longrightarrow \mathrm{Hom}(\mathcal{F}^\bullet, k(x)[p+q])$. Now, let m be the maximal integer such that $\mathcal{H}^m \neq 0$. Then all diffrentials from $E_r^{0,-m}$ are zero. Moreover, $\mathrm{Hom}(\mathcal{H}^{-q}, k(x)[p]) = \mathrm{Ext}^p(\mathcal{H}^{-q}, k(x)) = 0$ if $p < 0$. Therefore, all differentials in page r mapping to $E_r^{0,-m}$ are zero. Hence, $E_\infty^{0,-m} = E_2^{0,-m} = \mathrm{Hom}(\mathcal{H}^m, k(x) \neq 0$ for $x \in Supp(\mathcal{H}^m)$.

Moreover, since Φ is a equivalence, it is easy to check that, $k(f(x))$ spans $\mathrm{D}^b(Y)$. Therefore, for $y \in Y$ there is an integer m and $x \in X$ such that, $\mathrm{Hom}(k(f(x)), k(y)[m]) \neq 0$. This implies that, $y = f(x)$.
For injectivity, pick $x_1 \neq x_2$. then $\Phi(k(x_1)) \neq \Phi(k(x_2))$. Therefore, $f(x_1) \neq f(x_2)$.

We can use similar argument on the quasi inverse to Φ to show that f has an honest inverse.
Now, $\mathcal{P}|_{\mathrm{Supp}\mathcal{P}}$ has fibre of dimension 1, therefore, $\mathcal{P}|_{\mathrm{Supp}\mathcal{P}}$ is line bundle. Since, $Supp(\mathcal{P}) \simeq Y$ via the projection p, it gives rise to a line bundle $M = p_*\mathcal{P}$ on Y. From the formula for composition, it is possible to calculate and see that $\Phi_{\mathcal{P}} \simeq (\_ \otimes M) \circ f_*$. □

Lemma 1.3.5. *Consider a morphism $S \to X$. Suppose $\mathcal{P} \in D^b(S)$ and assume that for all closed points $x \in X$ the derived pull back $Li_x^*\mathcal{P} \in D^b(S_x)$, for $i_x : S_x \hookrightarrow S$, is a complex concentrated in degree 0, i.e. a sheaf. Then $\mathcal{P}$ is isomorphic to a coherent sheaf which is flat over X.*

Proof. For a proof, see [1, pg.82 Lemma 3.31] □

Corollary 1.3.6 (**Gabriel**). *If* $\Phi : \boldsymbol{Coh}(X) \to \boldsymbol{Coh}(Y)$ *is an equivalence of categories then* $\exists$ *a morphism* f *such that* $f : X \simeq Y$ *and* $\Phi \simeq (M \otimes \_) \circ f_*$, *for a line bundle* M *on* Y.

Proof. In order to apply the previous corollary, we need to check that, $\Phi(k(x)) \simeq k(f(x))$. □

Chapter 2

Passing to cohomology

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2.1 Grothendieck ring

Let $K(X)$ be the Grothendieck group of X. Recall that the elements are coherent sheaves on X with the following equivalence relation: if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence, then $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$. On a smooth projective variety any coherent sheaf admits a locally free resolution, hence any element in the Grothendieck group may be written asa linear combination of locally free sheaves.
We define a map $[\ ] : D^b(X) \to K(X)$ by $[\mathcal{F}^\bullet] = \sum (-1)^i [F^i]$. We also define a ring structure on $K(X)$ by $[\mathcal{E}_1] \cdot [\mathcal{E}_2] = [\mathcal{E}_1 \otimes \mathcal{E}_2]$.

Remark.

- $[\mathcal{F}^\bullet[k]] = (-1)^k [\mathcal{F}^\bullet]$.
- $[\mathcal{F}^\bullet{}_1 \oplus \mathcal{F}^\bullet{}_2] = [\mathcal{F}^\bullet{}_1] + [\mathcal{F}^\bullet{}_2]$.
- $[\mathcal{F}^\bullet] = \sum (-1)^i [\mathcal{H}^i(\mathcal{F}^\bullet)] \in K(X)$.
- $[\mathcal{F}^\bullet{}_1 \otimes \mathcal{F}^\bullet{}_2] = [\mathcal{F}^\bullet{}_1] \cdot [\mathcal{F}^\bullet{}_2]$.

So $[\ ]$ is a ring map.

We can define the pullback of a morphism $f : X \to Y$ at the Grothendieck ring in the usual way and it will be a ring homomorphism. Moreover, we will have:

$$\begin{tikzcd} D^b(Y) \arrow[r, "f^*"] \arrow[d, "[\ ]"'] & D^b(X) \arrow[d, "[\ ]"] \\ K(Y) \arrow[r, "f^*"'] & K(X) \end{tikzcd}$$

We also want to define a map of Grothendieck rings that commute with the pushforward. Let $f_! : K(X) \to K(Y)$ be defined by $f_![\mathcal{F}] = \sum(-1)^i[R^i f_*\mathcal{F}]$. Using the fact that for a spectral sequence $\sum(-1)^{p+q}[E_r^{p,q}] = \sum(-1)^{p+q}[E_{r+1}^{p,q}]$ is satisfied and using it for the spectral sequence $E_2^{p,q} = R^p f_*\mathcal{H}^q\mathcal{E}^\bullet \Rightarrow R^{p+q}f_*\mathcal{E}^\bullet$ we have that:

$$\begin{tikzcd} D^b(X) \arrow[r, "f_*"] \arrow[d, "[\ ]"'] & D^b(Y) \arrow[d, "[\ ]"] \\ K(X) \arrow[r, "f_!"'] & K(Y) \end{tikzcd}$$

Definition 2.1.1 (K-theoretic Fourier-Mukai transform)**.** Let $e \in K(X \times Y)$. The K-theoretic Fourier-Mukai transform is the map $\Phi_e^K : K(X) \to K(Y)$ defined by $\Phi_e^K(g) = p_!(e \cdot q^*(g))$.

Due to the previous remarks we have that:

$$\begin{tikzcd} D^b(X) \arrow[r, "\Phi_{\mathcal{P}}"] \arrow[d, "[\ ]"'] & D^b(Y) \arrow[d, "[\ ]"] \\ K(X) \arrow[r, "\Phi^K_{[\mathcal{P}]}"'] & K(Y) \end{tikzcd}$$

2.2 Cohomological Fourier-Mukai transform

We will assume from now that the ground field is $\mathbb{C}$.
Consider the ring $H^*(X, \mathbb{Q})$. The product of two classes $\alpha, \beta \in H^*(X, \mathbb{Q})$ will be denoted by $\alpha.\beta$ or simply $\alpha\beta$. Let $f : X \to Y$ be a morphism. The pullback is defined for cohomology rings in the usual way. We assume X and Y are projective varieties and hence Poincaré duality holds for them. The composition of the dual map of the pullback with the isomorphisms

of Poincaré duality give us a map $f_* : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$ and moreover we know that the map for degree k elements satisfy $f_* : H^k(X,\mathbb{Q}) \to H^{k+2\dim(Y)-2\dim(X)}(Y,\mathbb{Q})$. By definition, this map satisfies the Projection Formula, that is, $f_*(f^*\alpha.\beta) = \alpha.f_*\beta$.

Definition 2.2.1 (Cohomological Fourier-Mukai)**.** Let $\alpha \in H^*(X \times Y,\mathbb{Q})$. The cohomological Fourier-Mukai transform is the map $\Phi^H_\alpha : H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$ defined by $\Phi^H_\alpha(\beta) = p_*(\beta.q^*(\alpha))$.

We will now define a map $ch : K(X) \to H^*(X,\mathbb{Q})$ called the Chern character. Let $A^i(X)$ be the cycles of codimension i. First we define an element $c_i(\mathcal{E}) \in A^i(X)$ for a locally free sheaf $\mathcal{E}$. Recall that we also have a map $A^i(X) \to H^{2i}(X,\mathbb{Q})$. Notice that taking coefficients in $\mathbb{C}$ and using the Hodge decomposition, this map will land in $H^{i,i}(X)$. For the current purposes it is enough to give some conditions for $c_t(\mathcal{E}) = c_0(\mathcal{E}) + c_1(\mathcal{E})t + c_2(\mathcal{E})t^2 + \ldots$ to satisfy, as it will define the elements in a unique way. The conditions are:

1. If $\mathcal{E} \cong O_X(D)$ for a divisor D, then $c_t(\mathcal{E}) = 1 + Dt$.
2. For $f : Y \to X$, $c_i(f^*\mathcal{E}) = f^*c_i(\mathcal{E})$ for all i.
3. For an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, $c_t(\mathcal{E}) = c_t(\mathcal{E}').c_t(\mathcal{E}'')$.

It satisfies the following condition: suppose we have a filtration $\mathcal{E} = \mathcal{E}_0 \supseteq \mathcal{E}_1 \supseteq \ldots \supseteq \mathcal{E}_r = 0$ such that $\mathcal{E}_i/\mathcal{E}_{i+1} \cong \mathcal{L}_i$ an invertible sheaf. Then we have that $c_t(\mathcal{E}) = \prod c_t(\mathcal{L}_i)$ and for this we can use condition 1.

Suppose now that $c_t(\mathcal{E}) = \prod(1 + a_i t)$. We define $ch(\mathcal{E}) = \sum \exp(a_i)$. Notice that for $\mathcal{L} \in \mathrm{Pic}(X)$, we have $ch(\mathcal{L}) = \sum \frac{c_1(\mathcal{L})^i}{i!}$. We also define the Todd class as $\mathrm{td}(\mathcal{E}) = \prod \frac{a_i}{1-e^{-a_i}}$. We have that for an exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, $\mathrm{td}(\mathcal{E}) = \mathrm{td}(\mathcal{E}').\mathrm{td}(\mathcal{E}'')$.

Definition 2.2.2. Let X be a smooth variety. The Todd class of X is defined as $\mathrm{td}(X) = \mathrm{td}(\mathcal{T}_X)$.

Theorem 2.2.3 (Grothendieck-Riemann-Roch formula)**.** *Let $f : X \to Y$ be a projective morphism of smooth projective varieties. Then for any $e \in K(X)$*

$$ch(f_!(e)).td(Y) = f_*(ch(e).td(X))$$

For the case Y =Spec(k) we have that the pushforward may only be the nonzero map for $H^{2n}(X,\mathbb{Q})$ where $n=\dim(X)$. We denote by $\int_X$ the push-forward in this case. Notice that the map tells us the element in that degree. Also, notice that $f_![\mathcal{F}]=\sum(-1)^i[R^if_*\mathcal{F}]=\sum(-1)^i[H^i(X,\mathcal{F})]=\chi(\mathcal{F})$. Taking this case we recover the folowing theorem:

Theorem 2.2.4 (Hirzebruch-Riemann-Roch)**.** *For any $e\in K(X)$ we have $\chi(e)=\int_X(ch(e).td(X))$.*

Definition 2.2.5. The Mukai vector of $e\in K(X)$ is $v(e)=ch(e).\sqrt{\mathrm{td}(X)}$. For $\mathcal{E}^\bullet\in D^b(X)$ we define $v(\mathcal{E}^\bullet)=v([\mathcal{E}^\bullet])=ch(\mathcal{E}^\bullet).\sqrt{\mathrm{td}(X)}$.

Notice that it makes sense to write $\sqrt{\mathrm{td}(X)}$ as $\mathrm{td}(X)=1+\dots$ then we can construct an element such that its square is $\mathrm{td}(X)$.

The following is a Corollary of 2.2.3.

Corollary 2.2.6. *Let $e\in K(X\times Y)$. Then the following diagram commute:*

$$\begin{tikzcd}
K(X) \arrow[r, "\Phi_e^K"] \arrow[d, "v"'] & K(Y) \arrow[d, "v"] \\
H^*(X,\mathbb{Q}) \arrow[r, "\Phi_{v(e)}^H"'] & H^*(Y,\mathbb{Q})
\end{tikzcd}$$

Proof. We check that the following commute:

$$\begin{tikzcd}
K(X) \arrow[r, "q^*"] \arrow[d, "v"'] & K(X\times Y) \arrow[r, "\cdot e"] \arrow[d, "v\sqrt{\mathrm{td}(Y)}^{-1}"] & K(X\times Y) \arrow[r, "p_!"] \arrow[d, "v\sqrt{\mathrm{td}(X)}"] & K(Y) \arrow[d, "v"] \\
H^*(X,\mathbb{Q}) \arrow[r, "q^*"'] & H^*(X\times Y,\mathbb{Q}) \arrow[r, ".v(e)"'] & H^*(X\times Y,\mathbb{Q}) \arrow[r, "p_*"'] & H^*(Y,\mathbb{Q})
\end{tikzcd}$$

□

Given $\mathcal{P}\in D^b(X\times Y)$ we will denote by $\Phi_{\mathcal{P}}^H$ the induced cohomological Fourier-Mukai transform $\Phi_{v(\mathcal{P})}^H$. As characteristic classes ara in even degree as they come from cycles, we have that $\Phi_{\mathcal{P}}^H$ respects the parity. Indeed, $\Phi_{\mathcal{P}}^H$ is an intersection with an even element followed by a pushforward which respects parity, so we have: $\Phi_{\mathcal{P}}^H(H^{\mathrm{even}}(X))\subseteq H^{\mathrm{even}}(Y)$ and $\Phi_{\mathcal{P}}^H(H^{\mathrm{odd}}(X))\subseteq H^{\mathrm{odd}}(Y)$.//
Using the same notation as for Fourier-Mukai we have the following lemma.

Lemma 2.2.7. *Let $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ and $\Phi_{\mathcal{Q}} : D^b(Y) \to D^b(Z)$ be two Fourier-Mukai transforms and let $\Phi_{\mathcal{R}} : D^b(X) \to D^b(Z)$ be the composition. Then $\Phi^H_{\mathcal{R}} = \Phi^H_{\mathcal{Q}} \circ \Phi^H_{\mathcal{P}}$*

Proposition 2.2.8. *Suppose that $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is an equivalence for some $\mathcal{P} \in D^b(X \times Y)$. Then the induced cohomological Fourier-Mukai transform $\Phi^H_{\mathcal{P}} : H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ is an isomorphism of rational vector spaces.*

Proof. As $\Phi_{\mathcal{P}}$ is an equivalence then $\Phi_{\mathcal{P}_R} \circ \Phi_{\mathcal{P}} \cong \text{ id } \cong \Phi_{O_\Delta}$ and $\Phi_{\mathcal{P}} \circ \Phi_{\mathcal{P}_R} \cong \text{ id } \cong \Phi_{O_\Delta}$. Due to the previous lemma, we can conclude then that $\Phi^H_{\mathcal{P}_R} \circ \Phi^H_{\mathcal{P}} \cong \Phi^H_{O_\Delta}$ and $\Phi^H_{\mathcal{P}} \circ \Phi^H_{\mathcal{P}_R} \cong \Phi^H_{O_\Delta}$. So it is enough to show that $\Phi^H_{O_\Delta} \cong \text{ id }$.
Let $i : X \tilde{\to} \Delta \hookrightarrow X \times X$. Using Grothendieck-Riemann-Roch we have that $ch(O_\Delta).\text{td}(X \times X) = ch(i_! O_X).\text{td}(X \times X) = i_*(ch(O_X).\text{td}(X)) = i_*\text{td}(X)$. The last equality is because $ch(O_X) = 1$. So we get that $v(O_\Delta) = ch(O_\Delta).\sqrt{\text{td}(X \times X)} = i_*(\text{td}(X)).\sqrt{\text{td}(X \times X)}^{-1} =$
$= i_*(\text{td}(X).i^*\sqrt{\text{td}(X \times X)}^{-1}) = i_*(\text{td}(X).\text{td}(X)^{-1}) = i_*(1)$. We used that $i^*(\sqrt{\text{td}(X \times X)}) = \text{td}(X)$.
Finally, we get that $\Phi^H_{O_\Delta}(\beta) = p_*(q^*(\beta).v(O_\Delta)) = p_*(q^*(\beta).i_*(1)) =$
$= p_* i_*(i^* q^*(\beta)) = \beta$. □

As we are working with X smooth projective variety over $\mathbb{C}$, we have a Hodge structure $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ such that $\bar{H}^{p,q}(X) = H^{q,p}(X)$ and $H^{p,q}(X) = H^q(X, \Omega^p_X)$. As explained before all characteristic classes are of type (p,p), so $v() : K(X) \to \bigoplus H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$.

We have seen that the cohomological Fourier-Mukai does not respect the grading, only the parity. But we can improve that using the Hodge decomposition.

Proposition 2.2.9. *Suppose $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is an equivalence. Then the induced $\Phi^H_{\mathcal{P}}$ yields isomorphisms $\bigoplus_{p-q=i} H^{p,q}(X) \cong \bigoplus_{p-q=i} H^{p,q}(Y)$ $\forall i = -\dim(X), \ldots, \dim(X)$.*

Proof. $\Phi^H_{\mathcal{P}}$ defines an isomorphism between $H^*(x, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$ as we saw before. So we need to see that its $\mathbb{C}$-linear extension satisfies $\Phi^H_{\mathcal{P}}(H^{p,q}(X)) \subseteq$

$\bigoplus_{p-q=r-s} H^{r,s}(Y)$. Consider the Künneth decomposition of $v(\mathcal{P}) = \sum \alpha^{p',q'} \times \beta^{r,s}$ with $\alpha^{p',q'} \in H^{p',q'}(X)$ and $\beta^{r,s} \in H^{r,s}(Y)$. We know the element is a sum of element of type (t,t) we only consider the part of the sum such that $p' + r = q' + s$.
Let $\alpha \in H^{p,q}(X)$. Then $\Phi^H_{\mathcal{P}}(\alpha) = p_*(q^*(\alpha).\sum \alpha^{p',q'} \times \beta^{r,s}) =$
$= \sum (\int_X (\alpha.\alpha^{p',q'}))\beta^{r,s} \in \bigoplus H^{r,s}$. The last equality is by definition of pushforward and how the pushforward behaves in the Künneth decomposition. Then, it is needed that $(p+p', q+q') = (\dim(X), \dim(X))$. The result follows. □

Corollary 2.2.10. *Let E, E' be two elliptic curves. Then $D^b(E) \cong D^b(E')$ if and only if $E \cong E'$.*

Proof. Let $\Phi_{\mathcal{P}}$ be the equivalence. As $\Phi^H_{\mathcal{P}}$ respects parity, it induces and isomorphism $H^1(E) \cong H^1(E')$ and $H^0(E) \bigoplus H^2(E) \cong H^0(E') \bigoplus H^2(E')$. By the previous proposition we know it also induces an isomorphism on $H^{0,1}$ and $H^{1,0}$. We also know that $E \cong H^{1,0}(E)^*/H_1(E,\mathbb{Z}) \cong H^{0,1}(E)/H^1(E,\mathbb{Z})$. So we need to show that $\Phi^H_{\mathcal{P}}(H^1(E,\mathbb{Z})) \subseteq H^1(E',\mathbb{Z})$. As they are elliptic curves we have that $\mathrm{td}(E \times E') = 1$ and $ch(\mathcal{P}) = r + c_1(\mathcal{P}) + \frac{1}{2}(c_1^2 - 2c_2)(\mathcal{P})$. But the last term does not contribute to H^1. The result follows. □

Chapter 3

Kodaira dimension under derived equivalence

Lei Wu

Let X be a smooth projective variety over $\mathbb{C}$ and let Ω_X be its canonical bundle. The canonical ring of X is defined to be:

$$R(X) = \bigoplus_{i \geq 0} H^0(X, \omega_X^i),$$

and its kodaira dimension $\kappa(X)$ is defined to be the tanscendental degree of $R(X)$. The multiplicative structure of $R(X)$ is induced from the tensor product.

Proposition 3.0.11 (Orlov)**.** *Suppose X, Y are smooth projective varieties with $D^b(X) \simeq D^(Y)$. Then*

$$R(X) \simeq R(Y).$$

In particular,$\kappa(X) = \kappa(Y)$.

Proof. Assume the equivalence is giving by FM transformation,

$$\Phi_{\mathcal{P}} : D^b(X) \xrightarrow{\sim} D^b(Y),$$

with quasi-inverse

$$\Phi_{\mathcal{Q}} : D^b(Y) \xrightarrow{\sim} D^b(X),$$

for some $\mathcal{P}, \mathcal{Q} \in D^b(X \times Y)$. By uniqueness of right and left adjoint functor, we get

$$\Phi_{\mathcal{P}_L} \simeq \Phi_{\mathcal{Q}} \simeq \Phi_{\mathcal{P}_R}.$$

Hence $\mathcal{P}^V \otimes q * \omega_X[n] \simeq \mathcal{P}^V \otimes p * \omega_Y[n]$. Here $n = dimX = dimY$, because of the equivalence.
First, we want to prove

$$\Phi_{\mathcal{Q}} : D^b(X) \to D^b(Y)$$

is also an equivalence. Consider the composition

$$\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} = \Phi_{\mathcal{R}} : D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y) \xrightarrow{\Phi_{\mathcal{P}}} D^b(X).$$

As a composition, we know

$$\mathcal{R} = \pi_{13*}(\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{Q}) \simeq \mathcal{O}_\Delta,$$

because $\Phi_{\mathcal{R}}$ is the indentity. Denote τ_{12} to be the automorphism of $X \times X$ which interchanging the two factors, i.e. $\tau_{12}(x_1, x_2) = (x_2, x_1)$, and τ_{13} the automorphism of $X \times Y \times X$ interchanging the two X's. Then

$$\emptyset_\Delta \simeq = \tau_{12}^* \emptyset_\Delta) \simeq \tau_{12}^* \pi_{13*}(\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{Q}) \simeq \pi_{13*} \tau_{13}^* (\pi_{12}^* \mathcal{P} \otimes \pi_{23}^* \mathcal{Q}) \simeq \pi_{13*}(\pi_{23}^* \mathcal{P} \otimes \pi_{12}^* \mathcal{Q}).$$

But the last one is the kernal of $\Phi_{\mathcal{P}} \circ \Phi_{\mathcal{Q}}$. Hence

$$\Phi_{\mathcal{P}} \circ \Phi_{\mathcal{Q}} \simeq id_X.$$

Swiping $\mathcal{P}$ and $\mathcal{Q}$, we get

$$\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} \simeq id_Y.$$

Hence $\Phi_{\mathcal{Q}} : D^b(X) \to D^b(Y)$ is an equivalence with quasi-inverse $\Phi_{\mathcal{P}}$. Therefore,

$$\Phi_{\mathcal{Q} \boxtimes \mathcal{P}} : D^b(X \times X) \xrightarrow{\sim} D^b(Y \times)$$

is an equivalence.
Denote $\mathcal{S} := \Phi_{\mathcal{Q} \boxtimes \mathcal{P}}(\iota_* \omega_X^k)$, where ι is the diagonal embedding. Then the equivalence $\Phi_{\mathcal{S}}$ can be computed as the composition

$$D^b(Y) \xrightarrow{\Phi_{\mathcal{Q}}} D^b X \xrightarrow{\Phi_{\iota_* \omega_X^k}} D^b(X) \xrightarrow{\Phi_{\mathcal{P}}} D^b(Y)$$

Since $\Phi_{\iota_*\omega_X^k} \simeq S_X^k[-kn]$ (S_X is the Serre functor), and since the Serre functor commutes with any equivalence, we get

$$\Phi_{\mathcal{S}} \simeq S_Y^k[-kn] \simeq \Phi_{\iota_*\omega_Y^k}.$$

Hence, thanks to the uniqueness of the FM transformation

$$\mathcal{S} = \Phi_{\mathcal{Q}\boxtimes\mathcal{P}}(\iota_*\omega_X^k) \simeq \iota_*\omega_X^k.$$

Since $\Phi_{\mathcal{Q}\boxtimes\mathcal{P}}$ is an equivalence, we obtain

$$\begin{tikzcd}
\mathrm{Hom}_X(\omega_X^k, \omega_X^l) \arrow[r, "\Phi_{\mathcal{Q}\boxtimes\mathcal{P}}", "="'] \arrow[d, "\iota_*"', "\|"] & \mathrm{Hom}_Y(\omega_Y^k, \omega_Y^l) \arrow[d, "\iota_*", "\|"'] \\
\mathrm{Hom}_{X\times X}(\iota_*\omega_X^k, \iota_*\omega_X^l) \arrow[r, "="'] & \mathrm{Hom}_{Y\times Y}(\iota_*\omega_Y^k, \iota_*\omega_Y^l)
\end{tikzcd} .$$

Two vertical arrows are isomorphism because ι_* is fully faithful. Picking $k = 0, l \geq 0$, we have

$$H^0(X, \omega_X^l) \simeq H^0(Y, \omega_Y^l).$$

Since $H^0(X, \omega_X^l) \simeq \mathrm{Hom}_X(\mathcal{O}_X, \omega_X^l)$, multiplication of $R(X)$ is just composition of morphisms, which definitely compatible with any functors. So $R(X) \simeq R(Y)$. $\square$

Remark. Construct the bigraded ring containing $R(X)$,

$$HH(X) := \bigoplus_{i,l} HA_{i,l}(X),$$

where $HA_{i,l}(X) := Ext^i_{X\times X}(\iota_*\mathcal{O}_X, \iota_*(\omega_X^l))$. Define Hochschild cohomology as

$$HH^*(X) := \bigoplus_{i} HA_{i,0}(X),$$

and Hochschild homology as

$$HH_*(X) := \bigoplus_{i} HA_{i,1}(X).$$

Then exactly the same as the above proof, we get

$$HH(X) \simeq HH(Y)$$

Corollary 3.0.12. *If furthermore ω_X and ω_Y are ample (or anti-ample), then*

$$X \simeq Y.$$

In fact, only ampleness (or anti-ampleness) of ω_X is needed, because ampleness (or anti-ampleness) of ω_X will imply ampleness (or anti-ampleness) of ω_Y (See Proposition 4.11 in [1]).

Chapter 4

Nefness Under derived equivalence

Lei Wu

Let $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ be a F-M transformation with kernal $\mathcal{P} \in D^b(X \times Y)$. Then,

$$supp\mathcal{P} = supp\mathcal{P}^V = supp\mathcal{P}_R = supp\mathcal{P}_L.$$

From now on, we always assume $\Phi_{\mathcal{P}}$ is also an equivalence. Therefore,

$$\mathcal{P} \otimes q^*(\omega_X) \simeq \mathcal{P} \otimes p^*(\omega_Y),$$

and in particular,

$$\mathcal{H}^i(\mathcal{P}) \otimes q^*(\omega_X) \simeq \mathcal{H}^i(\mathcal{P}) \otimes p^*(\omega_Y).$$

Lemma 4.0.13. *The morphism $supp(\mathcal{P}) \to X$ is surjective.*

Proof. Assume $x \notin q(supp(\mathcal{P}))$. Hence $supp(q^*k(x))$ and $supp(\mathcal{P})$ are disjoint. The spectral sequence

$$E_2^{r,s} = \mathcal{T}or_{-r}(\mathcal{H}^s(\mathcal{P}), q^*k(x)) \Longrightarrow \mathcal{T}or_{-r-s}(\mathcal{P}, q^*k(x)).$$

Since $\mathcal{T}or$ is local, $\mathcal{T}or_i(\mathcal{H}^s(\mathcal{P}), q^*k(x))$ is 0 for all i. Hence

$$P \otimes^L q^*k(x) = 0.$$

So $\Phi_{\mathcal{P}}(k(x)) = 0$. But this can not happen because $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ is an equivalence. $\square$

Corollary 4.0.14. *There exists an integer $i \in \mathbb{Z}$ and an irreducible component Z of $supp(\mathcal{H}^i(\mathcal{P}))$ that projects onto X.*

Before we prove our main theorem in this section, let's look at some technical lemmas.

Lemma 4.0.15. *Let Z be a normal variety over k and let $\mathcal{F}$ be a coherent sheaf on Z generically of rank r. If $\mathcal{L}_1, \mathcal{L}_2 \in Pic(Z)$, such that*

$$\mathcal{F} \otimes \mathcal{L}_1 \simeq \mathcal{F} \otimes \mathcal{L}_2,$$

then $\mathcal{L}_1^r \simeq \mathcal{L}_2^r$.

Proof. By dividing out its torsion part we can assume $\mathcal{F}$ is torsion-free. Since Z is normal, $\mathcal{F}$ is locally free of rank r on some open U whose complement is of codimension at least 2. Hence

$$F|_U \otimes \mathcal{L}_1|_U = F|_U \otimes \mathcal{L}_2|_U.$$

By taking determinant,

$$(\mathcal{L}_1|_U)_r \simeq (\mathcal{L}_2|_U)^r.$$

Since Z is normal again, this isomorphism will extends to an isomorphism globally, i.e.

$$(\mathcal{L}_1)^r \simeq (\mathcal{L}_2)^r.$$

□

Corollary 4.0.16. *Let $Z \subset supp(\mathcal{P})$ be a closed irreducible subvariety with normalization $\mu : \tilde{Z} \to Z$. Then there exists an integer $r\,0$, such that*

$$\pi_X^* \omega_X^r \simeq \pi_Y^* \omega_Y^r,$$

where $\pi_X = q \circ \mu$, and $\pi_Y = p \circ \mu$.

Proof. Assume $Z \subset supp(\mathcal{H}^i(\mathcal{P}))$. Then $\mu^*(\mathcal{H}^i(\mathcal{P})|_Z)$ is coherent of generical rank > 0. By the above lemma, we are done. □

Immediately we get,

Corollary 4.0.17. $q^*\omega_X|_{supp(\mathcal{P})}$ *is numerically equivalent to* $p^*\omega_Y|_{supp(\mathcal{P})}$

By lemma 4.3, we can also get

Proposition 4.0.18 (Kawamata)**.** *Let X and Y be smooth complex projective varieties with equivalent derived categories $D^b(X)$ and $D^b(Y)$. Then the (anti)-canonical bundle of X is nef iff the (anti)-canonical bundle of Y is nef.*

Proof. We only need the following fact:
If $f : X \to Y$ is surjective morphism of projective scheme, then for a line bundle $\mathcal{L} \in Pic(Y)$ $\mathcal{L}$ is nef iff $f^*\mathcal{L}$ is nef. $\square$

Definition 4.0.19. Let $\mathcal{L}$ be a line bundle on X.

$$\nu(X, \mathcal{L}) := max\{m|[\psi^*\mathcal{L}]^m.[W] \neq 0, \text{ for some } \psi : W \to X \text{with } dimW = m\}$$

The following lemma is very basic in algebraic geometry.

Lemma 4.0.20. *Let $\pi : Z \to X$ be a morphism of projective schemes, and let $\mathcal{L} \in Pic(X)$.*

(i) Then $\nu(X, \mathcal{L}) \geq \nu(Z, \pi^*\mathcal{L})$

(ii) If π is surjective, then $\nu(X, \mathcal{L}) = \nu(Z, \pi^*\mathcal{L})$

With the help of the above lemma, similarly to Proposition 4.6 we get

Proposition 4.0.21 (kawamata)**.** *Let X and Y be smooth projective varieties with $D^b(X) \simeq D^b(Y)$. Then*

$$\nu(X, \omega_X) = \nu(Y, \omega_Y).$$

Chapter 5

More About Derived Equivalence

Yajnaseni Dutta

Throughout this chapter we will assume that k is algebraically closed and $\kappa(X)$ will denote the Kodaira dimension or Itaka dimension of ω_X.

5.1 Derived Equivalence and Birationality

Theorem 5.1.1 (Kawamata)**.** *Let X and Y be two smooth projective varieties over k with equivalent derived categories. If in addition, $\kappa(X) = \dim(X) = n$ or $\kappa(X, \omega_X^*) = n$, we get that X and Y are birational and there exists a normal variety Z such that there is a birational correspondence. Moreover, $p_X * \omega_X = p_Y * \omega_Y$.*

$$\begin{tikzcd} & Z \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

Proof. We will only treat the case when $\kappa(X) = \dim(X)$.
By Orlov's theorem in Chapter 1, we know, the derived equivalence is given by $\Phi_{\mathcal{P}} : D^b(X) \to D^b(Y)$ for some $\mathcal{P} \in D^b(X \times Y)$.

Let $H \subset X$ be a smooth hypersurface section in X. Then we have the following exact sequence:

$$0 \to \omega_X^l(-H) \to \omega_X^l \to \omega_X^l|_H \to 0$$

This induces a left exact sequece of global sections:

$$0 \to H^0(\omega_X^l(-H)) \to H^0(\omega_X^l) \to H^0(\omega_X^l|_H)$$

Since, $\kappa(X) = \dim(X)$, $H^0(\omega_X^l)$ grows as l^n and $H^0(\omega_X^l|_H)$ grows as l^{n-1} for large enough n, we get that, there is a section of $H^0(\omega_X^l(-H))$ defining an effective divisor D such that, $\omega_X^l = \mathcal{O}(D) \otimes \mathcal{O}(H)$.

Due to Corollary 4.0.14, there is a component in the $supp(\mathcal{P})$ surjecting onto X. Let Z be the normalisation of that component. Then, by corollary 4.0.17, we have that, $p_X^*\omega_X^r = p_Y^*\omega_Y^r$ for some integer r.

Birationality: We claim that $p_Y : Z \setminus p_X^{-1}(D) \to Y$ has finite fibres. Suppose on the contrary there is a curve C, not in $p_X^{-1}(D)$, contracted by p_Y. Then $\deg p_Y^*\omega_Y|_C = 0$. Since, $p_X^*\omega_X^r = p_Y^*\omega_Y^r$, we get, $\deg p_X^*\omega_X|_C = 0$. But, $\deg p_X^*\omega_X|_C > \frac{1}{l}\deg \mathcal{O}(H)|_C > 0$. The first inequality is a consequence of D being effective and therefore intersecting $p_X(C)$ in finitely many points. The second inequality is by the numerical criterion of ampleness.
Hence, $Z \to Y$ is generically finite and hence $\dim(Z) \leq \dim(Y)$. On the other hand, since $Z \to X$ is a surjection, $\dim(Z) \geq \dim(X)$. But, since X and Y have equivalent derived categories, $\dim(X) = \dim(Y)$ by corollary 1.3.2. Hence $X \xleftarrow{p_X} Z \xrightarrow{p_Y} Y$ maps generically finitely onto X and Y

In the following lemma, we will show that fibres of the surjection $\mathrm{Supp}(\mathcal{P}) \to X$ is connected. Assuming the lemma, let W be an irreducible component of $\mathrm{Supp}(\mathcal{P})$, other than Z, surjecting onto X. Then a fibre of W over $x \in X$ must contain the points in Z lying above X, by connectivity. Hence, Z is the only component dominating X and moreover the general fibre consists of one point. Therefore, Z is birational to X.

$p_X^*\omega_X = p_Y^*\omega_Y$: $\square$

Bibliography

[1] Huybrechts, Daniel, *Fourier Mukai Transforms in Algebraic Geometry,*