Extremal Curves

Yajnaseni Dutta

6th December, 2013

Abstract

This note is from my lecture at ACGH reading seminar at University of Illinois at Chciago during Fall 2013. The talk was about extremal curves and based mojorly on section on Castelnuovo's bound, Noether's theorem and extremal curves from *Geometry of Algebraic Curves: Volume I* by Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.D.

Theorem 1. Castelnuovo's Bound: *Let C be a smooth curve that admits a birational mapping onto a non-degenerate curve C_0 of degree d in $\mathbb{P}^r$. Then the genus of C satisfies the inequality*

$$g(C) \leq \pi(d, r)$$

where Castelnuovo's number $\pi(d, r)$ is defined by

$$\pi(d, r) = \frac{m(m-1)}{2}(r-1) + m\epsilon$$

where

$$m = \lfloor \frac{d-1}{r-1} \rfloor$$

$$d - 1 = m(r-1) + \epsilon$$

observe that for fixed r and large enough d, assymptotically,

$$\pi(d, r) \sim \frac{d^2}{2(r-1)}$$

Let $\phi : C \to C_0 \subset \mathbb{P}^r$ be the birational map and $D \in \phi^*\mathcal{O}_{C_0}(1)|$. Let E_l be the linear subsystem cut out on C by the pull back of the hypersurfaces of degree l in $\mathbb{P}^r$. Note that, $r < r(D)$ We will also use the following notations:

$$\alpha(l) = |lD| \text{ and } \beta(l) = \dim E_l$$

Definition 1. Extremal Curves: *A smooth curve C, that admits a birational mapping onto a non-degenrate curve of degree d in $\mathbb{P}^r$ is called extremal curve if*

$$g(C) = \pi(r, d)$$

Note, in this case, $\alpha(l) - \alpha(l-1) = \beta(l) - \beta(l-1) = min(d, l(r-1)+1)$. In particular, we have, $\alpha(k) = \beta(k)$ for all k. ([1], p. 114-115) Therefore, the mapping

$$H^0(\mathbb{P}^r, \mathcal{O}(l)) \to H^0(C, \mathcal{O}(lD))$$

is surjective for all $l \geq 1$.

We will understand the embedding of the extremal curves case by case.

Case 1: $d < 2r$**:** In this case $\frac{d}{2} < r < r(D)$ Thus, $r(D) > \frac{g}{2}$. Clifford's theorem ([1], p. 110) implies that D is non special. Therefore, $r(D) = d - g$. A simple computation will show that, $d > 2g$. Therefore, ϕ is an embedding.

Case 2: $d = 2r$ **:** In this case, $m = \lfloor \frac{2r-1}{r-1} \rfloor = 2$ and $\epsilon = 1$. Thus, $g(C) = \pi(r, 2r) = r+1$. Therefore, $r = g-1$. Hence, $C \subset \mathbb{P}^r$ is the canonical curve.

Theorem 2. Max Noether's Theorem: *If C is a nonhyperelliptic curve, then the homomorphisms:*

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}(l)) \to H^0(C, \mathcal{O}(K^l))$$

is surjective for all $l > 0$.

Proof: As observed before.

However, this theorem has strong consequences, Note that, $h^0(\mathbb{P}^{g-1}, \mathcal{O}(2)) = \frac{g(g+1)}{2}$ and $h^0(C, \mathcal{O}(K^2)) = 3g-3$ (using Riemann Roch on $-K$). Therefore, the canonical curve of genus g lies exactly on $\frac{(g-2)(g-3)}{2}$ linearly independent quadrics.

For $g = 4$, the above calculation shows that, non-hyperelliptic curve of genus 4 lies on a unique quadric in $\mathbb{P}^3$. A similar calculation will show that it lies on exactly five cubics, hence atleast one cubic that does not contain the quadric but contains the curve. Therefore, a non-hyperelliptic curve of genus 4 is the complete intersection of a quadric and a cubic in $\mathbb{P}^3$.

For $g = 4$ a similar ananlysis can be done. More interesting remarks can be found in [1] p.118.

Case 3: $d \geq 2r+1$**:** First when $r = 2$ Castelnuovo's bound gives,

$$g \leq \frac{(d-1)(d-2)}{2}$$

Therefore the extremal curves in this case are just the smooth plabe curves.

When $r = 3$, $d = 2k$ then, $m = k-1$ and $\epsilon = 1$, and when $d = 2k+1$, $m = k$ and $\epsilon = 0$. Therefore, for an extremal curve,

$$g(C) = \begin{cases} (k-1)^2 & d = 2k \\ k(k-1) & d = 2k+1 \end{cases} \tag{1}$$

On the other hand, if C is an extremal curve of degree $d \geq 5$, then

$$\alpha(2) = \alpha(1) + \min\ (d, 2(3-1)+1)$$

Now, since $\mathcal{O}_C(1) = \phi^*(\mathcal{O}_{\mathbb{P}^3}(1)$, we have, $h^0(C, \mathcal{O}_C(1)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$. Therefore,

$$h^0(C, \mathcal{O}_C(2)) = h^0(C, \mathcal{O}_C(1)) + 5 = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) + 5 = 9 = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) - 1$$

Thus C lies on a unique quadric. Further, assuming the quadric Q is smooth ruled surface. Let L and M are its ruling. Since, C is a divisor on Q, $C \in |mL + nM|$. Now, $K_Q \in |-2L-2M|$, and $L.M = 1$, $L.L = 0 = M.M$. Therefore, by the adjunction formula we get,

$$g(C) = \frac{K_Q.C + C.C}{2} + 1 = \frac{-2(m+n)+2mn}{2} + 1 = (m-1)(n-1)$$

Comparing this with equation (1), we get,

$$m = n = k \text{ when, } d = 2k$$

$$m - 1 = n = k \text{ when, } d = 2k+1$$

Thus we get the following lemma in this case of $r = 3$:

Lemma 1. *An extremal space curve of degree $d = 2k$ is the complete intersection of a quadric and a surface of degree k. When $d = 2k+1$ then, for some line L' lying on the quadric, the divisor $C + L'$ is the complete intersection of a quadric and a surface of degree $k+1$.*

Remark: We conclude that, the curves that are not of the type (k, k) or $k, k-1$ are not extremal curves. In other words, there exists $l \geq 1$ such that, the morphism, $H^0(\mathbb{P}^r, \mathcal{O}(l)) \to H^0(C, \mathcal{O}(lD))$ is not surjective. i.e. the space curves of unbalanced type (m, n), where, $|m-n| \geq 2$ are not projectively normal.

For the general r we will show that an extremal curve C of degree $d \geq 2r+1$ lie on a surface of very special type. Again similar calculation for extremal curves as before shows that, $\alpha(1)+1 = h^0(C, (O)_C(D)) = r+1$ and $\alpha(2)+1 = h^0(C, (O)_C(2D)) = r + min(d, 2r-1) + 1 = 3r$. But, $h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(2)) = \binom{r+2}{2}$. Therefore, we conclude that C lies on $\dfrac{(r-1)(r-2)}{2}$ many quadrics. We will now explore how these quadrics intersect in $\mathbb{P}^r$.

Consider Our irreducible non-degenerate extremal curve of degree d in $\mathbb{P}^r$. By the general position theorem, a general hyperplane H in $\mathbb{P}^r$ will intersect C in d point any r of which are linearly independent, in other words, those d points are in general position. Lets call the set of these d points, Γ. Then, those d points impose atleast $2r-1$ conditions on quadrics in H. Therefore, by Castelnuovo's lemma (stated below), Γ lies on a unique rational normal curve $X_\Gamma \subset H$.

Lemma 2. Castelnuovo's lemma: *Let d and r be intergers such that, $r \geq 3$ and $d \geq 2r+1$. Let, $\Gamma \subset \mathbb{P}^{r-1}$ be a collection of d points in general position which impose only $2r-1$ conditions on quadrics. Then, Γ lies on a unique rational normal curve X_Γ.*

Proof: Omitted. Ref: [2] p.528-531.

Since,

$$d = \deg(C) \geq 2r+1 > 2(r-1) = 2\deg(X_\Gamma$$

every quadric in $\mathbb{P}^r$ containing C, and hence Γ must also contain X_Γ. Conversely, since a rational normal curve is defined by vanishing of $\dfrac{(r-1)(r-2)}{2}$ number of 2×2 minors, it is the intersection of $\dfrac{(r-1)(r-2)}{2}$ quadrics in $\mathbb{P}^{r-1}$. Therefore, the intersection of quadrics containing C intersects H in exactly X_Γ. Since, X_Γ is a minimal variety, so is S, intersection of the quadrics in $\mathbb{P}^r$. Therefore, S is one of the following:

1. Veronese surface in $\mathbb{P}^5$.
2. rational normal scroll.

Let us consider the case of a smooth rational normal scroll S. Suppose, L be its line of ruling, then $S = \mathbb{P}(\mathcal{E})$ is a scheme over L such that $\mathcal{E}$ is the locally free sheaf $(O) \oplus \mathcal{O}(r-1)$ on L. Therefore, on S, $H.H = r-1, H.L = 1$ and $L.L = 0$.

In order to find the canonical divisor K_S on S we use the adjunction formula for the genus of L and H.

$$0 = p_a(L) = \frac{L.L + K_S.L}{2} + 1$$

$$0 = p_a(H) = \frac{H.H + K_S.H}{2} + 1$$

From these, we obtain, $K_S.L = -2$ and $K_S.H = -r-1$. Therefore, $K_S = -2H + (r-3)L$. From this we can use adjunction formula again to find out the arithmetic genus of C, assuming, $C \in |aH + bL|$. After computing,

$$p_a(C) = \frac{(a-1)(a-1)}{2}(r-1) + (r-2+b)(a-1)$$

We know, $d = \deg(C) = H.C = (r-1)a + b$. On the other hand, $d = m(r-1) + 1 + \epsilon$. Comparing these, we see that the maximal genus of C is attained when

$$a = m+1, \quad b = \epsilon - r + 2 \quad \text{for any } \epsilon$$

$$a = m \quad b = 1 \qquad \text{in case } \epsilon = 0$$

Putting everything together we get the following wonderful classification of the extremal curves in $\mathbb{P}^r$ of degree $d \geq 2r + 1$.

Theorem 3. *Let d and r be integers such that $r \geq 3$ an $d \geq 2r + 1$. Set $m = \lfloor \frac{d-1}{r-1} \rfloor$ and $d = m(r-1) + 1 + \epsilon$. Then extremal curves $C \subset \mathbb{P}^r$ of degree d exists and any such curve is one of the following:*

(i) The image of a smooth plane curve $C \in \mathbb{P}^2$ of degree k under the Veronese map $\mathbb{P}^2 \to \mathbb{P}^5$. In his case $r = 5$ and $d = 2k$.

(ii) A non-singular member of the linear system $|mH + L|$ on a rational normal scroll. In this case, $\epsilon = 0$

(iii) A non-singular member of the linear system $|(m+1)H + (r-2-\epsilon)L|$ on a rational normal scroll.

The book [1] remarks a lot more interesting points about the extremal curves. Also, it talks a little bit about Eisenbud and Harris' work on the classification in case of curves of genus very close to the Castelnuovo's number.

References

[1] *Geometry of Algebraic Curves: Volume I* by Arbarello, E., Cornalba, M., Griffiths, P., Harris, J.D.

[2] *Principles of Algebraic Geometry* by Phillip Griffiths and Joseph Harris.