Nearby and Vanishing D-modules

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1 V-filtration of $\mathfrak{D}_X$-mod

Let X be a complex manifold of dim n . Let $H \subset X$ be a smooth hypersurface I_H ideal sheaf of H.

Definition 1.1 (V-filtration)**.** The *Kashiwara-Malgrange* V-filtration of the D-module, $\mathfrak{D}_X$ is defined by

$$V^i\mathfrak{D} := \{P \in \mathfrak{D} | PI_H^j \subset I_H^{j+i} \text{for all } j \in \mathbb{Z}\}$$

where $I_H^j = \mathcal{O}_X$ for $j \leq 0$.

Locally on coordinates $(z_1, \ldots, z_{n-1}, t)$ of X, if $D = (t = 0)$ we can write the V-filtration as

$$V^0\mathfrak{D}_X = \mathcal{O}_X < \partial_1, \ldots, \partial_{n-1}, t\partial_t >$$

Similarly,

$$V^i\mathfrak{D} = t^i V^0\mathfrak{D}_X$$

and for $i > 0$

$$V^{-i}\mathfrak{D}_X = \partial_t V^{-i+1}\mathfrak{D}_X + V^{-i+1}\mathfrak{D}_X$$

From local calculation, it is easy to see that the section $t\partial_t$ is independent of choices of the defining function t, i.e. $t\partial_t$ is canonical and called the Euler vector field along H. For left $\mathfrak{D}_X$-module M we define V-filtration as follows:

Definition 1.2. The V-filtration of M along H is a $\mathbb{Z}$-indexed decreasing filtration $V^\bullet M$ such that

1. $V^k M$ is coherent over $V^0\mathfrak{D}_X$.
2. $tV^k M \subset V^{k+1}M$, $\partial_V^k M \subset V^{k-1}M$
3. $tV^k M = V^{k+1}M$ for $k > -1$
4. Eigenvalues of $t\partial_t$ on $g_V^k M$ have real part in $[k, k+1)$. The operator $t\partial_t$ is defined everywhere.

Example 1.3. $M = \mathcal{O}_X$ then $V^i = I_H^i$ is a V-filtration along H.

Such filtration may not exist but if it does it is unique. We will see later that for holonomic $\mathfrak{D}_X$-module this filtration always exists.

2 Example on the disc Δ:

Let $j : \Delta^* \hookrightarrow \Delta$ be the inclusion of the punctured disc in $\mathbb{C}$ to the disc. Here $X = \Delta$ and $H = \{0\} = (t = 0)$. let L be a local system on Δ^*. We want to understand $Rj_*L[1]$ and its nearby and vanishing cycles with the help of V-filtration.

Consider the universal covering map from the complex upper-half plane $(\mathbb{H}, w) \xrightarrow{\pi} (\Delta^*, t)$. Then set the fibres of L to be $V := H^0(\mathbb{H}, \pi^{-1}L) = \text{span}\{c_1, ..., c_m\}$. Let Γ be the monodromy acting on c_i. Then

$$c_i(w + 1) = \Gamma c_i(w)$$

with respect to the action of the monodromy. Note that c_i defines a multi-valued section of L and by definition of the associated holomorphic bundle with connection $\mathcal{V} := L \otimes_{\mathbb{C}} \mathcal{O}_X$ we have that

$$\nabla(c_i \otimes 1) = c_i \otimes \partial_t(1) = 0. \qquad (1)$$

Then $V = \oplus V_\lambda$ as generalised eigenspace decomposition also induces a decomposition of L downstairs. $log\Gamma = log(\Gamma_S\Gamma_U)$ is unique upto $\mathbb{Z}$. Γ_S is the semi-simple part and Γ_U is the unipotent part. Fix $\beta \in \mathbb{R}$, we can choose a branch of logarithm so that real part of the eigenvalues of $-(\log \Gamma)$ is in $[\beta, \beta + 1)$. Denote $\Lambda^\beta = -(\log \Gamma) \in \text{End}_{\mathbb{C}}(V)$ with this branch of logarithm. Define:

$$\{s_i\}_i := e^{\Lambda^\beta(w)}\{c_i\}_i.$$

Since $c_i(w + 1) = \Gamma c_i(w)$, we know s_i descend to holomorphic sections on Δ^*. Set

$$\mathcal{V}^\beta = \bigoplus_{j=1}^{m} \mathcal{O}_X s_j$$

with a logarithmic connection ∇^β satisfying (since $w = \log t$)

$$\nabla^\beta(\{s_i\}_i) = \frac{dt}{t} \otimes \Lambda^\beta e^{\Lambda^\beta w}\{c_i\}_i = \frac{dt}{t} \otimes \Lambda^\beta\{s_i\}_i$$

By construction we see that

$$(\mathcal{V}^\beta, \nabla^\beta)|_{\Delta^*} = (\mathcal{V}, \nabla),$$

and Λ^β is just the residue map of ∇^beta along 0, denoted by $\text{Res}_{\nabla^\beta}$. Since $\frac{dt}{t}$ generates $\Omega^1(\log H)$, set

$$DR_H(\mathcal{V}^\beta) : \mathcal{V}^\beta \xrightarrow{\nabla^\beta} \Omega^1(\log H) \otimes \mathcal{V}^\beta,$$

calling it the logarithmic de Rham complex of $\mathcal{V}^\beta$ along $H = \{0\}$. After identifying ∇^β with the action of $t\partial_t$ on $\mathcal{V}^\beta$, $DR_H(\mathcal{V}^\beta)$ is isomorphic to

$$\mathcal{V}^\beta \xrightarrow{t\partial_t} \mathcal{V}^\beta$$

Define

$$\mathcal{V}_* = \mathcal{V}^\beta \otimes_{\mathcal{O}_X} \mathcal{O}_X(*H).$$

∇^β induces a connection on $\mathcal{V}_*$, which make $\mathcal{V}_*$ a $\mathfrak{D}_X$-module. This definition does not depend on β.

If $\beta > \beta'$ then $\mathcal{V}^\beta \subset \mathcal{V}^{\beta'} \subset \mathcal{V}^*$. $\mathcal{V}_*$ has a $\mathfrak{D}$ structure via ∇. Although $\mathcal{V}^\beta$ is defined for all $\beta \in \mathbb{R}$, if we only looked at integer β, $\mathcal{V}^\beta$ gives the V-filtration of $\mathcal{V}_*$. We could also talk about an $\mathbb{R}$-indexed V-filtration for $\mathcal{V}_*$ and note that it is actually a discrete filtration. We will soon discuss an example of such $\mathcal{V}$. But before that we need a couple more definitions:

Definition 2.1. $\mathcal{V}^{>\beta} := \bigcap_{\beta>\beta'} \mathcal{V}^{\beta'}$ since V-filtration is discrete, for some $\beta'' > \beta$ we have $\mathcal{V}^{>\beta} = \mathcal{V}^{\beta''}$. Also, define, $gr^{\beta}_{\mathcal{V}}\mathcal{V}_* = \frac{\mathcal{V}^{\beta}}{\mathcal{V}^{>\beta}}$.

Clearly $gr^{\beta}_{\mathcal{V}}\mathcal{V}_*$ is supported on $\{0\}$. $t\partial_t$ induces a $\mathbb{C}$-linear transformation on $gr^{\beta}_{V}\mathcal{V}_*$ and the eigenvalues of this action have real parts β. Then we have

$$gr^{\beta}_{V}\mathcal{V}_* \xrightarrow{\partial_t} gr^{\beta-1}_{V}\mathcal{V}_* \xrightarrow{t} gr^{\beta}_{V}\mathcal{V}_*.$$

Note if $\beta \neq 0$, ∂_t is injective and t is surjective. Further since $\partial_t t = t\partial_t + 1$, if $\beta \neq -1$ then ∂_t is surjective. So when $\beta \neq 0, -1$ we have that $\partial_t : gr^{\beta}_{V}\mathcal{V}_* \to gr^{\beta-1}_{V}\mathcal{V}_*$ is an isomorphism. Indeed, it is always isomorphic when 1 is not an eigenvalue of Γ.

Lemma 2.2. *$\partial_t : \mathcal{V}^{\beta}_0 \to \mathcal{V}^{\beta-1}_0$ is surjective for $\beta > 0$ or for any β when 1 is not an eigenvalue of Γ.*

We only care about the stalk of the sheaves around 0. Using induction on the rank of L we only need to solve the problem when $\dim L = 1$. Localise and solve.

We observe the following $\mathcal{O}_X$-modules isomorphism: $\mathcal{V}_* = \sum_{p\geq 0} \partial_t^p \mathcal{V}^{-1}$. Also define the $\mathcal{O}_X$-module $\mathcal{V}_{\min} := \sum_{p\geq 0} \partial_t^p \mathcal{V}^{\beta}$ and $\beta > -1$. By the lemma $\mathcal{V}_{\min}$ is independent of β.

Example 2.3.

1. Let's assume $L = \mathbb{C}$ with trivial monodromy then $\mathcal{V}_* = \mathcal{O}_X(*H)$ and $\mathcal{V}_{\min} = \mathcal{O}_X$.
2. Let $L = \mathbb{C}t^{\lambda}$ with monodromy $e^{2\pi i(t-\lambda)}$ for some $\lambda \in \mathbb{C}^*$. Then formally, $\mathcal{V}_* = \mathcal{O}_X(*)t^{\lambda}$. But note that in this case $\mathcal{V}_{\min} = \mathcal{V}_*$.

The above two examples tell us that the eigenvalue 1 of Γ plays a crucial role for $\mathcal{V}_*$ In other words if 1 is not an eigenvalue then $\mathcal{V}_* = \mathcal{V}_{\min}$.

$DR(\mathcal{V}_{\min}) = j_{!*}L = j_*\mathbb{C}$ the last equality is not true in higher dimension. $DR(\mathcal{V}_*) = Rj_*L$.

Theorem 2.4. *We have a quasi-isomorphism*

$$DR_H(\mathcal{V}^{\beta}) \hookrightarrow DR(\mathcal{V}_*)$$

when $\beta \leq 0$ and the stalk $DR(\mathcal{V}^0)_0$ at 0 is quasi isomorphic to

$$0 \to gr^{0}_{V}\mathcal{V}_* \xrightarrow{\partial_t} gr^{-1}_{V}\mathcal{V}_*.$$

Moreover, $DR(\mathcal{V}_*) \simeq Rj_*L$

Proof. If β is slightly smaller than 0 we have

$$\mathcal{V}^{\beta}/\mathcal{V}^{>\beta} = \mathcal{V}^{\beta}/\mathcal{V}^{0}$$

Since $t\partial_t$ defines an isomorphism between $gr^{\beta}\mathcal{V}_*$ for $\beta \neq 0$, we have

$$\mathcal{V}^{\beta}/\mathcal{V}^{0} \xrightarrow{t\partial_t} \mathcal{V}^{\beta}/\mathcal{V}^{0}$$

is an isomorphism. Therefore,

$$DR_H(\mathcal{V}^{0}) \to DR_H(\mathcal{V}^{\beta})$$

is a quasi-isomorphism. Similarly for all $\beta'' < \beta' < 0$ we have quasi-isomorphisms

$$DR_H(\mathcal{V}^{\beta'}) \to DR_H(\mathcal{V}^{\beta''}).$$

Taking direct limit $\beta'' \to -\infty$ we get,

$$DR_H(\mathcal{V}^{\beta'}) \hookrightarrow DR(\mathcal{V}_*)$$

is a quasi-isomorphism.

For the second quasi-isomorphism in the statement, since

$$DR(\mathcal{V}^0) = \mathcal{V}^0 \xrightarrow{t\partial_t} \mathcal{V}^0,$$

it is enough to prove that

$$\mathcal{V}_0^{>0} \xrightarrow{t\partial_t} \mathcal{V}_0^{>0}$$

is an isomorphism. This is clear by induction on rank of L (see MHM project, chapter V). The third quasi-isomorphism can also be proved by induction on rank of L.

$\square$

2.1 Nearby and vanishing D-modules on Δ

Let $i : 0 \hookrightarrow \Delta$ be the inclusion. Then by the thorem $i^{-1}Rj_*L \simeq [gr_V^0\mathcal{V}_* \xrightarrow{t\partial_t} gr_V^0\mathcal{V}_*]$. Define the nearby cycle as

$$\Psi_t F := i^{-1}R\pi_*\pi^{-1}L.$$

Note that $(\Psi_t F)_0 = V$. Indeed, since $(\Psi_t F)_0 = \varinjlim_U R\Gamma(\pi^{-1}(U), \pi^{-1}F)$, $\pi^{-1}F$ is trivial and $\pi^{-1}U$ is contractible.

Denote by $(\Psi_t F)_{\text{unip}}$ as the generalized eigenspace of V corresponding to 1 under the monodromy operation. This corresponds to the 0^{th} generalised eigenspace of Λ^0. Therefore, But the stalk at 0 of $(\Psi_t F)_{\text{unip}_0} \simeq gr_V^0\mathcal{V}_*$. The specialisation map $sp : i^{-1}F \to \Psi_t F$ factors through $(\Psi_t F)_{\text{unip}}$. Then the specialisation map $i^{-1}F \xrightarrow{sp} \Psi_t F$ at 0 is given by $[gr_V^0\mathcal{V}_* \xrightarrow{t\partial_t} gr_V^0\mathcal{V}_*] \xrightarrow{sp} gr_V^0\mathcal{V}_*$. Therefore, the vanishing cycle $\Phi_t F$ defined as the cone of ϕ is given by $gr_V^{-1}\mathcal{V}_*$ and the canonical map can be seen as

$$gr_V^0\mathcal{V}_* \xrightarrow{can:=\partial_t} gr_V^{-1}\mathcal{V}_*$$

and

$$gr_V^{-1}\mathcal{V}_* \xrightarrow{var:=t} gr_V^0\mathcal{V}_*.$$

3 Comparison Theorems

We will talk about the existence of V-filtrations for holonomic $\mathfrak{D}$-modules. Let $H \subset X$. $gr_\bullet^V\mathfrak{D}$ is not commutative a-priori.

3.1 Some non-commutative algebra

Suppose A is a non-commutative Noetherian ring. Let $F^\bullet$ be a $\mathbb{Z}$-indexed filtration of A and M is a finitely generated left A-module. Then a filtration $\Gamma^\bullet M$ is a $\mathbb{Z}$-indexed filtration compatible with $(A, F^\bullet)$. This means that $F^l A.\Gamma^k M \subset \Gamma^{k+l} M$.

Definition 3.1 (Associated Rees ring)**.**

1. $R_F A := \oplus F^l A.T^l \subset A[T, \frac{1}{T}]$.
2. (M, Γ) is good if $R_\Gamma M$ is finitely generated

Lemma 3.2. *$(M, \Gamma^\bullet)$ is good iff there is $(k_1, \dots, k_s) \in \mathbb{Z}^s$ and $m_1, \dots, m_s$ generates M such that*

$$\Gamma^v M = F^{v-k_1} m_1 + \cdots + F^{v-k_s} m_s$$

for $v \gg 0$

Remark. $gr^\Gamma M$ finitely generated over $gr^F \mathfrak{D}$ does not imply $R_\Gamma M$ is finitely generated with respect to $\Gamma^\bullet$ unless M is complete with respect to Γ.

Define

$$j_A(M) := \max\{v | \operatorname{Ext}^i(M, A) = 0 \forall i < v\}.$$

Note that when A is a commutative local ring this notion is known as the depth of M.

Lemma 3.3. *If $gr^F_\bullet A$ is Auslander regular and $\Gamma^\bullet$ is good for M then*

$$j_A(M) \leq j_{gr^F A} gr^\Gamma M.$$

We are not going to define Auslander regular here. But below are some examples of AR rings. This theorem is crucial for proving the existence of V-filtrations for holonomic $\mathfrak{D}_X$-modules. This theorem is standard in non-commutative algebra.

Example 3.4 (AR rings)*.* .

1. A is commutative and regular then A is Auslander regular.
2. A_n the Weyl algebra is AR.
3. $\mathfrak{D}_{X,x}$ is Auslander regular.

If M is holonomic over $\mathfrak{D}_X$ then $j_{gr^F A} gr^\Gamma M = n$ with respect to the order filtration on $\mathfrak{D}_X$. This is because, $ht(\operatorname{Ann}(gr^{\Gamma^\bullet} M)) = n$ and since $\Gamma^\bullet$ is good, this is precisely the depth of the finitely generated module $gr^{\Gamma^\bullet} M$ over $gr^{F^\bullet}\mathfrak{D}$. From this we can also conclude that $j_A(M) = n$. Roughly this follows from the fact that the filtration on M induces a filtration on $\operatorname{Ext}^i(gr^\Gamma M, gr^F A)$.

Supose M is a coherent $\mathfrak{D}_X$-module. We can define a good filtration on M with respect to $V^\bullet \mathfrak{D}_X$. From Lemma 3.1, good filtrations exist for all coherent $\mathfrak{D}_X$-modules, locally at least.

3.2 Fuchsian filtration: Local description of V-filtration revisited

Let $A_1 = k < t, \partial_t >$ be the Weyl algebra. Set $\operatorname{ord}(t) = 1$ and $\operatorname{ord}(\partial_t) = -1$. This defines a a graded $\mathbb{C}$-algebra structure on A_1 via the filtration $V^\bullet A_1$ defined as follows: $f \in V^p$ if $\operatorname{ord}(f) \geq p$. For instance $\operatorname{ord}(t\partial_t) = 0$ but $\operatorname{ord}(t\partial_t + t) = 1$. We observe that $gr^{V^\bullet} A_1 \simeq A_1$ as rings. Hence we get $gr^{V^\bullet} \mathfrak{D}_{X,x} \simeq A_1 \otimes_{\mathbb{C}_H} \mathfrak{D}_{H,x}$. This is a graded ring isomorphism and the grading on the right hand side is governed by the Fuchsian grading on A_1.

Lemma 3.5. *If we give $T^*_H X$ a semi-Zariski and semi-anaytic topology (The base topology is analytic but the fibres are Zariski) and define*

$$\mathcal{O}_{T^*_H X}(\pi^{-1}(U)) = \mathcal{O}_H(U)[t].$$

Therefore we can identify

$$gr^V \mathfrak{D}_X \simeq \pi_* \mathfrak{D}_{T^*_H X},$$

where $\pi : T^*_H X \to H$.

Definition 3.6 (Specializability)**.** M is said to be specializable along H on an open subset $U \subset X$ for some good filtration $\Gamma^\bullet$ of M compatible with the V-filtration, if there is a polynomial $b(s) \in \mathbb{C}[s]$ such that $b(t\partial_t - k)$ acts on $\frac{\Gamma^k M}{\Gamma^{k+1} M}$ is trivially.

Define $\Phi := \oplus_k t\partial_t - k$ acting on $gr^V \mathfrak{D}_X$-linearly on $gr^\Gamma M$. Set

$$\widetilde{gr^\Gamma M} = \mathfrak{D}_{T^*_H X} \otimes_{\pi^{-1} gr^{V^\bullet} \mathfrak{D}_X} \pi^{-1} gr^\Gamma M$$

Therefore Φ lifts to $\widetilde{\Phi}$, a $\mathfrak{D}_{T^*_H X}$-linear map of $\widetilde{gr^\Gamma M}$.

Lemma 3.7. *Suppose M is specializable along H. Then there exists a good V-filtration $\Omega^\bullet M$ of M such that there is a polynomial $b'(s) \in \mathbb{C}[s]$ satisfying the condition that $b'(t\partial_t - k)$ acts on $\Omega^k M / \Omega^{k+1} M$ trivially and the $\Re(\text{roots of } b'(s)) \subset [0, 1)$.*

Proof. Set $\Omega^n = \Gamma^{n+k}$ then $b(t\partial_t - n - k)$ acts on Ω^n / Ω^{n+1} trivially. So we assume that the real part of the roots of b are larger than 0.

Now set $\Omega^n = \Gamma^{n-1} + (t\partial_t - \alpha_1 - n)^{n_1})\Gamma^n$ where α_1 is a root of $b(s)$. Recall that $\Re(\alpha_1) \geq 0$. Then

$$(t\partial_t - (\alpha - 1) - n) b_1(t\partial_t - n)$$

kills Ω^n / Ω^{n+1}. Repeating this, we are done. □

Corollary 3.8. *If M is specializable along H the V-filtration of M along H exists globally.*

4 holonomic $\mathfrak{D}_X$-modules

Let M be a coherent $(\mathfrak{D}_X, F^\bullet)$. Define

$$\mathrm{Char}(M) := \mathrm{cosupp}(\mathrm{Ann}(gr^{F^\bullet} M).$$

$\mathrm{Char}(M)$ is involutive. Therefore $\dim(\mathrm{Char}(M)) \geq n$. M is holonomic if $\dim(\mathrm{Char}(M)) = n$.

If M is holonomic $\mathrm{Char}(M)$ is Lagrangian analytic subset of $T^* X$. If $\dim(\operatorname{supp} M) = k \leq n$, then there is a Whitney stratification $\{X_\alpha\}$ of $\operatorname{supp}(M)$ such that

$$\mathrm{Char}(M) \subset \bigcup T^*_{X_\alpha} M.$$

This has been proved by Kashiwara. Set $U = X \setminus \bigcup_{\alpha \neq k} X_\alpha$ and $i : X_k \hookrightarrow U$ the closed embedding. By Kashiwara's equivalence, possibly after shrinking X_α and U we know

$$M|_U = i_+(\mathcal{O}_{X_\alpha} \otimes_{\mathbb{C}} L),$$

where L is a local system on X_α.

Lemma 4.1. *For $\phi \in Hom_{\mathfrak{D}_X}(M,M)$ and $X_0 \subset X$ relatively compact open neighbourhood, the there is $b(s) \in \mathbb{C}[s]$ such that $b(\phi)|_{X_0} = 0$. If X and M are algebraic, then $b(\phi) = 0$ globally.*

Proof. Let $U_0 = X_0 \cap U$ where U is as above. Then

$$\phi_0 = \phi|_{U_0} \in \mathrm{Hom}_{\mathfrak{D}_X}(M|_{U_0}, M|_{U_0}) = \mathrm{Hom}_{X_\alpha}(L,L).$$

Clearly ϕ_0 has a minimal polynomial $b(s)$ and $b(\phi_0)M$ is a holonomic $\mathfrak{D}$-module supported on an analytic set of dimension strictly smaller than k. Hence we are done by induction. $\square$

Theorem 4.2. *If M is holonomic, then M is specialisable along any $H \subset X$.*

Proof. Locally fix a good filtration $\Gamma^\bullet$ of M. Since M is holonomic, for all $x \in H$ $j_{\mathfrak{D}_{X,x}}(M_x) = n$. By lemma 3.3, we know that $j_{gr^F A} gr^\Gamma M \geq n$. Since $gr^V \mathfrak{D}_{X,x} \simeq A_1 \otimes_{\mathbb{C}} \mathfrak{D}_{H,x}$ we have that $gr^\Gamma M$ is holonomic over $A_1 \otimes_{\mathbb{C}} \mathfrak{D}_{H,x}$. So $gr^\Gamma M$ is hlonomic over $\mathfrak{D}_{T^*_H X}$. Recall that $\widetilde{\Phi}$ was defined to be a lift of $\Phi = \oplus_k t\partial_t - k$ and was $\mathfrak{D}_{T^*_H X}$-linear morphism of $gr^\Gamma M$. Now by Lemma 4.1 we have that there is a $b(s) \in \mathbb{C}[s]$ such that $b(\widetilde{\Phi}) = 0$ locally. Therefore, M is specialisable along H. $\square$

5 Perverse Sheaves: Nearby and Vanishing functors

We will talk about how the nearby and vanishing functors of constructible sheaves relate to the $\mathfrak{D}$-module case. Let X be a complex manifold of $\dim n$.

notations Let f be a morphism of complex manifolds Rf_* is the derived push-forward. f^{-1} is the sheaf pullback. $Rf_!$ is the derived push forward with compact support. $f^!$ is the adjoint of $Rf_!$. f^* is the underived $\mathcal{O}_X$-pullback. f_+ is derived $\mathfrak{D}_X$-pushforward.

5.1 Decomposition of nearby cycles for constructible sheaves

$$\Psi_f F^\bullet = \oplus \Psi_{f,\lambda}(F^\bullet)$$

Recall the following diagram:

$$\begin{tikzcd}
 & & \widetilde{X} \arrow[r] \arrow[dl, "\widetilde{p}"'] \arrow[d, "p"] & \widetilde{\mathbb{C}}^* \arrow[d, "\pi"] \\
X_0 \arrow[r, hook, "i"] & X & X^* \arrow[l, hook', "j"'] \arrow[r, "f"] & \mathbb{C}^*
\end{tikzcd}$$

for $f \in \Gamma(X, \mathcal{O}_X)$ and $X^* = f^{-1}(\mathbb{C}^*)$ and $X_0 = f^{-1}(0)$. We know $\Psi_f(\bullet)$ is functor between $D^b_c(X)$. This is well defined since $\Psi_f F^\bullet$ is constructible (Deligne). We will see that by definition, the nearby cycles only depend on $j^{-1}F$. We recall the definition here:

$$\Psi_f F^\bullet = i^{-1} Rj_* Rp_* p^{-1} j^{-1} F^\bullet = i^{-1} R\widetilde{p}_* \widetilde{p}^{-1} F^\bullet.$$

Since $\widetilde{p}^{-1} = \widetilde{p}^!$,

$$i^{-1} Rj_* Rp_* R\mathcal{H}om_{\mathbb{C}_{\widetilde{X}^*}}(\mathbb{C}_{\widetilde{X}^*}, p^!(j^{-1}F^\bullet))).$$

By Poincaré-Verdier duality, the later is isomorphic to

$$i^{-1} Rj_* R\mathcal{H}om_{\mathbb{C}_{X^*}}(Rp_! \mathbb{C}_{\widetilde{X}^*}, j^{-1}F^\bullet).$$

which is same as

$$i^{-1}Rj_*R\mathcal{H}om_{\mathbb{C}_{X^*}}(f^{-1}\pi_!\mathbb{C}_{\widetilde{\mathbb{C}^*}}, j^{-1}F^\bullet)$$

since the fibre of π is a discrete set. $\pi_!\mathbb{C}_{\widetilde{\mathbb{C}^*}}$ is a $\mathbb{C}[\mathbb{Z}]$-module. Let T (corresponding to $1 \in \pi_1(\mathbb{C}^*)$) acts on $\pi_!\mathbb{C}_{\widetilde{\mathbb{C}^*}}$. Thus there is an action of T on the nearby cycle. We have a short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \pi_!\mathbb{C}_{\widetilde{\mathbb{C}^*}} \arrow[r, "id"'] \arrow[d, "tr"'] & \pi_!\mathbb{C}_{\widetilde{\mathbb{C}^*}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{C}_{\mathbb{C}^*} \arrow[r] & \mathbb{C}_{\mathbb{C}^*} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

The trace map is defined via adjunction and $\mathbb{C}_{\widetilde{\mathbb{C}^*}} \to \pi^!\mathbb{C}_{\mathbb{C}^*}$. Stalkwise the trace map is the same as taking sum of all the entries in the orbit of action of T. Treat the vertical maps as complexes $A^\bullet$, $B^\bullet$ and $C^\bullet$ respectively. Therefore the above diagram is a short exact sequence of complexes:

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0.$$

Appying the following derived functor

$$G = i^{-1}Rj_*R\mathcal{H}om_{\mathbb{C}_{X^*}}\big(f^{-1}(\bullet), j^{-1}(F^\bullet)\big)$$

to the above short exact sequence we get,

$$G(C^\bullet) \to G(B^\bullet) \to G(A^\bullet) \xrightarrow{+1}$$

This is the same as:

$$\Psi_f F^\bullet[-1] \xrightarrow{\mathfrak{can}} \Phi_f F^\bullet \to i^{-1}F^\bullet \xrightarrow{+1}$$

Now consider

$$\begin{tikzcd} 0 \arrow[r] & \pi_!\mathbb{C}_{\widetilde{\mathbb{C}^*}} \arrow[r, "I-T"] \arrow[d] & \pi_!\mathbb{C}_{\widetilde{\mathbb{C}^*}} \arrow[r, "tr"] \arrow[d, "tr"'] & \mathbb{C}_{\widetilde{\mathbb{C}^*}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{C}_{\mathbb{C}^*} \arrow[r, "id"] & \mathbb{C}_{\mathbb{C}^*} \arrow[r] & 0 \end{tikzcd}$$

This gives us another map

$$i^!F^\bullet \to \Phi_f F^\bullet \xrightarrow{\text{var}} \Psi_f F^\bullet[-1].$$

This defines the canonical map. The T action on $\pi_!\mathbb{C}_{\widetilde{\mathbb{C}^*}}$ induces T action on $\Psi_f F^\bullet$ and $\Phi_f F^\bullet$. Clearly $\mathfrak{can} \circ \mathfrak{van} = T - I$ on $\Phi_f F^\bullet$ and $\text{var} \circ \mathfrak{can} = T - I$ on $\Psi_f F^\bullet$.

Theorem 5.1. *The nearby cycle $\Psi_f F^\bullet = \oplus \Psi_{f,\lambda}(F^\bullet)$ decomposes in $D^b_C(X)$ (more specifically in the category $Perv(X)$) as the Jordan decomposition of the T-action on the functor $\Psi_f F^\bullet$. This decomposition is canonical.*

Proof. Set

$$\psi(\bullet) = i^{-1}\widetilde{p}_*\widetilde{p}^{-1}(\bullet).$$

Then $\Psi(\bullet)$ is the right derived functor of $\psi(\bullet)$. For a constructible sheaf F, we know $\psi(F)$ is also constructible. So for any connected open neighborhood U, $\psi(F)(U)$ is finite dimensional $\mathbb{C}$-vector space. So the T action on $\psi(F)(U)$ has a Jordan decomposition. Therefore $\psi(F)$ has a functorial Jordan decomposition with respect to the T-action. So is $\Psi(\bullet)$ as right derived functor of $\psi(\bullet)$. $\square$

Corollary 5.2. *$\Phi_f F^\bullet$ also admit similar (non-canonical) decmposition into generalised eigen spaces.*

Proof. The eigen space corresponding to eigen-value 1, $\Psi_{f,1}(F^\bullet)$ is called the *unipotent nearby cycle.* Since T acts as identity on $i^{-1}F^\bullet$ we can define, $\Phi_{f,1}(F^\bullet)$ as the cone of the specialisation map

$$i^{-1}F^\bullet \to \Psi_{f,1}F^\bullet \to \Phi_{f,1}F^\bullet \xrightarrow{+1} .$$

Again since T acts on $i^{-1}F^\bullet$ as identity, the specialisation map $i^{-1}F^\bullet \xrightarrow{sp} \oplus_{\lambda\neq 1}\Psi_{f,\lambda}(F^\bullet)$ is trivial and the quotient of $\Phi_{f,1}(F^\bullet) \to \Phi_f F^\bullet$ is given by $\oplus_{i,\neq 1}\Psi_{f,1}F^\bullet$. Said differently,

$$\Phi_{f,\neq 1}F^\bullet = \Psi_{f,\neq 1}F^\bullet$$

$\square$

Consider $Y = X \times \mathbb{C}$ and denote the coordinate on $\mathbb{C}$ by t. For all $m \in \mathbb{Z}$ we are going to construct a vector bundles N_m with connection ∇_m and associated local system $\mathcal{L}_m$ on Y. These will be crucial objects in the purpose of making the Riemann-Hilbert correspondence between the nearby and vanishing fuctors and nearby and vanishing $\mathfrak{D}$-modules. Let t denote the coordinate of $\mathbb{C}^*$. Set $e_k := \dfrac{(logt)^k}{k!}$. Construct,

$$N_m = \bigoplus_{i=0}^{m} \mathcal{O}_{\mathbb{C}}\Big[\frac{1}{t}\Big]e_i.$$

This is a $\mathfrak{D}_{\mathbb{C}}$-module by the action $t\partial_t e_k = e_{k-1}$. Consider $DR(N_m)|_{\mathbb{C}^*}$ with connection $t\partial_t$. The kernel gives us a $\mathbb{C}^*$-local system $\mathcal{L}_{\mathbb{C},m}$. Denote the monodromy on this local system by $T = log\ t\partial_t$ and note that it is given by J_m, the m-dimensional Jordan block with eigenvalue 1. Set $\mathcal{L}_m = f^{-1}L_{\mathbb{C},m}$ with fibre L_m.

Lemma 5.3. *Let X and f be as before. If $\mathcal{L}$ is a local system on $\mathbb{C}^*$ with fiber L, then*

$$\Psi_f(F^\bullet \otimes_{\mathbb{C}} \mathcal{L}) = \Psi_f F^\bullet \otimes_{\mathbb{C}} L.$$

The action of T is componentwise.

Write $\mathcal{K}_\alpha$ is the rank 1 local system on $\mathbb{C}^*$ with the monodromy T given by multiplication by α. Therefor, by the above lemma,

$$\Psi_{f,1}(F^\bullet \otimes_{\mathbb{C}} \mathcal{K}_\alpha) \simeq \Psi_{f,\alpha}(F^\bullet)$$

Lemma 5.4. *Let W be a complex vector space and ϕ some $\mathbb{C}$-linear operator on W with a minimal polynomial. Set*

$$W_\infty = \bigoplus_{i=0}^{\infty} W \otimes_{\mathbb{C}} \mathbb{C}e_i$$

and define ϕ_∞ by

$$\phi_\infty(u \otimes e_i) = \phi(u) \otimes e_i + u \otimes e_{i-1}.$$

Assume that $e_{-1} = 0$. Then ϕ_∞ is surjective and $\mathrm{Ker}(\phi_\infty) \simeq W_0$ *where W_0 is the biggest subspace of W on which ϕ is nilpotent. Here W_0 identified with $W_0 \otimes \mathbb{C}e_0$ as a subspace of W_∞.*

From the above two lemmas, it is easy to conclude

Theorem 5.5.

$$i^{-1}\varinjlim Rj_*(j^{-1}F^\bullet \otimes \mathcal{L}_m) \simeq \Psi_{f,1}F^\bullet$$

6 Nearby and Vanishing $\mathfrak{D}_X$-module: Riemann-Hilbert Correspondence

Recall the Koszul complex in non-commutative setting. Suppose A is an abelian group. Let $\phi_i \in \operatorname{End}_{\mathbb{Z}} A$ be morphisms of A commuting pairwise. Let $K_1 = [A \xrightarrow{\phi_i} A]$ and $K_{i+1} = [K_i \xrightarrow{\phi_{i+1}} K_i]$. Then $K(\phi_1, \ldots, \phi_n; A) =: K_n$ is called the Koszul complex.

Lemma 6.1. *1. The order of ϕ_i is K does not matter.*

2. if one of ϕ_i is an isomorphism, then K is acyclic.

6.1 deRham functor

Let X be a complex manifold of dimension n. We have a canonical section $\sum dx_i \otimes \partial_i$ in Ω^1_X. Then

$$[\mathfrak{D}_X \xrightarrow{\nabla} \Omega^1_X \otimes \mathfrak{D}_X \to \ldots \to \Omega^n_X \otimes \mathfrak{D}_X] = DR_X(\mathfrak{D}_X)$$

$DR_X(\mathfrak{D}_X)$ is a complex of right $\mathfrak{D}_X$-module. It is a result that $DR_X(\mathfrak{D}_X)$ is a resolution of Ω^1_X as a right $\mathfrak{D}_X$-module. Now suppose M is a left $\mathfrak{D}_X$-module. Then define $DR_X(M) := DR_X\big(\mathfrak{D}_X\big) \bigotimes_{\mathfrak{D}_X} M$ This tensor product is derived since $DR_X(\mathfrak{D}_X)$ is locally free over $\mathfrak{D}_X$. So, $DR(\bullet) : D^b(coh(\mathfrak{D}_X)) \to D^b(\mathbb{C}_X)$ is an exact functor.

6.2 Local picture:

Pick coordinates $(x_1, \ldots, x_n)$, Then $DR(M) = K(\partial_1, \ldots, \partial_n; M)$. Here M considered as a $\mathbb{C}_X$-module. Now let $Z = (x_n = 0)$ Then from this inductive Koszul complex description of deRham complex, we can conclude that $DR(M) = [DR_Z(M) \xrightarrow{\partial_n} DR_Z(M)]$.

6.3 Comparison Theorem

Let $Y = X \times \mathbb{C}$. Let t be the coordinate on $\mathbb{C}$. Last time we defined $e_k = (\log t)^k/k!$ and N_m. So N_m is a regular holonomic $\mathfrak{D}_Y$-module. Denote $Y^* = Y \setminus (X \times \{0\})$. We have natural inclusion $N_m \hookrightarrow N_{m+1}$ compatiblle with $\mathfrak{D}_Y$-structure. Define, $N_\infty = \varinjlim N_m$. Last time we defined the local system corresponding to this vector bundle with connection as L_∞.

From now on assume M is regular holonomic.

Lemma 6.2. *With the notation as before, as a $\mathfrak{D}_X$-modules we have the decomposition*

$$\frac{V^k M}{V^{k+1} M} = \bigoplus_{roots\ of\ b_k(s)} gr^\alpha_V M$$

This decomposition is induced by action of $\partial_t t$ on $\frac{V^k M}{V^{k+1} M}$.

Definition 6.3. Nearby cycle of M is defined as follows:

$$\Psi_0(M) = gr^0_V M.$$

If 0 is not a root of $b_k(s)$ then $\Psi_0(M) = 0$.

Theorem 6.4. *We have a quasi-isomorphism* $DR_X(\Psi_0(M)) = \Psi_{t,1}\big(DR_Y(M)\big)$*, where* t *is the coordinate on* $\mathbb{C}$ *and*

$$DR_X\Big(\frac{V^kM}{V^{k+1}M}\Big) \simeq \Psi_t(DR(M)).$$

Direct sum is preserved under this quasi-isomorphism

Proof. Let $i : X \times 0 \hookrightarrow Y$ and $j : Y^* \hookrightarrow Y$. Then

$$i^{-1}DR_Y(M) = [DR_X(gr_V^0 M \xrightarrow{\partial_t} gr_V^{-1}M)$$

is a quai-isomorphism. We know locally the LHS is

$$K(\partial_1, \ldots, \partial_{n-1}, \partial_t; M) = i^{-1}[K(\partial_1, \ldots, \partial_{n-1}; M) \xrightarrow{\partial_t} K(\partial_1, \ldots, \partial_{n-1}, ; M)].$$

For the RHS, follow the following sequence of quasi-isomorphisms:

$$DR_X(V^0M \xrightarrow{\partial_t} V^{-1}M) \simeq DR(V^{-1}M \xrightarrow{\partial} V^{-2}M)$$

$$\simeq DR(V^{-k}M \xrightarrow{\partial} V^{-k-1}M).$$

By taking direct limit we can show that the later is quasi-isomorphic to $DR_X(M \xrightarrow{\partial_t} M)$.

By looking

Now, $i^{-1}DR_Y(M \otimes_{\mathcal{O}_X} N_\infty) = DR_X(gr_V^0 M)$ i.e. the unipotent part of the deRham complex can be obtained by direct limit over tensoring with N_m. Indeed, one can deduce that from the following calculations:

$$DR_Y(M \otimes_{\mathcal{O}_Y} N_m)\big|_{Y^*} = DR_{Y^*}(j^{-1}M \otimes_{\mathbb{C}_{Y^*}} N_m) = DR(j^{-1}M) \otimes_{\mathbb{C}_{Y^*}} L_m = j^{-1}DR_Y M \otimes_{\mathbb{C}_{Y^*}} L_m.$$

Then,

$$i^{-1}DR_Y(M \otimes N_m) = i^{-1}Rj_*(j^{-1}DR_Y M \otimes_{\mathbb{C}_{Y^*}} L_m).$$

By taking direct limit, the right hand side of this equality is same as $\Psi_{t,1}\big(DR_Y(M)\big)$. $\square$

Note that we can retrieve the the nearby and vanishing triangle by taking the $DR(\bullet)$ of the following triangle:

$$[gr_V^0 M \to gr_V^{-1}M] \to gr_V^0 M \xrightarrow{\partial_t} gr_V^{-1}M.$$

Then define the vanishing cycle by $DR_X(gr_V^{-1}M)$.