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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 10

Exercise 53. *Invertible sheaves* (4 points)
Let $\varphi\colon \mathcal{L} \to \mathcal{M}$ be a homomorphism of invertible sheaves on a scheme $(X, \mathcal{O}_X)$.

(i) Show that φ is an isomorphism if φ is surjective.

(ii) Give an example where φ is injective but not an isomorphism.

Exercise 54. *Blowing up points in affine space* (4 points)
Let k be a field. Let $\mathbb{A}^n_k = \text{Spec } k[x_1, \ldots, x_n]$, $\mathbb{P}^{n-1}_k = \text{Proj } k[y_1, \ldots, y_n]$, and $O = (0, \ldots, 0) \in \mathbb{A}^n_k(k)$. The *blow-up of* $\mathbb{A}^n_k$ *in* O is the closed subscheme $X \subseteq \mathbb{P}^{n-1}_k \times_{\text{Spec } k} \mathbb{A}^n_k = \mathbb{P}^{n-1}_{\mathbb{A}^n_k}$ determined by the homogeneous ideal

$$I = (\{x_i y_j - x_j y_i\}_{i,j=1}^n).$$

Note that X comes with two projections $\pi_1 : X \to \mathbb{A}^n_k$ and $\pi_2 : X \to \mathbb{P}^{n-1}_k$.

(i) Show that π_1 is an isomorphism over $\mathbb{A}^n_k \setminus \{O\}$.

(ii) Show that the fiber $\pi_1^{-1}(O)$ is isomorphic to $\mathbb{P}^{n-1}_k$.

(iii) Show that the fiber of π_2 over a k-rational point of $\mathbb{P}^{n-1}_k$ is isomorphic to $\mathbb{A}^1_k$.

Exercise 55. *The standard Cremona involution* (4 points)
Let k be a field and let

$$\begin{array}{rcl} \sigma : \mathbb{P}^2_k \setminus \{[0:0:1], [0:1:0], [1:0:0]\} & \to & \mathbb{P}^2_k \\ [a_0 : a_1 : a_2] & \mapsto & [a_1 a_2 : a_0 a_2 : a_0 a_1] \end{array}$$

be the morphism determined by the global sections $x_1x_2, x_0x_2, x_0x_1 \in \Gamma(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(2))$.

(i) Show that $\sigma(V_+(x_i))$ is a point for all i.

(ii) Show that σ^2 is well-defined on $\mathbb{P}^2_k \setminus (\cup_{i=1}^3 V_+(x_i))$ and that it extends to the identity on $\mathbb{P}^2_k$. Conclude that there is no morphism $\mathbb{P}^2 \to \mathbb{P}^2$ that restricts to σ on $\mathbb{P}^2_k \setminus \{[0:0:1], [0:1:0], [1:0:0]\}$.

(iii) Show that σ induces a morphism $X \to X$, where X is the blow-up of $\mathbb{P}^2$ at $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$, i.e., the scheme obtained by glueing the blow-ups of the three affine planes $\mathbb{A}^2_k \cong D_+(x_i) \subseteq \mathbb{P}^2_k$ in $(0,0)$. Conclude that σ defines an automorphism of X.

Due 16.12.2022, 2pm

Exercise 56. *Some examples of curves and surfaces* (4 points)
Let k be an algebraically closed field.

(i) Let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(2)$. Show that the morphism $\mathbb{P}^2 \to \mathbb{P}^5_k$ induced by the global sections $x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2 \in \Gamma(X, \mathcal{L})$ is the Veronese map of Exercise 48. In general, for integers $n, k \geq 0$ think about what kind of map $\mathcal{O}_{\mathbb{P}^n}(k)$ induces on $\mathbb{P}^n$.

(ii) Assume that $\mathrm{char}(k) \neq 2$. Show that the sections $x_0^2, x_1^2, x_2^2, x_1(x_0 - x_2), (x_0 - x_1)x_2 \in \Gamma(X, \mathcal{L})$ induce a closed immersion $X \hookrightarrow \mathbb{P}^4_k$.

(iii) Consider the subspace $V \subseteq \Gamma(X, \mathcal{L})$ in (i) spanned by those of the chosen basis above that vanish at the point $[0 : 0 : 1]$ and consider the morphism $\mathbb{P}^2 \setminus [0 : 0 : 1] \to \mathbb{P}^4$ induced by V. Describe the closure of the image in $\mathbb{P}^4$ of the cuspidal cubic curve given by $x_1^2 x_2 = x_0^3$ in $\mathbb{P}^2$ under this morphism.

Exercise 57. *Ramified coverings* (4 points)
Let k be an algebraically closed field of characteristic $\mathrm{char}(k) = p \geq 0$. Consider a homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ of degree d and the two closed subschemes $X = V_+(f) \subseteq \mathbb{P}^n_k$ and $Y = V_+(f - x_{n+1}^d) \subseteq \mathbb{P}^{n+1}_k$. Show that the projection $g\colon Y \to \mathbb{P}^n_k$ from $[0 : \ldots : 0 : 1]$ satisfies the following properties:

(i) Restricted to the intersection $Y \cap V_+(x_{n+1}) = V_+(f - x_{n+1}^d, x_{n+1}) \subseteq \mathbb{P}^{n+1}_k$, the morphism g yields an isomorphism with X.

(ii) If $p \nmid d$, then for every closed point in the complement of $X \subseteq \mathbb{P}^n_k$ the fiber of g is a reduced scheme consisting of exactly d k-rational points.

(iii) If $p \mid d$, then the fiber of g over every closed point of $\mathbb{P}^n_k$ is non-reduced.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 58. *Closed points exist on quasi-compact schemes* (+ 2 extra points)
Let X be a quasi-compact scheme. Show that for every $x \in X$, there exists a closed point in $\overline{\{x\}} \subseteq X$.