RESEARCH STATEMENT

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1. Introduction

My research lies in the area of complex algebraic geometry and more specifically in birational geometry and Hodge theory. Algebraic geometry involves studying the solution spaces, called *varieties*, of polynomial equations. Even though the techniques I use are largely algebraic, I am interested in the intersection of many areas of mathematics, primarily motivated by geometry. For example, over $\mathbb{C}$, the points of an algebraic variety form a complex analytic space and therefore numerous questions I study are also of central importance in analytic and differential geometry.

1.1. **Brief overview of past work.** More details to follow in §2.

1.1.1. *Relative Fujita conjecture.* In birational geometry, a great deal of interest lies in the study of space of global sections (i.e. *linear system*) associated to a line bundle and when it produces morphisms to projective spaces (i.e. are *globally generated*). For instance, Takao Fujita, in 1988, conjectured the following:

Conjecture 1.1 (Fujita conjecture.)**.** *Let $\mathcal{L}$ be an ample line bundle on a smooth projective variety X with $\dim X = n$. Then $\omega_X \otimes \mathcal{L}^{\otimes \ell}$ is globally generated for all $\ell \geq n+1$.*

In particular, the bound on ℓ is *effective*, or in other words it is independent of the choice of $\mathcal{L}$. For example, the canonical bundle $\mathcal{O}_{\mathbb{P}^n}(-n-1)$ of the projective space $\mathbb{P}^n$ needs to be tensored by $\mathcal{O}_{\mathbb{P}^n}(1)$ at least $n+1$ times to obtain global generation. Even though the conjecture remains unsolved as of today, partial progress was made by Angehrn–Siu [AS95], Ein–Lazarsfeld [EL93], Heier [Hei02], Helmke [Hel97, Hel99], Kawamata [Kaw97], Reider [Rei88], Ye–Zhu [YZ15] et al. Most of these proofs trace their origins back to Bombieri [Bom73], Kawamata [Kaw84], Shokurov [Sho85] et al., who showed global generation for large tensor powers of canonical bundles (i.e *the pluricanonical bundles*), when the canonical bundles were only nef and big. For a brief discussion on the main techniques involved in these proofs, see §3.2.

The notion of *global generation* can be generalised to arbitrary sheaves. Moreover, similar effective generation results for the pushforward $f_*\omega_Y^{\otimes k}$ of the k^{th}-pluricanonical bundle on a projective variety Y under a morphism f, can be applied to study a plethora of properties, such as *weak positivity*, *nefness* of pushforwards, the *Iitaka conjecture* for subadditivity of *log-Kodaira dimensions* etc. In this direction, I obtained effective generation results for such pushforwards (see Thm. A). This, in particular, ensured the existence of their global sections. My proofs also handle the case when the varieties involved have log canonical singularities[1]. As applications, I proved positivity of such pushforwards (see Thm. B) and found some effective results towards the vanishing of certain cohomology groups (see Thm. C).

1.1.2. *Hodge ideals.* In a different direction, I worked in the theory of *Hodge ideals*, studied thoroughly by Mustaţă and Popa in a series of papers (e.g. [MP16, MP18]) in order to understand the singularities and Hodge theoretic properties of hypersurfaces in smooth varieties. For instance, the zeroth Hodge ideals, $I_0(D)$ of a hypersurface D, coincide with certain multiplier ideals, a tool, conceived in the work of Demailly [Dem93], Nadel [Nad90] and Siu [Siu93] to study singularities of D. For the sake of applications, it is fundamental for such gadgets to satisfy vanishing theorems. Part of my work proved a vanishing result for hypersurfaces in toric varieties (see Thm. D). I applied these vanishing to study the classical problem of when isolated singular points on a hypersurface impose independent conditions on linear systems. In general these sorts of vanishing

[1]Moreover, when (Y, Δ) is a log canonical pair, I studied the coherent sheaves $f_*\mathcal{O}_Y(k(K_Y + \Delta))$ (when makes sense) as they replace the roles of the usual pushforward. Pairs are relevant in the context of the study of birational models of varieties.

statements for Hodge ideals may have many useful applications. For instance, vanishing results for Abelian varieities [MP16, Thm. 28.2] were already applied to study singularities of theta divisors. Despite some partial statements with rather strong assumptions in [MP16] and for certain types of $\mathbb{Q}$-divisors in [MP18], a full picture is far from being understood. I will continue to work towards proving similar vanishing statements and finding their applications (see §3.3 for more detail).

1.2. **On-going work/future directions.** Details to follow in §3.

1.2.1. *Relative generation in a more general setting.* In the course of studying varieties in families, I became interested in studying families in more general settings, for instance, families of *semi-log-canonical* (slc) varieties (see Definition 3.2). These varieties arise in the context of moduli problems for higher dimensional varieties. Recently Fujino [Fuj18] (and Kovács–Patakfalvi [KP17]) used positivity properties of pushforwards to show projectivity of moduli spaces of stable varieties (and stable pairs). Earlier Viehweg [Vie95] used weak positivity to show quasi-projectivity in a similar setting. Thus, it is interesting to extend the Fujita-type effective generation and the related problems described above, to families of varieties or pairs relevant to the moduli theory.

I plan to extend the Hodge theoretic techniques from Kawamata's paper [Kaw02], which is one of the most important advances towards the relative Fujita conjecture, to the case when (Y, Δ) is a reduced simple normal crossing pair (defined in §3.1). Once this is done, I will deal with the semi-log-canonical (see Defn. 3.2) and the pluricanonical cases. See §3.1 for a proposed generalisation in this direction. I would like to use these results to study weak positivity, Iitaka conjecture and their implications in moduli theory.

1.2.2. *Future work.* I will investigate further whether the relative Fujita conjecture can be established at least in the cases where the original conjecture is known. I will explore whether the theory of Hodge modules could provide any new insights towards the relative Fujita conjecture. Finally, I would like to understand the connection between my algebraic approach using weak positivity and Deng and Iwai's analytic techniques.

Hodge ideals arised out of the theory of Hodge modules, a vast generalisation of variations of mixed Hodge structures and their degenerations, pioneered by Morihiko Saito [Sai88, Sai90]. Later, these ideals were studied by Mustaţă and Popa using tools from birational geometry as well. I am very interested in the interplay between the theory of Hodge modules and birational geometry. While continuing to investigate foundational questions, such as vanishing theorems or the size of Hodge ideals (e.g. when the hypersurface has only isolated singularities), I would also like to explore the deeper connections they have to singularities, using tools from Hodge modules, birational geometry and perverse sheaves.

2. Past work

2.1. **Fujita-type conjecture for direct images of pluricanonical bundles.** ([Dut17, DM18]) The main motivation behind what follows stems from the Fujita-type conjecture proposed by Popa and Schnell:

Conjecture 2.1 ([PS14, Conjecture 1.3])**.** *Let $f\colon Y \to X$ be a morphism of smooth projective varieties, with $\dim X = n$, and let $\mathcal{L}$ be an ample line bundle on X. For each $k \geq 1$, the sheaf*

$$f_*\omega_Y^{\otimes k} \otimes \mathcal{L}^{\otimes \ell}$$

is globally generated for all $\ell \geq k(n+1)$.

In my paper [Dut17] and more recently, jointly with Takumi Murayama [DM18], I proved several effective generation statements. The following theorem partially encodes the current state of affairs regarding the relative Fujita conjecture:

Theorem A ([Kaw02, PS14, Dut17, Den17, Iwa18, DM18])**.** *Let $f\colon Y \to X$ be a fibration (i.e. a surjective morphism with irreducible general fibre) of projective varieties with $\dim X = n$. Let*

(Y,Δ) be a pair and $\mathcal{L}$ a line bundle on X. Suppose moreover that there is a Cartier divisor P on Y such that $P \sim_{\mathbb{Q}} k(K_Y+\Delta)$ for some integer $k \geq 1$. Then the sheaf

$$f_*\mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell}$$

is globally generated on a Zariski open set U in the following cases:

- *When X is smooth and $\mathcal{L}$ is ample*
  - [Kawamata] *and when $k=1$, $\Delta=0$, Y smooth and f is smooth outside a simple normal crossing divisor, then for every integer $\ell \geq n+1$ when $n \leq 4$ or more generally, for every integer $\ell \geq \frac{n^2+n}{2}+1$. In this case $U=X$.*
  - [**D.,** Deng, Iwai, **D.–Murayama**] *and (Y,Δ) has Kawamata log terminal (klt) singularities and Δ is a $\mathbb{Q}$-divisor on Y, then for every integer $\ell \geq k(n+1)$ when $n \leq 4$ or more generally, for every integer $\ell > k(n+1)+\frac{n^2-n}{2}$. (Deng and later Iwai, also obtained similar bounds using analytic techniques).*
  - [**D.–Murayama**] *and when (Y,Δ) is a log-canonical pair and Δ is a $\mathbb{R}$-divisor on Y, then for every integer $\ell \geq k(n+1)+n^2-n$.*
- [Popa–Schnell] *When (Y,Δ) is a log-canonical $\mathbb{R}$-pair as above and X is possibly singular and $\mathcal{L}$ is ample and globally generated. Then for every integer $\ell \geq k(n+1)$. Again in this case $U=X$.*
- [**D.–Murayama**] *When X, (Y,Δ) are as above and $\mathcal{L}$ is only big and nef, then for every integer $\ell \geq k(n^2+1)$*

Remark 2.2. (1) The results from [Dut17, DM18] stated above, include descriptions of the locus of global generation up to a birational modification of Y.
(2) The case $k=1$: in the log canonical case [DM18, Thm. C, Cor. 3.2], we proved *an extension theorem* using Seshadri constants and a cohomological injectivity theorem [Fuj17b, Theorem 5.4.1]. Therefore, in this case the locus of generation is additionally contained in the Zariski open set, on which there is an effective bound on the Seshadri constant of $\mathcal{L}$ [EKL95]. The klt-case [Dut17, Prop. 1.3] is dealt with by partially extending Kawamata's statement in Theorem A two ways; first, to allow klt-singularities on (Y,Δ) (using Kawamata covering [Laz04, Theorem 4.1.12]), and second, to allow arbitrary divisors in the non-smooth locus of f.
(3) In the pluricanonical case, the proofs follow the main idea from [PS14], namely to reduce the problem to $k=1$ along with a boundary divisor. This is done either by choosing a minimum value for ℓ and then arguing by minimality, or by using using the weak positivity result discussed below.

2.2. **Weak positivity.** ([DM18]) As a consequence of Popa–Schnell's statement in Thm. A, together with Murayama, I proved that the pushforwards of the relative log-pluricanonical sheaves are weakly positive. The result was an improvement of similar statements by Campana [Cam04, Thm. 4.13], and later more generally by Fujino [Fuj17a, Thm. 1.1], but using a slightly weaker notion of weak positivity (see [Fuj17a, Definition 7.3] and the comments thereafter).

Theorem B ([DM18, Weak positivity]). *Using the notations from Thm. A, assume that (Y,Δ) has log-canonical singularities, Δ is an $\mathbb{R}$-divisor and X is normal, Gorenstein. Then there exists a non-empty Zariski open set $U \subseteq X$ so that for any ample line bundle $\mathcal{L}$, given an integer $a>0$, there exists an integer $b>0$ such that*

$$\big(f_*\mathcal{O}_Y\big(k(K_{Y/X}+\Delta)\big)\big)^{[ab]} \otimes \mathcal{L}^{\otimes b}$$

is generated by global sections at every $x \in U$. Here $\mathcal{F}^{[s]}$ denotes the reflexive hull of $\mathcal{F}^{\otimes s}$.

One should note that the locus of global generation U does not vary with the choice of $\mathcal{L}$ or a. Furthermore, the proof gives a description of U in terms of the morphism f, the boundary divisor Δ and the sheaf $\mathcal{O}_Y(k(K_{Y/X}+\Delta))$ up to a birational modification of Y.

On the other hand, the proof of this statement, goes via showing an *effective version* (in the sense of [PS14, Thm. 4.2]); i.e. there exist choices of $\mathcal{L}$, for which the integer b does not depend on a. Such effective statements restricted to curves are akin to the notion of nefness.

2.3. **Effective vanishing theorems.** ([DM18]) Together with Murayama, I also deduced a Kollár-type vanishing theorem for pushforwards of pluricanonical bundles.

Theorem C ([DM18, Thm. 5.3])**.** *Let $f\colon Y \to X$ be a smooth fibration of smooth projective varieties with $\dim X = n$. Let $\mathcal{L}$ be an ample line bundle on X.smoot Assume also that for some fixed integer $k \geq 1$, $\omega_Y^{\otimes k}$ is relatively base point free. Then*

$$H^i\big(X, f_*\omega_Y^{\otimes k} \otimes \mathcal{L}^{\otimes \ell}\big) = 0$$

for all $i > 0$ and for all $\ell \geq k(n+1) - n$. Moreover, if ω_X is semiample, $\ell \geq 1$.

The main idea of the proof is that after reducing to the case $k = 1$, we apply Kollár's [Kol95, Thm. 10.19] or Ambro-Fujino's ([Amb03, Thm. 3.2], [Fuj11, Thm. 6.3]) vanishing statements. Here the description of loci of global generation from Thm. A and that of weak positivity Thm. B comes to our advantage.

2.4. **Hodge ideals.** ([Dut18]) Loosely speaking, the Hodge ideals $I_k(D)$, indexed by the non-negative integers and associated to a reduced, effective divisor D on a smooth variety X, are a measure of the deficit between the Hodge filtration and the pole-order filtration of the Hodge module $\mathcal{O}_X(*D)$ (i.e. the sheaf of holomorphic functions with arbitrary poles along D). Therefore the k^{th} Hodge ideals can be defined via the following formula:

$$F_k\mathcal{O}_X(*D) \simeq I_k(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X((k+1)D)$$

where $F_\bullet$ is the Hodge filtration on $\mathcal{O}_X(*D)$. In [MP18] these notions were extended to effective $\mathbb{Q}$-divisors.

I proved a vanishing theorem for Hodge ideals on toric varieties. For simplicity let us assume that D is a reduced hypersurface on a smooth projective toric variety X with $\dim X = n$ and torus invariant divisors D_i, $i = 1, \cdots, d$. For a fixed integer $k \geq 0$, consider an ample line bundle $\mathcal{L}$ on X such that the line bundles $\mathcal{L}(D_{\tau_1} + \cdots + D_{\tau_p})$ are ample for all $1 \leq p \leq k$ and for all $\tau_i \in \{1, \cdots, d\}$. When the D and D_i's are ample, we also allow $\mathcal{L} \simeq \mathcal{O}_X$.

Theorem D ([Dut18, Thm. A])**.** *With the notation as above, we have*

$$H^i\Big(X, \omega_X((k+1)D) \otimes \mathcal{L} \otimes I_k(D)\Big) = 0 \text{ for all } i > 0.$$

For a statement for effective $\mathbb{Q}$-divisors, see [Dut18, Thm. A]. For $k = 0$ this is nothing but the Nadel vanishing theorem for multiplier ideals on toric varieties. The key inputs in the proof are the de Rham complexes associated to left $\mathcal{D}$-modules [HTT95, §4.2], the Euler type cotangent sequence for smooth toric varieties [CLS11, Theorem 8.1.6] and the Eagon-Northcott complexes [Laz04, Theorem B.2.2.].

2.4.1. *Application.* As an application, I described the linear systems, on which $S_m(D)$, the set of all isolated singular points on a hypersurface D with multiplicity at least m, imposes independent conditions. For instance, the isolated singular points of a surface of type (c, d) in $\mathbb{P}^2 \times \mathbb{P}^3$ impose independent conditions on surfaces of type $(2c-2, 2d-1)$. This is reminiscent of Severi-type result $2d - 5$ [PW06, Main Thm.] for singular points on a surface of degree d in $\mathbb{P}^3$.

3. Ongoing/Future directions

In this section, I describe some of the problems that I am currently exploring, or am interested in looking into in the near future.

3.1. **Effective generation in more general settings.** I plan to extend Kawamata's effective global generation results in Thm. A to the case of *simple normal crossing (snc)*. A pair (Y, Δ) is said to be snc if for all $y \in Y$, there is a Zariski open set U containing y, such that U can be embedded in a smooth variety U' with local coordinates $(y_1, \cdots, y_s, t_1, \cdots, t_p)$ such that $U = (y_1 \cdots y_s = 0)$ and $\Delta|_U = \sum_i a_i(t_i = 0)$, $a_i \in \mathbb{R}$. We say that (Y, Δ) is a *reduced snc* pair if $a_i = 1$ for all i.

Problem 1 (snc case)**.** *Let (Y,Δ) be a reduced simple normal crossing pair and let $f\colon Y \to X$ be a projective surjective morphism onto a smooth projective variety X. Assume that every stratum of (Y,Δ) is dominant onto X and that outside of a simple normal crossing divisor on X, f restricted to every stratum of (Y,Δ) is smooth. Then for an ample line bundle $\mathcal{L}$ on X, is there an integer N, depending only on $\dim X$, such that $R^i f_*\omega_Y(\Delta)\otimes\mathcal{L}^{\otimes \ell}$ is globally generated for all $\ell \geq N$ and for all i?*

Problem 1 is motivated by the following theorem, which was first shown by Kawamata [Kaw11, Thm. 1.1] for unipotent monodromy and later by Fujino-Fujisawa [FF14, Thm 1.1]:

Theorem 3.1 ([FF14])**.** *Let $f:(Y,\Delta)\to X$ be as above. Then $R^i f_*\omega_{Y/X}(\Delta)$ is locally free $\forall i$.*

The essential tool used in the proof of this statement is the theory of admissible variations of mixed Hodge structures (vmHs). Inspired by the techniques involved in the proof of Kawamata's global generation result in Thm. A, and its generalisation to variations of Hodge structure in [Wu17], I will extend these techniques to vmHs to show an effective generation statement as in Problem 1.

As a follow up, it will be interesting to investigate the case of *semi-log-canonical* (slc) pairs. The underlying space of an slc variety is not normal in general, but it is so-called *demi-normal*, i.e. it satisfies the S_2 condition and has at worst normal crossing singuarities in codimension 1.

Definition 3.2 (slc pairs)**.** A pair (Y,Δ) is said to be semi-log-canonical, if Y is demi-normal and the boundary divisor Δ does not intersect the codimension 1 singular points. Moreover, once normalised (Y,Δ) "becomes" log-canonical.

Problem 2 (slc case)**.** *Can we make an effective (generic) global generation statement as in Problem 1 when (Y,Δ) is a projective slc pair?*

In this case, I will implement Bierstone–Vera-Pacheco type [BVP13] resolution of singularities, and an extension of Kawamata covering techniques to reduce to the case of reduced snc pairs. I hope to obtain effective generation statements at least generically this way.

Finally, I will extend these results to the pluri-canonical setting using minimality arguments as in [PS14] (see also [Dut17]), to answer the following question:

Problem 3 (pluricanonical case)**.** *Let (Y,Δ) be a projective slc pair and let $f\colon Y \to X$ be a surjective morphism onto a smooth projective variety X of dimension n. Assume that there is a $k\in\mathbb{Z}_{>0}$ such that $k(K_Y+\Delta)$ is a Cartier divisor. Let $\mathcal{L}$ be an ample line bundle on X. Is there an integer N, depending only on n and k, such that $f_*\mathcal{O}_Y(k(K_Y+\Delta))\otimes\mathcal{L}^{\otimes\ell}$ is globally generated for all $\ell\geq N$?*

In this direction Fujino [Fuj15, Thm. 1.1] gave a partial answer when $\mathcal{O}_Y(k(K_Y+\Delta))$ is f-generated and $\mathcal{L}$ is additionally globally generated. He showed that $N=k(n+1)$ in this case.

3.1.1. *Applications.* As before, a natural direction to go from here is the following:

Problem 4 (weak positivity and Iitaka conjecture)**.** *In the setting of Problem 3 assuming that the fibres are slc pairs, can we say that $f_*\mathcal{O}_X(k(K_{Y/X}+\Delta))$ is weakly positive? Is the weak positivity effective for some line bundle? Assume that the restriction $K_{Y_\eta}+\Delta_\eta$ to the fibre over a generic point η, is big, then does the Iitaka-type inequality hold? i.e.*

$$\kappa(K_Y+\Delta)\geq\kappa(K_X)+\kappa(K_{Y_\eta}+\Delta_\eta),$$

where $\kappa(K_X)$ denotes the Iitaka dimension of ω_X?

The last question is in the flavour of a more recent Iitaka-type statement for log canonical fibrations in [KP17, §9]. Once an effective generation is obtained, I will use Viehweg's fibre product techniques [Vie83] (also see [Hör10] for an excellent exposition) to show effective *weak positivity* for pushforwards of the relative log-pluricanonical sheaves. I used this technique previously in [DM18].

3.2. **Relative Fujita conjecture and Hodge modules.** The modern approaches to the Fujita conjecture use the Nadel vanishing for multiplier ideals. Briefly, the idea is to produce, for every $x \in X$, an effective $\mathbb{Q}$-divisor D, whose multiplier ideal $\mathcal{J}(D)$, is strictly co-supported at x around x and satisfies Nadel vanishing, namely for suitable choices of ℓ

$$H^i(X, \omega_X \otimes \mathcal{L}^{\otimes \ell} \otimes \mathcal{J}(D)) = 0 \text{ for all } i > 0$$

This implies the desired global generation for $\omega_X \otimes \mathcal{L}^{\otimes \ell}$ at x.

Unfortunately, unlike ω_X, Nadel-type vanishings are not known in general for pushforwards. However, it is known that $f_*\omega_Y$ is the lowest filtered piece of the Hodge module corresponding to the intermediate extension of the local system $f_*\mathbb{C}_{Y_0}$, where Y_0 is the preimage of the smooth locus of f. Deligne's extensions [Del70, §II.5] give an explicit way to understand such extensions when the morphism is smooth outside an snc divisor. As an upshot one obtains that $f_*\omega_Y$ is a locally free $\mathcal{O}_X$-module. These came to an advantage to Kawamata [Kaw02] and are not true in general.

In a more general setting, Wu, in his thesis [Wu17], akin to the multiplier ideals, defined *multiplier subsheaves* $\mathcal{J}(M; B)$ associated to a divisor B and a pure Hodge module M extending a (polarised) vHs on X_0 and he proved Nadel-type vanishing for these subsheaves. Furthermore, Wu interpreted some of Kawamata's constructions from [Kaw02], in terms of the V-filtrations on the $\mathcal{D}$-module underlying the minimal extension. These generalised constructions could be potential inputs in obtaining global generation results for arbitrary morphisms. With the goal of extending the relative Fujita conjecture to the cases where the original conjecture is known, I will explore the following:

Problem 5. *What are some useful properties that $\mathcal{J}(M; B)$ satisfies? Can one use these to obtain an effective global generation statement for $f_*\mathcal{O}_Y(K_Y)$, when X is a surface (since in this case $f_*\omega_Y$ is, in fact, locally free) or f is flat, or when the singularities of the fibres of f are rather special?*

3.2.1. *Relationship to the analytic techniques.* Deng and Iwai in [Den17, Iwa18] used analytic extension theorems for pluricanonical bundles to show an effective generation (see Thm. A) in the klt case. Even though results from [Dut17, DM18] in Theorem A use purely algebraic techniques and are apparently disjoint from the analytic methods, my longer term goal is to comprehend these deep analytic techniques to understand the relation between these two approaches and understand whether these methods could be used to improve results in the algebraic setting.

3.3. **Hodge ideals.** A sizable amount of foundational work is yet to be done in the theory Hodge ideals. Some possible frameworks where vanishing theorems could be shown are when the divisor has simple normal crossings or when it has ordinary isolated singularities. In the latter case, I plan to relate to the Hodge ideals, other geometric data like Milnor numbers or topological links of singularities in order to obtain a deeper understanding of these ideals.

On the other hand, the current approaches to the vanishing theorems work only for certain types of $\mathbb{Q}$-divisors. The main difficulty is that, in the $\mathbb{Q}$-divisor setting, there is no globally defined replacement for the Hodge module $\mathcal{O}_X(*D)$ when the effective divisor D is not reduced. Although the lack of such replacement does not pose any trouble to define the Hodge ideals globally, it does so while proving vanishing for them. In [MP18] it was shown that it is possible to find a globally defined Hodge module after taking a finite cover of X, however it is not clear how the Hodge ideals change under finite morphisms. With the hope of gaining some insights on how to get around this difficulty, I will find an answer to the following

Problem 6. *Let $p : X' \to X$ be a generically finite or finite morphism of smooth varieties. Let D be an effective $\mathbb{Q}$-divisor on X, then is there a morphism $\varphi : p_*I_k(p^*D) \to I_k(D)$, so that when D is reduced, φ is induced by the trace morphism $p_*\omega_{X'} \to \omega_X$?*

Similar problems in the case of multiplier ideals were dealt in [BST11].

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