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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 12

Exercise 65. *Geometric reducedness* (4 points)
Let k be a field, let A and B be k-algebras, and let $k \subseteq K$ be a field extension..

(i) Assume that A is non-reduced. Show that $A \otimes_k K$ is non-reduced.

(ii) Show that if $A \otimes_k B$ is non-reduced, then there exist finitely generated k-algebras $A' \subseteq A$ and $B' \subseteq B$ such that $A' \otimes_k B'$ is non-reduced.

(iii) Show that if K is finitely generated and separable over k, then A is reduced if and only if $A \otimes_k K$ is reduced.

(iv) Conclude that a scheme X over a perfect field is reduced if and only if it is geometrically reduced.

Exercise 66. *An alternative definition of the cotangent sheaf* (4 points)
Let A be a ring and B an A-algebra. Let $\rho : B \otimes_A B \to B$ be the multiplication map. Let $I = \mathrm{Ker}(\rho)$. We consider $B \otimes_A B$ as a B-module via multiplication on the right, so that I/I^2 becomes a B-module. Consider the map

$$\begin{array}{rcl} d : B & \to & I/I^2 \\ b & \mapsto & b \otimes 1 - 1 \otimes b. \end{array}$$

(i) Show that d is an A-linear derivation.

(ii) Show that the pair $(I/I^2, d)$ satisfies the universal property of the module of relative differentials of B over A.

(iii) Let $f : X \to Y$ be a morphism of schemes, let $\Delta : X \to X \times_Y X$ be the diagonal, and let $\mathcal{I}$ be the kernel of $\Delta^\sharp : \mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$. Show that $\Omega_{X/Y} \cong \Delta^*(\mathcal{I}/\mathcal{I}^2)$.

The following exercise uses the affine Jacobian criterion, which we will prove in the lecture on Monday.

Exercise 67. *Projective Jacobian criterion* (4 points)
Let k be a field and let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme given by the saturated ideal $(f_1, \ldots, f_m)$ with f_i homogeneous of degree d_i. We define the *Jacobian matrix* J_X of X as

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \cdots & \frac{\partial f_m}{\partial x_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

(i) Show the Euler identity: For every homogeneous polynomial f of degree d, one has $\sum_{i=0}^n x_i \frac{\partial f}{\partial x_i} = df$.

Due 20.01.2023, 2pm

(ii) Show that the rank of J_X at a point $x \in X$ is well-defined and equals the rank of the Jacobian matrix of the affine scheme $X \cap D_+(x_i)$ for any $D_+(x_i)$ containing x. Conclude that if $k(x)$ is separable over k, then X is smooth at $x \in X$ if and only if $J_X(x)$ has co-rank $\dim \mathcal{O}_{X,x} - \text{tr.deg.}(k(x)/k)$.

Exercise 68. *Some explicit computations* (4 points)
Let k be an algebraically closed field. Describe the non-smooth points of the following schemes over k:

(i) $V_+(y^2z - x^3 + xz^2) \subseteq \mathbb{P}^2_k$.

(ii) $V_+(\sum_{i=0}^r x_i^2) \subseteq \mathbb{P}^n_k$ for $1 \leq r \leq n$.

(iii) $X = V_+(f - x_{n+1}^d) \subseteq \mathbb{P}^{n+1}_k$ for $f \in k[x_0, \dots, x_n]_d$. Can X be regular if $\text{char}(k) \mid d$?

Exercise 69. *Exterior powers of sheaves* (4 points)
Recall that if A is a ring and M is an A-module, the *tensor algebra* of M is the graded (non-commutative) A-algebra $T(M) = \bigoplus_{n=0}^{\infty} T^n(M)$ where $T^n(M) = M^{\otimes n}$. The *exterior algebra* $\Lambda(M)$ is the quotient of $T(M)$ by the homogeneous ideal generated by the $x \otimes x$ with $x \in M$ and the n-th exterior power of M is $\Lambda(M)_n$.

Now, if $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$, we define the corresponding operations for $\mathcal{F}$ by applying them to each $\mathcal{F}(U)$ and passing to the sheafification.

(i) Assume that $\mathcal{F}$ is locally free of rank n. Show that $\Lambda^r(\mathcal{F})$ is locally free of rank $\binom{n}{r}$. In particular, $\det(\mathcal{F}) := \Lambda^n\mathcal{F}$ is an invertible sheaf which we call the *determinant* of $\mathcal{F}$.

(ii) Assume that $\mathcal{F}$ is locally free of rank n. Show that the natural map $\Lambda^r(\mathcal{F}) \otimes_{\mathcal{O}_X} \Lambda^{n-r}(\mathcal{F}) \to \Lambda^n(\mathcal{F})$ induces an isomorphism $\Lambda^r(\mathcal{F}) \cong \mathcal{H}om_{\mathcal{O}_X}(\Lambda^{n-r}(\mathcal{F}), \Lambda^n(\mathcal{F}))$

(iii) Let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a short exact sequence of locally free sheaves of finite rank. Show that, for any r, there exists a sequence of subsheaves

$$0 = F^{r+1} \subseteq F^r \subseteq \ldots \subseteq F^1 \subseteq F^0 = \Lambda^r(\mathcal{F})$$

such that $F^p/F^{p+1} \cong \Lambda^p(\mathcal{F}') \otimes \Lambda^{r-p}(\mathcal{F}'')$. Deduce that $\det(\mathcal{F}) \cong \det(\mathcal{F}') \otimes_{\mathcal{O}_X} \det(\mathcal{F}'')$.

(Analogously, one can define the symmetric algebra and the symmetric power of sheaves of $\mathcal{O}_X$-modules and then an analogous filtration exists for symmetric powers with respect to short exact sequences. We will come back to this in AG2.)

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 70. *Smoothness in characteristic* 0 (+ 4 extra points)
The goal of this exercise is to show that a scheme X of finite type over a field k of characteristic 0 is smooth if and only if $\Omega_{X/k}$ is locally free. For this, we have to show that if $\Omega_{X/k}$ is locally free, then it automatically has the correct rank.

(i) Show that $\Omega_{(\mathbb{F}_p[x]/x^p)/\mathbb{F}_p}$ is free of positive rank. In particular, the statement we want to prove is false in characteristic $p > 0$, even over perfect fields.

(ii) Let A be a k-algebra. Let $a \in A$ be an element such that Ada is a direct summand of $\Omega_{A/k}$. Show that a is not nilpotent.

(iii) Conclude that X is smooth if $\Omega_{X/k}$ is locally free.

(Remark: The Zariski–Lipman conjecture asks whether it is enough to assume that the dual of $\Omega_{X/k}$ is locally free to guarantee that X is smooth over k. This conjecture is known under certain assumptions on the singularities of X)