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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 13

Exercise 71. *Examples of smooth hypersurfaces* (3 points)
Let k be any field and d a natural number. Give an explicit example of a smooth hypersurface $V_+(f_d)$, where f_d is homogeneous of degree d.

(Remark: Next term, we will prove a theorem of Bertini, which implies that if k is infinite, then a "general choice" of f_d will yield a smooth hypersurface.)

Exercise 72. *Projective tangent spaces* (4 points)
Let k be a field, let $X = V_+(f_1, \ldots, f_m) \subseteq \mathbb{P}^n_k$ and let $x = [a_0 : \ldots : a_n] \in X$ be a k-rational point. The *projective tangent space of X at x* is the closed subscheme $T_x X \subseteq \mathbb{P}^n_k$ determined by the linear polynomials

$$\sum_{j=0}^{n} \frac{\partial f_i}{\partial x_j}(x) x_j.$$

(i) Prove that $T_x X$ does not depend on the choice of the f_i.

(ii) Show that $x \in X$ is a smooth point if and only if $\dim(T_x X) = \dim \mathcal{O}_{X,x}$.

(iii) Calculate the tangent line of the curve $C = V_+(y^2 z - x^3 + xz^2) \subseteq \mathbb{P}^2_k$ at $[0 : 1 : 0]$.

Exercise 73. *Slicing criterion for regularity* (4 points)
The goal of this exercise is to show that an effective Cartier divisor that passes through a non-regular point cannot be regular.

(i) Let A be a ring and let $x \in \operatorname{Spec} A$ be a closed point with maximal ideal $\mathfrak{m}$. Let $f \in \mathfrak{m}$. Show that the Zariski tangent space (from Exercise 17) of $\operatorname{Spec} A/(f)$ at x is isomorphic to the subspace of the Zariski tangent space of $\operatorname{Spec} A$ at x that is annihilated by f.

(ii) Let X be a Noetherian scheme and $D \subseteq X$ an effective Cartier divisor. Assume that $x \in D$ is regular. Show that $x \in X$ is regular.

Exercise 74. *Left-exactness of conormal sequence* (4 points)
Let k be a perfect field and X a smooth k-scheme of finite type. Let $Z \subseteq X$ be a reduced closed subscheme with ideal sheaf $\mathcal{I}_Z$. Assume that $\mathcal{I}_Z/\mathcal{I}_Z^2$ is locally free. Show that the conormal sequence

$$0 \to \mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{\delta} \Omega_{X/k} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \Omega_{Z/k} \to 0$$

is exact.

Please turn over

Due 27.01.2023, 2pm

Exercise 75. *Frobenius and differentials* (5 points)
All schemes considered in this exercise are schemes over $\mathbb{F}_p$. For a scheme X, the *absolute Frobenius* $F_X : X \to X$ is defined as $(\mathrm{id}_X, F^\sharp)$, where $F^\sharp(U)$ is the p-th power map (which is a ring homomorphism, since $\mathcal{O}_X(U)$ has characteristic p).

(i) Show that, for any $\mathbb{F}_p$-scheme S, base change along F_S determines a functor $(-)^{(p/S)} : (\mathrm{Sch}/S) \to (\mathrm{Sch}/S)$. For any S-scheme X, the morphism $F_{X/S} = (F_X, \pi_X) : X \to X^{(p/S)}$ is called *S-linear (or relative) Frobenius*. If S is clear from the context, we write $(-)^{(p)}$ instead of $(-)^{(p/S)}$.

(ii) Show that for any morphism $f : X \to Y$ of S-schemes, we have $f \circ F_X = F_Y \circ f$ and $f^{(p)} \circ F_{X/S} = F_{Y/S} \circ f$.

(iii) Show that $F_{X/S}$ is a universal homeomorphism.

(iv) If $S = \mathrm{Spec}\ A$ for some ring A and $X = \mathrm{Spec}\ A[x_1, \ldots, x_n]/I$ for an ideal I, show that $X^{(p/S)} \cong A[x_1, \ldots, x_n]/I^{(p)}$, where $I^{(p)} = (\{f^{(p)}\}_{f \in I})$, where $f^{(p)}$ denotes raising the coefficients of a polynomial f to the p-th power. Show that, under this identification, the map $F_{X/S}$ is induced by the A-linear map determined by $x_i \mapsto x_i^p$.

(v) Conclude that, for every morphism of $\mathbb{F}_p$-schemes $X \to Y$, there is an isomorphism $\Omega_{X/Y} \cong \Omega_{X/X^{(p/Y)}}$.

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 76. *Universally injective morphisms* (+ 4 extra points)
Recall that a field extension $k \subseteq K$ is called *purely inseparable* if $\mathrm{char}(k) = p \neq 0$ and for every $a \in K$, there exists $n > 0$ such that $a^{p^n} \in k$.

Let $f : X \to Y$ be a morphism of schemes. Show that the following properties are equivalent:

(i) f is universally injective.

(ii) f is *radicial*, i.e., it is injective and for all points $x \in X$, $k(x)$ is a trivial or purely inseparable extension of $k(f(x))$.

(iii) For every field k, the morphism $\mathrm{Hom}(\mathrm{Spec}\ k, X) \to \mathrm{Hom}(\mathrm{Spec}\ k, Y)$ is injective.

Conclude that, for every field k, the inclusion $k[x^2, x^3] \to k[x]$ induces a universal homeomorphism of affine schemes.

GROW@Bonn 2023

Dear students,

We are delighted to tell you about the upcoming conference "Graduate Research Opportunities for Women at Bonn (GROW@Bonn),"which we are organising with the support of the Hausdorff Center for Mathematics and Max Planck Institute for Mathematics. (Poster attached) https://www.hcm.uni-bonn.de/grow2023/ (Mar 30 - 31, 2023)

The conference is aimed at female, non-binary and gender-diverse bachelor and master students, with the objective of promoting graduate degrees, including a PhD in mathematics, as career paths. As Bonn locals you are invited to participate in the entirety of the conference irrespective of your gender. However, in the event of full seminar room we will give priorities to registered and accepted participants. If you have never considered doing a PhD, the confrence is just right for you!

We are following the model started in the US midwest by Bryna Kra.

https://www.ams.org/journals/notices/202005/rnoti-p724.pdf

The participation is free of cost. Reasonable cost of travel will be reimbursed and accommodation will be provided by HCM, Bonn to selected participants. So invite your female/non-binary/gender diverse math friends from other universities and encourage them to apply!

In order to be considered for full financial support mentioned above, please apply on our website by Feb 20, 2023.

Planned format:

Thu March 30, welcome, 2 panel discussions, research talk and Plenary talk by Karen Vogtmann (Warwick/Cornell).

Fri Mar 31: research talks, short research talks by PhD students, 1 panel discussion, open discussions.

The panels will be on:

- What to do with a PhD in mathematics - What is it like to do research in mathematics - Nuts and bolts of applying to masters and PhD programs in Europe and the US.

The conference will also offer some/all of the following networking aspects:

1) Exploring Bonn with local mathematicians and math students. 2) Interacting with and asking questions to local mathematicians over coffee and cookies. 3) conference dinner. 4) Having a lunch round table at Mensa on Friday with local mathematicians.

Please do not hesitate to reach out for any questions or concerns: grow-bonn@hcm.uni-bonn.de

Best wishes,

GROW@Bonn organization team