INTERCITY+ANTWERP READING SEMINAR ON HODGE CONJECTURE FOR CATEGORIES

SPRING 2024

The goal of this learning seminar is to learn about the integral Hodge conjecture for Calabi–Yau 2 categories after Alex Perry: https://arxiv.org/pdf/2004.03163.pdf.

Day 1: February 23 (Amsterdam)

Talk 1. *Motivation and state of the Hodge cojecture.*
Speaker: Mingmin Shen (Amsterdam)

Define Voisin group. Kollár's Trento example: failure of Hodge conjecture. Intermediate Jacobian. Variational Hodge conjecture. State of affairs. https://webusers.imj-prg.fr/~claire.voisin/Articlesweb/voisinhodge.pdf

Talk 2. *The Kuznetsov component of a cubic fourfold*
Speaker: Céline Fietz (Leiden)

The simplest example of a CY2 category is the derived category $D^b(S)$ of a K3 or abelian surface. The first "non-trivial" example is the Kuznetsov component $\mathcal{A}_X$ of a cubic fourfold X. In https://arxiv.org/pdf/1211.3758.pdf, which is an important inspiration for Perry's paper, Addington and Thomas proved that for X generic, $\mathcal{A}_X$ is equivalent to $D^b(S)$ for some K3 surface S if and only if there is a Hodge-theoretic relation between X and S. We spend 3 talks on reading Addington–Thomas.

- Recall semi-orthogonal decompositions, e.g. following §2.1 of https://arxiv.org/pdf/0808.3351.pdf, or §2.3 and §3.2 of https://arxiv.org/pdf/0904.4330v1.pdf. Introduce the Kuznetsov component $\mathcal{A}_X$ of a cubic fourfold X.
- State that there are cases for which $\mathcal{A}_X$ is equivalent to the derived category of a K3 surface or twisted K3 surface – recall what the second means. Two examples are given by §3 and §4 of https://arxiv.org/pdf/0808.3351.pdf; the second is more important for the proof of Addington–Thomas' main theorem.
- Explain the statement of Addington–Thomas' main theorem and the strategy of the proof.
- Recall the definition of Algebraic K-theory (and "numerical K-theory" as defined by Addington–Thomas).

Talk 3. *Topological K-theory and the Mukai lattice*
Speaker: Dion Leijnse (Amsterdam)

- Introduce topological K-theory (reference: Atiyah–Hirzebruch?)
- Define the Mukai lattice of a K3 surface and of a cubic fourfold, following §2 of Addington–Thomas. You can skip Proposition 2.5 (and possibly Proposition 2.4)
- State Theorem 3.1 of Addington–Thomas and the equivalent condition (1'). We skip the proof. Prove the "easy" direction of Addington–Thomas' main theorem.
- State Theorem 4.1 in Addington–Thomas. We skip the proof, but you may want to give the idea, explained between the statements of Theorem 4.1 and Lemma 4.2.

- Explain §5 of Addington–Thomas. The proofs of Propositions 5.1 and 5.2 can just be sketched.

Day 2: March 8 (Leiden)

Talk 4. *Hochschild (co)homology and deformations*
Speaker: Emma Brakkee (Leiden)

- Spend some time on Hochschild (co)homology, following §6 of Addington–Thomas (+more references?). Proposition 6.2 is used for the "inverse direction" of the main theorem – state it and if possible, prove it.
- State Theorem 7.1. The lines below the statement explain the main point of the proof which you may want to explain, but we skip the parts about Atiyah classes.
- Say that Theorem 3.1 can be extended by T1-lifting. You could state Proposition 6.6, but if it doesn't add any value, just skip it. State Theorem 7.7.
- Prove the "inverse direction" of the main theorem.

Talk 5. *Hochschild Homology for admissble subcategory*
Speaker: Francesca Leonardi / Márton Habliscek (Leiden)

By a result of Orlov, Hochschild Homology is a derived invariant. The goal of the next two talks is to discuss Hochschild (co)homology for an admissible subcategory $\mathcal{A}$ of the derived category of coherent sheaf $D^b(X)$ following §4-6 of https://arxiv.org/pdf/0904.4330v1.pdf and the summary slides: https://www.science.unitn.it/~pignatel/PoAV/talks/Kuznetsov.pdf. Define admissible subcategory. Existence of strong generators for $D^b(X)$ and its admissible subcategories (reference?). Present Lemma 4.3: re-intepretation of $\mathcal{A}$ as derived category of perfect complexes over dg-algebra of its strong generator (this is called "enhancement"). Define Hochschild (co)homology of a dg-algebra (beginning of §4.2). Use this to define Hochschild (co)homology of a enhanced admissible subcategory. Combining the results of Theorem 4.5 and Proposition 4.6 observe that Hochschild (c)ohomology of an admissible subcategory of $D^b(X)$ can be written in terms of the kernel of the projection $D^b(X) \to \mathcal{A}$. Functoriality of Hochschild (co)homology §6.

Talk 6. *HH for admissble subcategory and example*
Speaker: Francesca Leonardi / Márton Habliscek (Leiden)

Follow §7, 8 and 9 of https://arxiv.org/pdf/0904.4330v1.pdf and the summary slides https://www.science.unitn.it/~pignatel/PoAV/talks/Kuznetsov.pdf. Show additivity of Hochschild (co)homology over a given semi-orthogonal decomposition. Examples: 1) present Theorem 8.8 replacing Grassmannians by projective space. 2) Theorem 8.9: Fano threefold of index 2. State the non-vanishing conjecture and the Corollaries from §9. Remark that non-vanishing conjecture is true if the triangulated category is Calabi–Yau (define §6: https://arxiv.org/pdf/2004.03163.pdf) because in this case the HH is a free-module over Hochschild cohomology.

Day 3: April 5 (Utrecht)

Talk 7: *S-linear stable infinity categories*
Speaker: Lenny Taelman (Amsterdam)

Follow §2-4 of https://arxiv.org/pdf/2004.03163.pdf. If required, consult Part I of https://arxiv.org/pdf/1804.00132.pdf.

Talk 8. *Hodge theory of categories*
Speaker: Wouter Rienks (Amsterdam)

Follow §5.1 and 5.2 of https://arxiv.org/pdf/2004.03163.pdf. The main goal is to state the non-commutative (variational) Hodge conjecture (Conj. 5.11 and 5.21) and make sense of it. Define non-commutative Voisin group (Def. 5.15). Results needed to deal with the relative setting should be black-boxed or quoted from Talk 7, i.e. results from §2-4. Focus only on cubic 4-folds.

Talk 9. *CY2 categories and moduli of objects*
Speaker: Yagna Dutta (Leiden)

Follow §6 and 7 from https://arxiv.org/pdf/2004.03163.pdf. Focus only on cubic 4-folds.

Day 4: May (Antwerp)

Talk 10. *Proofs* **Speaker**:

Provide examples for the failure of the Hodge conjecture: Integral Hodge conjecture vs non-commutative integral Hodge conjecture (Prop. 5.16). Define non-commutative Intermediate Jacobian §5.3.State and prove the integral Hodge conjecture for CY2 categories. Follow Introduction and §8 of https://arxiv.org/pdf/2004.03163.pdf.

Further topics? Addington–Elmanto: https://arxiv.org/pdf/2312.06930.pdf (Dec 2023)?