

# SEMINAR ON HODGE MODULES

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## 1. INTRODUCTION

### 1.1. Notation.

- (1) For a map  $f : B \rightarrow \mathbb{C}$ ,  $B_t = f^{-1}(t)$  for  $t \in \mathbb{C}$ .
- (2)  $B_{\leq s} := \{z \in B \mid |f(z)| \leq s\}$
- (3)  $S^{n-1}$  and  $B^n$  will always denote a real sphere and a real ball respectively, no matter whether they come with a suffix or not.
- (4) For the sake of consistency, unlike in the talk, I am reverting to Voisin's notation of cone over vanishing sphere. Please replace  $B_{\leq s}^n$  by  $B_t^n$ .

**1.2. Goal.** The goal of the next two talks in this seminar is to understand nearby and vanishing cycles in the classical setting of Lefschetz degeneration. We will use that to show that if  $f : X \rightarrow \Delta$  is a proper holomorphic map from an  $n$ -dimensional complex variety to a disk, that is a submersion over  $\Delta^*$  and  $x_0$  is a non-degenerate critical point over  $0 \in \Delta$ , then there exists a deformation retraction of  $X$  onto  $X_t \cup B^n$  where  $B^n$  is an  $n$ -dimensional ball glued to  $X_t$  along the “vanishing” sphere  $S_t^{n-1}$  for  $t \in \Delta^*$ . Stealing the terminology from Morse theory we will refer to this phenomenon as homotopy type around critical points.

We will use this to give a proof of Lefschetz hyperplane theorem. For a smooth hyperplane section  $j : Y \hookrightarrow X$  of an  $n$ -dimensional smooth projective variety, the vanishing homology is defined by:

$$H_*(Y, \mathbb{Z})_{\text{van}} := \ker(j_* : H_*(Y, \mathbb{Z}) \rightarrow H_*(X, \mathbb{Z}))$$

We will eventually show the following theorem:

**Theorem 1.1.** *Let  $Y$  be a smooth hyperplane section  $j : Y \hookrightarrow X$  of an  $n$ -dimensional smooth projective variety. Then,*

- (1)  $H_n(X, Y, \mathbb{Z})$  is generated by classes of the cones on the vanishing spheres.
- (2)  $H_n(Y, \mathbb{Z})_{\text{van}}$  is generated by the classes of the vanishing spheres.
- (3)  $j_* : H_k(Y, \mathbb{Z}) \rightarrow H_k(X, \mathbb{Z})$  is an isomorphism for  $k < n - 1$ .

## 2. VANISHING SPHERES IN THE CONTEXT OF LEFSCHETZ DEGENERATION

Let  $f : \mathbb{C}^n \rightarrow \mathbb{C}$  be defined by  $f(z) = \sum z_i^2$ . Note that the point 0 is a non-degenerate (i.e.  $\text{Hess}_0 f$  is non singular) critical point or an ordinary double point. Let  $B$  be a ball of radius  $r$  around 0, then  $f(B)$  has values in a disk of radius  $r^2$ . Such a map  $f : B \rightarrow \Delta$  is called *Lefschetz degeneration*. Define,

$$S_t^{n-1} := \{z \in B \mid z_i = \sqrt{s} x_i e^{i\theta}, x_i \in \mathbb{R} \text{ and } \sum_{1 \leq i \leq n} x_i^2 \leq 1\} \subset B_t$$

where  $t = se^{i2\theta} \in \Delta$  and

$$B_t^n := \{z \in B \mid z_i = \sqrt{s}x_i e^{i\theta}, x_i \in \mathbb{R} \text{ and } \sum_{1 \leq i \leq n} x_i^2 \leq 1\} \subset B_{\leq s} = \{z \in B \mid |f(x)| \leq s\}$$

The sphere is called a vanishing sphere and the ball  $B_t^n$  is called the cone over the vanishing sphere.

**Definition 2.1** (vanishing cycle). The homology class  $\delta \in H_{n-1}(B_t, \mathbb{Z})$  with a choice of orientation on  $S_t^{n-1}$  is called the vanishing cycle of the Lefschetz degeneration  $f : B \rightarrow \Delta$ .

### 2.1. Retraction statement:

**Theorem 2.2.** *Let  $f : X \rightarrow \Delta$  be a proper holomorphic map from an  $n$ -dimensional complex variety to a disk, that is a submersion over  $\Delta^*$  and  $x_0$  is a non-degenerate critical point over  $0 \in \Delta$ , then there exists a deformation retraction of  $X$  onto  $X_t \cup B^n$  where  $B^n$  is an  $n$ -dimensional ball glued to  $X_t$  along the “vanishing” sphere  $S_t^{n-1}$  for  $t \in \Delta^*$ .*

*sketch:* Since  $f$  is a proper submersion over  $\Delta^*$ , by Ehresmann’s theorem it suffices to show the result for an arbitrary small disk centred around 0.

By holomorphic Morse lemma, we can assume that there is a ball  $B$  around  $x_0$  and holomorphic coordinates  $z_1, \dots, z_n$  such that  $f(z) = \sum z_i^2$  for  $f = f|_B : B \rightarrow \Delta$ . Now we are in situation of a Lefschetz degeneration. The homotopy type properties around critical points of a Morse function tells us that by further shrinking  $\Delta$  we have that  $B_\Delta := f^{-1}(\Delta)$ , the set  $B_\Delta$  deformation retracts onto  $B_t \cup B_{<s}^n$  where  $s, t$  and  $B^n$  are as above and  $B_{<s}^n$  is glued to  $B_t$  along  $S_t^{n-1}$ .

Now let  $B^0$  be the interior of  $B$ . Let,  $B_\Delta^0 := B^0 \cap B_\Delta$ . Then  $f : X_\Delta \setminus B_\Delta^0 \rightarrow \Delta$  is a fibration of varieties with boundary, by Ehresmann we there is a deformation of  $X_\Delta \setminus B_\Delta^0$  to  $X_t$  via the trivialisation  $X_t \times \Delta$ . The boundary,  $S_\Delta := B_\Delta \cap \partial B$  also retracts onto  $S_t$  via the same trivialisation. These two deformation processes are compatible because the deformation of  $B_\Delta$  can be chosen to preserve a specified retraction of  $S_\Delta$  to  $S_t$ . Hence the result.  $\square$

### 2.2. Theorem 1.1 for Lefschetz degeneration:

**Corollary 2.3.** *Let  $i : X_t \hookrightarrow X_\Delta$  be the inclusion. Then  $i_* : H_k(X_t, \mathbb{Z}) \rightarrow H_k(X_\Delta, \mathbb{Z})$  is an isomorphism for  $k < n - 1$  and the surjective for  $k = n - 1$ . Moreover, the kernel of  $i_*$  is generated by the class of ‘the’ vanishing sphere  $S_t^{n-1}$  of  $X_t$  for  $k = n - 1$ .*

*Proof.* By the discussion in the earlier section we know that  $X_\Delta$  has the homotopy type of  $X_t$  with  $B_t^n$  glued to it along  $S_t^{n-1}$  of  $X_t$ . Therefore,  $H_*(X_\Delta, \mathbb{Z}) \simeq H_*(X_t \cup_{S_t^{n-1}} B_t^n, \mathbb{Z})$ . Moreover, by excision we have

$$H_*(X_t \cup_{S_t^{n-1}} B_t^n, X_t, \mathbb{Z}) \simeq H_*(B^n, S^{n-1}, \mathbb{Z}).$$

We know,  $H_k(B^n, S^{n-1}, \mathbb{Z}) = 0$  for  $k \leq n - 1$ . Hence the isomorphism. For the second part, we need to write down the relative homology sequence for  $(B^n, S^{n-1})$  and  $(X_\Delta, X_t)$  along with the maps between them. This will show that the kernel is precisely given by  $\delta$  as defined in 2.1.  $\square$

## 3. LEFSCHETZ PENCILS:

A Pencil of hyperplane is a family of hyperplanes parametrised  $\mathbb{P}^1$ . Consider the setting of theorem 1.1. Suppose  $\{X_t\}$  is a pencil of hyperplane passing through  $Y = X_0 = (\sigma_0 = 0)$  i.e. for a hyperplane  $\sigma_\infty$ ,  $X_t := (\sigma_0 + t\sigma_\infty = 0)$ . Suppose also that the base locus  $B := Z(\sigma_0, \sigma_\infty)$  is smooth, or in other words  $(d\sigma_0, d\sigma_\infty)$  are linearly independent along  $B$ . Then the variety,

$$\tilde{X} := Bl_B X = \{(x, t) \in X \times \mathbb{P}^1 \mid \sigma_0(x) + t\sigma_\infty(x) = 0\}$$

is smooth and  $f : \tilde{X} \rightarrow \mathbb{P}^1$  is a proper surjective morphism submersion outside the singularities of  $X_t$ . If we would like to apply the techniques of Morse theory around singularities of  $X_t$  for the map  $f : X - X_\infty \rightarrow \mathbb{C}$  we would need to ensure that  $X_t$  do not have any worse than an ordinary double point singularity. Lefschetz proved that given a smooth hyperplane  $Y$  such a family always exists. Such a family is known as *Lefschetz Pencils*. More generally:

**Definition 3.1** (Lefschetz Pencils:). A Lefschetz pencil  $(X_t)_{\mathbb{P}^1}$  is a pencil of hypersurfaces satisfying the following conditions:

- (1) The base locus  $B$  is smooth of codimension 2
- (2) Every hypersurface  $X_t$  has at most one ordinary double point as singularity.

**Definition 3.2** (ordinary double point:). For a map  $f : X \rightarrow \mathbb{C}$ , we will say that  $X_t$  has an ordinary double point at 0 if 0 is a non-degenerate critical point of  $f$ .

**3.1. Existence of Lefschetz pencils:** Let  $X \subset \mathbb{P}^N$  be a non-degenerate (smooth??) subvariety of  $\mathbb{P}^N$  i.e. the map

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \rightarrow H^0(X, \mathcal{O}_X(1))$$

is injective. Then consider

$$Z := \{(x, H) \in X \times \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))) \mid X_H := X \cap H \text{ is singular at } x\}$$

Denote,  $N = h^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ . The first projection  $p_1 : Z \rightarrow X$  makes  $Z$  a fibre bundle over  $X$  with fibres of dimension  $\mathbb{P}^{N-n-1}$ . Therefore,  $Z$  is smooth and of dimension  $N - 1$ . It can be shown that,  $\dim p_2(Z) = N - 1$  iff there is a point  $(x, H) \in Z$  such that  $H \cap X$  has an ordinary double point  $x$ . For convenience, we denote  $k := H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ . More formally,

**Lemma 3.3.** *Let  $(x, H) \in Z$ . Then  $X_H$  has an ordinary double point at  $X$  iff  $p_2 : Z \rightarrow K$  is an immersion at the point  $(x, H)$ .*

*Proof.* local calculation see Voisin. □

Therefore, we have the following existence theorem for Lefschetz pencils:

**Theorem 3.4** (Existence:). *Let  $X \subset \mathbb{P}^N$  be a smooth subvariety. Then a line  $\Delta \subset K$  describes a Lefschetz pencil if and only if*

- (1)  $p_2(Z) =: \mathcal{D}_X$  is a hypersurface in  $K$  and  $\Delta$  meets the hypersurface  $\mathcal{D}_X$  transversally in the open dense set

$$\mathcal{D}_X^0 := \{H \mid X_H \text{ has at worst ordinary double points}\}$$

- (2)  $\dim \mathcal{D}_X < N - 1$  then  $\Delta$  does not intersect  $\mathcal{D}_X$ .

*Proof.* The second statement is clear since by the lemma, for all  $H \in \mathcal{D}_X$ ,  $X \cap H$  have worse than ordinary double points and by definition of a Lefschetz pencil  $\Delta$  cannot admit such a point. For the first statement similar argument says that  $\Delta$  must intersect  $\mathcal{D}_X$  in  $\mathcal{D}_X^0$ . Why transversally will require the fact that the base locus of  $\Delta$  is smooth and therefore cannot contain the double point. In other words, if  $\Delta$  passes through  $H \in \mathcal{D}_X$ , then since locally around  $H$   $\mathcal{D}_X$  looks like a hyperplane in  $K$ ,  $\Delta$  will be contained in  $\mathcal{D}_X$ . In other words,  $B$  will contain the double point.  $\square$