EMBEDDINGS INTO $\mathbb{P}^3$. PRESENTED BY: FELIX JÄGER

SCRIBED BY: RUNYU JING

Abstract. We first proved that any curve can be embedded in $\mathbb{P}^3$. Then we proved that a curve can be embedded into $\mathbb{P}^2$ with at most nodes.

1. Introduction

Notation 0.1. A curve will always be smooth prober integral over an algebraically closed field $k = \overline{k}$.

1. Embedding into $\mathbb{P}^3$.

Theorem 1.1. *Any curve embeds into $\mathbb{P}^3$.*

Fact 1. For any hyperplane $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ and any closed point $O \nsubseteq \mathbb{P}^{n-1}$, we have a unique morphsim $\mathbb{P}^n \setminus O \to \mathbb{P}^{n-1}$ that takes a point P to the intersection of $\overline{PO}$ with $\mathbb{P}^{n-1}$. This is called "projection from O".

Definition 1.2. A linear system $\mathfrak{D}$ of a divisor D_0 on X is a linear subspace of the complete linear system $|D_0| = \mathbb{P}(H^0(X, \mathscr{O}(D_0)))$. A linear system $\mathfrak{D}$ is called base point free(B.P. free) if there is no point $x \in X$ such that all sections of $\mathfrak{D}$ vanishes at x.

Fact 2. we have correspondences:

$$\{\text{B.P. free linear systems on } X\}$$

$$\longleftrightarrow$$

$$\{(\mathscr{L}, V) | \mathscr{L} \text{ a line bundle on } X \text{ and } V \text{ a linear system that generates } \mathscr{L}\}$$

$$\longleftrightarrow$$

$$\{\text{morphisms } X \to \mathbb{P}^n \text{ s.t. the image of X does not lie in any hyperplane}\} \setminus \{\text{automorphisms of } \mathbb{P}^n\}$$

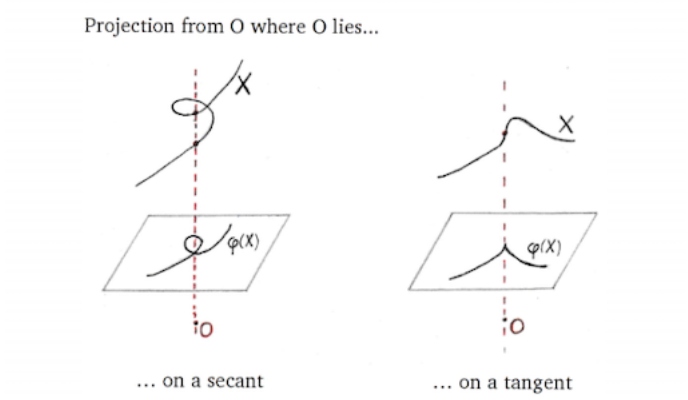
Definition 1.3. Let $X \subseteq \mathbb{P}^n$ be a curve. For $P \neq Q \in X$,

- define the line $\overline{PQ}$ as the secant line of P and Q. For $P \in X$,
- define the unique line $L_p \subseteq \mathbb{P}^n$ such that $T_P(X) = T_P(L_P)$ as the tangent line at P.

Proposition 1.4. *Let $X \subseteq \mathbb{P}^n$ be a curve, $O \in \mathbb{P}^n \setminus X$, $\varphi \colon X \to \mathbb{P}^{n-1}$ the projection. Then φ is a closed immersion if and only if:*

(1) O is not on any secant line of X;

(2) O is not on any tangent line of X.



Proof. φ corresponds to the linear system

$$\mathfrak{D} = \{H \cap X | H \subseteq \mathbb{P}^n \text{ a hyperplane}, O \in H, X \nsubseteq H\}$$

φ is a closed immersion if and only if $\mathfrak{D}$ separates points and tangents. But we know that

- $\mathfrak{D}$ separates points
  $\iff \forall P \neq Q \in X, \exists D \in \mathfrak{D}$ s.t. $P \in \operatorname{Supp} D, Q \notin \operatorname{Supp} D$
  $\iff \forall P \neq Q \in X, \exists H \subseteq \mathbb{P}^n$ s.t. $O, P \in H, Q \notin H \iff (1)$.
- $\mathfrak{D}$ separates tangents
  $\iff \forall P \in X, t \in T_P(X) - 0, \exists D \in \mathfrak{D}, P \in \operatorname{Supp} D, t \notin T_P D \subseteq T_P(X)$
  $\iff \forall P \in X, \exists H \subseteq \mathbb{P}^n$ s.t. $P, O \in H, T_P(H \cap X) = 0$
  $\iff \forall P \in X, \exists H \subseteq \mathbb{P}^n$ s.t. H intersects X at P with multiplicity > 1
  $\iff O$ is not on the tangent line of X at P.

$\square$

Definition 1.5. Let $X \subseteq \mathbb{P}^n$ be a curve.

- define $\operatorname{Sec}(X) = \cup$ secant lines of X = "secant variety",
- define $\operatorname{Tan}(X) = \cup$ tangent lines of X = "tangent variety".

Proof of 1.1 . First, there is a very ample line bundle on X. So we assume $X \subseteq \mathbb{P}^n$ for some big n. If $n \leq 3$, we are done. If not: notice that $\dim \operatorname{Sec} X \leq 3$ since $\operatorname{Sec} X$ is locally the image of a morphism:

$$(X \times X \backslash \Delta) \times \mathbb{P}^1 \to \mathbb{P}^n$$
$$(P, Q, t) \mapsto t \text{ on } \overline{PQ}$$

so we may compute that the dimension is no bigger than 3. Likewise we have that $\dim \operatorname{Tan} X \leq 2$: locally $\operatorname{Tan} X$ looks like the image of

$$X \times \mathbb{P}^1 \to \mathbb{P}^n$$
$$(P, t) \to t \text{ on } L_P$$

As $n > 3$, we can always project down by picking a point O away from $\operatorname{Sec} X$ and $\operatorname{Tan} X$ such that $X \hookrightarrow \mathbb{P}^{n-1}$ is a closed imbedding. After finite steps we arrive at that X embeds into $\mathbb{P}^3$. $\square$

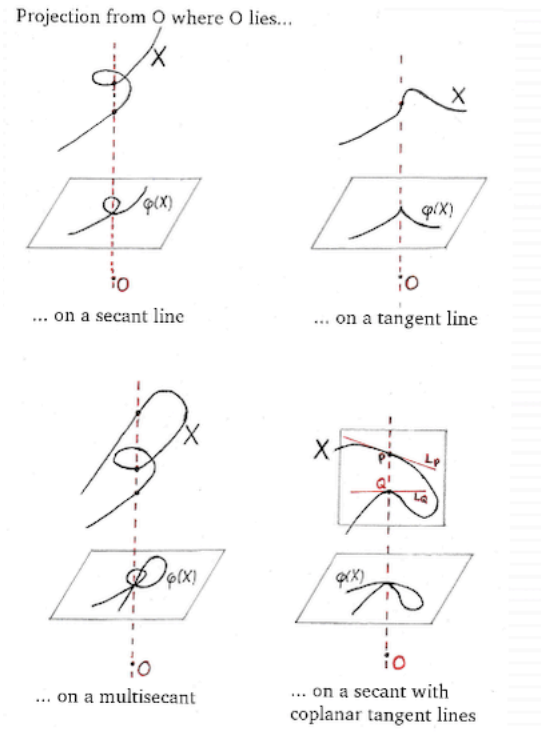
2. Maps to $\mathbb{P}^2$.

Definition 2.1. A node is a singular point of a plane curve of multiplicity 2 with distinct tangent directions.

Theorem 2.2. *Any curve is birationally equivalent to a plane curve with at most nodes.*

Proposition 2.3. *Let $X \subseteq \mathbb{P}^3$ be a curve, $O \in X$ a point and $\varphi\colon X \to \mathbb{P}^2$ a morphism determined by projection from O. Then φ is birational onto its image and $\varphi(X)$ has at most nodes if:*

(1) O lies on finitely many secants,
(2) O is not aon any tangent of X,
(3) O is not on any multi secants(secants that intersect X more than twice);
(4) O is not on any secant with coplane tangent lines(secant $\overline{PQ}$ such that L_P and L_Q lie on the same plane).



Proof. (1)+(2) guarantees that φ is birational onto its image. (2)+(3)+(4) will show that $\varphi(X)$ has at most nodes. $\square$

Proposition 2.4. *Let $X \subseteq \mathbb{P}^3$ be a curve not contained in any plane. Then not every sevant is a multisecant and not every two tangents are coplanar.*

Proof of 2.2. By 1.1 we may assume that $X \subseteq \mathbb{P}^3$ is not contained in a plane (or we are done!). We need to find O as above. Consider

$$\psi\colon (X \times X \backslash \Delta) \times \mathbb{P}^1 \to \mathbb{P}^3$$
$$(P, Q, t) \mapsto t \text{ on } \overline{PQ}$$

If $\dim \operatorname{im} \psi < 3$, then there is a nonempty open $U \subseteq \mathbb{P}^3 \setminus \operatorname{im} \psi$. If $\dim \operatorname{im} \psi = 3$, we must have that ψ^{-1}(the generic point) is not empty and thus a finite set. We may extend this to a nonempty open $U \subseteq \mathbb{P}^3$ such that $\psi^{-1}(x)$ is finite for every $x \in U$. Then any $O \in U$ fulfills (1) in 2.3. Now by 2.4, there exists $(P_0, Q_0) \in X \times X \backslash \Delta$ such that $\overline{PQ}$ is not a multi-secant and $(P_1, Q_1) \in X \times X \backslash \Delta$ such that L_{P_1} and L_{Q_1} are not coplanar. Both conditions hold for nearby points in $X \times X$. Thus we find out that the set $V \subseteq X \times X \backslash \Delta$ such that $\forall (P, Q) \in V$, $\overline{PQ}$ is no multi-secant and L_P is not coplanar with L_Q is open. Thus define

$$C \coloneqq \{(P, Q) \in X \times X \backslash \Delta \colon \overline{PQ} \text{ is a multi-sec or has coplanar tangents}\}$$

has dim < 1 in $X \times X \backslash \Delta$. Thus we have that

$$\dim \cup_{(P,Q) \in C} \overline{PQ} = \dim(C \times \mathbb{P}^1) \leq 2$$

On the other hand since we have that $\dim \operatorname{Tan} X \leq 2$, there must be a nonempty open $W \subseteq \mathbb{P}^3$ that consists of points satisfying (2)+(3)+(4). Then any point $O \in U \cap W$ fulfills (1)(2)(3)(4). □

It remains to prove 2.4, for which we need the following facts.

Fact 3. Let $X \subseteq \mathbb{P}^3$ be a curve, L, M lines in $\mathbb{P}^3$, and $L \cap X \neq \phi$. Then there exists a projection $X \to M$ sending $P \in X$ to the intersection point of M with the plane containing L and P.

Fact 4. Let $X \subseteq \mathbb{P}^3$ be a curve, $\varphi \colon X \to \mathbb{P}^2 (\varphi \colon X \to \mathbb{P}^1)$ a morphism determined by projection from $O \in \mathbb{P}^3 \backslash X$(from a line $L \subseteq \mathbb{P}^3 \backslash X$). Then φ is ramified exactly at the points $P \in X$ with $O \in L_P (L_P \cap L \neq \phi)$.

Definition 2.5. A finite morphism of curves $X \to Y$ is (purely in-/in-)separable if the corresponding field extension $K(Y) \hookrightarrow K(X)$ is so.

Fact 5 (Hurwitz)**.** Let $f \colon X \to Y$ be a finite separable morphism of curves. Then f is ramified at finitely many points $P_1, \cdots, P_n$. If u_i are local parameters at p_i and t_i are the local parameters at $f(P_i)$, we have that

$$2 \cdot g(X) - 2 = (\deg f)(2 \cdot g(Y) - 2) + \sum_{i=1}^{n} \nu_{P_i}(\frac{\mathrm{d}t_i}{\mathrm{d}u_i})$$

Fact 6. Let $f \colon X \to Y$ be a finite purely inseparable morphism of curves. Then $K(X) \cong K(Y)$ as abstract fields, and $K(Y) \hookrightarrow K(X)$ iis given by $x \mapsto x^{p^r}$ for some $r \in \mathbb{N}_+$ where char $(k) = p > 0$.

Corollary 2.6. *Let $f \colon X \to Y$ be a finite inseparable morphism of curves. Then $K(Y) \subseteq K(X)^p$(char $(k) = p > 0$).*

Corollary 2.7. *A finite inseparable morphism of curves is ramified everywhere.*

Fact 7 (Bézout)**.** Let $X \subseteq \mathbb{P}^3$ be a curve, $H \subseteq \mathbb{P}^3$ a plane, $X \nsubseteq H$, $P_1, \cdots, P_n$ the points in $H \cap X$ with intersection multiplicity $m_1, \cdots, m_n$. Then we have that

$$\sum_{i=1}^{n} m_i = \deg X$$

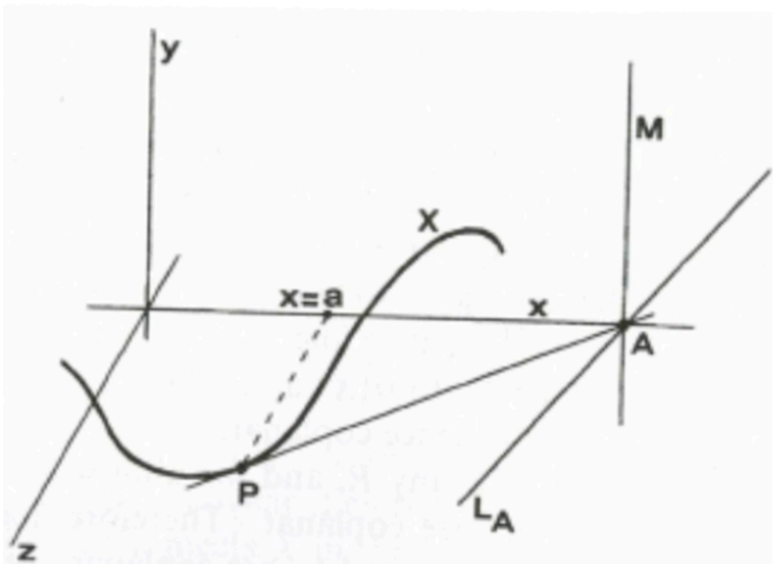
Lemma 2.8. [Har77, Prop IV.3.8] *Let $X \subseteq \mathbb{P}^3$ be a curve which is not contained in any plane. If any of the following is true:*

(a) *every secant is a multisecant, or*
(b) *every P, Q has coplanar tangents.*

Then there exists a point $A \in \mathbb{P}^3$ which lies on every tangent line of X.

Proof of 2.4. By Lemma 2.8, it suffices to show that for any curve $X \subseteq \mathbb{P}^3$ with such A as above, this *strange* curve is already in a plane(contradiction). Choose $\mathbb{A}^3$ on $\mathbb{P}^3$ with affine coordinate x, y, z such that

Figure 1. Picture from [Har77] p.312



(1) A is the point at ∞ on the X-axis,
(2) X does not meet the line at infinity of the XZ-plane, except at (possibly) A,
(3) If $A \in X$, then L_A is not in the XZ-plane,
(4) X does not meet Z-axis.

Let M be the line at ∞ of the XY-plane. Consider the projection $\psi\colon X \to M$ determined by projection from Z-axis. We assume that ψ is finite(otherwise is constant, so we are done). We have $\deg \psi = \deg X = d$ by Bézout's theorem 7. Let us find the ramification locus. As every tangent line also passes through A, ψ is ramified exactly at the points of X which lies in XZ-plane, but not at A by (3). By (2) these are exactly the points in the finite part of the XZ-plane. We may assume that the morphism ψ is separable(otherwise ψ ramifies everywhere so X is in XZ-plane). Let P be such a ramification point. We have $\psi(P) = A$. Take $u = x - a$(where a is the x-coordinate of P, $a \in k - \{0\}$ by (4)) as a local parameter at P and $t = \frac{y}{x}$ as a local parameter at A in M. We will calculate $\nu_P(\frac{\mathrm{d}t}{\mathrm{d}u})$. We have that

$$x = u + a \Rightarrow t = y(u + a)^{-1}$$

We may assume that $y \in K(X)^p$, where $p =$char(k). To see this, consider the morphism $\varphi\colon X \to \mathbb{P}^2$ corresponding to the projection from A to the YZ-plane. As A lies on every target line, φ is ramified everywhere, we have that $\operatorname{im} \varphi = \{P\}$(then $X = \overline{AP}$) or φ is inseparable. Then the function y restricted to X lies in $K(X)^p$, where char(k)=$p > 0$.

Thus we have that
$$\frac{\mathrm{d}y}{\mathrm{d}u} = 0 \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}u} = -y(u+a)^{-1},$$
where $u + a = x$ is a unit in $\mathscr{O}_P$ as $a \neq 0$. Hence we have that $\nu_P(\frac{\mathrm{d}t}{\mathrm{d}u}) = \nu_P(y)$, which is the multiplicity of the intersection of X and the XZ-plane at P. Let $P_1, \cdots, P_n$ be the finite points of $X \cap XZ$-plane. Thus Hurwitz's theorem 5 gives us that

$$2 \cdot g(X) - 2 = (\deg f)(2 \cdot g(Y) - 2) + \sum_{i=1}^{n} \nu_{P_i}(\frac{\mathrm{d}t_i}{\mathrm{d}u_i})$$
$$\iff 2 \cdot g(X) - 2 = -2d + \sum_{i=1}^{n} \nu_{P_i}(y)$$

Now, if $A \notin X$, then X meets the XZ-plane only at the P_i. Thus we have that:
$$\sum_{i=1}^{n} \nu_{P_i}(y) = d \Rightarrow 2g - 2 = -d \to g = 0, d = 2$$
Take 3 points on X and look at the plane H containing them. By Bézout's theorem 7 X is contained in H. If $A \in X$, X meets the XZ-plane at A with multiplicity 1, by (3) and Bézout's theorem 7, we have that
$$\sum_{i=1}^{n} \nu_{P_i}(y) + 1 = d \to 2g - 2 = -d - 1$$
$$\Rightarrow g = 1, d = 0$$
Thus X is again a line in $\mathbb{P}^3$. □

References

[Ara] D. Arapura. *The jacobian of a riemann surface.* preprint can be found at http://www.math.purdue.edu/~dvb/preprints/jacobian.pdf.

[ACGH85] E. Arbarello, M. Cornalba, P. Griffiths, J. D. Harris. *Geometry of Algebraic Curves.* vol. I, Springer, Grundlehren der mathematischen Wissenschaften, 1985.

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer, New York, 1977.

[Lan] A. Landesman. *The Torelli Theorem for Curves* preprint can be found at https://web.stanford.edu/~aaronlan/assets/torelli-theorem-notes.pdf.

[Mil86] J. S. Milne. *Jacobian varieties.* Arithmetic geometry (Storrs, Conn., 1984), pp. 167 212. Springer, New York, 1986

[Mum08] D. Mumford. *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics. vol. 5 TIFR, Bombay; by Hindustan Book Agency, New Delhi, 2008.

[Pet14] C. Peters. *Lectures on Torelli Theorems*, Spring School Rennes, https://www-fourier.ujf-grenoble.fr/~peters/ConfsAndSchools/Rennes/Torelli_Rennes.pdf, 2014.