On the moduli space of compact complex torus from Algebraic and Teichmüller viewpoint

Yajnaseni Dutta

Department of Mathematics, Northwestern University

Abstract

The aim of these notes is to define moduli spaces of compact Riemann surfaces from the Teichmüller point of view and then relate this point of view to the algebraic version of moduli problems, in case of elliptic curves.

The mapping class group

In order to understand the folklore that 'Moduli spaces are obtained as the quotient of the Teichmüller spaces via the action of the mapping class group', we first need to understand what mapping class groups are.

Definition 1. *Let S be a compact Riemann surface. The mapping class group of S, is defined by the group*

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S)/\,\mathrm{Homeo}_0^+(S),$$

where $\mathrm{Homeo}^+(S)$ *is a the group of all orientation preserving homeomorphisms of S and* $\mathrm{Homeo}_0^+(S)$ *is the connected component of identity in* $\mathrm{Homeo}^+(S)$*. In other words,* $\mathrm{Mod}(S)$ *is the set of all homeomorphism of S upto homotopy equivalence.*

It turns out that this group is isomorphic to the group of orientation preserving diffeomorphism up to isotopy i.e. $\mathrm{Mod}(S) \simeq \mathrm{Diff}^+(S)/\,\mathrm{Diff}_0^+(S)$. For the rest of the notes we will represent an element of $\mathrm{Mod}(S)$ as a class of orientation preserving diffeomorphism or a class of orientation preserving homeomorphism according to convenience.

Action of $\mathrm{Mod}(S)$ on the Teichmüller space

Let $\mathscr{T}(S)$ denote the Teichmüller space of the compact Riemann surface S. We now describe the action of $\mathrm{Mod}(S)$ on $\mathscr{T}(S)$. Given a representative element $(X, \varphi) \in \mathscr{T}(S)$, an element $\gamma \in \mathrm{Mod}(S)$ and representative diffeomorphism $h : S \to S$ of γ, define

$$\gamma[(X, \varphi)] := (X, \varphi \circ h^{-1}).$$

We have seen in class that this is a well defined action and that this action preserves distances. Therefore there is a morphism:

$$\Gamma : \mathrm{Mod}(S) \to \mathrm{Isom}(\mathscr{T}(S)).$$

defined via the above action. We have also seen in class that if $\Gamma(\gamma) = id$, then γ has a representative $h : S \to S$ such that h is homotopic to an isometry of S with respect to every hyperbolic metric on S.

Stabilizer of a point in $\mathscr{T}(S)$: We will see that the above action is far from being *free* and as a result the module space $\mathcal{M}(S)$ obtained by quotienting $\mathscr{T}(S)$ by this action is far from being a manifold. We will discuss this in detail later. The following lemma describes the stabiliser of this action.

Lemma 1. *For $(X,\varphi) \in \mathscr{T}(S)$. Then $\gamma[(X,\varphi)] = (X,\varphi)$ if and only if for any representative h of γ we have $\varphi h^{-1}\varphi^{-1} : X \to X$ is isotopic to an isometry $\tau_\gamma : X \to X$. In other words,*

$$\mathrm{Stab}[(X,\varphi)] \simeq \mathrm{Isom}(X).$$

Proof. This follows almost immediately from the definition Teichmüller space and the description of the action. Indeed, if $\gamma[(X,\varphi)] = (X,\varphi)$, then for a representative h of γ we have that $[(X,\varphi)] = [(X,\varphi h^{-1})]$ as points of $\mathscr{T}(S)$. In other words $\varphi h^{-1}\varphi^{-1} : X \to X$ is isotopic to an isometry $\tau_\gamma : X \to X$. This isometry does not depend of h since the free homotopy class of any closed curve contains a unique geodesic (see Lemma 1.3 [FM12]) and therefore no two distinct isometries can be isotopic to each other.

Conversely, if there is an isometry $\tau \in \mathrm{Isom}(X)$ then, $h = \varphi\tau\varphi^{-1}$ defines a diffeomorphism of S such that $\tau_h = \tau$. □

We will compute these stabilisers for compact Riemann surfaces of genus 1.

On the proper discontinuity of the action

A group G is said to act *properly discontinuously* on a topological spacce X if for all compact set $B \subset X$, the set $\{g \in G | gB \cap B \neq \emptyset\}$ is finite.

The action is called *free* if there are no nontrivial element of G that fixes an element of X, in other words $gx = x$ must mean that $g = id$.

The advantages of obtaining the moduli space $\mathcal{M}_g$ as a quotient is that, $\mathcal{M}_g$ obtains a pseudo-metric from $\mathscr{T}(S)$. The following theorem proves that the action is in fact, properly discontinuous and thereby proving that any metric on $\mathscr{T}(S)$ induces a metric on the orbifold $\mathcal{M}_g$. Further, if the action of $\mathrm{Mod}(S)$ were free, this would mean that $\mathcal{M}(S)$ is a manifold. We will see that this is not the case since the group of isometries, $\mathrm{Isom}(X)$, will not be trivial in most cases.

Theorem 1. *(Fricke) Let S_g be a compact Riemann surface of genus $g \geq 1$. Then the action of $\mathrm{Mod}(S_g)$ on $\mathscr{T}(S_g)$ is properly discontinuous.*

Proof. Let B be a compact set in $\mathscr{T}(S)$ of diameter D and consider a point $(X,\varphi) \in B$. Let $\gamma \in \mathrm{Mod}(S)$ satisfies $\gamma.B \cap B \neq \emptyset$, then the Teichmüller distance $d_{\mathscr{T}}[(X,\varphi),(X,\varphi h^{-1})] \leq 2D$, where $h : S \to S$ is a representative of γ. If K is the dilatation of $\varphi h^{-1}\varphi^{-1}$, then by definition of Teichmüller metric, $K \leq e^{4D}$.

For an essential closed curve $c \subset X$, denote by

$$l_X(c) = \text{ length of the unique geodesic in the homotopy class of } c.$$

Let c_1 and c_2 be a pair of essential closed curve that fill S. (Proposition 3.5 of [FM12] shows that we can pick exactly two such curves, although for the purpose of our proof it is enough that there are finitely many curves that fill S.) By Wolpert's lemma [Lemma 12.5, [FM12]], we have that,

$$l_X(h^{-1}(c_i)) \leq e^{4D} L,$$

where

$$L = \max\{l_X(c_1), l_X(c_2)\}.$$

We claim that, there are only finitely many isotopy classes of simple closed curve $c \subset S$ such that $l_X(c) \leq e^{4D} L$. This is the content of the following lemma. Assuming the claim, we have that, there are only finitely many distinct isotopy classes $h^{-1}(c_i)$ for $h \in \mathrm{Mod}(S)$. Represent these finitely many isotopy classes of curves by the curves $b_1, ..., b_n$. Note that, for any orientation preserving homeomorphism $\phi : S \to S$, $\phi(b_i)$ is isotopic to b_j for some $j \leq n$. Therefore, $\phi^r(b_i)$ is isotopic to b_i for some finite r. Moreover, for some bigger integer r, ϕ^r fixes $\{b_1, ..., b_n\}$ upto isotopy. Now since, $\{b_1, ..., b_n\}$ fills S and thereby, cutting along b_i gives a collection of closed discs, ϕ^r composed with the isotopy induces a homeomorphism of the closed disc D^2 that fixes the boundary. Such ϕ^r must be isotopic of identity on every such discs. Therefore, ϕ^r is isotopic to the identity on S. Since the permutation group of $\{b_1, ..., b_n\}$ is a finite group, there are only finitely many elements of $\mathrm{Mod}(S)$ that satisfy $\gamma.B \cap B \neq \emptyset$. $\square$

Lemma 2 (Discreteness of the length spectrum). *Let X be any closed hyperbolic surface. The set $rls(X) := \{l_X(c)\} \subset \mathbb{R}_+$ is a closed discrete subset of $\mathbb{R}$. Further, for each $L \in \mathbb{R}$, the set*

$$\{c : c \textit{ an isotopy class of simple closed curves in } X \textit{ with } l_X(c) \leq L\}$$

is finite.

Proof. We know that $X \simeq \mathbb{H}/\pi_1(X)$ where $\mathbb{H}$ is the upper half plane, $\pi_1(X)$ is the fundamental group of X and the action is free and properly discontinuous. Denote by $K \subset \mathbb{H}$ the compact fundamental domain (see Theorem 3) of this action. Let c be a simple closed geodesic in X with $l_X(c) \leq L$. Let $\tilde{c}$ be a lift of this curve in $\mathbb{H}$ such that intersects $\tilde{c} \cap K \neq \emptyset$. Then there is a unique closed loop $c_0 \in \pi_1(X)$ such that c_0 moves $\tilde{c}$ by L and c_0 is freely isotopic to c. Take a compact neighbourhood B of the compact set K of diameter $\mathrm{diam}(K) + 2R$, where R is slightly larger than L. Then $c_0 B \cap B \neq \emptyset$. Since $\pi_1(X)$ acts properly discontinuously on $\mathbb{H}$ there are only finitely many c_0. Moreover, since c is a geodesic, c_0 and c determine each other uniquely. Therefore are only finitely many c. $\square$

The moduli space of Torus

We start by describing the mapping class group of torus:

Theorem 2. *The homomorphism given by*

$$\sigma : \mathrm{Mod}(T^2) \to \mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{Aut}(H_1(T^2,\mathbb{Z}))$$

is an isomorphism.

We have discussed the proof of Theorem 2 in class. Moreover we have seen in class that $\mathrm{Mod}(T^2)$ acts on the Teichmüller space $\mathscr{T}(T^2) \simeq \mathbb{H}$ via the action of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})$$

on $\mathbb{H}$. The later action is described by the Möbius transformation

$$A.\tau = \frac{a\tau - b}{-c\tau + d}. \tag{1}$$

Let us denote a torus with a lattice generated by a fixed oriented basis $\{1, \tau\}$ for $\tau \in \mathbb{H}$ by T_τ. Then $A(T_\tau) = T_{\frac{a\tau-b}{-c\tau+d}}$.

First note that,

$$\frac{a\tau - b}{-c\tau + d} = \frac{-a\tau + b}{c\tau - d}$$

for all $\tau \in \mathbb{H}$. In other words, the kernel of the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathscr{T}(T^2)$ is ± 1. Therefore, we get an action of $\mathrm{PSL}(2,\mathbb{Z}) \simeq \mathrm{SL}(2,Z)/\pm I$ on $\mathscr{T}(T^2)$. Let us now understand the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathscr{T}(T^2)$.

Notice that for all $\tau \neq e^{\pi i/3}, i$ the stabilizer $\mathrm{Stab}(T_\tau) = I_{2\times 2} \in \mathrm{PSL}(2,\mathbb{Z})$. However,

$$\mathrm{Stab}(i) = \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \simeq \mathrm{Isom}(T_{\sqrt{-1}}).$$

and

$$\mathrm{Stab}(e^{i\pi/3}) = \left\{ \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \right\} \simeq \mathrm{Isom}(T_{e^{i\pi/3}})$$

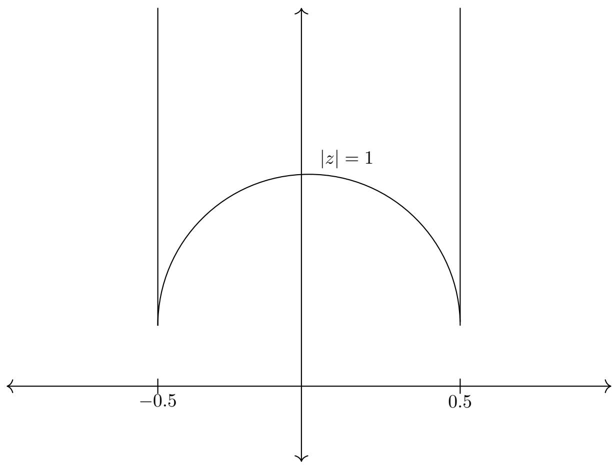

Figure 1: Fundamental domain of $\mathbb{H}$

Theorem 3 (Fundamental domain of uper half plane)**.** *Consider the following compact subset of the upper-half plane:*

$$F := \{z \in \mathbb{H} | |z| \geq 1 \textit{ and } -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2}\}.$$

Then every $\mathrm{PSL}(2, \mathbb{Z})$ *orbit of of* $\mathbb{H}$ *has at least one representative on* F *(See Figure).*

Proof. Note that if $|\tau| < 1$ then $|\frac{1}{\tau}| > 1$. Moreover the Möbius transformation $\tau \mapsto \tau \pm n$ can move τ in its orbit so that $|\Re(\tau)| \leq \frac{1}{2}$. $\square$

We now describe the quotient of F under the above action of $\mathrm{PSL}(2, \mathbb{Z})$ as a topological space. Recall that $\mathrm{PSL}(2, Z)$ acts freely everywhere but at i and $e^{i\pi/3}$.

It is known that $\mathrm{PSL}(2, \mathbb{Z})$ is generated by the matrices

$$\left\{ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

and they correspond to the actions $A\tau = -\frac{1}{\tau}$ and $B\tau = \tau - 1$.

Now, for $\tau = \frac{1}{2} + i\tau_2$, we have that, $B.\tau = -\frac{1}{2} + i\tau_2$. Therefore we need to identify the two long edges at 0.5 and -0.5 in Figure . After this identification the space looks like an infinite cylinder. However, recall that under the Teichmüller metric (which is equivalent to the hyperbolic metric on $\mathbb{H}$) the upper-half plane tapers up to a single point near infinity. Therefore, in fact our space now looks like an infinite cylinder pinched on the top. Now let $\tau = \tau_1 + i\tau_2 \in F$ such that $|\tau| = 1$. Note that, $A\tau = -\tau_1 + i\tau_2$. Therefore, A identifies two halves of the unit circle contained in F and thereby pinching the bottom of the cyclinder to a curve. Therefore $\mathcal{M}(T^2)$ topologically homeomorphic to a punctured

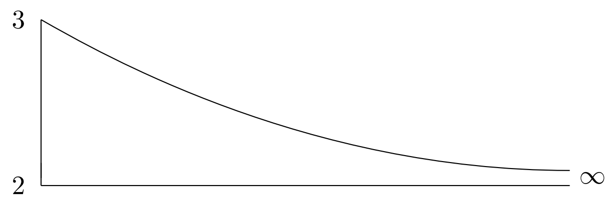

Figure 2: $\mathcal{M}(T^2)$ as an orbifold

sphere. However the points i and $e^{i\pi/3}$ are fixed by subgroups of $\mathrm{PSL}(2, Z)$ of order 2 and 3 respectively. Therefore $\mathcal{M}(T^2)$ can be seen as quotient of the puncture sphere by the action of the finite groups $\mathrm{Stab}(T_i)$ and $\mathrm{Stab}(T_{e^{i\pi/3}})$ and therefore the manifold structure of the upper half plane induces an orbifold structure on $\mathcal{M}(T^2)$ with signature $(2, 3)$. On the algebraic side of the story, this orbifold satisfies the properties of what is known as the Deligne-Mumford stack.

Compactness of $\mathcal{M}(T^2)$

We have noted that the Teichmüller metric on $\mathbb{H}$ descends to $\mathcal{M}(T^2)$. Consider the family of elliptic curves given by the lattice $ti \in \mathbb{H}$. Note that $d_{\mathscr{T}}(i, ti) = \frac{1}{2}\log t$ which goes to ∞ as t goes to ∞. Since, i and ti belong to different orbits under the action of $\mathrm{SL}(2, \mathbb{Z})$, we get that $d_{\mathcal{M}(T^2)}(T_i, T_{ti})$ also goes to ∞ as t goes to ∞. Therefore $\mathcal{M}(T^2)$ is not compact.

This issue is dealt by Deligne and Mumford in [DM69]. Their method is beyond the scope of our notes and involves the theory of algebraic stacks. Analytically this compactification can be seen by defining an L^2 metric on $\mathscr{T}(S)$, called the Weil-Peterson metric. This metric turns out to be incomplete. Deligne-Mumford compactification can be obtained by completion of this metric on one-point compactification of the punctured sphere. This treatment can be found in [HK14].

Complex Torus to Elliptic Curves with one marked point

Before we talk about Moduli spaces of complex torus from an algebraic viewpoint, we need to associate algebraic equations to complex torus. This chapter is devoted to understanding this connection via Weirestrass-- function.

Given a complex torus X with integral basis Λ spanned by $\{1,\tau\}$ for $\tau \in \mathbb{H}$, the Weirestrass $\wp$-function is defined as follows:

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{(0,0)\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$

A priori, $\wp(z)$ is defined on $\mathbb{C}$. It is known (can be shown) that $\wp(z)$ is an entire function. Note that

$$\wp'(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}$$

and hence $\wp'(z+w) = \wp'(z)$ for all $w \in \mathbb{Z} + \mathbb{Z}\tau$. We see that $\wp(z)$ is doubly periodic as follows: The derivative of $\wp(z+w) - \wp(z)$vanshes since $\wp'(z)$ is periodic. Since $\wp(z)$ is holomorphic everywhere on $\mathbb{C}$ and is bounded, it must be constant. Since, $\wp(-\frac{w}{2}) = \wp(\frac{w}{2})$, this constant must be 0.

Now lets look at its Laurent series centered at the origin with radius $|z| \leq \min\{1, |\tau|\}$:

$$\frac{1}{w^2(1-\frac{z}{w})^2} = \frac{1}{w^2}\left(1 + \sum_{n=1}^{\infty}(n+1)\left(\frac{z}{w}\right)^n\right).$$

Then $\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)G_{n+2}z^n$ where,

$$G_n = \sum_{w \in \Gamma} \frac{1}{w^n}.$$

These series are known as Eisenstein series and are convergent. Moreover,

$G_{2k+1} = 0$ for all $k \in \mathbb{Z}$. Therefore,

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)G_{2n+2}z^{2n}$$

and

$$\wp'(z) = -\frac{2}{z^3} + \sum_{n=1}^{\infty} 2n(n+1)G_{n+2}z^{2n-1}.$$

Therefore $[\wp'(z)]^2 - 4\wp^3(z) + 60G_4\wp(z) + 140G_6$ is periodic and holomorphic and thereby bounded and hence constant. Since, $O(z^2) \to 0$ as $z \to 0$, we have that the constant is 0. Therefore,

$$[\wp'(z)]^2 = 4\wp^3(z) - 60G_4\wp(z) - 140G_6.$$

The map $\mathbb{C} \to \mathbb{C}^2$ defined by,

$$z \mapsto (\wp(z), \wp'(z))$$

induces a bijection between $\mathbb{C}/\Gamma_\tau$ and the image defined by the algebraic equation

$$y^2 = 4x^3 - 60G_4(\tau)x - 140G_6(\tau).$$

The group law on $\mathbb{C}/\Lambda$ given by $(z_1 + \Lambda) + (z_2 + \Lambda) = z_1 + z_2 + \Lambda$ induces a group law on the elliptic curve and therefore the $p = (0 : 0 : 1)$ of the group law can be considered as a marked point.

Conversely, given an ellptic curve, we can see it as a torus via the so called Abel-Jacobi map defined below:

$$X \to H^1(X, \omega_X)$$

defined by

$$x \mapsto (\int_p^x \omega, \int_p^x \omega)$$

Here $p \in C$ is a fixed point and ω can be written locally as

$$\omega = \frac{dx}{\sqrt{4x^3 - 60G_4x - 140G_6}}.$$

This map turns out to be periodic with period $(\int_\alpha \omega, \int_\beta \omega)$, where, α and β are two based loops that generate $H_1(X, \mathbb{Z})$. One of course needs to check that this definition gives a homeomorphism between C and $\mathbb{C}/(\mathbb{Z}\alpha + \mathbb{Z}\beta)$ and that the Weirestrass polynomial obtained from the lattice $(\mathbb{Z}\alpha + \mathbb{Z}\beta)$ gives back the equation for X.

To summarize what we said above:

$\{$isomorphism classes of compact complex torus $\mathbb{C}/\Lambda$ with lattice based at $(0,0)\}$
$\leftrightarrow \{$isomorphism classes of elliptic curves with identity$\}$

The moduli space of elliptic curves with one marked point

The Moduli problem

Given a class $\mathbb{V}$ of objects (for instance, line bundles on a smooth curves, smooth curves of genus g, subvarieties of projective space of certain Hilbert polynomial, vector bundles of certain schemes etc) upto some specified equivalence relation on them (for instance: isomorphism, isometry etc), the moduli problem is to understand all "flat" families of such objects and to construct a variety $\mathcal{M}$(if it exists) whose points are in one-to-one correspondence with objects in $\mathbb{V}$. More concretely, we build a functor

$$\mathbb{F} : \mathrm{Sch} \to \mathrm{Sets}$$

defined by

$$\mathbb{F}(B) := \{\text{flat families over } B \text{ with fibres in } \mathbb{V} \text{ modulo ismorphisms over } B\}.$$

Then, $\mathcal{M}$ *represents* $\mathbb{F}$, i.e. $\mathbb{F}(B) \overset{\Psi_{\mathcal{M}}(B)}{=} \mathrm{Mor}(B, \mathcal{M})$ as a set. In other words, given a family $[\Upsilon \xrightarrow{f} B] \in \mathbb{F}(B)$ there is a unique morphism of schemes $\Psi_{\mathcal{M}}(B)(f) : B \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
\Upsilon \arrow[r] \arrow[d, "f"] & \mathscr{C} \arrow[d, "id"] \\
B \arrow[r, "\alpha"] & \mathcal{M}
\end{tikzcd}$$

where $\alpha = \Psi_{\mathcal{M}}(B)(f)$ and $\mathscr{C} \to \mathcal{M}_g$ is the *universal family* which is obtained by the pullback of the identity map. $\mathcal{M}_g \xrightarrow{id} \mathcal{M}_g$. In particular for $B = \mathrm{Spec}\,\mathbb{C}$, the above description implies that for a closed point $m \in \mathcal{M}$, the fibre/object over m is unique up to isomorphism over $\mathbb{C}$. Turns out, objects with non trivial automorphisms over $\mathbb{C}$, often creates obstructions in the existence of such fine moduli space. We will see an example of such a failure in section 3. In general, for curves of genus g there exists a quasi-projective scheme $\mathcal{M}_g$ such that the $\mathbb{C}$-points of $\mathcal{M}_g$ are in one-to-one correspondence with isomorphism classes of curves of genus g over $\mathbb{C}$ and for every flat family of curves over a scheme $f : \Upsilon \to B$, there is a morphism $\psi := \Psi_{\mathcal{M}}(B)(f) : B \to \mathcal{M}$ such that for every geometric point $b \in B$, $\psi(b)$ correspond to the isomorphism class of $f^{-1}(b)$. But there are no universal family. This prompts the following defintion:

Definition 2 (Coarse moduli space)**.** *A scheme $\mathcal{M}$ and a natural transformation*

$$\Psi_{\mathcal{M}} : \mathbb{F} \to \mathrm{Mor}(*, \mathcal{M})$$

are said to be a coarse moduli space for the functor $\mathbb{F}$ if

1. *The map $\Psi_{\mathbb{C}} : \mathbb{F}(\mathrm{Spec}\,\mathbb{C}) \to \mathrm{Mor}(\mathrm{Spec}\,\mathbb{C})$ is a set bijection.*

2. *Given another scheme $\mathcal{M}'$ and a natural transformation $\Psi_{\mathcal{M}'} : \mathbb{F} \to \mathrm{Mor}(*, \mathcal{M})$, there is a unique morphism $\pi : \mathcal{M} \to \mathcal{M}'$ such that the associated map of functors $\Pi : \mathrm{Mor}(*, \mathcal{M}) \to \mathrm{Mor}(*, \mathcal{M}')$ satisfies $\Psi_{\mathcal{M}} = \Pi \circ \Psi_{\mathcal{M}}$.*

In the subsequent two sections we will algebraically construct the coarse moduli space of elliptic curves with one marked point.

The isomorphism classes of ellptic curves

We have seen above that two compact complex torus with lattice based at $(0,0)$ defined by $\mathbb{C}/\Lambda_\tau$ and $\mathbb{C}/\Lambda_{\tau'}$ are isomorphic if and only if $A.\tau = \tau'$ for some $A \in \mathrm{SL}(2,\mathbb{Z})$ and the action as defined in Equation (1). The following theorem shows how this criterion translates from the point of view of elliptic curves and it's embedding the projective space $\mathbb{P}^2$.

Theorem 4.

1. *Curves given by Weierstrass equations $y^2 = 4x^3 - 60G_4x - 140G_6$ and $y^2 = 4x^3 60G_4'x - 140G_6'$ are isomorphic if and only if there exists $t \in C^*$ such that $G_4' = t^2G_4$ and $G_6 = t^3G_6$. There are only two curves with special automorphisms: the curve $y^2 = x^3 + 1$ gives $\mathbb{Z}_6$ and the curve $y^2 = x^3 + x$ gives $\mathbb{Z}_4$*

2. *Curves given by $y^2 = (x - x_1)(x - x_2)(x - x_3)$ and $y^2 = (x - x_1')(x - x_2')(x - x_3')$ are isomorphic if and only if there is a matrix $A \in \mathrm{PGL}(2,\mathbb{C})$ such that $A(x_1, 1) = (x_1', 1)$, $A(x_2, 1) = (x_2', 1)$, $A(x_3, 1) = (x_3', 1)$ and $A(1, 0) = A(1, 0)$. In particular, given $\lambda \in \mathbb{C} \setminus \{0, 1\}$, the roots $(0, 1, \lambda)$ determine a unique elliptic curve. Further, there are two cases with non-trivial (and non-involution) automorphisms, $\lambda = 1$ $(\mathrm{Aut}\,X = \mathbb{Z}_4)$ and $\lambda = \omega = e^{2\pi i/3}$. $(\mathrm{Aut}\,X = \mathbb{Z}_6)$.*

The j-invariant Given a compact Riemann surface $X_\tau \simeq \mathbb{C}/\Lambda_\tau$ with integral lattice based at $(0,0)$ and spanned by $\{1,\tau\}$, we have seen above that it corresponds to an elliptic curve in $\mathbb{P}^2$ defined on an affine chart by the equation:

$$y^2 = 4x^3 - 60G_4(\tau)x - 140G_6(\tau).$$

We can rewrite the above equation by

$$y^2 = (x - x_1^\tau)(x - x_2^\tau)(x - x_3^\tau).$$

Then define the j-invariant of the elliptic curve $X_\tau = (y^2 = (x-x_1^\tau)(x-x_2^\tau)(x-x_3^\tau))$ to be:

$$j(\tau) = 1728\frac{60^3G_4^3(\tau)}{[(x_1^\tau - x_2^\tau)(x_2^\tau - x_3^\tau)(x_3^\tau - x_1^\tau)]^2}.$$

When there is no room for confusion we will skip all the τ's.

One needs to justify that it is indeed a true invariant of an elliptic curve and that if two elliptic curves X and Y are isomorphic then, $j(X) = j(Y)$. Before we discuss that we present another way to write the same invariant. For this, we write the equation in an isomorphic coordinate so $x_1 \mapsto 0$, $x_2 \mapsto 1$ and therefore, $x_3 \mapsto \lambda$ for $\lambda = \frac{x_1 - x_3}{x_1 - x_2}$. Then,

$$j = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}. \tag{2}$$

Note that λ depends largely on the choice of roots that are sent to 0 and 1. In fact all the following choices of λ are isomorphic to X:

$$\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}. \tag{3}$$

It is clear that Equation (2) is independent of the above choices of λ. However, we need to keep in mind that these choices, induce non trivial automorphisms (isometries) of X. Moreover,

$$256(\lambda^2 - \lambda + 1)^3 - j(\lambda^2(\lambda - 1)^2$$

has 6 roots related by Equation (3). So the j-invariant uniquely determines isomorphism class of an elliptic curve. Further note that if $\lambda = 1$ or $\lambda = e^{2i\pi/3}$ is one of the roots of Equation (3), then these are multiple roots.

The coarse moduli space $\mathcal{M}_{1,1}$

We have seen that for two elliptic curves X and Y, $j(X) = j(Y)$ if and only if $X \simeq Y$. Therefore, j-line $\mathbb{A}^1$ is the coarse moduli space of Torus. However, there cannot be any "universal" family of curves over $\mathcal{M}_{1,1}$. For one, elliptic curves are invariant under involution, secondly the elliptic curves corresponding to $j = 0$ or $j = 1728$ admit other non-trivial automorphisms. The following example illustrates this failure:

Example 1. *Consider the family of elliptic curves $\Upsilon = \{y^2 = x^3 + t\}$ parametrised $S = \mathbb{A}^1_t$. Note that they all have the same $j = 0$. We can view this family as embedded in $\mathbb{A}^1_t \times \mathbb{P}^2$. Then the automorphism of $\mathbb{P}^2$ that takes the roots of $y^2 = x^3 + 1$ to $0, 1, e^{2\pi i/3}$ takes the roots of $y^2 = x^3 - 1$ to $0, 1, e^{\pi/3}$. Therefore Υ is not trivial. However if $\mathscr{C}$ were a universal elliptic curve over $\mathcal{M}_{1,1}$, we would be able to retrieve Υ as a pull-back of the constant map $0 : \mathbb{A}^1_t \to \mathbb{A}^1_j$. However, $0^*\mathscr{C} \simeq \mathscr{C}_0 \times \mathbb{A}^1_t$. This contradicts the non-triviality of Υ as a family over $\mathbb{A}^1_t$.*

The universal elliptic curve over $\mathbb{H}$: On the contrary to the above discussion the moduli space of complex torus with fixed lattice based at the origin is a fine moduli space. Consider the action of $\mathbb{Z}^2$ on $\mathbb{C} \times \mathbb{H}$ on the left by

$$(m, n)[(z, \tau)] = [(z + m + n\tau, \tau)]. \tag{4}$$

This action is properly discontinuous and free. Therefore the quotient $\Upsilon_{\mathbb{H}} \simeq \mathbb{C} \times \mathbb{H}/\mathbb{Z}^2$ is a complex manifold and it has a projection $p : \Upsilon_{\mathbb{H}} \to \mathbb{H}$. We claim the this is the universal family of elliptic curve with an oriented basis over $\mathbb{H}$. Note that the fibres are indeed $\mathbb{C}/\Lambda_\tau$. See [Hai11], Exercise 10 for a proof that this indeed is a fine moduli space.

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