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Summer term 2023

Exercises, Algebraic Geometry II – Week 2

Exercise 6. *Extension by zero* (4 points)
Let $(X, \mathcal{O}_X)$ be a ringed space and $j : U \hookrightarrow X$ an open subset considered as a ringed space with the sheaf of rings $\mathcal{O}_U := \mathcal{O}_X|_U$. For a sheaf of $\mathcal{O}_U$-modules $\mathcal{F}$, the *extension by zero* $j_!\mathcal{F}$ of $\mathcal{F}$ is the sheafification of the presheaf $(j_!)^{\mathrm{pre}}(\mathcal{F})$ defined by

$$(j_!)^{\mathrm{pre}}(\mathcal{F})(V) = \begin{cases} \mathcal{F}(V) \text{ if } V \subseteq U \\ 0 \text{ else .} \end{cases}$$

(1) Show that $(j_!\mathcal{F})_x = \mathcal{F}_x$ if $x \in U$ and 0 else. Conclude that $j_!$ defines an exact functor $j_! : \mathrm{Mod}(U, \mathcal{O}_U) \to \mathrm{Mod}(X, \mathcal{O}_X)$.

(2) Show that $j_!$ is left-adjoint to j^{-1}. Conclude that if $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then so is $\mathcal{I}|_U$.

(3) Give an example of X, U, and a quasi-coherent sheaf $\mathcal{F}$ on U such that $j_!\mathcal{F}$ is not quasi-coherent.

Exercise 7. *Cohomology with Supports* (5 points)
Let X be a topological space, $Y \subseteq X$ a closed subset, $U = X \setminus Y$, and $\mathcal{F}$ a sheaf of Abelian groups on X. Let $\Gamma_Y(X, \mathcal{F})$ be the group of sections of $\mathcal{F}$ with support on Y. Recall from Exercise 8 of Algebraic Geometry 1 that $\Gamma_Y(X, -) : \mathrm{Sh}(X) \to \mathrm{Ab}$ is a left-exact functor. We let $H^i_Y(X, -)$ be its right-derived functors.

(1) Show that if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of sheaves on X and and $\mathcal{F}'$ is flasque, then $0 \to \Gamma_Y(X, \mathcal{F}') \to \Gamma_Y(X, \mathcal{F}) \to \Gamma_Y(X, \mathcal{F}'') \to 0$ is exact.

(2) Show that flasque sheaves are $\Gamma_Y(X, -)$-acyclic.

(3) Show that if $\mathcal{F}$ is flasque, then there is an exact sequence

$$0 \to \Gamma_Y(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F}) \to 0.$$

(4) Show that, for every sheaf $\mathcal{F}$, there is a long exact sequence

$$\begin{array}{rl} 0 & \to \ H^0_Y(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \to \\ & \to \ H^1_Y(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}) \to \\ & \to \ H^2_Y(X, \mathcal{F}) \to \dots \end{array}$$

(5) (Excision) Let V be an open subset of X containing Y. Show that there are natural isomorphisms for all i and $\mathcal{F}$

$$H^i_Y(X, \mathcal{F}) \cong H^i_Y(V, \mathcal{F}|_V)$$

Due 21.04.2022, 2pm

Exercise 8. *Cohomology of the circle* (4 points)
Let S^1 be the circle considered as a topological space with the usual (analytic) topology.

(1) Show that $H^1(S^1, \underline{\mathbb{Z}}) \cong \mathbb{Z}$, where $\underline{\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}$.

(2) Show that $H^1(S^1, \mathcal{C}^0) = 0$, where $\mathcal{C}^0$ is the sheaf of continuous (real-valued) functions on S^1.

(Hint: Both sheaves are subsheaves of the flasque sheaf of (not necessarily continuous) real-valued functions on S^1)

Exercise 9. *Mayer–Vietoris Sequence* (3 points)
Let X be a topological space and let $Y_1, Y_2 \subseteq X$ be closed subsets. Show that, for every sheaf $\mathcal{F}$ of Abelian groups on X, there exists a long exact sequence

$$\ldots \to H^i_{Y_1\cap Y_2}(X,\mathcal{F}) \to H^i_{Y_1}(X,\mathcal{F}) \oplus H^i_{Y_2}(X,\mathcal{F}) \to H^i_{Y_1\cup Y_2}(X,\mathcal{F}) \to H^{i+1}_{Y_1\cap Y_2}(X,\mathcal{F}) \to \ldots$$

The next exercise is not necessary for the understanding of the lectures at this point.

Exercise 10. *Enough injectives in Grothendieck categories* (+ 4 extra points)
An Abelian category $\mathcal{C}$ is called *Grothendieck category*, if it satisfies the following properties:

(Ab3) Arbitrary coproducts in $\mathcal{C}$ exist.

(Ab4) Taking coproducts is exact.

(Ab5) For every object $A \in \mathcal{C}$, every subobject $A' \hookrightarrow A$, and every directed system of subobjects $\{A_i\}_{i\in I}$, we have [1]

$$\sum_{i\in I}(A_i \cap A') = (\sum_{i\in I} A_i) \cap A'.$$

(Generator) $\mathcal{C}$ has a generator, i.e., there exists an object $G \in \mathcal{C}$ such that for every object $A \in \mathcal{C}$ and every subobject $B \hookrightarrow A$ with $B \neq A$, there exists a morphism $G \to A$ that does not factor through B.

(1) Show that an object $A \in \mathcal{C}$ is injective if and only if for every subobject G' of the generator G, every morphism $G' \to A$ can be extended to a morphism $G \to A$.

(2) For $A \in \mathcal{C}$, let I be the set of all morphisms u_i from subobjects G_i of G to A. Consider the morphism

$$\varphi : \coprod_{i\in I} G_i \to A \times (\coprod_{i\in I} G),$$

where the first factor is $u_i : G_i \to A$ and the second is induced by the monomorphisms $G_i \hookrightarrow G$. Let $M_1(A) = \operatorname{coker}(\varphi)$. Show that the map $A \to M_1(A)$ given by the natural map $A \to A \times (\coprod_{i\in I} G)$ composed with the quotient map to $M_1(A)$ is a monomorphism.

(3) Set $M_i(A) = M_1(M_{i-1}(A))$ and use transfinite induction to define $M_\alpha(A)$ for all ordinals α. Show that if Ω is the smallest infinite ordinal that is larger than the set of subobjects of G, then $M_\Omega(A)$ is an injective object. Conclude that $\mathcal{C}$ admits enough injectives.

One can show that the category of sheaves of Abelian groups on a small *site*, i.e. on a small category with a Grothendieck topology as defined in Exercise 6 in AG 1 is a Grothendieck category. In particular, there is a notion of sheaf cohomology on every such site.

[1] Here, a subobject B of A is an object B together with a monomorphism $B \to A$, $\sum_{i\in I} A_i$ is the image of the natural map $\coprod_{i\in I} A_i \to A$, and, for subobjects B and C of A, $B \cap C$ is the kernel of $A \to A/B \times A/C$.