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Winter term 2022/23

Exercises, Algebraic Geometry I – Week 11

Exercise 59. *Tensor products with ample invertible sheaves* (4 points)
Let X be a scheme of finite type over a Noetherian ring A. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on X. Show the following:

(i) If $\mathcal{L}$ is ample and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is ample.

(ii) If $\mathcal{L}$ is ample, then there exists $n_0 \geq 0$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{M}$ is ample for all $n \geq n_0$.

(iii) if $\mathcal{L}$ and $\mathcal{M}$ are ample, then so is $\mathcal{L} \otimes \mathcal{M}$.

(iv) If $\mathcal{L}$ is very ample (over A) and $\mathcal{M}$ is globally generated, then $\mathcal{L} \otimes \mathcal{M}$ is very ample.

Exercise 60. *Extending coherent sheaves* (4 points)
The goal of this exercise is to extend a coherent sheaf defined on an open subscheme; in particular the following statement.

($*$) If X is a Noetherian scheme, $i : U \hookrightarrow X$ is an open subscheme of X, $\mathcal{F}$ is a coherent sheaf on U, and $\mathcal{G}$ is a quasi-coherent sheaf on X such that $\mathcal{F} \subseteq \mathcal{G}|_U$, then there exists a coherent subsheaf $\mathcal{F}' \subseteq \mathcal{G}$ such that $\mathcal{F}'|_U = \mathcal{F}$.

(i) Prove that every quasi-coherent sheaf on a Noetherian affine scheme is the union of its coherent subsheaves.

(Here, we say that a sheaf of abelian groups $\mathcal{F}$ on a topological space X is a union of subsheaves of abelian groups $\mathcal{F}_\alpha$ if for every open $U \subset X$ the group $\mathcal{F}(U)$ is the union of its subgroups $\mathcal{F}_\alpha(U)$.)

(ii) Show that ($*$) holds if X is affine.

(iii) Show that ($*$) holds.

Exercise 61. *Morphisms from projective spaces* (4 points)
Let k be a field and let $f : \mathbb{P}^n_k \to X$ be a morphism of k-schemes. Show the following:

(i) If X is affine, then $f(\mathbb{P}^n_k)$ is a (k-rational) point of X.

(ii) If $X = \mathbb{P}^m_k$, then f is the composition of a d-fold Veronese morphism for a unique $d \geq 0$ with iterated projections from points, inclusions of hyperplanes and an automorphism of $\mathbb{P}^m_k$.

(iii) If X is quasi-projective over k, then either $f(\mathbb{P}^n_k)$ is a (k-rational) point of X or the fibres of f over the k-rational points are finite.

(iv) (+1 extra point) Furthermore if $f(\mathbb{P}^n_k)$ is not a k-rational point then f is finite.

Please turn over

Due 13.01.2023, 2pm

Exercise 62. *Calculating divisor class groups* (4 points)
Let k be a field. Calculate the divisor class group of the affine cone over the Veronese image of $\mathbb{P}^1_k$ in $\mathbb{P}^2_k$.

Exercise 63. *Birational isomorphisms in small dimension* (4 points)
Two integral schemes X and Y are called *birational* (over a scheme S) if there exist non-empty open subschemes $U \subseteq X$ and $V \subseteq Y$ and an isomorphism $U \cong V$ (over S).

(i) Let X and Y be integral normal schemes of dimension 1 which are proper over a field k. Show that if X and Y are birational over k, then they are isomorphic over k.

(Hint: Use Exercise 36 (ii))

(ii) Let k be a field. Calculate $\mathrm{Pic}(\mathbb{P}^1_k \times_{\mathrm{Spec}\ k} \mathbb{P}^1_k)$. Deduce that $\mathbb{P}^1_k \times_{\mathrm{Spec}\ k} \mathbb{P}^1_k$ and $\mathbb{P}^2_k$ are not isomorphic over k (even though they are birational over k).

The last exercise is not necessary for the understanding of the lectures at this point.

Exercise 64. *The Grothendieck group of a scheme* (+ 5 extra points)
Let X be a Noetherian scheme. The *Grothendieck group* $K_0(X)$ of X is defined as the quotient of the free abelian group generated by all coherent sheaves on X by the subgroup generated by the expressions $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$, whenever there is an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

of coherent sheaves on X.

(i) If X is integral and $\mathcal{F}$ is a coherent sheaf on X, we define the *rank* of $\mathcal{F}$ as $\mathrm{rank}(\mathcal{F}) := \dim_{\mathcal{O}_{X,\eta}}(\mathcal{F}_\eta)$, where η is the generic point of X. Show that $\mathrm{rank}(-)$ defines a surjective homomorphism from $K_0(X)$ to $\mathbb{Z}$.

(ii) Let $Y \subseteq X$ be a closed subscheme and let $\mathcal{F}$ be a coherent sheaf on X with support on Y. Show that $\mathcal{F}$ admits a finite filtration by coherent sheaves

$$0 = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n = \mathcal{F}$$

such that each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is the pushforward of a coherent sheaf on Y.

(iii) Let $\iota : Y \hookrightarrow X$ be a closed immersion. Show that there is an exact sequence

$$K_0(Y) \xrightarrow{\alpha} K_0(X) \xrightarrow{\beta} K_0(X - Y) \to 0,$$

where α is induced by ι_* and β is induced by $(-)|_{X-Y}$.
(Hints: First, note that $\beta \circ \alpha = 0$, so that β induces a homomorphism

$$\overline{\beta} : K_0(X)/\alpha(K_0(Y)) \to K_0(X - Y).$$

Then, use Exercise 54 and Part (ii) of the current exercise to construct an inverse to $\overline{\beta}$.)

(iv) Let k be a field. Calculate $K_0(\mathrm{Spec}\ k), K_0(\mathbb{A}^1_k)$, and $K_0(\mathbb{P}^1_k)$.