Advanced Life Insurance Mathematics

Parametric mortality models and the closing of a mortality table

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Parametric models for mortality

- Ambition: (dates back to the 18th century with de Moivre, 1725)

  specify a parametric model for the death or mortality rates.

- A functional expression is suggested, depending on unknown parameters which should be estimated.

- When the parametric model is well chosen: very useful and elegant; dimensionality of problem is reduced.

- If it's not well chosen: completely wrong.

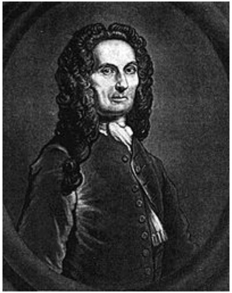
Parametric models for mortality

- Mortality law of Abraham de Moivre (1725):

$$S_0(x) = 1 - \frac{x}{\omega - 0} \quad \text{for } 0 \leq x < \omega.$$

- Following de Moivre (1725) we obtain:

$$\begin{aligned} {}_t p_x &= 1 - \frac{t}{\omega - x} \\ \mu_x &= \frac{1}{\omega - x} \\ \mathring{e}_x &= ml_x = \frac{\omega - x}{2}. \end{aligned}$$



Parametric models for mortality

- Gompertz (1825)'s law for μ_x:

$$\mu_x = \theta_2 \cdot \theta_3^x,$$

with $\theta_2 > 0$ and $\theta_3 > 1$.

- The law tries to describe the process of human ageing.
- An alternative formulation is:

$$\mu_x = \phi_1 \cdot \exp(\phi_2 x),$$

with $\phi_1,\ \phi_2 > 0$.

Benjamin Gompertz

Parametric models for mortality

- Makeham (1860) adjusted the Gompertz law.
- He includes an extra term, independent of age:

$$\mu_x = \theta_1 + \theta_2 \cdot \theta_3^x,$$

  with $\theta_1 \geq 0$, $\theta_2 > 0$ and $\theta_3 > 1$.
- This extra term takes accidental deaths and deaths due to illness (which can occur at any age) into account.

Parametric models for mortality

- Starting from Makeham's law, we see: (for any x)

$$\frac{d}{dx}\mu_x = \theta_2 \cdot \theta_3^x \cdot \ln \theta_3 > 0.$$

  Therefore, Makeham's law does not allow to take into account infant mortality and the accident hump.

- Makeham's law implies:

$$\begin{aligned} {}_tp_x &= \exp\left(-\theta_1 t - \frac{\theta_2\theta_3^x}{\ln \theta_3}(\theta_3^t - 1)\right) \\ \lambda_x &= \lambda_0 \exp\left(-\theta_1 x - \frac{\theta_2}{\ln \theta_3}(\theta_3^x - 1)\right). \end{aligned}$$

Parametric models for mortality

- A simpler specification of the Makeham law is then given by:

$$
\begin{aligned}
{}_t p_x &= \theta_5^t \cdot \theta_6^{(\theta_3^t - 1)\cdot \theta_3^x} \\
\lambda_x &= \theta_4 \cdot \theta_5^x \cdot \theta_6^{\theta_3^x},
\end{aligned}
$$

with θ_4, θ_5 and θ_6 identified as:

$$
\begin{aligned}
\theta_4 &= \lambda_0 \exp\left(\frac{\theta_2}{\ln \theta_3}\right) && > 0 \\
\theta_5 &= \exp\left(-\theta_1\right) && \leq 1 \\
\theta_6 &= \exp\left(-\frac{\theta_2}{\ln \theta_3}\right) && < 1.
\end{aligned}
$$

Parametric models for mortality

- Several methods have been proposed to estimate the parameters in Makeham's law.
- We consider:
  - Ballegeer (1973)
  - De Vylder (1975).

Parametric models for mortality

Method of Ballegeer (1973):

- minimize the following least squares problem

$$\sum_{x=x_{min}}^{x_{max}} \left(\ln(\theta_5) + (\theta_3 - 1)\theta_3^x \ln(\theta_6) - \ln p_x \right)^2 ,$$

- numerical techniques, like Newton-Raphson, are available to solve this optimization problem.

Parametric models for mortality

Method of De Vylder (1975):

- consider the d_x as realizations from a random variable $\mathcal{D}_x$ following a Binomial law

$$\mathcal{D}_x \sim \text{Bin}(\lambda_x, q_x),$$

- use maximum likelihood to estimate θ_1, θ_3 and θ_7 from

$$\prod_{x=x_{min}}^{x_{max}} \frac{\lambda_x!}{(\lambda_x - d_x)!d_x!} p_x^{\lambda_x - d_x} q_x^{d_x},$$

where $-\ln(p_x) = \theta_1 + \theta_7 \cdot \theta_3^x$. This problem reduces to maximizing

$$\sum_{x=x_{min}}^{x_{max}} ((\lambda_x - d_x)\ln(p_x) + d_x \ln(q_x)).$$

Parametric models for mortality

How to simulate the remaining lifetimes for a group of n individuals aged x, according to Makeham's survival model?

- if $V_1 \sim \text{EXP}(1)$ the random variable

$$T_1 = \frac{\ln(\theta_2\theta_3^x + V_1 \ln\theta_3) - \ln\theta_2}{\ln\theta_3} - x$$

follows a Gompertz law, such that

$$P[\,T_1 > t\,] = {}_tp_x = \exp\left(-\frac{\theta_2\theta_3^x}{\ln\theta_3}(\theta_3^t - 1)\right)$$

- the remaining lifetime according to Makeham's law is then given by

$$T = \min\,(\,T_1, T_2),$$

where $T_2 = V_2/\theta_1$ and V_2 is a second, independent $\sim \text{EXP}(1)$.

Parametric models for mortality

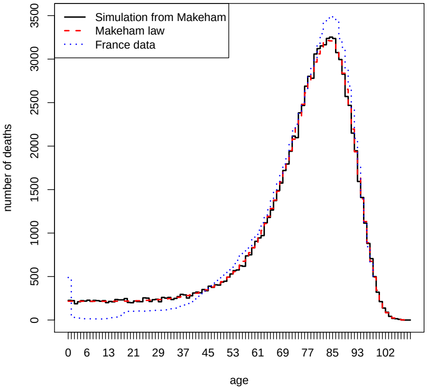
(Continued)

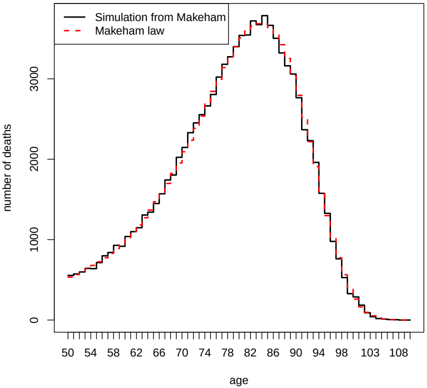
- indeed,

$$\begin{aligned} P[T > t] &= P[\min(T_1, T_2) > t] \\ &= P[T_1 > t] \cdot P[T_2 > t] \\ &= P[T_1 > t] \cdot P[V_2 > \theta_1 t] \\ &= \exp\left(-\theta_1 t - \frac{\theta_2 \theta_3^x}{\ln \theta_3}(\theta_3^t - 1)\right), \end{aligned}$$

which is the survival probability for an x–year old, according to Makeham's law.

Illustrating Makeham's law





Parametric models for mortality

- Previous laws use a small number of parameters, but: not able to fit mortality over all ages.

- The law of Heligman & Pollard (1980) is popular among Anglo-Saxon actuaries (see McDonald et al., 1998).

- Heligman & Pollard:

$$\frac{q_x}{p_x} = \theta_1^{(x+\theta_2)^{\theta_3}} + \theta_4 \exp\left(-\theta_5(\ln x - \ln \theta_6)^2\right) + \theta_7\theta_8^x.$$

- First term = mortality of infants, second term = accident hump, third term = Gompertz law for adult mortality.

Parametric models for mortality

- To estimate the parameters in this model, Heligman & Pollard (1980) use least squares: minimize

$$\sum_{x=x_{min}}^{x_{max}} \left(\frac{q_{x,HP} - q_x}{q_x} \right)^2,$$

where the q_x are observed probability estimates and

$$q_{x,HP} = \frac{\theta_1^{(x+\theta_2)^{\theta_3}} + \theta_4 \exp\left(-\theta_5 (\ln x - \ln \theta_6)^2\right) + \theta_7 \theta_8^x}{1 + \theta_1^{(x+\theta_2)^{\theta_3}} + \theta_4 \exp\left(-\theta_5 (\ln x - \ln \theta_6)^2\right) + \theta_7 \theta_8^x}.$$

- Alternatively, use maximum likelihood estimation.

Parametric models for mortality

- How to simulate random lifetimes from the Heligman & Pollard law?
- We do not have μ_x or ${}_t p_x$ at our disposal.
- Only p_x is available:

$$p_x = \frac{1}{1 + \theta_1^{(x+\theta_2)^{\theta_3}} + \theta_4 \exp\left(-\theta_5(\ln x - \ln \theta_6)^2\right) + \theta_7 \theta_8^x}.$$

- Solution:

use e.g. uniform or piecewise constant assumption.

Parametric models for mortality

- Simulation algorithm for Heligman & Pollard law:
  - simulate u from $U \sim \text{UN}(0, 1)$
  - find t_1 such that

$$\prod_{j=0}^{t_1-1} p_{x+j} \geq u > \prod_{j=0}^{t_1} p_{x+j}.$$

Parametric models for mortality

- Simulation algorithm for Heligman & Pollard law (continued):
  - calculate t_2 such that

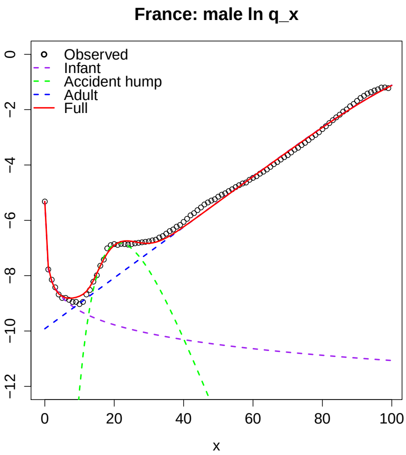
$$\left(\prod_{j=0}^{t_1-1} p_{x+j} \right) \, {}_{t_2}p_{x+t_1} = u$$

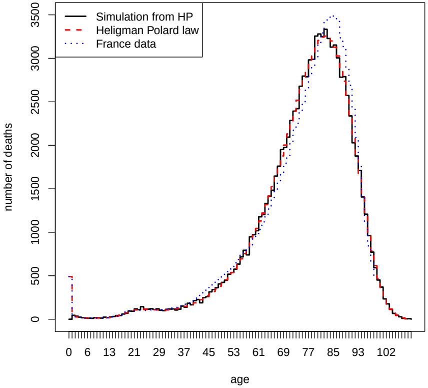
e.g. with the piecewise constant assumption:

$$t_2 = \frac{\ln u - \sum_{j=0}^{t_1-1} \ln p_{x+j}}{\ln p_{x+t_1}},$$

the simulated remaining lifetime is then $t = t_1 + t_2$.

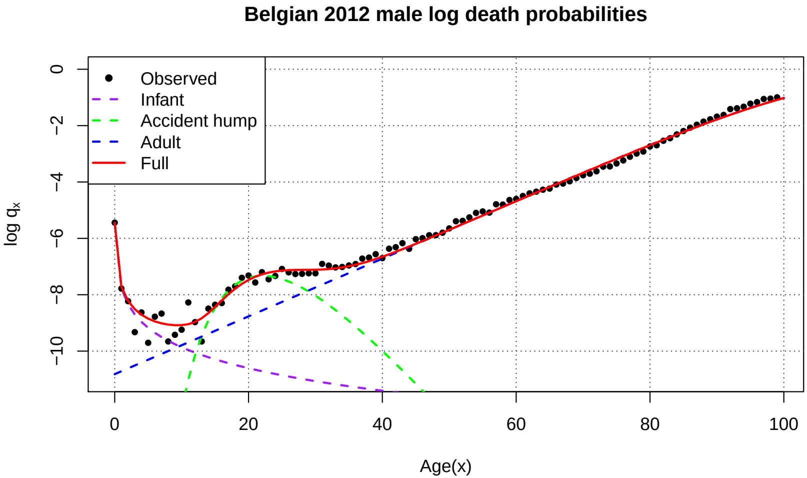
Illustrating Heligman & Polard's law





Illustrating Heligman & Polard's law

Belgian male q_x (from 2012)



Mortality at old ages

- Very often mortality data at old ages are very volatile.
- Techniques have been developed to extrapolate mortality at old ages

  i.e. the closing of mortality tables.
- For instance, Denuit & Goderniaux (2005):

$$\ln q_x = \theta_1 + \theta_2 x + \theta_3 x^2 + \epsilon_x,$$

  with independent error terms $\epsilon_x \sim N(0, \sigma^2)$ and constraints (when error terms put to zero):

$$\begin{aligned} q_{130} &= 1 \\ \frac{\partial q_x}{\partial x}\Big|_{x=130} &= 0. \end{aligned}$$

Mortality at old ages

- Denuit & Goderniaux (2005) (DG) (continued).
- Implementing these constraints:

$$\theta_1 + \theta_2 x + \theta_3 x^2 \quad = \quad \theta_3(130^2 - 260x + x^2),$$

and the regression model can be rewritten as

$$\ln q_x \quad = \quad \theta_3(130^2 - 260x + x^2) + \epsilon_x.$$

This is an example of a linear regression model (without intercept)!

Mortality at old ages

- How to find x_0 from which the observed/fitted q_x will be replaced by the $\hat{q}_x$ from DG?
  - let x_0 vary in the interval $50 - 85$
  - fit the linear model to ages $\geq x_0$ and get the R^2
  - the x_0 that realized the largest R^2 will be used to close the mortality table.
- Be careful!: don't let x_0 become too small.

Mortality at old ages

- Kannistö (1992) proposed

$$\mu_x = \frac{\phi_1 \exp(\phi_2 x)}{1 + \phi_1 \exp(\phi_2 x)}.$$

- Those specifications imply (for Kannistö):

$${}_t p_x = \left(\frac{1 + \phi_1 \exp(\phi_2 x)}{1 + \phi_1 \exp(\phi_2 (x + t))} \right)^{1/\phi_2}.$$

Mortality at old ages

- For calibration: use De Vylder (1975).
- Alternative: (see Doray, 2008)

$$\text{logit}(\mu_x) \quad = \quad \log(\phi_1) + \phi_2 x,$$

and apply least squares.

Mortality at old ages

- Use Kannistö to close the mortality table for old ages, say $x \in \{91, 92, \ldots, 120\}$:
  - estimate (ϕ_1, ϕ_2) using the relation (see Doray, 2008)

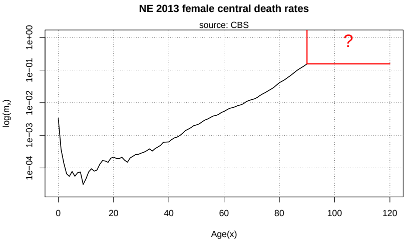
    $$\text{logit}(\mu_x) = \log(\phi_1) + \phi_2 x,$$

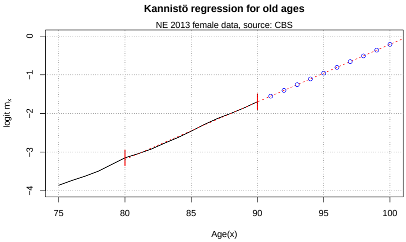
    with OLS on the ages - say - $x \in \{80, 81, \ldots, 90\}$
  - use these $(\hat{\phi}_1, \hat{\phi}_2)$ in the parametric law for μ_x as proposed by Kannistö:

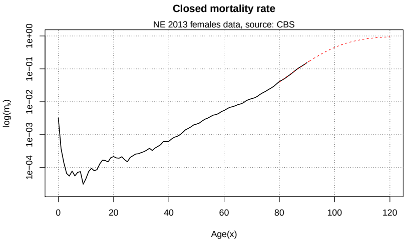
    $$\mu_x = \frac{\phi_1 \exp(\phi_2 x)}{1 + \phi_1 \exp(\phi_2 x)},$$

    for ages $x > 90$
  - p_x follows from the connection: $p_x = \exp\left(-\int_0^1 \mu_{x+r} dr\right) = \exp(-\mu_x)$ with piecewise constant μ_x (see further).

Illustrating Kannistö on Dutch female period life table







Wrap–up

After this class you are able to:

- identify parametric mortality laws, their use and interpretation
- identify and apply methods to estimate the unknown parameters in a parametric mortality law
- explain the idea of ‘closing a mortality table’
- identify and apply parametric mortality models designed for old ages.