Homework 2

ECE-GY 6223 – System Optimization Methods

Fall 2021, New York University

Due Date: 5PM on Oct 7, 2021

Problem 1 (Convex Sets)**.**

(i) Let $C \subseteq \mathbb{R}^n$ be a convex set, with $x_1, \cdots, x_k \in C$, and let $\theta_1, \cdots, \theta_k \in \mathbb{R}$ satisfy $\theta_i > 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$.

(ii) Let $C \subseteq \mathbb{R}^n$ be the solution set of quadratic inequality

$$C = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} + \mathbf{c} \leq 0\}$$

with $\mathbf{A} \in \mathbb{R}^n \times \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^n, \mathbf{c} \in \mathbb{R}$. Show that C is convex if $\mathbf{A}$ is semi-positive definite.

(iii) Show that the hyperbolic set

$$\{\mathbf{x} \in \mathbf{R}^2_+ \mid x_1 x_2 \geq 1\}$$

is convex. As a generalization, show that the set

$$\left\{\mathbf{x} \in \mathbf{R}^n_+ \;\middle|\; \prod_{i=1}^n x_i \geq 1\right\}$$

is convex.

Problem 2 (Convex Functions)**.**

(i) Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $f : \mathbb{R} \to \mathbb{R}$ be an increasing convex function. Show that $f(g(\mathbf{x}))$ is a convex function.

(ii) Show that the log-exp function $f : \mathbb{R}^n \to \mathbb{R}$,

$$f(\mathbf{x}) = \log\left(\sum_{i=1}^n e^{x_i}\right)$$

is convex.

(iii) Let $p < 1, p \neq 0$. Show that the following function $f : \mathbb{R}^n_{++} \to \mathbb{R}$,

$$f(\mathbf{x}) = \left(\sum_{i=1}^n x_i^p\right)^{1/p}.$$

is concave.

(iv) Show that the following function $f : \mathbb{R}^n \to \mathbb{R}$ is convex.

$$f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|,$$

where $\mathbf{A} \in \mathbf{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$, and $\|\cdot\|$ is a norm on $\mathbb{R}^m$.