System Optimization Method (ECE 6233) October 1, 2020
NYU Tandon School of Engineering, Fall 2021
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Solution to Homework 1

1. The characteristic polynomial of A is

$$\begin{aligned}
\det(A-\lambda I) =& \det\begin{pmatrix} 1-\lambda & 0 & -1 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ -1 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix} \\
=&(1-\lambda)\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} + (-1)\det\begin{pmatrix} 0 & 1-\lambda & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \\
=&(1-\lambda)^4 - (1-\lambda)^2 \\
=&\lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda.
\end{aligned}$$

We can find the eigenvalues of the matrix by computing

$$\det(A-\lambda I) = 0$$
$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 1, \quad \lambda_4 = 2$$

Eigenvectors $v_i \in \mathbb{R}^4$ is a non-zero vector which should satisfy

$$(A-\lambda I)v = 0$$

- $\lambda = 0$:

$$v \in span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- $\lambda = 1$:

$$v \in span\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- $\lambda = 2$:

$$v \in span\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Matrix A is not positive definite. For real symmetric matrices, the matrix is positive definite if and only if all of its eigenvalues are positive.

2. Let

$$\nabla_x f(x) = \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - 3 \end{pmatrix} = 0,$$

which gives

$$x_1 = 1 \qquad x_2 = 2.$$

Hessian matrix of the function is

$$H_f = \nabla_x^2 f(x^*) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Since $H_{11} > 0$ and $\det(H) = 6 > 0$, the Hessian matrix is positive definite. Thus, the point x^* is the local minimum point

$$x^* = (1, 2) \quad \Rightarrow \quad f(x^*) = -3$$

We can rewrite the function as

$$f(x) = x^T Q x - 3x_2$$

where

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}$$

The matrix Q is positive definite and $\lambda_{min} > 0$. Thus,

$$f(x) \geq \lambda_{min} \|x\|^2 - 3x_2 = \lambda_{min}(x_1^2 + x_2^2) - 3x_2$$

The function $f(x) \to \infty$ as $\|x\|^2 \to \infty$. Thus, the point is also a global minimum.

3. (a) Let

$$\nabla_x f(x) = \begin{pmatrix} -4x_1 - 2x_2 + 1 \\ -2x_1 - 4x_2 + 1 \end{pmatrix} = 0,$$

which gives

$$x_1 = x_2 = \frac{1}{6}$$

Hessian matrix of the function is

$$H_f = \nabla_x^2 f(x^*) = \begin{pmatrix} -4 & -2 \\ -2 & -4 \end{pmatrix}$$

It is easy to check that $-H_f$ is positive definite. Thus, H_f is negative definite. There exits a local maximum point

$$x^* = \left(\frac{1}{6}, \frac{1}{6}\right) \quad \Rightarrow \quad f(x^*) = \frac{19}{6}$$

Since this is the only solution to $\nabla_x f(x) = 0$, there is no local minimum.

(b) We can rewrite the function as

$$f(x) = -x^T Q x + x_1 + x_2 + 3$$

where

$$Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

is a positive definite matrix and $\lambda_{min} > 0$. Thus, $x^T Q x \geq \lambda_{min} \|x\|^2$ and

$$f(x) \leq -\lambda_{min} \|x\|^2 + x_1 + x_2 + 3$$

The function $f(x) \to -\infty$ as $\|x\|^2 \to \infty$.