System Optimization Method (ECE 6233) October 29, 2020
NYU Tandon School of Engineering, Fall 2021
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Solution to Homework 3

1. Problem 5.21 (a)-(c)

Solution.

(a) $p^\star = 1$.

(b) The Lagrangian is $L(x, y, \lambda) = e^{-x} + \lambda x^2/y$. The dual function is

$$g(\lambda) = \inf_{x,y>0}(e^{-x} + \lambda x^2/y) = \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0, \end{cases}$$

so we can write the dual problem as

$$\begin{array}{ll} \text{maximize} & 0 \\ \text{subject to} & \lambda \geq 0, \end{array}$$

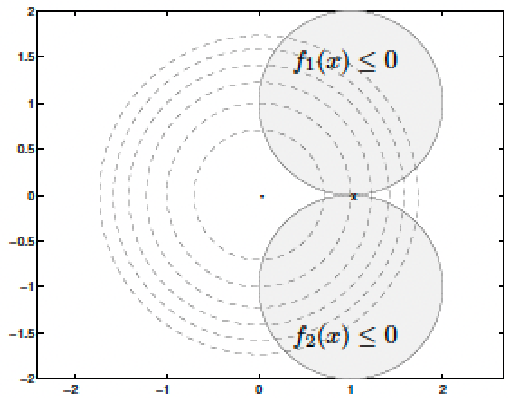
with optimal value $d^\star = 0$. The optimal duality gap is $p^\star - d^\star = 1$.

(c) Slater's condition is not satisfied.

2. Problem 5.26

Solution.

(a) The figure shows the feasible set (the intersection of the two shaded disks) and some contour lines of the objective function. There is only one feasible point, $(1,0)$, so it is optimal for the primal problem, and we have $p^\star = 1$.



(b) The KKT conditions are

$$\begin{gathered}
(x_1-1)^2 + (x_2-1)^2 \leq 1, \quad (x_1-1)^2 + (x_2+1)^2 \leq 1, \\
\lambda_1 \geq 0, \qquad \lambda_2 \geq 0 \\
2x_1 + 2\lambda_1(x_1-1) + 2\lambda_2(x_1-1) = 0 \\
2x_2 + 2\lambda_1(x_2-1) + 2\lambda_2(x_2+1) = 0 \\
\lambda_1((x_1-1)^2 + (x_2-1)^2 - 1) = \lambda_2((x_1-1)^2 + (x_2+1)^2 - 1) = 0.
\end{gathered}$$

At $x = (1,0)$, these conditions reduce to

$$\lambda_1 \geq 0, \qquad \lambda_2 \geq 0, \qquad 2 = 0, \qquad -2\lambda_1 + 2\lambda_2 = 0,$$

which (clearly, in view of the third equation) have no solution.

(c) The Lagrange dual function is given by

$$g(\lambda_1, \lambda_2) = \inf_{x_1, x_2} L(x_1, x_2, \lambda_1, \lambda_2)$$

where

$$\begin{aligned}
&L(x_1, x_2, \lambda_1, \lambda_2) \\
&\quad = \quad x_1^2 + x_2^2 + \lambda_1((x_1-1)^2 + (x_2-1)^2 - 1) + \lambda_2((x_1-1)^2 + (x_2+1)^2 - 1) \\
&\quad = \quad (1+\lambda_1+\lambda_2)x_1^2 + (1+\lambda_1+\lambda_2)x_2^2 - 2(\lambda_1+\lambda_2)x_1 - 2(\lambda_1-\lambda_2)x_2 + \lambda_1 + \lambda_2.
\end{aligned}$$

L reaches its minimum for

$$x_1 = \frac{\lambda_1 + \lambda_2}{1 + \lambda_1 + \lambda_2}, \qquad x_2 = \frac{\lambda_1 - \lambda_2}{1 + \lambda_1 + \lambda_2},$$

and we find

$$g(\lambda_1, \lambda_2) \quad = \quad \begin{cases} -\frac{(\lambda_1+\lambda_2)^2 + (\lambda_1-\lambda_2)^2}{1+\lambda_1+\lambda_2} + \lambda_1 + \lambda_2 & 1 + \lambda_1 + \lambda_2 \geq 0 \\ -\infty & \text{otherwise,} \end{cases}$$

where we interpret $a/0 = 0$ if $a = 0$ and as $-\infty$ if $a < 0$. The Lagrange dual problem is given by

$$\begin{array}{ll} \text{maximize} & (\lambda_1 + \lambda_2 - (\lambda_1 - \lambda_2)^2)/(1 + \lambda_1 + \lambda_2) \\ \text{subject to} & \lambda_1, \lambda_2 \geq 0. \end{array}$$

Since g is symmetric, the optimum (if it exists) occurs with $\lambda_1 = \lambda_2$. The dual function then simplifies to

$$g(\lambda_1, \lambda_1) = \frac{2\lambda_1}{2\lambda_1 + 1}.$$

We see that $g(\lambda_1, \lambda_2)$ tends to 1 as $\lambda_1 \to \infty$. We have $d^\star = p^\star = 1$, but the dual optimum is not attained.

Recall that the KKT conditions only hold if (1) strong duality holds, (2) the primal optimum is attained, and (3) the dual optimum is attained. In this example, the KKT conditions fail because the dual optimum is not attained.

3. Problem 5.29

Solution.

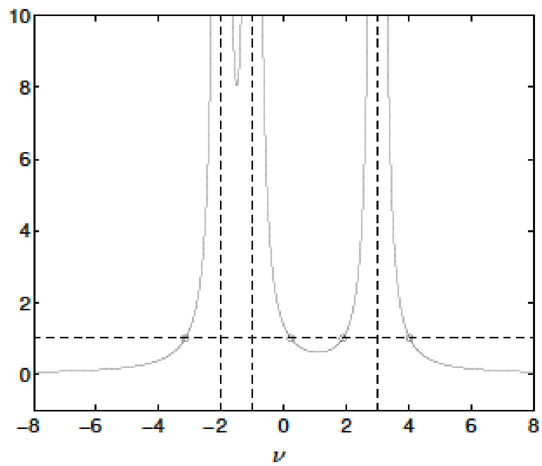
(a) The KKT conditions are

$$x_1^2+x_2^2+x_3^2=1, \qquad (-3+\nu)x_1+1=0, \qquad (1+\nu)x_2+1=0, \qquad (2+\nu)x_3+1=0.$$

(b) A first observation is that the KKT conditions imply $\nu \neq 2$, $\nu \neq -1$, $\nu \neq 3$. We can therefore eliminate x and reduce the KKT conditions to a nonlinear equation in ν:

$$\frac{1}{(-3+\nu)^2}+\frac{1}{(1+\nu)^2}+\frac{1}{(2+\nu)^2}=1$$

The lefthand side is plotted in the figure.



There are four solutions:

$$\nu=-3.15, \qquad \nu=0.22, \qquad \nu=1.89, \qquad \nu=4.04,$$

corresponding to

$$x=(0.16,0.47,-0.87), \qquad x=(0.36,-0.82,0.45),$$

$$x=(0.90,-0.35,0.26), \qquad x=(-0.97,-0.20,0.17).$$

(c) $\nu^\star$ is the largest of the four values: $\nu^\star = 4.0352$. This can be seen several ways. The simplest way is to compare the objective values of the four solutions x, which are

$$f_0(x) = 1.17, \qquad f_0(x) = 0.67, \qquad f_0(x) = -0.56, \qquad f_0(x) = -4.70.$$

We can also evaluate the dual objective at the four candidate values for ν. Finally we can note that we must have

$$\nabla^2 f_0(x^\star) + \nu^\star \nabla^2 f_1^\star(x^\star) \succeq 0,$$

because $x^\star$ is a minimizer of $L(x, \nu^\star)$. In other words

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \nu^\star \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \succeq 0,$$

and therefore $\nu^\star \geq 3$.