CLASSICAL CALCULUS OF VARIATIONS

3

3-1. INTRODUCTION

In its most general form, calculus of variations is the body of theory with which a set of functions of several independent variables is selected so as to extremize a given measure of the set of functions, subject to known constraints imposed on the functions. The phrase "calculus of variations" was first used to describe theory associated with the solution of such problems as a result of notation used by Lagrange about the year 1760. The word "variation" is a key word in the classical solution of such problems, as is shown in this chapter.

In Chapter 2, the problem of determining the *points of extrema* of a function was of interest. Here the object is to find the *functions of extrema* of a *functional*. A functional, then, has functions and perhaps derivatives of functions for its arguments. For a specific assignment of the functions, a scalar functional attains a scalar value.

The functionals treated in this chapter are restricted to those which can be represented in the form of an integral of a function of several dependent functions. Only one *independent* variable of integration is considered. The majority of variational problems treated in the literature are or can be put in the above form. For those which cannot, the attainment of necessary conditions for the optimum usually follows by reasoning which is analogous to that presented in the following sections.

In some problems, the functional to be extremized is an obvious facet of the problem, e.g., a functional representing expended energy which is to be minimized. In other problems, however, the functional to be used is a matter of choice. This is especially true for problems in which an error as

a continuous function of time is to be minimized; the functional for such a problem could be the integral of the squared error (ISE criterion); it could be the integral of the absolute value of the error; it could be the integral of the time-weighted absolute value of the error (ITAE criterion); and so on, there being an unlimited number of such measures which could be used. For a particular design problem, one choice may be preferable to the others on the basis of practical considerations, and a different choice may be preferable to others because of the mathematical finesse with which it leads to a solution of the problem. For example, the ISE criterion is a tractable one with which emphasis is placed on the reduction of large error magnitudes; whereas the ITAE criterion is a less tractable one, but sometimes a more desirable one, with which emphasis on error magnitude reduction is increased linearly with time.

The theory developed in this chapter is the Euler-Lagrange formalism of the calculus of variations, in contrast to the Hamiltonian approach and the Hamilton-Jacobi approach. These last two approaches are considered further in the chapters which are primarily concerned with Pontryagin's maximum principle and dynamic programming. In Section 3-2, fundamental concepts are presented, and equivalence relations are established between three different forms of functionals. In Sections 3-3 through 3-6, fundamental necessary conditions for the optimum are developed for integral functionals with one dependent function to be selected for the attainment of an optimal solution. In Sections 3-7 and 3-8, these necessary conditions are generalized to the case of n dependent functions, and various constraint conditions are incorporated in the theory by use of Lagrange multipliers. These Lagrange multipliers are often functions of the independent variable of integration. Various constraints can be imposed on the end-point values of independent and dependent variables.

Design examples are given throughout to illustrate applications of the theory; and in Section 3-9, a general Euler-Lagrange formulation for a class of control problems is given with its limitations.

A solution which is obtained on the basis of "necessary conditions" should be tested by "sufficient conditions" to establish with certainty the optimality of the solution. In many problems, sufficiency can be established by considering the physical nature of the process involved. When this is not feasible, mathematical sufficiency conditions may be used to establish the optimality of a solution. Such conditions are considered in Section 3-10.

In Section 3-11, attention is centered on *direct methods* of finding the optimal solutions to variational problems—direct methods are those which do not depend on the application of classical necessary-and-sufficient conditions. And in Section 3-12, functionals are examined from the standpoint of sensitivity with respect to changes in their argument functions.

3-2. PRELIMINARY CONCEPTS

3-2a. Continuity, Extrema, and Variations

Pertinent properties of functionals are introduced by considering the single dependent function case:

$$J = P(x,t) \tag{3-1}$$

where for a specified range of the real independent variable t and for a given real-valued function $x \equiv x(t)$, the performance measure $P(x,t)$ yields a scalar value.

In this chapter, interest is centered primarily on those $x(t)$'s which are continuous and for which derivatives are uniquely defined, except possibly at a finite number of points within the range associated with the independent variable t. Curves (functions) which exhibit these properties are often referred to as *admissible curves* in the classical literature. Of course, we should not conclude that only this type of "admissible" curve results in the extremization of functionals. For example, consider the particular functional

$$\int_{-c}^{c} t^2\dot{x}^2\,dt$$

where c is a positive constant, $x(-c) = -1$, and $x(c) = 1$. This functional does not contain a relative minimum within the above-noted class of admissible curves; the minimum of zero is obtained when x equals

$$x^* = \lim_{k\to\infty} (\tan^{-1} kt)/(\tan^{-1} kc),$$

which has a jump discontinuity at $t = 0$.

Consider the case in which a function $x_\alpha \equiv x_\alpha(t)$ results in a relative minimum $J(x_\alpha)$, Equation 3-1, for the range of t between given values t_a and t_b, $t_a < t_b$. The implications of the above statement are not as straightforward as those of the analogous statement for ordinary min-max problems. Here the functional $J(x_\alpha)$ may be a minimum relative to functions x not identically equal to x_α in the sense that $J(x_\alpha) - J(x) < 0$ whenever $|x_\alpha - x| < \Delta_1$ for some real value Δ_1 and all t contained in $[t_a,t_b]$, in which case $J(x_\alpha)$ is said to be a *strong relative minimum*. On the other hand, $J(x_\alpha)$ may be a minimum relative to functions x not identically equal to x_α in the sense that $J(x_\alpha) - J(x) < 0$ whenever $|x_\alpha - x| < \Delta_1$ and $|\dot{x}_\alpha - \dot{x}| < \Delta_2$ for some real values Δ_1 and Δ_2 and all t contained in $[t_a,t_b]$, in which case $J(x_\alpha)$ is said to be a *weak relative minimum*.[1] A strong relative minimum is therefore a special case of a

[1] These defining conditions can be expressed more compactly in terms of a norm on the space $C(t_a,t_b)$ of continuous functions (see Appendix D). The definitions of strong and weak relative extrema given here are to be distinguished from those of Chapter 2.

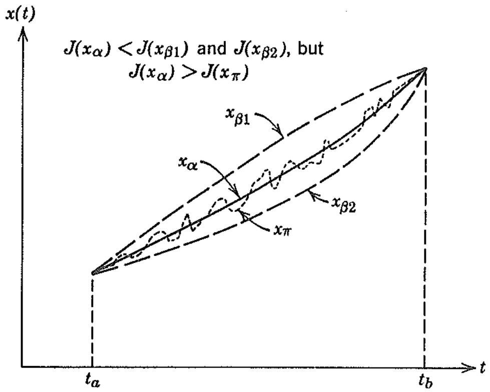

Figure 3-1. Example of a weak relative minimum.

weak relative minimum in which Δ_2 can be assigned an arbitrarily large value. Similar statements apply to strong and weak relative maxima.

The curves in Figure 3-1 illustrate a hypothetical weak relative minimum. The curves $x_{\beta 1}$ and $x_{\beta 2}$ are representative functions of a class of continuous functions for which $|\dot{x}_\beta - \dot{x}_\alpha|$ is less than some number, say 10, for all t in $[t_a,t_b]$; whereas the curve x_π is representative of a class of continuous functions for which $|\dot{x}_\pi - \dot{x}_\alpha| \geq 10$ for some nontrivial subset of $[t_a,t_b]$. Corresponding to the curve $x_\alpha(t)$, the value $J(x_\alpha)$ is less than $J(x_{\beta 1})$ or $J(x_{\beta 2})$. On the other hand, $J(x_\alpha)$ is greater than $J(x_\pi)$, this being the case for some x_π even when $|x_\pi(t) - x_\alpha(t)|$ is arbitrarily small over $[t_a,t_b]$. On the basis of this example, it should be evident that relative extrema associated with variational problems of a physical origin are generally strong relative extrema.

The functional $J(x)$ is said to be *strongly continuous* (or, simply, *continuous*) if a small modification in the function x always results in a small change in the scalar value of the functional $J(x)$. Specifically, let $x_\alpha = x_\alpha(t)$ and $x_\beta = x_\beta(t)$ denote typical functions. Then the functional $J(x)$ is termed strongly continuous about $x_\alpha(t)$ if for any real number ϵ_1, $\epsilon_1 > 0$, there exists a real number ϵ_2 such that, for every function x_β which satisfies $|x_\alpha - x_\beta| < \epsilon_2$ throughout the range of t of interest, the value of $|J(x_\alpha) - J(x_\beta)|$ is less than ϵ_1.

The concept of a *variation* is paramount to the calculus of variations. When a function $x = x(t)$ is modified by an amount $\delta x = \delta x(t)$ to $x + \delta x$, the modification δx in the function is called a *variation of the function x*. Any variation δx can be viewed as a special case of a variation $\epsilon\,\delta x$ where ϵ is a real independent variable. Corresponding to a variation $\epsilon\,\delta x$ in the function x, the functional J of Equation 3-1 assumes the value $J(x + \epsilon\, \delta x)$. Let the *increment* ΔJ be defined by

$$\Delta J \triangleq J(x + \epsilon\, \delta x) - J(x) \tag{3-2}$$

and assume that the range of t of interest is fixed. Also, for a given x and δx, suppose that $J(x + \epsilon\, \delta x)$ possesses finite derivatives of all orders with respect to ϵ in an open neighborhood of $\epsilon = 0$. It follows that a Maclaurin's series expansion with respect to ϵ can be made of the first term in the right-hand member of 3-2 with the result that

$$\Delta J = \frac{\partial J(x + \epsilon\, \delta x)}{\partial \epsilon}\bigg|_{\epsilon=0} \epsilon + \frac{\partial^2 J(x + \epsilon\, \delta x)}{\partial \epsilon^2}\bigg|_{\epsilon=0} \frac{\epsilon^2}{2} + \cdots \tag{3-3a}$$

By convention (dating back to Lagrange) the first term $[\partial J(x + \epsilon\, \delta x)/\partial\epsilon]_{\epsilon=0}\, \epsilon$ is called the *first variation of the functional* J evaluated at x and is denoted by δJ. As is shown in Section 3-3, a necessary condition for a relative extremum of J evaluated at a particular x, whether it be strong or weak, is that δJ must equal zero for that x when $\epsilon\, \delta x$ is an admissible variation.[2] In like manner, the term $[\partial^2 J(x + \epsilon\, \delta x)/\partial\epsilon^2]_{\epsilon=0}\, \epsilon^2$ of 3-3a is called the *second variation of the functional* J evaluated at x and is denoted by $\delta^2 J$. The second variation plays an important role in the establishment of sufficient conditions for weak relative extrema. In general, the increment ΔJ can be expressed in the form

$$\Delta J = \delta J + \frac{\delta^2 J}{2!} + \frac{\delta^3 J}{3!} + \cdots \tag{3-3b}$$

in which the first variation δJ is linear in ϵ, the second variation $\delta^2 J$ is linear in ϵ^2, etc. The evaluation of the series 3-3 at $\epsilon = 1$ is a special case of particular interest.

3-2b. Classes of Problems and Equivalence Relations

In this subsection, three related variational problems are examined. The distinguishing feature of these problems is the form of the functional employed—the three functionals are J, J_μ, and J_β, as follows:

$$J = \int_{t_a}^{t_b} f(\mathbf{x},\dot{\mathbf{x}},t)\, dt \qquad \text{(the problem of Lagrange)} \tag{3-4}$$

$$J_\mu = f_\mu(\mathbf{x},t)\Big|_{t_a}^{t_b} = f_\mu[\mathbf{x}(t_b),t_b] - f_\mu[\mathbf{x}(t_a),t_a] \qquad \text{(the problem of Mayer)} \tag{3-5}$$

and

$$J_\beta = f_{\beta 1}(\mathbf{x},t)\Big|_{t_a}^{t_b} + \int_{t_a}^{t_b} f_{\beta 2}(\mathbf{x},\dot{\mathbf{x}},t)\, dt \qquad \text{(the problem of Bolza)} \tag{3-6}$$

[2] An admissible variation is one that satisfies the properties of an admissible curve.

where $\mathbf{x} \equiv \mathbf{x}(t)$ represents the set $\{x_1(t),x_2(t),\ldots,x_n(t)\}$ of real-valued functions of time, $\dot{\mathbf{x}}$ represents the set $\{\dot{x}_1(t),\dot{x}_2(t),\ldots,\dot{x}_n(t)\}$ of derivatives of $\mathbf{x}$, and $f(\mathbf{x},\dot{\mathbf{x}},t)$, $f_\mu(\mathbf{x},t)$, $f_{\beta 1}(\mathbf{x},t)$, and $f_{\beta 2}(\mathbf{x},\dot{\mathbf{x}},t)$ are given real-valued functions of class C^2 with respect to their arguments, that is, partial derivatives of these functions up to and including the second are assumed to exist and to be continuous. It is convenient to express $\mathbf{x}$ in column matrix form,

$$\mathbf{x} = [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]' \tag{3-7}$$

and $\mathbf{x}$ is called a state vector.

The functionals of 3-4, 3-5, and 3-6 are typically used as performance measures for the design and/or operation of physical systems. The general problem associated with the extremization of J, Equation 3-4, is known as the problem of Lagrange; that associated with J_μ, Equation 3-5, is known as the problem of Mayer; and that associated with J_β, Equation 3-6, is known as the problem of Bolza [3.7]. In a given problem, the objective is to find a trajectory $\mathbf{x}^*(t)$ which when substituted for $\mathbf{x}(t)$ in a particular performance measure results in the optimum of the performance measure. Of course, certain of the x_i's and $\dot{x}_i$'s may be constrained to satisfy known physical laws; such constraints typically take the form of differential equations—these and other constraint forms are considered later in this chapter. Similarly, the *end conditions* $\mathbf{x}(t_a)$, $\mathbf{x}(t_b)$, t_a, and t_b may be constrained to satisfy prescribed relationships; these may either partially or completely prescribe $\mathbf{x}(t_a)$, $\mathbf{x}(t_b)$, t_a, and t_b. When not completely prescribed, the optimal end conditions may be found by use of *transversality conditions* (Sections 3-5 and 3-7).

In this section, the objective is simply to show that the problems of Bolza and Mayer can be placed in the form of the problem of Lagrange. Thus, the theory that is developed in this chapter for solution of Lagrange's problem can be applied to the solution of the other two problems as well.

Consider first the problem of Mayer associated with 3-5. Because of the assumed differentiability of $f_\mu(\mathbf{x},t)$, J_μ of 3-5 can be placed in the form

$$\begin{aligned} J_\mu &= \int_{t_a}^{t_b} \frac{df_\mu(\mathbf{x},t)}{dt}\, dt \\ &= \int_{t_a}^{t_b} \left\{ \frac{\partial f_\mu(\mathbf{x},t)}{\partial t} + \left[\frac{\partial f_\mu(\mathbf{x},t)}{\partial \mathbf{x}}\right]' \dot{\mathbf{x}} \right\} dt \end{aligned} \tag{3-8}$$

in which $\partial f_\mu(\mathbf{x},t)/\partial\mathbf{x}$ is the *gradient* of $f_\mu(\mathbf{x},t)$ with respect to $\mathbf{x}$ and is defined, as expressed in column matrix form, by

$$\frac{\partial f_\mu(\mathbf{x},t)}{\partial \mathbf{x}} \triangleq \left[\frac{\partial f_\mu(\mathbf{x},t)}{\partial x_1} \quad \frac{\partial f_\mu(\mathbf{x},t)}{\partial x_2} \quad \cdots \quad \frac{\partial f_\mu(\mathbf{x},t)}{\partial x_n}\right]' \tag{3-9}$$

Obviously, J_μ of 3-8 is of the form of J of 3-4, and the desired equivalence is established.

For the problem of Bolza associated with 3-6, the same approach as followed in the preceding paragraph can be applied to the first term of the right-hand member of 3-6, with the result that

$$J_\beta = \int_{t_a}^{t_b} \left\{ f_{\beta 2}(\mathbf{x},\dot{\mathbf{x}},t) + \frac{\partial f_{\beta 1}(\mathbf{x},t)}{\partial t} + \left[\frac{\partial f_{\beta 1}(\mathbf{x},t)}{\partial \mathbf{x}}\right]' \dot{\mathbf{x}} \right\} dt \tag{3-10}$$

As before, this expression is in the form of J of 3-4, and the desired relationship between Bolza's problem and Lagrange's problem is thereby established.

3-3. THE PROBLEM OF LAGRANGE: SCALAR CASE

3-3a. Problem Statement and the First Variation

Consider the problem of determining the particular real curve $x^* \equiv x^*(t)$ which, when substituted for x, yields the minimum (maximum) of the functional J:

$$J \equiv J(x) = \int_{t_a}^{t_b} f(x,\dot{x},t)\, dt \tag{3-11}$$

where t_a, t_b, $x(t_a) = c_a$, and $x(t_b) = c_b$ are fixed. The necessary and the sufficient conditions that are developed in this chapter are applicable to the extremization of J, Equation 3-11, provided that the real-valued function $f \equiv f(x,\dot{x},t)$ is of class C^2 with respect to all of its arguments.

The preceding statements constitute the most basic form of the problem known as the problem of Lagrange in the calculus of variations. All of the results that are derived for the solution of this problem in this section are extendible to more complex situations.

The optimal $x(t)$, $x^*(t)$, is assumed to belong to a family of functions with certain properties in common. One property of this family is that the end conditions must be satisfied in each case, i.e., for each x of the family, $x(t_a) = c_a$ and $x(t_b) = c_b$. A second property of the family is that $x_\alpha = x + \epsilon(x_\beta - x)$ is a member of the family if x and x_β are members of the family where ϵ is any real number. Notice in particular that if x^* is the optimal curve, then $x_\alpha = x^* + \epsilon\,\delta x$ is a member of the family where $\epsilon\,\delta x = \epsilon(x_\beta - x^*)$ is a *variation* of x^*, as defined in the preceding section.

Consider the nonoptimal value of J given by

$$J(x^* + \epsilon\,\delta x) = \int_{t_a}^{t_b} f(x^* + \epsilon\,\delta x,\ \dot{x}^* + \epsilon\,\delta\dot{x},\ t)\, dt \tag{3-12}$$

where $\delta\dot{x}$ is used to denote $d\,\delta x/dt$.

For the sake of clarity, assume that $J(x^*)$ is the minimum, rather than the maximum of $J(x)$. Then for any particular δx, a plot of J versus ϵ similar to that of Figure 3-2a can be drawn. Corresponding to the definition given in the preceding section, the quantity ϵ multiplied by the slope of the curve at $\epsilon = 0$ in Figure 3-2a is the *first variation of J* for the particular case that x equals x^*. In general, the first variation δJ is given by

$$\delta J = \left.\frac{\partial J(x + \epsilon\,\delta x)}{\partial \epsilon}\right|_{\epsilon=0} \epsilon \tag{3-13}$$

Observe that the slope $[\partial J(x^* + \epsilon\,\delta x)/\partial\epsilon]|_{\epsilon=0}$ in Figure 3-2a must equal zero for any allowable δx if $J(x^*)$ is to be truly the minimum; therefore, a *necessary condition* for the minimum of J is that $[\partial J(x + \epsilon\,\delta x)/\partial\epsilon]|_{\epsilon=0}$ must equal zero when x equals x^*, where

$$\left.\frac{\partial J(x + \epsilon\,\delta x)}{\partial \epsilon}\right|_{\epsilon=0} = \left\{\frac{\partial}{\partial\epsilon}\int_{t_a}^{t_b} f(x + \epsilon\,\delta x,\ \dot{x} + \epsilon\,\delta\dot{x},\ t)\, dt\right\}_{\epsilon=0} \tag{3-14a}$$

The integrand of this equation can be expanded in a Taylor's series about x, $\dot{x}$, and t; thus,

$$\begin{aligned}\left.\frac{\partial J(x + \epsilon\,\delta x)}{\partial \epsilon}\right|_{\epsilon=0} &= \left\{\frac{\partial}{\partial\epsilon}\int_{t_a}^{t_b}\left[f(x,\dot{x},t) + \epsilon\,\delta x\,\frac{\partial f(x,\dot{x},t)}{\partial x}\right.\right.\\ &\qquad\left.\left. + \epsilon\,\delta\dot{x}\,\frac{\partial f(x,\dot{x},t)}{\partial \dot{x}} + \epsilon o(\epsilon)\right] dt\right\}_{\epsilon=0}\\ &= \left\{\int_{t_a}^{t_b}\left[\frac{\partial f(x,\dot{x},t)}{\partial x}\,\delta x + \frac{\partial f(x,\dot{x},t)}{\partial \dot{x}}\,\delta\dot{x} + o_1(\epsilon)\right] dt\right\}_{\epsilon=0}\\ &= \int_{t_a}^{t_b} (f_x\,\delta x + f_{\dot{x}}\,\delta\dot{x})\, dt \end{aligned} \tag{3-14b}$$

in which f_x is the partial derivative of $f(x,\dot{x},t)$ with respect to x (meaning that $\dot{x}$ and t are treated as constants in the differentiation), and $f_{\dot{x}}$ is the partial derivative of $f(x,\dot{x},t)$ with respect to $\dot{x}$ (meaning that x and t are

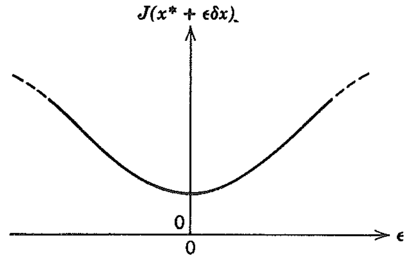

Figure 3-2a. Plot of $J(x^* + \epsilon\,\delta x)$ versus ϵ: $J(x^*)$ minimum, and δx fixed but typical.

treated as constants in the differentiation). Both $o(\epsilon)$ and $o_1(\epsilon)$ in 3-14b are functions of order ϵ, meaning that $\lim_{\epsilon\to 0+} o_1(\epsilon) = 0$. The fact that the right-hand member of 3-14b must equal zero when x equals x^* is of little use in the present form; what is desired is an expression which is independent of δx and which can be used with other facts to find $x^*(t)$. Such an expression is derived as follows.

The first term of the integrand of 3-14b is integrated by parts to obtain

$$\left.\frac{\partial J(x + \epsilon\, \delta x)}{\partial \epsilon}\right|_{\epsilon=0} = \left[\delta x \int f_x\, dt\right]_{t_a}^{t_b} + \int_{t_a}^{t_b} \left(f_{\dot{x}} - \int f_x\, dt\right) \delta\dot{x}\, dt \qquad (3\text{-}15)$$

In this section, the end conditions are assumed to be fixed, and therefore $\delta x(t_a) = \delta x(t_b) = 0$. Thus,

$$\left.\frac{\partial J(x + \epsilon\, \delta x)}{\partial \epsilon}\right|_{\epsilon=0} = \int_{t_a}^{t_b} N\, \delta\dot{x}\, dt \qquad (3\text{-}16)$$

where

$$N \triangleq f_{\dot{x}} - \int f_x\, dt \qquad (3\text{-}17)$$

The condition on N which assures that $\delta J = 0$ is embodied in the following lemma which is known as the *fundamental lemma* of the calculus of variations.

3-3b. Fundamental Lemma

Of all bounded, single-valued functions $N = N(t)$ which are continuous on the interval $[t_a,t_b]$, except possibly at a finite number of jump-type discontinuities, only those $N(t)$ which equal a constant on $[t_a,t_b]$ result in zero values of the integral

$$\int_{t_a}^{t_b} N(t)\, \delta\dot{x}(t)\, dt \qquad (3\text{-}18)$$

for every admissible $\delta x(t)$ where $\delta x(t_a) = \delta x(t_b) = 0$.

In other words, of the class of functions considered for $N(t)$, it is necessary and sufficient that $N(t)$ equal a constant c to insure that $\delta J = 0$. That it is sufficient for $N(t)$ to equal c is established by integrating 3-18 by parts,

$$\int_{t_a}^{t_b} N(t)\, \delta\dot{x}(t)\, dt = [N(t)\, \delta x]\Big|_{t_a}^{t_b} - \int_{t_a}^{t_b} \frac{dN(t)}{dt}\, \delta x\, dt \qquad (3\text{-}19)$$

The first term of the right-hand member of 3-19 is zero because $\delta x(t_a) = \delta x(t_b) = 0$; and if $N(t) = c$, a constant, the second term of the right-hand member of 3-19 is also zero.

The necessary part of the lemma is established by considering a particular variation which is given by

$$\delta\dot{x}(t) = N(t) - c \qquad (3\text{-}20)$$

and by noting that

$$\int_{t_a}^{t_b} [N(t) - c]\, \delta\dot{x}(t)\, dt \qquad (3\text{-}21)$$

must vanish for all admissible $\delta x(t)$ if 3-18 is to vanish also. It follows, upon substitution of the right-hand member of 3-20 for $\delta\dot{x}(t)$ in 3-21, that

$$\int_{t_a}^{t_b} [N(t) - c]^2\, dt = 0 \qquad (3\text{-}22)$$

only if $N(t) = c$.

3-3c. First Necessary Condition and First-Variational Curves

The important result thus far is that the equation

$$f_{\dot{x}} = \int f_x\, dt + c \qquad (3\text{-}23)$$

must be satisfied by the optimal $x = x^*(t)$ which results in the minimum (maximum) of J. Unfortunately, more than one function may satisfy 3-23. For example (see Figure 3-2b), the first variation δJ of J about the curve x_s is zero, but x_s is obviously not optimum. However, in the case that the solution of 3-23 is unique and the optimal $x = x^*$ is from the class of continuous functions considered in this chapter, it follows necessarily that the solution is optimal. In short, Equation 3-23 constitutes a necessary condition—often called the *first necessary condition*—for the optimal x, but not a sufficient one.

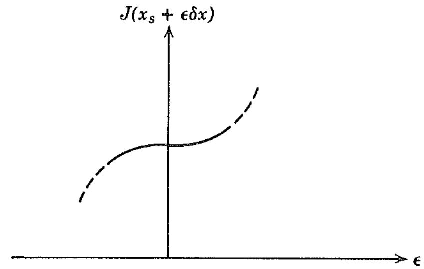

Figure 3-2b. Plot of $J(x_s + \epsilon\, \delta x)$ versus ϵ: $J(x_s)$ nonoptimal, δx fixed but typical, and $[\partial J(x_s + \epsilon\, \partial x)/\partial\epsilon]|_{\epsilon=0}$ equals zero.

Curves $x(t)$ which satisfy the first necessary condition are named *extremal curves* in the classical literature, an unfortunate designation in that all such curves do not result in relative extrema of J. In analogy with the designation of stationary points in Chapter 2, we could conceivably refer to such curves as stationary curves, but this too is nondescriptive, if not altogether misleading. Instead, curves which satisfy 3-23 are designated as *first-variational curves* in this book because they are obtained by operating on the first variation of the functional J.

3-3d. A Corner Condition

Equation 3-23 is now employed in developing two important results which are used in the determination of first-variational curves. The first of these results involves the concept of a *corner point* of a function. In Figure 3-3, the function $x(t)$ has a corner point at $t = t_c$. A corner point of $x(t)$, therefore, is a point at which the derivative dx/dt is not uniquely defined; i.e., a point at which dx/dt possesses a jump discontinuity. Corner points of functions are found in problems which involve reflection or refraction, for example.

Suppose $x(t)$ is a continuous first-variational curve; the question is, "Does $x(t)$ possess corner points, and if so, where are they?" As a first step toward the answer to this question, observe that the integrand of the indefinite integral in 3-23 exhibits, at most, bounded jump discontinuities as a function of t when $x(t)$ is a continuous first-variational curve. The integral of such a function is continuous, so the right-hand member of 3-23 is continuous. But then the left-hand member of 3-23 must be continuous too. Thus, if $x(t)$ has a corner point at $t = t_c$, it must still be true that[3]

$$f_{\dot{x}}|_{t=t_c-0} = f_{\dot{x}}|_{t=t_c+0} \tag{3-24}$$

The above condition is one part of the Erdmann-Weierstrass corner conditions which are treated in more detail in Section 3-6. In conjunction with the results that follow, the Erdmann-Weierstrass corner conditions can be used to determine locations of corner points, if they exist.

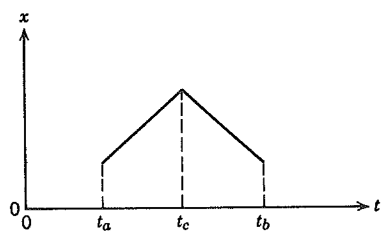

Figure 3-3. Corner point at $t = t_c$.

[3] The expressions $t_c - 0$ and $t_c + 0$ denote, respectively, $\lim_{\epsilon \to 0+} t_c - \epsilon$ and $\lim_{\epsilon \to 0+} t_c + \epsilon$.

3-3e. The Euler-Lagrange Equation

The fact that the left- and right-hand members of 3-23 are continuous enables us to differentiate and obtain

$$\frac{d}{dt}(f_{\dot{x}}) - f_x = 0 \tag{3-25}$$

which is known as the *Euler-Lagrange equation*—unquestionably the most famous result of the calculus of variations.

At values of t between corners of $x(t)$, differentiation of the first term in 3-25 can be effected to yield

$$\frac{d^2x}{dt^2}f_{\dot{x}\dot{x}} + \frac{dx}{dt}f_{\dot{x}x} + f_{\dot{x}t} = f_x \tag{3-26}$$

where $x(t_a) = c_a$ and $x(t_b) = c_b$ are the given boundary conditions. Because of the specified end conditions, the problem of solving 3-26 is called a *two-point boundary-value problem*. In some cases, e.g., the detailed examples in this chapter, the Euler-Lagrange equations can be solved in closed form, and two-point boundary conditions are easily incorporated in the solution. In other cases, however, special techniques ([3.5, 3.11, 3.15, 3.17, and 3.26] for example) may be required for solution of corresponding two-point boundary-value problems. Dynamic programming (Chapter 7), direct methods (Section 3-11), and search techniques (Chapter 6 and Section 8-8) are of use in this regard. Several important cases of Euler-Lagrange equations are given below.

Example 3-1. ($f(x,\dot{x},t)$ quadratic in x and $\dot{x}$.)
Suppose that

$$f = a_1\dot{x}^2 + a_2\dot{x}x + a_3x^2 + g_1(t)\dot{x} + g_2(t)x \tag{3-27}$$

is the integrand of 3-11 where a_1, a_2, and a_3 are real constants, and $g_1(t)$ and $g_2(t)$ are given, differentiable, real-valued functions. The occurrences of this type of problem and related problems in technical fields are many and varied ([3.13, 3.19, and 3.20] for example). In this case, the Euler-Lagrange equation 3-26 between corners reduces to

$$2a_1\ddot{x} - 2a_3x = g_2(t) - \dot{g}_1(t) \tag{3-28a}$$

which is a nonhomogeneous, second-order, linear differential equation.

Assume that a_1 and a_3 are greater than zero; and set $\alpha^2 \triangleq (a_3/a_1)$ and $g_0(t) \triangleq [g_2(t) - \dot{g}_1(t)]/2a_1$. The solution of 3-28a for $t \geq t_a$ is

$$x(t) = c_1 e^{-\alpha t} + c_2 e^{\alpha t} + \frac{1}{2\alpha}\int_0^{t-t_a} (e^{\alpha\tau} - e^{-\alpha\tau}) g_0(t - \tau)\, d\tau \quad (3\text{-}28b)$$

where c_1, c_2, and t_a are constants that are determined so as to satisfy boundary conditions in any given case.

●

Example 3-2. (f formally independent of t.)

Assume that f is formally independent of t, meaning that t is not an explicit argument of f, whereas $x(t)$ and $\dot{x}(t)$ may be, and therefore $\partial f/\partial t$ is zero. Consider the identity

$$\begin{aligned}\frac{d}{dt}(\dot{x}f_{\dot{x}} - f) &= \dot{x}\ddot{x}f_{\dot{x}\dot{x}} + \dot{x}^2 f_{x\dot{x}} + \ddot{x}f_{\dot{x}} - \dot{x}f_x - \ddot{x}f_{\dot{x}} \\ &= \dot{x}(\ddot{x}f_{\dot{x}\dot{x}} + \dot{x}f_{x\dot{x}} - f_x) \quad (3\text{-}29)\end{aligned}$$

The term within the parentheses of the right-hand member of 3-29 equals zero for any x which satisfies the Euler-Lagrange equation 3-26 with $f_{\dot{x}t} = 0$. Hence, in this case the Euler-Lagrange equation has a *first integral* given by

$$\dot{x}f_{\dot{x}} - f = c \quad (3\text{-}30)$$

where c is a constant. This first integral may be solved by solving 3-30 for $\dot{x}$ in terms of x and c followed by application of the method of separation of variables.

●

Example 3-3. (f formally independent of $\dot{x}$.)

If $f_{\dot{x}\dot{x}} = f_{\dot{x}x} = f_{\dot{x}t} = 0$, the Euler-Lagrange equation reduces to

$$f_x = 0 \quad (3\text{-}31)$$

The solution of this equation involves no arbitrary constants, and therefore, boundary conditions cannot be specified arbitrarily, unless of course the class of functions of which the optimal x is a member is broadened to include functions with jump-type discontinuities at the ends of the interval.

In many practical problems of the preceding type, end conditions on x are not given explicitly. For example, consider the problem of minimizing $\overline{e^2}$, the mean-square error between a pulse-code-modulated signal and its detected interpretation [3.2]:

$$\overline{e^2} = \frac{1}{2}\int_{-\infty}^{\infty} [(1 - x)^2 p(\eta - a) + x^2 p(\eta + a)]\, d\eta \quad (3\text{-}32)$$

where $p(\eta - a)$ denotes the probability density function for noise associated with a "positive" pulse when received in the presence of additive Gaussian noise at a signal-to-noise ratio of a^2, $p(\eta + a)$ is defined correspondingly but for a "negative" pulse, and $x = x(\eta)$ is the desired detector characteristic. The Euler-Lagrange equation is

$$f_x = -(1 - x)p(\eta - a) + xp(\eta + a) = 0 \quad (3\text{-}33)$$

from which the optimal $x(\eta)$, $x^*(\eta)$, is obtained:

$$x^*(\eta) = \frac{p(\eta - a)}{p(\eta - a) + p(\eta + a)} \quad (3\text{-}34)$$

●

3-4. ISOPERIMETRIC CONSTRAINTS

As an additive feature to the problem given in Section 3-2a, consider a constraint equation of the form

$$K_1 = \int_{t_a}^{t_b} f_1(x,\dot{x},t)\, dt \quad (3\text{-}35)$$

where K_1 is a constant and $f_1 \equiv f_1(x,\dot{x},t)$ is a known real-valued function of its arguments with properties equivalent to those assumed for f in 3-11. Numerous examples of such constraints occur in practice because of limited resources (e.g., limited energy, limited fuel, limited surface area, etc.) available. In the classical literature, the first problems of this sort to be considered were problems concerning the determination of the maximum area of an enclosed planar surface where the "perimeter" of the area was constrained; hence, the term "isoperimetric" constraint was used, and this terminology remains in use today for constraints of the form of 3-35.

Isoperimetric constraints are easily incorporated in the problem of Section 3-3 by use of Lagrange multipliers—the Lagrange multiplier is used in the same manner as in Chapter 2.

Isoperimetric Theorem. *Assume that the function $x^*(t)$ is a first-variational curve which results in the maximum of the functional*

$$J_a(x) = \int_{t_a}^{t_b} [f(x,\dot{x},t) + h_1 f_1(x,\dot{x},t)]\, dt \quad (3\text{-}36)$$

That is,

$$J_a(x) \leq J_a(x^*) = \int_{t_a}^{t_b} [f(x^*,\dot{x}^*,t) + h_1 f_1(x^*,\dot{x}^*,t)]\, dt \quad (3\text{-}37)$$

where h_1 is independent of x and t. Also, assume that constraint 3-35 *is satisfied by*

$x^*(t)$. Then $x^*(t)$ gives rise to the maximum of J in 3-11 *subject to the isoperimetric constraint* 3-35.

The proof of the theorem follows by considering the result which would hold if the theorem were false, i.e., assume that there exists a function $r(t)$ which has the property

$$\int_{t_a}^{t_b} f(r,\dot{r},t)\, dt > \int_{t_a}^{t_b} f(x^*,\dot{x}^*,t)\, dt \tag{3-38}$$

and which satisfies the constraint equation

$$\int_{t_a}^{t_b} f_1(r,\dot{r},t)\, dt = K_1 \tag{3-39}$$

Then

$$\int_{t_a}^{t_b} f(r,\dot{r},t)\, dt + h_1 \int_{t_a}^{t_b} f_1(r,\dot{r},t)\, dt = \int_{t_a}^{t_b} f(r,\dot{r},t)\, dt + h_1 K_1 \tag{3-40}$$

But now the right-hand member of 3-40 is greater than that of 3-37, hence, a contradiction! The assumption that the theorem is not true is invalid, and the validity of the theorem follows.

●

With a change in inequality signs, the above argument applies equally well to the case that the minimum, rather than the maximum, of J is desired. Also, if two or more constraints of the form 3-35 exist, they can be treated by assigning a separate Lagrange multiplier to each constraint and by proceeding as before. Thus, the procedure to be used in solving such problems is similar to that used in Chapter 2 to treat equality constraints associated with classical min-max problems.

To utilize the theorem, one obtains the Euler-Lagrange equation corresponding to the *augmented functional* 3-36 and solves for first-variational curves in terms of t and the Lagrange multiplier h_1. These are then substituted into the constraint Equation 3-35 which is integrated to obtain an explicit value of h_1 for each such curve in terms of the constant K_1. In some cases, it is easier to alter this procedure and to treat K_1 as a parameter. When this is true, various values of h_1 are assumed, and corresponding values of K_1 are evaluated. Finally, the maximum (minimum) $J(x^*)$ of J may be found by direct comparison of those values (of J) which correspond to first-variational curves (of J_a) that satisfy 3-35.

Example 3-4. In the circuit shown in Figure 3-4, the input $v_i(t)$ is a voltage pulse of width T. The pulse shape $v_i \equiv v_i(t)$ is to be selected such that the average value J of the output voltage $v_o \equiv v_o(t)$ is maximized over the

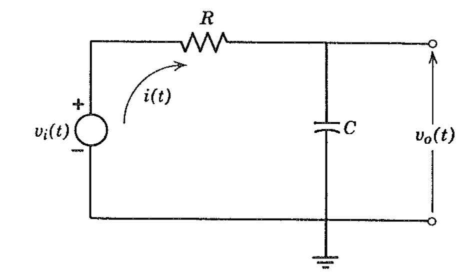

Figure 3-4. Voltage $v_i(t)$ to be selected for the optimum.

interval $0 \le t \le T$.

$$J = \int_0^T \frac{v_o}{T}\, dt \tag{3-41}$$

The maximum value of J is limited, however, by the limited energy K which can be dissipated in the resistor R over the period $0 \le t \le T$. Thus, a constraint is

$$K = \int_0^T i^2 R\, dt \tag{3-42}$$

where $i \equiv i(t)$ is the current through the resistor. End-point conditions are given at $t = 0$ and $t = T$: $v_o(0) = 0$, and $v_o(T) = V$ where V is a given constant.

Problem: Find the optimal $v_o(t)$ and the corresponding $v_i(t)$.

As a first step in the solution, the identity $C\dot{v}_o = i$ is used in place of i in 3-42 to reduce the problem to one involving one dependent function. Thus,

$$K = \int_0^T RC^2\dot{v}_o^2\, dt \tag{3-43}$$

and the augmented functional J_a is formed,

$$J_a = \frac{1}{T}\int_0^T (v_o + h_1 RC^2 T\dot{v}_o^2)\, dt \tag{3-44}$$

from which the Euler-Lagrange equation is obtained as a special case of 3-28:

$$\frac{\ddot{v}_o}{\beta} - 1 = 0, \qquad v_o(0) = 0 \quad \text{and} \quad v_o(T) = V \tag{3-45}$$

where $\beta = 1/(2h_1RC^2T)$.

The solution to 3-45 is

$$v_o = c_2t^2 + c_1t + c_0 \tag{3-46}$$

where the constants c_0, c_1, and c_2 must be selected to satisfy the prescribed boundary conditions and to satisfy constraint 3-43. (Note that the Lagrange multiplier h_1 is absorbed in c_0, c_1, and c_2.) At $t = 0$, $v_o(0)$ equals 0, and c_0 must therefore be zero. At $t = T$, $v_o = V$ and from 3-46

$$V = c_2T^2 + c_1T \tag{3-47}$$

A second relationship between c_1 and c_2 is obtained by using the identity $\dot{v}_o = 2c_2t + c_1$ in constraint equation 3-43:

$$\begin{aligned} \frac{K}{RC^2} &= \int_0^T (2c_2t + c_1)^2\, dt \\ &= \tfrac{1}{3}(4c_2{}^2T^3 + 6c_2c_1T^2 + 3c_1{}^2T) \end{aligned} \tag{3-48}$$

Simultaneous solution of 3-47 and 3-48 results in

$$c_1 = \frac{1}{T}\left\{V \pm \left[\frac{3KT}{RC^2} - 3V^2\right]^{1/2}\right\} \tag{3-49}$$

and

$$c_2 = \mp\frac{1}{T^2}\left[\frac{3KT}{RC^2} - 3V^2\right]^{1/2} \tag{3-50}$$

In Equations 3-49 and 3-50, the choice of the upper sign in the $\pm$ and $\mp$ terms corresponds to a maximum of J.

It should be observed that a real answer is obtainable if and only if the terms under the square-root signs in 3-49 and 3-50 are non-negative, that is, K must be greater than or equal to V^2RC^2/T if c_1 and c_2 are to be real. This means, in essence, that if $K < V^2RC^2/T$, the output voltage cannot be changed from zero volts at $t = 0$ to V volts at $t = T$ because of the lack of allowable dissipated energy.

Note that $\dot{v}_o{}^2$ appears as an additive term in the integrand of 3-44. If v_o should exhibit any radical changes, such as jump-type discontinuities, $\dot{v}_o$ would exhibit even more pronounced changes, such as impulse-type discontinuities, and the effect of these on the integral of $\dot{v}_o{}^2$ would be to radically increase K.[4] Thus, the optimal $v_o(t)$, $v_o{}^*(t)$, must be continuous and a first-variational curve. Because the derived first-variational curve (3-46) satisfies all prescribed conditions, it must be the optimal curve $v_o{}^*(t)$.

The required optimal $v_i(t)$, $v_i{}^*(t)$, is determined by using 3-46 in conjunction with the relation

$$\begin{aligned} v_i{}^* &= RC\dot{v}_o{}^* + v_o{}^* \\ &= c_2t^2 + (c_1 + 2RCc_2)t + RCc_1 \end{aligned} \tag{3-51}$$

where, as before, c_1 and c_2 are given by 3-49 and 3-50, respectively.

[4] Over an interval $[t_1,t_1 + \epsilon]$ contained in $[0,T]$, suppose that $v_o(t) = v_o(t_1) + [(t - t_1)/\epsilon][v_o(t_1 + \epsilon) - v_o(t_1)]$, and therefore $\dot{v}_o(t) = [v_o(t_1 + \epsilon) - v_o(t_1)]/\epsilon$ for $t \in (t_1,t_1 + \epsilon)$. The integral of $\dot{v}_o{}^2$ from t_1 to $t_1 + \epsilon$ is $[v_o(t_1 + \epsilon) - v_o(t_1)]^2/\epsilon$. This integral can be made arbitrarily large through selection of ϵ, $0 < \epsilon \ll 1$, if $v_o(t_1 + \epsilon)$ differs from $v_o(t_1)$ by a fixed amount, say, 1.

Finally, the maximum value of J is found by substituting the right-hand member of 3-46 for $v_o(t)$ in 3-41 as follows:

$$\begin{aligned} J &= \frac{1}{T}\int_0^T (c_2t^2 + c_1t)\, dt \\ &= \frac{c_2T^2}{3} + \frac{c_1T}{2} \\ &= \frac{1}{6}\left\{3V + \left[\frac{3KT}{RC^2} - 3V^2\right]^{1/2}\right\} \end{aligned} \tag{3-52}$$

Example 3-5. Consider the regulating control system shown in block-diagram form in Figure 3-5. The purpose of the controller is to generate a control signal $v(t)$ (volts) so that the difference between the desired output θ_d (degrees) and the actual output θ (degrees) is minimized in some sense over time. The function $G(s)$ is a transfer function; $G(s)$ is the ratio of the one-sided Laplace transform of $\theta(t)$ to that of $v(t)$ under the condition that both $\theta(0)$ and $\dot{\theta}(0)$ are zero. $G(s)$ characterizes an electromechanical, rotational system. The constant J_m (newton-meter-sec^2/degree) in Figure 3-5 is the effective inertia of the motor and mechanical load, as reflected to the output shaft; B_m (newton-meter-sec/degree) is the effective viscous friction; and K_m (newton-meter/volt) is a torque conversion factor.

The following simplifying conditions are assumed: $K_m = 1$; $\theta(0) = 0$; for $-\infty < t < 0$, $\theta_d = 0$; and for $0 \le t \le \infty$, θ_d equals the real constant b. It is desired that the functional J be minimized:

$$J = \int_0^\infty (\theta - \theta_d)^2\, dt \tag{3-53}$$

which is the *integral of the squared error* (ISE).

Problem: Find the transfer function of the controller which minimizes J.

For J of Equation 3-53 to be finite, $\theta(t)$ must approach b as t approaches infinity; and since $\theta(0) = 0$, both end-point conditions are specified.

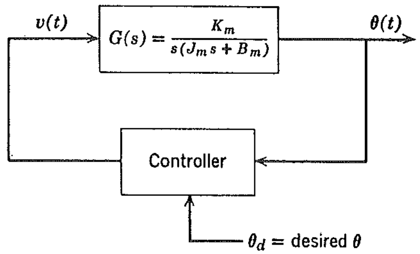

Figure 3-5. A feedback controller.

A limitation to be imposed on the minimization process is that the viscous frictional energy E_B divided by b^2 is not to exceed a specified value K; that is, $K = E_B/b^2$ where E_B is the energy which is dissipated as heat at the output of the controlled system as a result of a step change b of the desired output:

$$K = \int_0^\infty \frac{B_m\dot{\theta}^2}{b^2}\,dt \tag{3-54}$$

The augmented functional J_a is obtained by combining the right-hand members of 3-53 and 3-54 as follows:

$$J_a = \int_0^\infty \left[(\theta - \theta_d)^2 + \frac{h_1 B_m\dot{\theta}^2}{b^2}\right] dt \tag{3-55}$$

From the integrand of the above equation, the Euler-Lagrange equation is obtained in the form

$$\ddot{\theta} - \alpha^2(\theta - \theta_d) = 0 \tag{3-56}$$

where $\alpha^2 = b^2/h_1 B_m$. The general solution of this equation is of the form

$$\theta - \theta_d = c_1 e^{-\alpha t} + c_2 e^{\alpha t} \tag{3-57}$$

where c_1 and c_2 must be specified to satisfy the boundary conditions imposed on θ; thus, $c_2 = 0$, and $c_1 = -b$ with the result that

$$\theta = b(1 - e^{-\alpha t}) \tag{3-58}$$

In the above solution for θ, the parameter $\alpha^2 = b^2/h_1 B_m$ is yet to be evaluated. This is accomplished by substituting $\alpha b e^{-\alpha t}$ for $\dot{\theta}$ in the constraint equation (3-54) as follows:

$$K = \int_0^\infty B_m \alpha^2 e^{-2\alpha t}\,dt = \frac{B_m \alpha}{2} \tag{3-59}$$

from which

$$\alpha = \frac{2K}{B_m} \tag{3-60}$$

As in Example 3-4, jump-type discontinuities in the optimal solution are ruled out by the particular way in which the derivative of this solution appears in the integrand of the functional J_a. Thus, the optimal θ, θ^*, must be a first-variational function; and since 3-58 is the only such function which satisfies all prescribed conditions, it must be optimal. Thus,

$$\theta^*(t) = b[1 - e^{-(2K/B_m)t}] \tag{3-61}$$

Next, a specific form for the controller is found by noting that the Laplace transform $V^*(s)$ of the optimal control signal $v^*(t)$ is

$$V^*(s) = \frac{\Theta^*(s)}{G(s)} = b\left[\frac{1}{s} - \frac{1}{s+\alpha}\right]s(J_m s + B_m)$$
$$= \frac{b\alpha(J_m s + B_m)}{s+\alpha} \tag{3-62}$$

The above can be realized as a feedback signal by multiplying the error signal $\Theta_d(s) - \Theta^*(s)$ by $\alpha(J_m s + B_m)$. Hence, the block diagram of the optimal system is shown in Figure 3-6.

Several factors limit the practicality of the solution obtained. First, the controller of Figure 3-6 requires a "pure" differentiator which can be obtained only approximately in practice. Second, an instantaneous velocity change of the mechanical load takes place at $t = 0$. This instantaneous change would require an infinite acceleration at $t = 0$, an acceleration which can only be approached in practice and one which is undesirable because of the mechanical stresses which would result. And third, the question arises as to whether or not the frictional power absorbed by the controlled system is always greater than the inertial power released; if not, energy is being supplied over a period of time from the mechanical part of the system back to the electrical part, and this energy is usually wasted in the form of heat. Thus, in order to determine when this third consideration is not important, the condition under which

$$P_B(t) \geq P_J(t) \tag{3-63}$$

must be established: $P_B(t) = B_m\dot{\theta}^2$ is the power supplied to the frictional dissipating element, and $P_J(t) = -J_m\dot{\theta}\ddot{\theta}$ is the inertial power which is released. Thus, condition 3-63 is satisfied if

$$B_m\dot{\theta}^2 \geq -J_m\dot{\theta}\ddot{\theta} \tag{3-64}$$

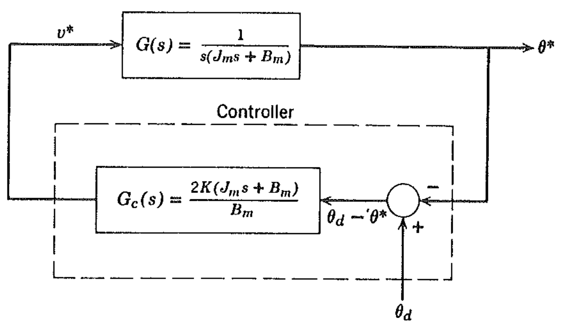

Figure 3-6. The "optimum" system of Example 3-5.

or because $\dot{\theta}^*$ is greater than zero for all $t > 0$,

$$B_m\dot{\theta}^* \geq -J_m\ddot{\theta}^* \tag{3-65}$$

With values of $\dot{\theta}^*$ and $\ddot{\theta}^*$ obtained from 3-58, the above reduces to

$$\frac{B_m^2}{2J_m} \geq K \tag{3-66}$$

If inequality 3-66 is not satisfied, we may wish to include additional constraints in the statement of the problem.

More complex forms of the above problem, and related problems, are of interest [3.13]; and another version of this problem is considered in Example 3-10.

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3-5. VARIABLE END-POINT CONDITIONS

End-point conditions are conditions imposed on the limits of integration t_a and t_b of the functional J and on the boundary values $x(t_a)$ and $x(t_b)$ of first-variational curves. When some or all of the end-point conditions are not fixed in advance, additional relations are needed to determine the particular end-point conditions, of those allowable, which result in the optimal solution. In the first part of this section, the most general case is treated; the values of t_a, t_b, $x(t_a)$, and $x(t_b)$ are assumed to satisfy arbitrary constraint equations, and conditions which the optimal values of the above must satisfy are found. In the latter part of this section, various special forms of constraint relationships are assumed to exist between the end-point values, and the corresponding end-point conditions which the optimal end-point values must satisfy are obtained.

The key result is given first. The *transversality condition* for the functional 3-11 is

$$[(f - \dot{x}f_{\dot{x}})\,\delta t]_{t_a}^{t_b} + [f_{\dot{x}}\,\delta x(t)]_{t_a}^{t_b} = 0 \tag{3-67}$$

where the meanings of the end-point variations δt_a, δt_b, $\delta x(t_a)$, and $\delta x(t_b)$ are shown in the following. Optimal values of t_a, t_b, $x(t_a)$, and $x(t_b)$ must satisfy 3-67. In fact, Equation 3-67, in conjunction with other necessary conditions, is used to obtain the optimal values of t_a, t_b, $x(t_a)$, and $x(t_b)$.

Equation 3-67 is sometimes denoted by

$$[(f - \dot{x}f_{\dot{x}})\,dt + f_{\dot{x}}\,dx]_{t_a}^{t_b} = 0 \tag{3-68}$$

It is to be emphasized, however, that the differentials dx and dt in 3-68 are to be interpreted as small variations which approach differentials if taken

sufficiently small in magnitude (see Case B of the latter part of this section).

If the conditions at one end point are fixed—for example, if t_a and $x(t_a)$ equal specified constants—then the variations δt_a and $\delta x(t_a)$ are zero, and 3-67 reduces to

$$[(f - \dot{x}f_{\dot{x}})\,\delta t]_{t_b} + [f_{\dot{x}}\,\delta x(t)]_{t_b} = 0 \tag{3-69}$$

The derivation of this equation is given below. It is clear that the proof of 3-67 is a simple extension of the following.

If a solution exists to the stated problem, at least one pair of optimal end points must exist in the x,t plane. If an optimal curve which connects these points has a time derivative that is piecewise continuous, it is necessarily a first-variational curve, as shown in preceding sections. Thus, classical admissible candidates for the optimum in variable end-point cases can be restricted to first-variational curves.

Consider the curves shown in Figure 3-7a. These curves are *assumed to be two of the first-variational curves* of the functional in question. In general, there exists a family of such curves, each member of the family corresponding to one particular value of $x(t_b)$ at $t = t_b$. For convenience in this derivation, it is assumed that the members of the family intersect at $t = t_a$ only. If other points of intersection exist in a given case, they can be detected by using the Jacobi condition (Section 3-10d).

The increment ΔJ in the value of J for the two curves shown in Figure 3-7a

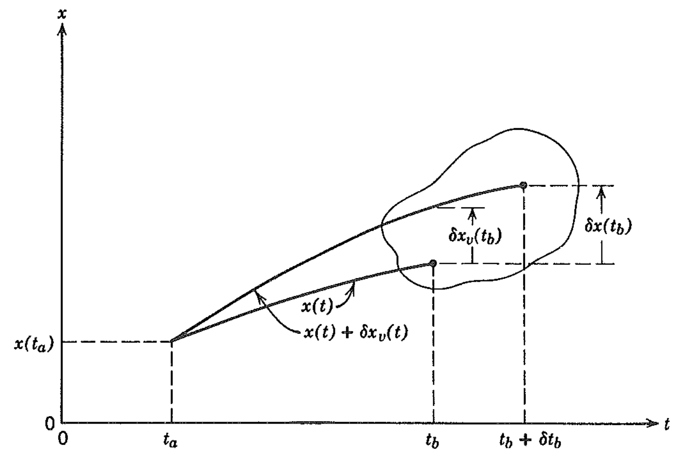

Figure 3-7a. Two first-variational curves; t_b and $x(t_b)$ to be selected for the optimum.

is of interest:

$$\Delta J = \int_{t_a}^{t_b} [f(x + \delta x_v, \dot{x} + \delta \dot{x}_v, t) - f(x,\dot{x},t)]\, dt$$
$$+ \int_{t_b}^{t_b+\delta t_b} f(x + \delta x_v, \dot{x} + \delta \dot{x}_v, t)\, dt \tag{3-70}$$

Excluding terms *of the order* of $(\delta t_b)^2$ or smaller, the value of the second integral in 3-70 is approximated by

$$f(x,\dot{x},t)|_{t=t_b}\, \delta t_b \tag{3-71}$$

As for the first integral in 3-70,

$$\int_{t_a}^{t_b} [f(x + \delta x_v, \dot{x} + \delta \dot{x}_v, t) - f(x,\dot{x},t)]\, dt \cong \int_{t_a}^{t_b} [f_x\, \delta x_v + f_{\dot{x}}\, \delta \dot{x}_v]\, dt \tag{3-72}$$

is obtained by a Taylor's series expansion of $f(x + \delta x_v, \dot{x} + \delta \dot{x}_v, t)$ about x, $\dot{x}$, t. Furthermore, integrating by parts reduces the right-hand member of 3-72 to

$$[f_{\dot{x}}\, \delta x_v]_{t_a}^{t_b} + \int_{t_a}^{t_b} \left(f_x - \frac{d}{dt} f_{\dot{x}}\right) \delta x_v\, dt \tag{3-73}$$

in which the term $f_x - df_{\dot{x}}/dt = 0$ because x is assumed to be a first-variational curve which satisfies 3-25, and $\delta x_v(t_a) = 0$ because $x(t_a)$ is assumed to be fixed. Thus, 3-71 and 3-73 lead to the following value of ΔJ for small (in magnitude) variations δt_b and $\delta x_v(t_b)$.

$$\Delta J \cong \delta J = [f(x,\dot{x},t)\, \delta t]_{t=t_b} + [f_{\dot{x}}\, \delta x_v]_{t=t_b} \tag{3-74}$$

Notice from Figure 3-7b that

$$\delta x_v(t_b) \cong \delta x(t_b) - \dot{x}(t_b)\, \delta t_b \tag{3-75}$$

which is used in 3-74 to obtain

$$\delta J = [(f - \dot{x} f_{\dot{x}})\, \delta t]_{t=t_b} + [f_{\dot{x}}\, \delta x]_{t=t_b} \tag{3-76}$$

If the above variation of the functional J is different from zero, some incremental changes in t_b or $x(t_b)$ or both would result in an incremental increase in J, while other incremental changes would result in decreases in J. Thus, a necessary condition for an extremum of J is that δJ of 3-76 equals zero, in which case 3-76 reduces to 3-69 which was to be proved. Important special cases are considered next.

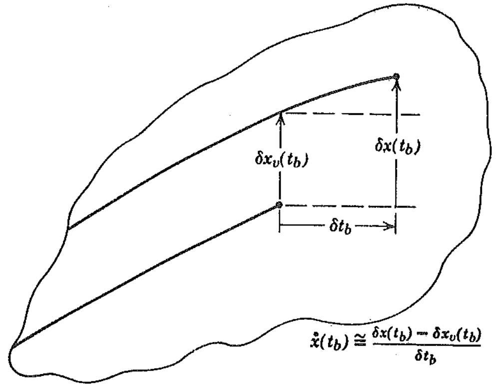

Figure 3-7b. Enlarged view of part of Figure 3-7a.

Case A. (All variations independent.)

When δt_a, δt_b, $\delta x(t_a)$, and $\delta x(t_b)$ are independent of one another, the only way in which 3-67 can be satisfied is that

$$[f - \dot{x} f_{\dot{x}}]_{t=t_a} = [f - \dot{x} f_{\dot{x}}]_{t=t_b} = [f_{\dot{x}}]_{t=t_a} = [f_{\dot{x}}]_{t=t_b} = 0 \tag{3-77}$$

and these four relations, in conjunction with the fact that x is a first-variational curve, are used in the determination of optimal values of t_a, t_b, $x(t_a)$, and $x(t_b)$.

Case B. (t_a and $x(t_a)$ fixed; $x(t)$ intersects $\phi(t)$ at $t = t_b$.)

In this case the right-hand end point of x is constrained to lie on a given curve $\phi(t)$ where $\dot{\phi}(t)$ exists and is continuous. Note from Figure 3-8a that

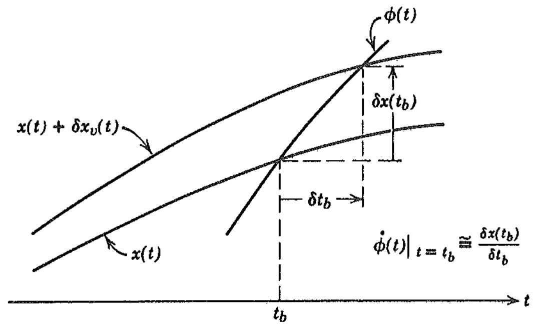

Figure 3-8a. The optimal curve constrained to intersect a given curve $\phi(t)$ at $t = t_b$.

$\delta x(t_b)/\delta t_b$ approaches $\dot{\phi}(t_b)$ as δt_b is decreased toward zero. It follows that $\dot{\phi}(t_b)\,\delta t_b$ can be substituted for $\delta x(t_b)$ in 3-69 with the result that

$$[f - \dot{x}f_{\dot{x}} + f_{\dot{x}}\dot{\phi}]_{t=t_b}\,\delta t_b = 0 \tag{3-78}$$

and since $\delta t_b \neq 0$ in general,

$$[f - \dot{x}f_{\dot{x}} + \dot{\phi}f_{\dot{x}}]_{t=t_b} = 0 \tag{3-79}$$

This equation and the given relation $x(t_b) = \phi(t_b)$ are used in the determination of the best values of t_b and $x(t_b)$ corresponding to the first-variational curves.

Case C. ($x(t_a)$, $x(t_b)$, and t_a fixed; t_b variable.)

With $x(t_a)$, $x(t_b)$, and t_a fixed, the variations $\delta x(t_a)$, δt_a, and $\delta x(t_b)$ are constrained to be zero; but note that $\delta x_v(t_b)$ in addition to δt_b need not be zero (Figure 3-8b). For sufficiently small $|\delta t_b|$, the ratio of $\delta x_v(t_b)$ to $-\delta t_b$ approaches $\dot{x}(t_b)$.[5] With this fact and Equation 3-74,

$$0 = [f - f_{\dot{x}}\dot{x}(t)]_{t=t_b}\,\delta t_b \tag{3-80}$$

and since δt_b need not be zero,

$$\left[f(x,\dot{x},t) - \dot{x}(t)\,\frac{\partial f(x,\dot{x},t)}{\partial \dot{x}}\right]_{t=t_b} = 0 \tag{3-81}$$

Observe that this result is the special case of 3-79 in which $\dot{\phi}$ is zero and, therefore, ϕ is a constant.

Case D. ($x(t_a)$, $x(t_b)$ fixed; $t_b - t_a = c$.)

With $x(t_a)$ and $x(t_b)$ fixed, $\delta x(t_a)$ and $\delta x(t_b)$ are 0. The end points t_b and t_a are related by the equation $t_b - t_a = c$, where c is a constant. Hence, the

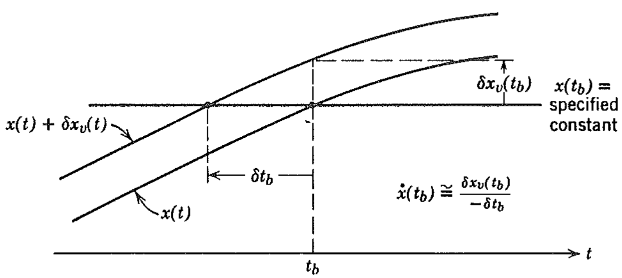

Figure 3-8b. The variations $\delta x(t_a)$, δt_a, and $\delta x(t_b)$ are constrained to be zero.

[5] The counter direction of the arrow associated with δt_b to that associated with the t axis is indicative that δt_b itself is a negative entity in Figure 3-8b.

variations δt_a and δt_b are related by

$$(t_b + \delta t_b) - (t_a + \delta t_a) = c \tag{3-82}$$

which requires that $\delta t_b = \delta t_a$. The transversality condition 3-67 therefore reduces to

$$(f - \dot{x}f_{\dot{x}})_{t_a}^{t_b}\,\delta t_a = 0 \tag{3-83}$$

and since $\delta t_a \neq 0$ in general,

$$(f - \dot{x}f_{\dot{x}})_{t_a}^{t_b} = 0 \tag{3-84}$$

The fact that $t_b - t_a = c$ and knowledge of Equations 3-25 and 3-84 enable the formal determination of a subclass of the class of first-variational curves and, hopefully, the optimal curve $x^*(t)$.

Example 3-6. In Example 3-4, the final output voltage $v_o(T)$ is assumed to be fixed. In this example, $v_o(T) = V$ is to be selected so as to maximize J_a of 3-44. With T and $v_o(0)$ fixed, the transversality condition 3-67 reduces to

$$\left[\frac{\partial f_a}{\partial \dot{v}_o}\right]_{t=T}\delta v_o(T) = 0 \tag{3-85}$$

which is satisfied when $[\partial f_a/\partial \dot{v}_o]_{t=T} = 0$. From 3-44, $[\partial f_a/\partial \dot{v}_o] = 2h_1RC^2\dot{v}_o$. Thus, Equation 3-46 and the above statements give

$$\dot{v}_o(T) = 2c_2T + c_1 = 0 \tag{3-86}$$

which is solved simultaneously with the energy constraint equation (3-48) to yield

$$c_1 = 2T\left[\frac{3K}{4RC^2T^3}\right]^{1/2} \quad \text{and} \quad c_2 = -\left[\frac{3K}{4RC^2T^3}\right]^{1/2} \tag{3-87}$$

The optimal value V^* of $v_o(T)$, obtained by substituting the above identities appropriately into 3-47, is

$$V^* = \left[\frac{3KT}{4RC^2}\right]^{1/2} \tag{3-88}$$

The reader can check this solution by finding the maximum of J in 3-52 with respect to V.

●

3-6. CORNER CONDITIONS

The concept of a corner point is introduced in Section 3-3d. It is important to note that the occurrence of corner points is not always evident from the form of the functional J. There are certain necessary conditions, however, which are derived here and which can be used to determine the locations of corner points, if any exist. The derivation is given for fixed end-point conditions, but it should be clear that the conditions also apply for variable end-point problems.

The functional J of 3-11 can be expressed in the form

$$J = \int_{t_a}^{t_c} f\,dt + \int_{t_c}^{t_b} f\,dt \tag{3-89}$$

where t_c, as yet unspecified, is to be examined for corner-point properties. If more than one corner point exist, the integral could be subdivided further, but this is not essential for the derivation that follows.

A typical corner condition is shown in Figure 3-3. Once t_c is determined, it is, of course, necessary that both segments of the curve in Figure 3-3 be first-variational curves. Following the line of reasoning of the preceding section, the values t_c and $x(t_c)$ can be viewed as variable end conditions on each of the integrals in 3-89. A transversality condition similar to 3-67 is applicable, therefore, as follows:

$$[(f - \dot{x}f_{\dot{x}})\,\delta t]_{t=t_c+0}^{t=t_c-0} + [f_{\dot{x}}\,\delta x]_{t=t_c+0}^{t=t_c-0} = 0 \tag{3-90}$$

But since the variations δt_c and $\delta x(t_c)$ are independent, it follows that the conditions

$$[f - \dot{x}f_{\dot{x}}]_{t=t_c-0} = [f - \dot{x}f_{\dot{x}}]_{t=t_c+0} \tag{3-91}$$

and

$$f_{\dot{x}}|_{t=t_c-0} = f_{\dot{x}}|_{t=t_c+0} \tag{3-92}$$

must be satisfied at a corner point of any first-variational curve. Equations 3-91 and 3-92 are known as the *Erdmann-Weierstrass corner conditions.*

Example 3-7. Suppose the integrand f of the functional J, Equation 3-11, is given by

$$f(x,\dot{x},t) = \dot{x}^2 f_1(x,t) + f_2(x,t) \tag{3-93}$$

where $f_1 = f_1(x,t)$ and $f_2 = f_2(x,t)$ are of class C^2.

Problem: Determine the locations of corner points if they exist.

To apply condition 3-92, $f_{\dot{x}} = 2\dot{x}f_1(x,t)$ is obtained from 3-93 and the result is used in 3-92 as follows:

$$2\dot{x}f_1|_{t=t_c-0} = 2\dot{x}f_1|_{t=t_c+0} \tag{3-94}$$

Because f_1 is continuous in t and x, and assuming that $f_1(x,t)$ is nonzero over the time interval $[t_a,t_b]$, it follows from 3-94 that $\dot{x}(t_c - 0) = \dot{x}(t_c + 0)$, which fact obviates the possibility that any corners exist.

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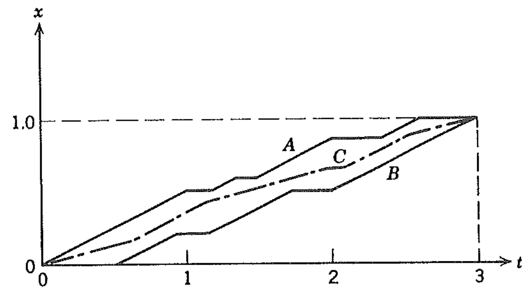

Figure 3-9. An example of first-variational curves with corner points.

Example 3-8. Suppose the integrand f of the functional J, Equation 3-11, is given by

$$f = \dot{x}^2(2\dot{x} - 1)^2 \tag{3-95}$$

In this case, f is formally independent of t, and the first integral—Equation 3-30—insures that condition 3-91 is satisfied. The second condition, Equation 3-92, is satisfied if

$$p(2p - 1)(4p - 1) = q(2q - 1)(4q - 1) \tag{3-96}$$

where p is identified with $\dot{x}(t_c - 0)$ and q is identified with $\dot{x}(t_c + 0)$. There are six ways in which 3-96 can be satisfied without p being equal to q; namely, $p = 0$ and $q = \frac{1}{2}$, or $p = 0$ and $q = \frac{1}{4}$, or $p = \frac{1}{2}$ and $q = 0$, or $p = \frac{1}{2}$ and $q = \frac{1}{4}$, or $p = \frac{1}{4}$ and $q = 0$, or finally, $p = \frac{1}{4}$ and $q = \frac{1}{2}$. Not all of these pairs result in the optimal value of J since 3-96 is a necessary condition, but not a sufficient one.

It is left to the reader (Problem 3.12) to show that the first-variational curves in this case are of the linear form

$$x = c_1 t + c_0 \tag{3-97}$$

between corners, where c_1 and c_0 are constants.

For the first-variational curves drawn in Figure 3-9, the preceding statements are used in verifying that curves labeled A and B (and any other curve which has either the slope $\frac{1}{2}$ or the slope zero on any given interval and which satisfies the indicated boundary values) result in the minimum of the functional in which case $J = 0$; whereas curves such as C, even though it is a first-variational curve between corners, do not minimize J.

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3-7. THE PROBLEM OF LAGRANGE: STATE-VECTOR CASE

The results of the preceding sections of this chapter are generalized here, using argumentative proofs only, to account for the case in which $f \equiv f(\mathbf{x},\dot{\mathbf{x}},t)$ is a given real-valued function of its arguments: $\mathbf{x} \equiv \{x_1,x_2,\ldots,x_n\}$ and $\dot{\mathbf{x}} \equiv \{\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_n\}$ are sets of real-valued functions of the independent variable t. As before, f is assumed to be of class C^2 with respect to its arguments.

The functional to be extremized with respect to $\mathbf{x}$ is

$$J \equiv J(\mathbf{x}) = \int_{t_a}^{t_b} f(\mathbf{x},\dot{\mathbf{x}},t)\, dt \tag{3-98}$$

The optimal $\mathbf{x}$, $\mathbf{x}^*$, must satisfy certain necessary conditions. The principal one of these is that the first variation of J with respect to a variation in any x_k must be zero. This follows from the fact that if all of the x_i's except x_k have been selected in terms of t in an optimal fashion, then 3-98 reduces to 3-11 in terms of the one remaining $x_k(t)$ to be selected. Thus, the above condition imposed on the first variation leads to a set of Euler-Lagrange equations, each of which is similar to that obtained in Section 3-3, which must be satisfied by the optimal $\mathbf{x}$. The Euler-Lagrange equations have the simple form

$$f_{x_k} - \frac{d}{dt} f_{\dot{x}_k} = 0, \qquad k = 1, 2, \ldots, n \tag{3-99}$$

and this set of equations can be solved for sets of first-variational curves.

When some of the end-point conditions on the first-variational curves are unspecified, or when certain end-point conditions are related by given functions, the transversality condition

$$\left[\left(f - \sum_{k=1}^{n} f_{\dot{x}_k}\dot{x}_k\right) dt + \sum_{k=1}^{n} f_{\dot{x}_k}\, dx_k\right]_{t=t_a}^{t=t_b} = 0 \tag{3-100}$$

must be satisfied by the optimal solution. Again, as in Section 3-5, it is emphasized that dt and dx_k in 3-100 denote infinitesimal *variations* of the variables t and x_k, respectively. For example, if $x_k(t)$ must equal a given differentiable function $\phi(t)$ at $t = t_b$, then the *ratio* $[dx_k(t_b)/dt_b]$ equals $[d\phi(t)/dt]_{t=t_b}$. Most emphatically, this ratio of $dx_k(t_b)$ to dt_b should not be identified with $\dot{x}_k(t_b)$! Here as in Section 3-5, special forms of 3-100 can be derived in special cases (Problems 3.14 and 3.15, for example). Also note that when Euler-Lagrange equations are formed on the basis of an augmented performance measure, the f in 3-100 must be replaced by the integrand of the augmented performance measure. Additional insight into transversality conditions can be obtained from the deliberations in Sections 3-9 and 8-6.

Similarly, the corner conditions for the n-dependent-function case are listed:

$$\left.\frac{\partial f}{\partial \dot{x}_k}\right|_{t=t_c-0} = \left.\frac{\partial f}{\partial \dot{x}_k}\right|_{t=t_c+0}, \qquad k = 1, 2, \ldots, n \tag{3-101}$$

and

$$\left[-f + \sum_{k=1}^{n} f_{\dot{x}_k}\dot{x}_k\right]_{t=t_c-0} = \left[-f + \sum_{k=1}^{n} f_{\dot{x}_k}\dot{x}_k\right]_{t=t_c+0} \tag{3-102}$$

Proofs of conditions 3-100, 3-101, and 3-102 can be supplied by straightforward extensions of the corresponding proofs in Section 3-5.

One additional relation is given below without proof (but see Problem 3.16). This relation applies to the case in which one (or more) of the x_k's has derivatives, of order higher than the first, which appear as arguments of f in 3-98. If $x_j^{(m)}$ is the highest-order derivative of x_j which appears in f, the Euler-Lagrange equation associated with x_j is

$$f_{x_j} - \frac{d}{dt}(f_{\dot{x}_j}) + \cdots + (-1)^m \frac{d^m}{dt^m}[f_{x_j^{(m)}}] = 0 \tag{3-103}$$

3-8. CONSTRAINTS

The theory developed thus far in this chapter is applicable to a rather limited class of practical problems because methods for treating constraint relationships—which almost always exist in problems with physical origins—between the x_k's have been neglected almost entirely. Fortunately, only minor modifications of the theory are required in order to treat more involved problems, and these are considered in the following four subsections, the first three being devoted to equality constraints of various forms, and the fourth to inequality constraints. Proofs given in this section are indicative in nature; the reader who desires to study rigorous proofs is referred to any of several books ([3.1, 3.6, 3.10, or 3.12] for example). Though the various constraints are treated separately in the following subsections, *problems having several different types of constraints are handled by treating each constraint semi-independently of the others*, as illustrated in Examples 3-9 and 3-10.

3-8a. Isoperimetric Constraints

Consider constraint equations of the form

$$K_i = \int_{ta}^{t_b} f_i(\mathbf{x},\dot{\mathbf{x}},t)\, dt, \qquad i = 1, 2, \ldots, m \tag{3-104}$$

where, as before, $\mathbf{x}$ denotes $\{x_1,x_2,\ldots,x_n\}$, K_i is a constant, and f_i is a given real-valued function of class C^2 with respect to its arguments. The constraints of 3-104 are called isoperimetric constraints, a basic form of which is treated in Section 3-4. As in Section 3-4, an isoperimetric theorem applies.

Isoperimetric Theorem. *Assume that $\mathbf{x}^*(t)$ results in the maximum of the functional $J_a(\mathbf{x})$:*

$$J_a(\mathbf{x}) = \int_{t_a}^{t_b} \left[f + \sum_{i=1}^{m} h_i f_i \right] dt$$

$$= \int_{t_a}^{t_b} f_a(\mathbf{x},\dot{\mathbf{x}},t,h_1,h_2,\ldots,h_m)\, dt \tag{3-105}$$

That is,

$$J_a(\mathbf{x}) \leq J_a(\mathbf{x}^*) \tag{3-106}$$

for any admissible $\mathbf{x}$, where the h_i's are called Lagrange multipliers and are independent of $\mathbf{x}$ and t. Also, assume that 3-104 is satisfied by $\mathbf{x}^*$. Then $\mathbf{x}^*$ yields a maximum of 3-98 subject to the isoperimetric constraints (3-104).

A proof of this theorem can be obtained on the basis of an expanded version of that given in Section 3-4 and is not considered here. An analogous theorem applies for the case that a minimum of J is desired. To apply the theorem, we first obtain the first-variational curves corresponding to the functional J_a in 3-105. These curves are functions of t and of the constant h_i's. The h_i's are evaluated by the requirement that Equations 3-104 must be satisfied. When applicable, corner conditions, transversality conditions, and sufficiency conditions should also be applied in the determination of the optimal first-variational curves.

3-8b. Constraints of the Form $g_i(\mathbf{x},\dot{\mathbf{x}},t) = 0$

Consider constraint equations of the form

$$g_i \equiv g_i(\mathbf{x},\dot{\mathbf{x}},t) = 0, \qquad i = 1, 2, \ldots, j \tag{3-107}$$

The g_i's are known real-valued functions of class C^2 with respect to the arguments $\mathbf{x}$, $\dot{\mathbf{x}}$, and t; and the Equations 3-107 must be satisfied by any $\mathbf{x} = \{x_1,x_2,\ldots,x_n\}$ which extremizes J of 3-98. If a Lagrange multiplier $\lambda_i = \lambda_i(t)$ is associated with each g_i, an augmented functional J_a can be written in the form

$$J_a = \int_{t_a}^{t_b} \left[f + \sum_{i=1}^{j} \lambda_i(t) g_i \right] dt$$

$$= \int_{t_a}^{t_b} f_a(\mathbf{x},\dot{\mathbf{x}},t,\lambda_1,\lambda_2,\ldots,\lambda_j)\, dt \tag{3-108}$$

Because an optimal $\mathbf{x}$, $\mathbf{x}^*$, must satisfy Equations 3-107, the above functional is formally equivalent to that of 3-98. To find first-variational curves, therefore, Euler-Lagrange equations are derived by using the integrand $f_a \equiv f_a(\mathbf{x},\dot{\mathbf{x}},t,\lambda_1,\lambda_2,\ldots,\lambda_j)$ of 3-108, as follows:

$$\frac{\partial f_a}{\partial x_i} - \frac{d}{dt}\frac{\partial f_a}{\partial \dot{x}_i} = 0, \qquad i = 1, 2, \ldots, n \tag{3-109a}$$

and

$$\frac{\partial f_a}{\partial \lambda_i} - \frac{d}{dt}\frac{\partial f_a}{\partial \dot{\lambda}_i} = 0, \qquad i = 1, 2, \ldots, j \tag{3-109b}$$

where the latter set of Euler-Lagrange equations (3-109b) reduces to the original constraint equations (3-107) because f_a is formally independent of $\dot{\lambda}_i$, $i = 1, 2, \ldots, j$.

It is emphasized that, in contrast to the Lagrange multipliers of Section 3-8a, the Lagrange multipliers in 3-108 and 3-109a are functions of t, in which case the differentiation with respect to t in 3-109a results in terms which contain $\dot{\lambda}_i$ as a factor. The indicated differentiation in 3-109a is effected to obtain

$$(f_a)_{x_i} = (f_a)_{\dot{x}_i t} + \sum_{k=1}^{n} [(f_a)_{\dot{x}_i \dot{x}_k}\ddot{x}_k + (f_a)_{\dot{x}_i x_k}\dot{x}_k] + \sum_{k=1}^{j} (f_a)_{\dot{x}_i \lambda_k}\dot{\lambda}_k \tag{3-110}$$

which is determined for $i = 1, 2, \ldots, n$.

Equations 3-107 and 3-110 constitute a set of $n + j$ differential equations, associated with which there are $n + j$ unknown functions of t, the λ_i's and the x_i's. These equations are solved for the x_i's which are first-variational curves; as before, the first-variational curves are subject to corner conditions, transversality conditions, and sufficiency tests before an optimal solution is established.

An important special case of 3-110 is that in which Equations 3-107 are or can be placed in the form

$$\dot{x}_i - q_i(\mathbf{x},t) = 0, \qquad i = 1, 2, \ldots, j \tag{3-111a}$$

In this case, the Euler-Lagrange equations (3-110) reduce to

$$f_{x_i} - \sum_{k=1}^{j} \lambda_k \frac{\partial q_k}{\partial x_i} = f_{\dot{x}_i t} + \sum_{k=1}^{n} (f_{\dot{x}_i \dot{x}_k}\ddot{x}_k + f_{\dot{x}_i x_k}\dot{x}_k) + \dot{\lambda}_i \tag{3-111b}$$

for $i = 1, 2, \ldots, n$.

There are several ways in which we can justify the preceding approach. One of these is given in this paragraph for the case that $t_a = 0$ and $j = 1$ in

3-107. The problem is then to find the minimum (maximum) of

$$J = \int_0^{t_b} f(\mathbf{x},\dot{\mathbf{x}},t)\,dt \tag{3-112a}$$

subject to satisfying the constraint that

$$g_1(\mathbf{x},\dot{\mathbf{x}},t) = 0 \tag{3-112b}$$

An approximate form of 3-112a is

$$J_a = \sum_{k=0}^{N} f\left[\mathbf{x}(k\,\Delta t), \frac{\Delta\mathbf{x}(k\,\Delta t)}{\Delta t}, k\,\Delta t\right]\Delta t \tag{3-113a}$$

where $\Delta\mathbf{x}(k\,\Delta t) = \mathbf{x}[(k+1)\,\Delta t] - \mathbf{x}(k\,\Delta t)$, Δt is a small increment, and $N\,\Delta t = t_b - \Delta t$. Similarly, an approximate form of 3-112b is

$$g_1\left[\mathbf{x}(k\,\Delta t), \frac{\Delta\mathbf{x}(k\,\Delta t)}{\Delta t}, k\,\Delta t\right] = 0, \qquad k = 0, 1, \ldots, N \tag{3-113b}$$

To solve the approximate problem associated with 3-113a and 3-113b, classical min-max theory can be applied. Thus, a Lagrange multiplier h_k is introduced for each constraint relationship of 3-113b, and an augmented performance measure J_a is formed,

$$J_a = \sum_{k=0}^{N}\left\{f\left[\mathbf{x}(k\,\Delta t), \frac{\Delta\mathbf{x}(k\,\Delta t)}{\Delta t}, k\,\Delta t\right]\Delta t + h_k g_1\left[\mathbf{x}(k\,\Delta t), \frac{\Delta\mathbf{x}(k\,\Delta t)}{\Delta t}, k\,\Delta t\right]\right\}$$

Let $\lambda_1(k\,\Delta t) \triangleq h_k/\Delta t$ in the above expression with the result that

$$J_a = \sum_{k=0}^{N}\left\{f\left[\mathbf{x}(k\,\Delta t), \frac{\Delta\mathbf{x}(k\,\Delta t)}{\Delta t}, k\,\Delta t\right] + \lambda_1(k\,\Delta t) g_1\left[\mathbf{x}(k\,\Delta t), \frac{\Delta\mathbf{x}(k\,\Delta t)}{\Delta t}, k\,\Delta t\right]\right\}\Delta t$$

and for arbitrarily small Δt, this augmented performance measure to be minimized (maximized) approaches

$$J_a = \int_0^{t_b} [f(\mathbf{x},\dot{\mathbf{x}},t) + \lambda_1(t) g_1(\mathbf{x},\dot{\mathbf{x}},t)]\,dt$$

which is of the form expressed by 3-108.

3-8c. Constraints of the Form $z_i(\mathbf{x},t) = 0$

Consider constraint equations of the form

$$z_i(\mathbf{x},t) = 0, \qquad i = 1, 2, \ldots, m < n \tag{3-114}$$

These constraints are actually a special case of those considered in subsection 3-8b. Here, the constraint equations are independent of $\dot{\mathbf{x}} = \{\dot{x}_1,\dot{x}_2,\ldots,\dot{x}_n\}$, and it may be possible to solve 3-114 for m of the x_i's in terms of t and the remaining $n - m$ of the x_i's. The x_i's thus obtained could be substituted appropriately into 3-98, and optimization could then be effected without further regard to the original constraint equations.

On the other hand, it may be difficult at the outset to solve 3-114 for some of the x_i's in terms of t and other x_i's. If so, exactly the same procedure as given in Section 3-8b can be employed to obtain the following set of Euler-Lagrange equations:

$$f_{x_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial z_k}{\partial x_i} = f_{\dot{x}_i t} + \sum_{k=1}^{n} (f_{\dot{x}_i \dot{x}_k}\ddot{x}_k + f_{\dot{x}_i x_k}\dot{x}_k) \tag{3-115}$$

where $i = 1, 2, \ldots, n$. Although the λ_k's in 3-115 are still functions of t in general, no $\dot{\lambda}_k$'s appear; and therefore the set of Equations 3-114 and 3-115 are usually somewhat easier to solve than the corresponding equations in Section 3-8b.

3-8d. Inequality Constraints

When the equality symbols in Equations 3-104, 3-107, 3-111a, and/or 3-114 are replaced by inequality symbols, the resulting relations are inequality constraints. For example, a particular inequality constraint might assume the form

$$0 \le g(\mathbf{x},\dot{\mathbf{x}},t) \tag{3-116}$$

which is an inequality constraint corresponding to the equality constraints of 3-107. Quite often, such inequality constraints define a connected region of allowable x_i's and $\dot{x}_i$'s in the space of possible x_i's and $\dot{x}_i$'s, and this region is generally a function of t. In some problems, the optimal value of the functional is obtained when the x_i's and $\dot{x}_i$'s assume values on the boundary of the allowable region for all t in the closed interval $[t_a,t_b]$, and in this case the optimum is obtained when the equality, rather than the inequality, is used in 3-116. If the preceding statement holds, the appropriate method of treating the equality constraint can be used. On the other hand, if it is questionable that the optimal x_i's and $\dot{x}_i$'s correspond to boundary values of a connected region, more devious methods must be employed. Three of these approaches are given in the following paragraphs of this section.

For differential inequality constraints of the form

$$c_1 \le g \equiv g(\mathbf{x},\dot{\mathbf{x}},t) \le c_2 \tag{3-117}$$

or for inequality constraints of the form

$$c_1 \leq z \equiv z(\mathbf{x},t) \leq c_2 \tag{3-118}$$

the first two methods that follow are applicable, at least in theory. For illustrative purposes, only one constraint of the form given in 3-117 is considered.

The *first method* is the *slack-variable method* suggested by Valentine [3.29]. In applying this method, we introduce a new variable $x_{n+1} \equiv x_{n+1}(t)$ which is defined by

$$x_{n+1}^2 \triangleq (g - c_1)(c_2 - g) \tag{3-119}$$

where g, c_1, and c_2 are the same entities that appear in 3-117. Assuming that $x_{n+1}(t)$ is a real-valued function of t, the right-hand member of 3-119 must be non-negative over the range of t under consideration. But of course this implies that the constraint is satisfied; i.e., $g \geq c_1$ and $g \leq c_2$. Thus, Equation 3-119 is rearranged, and a new equality constraint (3-120) is defined to replace the old inequality constraint (3-117).

$$g_e \triangleq (g - c_1)(c_2 - g) - x_{n+1}^2 = 0 \tag{3-120}$$

Because this new constraint equation is of the form considered in Section 3-8b, an augmented functional J_a is formed in accord with the theory developed in Section 3-8b,

$$J_a = \int_{t_a}^{t_b} \{f + \lambda(t)[(g - c_1)(c_2 - g) - x_{n+1}^2]\}\, dt \tag{3-121}$$

from which the following Euler-Lagrange equations are derived in the ordinary way:

$$f_{x_i} + \lambda g_{x_i}(c_1 + c_2 - 2g) - \frac{d}{dt}[f_{\dot{x}_i} + \lambda g_{\dot{x}_i}(c_1 + c_2 - 2g)] = 0 \tag{3-122}$$

where $i = 1, 2, \ldots, n$; and an additional Euler-Lagrange equation is

$$-2x_{n+1}(t)\lambda(t) = 0 \tag{3-123}$$

Note from the above equation that the nonzero domains of $x_{n+1}(t)$ and $\lambda(t)$ are mutually exclusive. The problem, therefore, is to determine over which intervals of t the Lagrange multiplier $\lambda(t)$ is zero, in which case Equations 3-122 reduce to the unconstrained Euler-Lagrange equations; and over which intervals of t the variable $x_{n+1}(t)$ is zero, in which case the function g equals either c_1 or c_2. Unfortunately, the computations that are required to obtain numerical solutions to these equations are quite difficult in general. But as a means of determining properties of solutions and of solving certain special cases, the above development and similar developments are quite useful (see Problem 3.19 and references [3.3, 3.4, and 3.29]).

The *second method* to be considered here is a *penalty-function method* known as the *method of elastic stops.* This method is based on the premise that if the performance measure is severely penalized when the constraint 3-117 *is not* satisfied but the performance measure is not penalized when the same constraint *is* satisfied, then the optimal solution will be forced to satisfy the constraint. More precisely, suppose that the functional J of 3-98 is to be minimized with respect to the selection of the x_i's, and that the constraint 3-117 has to be satisfied by the optimal solution. If a function $f_1 = f_1(g,c_1,c_2)$ can be found with the property that

$$f_1 \cong 0 \quad \text{for} \quad c_1 \leq g \leq c_2 \tag{3-124}$$

and

$$f_1 \gg 0 \quad \text{for} \quad c_1 > g \text{ or } g > c_2 \tag{3-125}$$

then this function can be added to the integrand of 3-98 to obtain an augmented functional J_a,

$$J_a = \int_{t_a}^{t_b} (f + f_1)\, dt \tag{3-126}$$

and the minimum of J_a, with no further regard to the constraint, will be almost the same as the *constrained minimum* of J in 3-98.

Consider a particular f_1 function,

$$f_1 = \left(\frac{2g - c_1 - c_2}{c_2 - c_1}\right)^{2k} \tag{3-127}$$

where k is a positive integer. Typical curves of this f_1 versus g are given in Figure 3-10 for various values of k. It is observed that if k is taken sufficiently large, f_1 of 3-127 satisfies the requirements of 3-124 and 3-125.

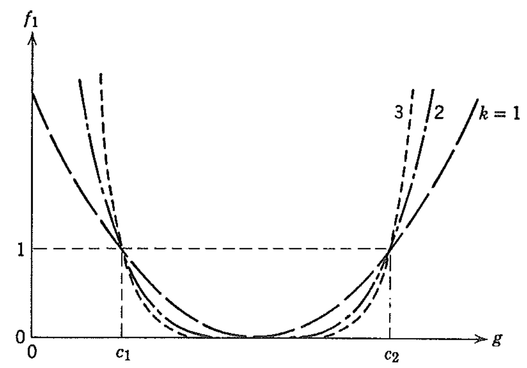

Figure 3-10. Elastic stops at $g = c_1$ and $g = c_2$.

The Euler-Lagrange equations corresponding to 3-126 are obtained in the ordinary way with f_1 as given by 3-127; thus,

$$f_{x_i} + \frac{4kg_{x_i}(2g - c_1 - c_2)^{2k-1}}{(c_2 - c_1)^{2k}} = \frac{d}{dt}\left[f_{\dot{x}_i} + \frac{4kg_{\dot{x}_i}(2g - c_1 - c_2)^{2k-1}}{(c_2 - c_1)^{2k}}\right] \quad (3\text{-}128)$$

for $i = 1, 2, \ldots, n$. Even though Equations 3-128 appear to be unwieldy, they have the desirable feature that they possess no unknown Lagrange multiplier. Once k is selected in 3-128, the problem is the same as that treated in Section 3-7. Typically, we assume a small value of k at the start to obtain an approximate solution; and then, for an accurate solution, we assume as large a value of k as can be handled with the computational facilities available.

The *third approach* of this section concerns isoperimetric inequality constraints only. To apply the method in the case of a single isoperimetric constraint, assume some particular equality which is consistent with the original inequality constraint and use this in place of the original inequality constraint. Next, solve the problem as though it were of the ordinary equality constraint type and evaluate the performance measure J. Continue by repeating the preceding step using different equalities for the equality constraint; and finally, with the aid of systematic search techniques (Chapter 6), converge to an optimal solution.

It is all too apparent that the above methods of treating inequality constraints leave much to be desired. Another approach which is useful at times is based on the parametric representation of the functions involved [3.12]. Still other approaches have been developed in the literature for certain types of inequality constraints which often occur in practice—see Chapters 7 and 8 in this regard.

Example 3-9. Reconsider Examples 3-4 and 3-6, but assume here that the input energy E_i,

$$E_i = \int_0^T iv_i\,dt = \int_0^T C\dot{v}_o v_i\,dt \quad (3\text{-}129)$$

is fixed, rather than the energy K dissipated in the resistor R of Figure 3-4. The voltages v_o and v_i in 3-129 are related by the differential equation

$$\dot{v}_o - \frac{v_i - v_o}{RC} = 0, \qquad v_o(0) = 0 \quad (3\text{-}130)$$

As in Example 3-4, the problem is to maximize the average value 3-41 of the output $v_o(t)$. Here, however, constraint equations 3-129 and 3-130 are in effect. One way to proceed would be to solve 3-130 for v_i in terms of v_o and $\dot{v}_o$, in which case the result could be used to eliminate v_i from 3-129, but this approach would not illustrate application of the theory of Section

3-8b. In order to illustrate use of the theory of both Sections 3-8a and 3-8b, the approach employed here is as follows. First, an augmented functional J_a and Lagrange multipliers h_1 and $\lambda_1 \equiv \lambda_1(t)$ are introduced:

$$\begin{aligned} J_a &= \int_0^T \left[\frac{v_o}{T} + h_1 C\dot{v}_o v_i + \lambda_1\left(\dot{v}_o - \frac{v_i - v_o}{RC}\right)\right] dt \\ &= \int_0^T f_a\,dt \end{aligned} \quad (3\text{-}131)$$

Next, v_o is associated with x_1, v_i with x_2, $[(v_o/T) + h_1 C\dot{v}_o v_i]$ with f, and $[(v_i - v_o)/RC]$ with q_1. The Euler-Lagrange equations 3-111b are applicable here and assume the particular form

$$\frac{1}{T} + \frac{\lambda_1}{RC} = h_1 C\dot{v}_i + \dot{\lambda}_1 \quad (3\text{-}132)$$

and

$$h_1 C\dot{v}_o - \frac{\lambda_1}{RC} = 0 \quad (3\text{-}133)$$

Simultaneous solution of 3-130, 3-132, and 3-133 gives

$$v_o(t) = c_1 t^2 + c_2 t \quad (3\text{-}134)$$

$$v_i(t) = c_1 t^2 + (2RCc_1 + c_2)t + RCc_2 \quad (3\text{-}135)$$

and

$$\lambda_1(t) = \frac{t}{2T} + \frac{c_2}{4c_1 T} \quad (3\text{-}136)$$

where the Lagrange multiplier h_1 has been absorbed by the constants c_1 and c_2, $c_1 = 1/(4h_1 TRC^2)$. The transversality condition associated with $v_o(T)$ and the isoperimetric constraint 3-129 are the two conditions which enable explicit determination of c_1 and c_2. With details left to the reader (Problem 3.17), the results are

$$h_1 C v_i(T) + \lambda_1(T) = 0 \quad (3\text{-}137)$$

$$c_1(T^2 + 4RCT) + c_2(T + 2RC) = 0 \quad (3\text{-}138)$$

$$c_1 = (3E_i/RC^2T^3)^{1/2}[(T + 2RC)/(T + 8RC)]^{1/2} \quad (3\text{-}139)$$

and

$$c_2 = [-T(T + 4RC)/(T + 2RC)]c_1 \quad (3\text{-}140)$$