System Optimization Method (ECE 6233) December 10, 2020
NYU Tandon School of Engineering, Fall 2021
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Solution to Homework 5

1. **Solution:**

(a) Since $F(x, y, y') = F(y, y')$, we use the special case of Euler–Lagrange equation

$$F - \frac{\partial F}{\partial y'} y' = C$$

where C is a constant.

$$\frac{\partial F}{\partial y'} = \frac{y'}{y\sqrt{1 + (y')^2}}$$

Hence the Euler–Lagrange equation becomes

$$\frac{\sqrt{1 + (y')^2}}{y} - \frac{(y')^2}{y\sqrt{1 + (y')^2}} = C$$

This is equivalent to

$$\frac{1}{y\sqrt{1 + (y')^2}} = C$$

which implies

$$y' = \frac{dy}{dx} = \pm \frac{\sqrt{1/C^2 - y^2}}{y} \quad \Rightarrow \quad dx = \pm \frac{y}{\sqrt{1/C^2 - y^2}} dy$$

Integrate on both sides,

$$\int dx = \pm \int \frac{y}{\sqrt{\hat{C}^2 - y^2}} dy$$

Thus,

$$x + B = \pm\sqrt{1/C^2 - y^2} \quad \Rightarrow \quad (x + B)^2 + y^2 = \frac{1}{C^2}$$

where B and C are determined by the boundary conditions.

(b) Since $F(x, y, y') = F(y, y')$, we use the special case of Euler–Lagrange equation

$$F - \frac{\partial F}{\partial y'} y' = C$$

where C is a constant.

$$\frac{\partial F}{\partial y'} = \frac{y^{1/2} y'}{\sqrt{1 + (y')^2}}$$

Hence the Euler–Lagrange equation becomes

$$y^{1/2} \sqrt{1 + (y')^2} - \frac{y^{1/2} (y')^2}{\sqrt{1 + (y')^2}} = C$$

This is equivalent to

$$\frac{y}{1 + (y')^2} = C^2$$

which implies

$$y' = \frac{dy}{dx} = \pm \sqrt{y/C^2 - 1} \quad \Rightarrow \quad dx = \pm \frac{C}{\sqrt{y - C^2}} dy$$

Integrate on both sides,

$$\int dx = \pm \int \frac{C}{\sqrt{y - C^2}} dy$$

Thus,

$$x + B = \pm 2C \sqrt{y - C^2} \quad \Rightarrow \quad y = \frac{1}{4C} (x + B)^2 + C^2$$

where B and C are determined by the boundary conditions.

(c) The function $F(x, y, y') = F(x, y')$ does not depend on y, the original Euler-Lagrange equation can be reduced to

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0 \quad \Rightarrow \quad \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = 0$$

After integrating with respect to x, we obtain

$$\frac{\partial F}{\partial y'} = C$$

where C is a constant.

$$\frac{\partial F}{\partial y'} = 1 + 2x^2 y'$$

The E-L equation leads to

$$x^2 y' = \frac{C-1}{2} = B$$

which implies

$$y' = \frac{dy}{dx} = \frac{B}{x^2} \quad \Rightarrow \quad dy = \frac{B}{x^2} dx$$

Integrate on both sides,

$$\int dy = \int \frac{B}{x^2} dx$$

Thus,

$$y + A = -\frac{B}{x}$$

where A and B are determined by the boundary conditions.

2. **Solution:** The Lagrangian is

$$\begin{aligned} L &= -\int_a^b \phi(x) \ln \phi(x) dx + \lambda \left(\int_a^b \phi(x) dx - 1 \right) \\ &= \int_a^b -\phi(x) \ln \phi(x) + \lambda \phi(x) dx - \lambda \\ &= \int_a^b F(\phi(x), \phi'(x), x) dx - \lambda \end{aligned}$$

The extrema of the functional can be found through E-L equation.

$$\frac{\partial F}{\partial \phi} = -\ln \phi(x) - 1 + \lambda, \quad \frac{\partial F}{\partial \phi'} = 0$$

which implies

$$\ln \phi(x) + 1 - \lambda = 0 \quad \Rightarrow \quad \phi(x) = e^{-1+\lambda}$$

Apply this into the constraint

$$\int_a^b \phi(x) dx = \int_a^b e^{-1+\lambda} dx = e^{-1+\lambda} \int_a^b dx = 1$$

which yields

$$\phi^*(x) = e^{-1+\lambda} = \frac{1}{b-a}$$

and the associated entropy is

$$H^*[X] = -\int_a^b \phi(x) \ln \phi(x) dx = \ln(b-a)$$

3. **Solution:**

   (a)

$$\begin{aligned} x_1 =& x_0 + u_0 + w_0 \\ x_2 =& x_1 + u_1 + w_1 = x_0 + (u_1 + u_0) + (w_1 + w_0) \\ & \dots \\ x_k =& x_0 + \sum_{i=0}^{k-1} u_i + \sum_{i=0}^{k-1} w_i \end{aligned}$$

Thus,

$$\begin{aligned} x_k^2 =& \left(x_0 + \sum_{i=0}^{k-1} u_i + \sum_{i=0}^{k-1} w_i \right)^2 \\ =& x_0^2 + \left(\sum_{i=0}^{k-1} u_i \right)^2 + \left(\sum_{i=0}^{k-1} w_i \right)^2 + 2x_0 \sum_{i=0}^{k-1} u_i + 2x_0 \sum_{i=0}^{k-1} w_i + 2 \left(\sum_{i=0}^{k-1} u_i \right) \left(\sum_{i=0}^{k-1} w_i \right) \end{aligned}$$

The cost criterion is

$$J = \mathbb{E}\left[\sum_{k=0}^{N} x_k^2 \right] = \sum_{k=0}^{N} \mathbb{E}\left[x_k^2 \right]$$

Because $\mathbb{E}[x_0] = 0$, $\mathbb{E}[x_0^2] = 1$, $\mathbb{E}[w_k] = 0$, and $\mathbb{E}[w_k^2] = 1$,

$$J = \sum_{k=0}^{N} \left(1 + \mathbb{E}\left(\sum_{i=0}^{k-1} u_i \right)^2 + \mathbb{E}\left(\sum_{i=0}^{k-1} w_i \right)^2 \right)$$

Since

$$\left(\sum_{i=0}^{k-1} a_i \right)^2 = \sum_{i=0}^{k-1} a_i^2 + \sum_{i \neq j} a_i a_j$$

and w_k is an independent sequence ($\mathbb{E}[w_i w_j] = \mathbb{E}[w_i]\mathbb{E}[w_j]$), we have

$$\mathbb{E}\left(\sum_{i=0}^{k-1} w_i\right)^2 = \sum_{i=0}^{k-1} 1 = k$$

On the other hand, u_k is a deterministic sequence, the expectation is itself.

$$\mathbb{E}\left(\sum_{i=0}^{k-1} u_i\right)^2 = \left(\sum_{i=0}^{k-1} u_i\right)^2$$

Thus,

$$\begin{aligned} J &= \sum_{k=0}^{N}\left(1 + k + \left(\sum_{i=0}^{k-1} u_i\right)^2\right) \\ &= \frac{(N+2)(N+1)}{2} + \sum_{k=0}^{N}\left(\sum_{i=0}^{k-1} u_i\right)^2 \end{aligned}$$

(b) If $u_k = -x_k$, we have

$$x_k = w_k$$

The cost criterion is now

$$\hat{J} = \mathbb{E}\left[\sum_{k=0}^{N} x_k^2\right] = \sum_{k=0}^{N} \mathbb{E}\left[w_k^2\right] = \sum_{k=0}^{N} 1 = N + 1$$

From previous problem, the cost criterion in (a) is

$$J = 1 + N + \frac{N^2}{2} + \sum_{k=0}^{N}\left(\sum_{i=0}^{k-1} u_i\right)^2$$

Hence,

$$J - \hat{J} = \frac{N^2}{2} + \sum_{k=0}^{N}\left(\sum_{i=0}^{k-1} u_i\right)^2 \geq 0$$

The cost of open-loop control J is always greater than closed-loop control cost $\hat{J}$.