System Optimization Method (ECE 6233) October 11, 2020
NYU Tandon School of Engineering, Fall 2021
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Solution to Midterm 1

1. **Problem:**

Problem 1 (20pts). For each value of the scalar β, find the set of all stationary points of the following function of the two variables x and y:

$$f(x,y) = x^2 + y^2 + \beta xy + x + 2y$$

Which of these stationary points are global minima?

Solution:

We have

$$\nabla f(x,y) = \begin{pmatrix} 2x + \beta y + 1 \\ 2y + \beta x + 2 \end{pmatrix}$$

Setting $\nabla f(x,y) = 0$, we obtain the system of equations

$$\begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This system has a unique solution (a unique stationary point) except when

$$\beta^2 = 4.$$

If $\beta^2 = 4$, it can be verified that there is no solution to the above system (no stationary point). Assuming $\beta^2 \neq 4$, for the stationary point to be a local minimum, the Hessian matrix of f, which is

$$Q = \begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix},$$

must be positive semidefinite. But if this is so, $f(x,y)$ will be a convex quadratic function and each local minimum will be global.

The Hessian Q will be positive definite if and only if $\beta^2 < 4$ and positive semidefinite if $\beta^2 = 4$, in which case there is no stationary point by the preceding discussion.

Thus, if $\beta^2 < 4$, there is a unique stationary point which is a global minimum. If $\beta^2 = 4$, there is no stationary point. If $\beta^2 > 4$, there is a unique stationary point which, however, is not a local minimum.

The stationary points are

$$x = \frac{2\beta - 2}{4 - \beta^2} \qquad y = \frac{\beta - 4}{4 - \beta^2} \qquad (\beta^2 \neq 4)$$

2. **Problem:**

Problem 2 (30pts). Consider the function $f : \mathbb{R}^n \to \mathbb{R}$:

$$f(x) = \log\left(\sum_{i=1}^{m} \exp(a_i^T x + b_i)\right),$$

where $a_1, \cdots, a_m \in \mathbb{R}^n$, and $b_1, \cdots, b_m \in \mathbb{R}$.

(i) Find the gradient ∇f

(ii) Find the Hessian $\nabla^2 f$

(iii) Consider a simplified case where all entries of vectors a_i^T and b_i are positive. Determine whether the function is convex or not.

Solution:

(i) Find the gradient

where $a_1, \ldots, a_m \in \mathbf{R}^n$, and $b_1, \ldots, b_m \in \mathbf{R}$. We can find a simple expression for its gradient by noting that it is the composition of the affine function $Ax + b$, where $A \in \mathbf{R}^{m \times n}$ with rows $a_1^T, \ldots, a_m^T$, and the function $g : \mathbf{R}^m \to \mathbf{R}$ given by $g(y) = \log(\sum_{i=1}^m \exp y_i)$. Simple differentiation (or the formula (A.6)) shows that

$$\nabla g(y) = \frac{1}{\sum_{i=1}^m \exp y_i} \begin{bmatrix} \exp y_1 \\ \vdots \\ \exp y_m \end{bmatrix}, \tag{A.7}$$

so by the composition formula we have

$$\nabla f(x) = \frac{1}{\mathbf{1}^T z} A^T z$$

where $z_i = \exp(a_i^T x + b_i)$, $i = 1, \ldots, m$.

Alternative form:

$$\nabla_x f(x) = \frac{\sum_{i=1}^m a_i^T e^{a_i^T x + b_i}}{\sum_{i=1}^m e^{a_i^T x + b_i}}$$

(ii) Find the Hessian

Example A.4 We consider the function $f : \mathbf{R}^n \to \mathbf{R}$ from example A.2,

$$f(x) = \log \sum_{i=1}^{m} \exp(a_i^T x + b_i),$$

where $a_1, \ldots, a_m \in \mathbf{R}^n$, and $b_1, \ldots, b_m \in \mathbf{R}$. By noting that $f(x) = g(Ax + b)$, where $g(y) = \log(\sum_{i=1}^m \exp y_i)$, we can obtain a simple formula for the Hessian of f. Taking partial derivatives, or using the formula (A.8), noting that g is the composition of log with $\sum_{i=1}^m \exp y_i$, yields

$$\nabla^2 g(y) = \mathbf{diag}(\nabla g(y)) - \nabla g(y) \nabla g(y)^T,$$

where $\nabla g(y)$ is given in (A.7). By the composition formula we have

$$\nabla^2 f(x) = A^T \left(\frac{1}{\mathbf{1}^T z} \mathbf{diag}(z) - \frac{1}{(\mathbf{1}^T z)^2} z z^T \right) A,$$

where $z_i = \exp(a_i^T x + b_i)$, $i = 1, \ldots, m$.

Alternative form:

$$\nabla_x^2 f(x) = \frac{\sum_{i=1}^m a_i^T e^{a_i^T x + b_i} a_i}{\sum_{i=1}^m e^{a_i^T x + b_i}} - \frac{\left(\sum_{i=1}^m a_i^T e^{a_i^T x + b_i}\right) \left(\sum_{i=1}^m e^{a_i^T x + b_i} a_i\right)}{\left(\sum_{i=1}^m e^{a_i^T x + b_i}\right)^2}$$

(iii) If the function is convex, we need to show that for any arbitrary vector v, we have $v^T \bigtriangledown^2 f(x) v \geq= 0$, i.e.,

$$v^T \bigtriangledown^2 f(x) v = v^T A^T \left(\frac{1}{\mathbf{1}^T z} diag(z) - \frac{1}{(\mathbf{1}^T z)^2} z z^T \right) Av$$

Since v is arbitrary, we can let $u = Av$ as a new vector, which is also arbitrary. The proof will then becomes the same as to prove the convexity of Log-sum-exp.

Log-sum-exp. The Hessian of the log-sum-exp function is

$$\nabla^2 f(x) = \frac{1}{(\mathbf{1}^T z)^2} \left((\mathbf{1}^T z)\, \mathbf{diag}(z) - zz^T \right),$$

where $z = (e^{x_1}, \ldots, e^{x_n})$. To verify that $\nabla^2 f(x) \succeq 0$ we must show that for all v, $v^T \nabla^2 f(x) v \geq 0$, *i.e.*,

$$v^T \nabla^2 f(x) v = \frac{1}{(\mathbf{1}^T z)^2} \left(\left(\sum_{i=1}^n z_i \right) \left(\sum_{i=1}^n v_i^2 z_i \right) - \left(\sum_{i=1}^n v_i z_i \right)^2 \right) \geq 0.$$

But this follows from the Cauchy-Schwarz inequality $(a^T a)(b^T b) \geq (a^T b)^2$ applied to the vectors with components $a_i = v_i \sqrt{z_i}$, $b_i = \sqrt{z_i}$.

3. **Problem:**

Problem 3 (20pts)**.** The epigraph epi(f) of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of $\mathbb{R}^{n+1}$ defined as

$$\text{epi}(f) = \{(x, t) : x \in \mathbb{R}^n, f(x) \leq t\}$$

Show that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph is a convex set. Note that you have to show both "if" and "only if".

Solution:

(a) f is convex $\Rightarrow$ epigraph is a convex set.

Assume the point $(x_1, t_1) \in epi(f), (x_2, t_2) \in epi(f)$. We would like to show the point (x_3, t_3) is also in the $epi(f)$, where

$$x_3 = \lambda x_1 + (1 - \lambda) x_2, \quad t_3 = \lambda t_1 + (1 - \lambda) t_2 \qquad (\lambda \in [0, 1])$$

Since the function f is convex, we have

$$f(\lambda x_1 + (1 - \lambda) x_2) \leq \lambda f(x_1) + (1 - \lambda) f(x_2).$$

Besides, $(x_1, t_1) \in epi(f), (x_2, t_2) \in epi(f)$ implies

$$f(x_1) \leq t_1 \qquad f(x_2) \leq t_2.$$

With $\lambda \in [0, 1]$, we have

$$\begin{aligned} f(\lambda x_1 + (1 - \lambda) x_2) &\leq \lambda f(x_1) + (1 - \lambda) f(x_2) \\ &\leq \lambda t_1 + (1 - \lambda) t_2 \end{aligned}$$

Thus, the point (x_3, t_3) is also in the epigraph, and the epigraph is a convex set.

(b) epigraph is a convex set $\Rightarrow f$ is convex
Since epigraph is a convex set, we know that if $(x_1, t_1) \in epi(f), (x_2, t_2) \in epi(f)$,

$$f(x_1) \leq t_1 \qquad f(x_2) \leq t_2. \tag{1}$$

the point (x_3, t_3) as defined in the previous problem will also be in the epigraph,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda t_1 + (1-\lambda)t_2 \tag{2}$$

We can pick $t_1 = f(x_1)$ and $t_2 = f(x_2)$, which still satisfy the inequalities in (1). Apply the value to the RHS of (2), we get

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

with $\lambda \in [0, 1]$. Thus, the function f is a convex function.