ULTRA-LOW ELEMENTS AND JOIN IRREDUCIBLE GATES IN COXETER GROUPS

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Abstract. In this note, we study the *tight gates* of the cone type partition of a Coxeter group W; a partition giving rise to the minimal automaton recognising the language of reduced words of W recently studied by Parkinson and Yau ([PY22]) and Przytycki and Yau ([PY23]). We show that the *tight gates* are the join-irreducible elements of the smallest Garside shadow of W and more generally, that *tight* elements are the join-irreducibles of convex Garside shadows. We study the properties of the tight gates and show that they also act as a *gate* of the witnesses of boundary roots of cone types and that they are closed under suffix. As an application of these results, we give a very efficient method of determining whether two elements have the same cone type without having to compute the minimal automaton. We also apply our results to explicitly describe the set of ultra-low elements in some select classes of Coxeter groups, thereby verifying a conjecture of Parkinson and Yau for these groups. Furthermore, we explicitly describe the set of tight gates in these cases.

Contents

1. Introduction

Since Brink and Howlett showed in [BH93] how to explicitly construct an automaton recognising the language of reduced words in Coxeter groups, a number of other constructions of automata have recently been studied ([HNW16], [PY22], [OP22]). The study of these *automatic structures* has led to many rich results illuminating the combinatorial and geometric structure of Coxeter groups, including the resolution of the long standing question of biautomaticity of Coxeter groups (see [OP22]).

A key component of Brink and Howlett's construction was the notion of elementary roots $\mathscr{E}$ and elementary inversion sets $\mathscr{E}(W)$. They showed that in every finitely generated Coxeter group W, the set $\mathscr{E}$ is finite and thus the partition of W into elementary inversion sets is finite. The partition of W into the sets of $\mathscr{E}(W)$ generalises the well known *Shi-arrangement* to all Coxeter groups.

Another construction of automata recognising reduced words of W was introduced by Hohlweg, Nadeau and Williams ([HNW16]) using the notion of *Garside shadows*; finite subsets of W closed under join and suffix which were shown to exist in every Coxeter group by Dehornoy, Dyer and Hohlweg in [DDH15]. In particular, an important class of Garside shadows, the *low elements* (and more generally *m-low elements*) have recently been extensively studied and have been shown to be in some sense the canonical representatives of elementary inversion sets $\mathscr{E}(W)$ (resp. m-elementary inversion sets) (see [Hoh16], [CLH22] and [DFHM24]).

Hohlweg, Nadeau and Williams work in [HNW16] not only gave a new method of constructing automata but established some important conjectures as well. In particular, they conjectured that the automaton constructed from the smallest Garside shadow $\widetilde{S}$ is the minimal automaton (in terms of number of states) recognising the language of reduced words of W [HNW16, Conjecture 2].

Motivated by this conjecture, in [PY22] we studied the minimal automaton via the study of the set of *cone types* $\mathbb{T}$ of W. In [PY22], we introduced the notion of a *regular partition* $\mathscr{R}$; a class of partitions which characterises the (reduced) automata recognising the language of reduced words of W. Furthermore, we showed that each part P of the regular partition corresponding to the cone type automaton, $\mathscr{T}$ (called the *cone type arrangement* or *cone type partition*) contains a unique minimal length element g_p (called the *gate* of P) which is a prefix of all elements in P. The set of these minimal length cone type representatives Γ, called the *gates* of the cone type arrangement were conjectured to be the smallest Garside shadow [PY22, Conjecture 1]. In addition, we conjectured that there is a characterisation of the elements of Γ in terms of their inversion set and their cone type [PY22, Conjecture 2]. This motivated the introduction of a new set of elements analogous to the low-elements called the *ultra-low* elements $\mathscr{U}$. Then [PY22, Conjecture 2] is equivalent to the sets Γ and $\mathscr{U}$ being equal.

In [PY23], we verified that the set Γ in closed under join (see [PY23, Theorem 1.4]) and it was shown that Γ is closed under suffix in [PY22]; thus Γ is the smallest Garside shadow, verifying [PY22, Conjecture 1]. However, whether the set $\Gamma = \mathscr{U}$ remains a seemingly challenging open question and hence motivates much of our work here. Indeed, one of the contributions of the work in this note is resolving this conjecture for rank 3 irreducible Coxeter groups, potentially simplifying the argument for a general resolution of the conjecture by reducing it to this case.

Further motivation for our study of cone types is inspired by Osada and Przytycki's recent breakthrough work in [OP22] showing that all Coxeter groups are biautomatic, finally confirming a long held conjecture. Remarkably, cone types play a role in this biautomatic structure as well.

In this article, we study an important subset of Γ, which we call the *tight gates* (of the cone type arrangement) and denote these elements by Γ^0. We first fix some terminology and notation before outlining our main results below.

Let (W, S) be a Coxeter system with Φ the associated root system. For $w \in W$ denote $\Phi(w)$ to be the *inversion set* of w. The *right weak order* on W is defined by $x \preceq y$ if and only if $\ell(y) = \ell(x) + \ell(x^{-1}y)$, equivalently, this means x is a *prefix* of y. Similarly, an element $x \in W$ is a *suffix* of y if $\ell(y) = \ell(yx^{-1}) + \ell(x)$. In terms of reduced words, x is a prefix of y if there is a reduced word for y which begins with a reduced word for x. Dually, x is a suffix of y if there is a reduced word for y which ends with a reduced word for x. The *left descent set* of w is the set $D_L(w) = \{s \in S \mid \ell(sw) = \ell(w) - 1\}$. For $w \in W$ we denote $T(w^{-1})$ to be the *cone type* of w^{-1} and the set of *boundary roots* of $T(w^{-1})$ is

$$\partial T(w^{-1}) = \{\beta \in \Phi^+ \mid \exists\, y \in W \text{ with } \Phi(w) \cap \Phi(y) = \{\beta\}\}$$

The boundary roots ∂T is the most precise set of roots which determines the cone type T [PY22, Theorem 2.8]. We denote $\mathbb{T}$ to be the set of cone types of W. We say a root $\beta \in \Phi^+$ is *super-elementary* if there exists $x, y \in W$ with $\Phi(x) \cap \Phi(y) = \{\beta\}$ and denote the set of super-elementary roots by $\mathcal{S}$. In the course of studying the tight gates, we also examine the set of *witnesses* of boundary roots of cone types.

Definition 1.1. Let $T \in \mathbb{T}$ be a cone type and $\beta \in \partial T$. An element $y \in W$ is a *witness of β with respect to T* if

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

for any $x \in W$ with $T(x^{-1}) = T$. We denote ∂T_β to be the set of *witnesses of β with respect to T*. More precisely

$$\partial T_\beta = \{y \in W \mid \Phi(x) \cap \Phi(y) = \{\beta\} \text{ for } x \in W \text{ with } T(x^{-1}) = T\}$$

Remark 1.1. By Theorem 7 a witness y of β with respect to T is independent of the elements x with $T(x^{-1}) = T$. That is, if $\Phi(y) \cap \Phi(x) = \{\beta\}$ for some x with $T(x^{-1}) = T$ then $\Phi(y) \cap \Phi(x) = \{\beta\}$ for all x with $T(x^{-1}) = T$.

For a subset $X \subseteq W$, we say X is *gated* if there is a unique minimal length element $x \in X$ such that $x \preceq y$ for all $y \in X$. We say the subset X is *convex* if for all $x, y \in X$ and all reduced expressions $x^{-1}y = s_1 \cdots s_n$, the elements $xs_1 \cdots s_j$ with $0 \leq j \leq n$ is in X. In terms of the Cayley graph X^1 of W this means that all elements on geodesics between x and y are in X.

A main result in this article is the following (a consequence of Lemma 4.2 and Theorem 11).

Theorem 1. *For each cone type T and $\beta \in \partial T$ there exists a unique minimal length element $y \in W$ such that*

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

for all $x \in W$ with $T(x^{-1}) = T$. Furthermore, the set of witnesses ∂T_β of β with respect to T is a convex, gated set in W.

This theorem reveals that each cone type T is associated with a number of convex, gated subsets of W including its cone type part $P_T \in \mathscr{T}$ and the sets ∂T_β for each $\beta \in \partial T$ (as well as the cone type T itself with the identity as the gate). We illustrate Theorem 1 in the case of a cone type in the Coxeter group of type $\widetilde{G}_2$ in Figure 1.

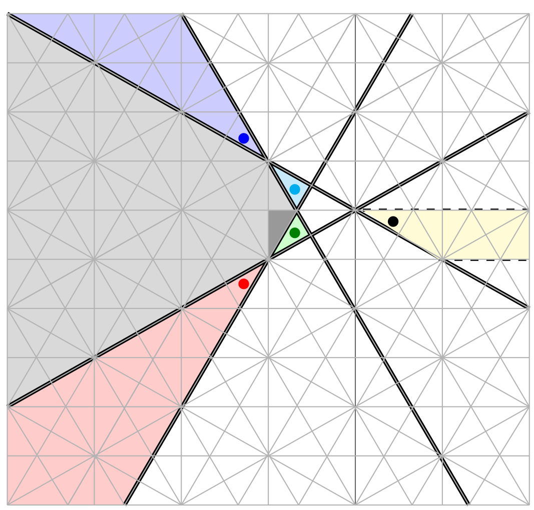

FIGURE 1. Let the element x be represented by the alcove with the black dot. The grey shaded region is the cone type $T := T(x^{-1})$ (the darker alcove represents the identity). The elements w in the yellow shaded region are the elements such that $T(w^{-1}) = T$ (i.e. the cone type part of $\mathscr{T}$ corresponding to T). The remaining coloured shaded regions are the sets ∂T_β for each $\beta \in \partial T$ and the corresponding coloured dots are the gates of those regions.

In [PY22] we showed that for each boundary root $\beta \in \partial T$ and $x \in W$ with $T(x^{-1}) = T$ there is *a* witness $y \in W$ such that $\beta \in \Phi^0(y)$, where $\Phi^0(y) = \{-y(\alpha_s) \mid s \in D_L(y)\}$ is the set of *final roots* of y. Moreover $y \in \Gamma$ and $\Phi^0(y) = \{\beta\}$ (see [PY22, Proposition 4.34]), however, it was not known whether such an element

y is unique, let alone that it is the minimal length representative of ∂T_β. We call $\Gamma^0 = \{w \in \Gamma \mid |\Phi^0(w)| = 1\}$ the set of *tight gates* of the cone type partition $\mathscr{T}$.

As an application of Theorem 1 we obtain a correspondence between Γ^0 and the set of super-elementary roots $\mathcal{S}$ for a few classes of Coxeter groups, giving a precise size and description of Γ^0 in these cases. Another main result in this work shows that for any *convex* Garside shadow G the set of its *tight* elements $G^0 = \{g \in G \mid |\Phi^0(g)| = 1\}$ are the most fundamental elements of G in the following sense (see Theorem 9 and Theorem 10).

Theorem 2. *Let G be a convex Garside shadow. Then*
$$G^0 = \{g \in G \mid |\Phi^0(g)| = 1\}$$
is the set of join-irreducible elements of the partially ordered set $(G, \preceq)$.

Since Γ is a convex Garside shadow, this means that the set of tight gates Γ^0 completely determines the cone type arrangement (see Corollary 3.4). In addition, we obtain the following (see Proposition 4.7).

Corollary 1.2. *The set Γ^0 is closed under suffix.*

Corollary 1.2 then leads to a very efficient algorithm to compute the set Γ^0 and $\mathcal{S}$ without having to compute the entire set Γ first. A consequence of this result is that to determine whether two elements x, y have the same cone type, one is only required to compute a set smaller than Γ. Our computations using Sagemath ([The24]) show that this set is in many cases much smaller than Γ as the rank of W grows (see Algorithm 1 and Figure 4).

In Section 6 we introduce the notion of *set dominance* of roots, generalising the concept of dominance for positive roots. The main utility of the concept at this stage is to give an alternative characterisation of boundary roots and will be explored further in subsequent work. We record the idea here for future reference. In Section 7 we prove [PY22, Conjecture 2] for a few select classes of Coxeter groups (see Theorem 12 and Section 7.2).

Theorem 3. *Let W be a Coxeter group of one of the following types*
(i) W is finite dihedral,
(ii) W is irreducible rank 3,
(iii) W is right-angled, or;
(iv) The Coxeter graph Γ_W is a complete graph.

Then $\mathscr{U} = \Gamma = \widetilde{S}$.

The following is then a consequence of Corollary 8.1 and Corollary 8.2.

Theorem 4. *Let (W, S) be a Coxeter group of one of the following types*
(i) W is finite dihedral,
(ii) W is irreducible rank 3,
(iii) W is right-angled, or;
(iv) The Coxeter graph Γ_W is a complete graph.

Then for each non-simple super-elementary root β there is a unique pair of tight gates x, y such that
$$\Phi(x) \cap \Phi(y) = \{\beta\}$$
and $\Phi^0(x) = \Phi^0(y) = \{\beta\}$. Furthermore,
$$|\Gamma^0| = 2|\mathscr{E}| - |S|$$

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2. Preliminaries

We recall the necessary facts on Coxeter systems, Cayley graphs, root systems, Garside shadows, cone types, gates and ultra-low elements. For general Coxeter group theory the primary references are [Hum90], [BB05] and [Ron]. For background on Garside shadows our primary references are [Hoh16], [HL16], and for cone types and ultra-low elements see [PY22].

2.1. **Coxeter Groups.** Let (W,S) be a Coxeter system with $|S|<\infty$. The *length* of $w\in W$ is

$$\ell(w)=\min\{n\geq 0\mid w=s_1\dots s_n \text{ with } s_1,\dots,s_n\in S\},$$

and an expression $w=s_1\dots s_n$ with $n=\ell(w)$ is called a *reduced expression* or *reduced word* for w. We denote the *left (resp. right) descent set* of w by $D_L(w)$ and $D_R(w)$. It is well known that the set

$$R=\{wsw^{-1}\mid w\in W, s\in S\}$$

is the set of *reflections* of W (see [BB05, Chapter 1]). For $J\subseteq S$, we denote $W_J\leq W$ to be the standard parabolic subgroup of W. For each $w\in W$ there is a unique reduced decomposition $w=w_J\cdot w^J$ where $w_J\in W_J$ and $w^J\in W^J$, and $W^J=\{v\in W\mid D_L(v)\cap J=\emptyset\}$.

2.1.1. *Root System of W.* Let V be an $\mathbb{R}$-vector space with basis $\{\alpha_s\mid s\in S\}$. Define a symmetric bilinear form on V by linearly extending $\langle\alpha_s,\alpha_t\rangle=-\cos(\pi/m(s,t))$. The Coxeter group W acts on V by the rule $sv=v-2\langle v,\alpha_s\rangle\alpha_s$ for $s\in S$ and $v\in V$, and the root system of W is $\Phi=\{w\alpha_s\mid w\in W,\, s\in S\}$. The elements of Φ are called *roots*, and the *simple roots* are the roots α_s with $s\in S$.

Each root $\alpha\in\Phi$ can be written as $\alpha=\sum_{s\in S}c_s\alpha_s$ with either $c_s\geq 0$ for all $s\in S$, or $c_s\leq 0$ for all $s\in S$. In the first case α is called *positive* (written $\alpha>0$), and in the second case α is called *negative* (written $\alpha<0$). For $\beta\in\Phi$ we denote the coefficient of the simple root α_s by $\mathrm{Coeff}_{\alpha_s}(\beta)$. The *support* of α is the set $J(\alpha)=\{s\in S\mid c_s\neq 0\}$. We denote the subgraph of the Coxeter graph Γ_W corresponding to $J(\alpha)$ by $\Gamma(\alpha)$. We say a root α has *non-spherical* support if $J(\alpha)$ generates an infinite Coxeter group. Denote Φ^+ (resp. Φ^-) to be the positive roots (resp. negative roots) of Φ. By definition of Φ, each root β can be written $\beta=w\alpha_s$ for some $s\in S$. Then the reflection $s_\beta=wsw^{-1}$ separates W into two sets

$$H^-_\beta=\{w\in W\mid \ell(s_\beta w)<\ell(w)\}\text{ and }H^+_\beta=\{w\in W\mid \ell(s_\beta w)>\ell(w)\}$$

For $A\subset\Phi^+$, $cone(A)$ is the set of non-negative linear combinations of roots in A and $cone_\Phi(A)=cone(A)\cap\Phi^+$. The *(left) inversion set* of $w\in W$ is

$$\Phi(w)=\{\alpha\in\Phi^+\mid w^{-1}(\alpha)<0\}.$$

An important subset of an inversion set $\Phi(w)$ is the *base* $\Phi^1(w)$. These roots are used to define *low* elements and *ultra-low* elements (see Section 2.1.2 and Section 2.1.3).

Definition 2.1. [Hoh16, Proposition 4.6] Let $w\in W$. The *base* of the inversion set $\Phi(w)$ is

$$\Phi^1(w)=\{\beta\in\Phi(w)\mid \ell(s_\beta w)=\ell(w)-1\}$$

The set $\Phi^1(w)$ determines the inversion set $\Phi(w)$ in the following way (see [Dye19, Lemma 1.7] and [HL16, Corollary 2.13]).

Proposition 2.2. *For $w\in W$ we have $\Phi(w)=cone_\Phi(\Phi^1(w))$ and if $A\subset\Phi^+$ is such that $\Phi(w)=cone_\Phi(A)$ then $\Phi^1(w)\subseteq A$.*

The following result will be used to compute $\Phi^1(sw)$ from $\Phi^1(w)$ whenever $\ell(sw)>\ell(w)$.

Proposition 2.3. [Hoh16, Theorem 4.10(2)] *Let $s\in S$ and $w\in W$. If $\ell(sw)>\ell(w)$ then*

$$\Phi^1(sw)=\{\alpha_s\}\sqcup s\big(\{\beta\in\Phi^1(w)\mid s_\beta w<ss_\beta w\}\big)$$

Another important subset of $\Phi(w)$ is the following.

Definition 2.4. Let $w\in W$. The set of *final roots* of w is

$$\Phi^0(w)=\{-w\alpha_s\mid s\in D_R(w)\}$$

A root $\beta\in\Phi^0(w)$ if and only if $s_\beta w=ws$ for some $s\in D_R(w)$. In terms of the Cayley graph X^1 the final roots correspond to the set of *final* or *last* walls crossed for geodesics from the identity to w.

We collect some further useful facts regarding joins in the right weak order.

Proposition 2.5. [Hoh16, Proposition 2.8] *If $X\subset W$ is bounded (in the right weak order), then*

$$\Phi(\bigvee X)=cone_\Phi(\bigcup_{x\in X}\Phi(x))$$

Corollary 2.6. [Hoh16, Corollary 4.7 (2)] *If $x\vee y$ exists and $\ell(s_\beta(x\vee y))=\ell(x\vee y)-1$, then either $\ell(s_\beta x)=\ell(x)-1$ or $\ell(s_\beta y)=\ell(y)-1$.*

2.1.2. *Elementary roots and low-elements.* We recall the notion of *dominance* of positive roots. A root $\beta \in \Phi^+$ dominates $\alpha \in \Phi^+$ if whenever $w^{-1}\beta < 0$ we have $w^{-1}\alpha < 0$. The set of elementary roots $\mathscr{E}$ is the set of positive roots which dominate no roots (other than itself). The following result of Brink provides very useful information about the elementary roots in terms of their support.

Lemma 2.7. [Bri98, Lemma 4.1] *Suppose that $\alpha \in \Phi^+$ and $\Gamma(\alpha)$ contains a curcuit or an infinite bond. Then $\alpha \notin \mathscr{E}$.*

An element $w \in W$ is a *low*-element if $\Phi^1(w) \subset \mathscr{E}$ (or more generally, these are known as the 0-low elements). The set of low-elements of W is denoted L. A recent result of Dyer, Fishel, Hohlweg and Marks shows that the set L is the set of minimal length elements of the *Shi*-partition (also called 0-*Shi*-partition, see [DFHM24, Theorem 1.1]. It should also be noted that they show more generally that the m-low elements are the minimal elements of the m-Shi partition). The $m = 0$ case was also independently proven in [PY23].

2.1.3. *Super-elementary roots and ultra-low elements.*

Definition 2.8. A root $\beta \in \Phi^+$ is *super-elementary* if there exists $x, y \in W$ with

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

We let $\mathcal{S}$ denote the set of super-elementary roots. By the following result we have that $\mathcal{S} \subseteq \mathscr{E}$ and hence the set $\mathcal{S}$ is finite for every Coxeter group. For a number of classes of Coxeter groups, including those of affine type, we have $\mathcal{S} = \mathscr{E}$ (see [PY22, Section 7]).

Lemma 2.9. [PY22, Lemma 2.4] *Let $x, y \in W$ and $\beta \in \Phi^+$. Suppose that $\Phi(x) \cap \Phi(y) = \{\beta\}$. Then:*

(1) $\beta \in \mathscr{E}$, and;
(2) $\beta \in \Phi^1(x) \cap \Phi^1(y)$

Definition 2.10. An element $w \in W$ is *ultra-low* if for each $\beta \in \Phi^1(x)$ there exists $y \in W$ with

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

We denote the set of all ultra-low elements by $\mathscr{U}$.

2.1.4. *Garside shadows.* A subset $B \subseteq W$ is a *Garside shadow* if B contains S and

(1) B is closed under join in the (right) weak order: if $X \subseteq B$ is bounded then $\bigvee X \in B$.
(2) B is closed under taking suffixes. If $w \in B$ then any suffix of w is in B.

An intersection of Garside shadows in (W, S) is a Garside shadow and for any $X \subseteq W$ there is a smallest Garside shadow $Gar_S(X)$ containing X ([Hoh16, Proposition 2.2]). For every Coxeter system (W, S) with S finite, the smallest Garside shadow $\tilde{S} := Gar_S(S)$ in W is finite ([Hoh16, Corollary 1.2]). For a Garside shadow G and an element $x \in W$ we denote $\pi_G(x)$ to be the longest prefix of x contained in G.

2.1.5. *Cone Types and Boundary roots of W.* We recall here the definition of cone types as well as some key properties that will be useful for our work here.

Definition 2.11. For $x \in W$, the *cone type* of x is

$$T(x) = \{y \in W \mid \ell(xy) = \ell(x) + \ell(y)\}$$

For each finitely generated Coxeter system (W, S) the set of its cone types $\mathbb{T}$ is finite ([PY22, Corollary 1.26]). The *cone type arrangement* or *cone type partition* $\mathscr{T}$ is the partition of W into the sets

$$P_T = \{x \in W \mid T(x^{-1}) = T\}$$

for each $T \in \mathbb{T}$. The partition $\mathscr{T}$ is the minimal *regular partition* of W as in discussed in some depth in [PY22]. The parts P_T are in bijection with the cone types and each part contains a unique minimal length representative g_T (called the *gate* of P_T) which is a prefix of all elements in P_T.

Theorem 5. [PY22, Theorem 1] *For each cone type T there is a unique element $m_T \in W$ of minimal length such that $T(m_T) = T$. Moreover, if $w \in W$ with $T(w) = T$ then m_T is a suffix of w.*

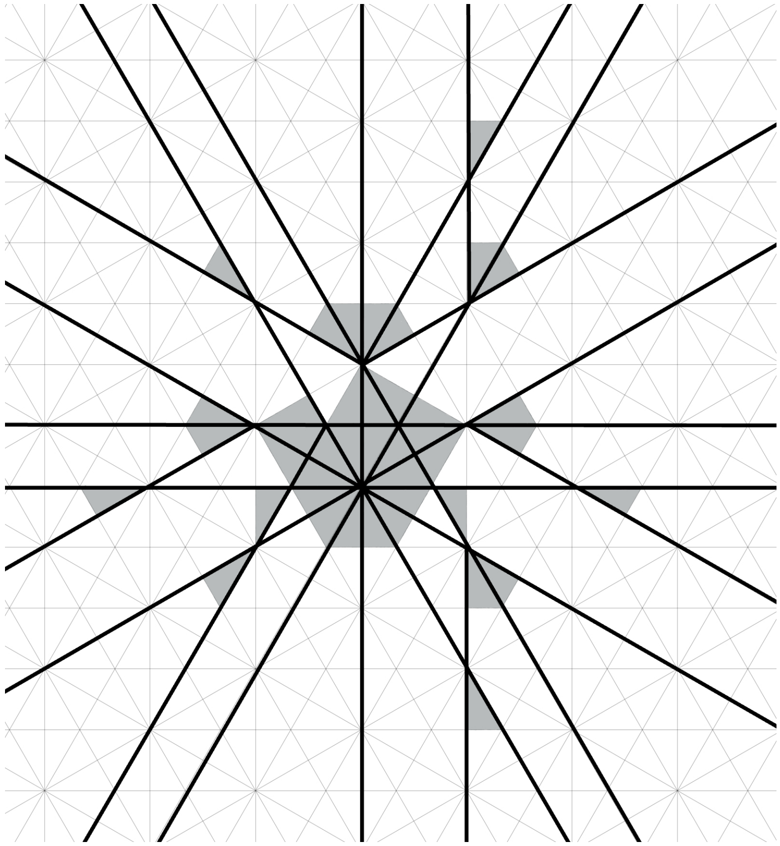


FIGURE 2. The cone type partition $\mathscr{T}$ for the Coxeter group of type $\widetilde{G}_2$. For each cone type T, the part P_T contains the elements x such that $T(x^{-1}) = T$. The shaded in alcove of each part represents the inverse of the unique minimal length cone type representatives m_T^{-1}. These are the *gates* of $\mathscr{T}$.

By Theorem 5 and as illustrated in Figure 2 from the point of view of the cone type partition $\mathscr{T}$, for $x \in W$, it is in some sense more natural for one to consider the cone type $T(x^{-1})$. The *gates of* $\mathscr{T}$ is the set

$$\Gamma = \{m_T^{-1} \mid m_T \text{ is the unique minimal length element with } T(m_T) = T\}$$

and the *tight gates* of $\mathscr{T}$ is the set

$$\Gamma^0 = \{g \in \Gamma \mid |\Phi^0(g)| = 1\}$$

Remark 2.1. In general, we say a partition $\mathscr{P}$ of W, is *gated* if for each part $P \in \mathscr{P}$ there is a unique minimal length element $g \in P$ such that $g \preceq y$ for all $y \in P$. For the vast majority of this article, we use the term *gates* to mean Γ. It will be made clear when this is not the case.

The following theorem gives a characterisation of a gate x in terms of $\Phi^0(x)$.

Theorem 6. [PY22, Theorem 4.33] *Let $x \in W$. Then $x \in \Gamma$ if and only if for each $\beta \in \Phi^0(x)$ there exists $w \in W$ with $\Phi(x) \cap \Phi(w) = \{\beta\}$.*

Theorem 7. [PY22, Theorem 2.6] *Let T be a cone type. If $T = T(x^{-1})$ then $\beta \in \partial T$ if and only if there exists $w \in W$ with*

$$\Phi(x) \cap \Phi(w) = \{\beta\}$$

Moreover, if $\beta \in \partial T$ then there exists $w \in W$, independent of x, such that $\Phi(x) \cap \Phi(w) = \{\beta\}$ whenever $T = T(x^{-1})$.

We list some further relevant results regarding cone types and their boundary roots.

Lemma 2.12. [PY22, Lemma 1.15] *If $x \preceq y$ then $T(y^{-1}) \subseteq T(x^{-1})$.*

Theorem 8. [PY22, Corollary 2.9] *Let T be a cone type. If $T = T(x^{-1})$ Then*

$$T = \bigcap_{\Phi(x)} H_\alpha^+ = \bigcap_{\mathscr{E}(x)} H_\alpha^+ = \bigcap_{\Phi^1(x)} H_\alpha^+ = \bigcap_{\Phi^1(x)\cap\mathscr{E}(x)} H_\alpha^+ = \bigcap_{\partial T(x^{-1})} H_\alpha^+$$

Proposition 2.13. [PY22, Proposition 1.19] *If $X \subseteq W$ is bounded with $y = \bigvee X$ then $T(y^{-1}) = \bigcap_{x\in X} T(x^{-1})$.*

Proposition 2.14. [Yau21, Proposition 3.2.4] *Let T be a cone type.*

(i) *If $\beta \in \Phi^+$ is a boundary root of T then there exists $w \in W$ such that $\Phi(x) \cap \Phi(w) = \{\beta\}$ for all $x \in W$ with $T(x^{-1}) = T$.*
(ii) *For $x, w \in W$, if $T(x^{-1}) = T$ and $\Phi(x) \cap \Phi(w) = \{\beta\}$ then $\beta \in \partial T$.*

The following is then a consequence of Theorem 7 above and the definition of the minimal length cone type representatives, low-elements and ultra-low elements.

Proposition 2.15. [PY22, Proposition 6.2] *For each Coxeter group W,*

$$\mathscr{U} \subseteq \Gamma \subseteq L$$

We end this section with a few comments on convexity in W. Consider W in terms of it's Cayley graph X^1. Recall that a subset $X \subseteq W$ is *convex* if for all $x, y \in X$ all geodesics from x to y lies in X. A characterisation of convexity is given in the following well known result.

Lemma 2.16. [AB08, Proposition 3.94] *A subset $A \subseteq W$ is convex if and only if it is an intersection of half spaces.*

2.2. **Automata and Regular Partitions.** It will be helpful to briefly mention the connection between partitions of W and automata for recognising the language of reduced words of (W, S).

For our purposes, one may consider an automaton $\mathscr{A}$ as a directed graph with edges labelled by elements of S together with a transition function $\mu : Y \times S \to Y$ where Y are the vertices or *states* of $\mathscr{A}$. A state Y_o is designated as the *initial* or *start* state and if $x, y \in Y$ and $\mu(x, s) = y$ then there is a directed edge from x to y labelled s. A word $\widetilde{w} = s_1s_2 \dots s_n$ is reduced if and only if there is a path in $\mathscr{A}$ with a sequence of labels $s_1, s_2, \dots, s_n$ starting from the initial state (see [PY22, Section 1.8] for further details).

In [PY22] we characterise (the states of) automata recognising the language of reduced words of (W, S) in terms of the notion of a *regular partition* (see [PY22, Theorem 2]). In brief, each automaton arises from a regular partition and for each regular partition there exists an explicitly defined automaton. Primary examples of regular partitions are given by the partition induced by the low elements L, known as the *Shi-partition*, and the partition induced by the inverses of the minimal length cone type representatives (the set of *gates* Γ) with parts defined in the same way. The corresponding automata associated with these partitions are known as the *Brink-Howlett* automaton and the *cone type* automaton in the literature. The elements L and Γ are Garside shadows and induce partitions in the following way.

For a Garside shadow G, the partition $\mathscr{G}$ of W with parts defined by the relation $x \sim_G y$ if and only if $\pi_G(x) = \pi_G(y)$ (then $T(x^{-1}) = T(y^{-1})$) is an example of a (gated) regular partition. We say a partition $\mathscr{P}$ is *convex* if each part $P \in \mathscr{P}$ is convex. We say a Garside shadow G is convex if the partition it induces is convex.

2.2.1. *The Cayley graph of (W, S).* We conclude our preliminary section with a brief review of the Cayley graph.

We denote X^1 to be the Cayley graph of W with vertex set $X^0 = W$ and edge-set $\{(w, ws) \mid w \in W, s \in S\}$ where each edge is of length 1. For vertices $x, y \in X^1$ a *geodesic* between x and y is a minimal length path between x and y in X^1. In the setting of X^1, for $w \in W$, $\ell(w)$ is the length of geodesics from the identity to w.

We consider the left action of W on X^0 which induces an action of W on X^1. For $r \in R$ the *wall* H_r is the fixed point set of r in X^1 and an edge (w, ws) *crosses* a wall H_r if and only if $r = wsw^{-1}$. Each wall H_r separates X^1 into two *half spaces*. We denote H_r^+ to be the half space containing the identity and H_r^- the half space not containing the identity. By the bijective correspondence between roots and reflections, as subsets of W, we have $H_\beta^- = H_{s_\beta}^-$ (and $H_\beta^+ = H_{s_\beta}^+$). We will use the terms *walls* and *roots* interchangably.

Two distinct walls H_r and H_q *intersect* if H_r is not contained in a half-space for H_q (this relation is symmetric), or equivalently, $\langle r, q\rangle$ is a finite group. We say r, q are *sharp-angled* if r and q do not commute and there is $w \in W$ such that wrw^{-1} and wqw^{-1} are in S. Then there is a component of $X^1 \setminus (H_r \cup H_q)$ whose intersection with X^0 is a *geometric fundamental domain* for the action of $\langle r, q\rangle$ on X^0.

For $J \subseteq R$ and $p \in X^0$ the $J(p)$-*residue* is the subgraph of X^1 induced by the action of the subgroup generated by J on p. In particular, note that for $w \in W$ if $\alpha_r, \alpha_q \in \Phi^0(w)$ then $w^{-1}rw = s$ and $w^{-1}qw = t$ are in S and $\langle r, q\rangle$ is a finite dihedral reflection subgroup with $2m(s,t)$ elements. Then the residue $\langle r, q\rangle(w) = wW_{\langle s,t\rangle}$ with $w\Phi^+_{\langle s,t\rangle}$ the set of $m(s,t)$ corresponding roots separating the $2m(s,t)$ elements of $\langle r, q\rangle(w)$.

3. The tight gate partition and Join-irreducible elements of L and Γ

In this section, we initiate the study of some properties of the *tight* elements of L and $\mathscr{T}$. We show that the partition induced by the tight gates $\mathscr{T}^0$ produces the cone type arrangement $\mathscr{T}$. While we mainly focus on the tight elements of $\mathscr{T}$, another primary result in this section is that the *tight* elements of convex Garside shadows G are the join-irreducible elements of the poset $(G, \preceq)$. In particular, in follows from [PY23, Theorem 1.4] that the set Γ^0 are the most fundamental elements of the smallest Garside shadow of W.

The following result is an obvious, but rather useful fact.

Lemma 3.1. *Let $x \in W$. If $\alpha \in \Phi(x)$ then there is $y \preceq x$ with $\Phi^0(y) = \{\alpha\}$*

Proof. Let x_0 be a prefix of x with $\alpha \in \Phi^0(x_0)$. If there is $\beta \in \Phi^0(x_0)$ with $\beta \neq \alpha$ then consider the prefix $x_1 := s_\beta x_0$ of x. Repeating this process, there must be some prefix x_k of x with $\Phi^0(x_k) = \{\alpha\}$. □

As noted in the introduction, the following result is known. It is a direct consequence of Lemma 3.1 and Theorem 6.

Proposition 3.2. [PY22, Proposition 4.34] *Let $x \in W$ and $\beta \in \Phi^+$. Suppose there exists $w \in W$ such that $\Phi(x) \cap \Phi(w) = \{\beta\}$, and let w be of minimal length subject to this property. Then $\Phi^0(w) = \{\beta\}$ and w is a gate.*

The followng result shows that every gate of $\mathscr{T}$ is a join of tight gates.

Theorem 9. *Let $x \in \Gamma$. Then*

$$x = \bigvee X$$

where $X = \{z \in W \mid z \preceq x$ with $\Phi^0(z) = \{\alpha\}$ and $\alpha \in \partial T(x^{-1})\}$. Hence, every gate is a join of tight gates.

Proof. Let $x \in \Gamma$. By Lemma 3.1, for each $\alpha \in \partial T(x^{-1})$, there is $z_\alpha \preceq x$ with $\Phi^0(z_\alpha) = \{\alpha\}$. Since $\alpha \in \partial T(x^{-1})$, there exists $w_\alpha \in \Gamma^0$ such that $\Phi(x) \cap \Phi(w_\alpha) = \{\alpha\}$. Therefore, we also have $\Phi(z_\alpha) \cap \Phi(w_\alpha) = \{\alpha\}$. By Theorem 6 then we have $z_\alpha \in \Gamma^0$ and x is an upper bound of

$$X := \{z \in W \mid z \preceq x \text{ with } \Phi^0(z) = \{\alpha\} \text{ and } \alpha \in \partial T(x^{-1})\} \subset \Gamma^0$$

Now let $y = \bigvee X$ and let $\Phi(X) := \bigcup_{z \in X} \Phi(z)$. By Proposition 2.13 and Theorem 8 we have

$$T(y^{-1}) = \bigcap_{z \in X} T(z^{-1}) = \bigcap_{\Phi(X)} H^+_\beta$$

Since $y \preceq x$ and $\partial T(x^{-1}) \subseteq \Phi(X)$ it follows by Theorem 8 again that

$$T(x^{-1}) \subseteq T(y^{-1}) = \bigcap_{\Phi(X)} H^+_\beta \subseteq \bigcap_{\partial T(x^{-1})} H^+_\beta = T(x^{-1})$$

Since $x \in \Gamma$ we must have $y = x$. □

A very similar proof shows that

Proposition 3.3. *Let $x \in \Gamma$. Then*

$$x = \bigvee X_1$$

where $X_1 = \{z \in W \mid z \preceq x$ with $\Phi^0(z) = \{\alpha\}$ and $\alpha \in \Phi^1(x)\}$.

Proof. Clearly x is an upper bound of X_1. Let $\Phi^1(X_1) = \{\alpha \in \Phi^1(z) \mid z \in X_1\}$. Let $y = \bigvee X_1$. Then by Theorem 8

$$T(y^{-1}) = \bigcap_{z \in X_1} T(z^{-1}) = \bigcap_{\Phi^1(X_1)} H^+_\alpha$$

Now since $\Phi^1(x) \subseteq \Phi^1(X_1)$ we have

$$T(y^{-1}) = \bigcap_{\Phi^1(X_1)} H^+_\alpha \subseteq T(x^{-1}) = \bigcap_{\Phi^1(x)} H^+_\alpha$$

But since $y \preceq x$ we have $T(x^{-1}) \subseteq T(y^{-1})$ and since $x \in \Gamma$ this means $x = y$. □

Corollary 3.4. *Each cone type $T \in \mathbb{T}$ is expressible as an intersection of cone types of tight gates.*

Proof. By Theorem 9 it follows that every gate is a join of a set of tight gates X. Then by Proposition 2.13 we have that $T(x^{-1}) = \bigcap_{z \in X} T(z^{-1})$. ☐

Corollary 3.5. *Define $x \sim_P y$ if the following holds: $x \in T(w^{-1})$ if and only if $y \in T(w^{-1})$ for all $w \in \Gamma^0$.*

Let $\mathscr{T}^0$ be the partition of W with parts P defined by the relation $x \sim_P y$. Then $\mathscr{T}^0 = \mathscr{T}$ and hence $\mathscr{T}^0$ is a regular partition.

Proof. It suffices to show that if x, y are in the same part of $\mathscr{T}^0$ then x, y are in the same part of $\mathscr{T}$. Suppose $x \in T(z^{-1})$ if and only if $y \in T(z^{-1})$ for all $z \in \mathscr{T}^0$. Let $x \in T(g^{-1})$ for $g \in \Gamma \setminus \Gamma^0$. By Theorem 9 we then have $x \in T(w^{-1})$ for all $w \in X$ where

$$X = \{w \in W \mid w \preceq g \text{ with } \Phi^0(w) = \{\alpha\} \text{ and } \alpha \in \partial T(g^{-1})\}$$

Since $X \subset \Gamma^0$ this implies $y \in T(w^{-1})$ for all $w \in X$ and hence by Theorem 9 and Proposition 2.13 this implies $y \in T(g^{-1})$. Similarly, if $x \notin T(g^{-1})$ for some $g \in \Gamma \setminus \Gamma^0$ then by Theorem 9, there must be some $w \preceq g$ with $\Phi^0(w) = \{\alpha\}$ and $\alpha \in \partial T(g^{-1}) \cap \partial T(w^{-1})$ such that $x \notin T(w^{-1})$. Therefore, also $y \notin T(w^{-1})$. ☐

We could replace the set X in the proof of Corollary 3.5 by the set X_1 from Proposition 3.3 if the following were true.

Conjecture 1. *Let $x \in \Gamma$ and $\alpha \in \Phi^1(x)$. Suppose $g \preceq x$ and $\Phi^0(g) = \{\alpha\}$ then $g \in \Gamma^0$.*

Remark 3.1. A consequence of Corollary 3.5 is that to determine whether two elements x, y have the same cone type (or equivalently, whether x^{-1} and y^{-1} live in the same part of the cone type partition), one only needs to compute the set of tight gates Γ^0 and then check whether $x \in T(w^{-1})$ if and only if $y \in T(w^{-1})$ for all $w \in \Gamma^0$. Our computations in Sagemath in Section 5.1 show that often, the set Γ^0 is much smaller than Γ. We illustrate the $\mathscr{T}^0$ arrangements for the rank 3 Coxeter groups of affine type in Figure 3.

Recall that for a poset P an element $x \in P$ is *join-irreducible* if x is not the minimal element in P and if $x = a \vee b$ then either $x = a$ or $x = b$. Equivalently, if $X \subseteq P$ and $x = \bigvee X$ then $x \in X$.

Theorem 10. *Let G be a convex Garside shadow. Then the set*

$$G^0 = \{x \in G \mid |\Phi^0(x)| = 1\}$$

is the set of join-irreducible elements of $(G, \preceq)$.

Proof. Suppose $x \in G^0$ and let $\Phi^0(x) = \{\beta\}$. Let $x = \bigvee X$ for some $X \subseteq W$. Since $\beta \in \Phi^1(x)$ by Corollary 2.6 $\beta \in \Phi^1(z)$ for some $z \in X$. If $z \neq x$ then we have $z \prec x$ with $\beta \in \Phi(z)$. This then contradicts that $\Phi^0(x) = \{\beta\}$.

Now let $x \in G$ be join-irreducible and suppose $|\Phi^0(x)| > 1$. Let $\{\alpha_1, \ldots \alpha_k\} = \Phi^0(x)$ and $x_i := s_{\alpha_i} x$. Let $\pi(x_i) \in G$ be the G-projection of x_i. We claim that $x = \bigvee \pi(x_i)$. Clearly, $\{\pi(x_i)\}$ is bounded since $\pi(x_i) \preceq x_i \prec x$. Now let $y = \bigvee \pi(x_i)$. If $y \neq x$ then we have $y \preceq x_i \preceq x$ and by [HNW16, Proposition 2.5 (iii)] we have $y = \pi(y) \preceq \pi(x_i)$. Hence y is also a lower bound of $\{\pi(x_i)\}$ and thus $y \preceq \pi(x_i) \preceq y$. Therefore we must have $y = \pi(x_1) = \ldots = \pi(x_k)$. But x lies on a geodesic from x_i to x_j for any $1 \leq i, j \leq k$ and so by convexity, we must have $x = \pi(x) = \pi(x_1) = \ldots = \pi(x_k) = y$, a contradiction. Therefore we have $x = \bigvee \pi(x_i)$ which contradicts that x is join-irreducible. ☐

Remark 3.2. It follows by Lemma 2.16 that Garside shadows induced by hyperplane arrangements are convex, so by [DFHM24, Theorem 1.1], Theorem 10 applies to the m-low elements. It also applies to Γ by [PY22, Proposition 4.3].

4. Witnesses of boundary roots

In this section, we study the *witnesses* of boundary roots of cone types. We show that for each cone type T and boundary root $\beta \in \partial T$, the set ∂T_β is a convex, gated subset of W. We begin with a direct observation.

Lemma 4.1. *A tight gate w is it's own witness if and only if $w \in S$.*

Lemma 4.2. *Let $x \in W$ and $T := T(x^{-1})$. Then*

$$\partial T_\beta = H_\beta^- \cap \Big(\bigcap_{\alpha \in \Phi(x) \setminus \{\beta\}} H_\alpha^+ \Big)$$

and the set ∂T_β is convex.

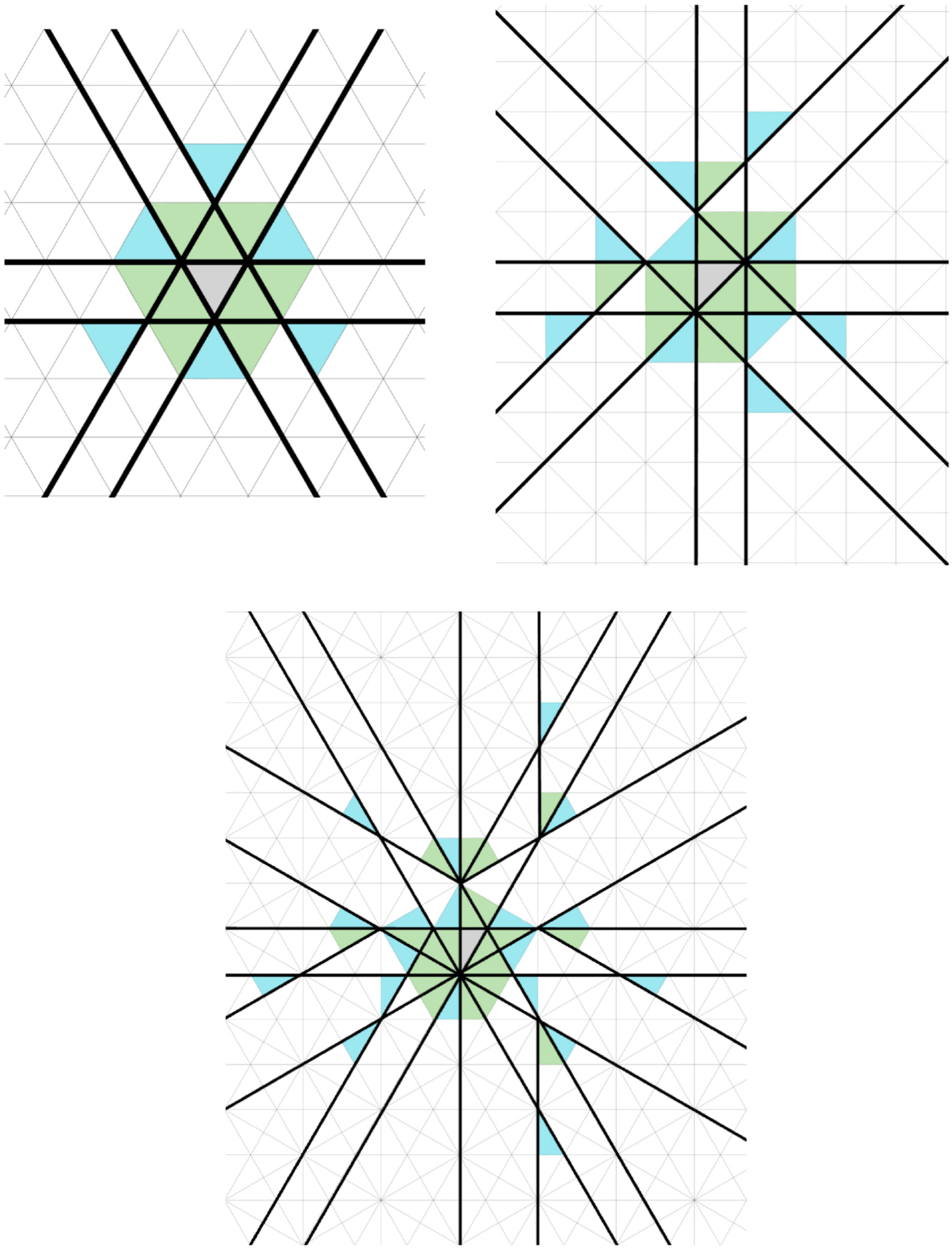


FIGURE 3. The partition $\mathscr{T}^0 = \mathscr{T}$ for the rank 3 Coxeter groups of affine type. The green alcoves represent the tight gates and the cyan alcoves are the gates which are non-trivial joins of tight gates.

Proof. The proof is straightforward. If $w \in \partial T_\beta$ then $\alpha \notin \Phi(w)$ for all $\alpha \neq \beta \in \Phi(x)$, so $w \in H_\alpha^+$ for all $\alpha \in \Phi(x) \setminus \{\beta\}$. Since $\beta \in \Phi(w)$ we have $w \in H_\beta^-$. The reverse inclusion is also clear. Convexity follows by the fact that ∂T_β is an intersection of half spaces. □

Corollary 4.3. *Let T be a cone type and $\beta \in \partial T$. The set ∂T_β is a union of cone type parts.*

Proof. Let $y \in \partial T_\beta$. Then by Proposition 2.14 for $x \in W$ if $T(x^{-1}) = T$ we have $\Phi(x) \cap \Phi(y) = \{\beta\}$ and hence $\beta \in \partial T(y^{-1})$. Let $z \preceq y$ be the gate of the cone type part P containing y, so $\partial T(z^{-1}) = \partial T(y^{-1})$. Then $\beta \in \Phi(z)$ and $\Phi(x) \cap \Phi(z) = \{\beta\}$, so $z \in \partial T_\beta$ and hence $P \subseteq \partial T_\beta$. □

The following two results are not directly required for our proof of Theorem 1. They are included to demonstrate how the boundary roots and witnesses can be inductively computed.

Lemma 4.4. [Yau21, Lemma 3.3.4] *Let $w \in W$ and $s \in S$ where $s \notin D_L(w)$. Then*

$$\partial T(w^{-1}s) = \{\alpha_s\} \sqcup s\big(\{\alpha \in \partial T(w^{-1}) \mid \exists\, y \in \partial T(w^{-1})_\alpha \text{ and } s \in D_L(y)\}\big)$$

Proof. We first show the left to right inclusion. If $\alpha \neq \alpha_s \in \partial T(w^{-1}s)$ then by Proposition 2.14 there is $x \in W$ such that $\Phi(sw) \cap \Phi(x) = \{\alpha\}$. Thus $\alpha_s \notin \Phi(x)$ and hence $\Phi(sx) = \{\alpha_s\} \sqcup s\Phi(x)$. Since $\alpha_s \in \Phi(sw)$ we have $\Phi(w) = s\big(\Phi(sw) \setminus \{\alpha_s\}\big)$. Therefore

$$\begin{aligned}\Phi(w) \cap \Phi(sx) &= s\big(\Phi(sw) \setminus \{\alpha_s\}\big) \cap \big(\{\alpha_s\} \sqcup s\Phi(x)\big)\\ &= s\big(\Phi(sw) \cap \Phi(x)\big)\\ &= s(\{\alpha\})\end{aligned}$$

We now show the right to left inclusion. It is clear that $\alpha_s \in \partial T(w^{-1}s)$. For a root $\alpha \in \partial T(w^{-1})$ where there is $y \in W$ with $\Phi(w) \cap \Phi(y) = \{\alpha\}$ and $s \in D_L(y)$ we aim to show that $\Phi(sw) \cap \Phi(sy) = \{s\alpha\}$. It then follows by Proposition 2.14 (ii) that $s\alpha$ is a boundary root of $T(w^{-1}s)$. Since $\ell(sw) > \ell(w)$ we have that $\Phi(sw) = \{\alpha_s\} \sqcup s\Phi(w)$. If $y \in W$ is such that $s \in D_L(y)$ then $\Phi(sy) = s\big(\Phi(y) \setminus \{\alpha_s\}\big)$. Hence we have

$$\Phi(sw) \cap \Phi(sy) = \big(\{\alpha_s\} \sqcup s\Phi(w)\big) \cap s\big(\Phi(y) \setminus \{\alpha_s\}\big)$$

Since $\Phi(w) \cap \Phi(y) = \{\alpha\}$ by the formula for $\Phi(sw) \cap \Phi(sy)$ above it follows that $s\alpha \in \Phi(sw) \cap \Phi(sy)$. We now show that $\Phi(sw) \cap \Phi(sy) = s\alpha$. If $\beta \in \Phi(sw) \cap \Phi(sy)$ then $\beta \in s\Phi(w) \cap s\Phi(y) = s\big(\Phi(w) \cap \Phi(y)\big) = s(\{\alpha\})$. So $\beta = s\alpha$ as required. □

The next result shows how the set $\partial T'_{s\beta}$ is obtained from ∂T_β if $\beta \in \partial T$ and $s\beta \in \partial T'$.

Corollary 4.5. *Let $x \in W$, $s \in S$ with $s \notin D_L(x)$. Let $y = sx$ and denote $T' = T(y^{-1})$ and $T = T(x^{-1})$. If $\beta \in \partial T$ and $s\beta \in \partial T'$ then*

$$\partial T'_{s\beta} = \{w \in \partial T_\beta \mid s \in D_L(w)\}$$

Proof. If $v \in \partial T'_{s\beta}$ then $\Phi(y) \cap \Phi(v) = \{s\beta\}$. Since $\alpha_s \in \Phi(y)$ this implies that $\alpha_s \notin \Phi(v)$ and so $\Phi(sv) = \{\alpha_s\} \sqcup s\Phi(v)$. Since $\Phi(x) = s\Phi(y) \setminus \{\alpha_s\}$ we have

$$\Phi(x) \cap \Phi(sv) = \big(\{\alpha_s\} \sqcup s\Phi(v)\big) \cap \big(s\Phi(y) \setminus \{\alpha_s\}\big)$$

Therefore, as in the proof of Lemma 4.4, we have $\Phi(x) \cap \Phi(sv) = \{\beta\}$. The reverse inclusion directly follows from the second half of the proof of Lemma 4.4. □

We now prove the main result of this section.

Theorem 11. *For each cone type T and $\beta \in \partial T$ there exists a unique minimal length element $y \in W$ such that*

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

for all $x \in W$ with $T(x^{-1}) = T$. Furthermore, the set of witnesses ∂T_β of β with respect to T is a convex, gated set in W.

Proof. The proof is very similar to the proof of [PY23, Theorem 1.1] and [PY23, Theorem 1.3]. We pass between reflections in the Cayley graph X^1 of W and their corresponding roots in Φ^+. Hence consider ∂T_β to be the intersection of half spaces H_r^+ in X^1 of the corresponding reflections to the set of roots $\{\alpha \in \Phi^+ \mid \alpha \in \Phi(x) \setminus \{\beta\}\}$.

It suffices to show that for each $p_0, p_n \in \partial T_\beta$ there is $p \in \partial T_\beta$ with $p_0 \succeq p \preceq p_n$. Let $\pi = (p_0, p_1, \ldots, p_n)$ be a geodesic edge path between p_0 and p_n. By Lemma 4.2, each $p_i \in \partial T_\beta$ for $0 \leq i \leq n$. We let

$L = max\{\ell(p_i) \mid 1 \leq i \leq n\}$ and define the complexity of the path π as the tuple $(n_L, , n_{L-1}, \dots n_2, n_1)$ where n_j is the number of elements p_i in π with $\ell(p_i) = j$.

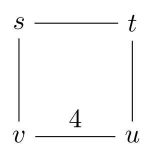
Our aim is to modify π to obtain another embedded edge path between p_0 and p_n so that there is no $p_{i-1} \prec p_i \succ p_{i+1}$. Now suppose there is $p_{i-1} \prec p_i \succ p_{i+1}$. Let H_r, H_q be the walls separating p_{i-1} from p_i and p_i from p_{i+1} respectively and let α_r, α_q be the corresponding roots. Note that $\alpha_r \neq \beta$ and $\alpha_q \neq \beta$ since by convexity $p_{i-1}, p_i, p_{i+1} \in \partial T_\beta$. Also, p_{i-1}, p_{i+1} are elements of the *residue* $R = \langle r, q\rangle(p_i)$ and $\alpha_r, \alpha_q \in \Phi^0(p_i)$. If r and q do not commute then r, q are sharp angled and the identity e is in a geometric fundamental domain for the finite dihedral reflection subgroup $W_{\langle r,q\rangle}$ generated by the reflections r, q. We claim that all the elements of the residue $R = \langle r, q\rangle(p_i)$ lie in ∂T_β.

Since $p_i \in \partial T_\beta$ this implies that $\alpha_r, \alpha_q \notin \Phi(x) \setminus \{\beta\}$. We then need to show that $\alpha' \notin \Phi(x) \setminus \{\beta\}$ for any $\alpha' \in \Phi^+_{\langle r,q\rangle}$. Hence we can assume that r and q do not commute, otherwise there is no other root in $\Phi^+_{\langle r,q\rangle}$. If there is $\alpha' \in \Phi^+_{\langle r,q\rangle} \cap \Phi(x)$ then since $\alpha' = k\alpha_r + j\alpha_q$ for some $k, j \in \mathbb{R}_{>0}$ this implies that either $\alpha_r \in \Phi(x) \cap \Phi(p_i)$ or $\alpha_q \in \Phi(x) \cap \Phi(p_i)$. Since $\beta \in \Phi(x) \cap \Phi(p_i)$, this then implies that $|\Phi(x) \cap \Phi(p_i)| > 1$, a contradiction.

We now modify π and replace the subpath (p_{i-1}, p_i, p_{i+1}) with the other embedded edge-path in the residue R from p_{i-1} to p_{i+1}. Note that by [Ron, Theorem 2.9] all the elements of R are $\preceq p_i$, since the element $\mathrm{Proj}_R(e)$ of R is opposite to p_i. It then follows by [Ron, Theorem 2.15] that all elements of R lie on a geodesic edge path from $\mathrm{Proj}_R(e)$ to p_i and hence on a geodesic edge path from e to p_i through $\mathrm{Proj}_R(e)$. Thus this path modification decreases the complexity of π. After finitely many such modifications we obtain the desired path. $\square$

For each super-elementary root β there are potentially multiple "pairs" of minimal length tight gates x, y with $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^0(x) = \Phi^0(y) = \{\beta\}$. Consider the following example.

Example 4.6. Let W be the rank 4 compact hyperbolic Coxeter group whose graph is given by



Let $\beta = \alpha_t + \sqrt{2}\alpha_u + 2\alpha_v$. Denote the elements $a = tvutv, b = uvutv, c = utvutv, d = vutv$. Then these elements are all tight gates with $\Phi^0(a) = \Phi^0(b) = \Phi^0(c) = \Phi^0(d) = \{\beta\}$ and we have

$$\Phi(a) \cap \Phi(b) = \{\beta\}$$
$$\Phi(c) \cap \Phi(d) = \{\beta\}$$

But d is not a witness of β with respect to $T(a^{-1})$ and b is not a witness of β with respect to $T(c^{-1})$.

The following result then implies that the set of tight gates is closed under suffix.

Proposition 4.7. *Let $x \in \Gamma$ and $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^0(y) = \{\beta\}$ where y is of minimal length with this property. If $s \in D_L(y)$ then*

$$\Phi(sx) \cap \Phi(sy) = \{s\beta\}$$

and $sy \in \Gamma^0$.

Proof. Let $x, y \in W$ possess the properties as stated. Since $\alpha_s \in \Phi(y)$ it follows that $\alpha_s \notin \Phi(x)$. Denote $v = sx$. It then follows that $\Phi(v) = \{\alpha_s\} \sqcup s\Phi(x)$ and $\Phi(sy) = s\Phi(y) \setminus \{\alpha_s\}$. Therefore $\Phi(v) \cap \Phi(sy) = \{s\beta\}$ and it is clear that $s\beta \in \Phi^0(sy)$. Now suppose there is $\alpha \neq s\beta \in \Phi^0(sy)$. Then we have $s_\alpha sy \preceq sy$ and $\alpha_s \notin \Phi(s_\alpha sy) \cap \Phi(sy)$. Thus $ss_\alpha sy \preceq y$ with $\ell(s_\alpha sy) = \ell(sy) - 1 = \ell(y) - 2$ and $s\beta \in \Phi(s_\alpha sy)$. Then $\Phi(ss_\alpha sy) = \{\alpha_s\} \sqcup \Phi(s_\alpha sy)$ and $\beta \in \Phi(ss_\alpha sy)$. This then implies that

$$\Phi(x) \cap \Phi(ss_\alpha sy) = \{\beta\}$$

and $\ell(ss_\alpha sy) = \ell(y) - 1$ contradicting the minimality of y. $\square$

Corollary 4.8. *The set Γ^0 is closed under suffix.*

Proof. This follows immediately from Theorem 11 and Proposition 4.7. $\square$

5. Computing the sets $\mathcal{S}$ and Γ^0

Super elementary roots and tight gates are intimately connected by definition. In this section, we highlight some results connecting the relationship between the two.

Since $\mathcal{S} \subseteq \mathscr{E}$, the set $\mathcal{S}$ can be determined in finite time by considering each root $\beta \in \mathscr{E}$ and checking whether there exists a pair of gates or low-elements $\{x, y\}$ with $\Phi(x) \cap \Phi(y) = \{\beta\}$. An application of Theorem 11 and Corollary 4.8 gives an alternative method which concurrently computes the set $\mathcal{S}$ and Γ^0 without first having to compute the set L or the set Γ. It only requires that the elementary roots $\mathscr{E}$ is computed first (which is quite efficient).

Algorithm 1 Compute tight gates and super-elementary roots

Let $L' = S$, $\mathcal{S}' = \Delta$, $\Gamma' = \emptyset$
Initialize $L'_i = \{x \in L' \mid \ell(x) = i\}$ with $i = 1$
while $L'_i \neq \emptyset$ **do**
  for $x \in L'_i$ **do**
    for $s \notin D_L(x)$ **do**
      Set $y = s \cdot x$
      if $|\Phi^0(y)| = 1$ and $\{\beta\} = \Phi^0(y) \in \mathcal{E}$ **then**
        Add y to L'
        if there is $z \neq y \in L'$ with $\Phi^0(z) = \{\beta\}$ and $\Phi(z) \cap \Phi(y) = \{\beta\}$ **then**
          Add β to $\mathcal{S}'$
          Add y, z to Γ'
        end if
      end if
    end for
  end for
  Set $i = i + 1$
end while

The algorithm terminates in finite time and upon termination $\Gamma' = \Gamma^0$ and $\mathcal{S}' = \mathcal{S}$

Proof. By Corollary 4.8 it follows that the tight gates form saturated chains in the partial order $(L, \preceq_L)$. The set L' in the algorithm iteratively records the set of *tight* low-elements, that is the set $L' = \{w \in L \mid |\Phi^0(w)| = 1\}$. Since $\Gamma^0 \subseteq \Gamma \subseteq L$ and $|L| < \infty$, the algorithm terminates in finite time. Inductively, if $x \in \Gamma'$, then $x \in \Gamma^0$ and if $y = s \cdot x \in \Gamma^0$ then there is some $z \in L'$ such that $\Phi(y) \cap \Phi(z) = \{\beta\}$ with $\Phi^0(y) = \{\beta\} = \Phi^0(z)$. So $\beta \in \mathcal{S}$ and by Theorem 11, this element z is unique and is a tight gate (by symmetry, y is also the unique tight gate such that $\Phi(z) \cap \Phi(y) = \{\beta\}$). $\square$

We also note the following consequence of Theorem 11 which shows that the number of non-simple tight gates is always even.

Corollary 5.1. *Let $|\Gamma^0 \setminus S| = K$. Then K is even, and $|\mathcal{S} \setminus \Delta| \leq K/2$.*

Proof. Theorem 11 shows that tight gates $g \in \Gamma^0 \setminus S$ are naturally paired according to their final roots $\Phi^0(g)$ in the following way: For each $\beta \in \mathcal{S} \setminus \Delta$, there is some $x \in \Gamma^0$ with $\beta \in \partial T(x^{-1}) \cap \Phi^0(x)$. Hence by Theorem 11 there is a unique element $y \in \Gamma^0 \setminus S$ such that $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^0(y) = \{\beta\}$. By symmetry, x is also the unique element of Γ^0 with $\Phi(x) \cap \Phi(y) = \{\beta\}$. $\square$

5.1. Data for select Coxeter groups.

We utilise Algorithm 1 to compute the size of $\mathcal{S}$ and Γ^0 for some Coxeter systems in low rank. We note that as the size of Γ increases, in general, the ratio $|\Gamma^0|/|\Gamma|$ tends to decrease, making the computation of Γ^0 generally much more efficient to determine whether two elements have the same cone type.

W	$\|\mathcal{E}\|$	$\|\mathcal{S}\|$	$\|L\|$	$\|L^0\|$	$\|\Gamma\|$	$\|\Gamma^0\|$
$\widetilde{A}_2$	6	6	16	9	16	9
$\widetilde{B}_2$	8	8	25	14	24	13
$\widetilde{G}_2$	12	12	49	26	41	21
$\widetilde{A}_3$	12	12	125	28	125	28
$\widetilde{B}_3$	18	18	343	66	315	58
$\widetilde{C}_3$	18	18	343	66	317	58
$\widetilde{A}_4$	20	20	1296	75	1296	75
$\widetilde{B}_4$	32	32	6561	270	5789	227
$\widetilde{C}_4$	32	32	6561	270	5860	227
$\widetilde{D}_4$	24	24	2401	140	2400	139
$\widetilde{F}_4$	48	48	28561	1054	22428	715
$\widetilde{A}_5$	30	30	16807	186	16807	186
$\widetilde{B}_5$	50	50	161051	1030	137147	836
$\widetilde{C}_5$	50	50	161051	1030	139457	836
$\widetilde{D}_5$	40	40	59049	608	58965	596

W	$\|\mathcal{E}\|$	$\|\mathcal{S}\|$	$\|L\|$	$\|L^0\|$	$\|\Gamma\|$	$\|\Gamma^0\|$
$X_4(4)$	25	25	438	112	392	98
$X_4(5)$	32	32	516	158	462	138
Y_4	32	32	687	166	578	150
Z_4	30	30	513	142	473	132
$X_5(3)$	114	114	101412	5767	52542	4071
$X_5(4)$	83	83	25708	3128	22886	2871
$X_5(5)$	135	135	42064	6014	37956	5523
Z_5	120	120	41385	5476	39138	5391

FIGURE 4. Data for low rank affine and compact hyperbolic Coxeter groups.

6. Set Dominance

In this section, to further study the set of boundary roots of a cone type, we introduce the notion of a positive root β dominating a set of positive roots, generalising the notion of *dominance* of roots from [BH93, Definition 2.1]. The main result in this section is an alternative characterisation of boundary roots.

Definition 6.1. Let $\Lambda \subset \Phi^+$ be non-empty and let $\beta \in \Phi^+$. Then β *dominates* the set Λ if $\beta \notin \Lambda$ and whenever $w^{-1}\beta < 0$ we have $w^{-1}\alpha < 0$ for some $\alpha \in \Lambda$.

Remark 6.1. (i) By the above definition, it is clear that if β dominates $\Lambda \subset \Phi^+$ then β dominates $\Lambda' := \Lambda \cup \{\alpha_1, \dots, \alpha_k\}$ for any arbitary set of roots $\{\alpha_1, \dots, \alpha_k\}$. We say that β *properly dominates* $\Lambda \subseteq \Phi^+$ if there is no $\alpha \in \Lambda$ which is *redundant*, i.e. for each $\alpha \in \Lambda$ there exists some $w \in W$ such that $w^{-1}\beta < 0$ and $w^{-1}\alpha < 0$.

(ii) If β dominates $\Lambda \subseteq \Phi^+$ then β properly dominates some subset $\Lambda' \subseteq \Lambda$.

(iii) We say that β *tightly dominates* Λ if β does not dominate any subset $\Lambda' \subset \Lambda$.

(iv) If β dominates Λ and $|\Lambda| = 1$ then the notion of set dominance is the same as dominance in the sense of [BH93, Definition 2.1].

We collect some further basic properties of set dominance.

Lemma 6.2. *Let $\Lambda \subseteq \Phi^+$*

(i) If $\beta \in Cone_{\Phi^+}(\Lambda)$ and $\beta \notin \Lambda$ then β dominates Λ.

(ii) For $w \in W$ and $\beta \in \Phi(w) \setminus \Phi^1(w)$, β dominates $\Phi^1(w)$.

(iii) If $\beta \in \mathcal{E}$ and β dominates Λ, then $|\Lambda| > 1$.

(iv) β dominates Λ if and only if $H_\beta^- \cap \big(\cap_\Lambda H_\alpha^+\big) = \emptyset$

Proof. (i) If $\beta \in Cone_{\Phi^+}(\Lambda)$ then we can write $\beta = \sum c_\alpha \alpha$ for $\alpha \in \Lambda$ with $c_\alpha \geq 0$. Hence for any $w \in W$, if $w^{-1}\beta < 0$ then $w^{-1}\alpha < 0$ for some $\alpha \in \Lambda$.

(ii) This follows by (i) since $\Phi(w) = Cone_\Phi(\Phi^1(w))$.

(iii) This is clear since elementary roots dominate no singleton sets of roots.

(iv) Suppose there is $w \in H_\beta^- \cap \big(\cap_\Lambda H_\alpha^+\big)$ then $w^{-1}\beta < 0$ and $w^{-1}\alpha > 0$ for all $\alpha \in \Lambda$. Conversely, if $H_\beta^- \cap \big(\cap_\Lambda H_\alpha^+\big) = \emptyset$ then for any $w \in W$ with $\beta \in \Phi(w)$ we have that $w \in H_\alpha^-$ for some $\alpha \in \Lambda$, so $w^{-1}\alpha < 0$. Hence β dominates Λ.

□

The following straightforward result gives an alternative characterisation of boundary roots in terms of the notion of set dominance.

Lemma 6.3. *Let $x \in W$ and denote $T := T(x^{-1})$. Then $\beta \in \partial T$ if and only if $\beta \in \Phi^1(x)$ and β does not dominate any non-empty subset $\Lambda \subseteq (\Phi^1(x) \setminus \{\beta\})$.*

Proof. Suppose $\beta \in \partial T$. Then $\beta \in \Phi^1(x)$ and by Proposition 2.14 there exists $w \in W$ with $\Phi(x) \cap \Phi(w) = \{\beta\}$. Hence $w^{-1}\beta < 0$ and $w^{-1}\alpha > 0$ for all $\alpha \in \Phi^1(x) \setminus \{\beta\}$. Therefore β does not dominate any non-empty subset $\Lambda \subseteq (\Phi^1(x) \setminus \{\beta\})$. Conversely, since $\beta \in \Phi^1(x)$ and β does not dominate any subset $\Lambda \subseteq (\Phi^1(x) \setminus \{\beta\})$ this implies that there exists $w \neq x \in W$ with $\beta \in \Phi(w)$ and $w^{-1}\alpha > 0$ for all $\alpha \in \Phi^1(x) \setminus \{\beta\}$. Therefore, $\Phi(x) \cap \Phi(w) = \{\beta\}$. □

7. Ultra-Low Elements

The remainder of this paper is devoted to proving Theorem 3.

7.1. **Dihedral groups, right-angled and complete graph Coxeter groups.** We begin with some basic observations and useful results regarding inversion sets of finite Coxeter groups and dihedral groups.

Lemma 7.1. *Let W be a finite Coxeter group and w_o the longest element of W. Then*

$$T(w_o) = \bigcap_{s \in S} H_s^+ = \{e\}$$

and $w_o \in \mathscr{U}$.

Proof. It is clear that $\Phi^1(w_o) = \Delta$ and since $\Phi(w_o) \cap \Phi(s) = \{\alpha_s\}$ for all $s \in S$ we have $\Phi^1(w_o) = \partial T(w_o)$. □

Lemma 7.2. *Let $x \in W'$ be ultra-low and W' a standard parabolic subgroup of W. Then x is ultra-low in W.*

Proof. This is clear by the definition of ultra-low elements. □

The following technical lemma will be useful for working with finite dihedral groups and shows that $\mathscr{U} = \Gamma = \widetilde{S}$ for these groups. We denote $W_{\langle s,t \rangle}$ to be a dihedral group generated by the reflections s, t. Denote the alternating word $stst\ldots$ of length n by $[s,t]_n$.

Lemma 7.3. *Let $W_{\langle s,t \rangle} = \langle s, t \mid (st)^l = s^2 = t^2 = 1 \rangle$ with $l < \infty$. Then*

(i) *When l is odd,*
   (1) $[s,t]_n(\alpha_t) = [t,s]_{l-1-n}(\alpha_s)$ for n odd and $1 \leq n \leq l-1$.
   (2) $[s,t]_n(\alpha_s) = [t,s]_{l-1-n}(\alpha_t)$ for n even and $1 \leq n \leq l-1$.

(ii) *When l is even,*
   (1) $[s,t]_n(\alpha_t) = [t,s]_{l-1-n}(\alpha_t)$ for n odd and $1 \leq n \leq l-1$.
   (2) $[s,t]_n(\alpha_s) = [t,s]_{l-1-n}(\alpha_s)$ for n even and $1 \leq n \leq l-1$.

(iii) *Let*
$$L = \big(\alpha_s, s(\alpha_t), st(\alpha_s), \ldots, [s,t]_{l-1}(\alpha_u)\big)$$
where $u = s$ for l odd and $u = t$ for l even. Then for $j \leq l-1$
$$\Phi([s,t]_j) = \{\gamma_1, \ldots, \gamma_j\}$$
$$\Phi([t,s]_j) = \{\gamma_l, \gamma_{l-1}, \ldots, \gamma_{l-j}\}$$
where γ_i is the ith root in L. In particular,
$$\Phi([s,t]_n) \cap \Phi([t,s]_m) = \{[s,t]_{n-1}(\delta)\} = \{[t,s]_{m-1}(\delta')\}$$
for $n + m = l + 1$ and $\delta = \alpha_s$ if $[s,t]_{n-1}$ ends in t and $\delta = \alpha_t$ if $[s,t]_{n-1}$ ends in s (similarly for $[t,s]_{m-1}$) and δ' is the other simple root.

(iv)
$$\Phi^1([s,t]_j) = \{\gamma_1, \gamma_j\} = \partial T([s,t]_j^{-1}) \text{ and} \tag{7.1}$$
$$\Phi^1([t,s]_j) = \{\gamma_l, \gamma_{l-j}\} = \partial T([t,s]_j^{-1}) \qquad \text{for } j \leq l-1 \tag{7.2}$$

Proof. $(i)-(iii)$ are straightforward calculations. To prove (iv) consider the following (we discuss the case of (7.1) with (7.2) being entirely similar). The first equality is clear to see since for γ_i with $i \notin \{1, j\}$ we have $\ell(s_{\gamma_i}[s,t]_j) < \ell([s,t]_j) - 1$ and so $\gamma_i \notin \Phi^1([s,t]_j)$ and thus $\gamma_i \notin \partial T([s,t]_j^{-1})$ since $\partial T([s,t]_j^{-1}) \subseteq \Phi^1([s,t]_j)$. It remains to justify that $\Phi^1([s,t]_j) \subset \partial T([s,t]_j^{-1})$. Since γ_1 is simple it is a boundary root. Then the root γ_j is a boundary root since $\gamma_j \in \Phi^0([s,t]_j)$ and $W = \Gamma$ when W is finite, hence the result follows by Theorem 6. □

Corollary 7.4. *Let $W_{\langle s,t\rangle} = \langle s, t \mid (st)^l = s^2 = t^2 = 1\rangle$ with $l < \infty$ and denote w_o the longest element of $W_{\langle s,t\rangle}$. Then*

(i) $\Gamma^0 = W_{\langle s,t\rangle} \setminus \{e, w_o\}$
(ii) $W_{\langle s,t\rangle} = \mathscr{U} = \Gamma = \widetilde{\mathcal{S}}$
(iii) *If* $W_{\langle s,t\rangle} \leq W$ *then* $W_{\langle s,t\rangle} \subseteq \mathscr{U}$

Proof. (i) For W a finite dihedral group, we have $W = \Gamma$ and it is clear that $|\Phi^0(w)| = 1$ for all $w \neq e, w_o$. Then (ii) follows by Lemma 7.3 (iv) and Lemma 7.1. Then (iii) follows by Lemma 7.2. □

The following result will greatly simplify calculations in this section and shows that any root β whose support $\Gamma(\beta)$ is a standard finite parabolic dihedral root sub system is a boundary root.

Lemma 7.5. *Let $x \in W$ and $\beta \in \Phi^1(x)$ where $\Gamma(\beta) = \{s, t\}$ and $W' = \langle s, t \mid (st)^l = s^2 = t^2 = 1\rangle$ is a standard finite parabolic dihedral subgroup. Then there exists a unique $y \in W'$ with $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^0(y) = \{\beta\}$. Thus $\beta \in \partial T(x^{-1})$.*

Proof. If β is a simple root the result is clear. Hence assume β is non-simple (note that since $\beta \in \mathscr{E}$ by Lemma 2.7 this then implies that W' is finite). Then $\beta = [s, t]_{n-1}(\delta)$ for some $1 \leq n \leq l$ and where $\delta = \alpha_s$ if $[s, t]_{n-1}$ ends in t and $\delta = \alpha_t$ if $[s, t]_{n-1}$ ends in s. It then follows from Lemma 7.3 (iv) that

$$\Phi^0([s, t]_n) = \{[s, t]_{n-1}(\delta)\} = \{[t, s]_{m-1}(\delta')\} = \Phi^0([t, s]_m)$$

and $\Phi([s, t]_n) \cap \Phi([t, s]_m) = \{[s, t]_{n-1}(\delta)\}$ where $m + n = l + 1$ and δ' is the other simple root. Since $\beta \in \Phi^1(x) \cap \Phi^+_{W'}$ we have $\beta \in \Phi^1(x_J)$ with respect to the unique reduced decomposition $x = x_J \cdot x^J$, where $x_J \in W'$ and $x^J \in X_{W'}$. Then by Lemma 7.3 (iii) it must follow that either $x_J = [s, t]_n$ or $x_J = [t, s]_m$. Without loss of generality, suppose $x_J = [s, t]_n$. We claim that $\Phi(x) \cap \Phi([t, s]_m) = \{\beta\}$. Clearly, $\Phi(x_J) \cap \Phi([t, s]_m) = \{\beta\}$ and since $\Phi(x) = \Phi(x_J) \sqcup x_J\Phi(x^J)$ it remains to justify that $\Phi([t, s]_m) \cap x_J\Phi(x^J) = \emptyset$. If $x^J = e$ then we are done, otherwise for each $\alpha \in x_J\Phi(x^J)$ it follows that $\mathrm{Coeff}_\alpha(\alpha_u) > 0$ for some simple root $\alpha_u \notin \{\alpha_s, \alpha_t\}$. If $x_J = [t, s]_m$ then swapping the roles of $[s, t]_n$ and $[t, s]_m$ in the above argument, we have $\Phi(x) \cap \Phi([s, t]_n) = \{\beta\}$. This completes the proof of our claim. □

In [DDH15] the authors provide an explicit description of $\widetilde{S}$ when W is right-angled (i.e. $m(s, t) \in \{2, \infty\}$ for all $s \neq t$) or when Γ_W is a complete graph (i.e. $m(s, t) \geq 3$ for all $s \neq t$). This description allows us to directly show that these elements are also ultra-low.

Proposition 7.6. [DDH15, Proposition 5.1]

(i) *When W is right-angled then*

$$\widetilde{\mathcal{S}} = \bigcup_{J \subseteq S} W_J$$

where $J \subseteq S$ is a set of pairwise commuting simple reflections.

(ii) *When Γ_W is the complete graph then*

$$\widetilde{\mathcal{S}} = \Big(\cup_{\{(s,t) \in W|\ m(s,t) < \infty\}} W_{\langle s,t\rangle} \Big) \bigcup \Big(\cup_{(s,t,r) \in X} \big(t[s, r]_{m(s,r)}\big)$$

where X consists of all triples $(s, t, r) \in S$ with s, t, r distinct and $m(s, t), m(s, r), m(t, r)$ all finite.

Given the explicit description of $\widetilde{\mathcal{S}}$ in these cases, our goal is then to show that $\widetilde{\mathcal{S}} \subseteq \mathscr{U}$. It then follows from [PY22, Proposition 6.2] that $\widetilde{\mathcal{S}} \subseteq \mathscr{U} \subseteq \Gamma \subseteq \widetilde{\mathcal{S}}$ and thus $\widetilde{\mathcal{S}} = \mathscr{U} = \Gamma$.

The following relates to Theorem 12 (ii).

Lemma 7.7. *Let Γ_W be the complete graph, with notation as in Proposition 7.6. Then*

$$\Phi^1(t[s, r]_{m(s,r)}) = \{\alpha_t, t(\alpha_s), t(\alpha_r)\} = \partial T(t[s, r]^{-1}_{m(s,r)})$$

for r, s, t are distinct and $m(s, t), m(s, r), m(t, r)$ are all finite.

Proof. We prove the first equality. Note that $w_o := [s, r]_{m(s,r)}$ is the longest element of $W_{\langle s,r\rangle}$ and hence $\Phi^1(w_o) = \{\alpha_s, \alpha_r\}$ by Lemma 7.1. The first equality then follows using Proposition 2.3 by direct calculation. The second equality then follows by Lemma 7.5. □

Theorem 12. *Let (W, S) be a Coxeter system such that*

(i) W is right-angled, or;
(ii) The Coxeter graph is a complete graph

Then $\mathscr{U} = \Gamma = \widetilde{\mathcal{S}}$.

Proof. (i) By Proposition 7.6, $\widetilde{S}$ is the union of W_J where $J \subseteq S$ is a set of pairwise commuting reflections. We show that each W_J in this union consists of ultra-low elements. If $|J| = 1$ then the only element of W_J is the simple reflection s_J, which is ultra-low. If $|J| = 2$ then the result follows by Corollary 7.4. For $|J| > 2$ then

$$W_J = \{s_i \mid i \in J\} \cup \{st \mid st = ts \text{ and } s, t \in J\} \cup \{w_o\}$$

where w_o is the product of distinct simple reflections in J (the longest element of W_J). Hence the result follows from Corollary 7.4 and Lemma 7.1.
(ii) In the case of Γ_W being the complete graph, the elements contained in finite dihedral subgroups are ultra-low by Corollary 7.4 and the elements of the form $t[s,r]_{m(s,r)}$ with s,t,r distinct and $m(s,t), m(s,r), m(t,r)$ all finite are ultra-low by Lemma 7.7.

This proves Theorem 3 for the cases (i) and (ii). □

Corollary 7.8. *Let* (W,S) *be a right-angled Coxeter system. Then* $\Gamma^0 = S$.

Proof. By Theorem 12 we have $\widetilde{S} = \mathscr{U}$ and by Proposition 7.6 if $w \in \widetilde{S}$ then $w \in W_J$ where $|J| > 2$ is a set of pairwise commuting reflections. It is then clear to see that if $w \notin S$ then $|\Phi^0(w)| > 1$. □

Corollary 7.9. *Let* W *be a Coxeter group such that* Γ_W *is a complete graph. Then* $\Gamma^0 = \bigcup W_J \setminus \{e, w_J\}$ *where* W_J *is a standard finite dihedral parabolic subgroup of* W *and* w_J *is the longest element of* W_J.

Proof. By Theorem 12 we have that

$$\mathscr{U} = \Gamma = \widetilde{\mathcal{S}} = \left(\cup_{\{(s,t) \in W \mid \ m(s,t) < \infty\}} W_{\langle s,t \rangle} \right) \bigcup \left(\cup_{(s,t,r) \in X} \left(t[s,r]_{m(s,r)} \right) \right)$$

where X consists of all triples $(s,t,r) \in S$ with s,t,r distinct and $m(s,t), m(s,r), m(t,r)$ all finite. By Corollary 7.4 the elements in the left side of the above union (with the exception of e and w_J) are tight.

It is then straightforward to verify that the elements of the form $t[s,r]_{m(s,r)}$ are not tight gates. Since $[s,r]_{m(s,r)}\alpha_s = -\alpha_r$ and $[s,r]_{m(s,r)}\alpha_r = -\alpha_s$ it follows by Definition 2.4 that $t(\alpha_s), t(\alpha_r) \in \Phi^0(t[s,r]_{m(s,r)})$. □

7.2. **Rank 3 irreducible Coxeter groups.** We now discuss the case of irreducible rank 3 Coxeter groups.

When Γ_W is a complete graph, then the result follows by Theorem 12. When W is affine of type $\widetilde{B}_2$ or $\widetilde{G}_2$ then $\mathscr{U} = \Gamma$ can be directly verified from the illustration of their cone type arrangements in Figure 3 in the following way: for each gate $g \in \Gamma$ and each hyperplane H separating g from the identity such that $\ell(s_H g) = \ell(g) - 1$ (i.e. the hyperplanes corresponding to the roots $\Phi^1(g)$) there is a tight gate h such that the only hyperplane separating g and h is H. This then directly shows that $\Phi^1(g) = \partial T(g^{-1})$.

Hence we are left with the (non-affine) linear diagrams. We separate these diagrams into three types.

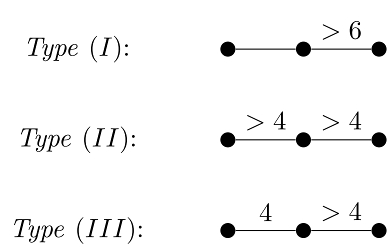


FIGURE 5. Throughout this chapter let the simple reflections s, t, u correspond to the vertices reading left to right and denote $a = m_{s,t}$ and $b = m_{t,u}$. The corresponding cone type automata is illustrated in Figure 6, Figure 7, Figure 8 respectively.

The following result allows us to only consider the above diagrams with all labels finite.

Theorem 13. *Let* W *be a rank 3 Coxeter group with* Γ_W *a linear diagram and such that one of the standard parabolic dihedral subgroups of* W *is an infinite dihedral group (i.e. either* $W_{\langle s,t \rangle} = W_\infty$ *or* $W_{\langle t,u \rangle} = W_\infty$ *in the labelling convention of Figure 5). Then*

$$W_{\langle s,t \rangle} \cup W_{\langle s,u \rangle} = \Gamma = \widetilde{S} = L = \mathscr{U}$$

.

Proof. By symmetry we may suppose that the right label in one of the diagrams in Figure 5 is ∞, then by [PY19, Theorem 1], the automaton $\mathcal{A}_{BH}$ is minimal and hence $\mathcal{A}_{\widetilde{S}}$ is minimal since $\widetilde{S} \subseteq L$. Therefore we have $\Gamma = \widetilde{S} = L$.

We claim that $\Gamma = W_{\langle s,t\rangle} \cup W_{\langle s,u\rangle}$. By Corollary 7.4 we have $W_{\langle s,t\rangle} \cup W_{\langle s,u\rangle} \subseteq \Gamma$ and note that by Lemma 2.7 we have $\mathscr{E} = \Phi^+_{\{s,t\}} \sqcup \{\alpha_u\}$. First note that if $w \in W_{\langle t,u\rangle}$ and $w \neq u$ then $\Phi^0(w) = \beta = a\alpha_t + b\alpha_u$ for some $a, b > 0$. Since $\beta \notin \mathscr{E}$, β cannot be a boundary root. Hence by Theorem 6 $w \notin \Gamma$. Now if there is $g \in \Gamma$ such that g is not contained in either $W_{\langle s,t\rangle}$ or $W_{\langle s,u\rangle}$ then g must have a suffix, which is also in Γ, in the form of one of the following.

(i) $u[t,s]_k$ for some $1 \leq k \leq m(s,t)$
(ii) $u[s,t]_k$ for some $1 \leq k \leq m(s,t)$
(iii) tu, tsu.

Suppose g has a suffix in the form of (i). Now

$$\partial T([t,s]_k u) \subseteq \Phi^1(u[t,s]_k) \subseteq \{\alpha_u\} \cup u\Phi^1([t,s]_k) = \{\alpha_u\} \cup u\big(\{\alpha_t, c_1\alpha_t + c_2\alpha_s\}\big)$$

for some $c_1, c_2 > 0$, where the first inclusion follows by Lemma 2.9 and the second inclusion follows by Proposition 2.3. The equality then follows by Lemma 7.3. Since $u(\alpha_t) \notin \mathscr{E}$ and $u(c_1\alpha_t + c_2\alpha_s) \notin \mathscr{E}$ it follows that $\partial T([t,s]_k u) = \{\alpha_u\} = \partial T(u)$ and thus $u[t,s]_k$ is not a gate, which contradicts suffix closure. The same argument applies for (ii) and (iii), completing the proof of our claim. Then the fact that $\Gamma = \mathscr{U}$ follows by Corollary 7.4 (iii). ☐

We now provide complete details for the proof of type I and be a little more brief for types II and III since the majority of the proofs are exactly the same (and easier) as in the case of type I and the computations are straightforward. We note that the automata illustrated in Figure 6, Figure 7, and Figure 8 was first studied in [GMP17]. We illustrate them here for the purpose of proving $\mathscr{U} = \Gamma$ and to indicate the tight gates in these groups.

In the remainder of the paper, it will be helpful to record the elementary roots in rank 3 with full support, which can be directly computed following the results in [Bri98].

Lemma 7.10. *Let W be a rank 3 Coxeter group whose graph Γ_W is linear with finite labels (see Figure 5). Let $c_1 = 2\cos\frac{\pi}{m_{st}}$ and $c_2 = 2\cos\frac{\pi}{m_{tu}}$.*

(1) For W of type I. Then

$$\mathscr{E}_S = \{(c_1, c_1, 1), (c_1, c_1, c_1^2 - 1), (1, 1, c_1)\}$$

(2) For W of type II or III then

$$\mathscr{E}_S = \{(c_1, 1, c_2)\}$$

Proof. See [Bri98, Lemma 4.7] and [Bri98, Proposition 6.7]. ☐

Remark 7.1. Our computations in this section is greatly simplified by Lemma 7.5. For any element x and each $\beta \in \Phi^1(x^{-1})$, if $\Gamma(\beta) \in \Phi^+_{W'}$ for $W' \in \{W_{\langle s,t\rangle}, W_{\langle t,u\rangle}\}$ then we immediately have $\beta \in \partial T(x)$. Thus it suffices to only consider the roots whose support is all of S.

7.3. **Type I.** In this section, we detail the results specific to Type I graphs.

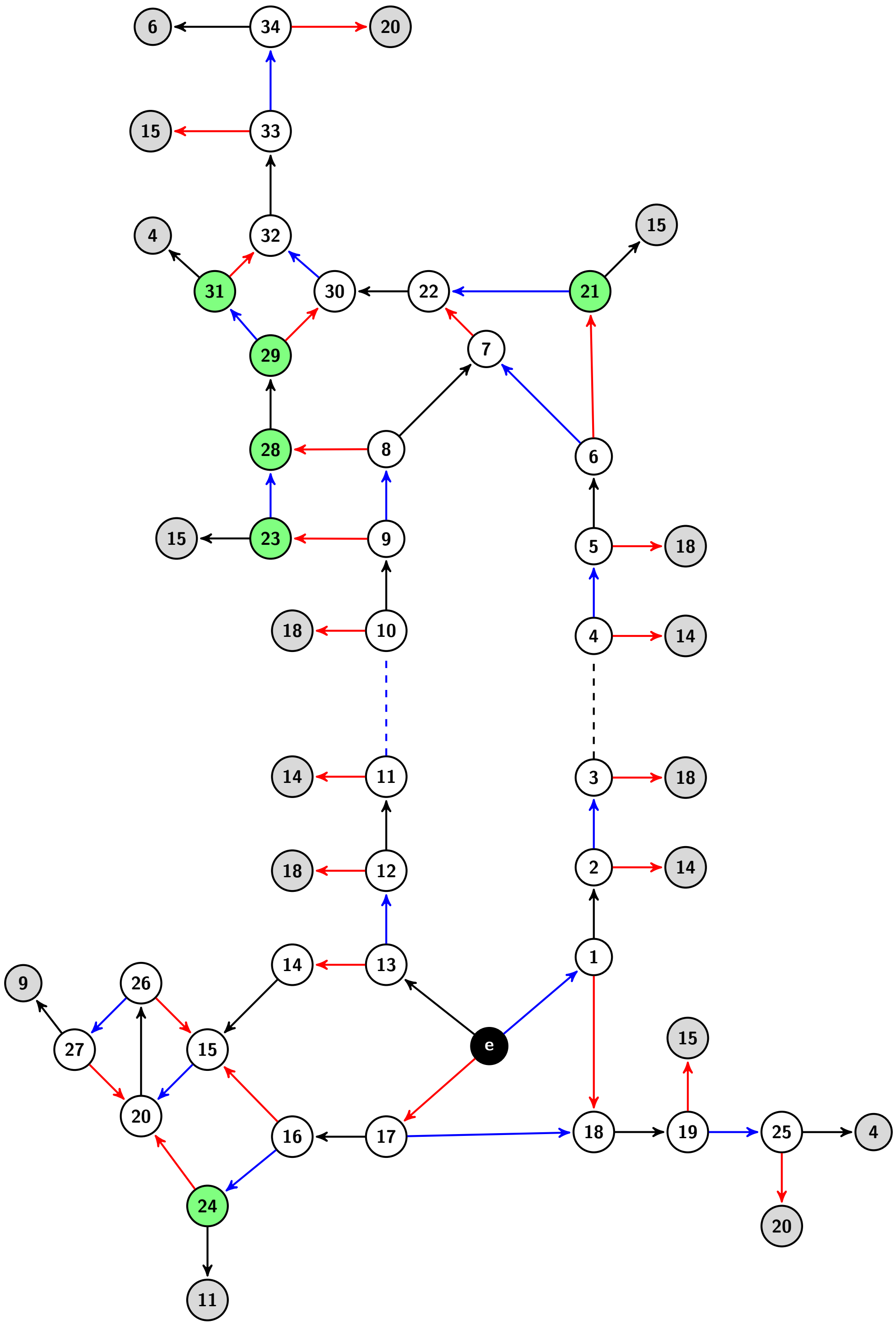


FIGURE 6. The automaton $\mathcal{A}_o$ for (W, S) of Type I with b odd (the illustration includes all states for the case $b = 7$ with the trailing blue and black dots indicating the additional elements of $W_{\langle t,u\rangle}$ when $b > 7$). The node coloured black denotes the identity (start state) and the red, black and blue arrows correspond to s, t and u transitions respectively. Transitions into gray filled nodes informs the reader to continue the reading of a word by continuing the path at the corresponding white filled node with the same label. The nodes highlighted green are the inverses of the tight gates whose final root has non-spherical support (see Appendix A).

Lemma 7.11. *Let W be of type I. Then*

(i) $T(st) \xrightarrow{u} T(stu)$
(ii) $T(sts) \xrightarrow{u} T(stsu)$
(iii) $T(stsu) \xrightarrow{t} T(stsut)$
(iv) $T(stsut) \xrightarrow{u} T(stsutu)$

are transitions in $\mathcal{A}_o$, and each element above is ultra-low.

(Note that this result corresponds to transitions leaving nodes $15, 16, 20, 26$ *in Figure 6)*

Proof. (i) By Lemma 7.3 (iv) we have $\Phi^1(ts) = \partial T(st) = \{\alpha_t, s\alpha_t\}$ and using Proposition 2.3 we compute $\Phi^1(uts) = \{\alpha_u, u\alpha_t, ut\alpha_s\}$ (note that by the calculations in Appendix A.1 uts is a tight gate with $\Phi^0(uts) = \{ut\alpha_s\}$ and so $ut\alpha_s \in \partial T(stu)$). It then follows by Theorem 11 and the calculations in Appendix A.1 that we must have

$$\Phi^1(uts) \cap \Phi(s[t,u]_{b-1}) = ut\alpha_s = (1, 1, c_1)$$

(ii) By Lemma 7.1 we have $\Phi^1(sts) = \{\alpha_s, \alpha_t\}$ and using Proposition 2.3 we compute $\Phi^1(usts) = \{\alpha_s, \alpha_u, u\alpha_t\}$. So the result is immediate by Lemma 7.5.

(iii) We compute $\Phi^1(tusts) = \{\alpha_t, t\alpha_s, tust\alpha_s\}$. Note that $tust\alpha_s = tu\alpha_t$, then again we are done by Lemma 7.5.

(iv) Finally, we compute $\Phi^1(utusts) = \{\alpha_u, utu\alpha_t, ut\alpha_s\}$. We claim that

$$\Phi^1(utusts) \cap \Phi(s[t,u]_{b-1}) = ut\alpha_s = (1, 1, c_1)$$

It suffices to show that $\alpha_u, utu\alpha_t \notin \Phi(s[t,u]_{b-1})$. Since $s \notin D_L([t,u]_{b-1})$ we have $\Phi(s[t,u]_{b-1}) = \{\alpha_s\} \sqcup s\Phi([t,u]_{b-1})$ and hence $\mathrm{Coeff}_{\alpha_s}(\beta) > 0$ for each $\beta \in \Phi(s[t,u]_{b-1})$. □

For the remaining results we will use Lemma 7.5 without further direct reference.

Lemma 7.12. *Let W be of type I. Then*

(i) $T(su) \xrightarrow{t} T(sut)$
(ii) $T(sut) \xrightarrow{u} T(sutu)$

and each element above is ultra-low.

(Note that this result corresponds to the transitions leaving the nodes $18, 19$ *in Figure 6).*

Proof. For (i), su is ultra-low by Corollary 7.4 and we compute $\Phi^1(tus) = \{\alpha_t, t\alpha_u, t\alpha_s\}$, so the result is immediate.

For (ii) we compute from (i) to obtain $\Phi^1(utus) = \{\alpha_u, ut\alpha_u, ut\alpha_s\}$. By the same argument as in Lemma 7.11 (iv), we have $\Phi(utus) \cap \Phi(s[t,u]_{b-1}) = ut\alpha_s$. □

Lemma 7.13. *Let W be of type I with $k = b - 2$.*

(i) *If b is odd, then $T([t,u]_k) \xrightarrow{s} T([t,u]_k s)$*
(ii) *If b is even, then $T([u,t]_k) \xrightarrow{s} T([u,t]_k s)$*

and the elements above are ultra-low.

(Note that this result corresponds to the s-transition from the nodes 9 *to* 23 *in Figure 6).*

Proof. The proof is entirely the same for cases (i) and (ii), so we will just prove (i). Note that in both cases, the alternating words $[t,u]_k$ and $[u,t]_k$ end in t and by Lemma 7.3 (iv) we have that

$$\Phi^1([u,t]_k^{-1}) = \Phi^1([t,u]_k) = \{\alpha_t, ut\alpha_u\}$$

We then compute $\Phi^1(s[u,t]_k^{-1}) = \Phi^1(s[t,u]_k) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$. Note that $sut\alpha_u = (c_1, c_1, c_1^2 - 1)$. We claim that

$$\Phi^1(s[t,u]_k) \cap \Phi(utus[t,u]_{b-2}) = \{sut\alpha_u\}$$

Since $utus[t,u]_{b-2}$ is reduced, we have $\Phi(utus[t,u]_{b-2}) = \Phi(utus) \sqcup utus\Phi([t,u]_{b-2})$. Clearly, $sut\alpha_u \in \Phi(utus)$ and $\{\alpha_s, s\alpha_t\} \not\subset \Phi(utus)$. Then note that $\mathrm{Coeff}_{\alpha_u}(\beta) > 0$ for $\beta \in utus\Phi([t,u]_{b-2})$, hence $\alpha_s, s\alpha_t \notin \Phi(utus[t,u]_{b-2})$. □

Lemma 7.14. *Let W be of type I with $k = b - 2$.*

(i) *If b is odd then $T([t,u]_k s) \xrightarrow{u} T([t,u]_k su)$*
(ii) *If b is even then $T([u,t]_k s) \xrightarrow{u} T([u,t]_k su)$*

and the elements above are ultra-low.

(Note that this result corresponds to the u-transition from the nodes 23 *to* 28 *in Figure 6).*

Proof. The argument for both cases is again identical and so we will prove (i) (the proof of (ii) is exactly the same, by replacing $[t,u]_k$ with $[u,t]_k$).

In Lemma 7.13 we computed $\Phi^1(s[t,u]_k)$, and hence we compute $\Phi^1(us[t,u]_k) = \{\alpha_u, \alpha_s, st\alpha_u\}$. We show that $st\alpha_u = (c_1, c_1, 1)$ is a boundary root of $T([t,u]_k su)$. We claim that $\Phi^1(us[t,u]_k) \cap \Phi(tus[t,u]_k) = \{(c_1, c_1, 1)\}$. We have $\Phi(tus[t,u]_k) = \Phi(tus) \sqcup tus\Phi([t,u]_k)$. Clearly, $(c_1, c_1, 1) \in \Phi(tus)$ and since $tus[t,u]_k$ is a reduced expression and is a tight gate by Appendix A.1 it follows that $\alpha_u, \alpha_s \notin \Phi(tus[t,u]_k)$. ☐

Lemma 7.15. *Let W be of type I with $b = m_{ut}$ odd and let $k = b-2$. Then*

(i) $T([t,u]_k us) \xrightarrow{t} T([t,u]_k ust)$
(ii) $T([t,u]_k ust) \xrightarrow{s} T([t,u]_k usts)$
(iii) $T([t,u]_k ust) \xrightarrow{u} T([t,u]_k ustu)$
(iv) $T([t,u]_k ustu) \xrightarrow{s} T([t,u]_k ustus)$
(v) $T([t,u]_k ustus) \xrightarrow{t} T([t,u]_k ustust)$
(vi) $T([t,u]_k ustust) \xrightarrow{u} T([t,u]_k ustustu)$

and the elements above are ultra-low.

When b is even, replace $[t,u]_k$ with $[u,t]_k$ in the above list, then the same results hold.

(Note that this result corresponds to the transitions leaving the nodes ($28, 29, 30, 31, 32, 33$ *and* 34*) in Figure 6).*

Proof. Again, the case for b even is entirely the same and hence we show the case of b odd. For b even, replace $[t,u]_k$ with $[u,t]_k$ in the below arguments.

(i) From Lemma 7.14 we computed $\Phi^1(su[t,u]_k)$, hence we compute $\Phi^1(tsu[t,u]_k) = \{\alpha_t, t\alpha_u, t\alpha_s, st\alpha_u\}$. By Appendix A the element $tsu[t,u]_k$ is a tight gate and then by Theorem 11 we have $\Phi^1(tsu[t,u]_k) \cap \Phi(tsu) = \{st\alpha_u\}$.
(ii) We then compute $\Phi^1(stsu[t,u]_k) = \{\alpha_s, \alpha_t, t\alpha_u\}$.
(iii) Compute $\Phi^1(utsu[t,u]_k) = \{\alpha_u, ut\alpha_u, sut\alpha_u\}$. Then note from Lemma 7.13 (ii) we had
$$\Phi^1(s[t,u]_k) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$$
Hence $\Phi^1(utsu[t,u]_k) \cap \Phi(s[t,u]_k) = \{sut\alpha_u\}$.
(iv) We compute $\Phi^1(sutsu[t,u]_k) = \{\alpha_s, \alpha_u, ut\alpha_u\}$. So the result is immediate and similarly;
(v) We compute $\Phi^1(tsutsu[t,u]_k) = \{\alpha_t, t\alpha_s, tut\alpha_u\}$.
(vi) Finally, compute $\Phi^1(utsutsu[t,u]_k) = \{\alpha_u, ut\alpha_s, utut\alpha_u\}$. And by the computations in Lemma 7.11 (i) we have $\Phi^1(utsutsu[t,u]_k) \cap \Phi(s[t,u]_{b-1}) = \{ut\alpha_s\}$.

☐

Lemma 7.16. *Let W be of type I with $b = m_{ut}$ odd and $k = b-1$. Then*

(i) $T(w_{ut}) \xrightarrow{s} T(w_{ut}s)$
(ii) $T([u,t]_k) \xrightarrow{s} T([u,t]_k s)$

and the elements above are ultra-low.

When b is even, replace $[u,t]_k$ with $[t,u]_k$ in (ii), then both results hold.

(Note that this result corresponds to the s-transitions leaving the nodes 7 *and* 6 *in Figure 6).*

Proof. By Lemma 7.1 we have that w_{ut} is ultra-low then compute $\Phi^1(sw_{ut}) = \{\alpha_u, \alpha_s, \alpha_s + \alpha_t\}$ and hence the result is immediate.

Similarly, we have $\Phi^1([t,u]_k) = \{\alpha_t, u\alpha_t\}$ and compute
$$\Phi^1(s[t,u]_k) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$$
Then by Lemma 7.15 (iii) we have $\Phi^1(s[u,t]_k) \cap \Phi(utus[t,u]_{b-2}) = \{sut\alpha_u\}$. ☐

7.3.1. *Remaining Transitions.* The previous results together show that the inverses of the minimal length representatives of the states (gates of $\mathscr{T}$) coloured white and green in Figure 6 are indeed ultra-low. The final step is then to show that there are not any more gates. The following sets of results prove the transitions to the gray nodes in Figure 6.

Lemma 7.17. *Let (W,S) be a Coxeter system whose graph is contained in Figure 5 and let $b = m_{ut}$.*

(1) For W of Type I:
  (i) *If b is odd, let w be an element of the form:*
    (a) $[t,u]_n$ *for* $2 \leq n \leq b-3$ *for n even.*
    (b) $[u,t]_n$ *for* $3 \leq n \leq b-2$ *for n odd.*
  (ii) *If b is even, let w be an element of the form:*
    (a) $[t,u]_n$ *for* $2 \leq n \leq b-2$ *for n even.*
    (b) $[u,t]_n$ *for* $3 \leq n \leq b-3$ *for n odd.*

Then $T(w) \to T(us)$ is an s-transition in $\mathcal{A}_o$.

(Note that this result corresponds to the s-transitions leaving the nodes, $3, 5, 10, 12$ *in Figure 6 and* $13, 16, 2, 4, 7$ *in Figure 7).*

Proof. Note that in each case, the elements $[t,u]_n$ and $[u,t]_n$ end in u. Clearly $\ell(ws) > \ell(w)$ so ws is reduced. We now show $T(us) = T(ws)$. We prove (i) with (ii) being entirely the same argument. Hence let b be odd and $n = b-2$. Note that this means $[u,t]_n = [u,t]_n^{-1}$. By Lemma 7.3 (iv) we have

$$\Phi^1(w) = \Phi^1([u,t]_n) = \partial T([u,t]_n) = \{\alpha_u, tu\alpha_t\}$$

Using Proposition 2.3 we compute $\Phi^1(sw) = \{\alpha_s, \alpha_u, stu\alpha_t\}$. By direct calculation it follows that $stu\alpha_t \notin \mathscr{E}$ (since the coefficient of α_s in $stu\alpha_t$ is $c_1^2 - 1$ in the notation of Lemma 7.10. Thus by Lemma 2.9, $stu\alpha_t \notin \partial T(ws)$. Since simple roots are always boundary roots, we have $\partial T(ws) = \{\alpha_s, \alpha_u\} = \partial T(us)$. Therefore by Lemma 2.12 we have the following chain of inclusions:

$$T(us) \subseteq T([u,t]_n s) \subseteq T([t,u]_{n-1} s) \subseteq \ldots \subseteq T(us)$$

which includes all the elements in (a) and (b). □

Lemma 7.18. *Let (W,S) be a Coxeter system of type (I) and let $b = m_{ut}$.*
  (i) *If b is odd, let w be an element of the form:*
    (1) $[t,u]_n$ *for* $3 \leq n \leq b-4$ *for n odd.*
    (2) $[u,t]_n$ *for* $2 \leq n \leq b-3$ *for n even.*
  (ii) *If b is even, let w be an element of the form:*
    (1) $[t,u]_n$ *for* $3 \leq n \leq b-3$ *for n odd.*
    (2) $[u,t]_n$ *for* $2 \leq n \leq b-4$ *for n even.*

then $T(w) \to T(ts)$ is a transition in $\mathcal{A}_o$.

(We note that this result corresponds s-transitions leaving the nodes, $2, 4, 11$ *in Figure 6).*

Proof. The proof is the same as in Lemma 7.17. We again prove (i) with (ii) being the exact same calculation. Let $n = b-3$, so $[u,t]_n = [t,u]_n^{-1}$ then by Lemma 7.3 (iv) we have $\partial T([u,t]_n) = \Phi^1([t,u]_n) = \{\alpha_t, tut\alpha_u\}$. Using Proposition 2.3, we compute $\Phi^1(s[t,u]_n) = \{\alpha_s, s\alpha_t, stut\alpha_u\}$. Since $stut\alpha_u \notin \mathscr{E}$ (which can be seen from Lemma 7.10), because the coefficient of α_s is not c_1 or 1) it follows by Lemma 2.9 that $stut\alpha_u \notin \partial T([u,t]_n s)$. We can then immediately conclude that $\partial T([u,t]_n s) = \{\alpha_s, s\alpha_t\} = \partial T(ts)$ and using Lemma 2.12 we have the chain of inclusions

$$T(ts) \subseteq T([u,t]_n s) \subseteq T([t,u]_{n-1} s) \subseteq \ldots \subseteq T(ts)$$

which includes all the elements in (a) and (b). □

Lemma 7.19. *Let W be of type I with $b = m_{ut}$ odd and $k = b-1$. Then*
  (i) $T([t,u]_{k-1} s) \xrightarrow{t} T(sts)$
  (ii) $T([t,u]_{k-1} ustust) \xrightarrow{s} T(sts)$
  (iii) $T(stsut) \xrightarrow{s} T(sts)$
  (iv) $T([u,t]_k s) \xrightarrow{t} T(sts)$

When b is even, swap $[t,u]_k$ with $[u,t]_k$ in the above list, then the same results hold.

(Note that this result corresponds to the transitions to the gray coloured nodes labelled 15 *from the nodes* $19, 21, 23, 33$ *and the s-transition from* 26 *to* 15 *in Figure 6).*

Proof. Cases (i)-(iv) are all similar. Again, swap $[t,u]_k$ with $[u,t]_k$ in all calculations below for b even.

From Lemma 7.13 we have $\Phi^1(s[t,u]_{k-1}) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$. Then using Proposition 2.3 we compute $\Phi^1(ts[t,u]_{k-1}) = \{\alpha_t, \alpha_s, tsut\alpha_u\}$. However, by Lemma 7.10,we have $tsut\alpha_u \notin \mathscr{E}$ and thus by the fact that simple roots are always boundary roots, it follows that $\partial T([t,u]_k st) = \{\alpha_s, \alpha_t\} = \partial T(sts)$.

(ii), we have

$$\Phi^1(tsutsu[t,u]_{k-1}) = \{\alpha_t, t\alpha_s, tut\alpha_u\}$$

and

$$\Phi^1(stsutsu[t,u]_{k-1}) = \{\alpha_s, \alpha_t, stut\alpha_u\}$$

Now $stut(\alpha_u) \notin \mathscr{E}$ and hence $\partial T([t,u]_k ustust) = \{\alpha_s, \alpha_t\} = \partial T(sts)$.

(iii) we computed in Lemma 7.11 that $\Phi^1(tusts) = \{\alpha_t, t\alpha_s, tu\alpha_t\}$. Again, it can be directly checked from Lemma 7.10 that $stut\alpha_t \notin \mathscr{E}$. Therefore, $\partial T(stsuts) = \{\alpha_s, \alpha_t\}$.

(iv) From Lemma 7.16 we computed $\Phi^1(s[t,u]_k) = \{\alpha_s, s\alpha_t, ut\alpha_s\}$ and we now compute $\Phi^1(ts[t,u]_k) = \{\alpha_s, \alpha_t, tut\alpha_s\}$. Again $tut\alpha_s \notin \mathscr{E}$, so $\partial T([u,t]_k st) = \{\alpha_s, \alpha_t\}$. □

There are just a few remaining cases for W of type I.

Lemma 7.20. *Let W be of type I with $b = m_{ut}$ odd and $k = \ell(w_{ut}) - 2$. Then*

(i) $T([t,u]_k sutu) \xrightarrow{t} T(utut)$
(ii) $T([t,u]_k ustustu) \xrightarrow{s} T(stsu)$
(iii) $T([t,u]_k sutustu) \xrightarrow{t} T([u,t]_6)$
(iv) $T(w_{ut}s) \xrightarrow{t} T([t,u]_k usts)$
(v) $T(stu) \xrightarrow{t} T(tut)$
(vi) $T(stsutu) \xrightarrow{t} T(tutut)$
(vii) $T(sutu) \xrightarrow{t} T(utut)$
(viii) $T(sutu) \xrightarrow{s} T(stsu)$

When b is even, swap $[t,u]_k$ with $[u,t]_k$ in the above list, then the same results hold.

(Note that this result corresponds to the following transitions in Figure 6: $31 \to 4$*,* $34 \to 20$*,* $34 \to 6$*,* $22 \to 30$*,* $24 \to 11$*,* $27 \to 9$*,* $25 \to 4$ *and* $25 \to 20$*).*

Proof. Again, swap $[t,u]_k$ with $[u,t]_k$ in all calculations below for b even.

(i) We have from Lemma 7.15 (iii) that

$$\Phi^1(utsu[t,u]_k) = \{\alpha_u, ut\alpha_u\, sut\alpha_u\}.$$

Then using Proposition 2.3 we compute

$$\Phi^1(tutsu[t,u]_k) = \{\alpha_t, tut\alpha_u, tsut\alpha_u\}$$

Since $tsut\alpha_u \notin \mathscr{E}$ by Lemma 7.5 we immediately have

$$\partial T([t,u]_k ustu) = \{\alpha_t, tut\alpha_u\} = \partial T(utut).$$

(ii) From Lemma 7.15 (vi) we had

$$\Phi^1(utsutsu[t,u]_k) = \{\alpha_u, ut\alpha_s, utut\alpha_u\}$$

and thus compute $\Phi^1(sutsutsu[t,u]_k) = \{\alpha_s, \alpha_u, u\alpha_t, sutut\alpha_u\}$. Then since $sutut\alpha_u \notin \mathscr{E}$ by Lemma 7.11 (ii) and Lemma 7.5 again, we have

$$\partial T([t,u]_k sutustsu) = \{\alpha_u, \alpha_s, u\alpha_t\} = \partial T(stsu)$$

(iii) Following from (ii) we compute $\Phi^1(tutsutsu[t,u]_k) = \{\alpha_t, tut\alpha_s, tutut\alpha_u\}$. Note that $tut\alpha_s \notin \mathscr{E}$ so

$$\partial T([t,u]_k sutustut) = \{\alpha_t, tutut\alpha_u\} = \partial T([u,t]_6).$$

(iv) From Lemma 7.16 (i) we had $\Phi^1(sw_{ut}) = \{\alpha_u, \alpha_s, \alpha_s + \alpha_t\}$ and thus we compute $\Phi^1(tsw_{u,t}) = \{\alpha_t, \alpha_s, t\alpha_u\} = \partial T([t,u]_k usts)$ from Lemma 7.15 (ii).

(v) From Lemma 7.11 (i) we have $\Phi^1(uts) = \{\alpha_u, u\alpha_t, ut\alpha_s\}$. Then using Proposition 2.3 we compute $\Phi^1(tuts) = \{\alpha_t, tu\alpha_t, tut\alpha_s\}$. Since $tut\alpha_s \notin \mathscr{E}$, we must have $\partial T(stut) = \{\alpha_t, tu(\alpha_t)\} = \partial T(tut)$.

(vi) Similarly, from Lemma 7.11 (iv) we have $\Phi^1(utusts) = \{\alpha_u, utu\alpha_t, ut\alpha_s\}$. Then we compute $\Phi^1(tutusts) = \{\alpha_t, tutu\alpha_t, tut(\alpha_s)\}$ and $tut(\alpha_s) \notin \mathscr{E}$ so $\partial T(stsutut) = \{\alpha_t, tutu\alpha_t\} = \partial T(tutut)$.

(vii) From Lemma 7.12 (ii) we have $\Phi^1(utus) = \{\alpha_u, ut\alpha_u, ut\alpha_s\}$. Then we compute $\Phi^1(tutus) = \{\alpha_t, tut\alpha_u, tut\alpha_s\}$. Again $tut\alpha_s \notin \mathscr{E}$ and the result follows.

(viii) Finally compute $\Phi^1(sutus) = \{\alpha_s, \alpha_u, sut\alpha_u, u\alpha_t\}$. Note that $stsu$ is a suffix of $sutus = ustsu$, so $T(sutus) \subseteq T(stsu)$, and by Lemma 7.11 (ii) we have $\partial T(stsu) = \{\alpha_s, \alpha_u, u\alpha_t\}$. To show equality, we just need to show that $sut\alpha_u \notin \partial T(sutus)$. Note that $sut\alpha_u = (c_1, c_1, c_1^2 - 1)$. By Theorem 11 and Appendix A.1, if $sut\alpha_u \in \partial T(sutus)$, then we must have either $\Phi(sutus) \cap \Phi(s[t,u]_{b-2}) = (c_1, c_1, c_1^2 - 1)$ or $\Phi(sutus) \cap \Phi(utus[t,u]_{b-2}) = (c_1, c_1, c_1^2 - 1)$. But clearly, $\alpha_s \in \Phi^1(s[t,u]_{b-2})$ and $\alpha_u \in \Phi^1(utus[t,u]_{b-2})$. Therefore, $sut\alpha_u \notin \partial T(sutus)$.

□

The above results complete the proof for W of type I.

Theorem 14. *Let (W,S) be the Coxeter system of Type I with Γ_W as illustrated in Figure 5. Let a be the left edge label and b the right edge label in Γ_W. Let W' be the Coxeter system obtained from W by replacing b with $b+1$ in Γ_W and denote $\mathscr{U}_{W'}$ and $\mathscr{U}_W$ to be the ultra-low elements respectively. Then*

$$|\mathscr{U}_{W'}| = |\mathscr{U}_W| + 2$$

The effect of increasing the label b by 1 adds only two additional ultra low elements corresponding to the two additional elements of the finite dihedral group $W_{\langle t,u\rangle}$. The automaton is illustrated in Figure 6 for b odd (the automaton for b even is entirely similar with only a swapping of the alternating word $[t,u]$ with $[u,t]$ in some instances).

Corollary 7.21. *Let W be of type I with $7 \leq b = m_{ut} < \infty$. Then*

$$|\mathscr{U}| = 35 + 2(b-7) = 21 + 2b$$

7.4. **Type II.** We illustrate the cone type automaton in Figure 7 below. A number of results in this section apply to both Types II and III.

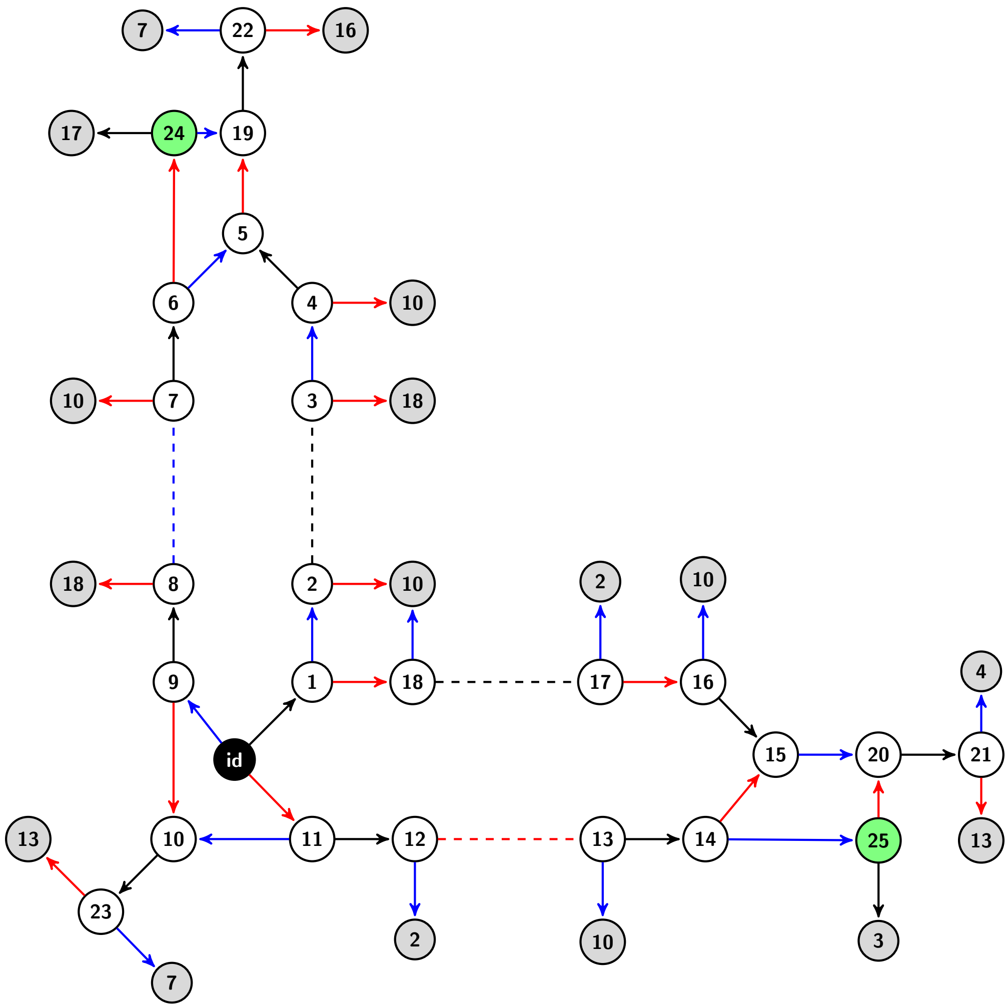


FIGURE 7. The automaton $\mathcal{A}_o$ for (W,S) of Type II with a and b odd (the illustration includes all states for the case $a = b = 5$ with the trailing dashes indicating additional elements of $W_{\langle t,u\rangle}$ and $W_{\langle t,s\rangle}$ when $a, b > 5$). One may note that the automaton here is similar to Type *I* but less complicated in the sense that there are less nodes at the top of the figure. Again, the states highlighted green indicate the (inverses of) tight gates whose final root has non-spherical support.

Lemma 7.22. *Let W be of type II or III. Then $T(su) \xrightarrow{t} T(sut)$ is a transition in $\mathcal{A}_o$ and sut is ultra-low. (Note that this result corresponds to the transition $10 \to 23$ in Figure 7 and $10 \to 22$ in Figure 8).*

Proof. The proof is exactly the same computation as in Lemma 7.12 (i). $\square$

Lemma 7.23. *Let W be of type II with a, b odd. Then*

(i) $T([s,t]_{a-1}) \xrightarrow{u} T([s,t]_{a-1}u)$
(ii) $T(w_{st}) \xrightarrow{u} T(w_{st}u)$
(iii) $T(w_{st}u) \xrightarrow{t} T(w_{st}ut)$

and

(iv) $T([u,t]_{b-1}) \xrightarrow{s} T([u,t]_{b-1}s)$
(v) $T(w_{ut}) \xrightarrow{s} T(w_{ut}s)$
(vi) $T(w_{ut}s) \xrightarrow{t} T(w_{ut}st)$

where each element w above is ultra-low.

If a is even then swap $[s,t]_{a-1}$ in (i) with $[t,s]_{a-1}$ and if b is even then swap $[u,t]_{b-1}$ in (iv) with $[t,u]_{a-1}$, then the same results hold.

(Note that this result corresponds to the transitions $14 \to 25, 15 \to 20, 20 \to 21, 6 \to 24, 5 \to 19, 19 \to 22$, in Figure 7).

Proof. Note that (iv)-(vi) is the same as (i)-(iii) with the roles of s and u swapped, hence we will just provide the details for (i)-(iii). If a is even, then replace $[s,t]_{a-1}$ with $[t,s]_{a-1}$ in the proofs below, and the same calculations hold. Similarly if b is even, replace $[u,t]_{b-1}$ with $[t,u]_{b-1}$.

(i) By Lemma 7.3 we have $\Phi^1([t,s]_{a-1}) = \{\alpha_t, s\alpha_t\}$ and thus using Proposition 2.3 we compute $\Phi^1(u[t,s]_{a-1}) = \{\alpha_u, u\alpha_t, us\alpha_t\}$. By Lemma 7.5 again we just need to justify that $us\alpha_t = (c_1, 1, c_2) \in \partial T([s,t]_{a-1}u)$. Now $\Phi^1([t,u]_{b-1}) = \{\alpha_t, u(\alpha_t)\}$ and by Proposition 2.3 we compute $\Phi^1(s[t,u]_{b-1}) = \{\alpha_s, s\alpha_t, us\alpha_t\}\}$. Therefore,
$$\Phi^1(u[t,s]_{a-1}) \cap \Phi(s[t,u]_{b-1}) = \{us\alpha_t\}$$
(ii) By Lemma 7.1 we have $\Phi(w_{st}) = \{\alpha_s, \alpha_t\}$ and compute
$$\Phi^1(uw_{st}) = \{\alpha_u, \alpha_s, u\alpha_t\}$$
hence the result by Lemma 7.5.
(iii) We compute $\Phi^1(tuw_{st}) = \{\alpha_t, t\alpha_s, tu\alpha_t\}$, and hence the result.

□

This proves that each of the states indicated by black outlined nodes in Figure 7 are ultra-low. We now prove that there are not any more (and prove the remaining transitions into the gray coloured states).

Lemma 7.24. *Let W be of type II with a, b odd. If w be an element of the form*

(i) *(1)* $[t,s]_n$ *for* $2 \le n \le a-1$ *for* n *even*
(2) $[s,t]_n$ *for* $3 \le n \le a-2$ *for* n *odd*

Then $T(w) \xrightarrow{u} T(us)$ *is a transition in* $\mathcal{A}_o$*, and if* w *is an element of the form;*

(ii) *(1)* $[t,u]_n$ *for* $2 \le n \le b-1$ *for* n *even*
(2) $[u,t]_n$ *for* $3 \le n \le b-2$ *for* n *odd*

Then $T(w) \xrightarrow{s} T(us)$ *is a transition in* $\mathcal{A}_o$.

If a is even then (1) and (2) in (i) should be replaced with
(1) $[s,t]_n$ *for* $3 \le n \le a-1$ *for* n *odd*
(2) $[t,s]_n$ *for* $2 \le n \le a-2$ *for* n *even*

Similarly if b is even then (1) and (2) in (ii) should be replaced with
(1) $[u,t]_n$ *for* $3 \le n \le b-1$ *for* n *odd*
(2) $[t,u]_n$ *for* $2 \le n \le b-2$ *for* n *even*

(Note that this result corresponds to the u and s transitions from the nodes labelled $13, 16, 18, 2, 4, 7$ in Figure 7).

Proof. The proof is the same Lemma 7.17. We again prove (i) in the case of a odd and omit the details for (ii).

Let $n = a-1$, then $\Phi^1([t,s]_n^{-1}) = \{\alpha_s, t\alpha_s\}$. Using Proposition 2.3 we compute
$$\Phi^1(u[t,s]_n^{-1}) = \{\alpha_u, \alpha_s, ut\alpha_s\}$$
then since $ut\alpha_s \notin \mathscr{E}$, we have $\partial T([t,s]_n u) = \{\alpha_u, \alpha_s\} = \partial T(us)$. Then by Lemma 2.12 we have the chain of inclusions
$$T(us) \subseteq T([t,s]_n u) \subseteq T([s,t]_{n-1}u) \subseteq \ldots \subseteq T(tsu) \subseteq T(us)$$
which includes all the elements in (1) and (2). □

The next result is very similar to Lemma 7.24.

Lemma 7.25. *Let W be of type II. When b is odd and w is an element of the form*

(1) $[t,u]_n$ *for* $3 \leq n \leq b-2$ *for* n *odd*
(2) $[u,t]_n$ *for* $2 \leq n \leq b-3$ *for* n *even*

Then $T(w) \xrightarrow{s} T(ts)$ *is a transition in* $\mathcal{A}_o$. *If* b *is even then* (1) *and* (2) *above should be replaced with*

(1) $[u,t]_n$ *for* $2 \leq n \leq b-2$ *for* n *even*
(2) $[t,u]_n$ *for* $3 \leq n \leq b-3$ *for* n *odd*

(Note that this result corresponds to the s*-transitions from the nodes labelled* $3, 8$ *in Figure 7). When* a *is odd and* w *is an element of the form*

(1) $[t,s]_n$ *for* $3 \leq n \leq a-2$ *for* n *odd*
(2) $[s,t]_n$ *for* $2 \leq n \leq a-3$ *for* n *even*

Then $T(w) \xrightarrow{u} T(tu)$ *is a transition in* $\mathcal{A}_o$. *If* a *is even then* (1) *and* (2) *above should be replaced with*

(1) $[s,t]_n$ *for* $2 \leq n \leq a-2$ *for* n *even*
(2) $[t,s]_n$ *for* $3 \leq n \leq a-3$ *for* n *odd*

(Note that this result corresponds to the u*-transitions from the nodes labelled* $12, 17$ *in Figure 7).*

Proof. The proof is essentially the same as Lemma 7.24. We again show the case of b odd and leave the computation for the other cases to the reader. Let $n = b-2$, then $\Phi^1([t,u]_n^{-1}) = \Phi^1([t,u]_n) = \{\alpha_t, ut\alpha_u\}$. Using Proposition 2.3 we compute

$$\Phi^1(s[t,u]_n) = \{\alpha_s, s\alpha_t, sut\alpha_u\}$$

then since $sut\alpha_u \notin \mathscr{E}$, we have $\partial T([t,u]_n s) = \{\alpha_s, s\alpha_t\} = \partial T(ts)$. Then by Lemma 2.12 we have the chain of inclusions

$$T(ts) \subseteq T([t,u]_n s) \subseteq T([u,t]_{n-1}s) \subseteq \ldots \subseteq T(uts) \subseteq T(ts)$$

which includes all the elements in (1) and (2). $\square$

The final result of this section covers the remaining transitions.

Lemma 7.26. *Let W be of type II with a, b odd. Then*

(i) $T([s,t]_{a-1}u) \xrightarrow{t} T(tut)$
(ii) $T(w_{st}ut) \xrightarrow{u} T(tutu)$
(iii) $T(w_{st}ut) \xrightarrow{s} T(sts)$
(iv) $T([u,t]_{b-1}s) \xrightarrow{t} T(tst)$
(v) $T(w_{ut}st) \xrightarrow{u} T(utu)$
(vi) $T(w_{ut}st) \xrightarrow{s} T(tsts)$
(vii) $T(ust) \xrightarrow{s} T(sts)$
(viii) $T(ust) \xrightarrow{u} T(utu)$

If a is even then swap $[s,t]_{a-1}$ in (i) with $[t,s]_{a-1}$ and if b is even then swap $[u,t]_{b-1}$ in (iv) with $[t,u]_{b-1}$, then the same results hold.

(Note that this result corresponds to the transitions from the nodes labelled $25 \to 3, 21 \to 4, 21 \to 13, 24 \to 17, 22 \to 7, 22 \to 16, 23 \to 13, 23 \to 7$ *in Figure 8).*

Proof. Again, if a is even, then replace $[s,t]_{a-1}$ with $[t,s]_{a-1}$ in the proofs below, and the same calculations hold. Similarly if b is even, replace $[u,t]_{b-1}$ with $[t,u]_{b-1}$.

(i) From Lemma 7.24 we have $\Phi^1(u[t,s]_{a-1}) = \{\alpha_u, u\alpha_t, us\alpha_t\}$ and compute $\Phi^1(tu[t,s]_{a-1}) = \{\alpha_t, tu\alpha_t, tus\alpha_t\}$. Then since $tus\alpha_t \notin \mathscr{E}$ it follows by Lemma 7.5 that we must have

$$\partial T([s,t]_{a-1}ut) = \{\alpha_t, tu(\alpha_t)\} = \partial T(tut)$$

(ii) By Lemma 7.23 (iii) we have $\Phi^1(tuw_{st}) = \{\alpha_t, t\alpha_s, tu\alpha_t\}$. Then we compute $\Phi^1(utuw_{st}) = \{\alpha_u, ut\alpha_s, utu\alpha_t\}$. Again since $ut\alpha_s) \notin \mathscr{E}$ the result follows.

The proofs of (iii)-(viii) again all have the exact same structure and can be easily verified. For each element w on the left hand side of the arrow, we compute $\Phi^1(s_i w)$. Then there is a single $\beta \in \Phi^1(s_i w) \setminus \mathscr{E}$ with

$\beta \in \Phi^0(s_i w)$. Hence $\beta \notin \partial T(w^{-1}s_i)$ by Lemma 2.9. The remaining roots of $\Phi^1(s_i w)$ have dihedral support and therefore are boundary roots by Lemma 7.5. $\square$

This completes the case for W of type II. We have the following consequence combining the results in this section.

Theorem 15. *Let (W, S) be the Coxeter system of Type II with Γ_W as illustrated in Figure 5. Let a be the label of the left edge and b the label of the right edge in Γ_W. Let W' be the Coxeter system obtained from W by replacing b with $b+1$ or by replacing a with $a+1$ in Γ_W. Then*

$$|\mathscr{U}_{\mathscr{W}'}| = |\mathscr{U}| + 2$$

Corollary 7.27. *Let W be of type II with $4 < a = m_{st} < \infty$ and $4 < b = m_{ut} < \infty$. Then*

$$|\mathscr{U}| = 26 + 2(a-5) + 2(b-5) = 6 + 2(a+b)$$

7.5. **Type III.** We end this work with a complete description of the ultra-low elements for type III. Type III is very similar to type II, as illustrated in Figure 8.

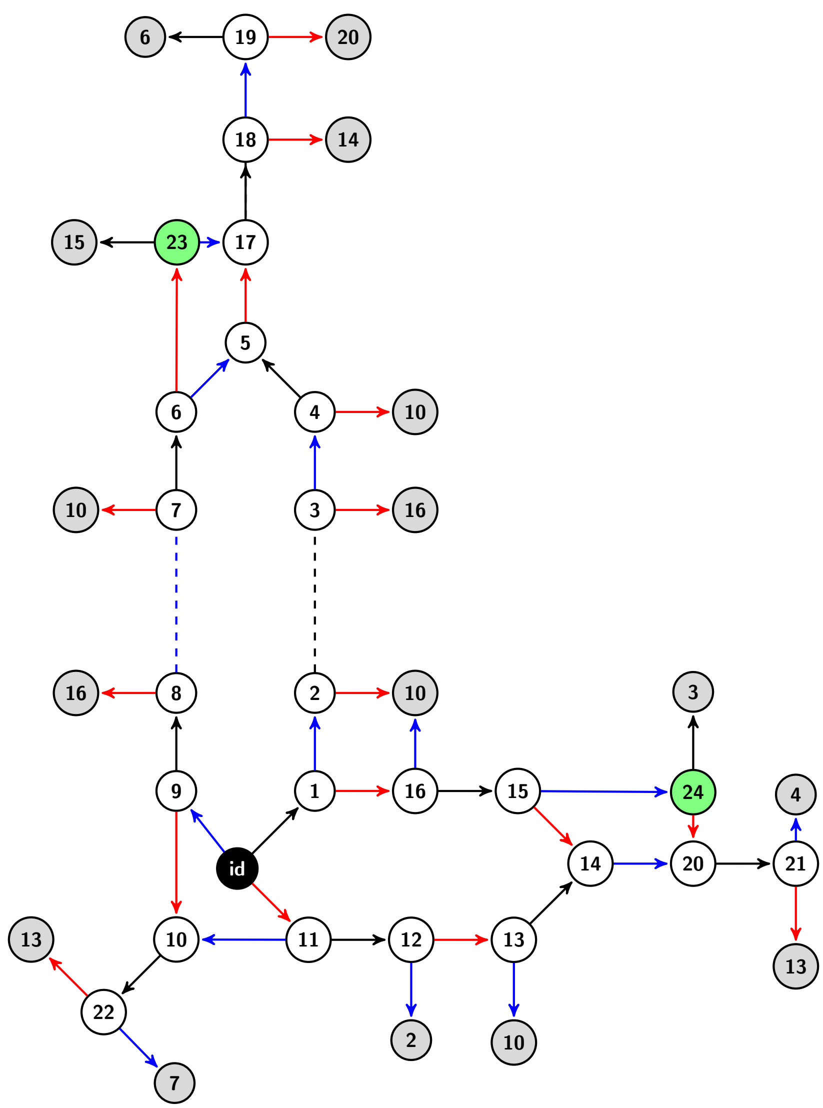


FIGURE 8. The automaton $\mathcal{A}_o$ for (W, S) of Type III. All transitions and ultra-low elements remain the same as in type II, with the exception of the states at the "top" of the diagram (compare the top part of Figure 7 with this automaton to see that there is an "extra" non-spherical state (labelled 18) here.

Specifically, the following results from type II carry over to type III with of course, $a = 4$ being even. The proofs are exactly the same, and so we leave the computations to the reader. These results correspond to all the same transitions in Figure 7 as in Figure 8 with the exception of the "extra" state and it's transitions at the top of the diagram (transitions $18 \to 19, 19 \to 6, 19 \to 20$).

(i) Lemma 7.22 (corresponding to the transition $10 \to 23$ in Figure 8).
(ii) Lemma 7.23 (for the transitions $5 \to 17, 17 \to 18, 6 \to 23, 14 \to 20, 20 \to 21$ and $15 \to 24$ in Figure 8).
(iii) Lemma 7.24 (for the transitions $2 \to 10, 16 \to 10, 7 \to 10, 13 \to 10$ and $4 \to 10$ in Figure 8).
(iv) Lemma 7.25 (for the transitions $3 \to 16, 8 \to 16, 12 \to 2$ in Figure 8).
(v) Lemma 7.26 excluding (v) (for the transitions $24 \to 3, 21 \to 4, 21 \to 13, 23 \to 15, 18 \to 14, 22 \to 13, 22 \to 7$ in Figure 8).

The following results are specific to type III and correspond to the transitions $18 \to 19, 19 \to 6$ and $19 \to 20$.

Lemma 7.28. *Let W be of type III. Then $T(w_{ut}st) \xrightarrow{u} T(w_{ut}stu)$ is a transition in $\mathcal{A}_o$ and $w_{ut}stu$ is ultra-low.*

Proof. We compute $\Phi^1(tsw_{ut}) = \{\alpha_t, s\alpha_t, t\alpha_u\}$ and using Proposition 2.3 we compute $\Phi^1(utsw_{ut}) = \{\alpha_u, us\alpha_t, ut\alpha_u\}$. Then by what was computed in Lemma 7.23 (i) we have

$$\Phi^1(utsw_{ut}) \cap \Phi(s[t,u]_{b-1}) = \{us\alpha_t\}$$

□

The next result includes the remaining transitions.

Lemma 7.29. *Let W be of type III. Then*

(i) $T(w_{ut}stu) \xrightarrow{t} T(utut)$
(ii) $T(w_{ut}stu) \xrightarrow{s} T(w_{st}u)$

are transitions in $\mathcal{A}_o$.

Proof. (i) From Lemma 7.28 we have $\Phi^1(utsw_{ut}) = \{\alpha_u, us\alpha_t, ut\alpha_u\}$ and compute $\Phi^1(tutsw_{ut}) = \{\alpha_t, ut\alpha_u, tuts\alpha_t\}$. Since $tuts\alpha_t \notin \mathscr{E}$ The result is immediate. Similarly, for (ii) we compute $\Phi^1(sutsw_{ut}) = \{\alpha_s, \alpha_u, u\alpha_t, sut(\alpha_u)\}$ and $sut(\alpha_u) \notin \mathscr{E}$ by Lemma 7.10. By Lemma 7.23 (ii) we then have $\partial T(w_{ut}stus) = \partial T(w_{st}u)$. □

This concludes the proof for W of type III.

Corollary 7.30. *Let W be of type (II) with $a = 4$ and $5 \le b = m_{ut} < \infty$. Then*

$$|\mathscr{U}| = 25 + 2(b - 5) = 15 + 2b$$

8. Further results on tight gates

We conclude this note by summarising some results on tight gates which follow by of our calculations in Section 7 and Appendix A.

Corollary 8.1. *Let W be a Coxeter group of one of the following types*

(i) W is finite dihedral,
(ii) W is irreducible rank 3
(iii) W is right-angled, or;
(iv) The Coxeter graph Γ_W is a complete graph.

Then for each non-simple super-elementary root β there is a unique pair of tight gates x, y such that

$$\Phi(x) \cap \Phi(y) = \{\beta\}$$

and $\Phi^0(x) = \Phi^0(y) = \{\beta\}$.

Proof. (i) This follows by Lemma 7.3 (iii).
(ii) For irreducible rank 3 Coxeter groups with a linear graph, see the computations in Appendix A. For the rank 3 groups of affine type the results can be visually verified by the illustrations of the cone type partition $\mathscr{T}$ in Figure 3. Rank 3 groups with a complete graph are addressed in (iv).
(iii) This follows trivially, since there are no non-simple super-elementary roots in this case.

(iv) This follows by the discussion in Corollary 7.9.

□

Corollary 8.2. *Let W be a Coxeter group of a type listed in Corollary 8.1. Then*
$$|\Gamma^0| = 2|\mathscr{E}| - |S|$$

Proof. Note that in each case, we have $\mathscr{E} = \mathcal{S}$, since:

(i) When W is finite dihedral, $\mathscr{E} = \Phi^+ = \mathcal{S}$ by Lemma 7.3 (iii).
(ii) By [Yau21, Proposition 6.0.7] for rank 3 we have $\mathscr{E} = \mathcal{S}$.
(iii) When W is right-angled, by Lemma 2.7 we have $\mathscr{E} = \mathcal{S} = \Delta$.
(iv) When Γ_W is a complete graph, then again by Lemma 2.7 if $\beta \in \mathscr{E}$ the subgraph $\Gamma(\beta)$ cannot contain a curcuit or an infinite bond, hence $\mathscr{E} = \bigcup_{\{s,t\}\subseteq S} \Phi^+_{\langle s,t\rangle}$ with $m(s,t) < \infty$. Thus again by Lemma 7.3 (iii) $\mathscr{E} = \mathcal{S}$.

Then for each simple root, the corresponding simple reflection is a tight gate and it's own witness. By Corollary 8.1 for each non-simple super elementary root there is a unique pair of tight gates x, y such that $\Phi(x) \cap \Phi(y) = \{\beta\}$ and $\Phi^0(x) = \Phi^0(y) = \{\beta\}$. Hence
$$|\Gamma^0| = 2 * (|\mathscr{E}| - |S|) + |S| = 2|\mathscr{E}| - |S|$$

□

Appendix A. Tight gates in rank 3

We utilise Algorithm 1 to compute and record the tight gates in rank 3 and their final roots. Retaining the labelling convention as described in Figure 5 let
$$X = \Big(W_{\langle s,t\rangle} \setminus \{e, w_{s,t}\}\Big) \cup \Big(W_{\langle t,u\rangle} \setminus \{e, w_{t,u}\}\Big)$$
where $w_{s,t}$ and $w_{t,u}$ are the longest elements of their respective dihedral groups. Let $b = m_{u,t}$. The elements X are tight gates by Corollary 7.4.

A.1. **Type I.** The tight gates are:

$$\Gamma^0 = X \cup \{uts, s[t,u]_{b-2}, us[t,u]_{b-2}, tus[t,u]_{b-2}, utus[t,u]_{b-2}, s[t,u]_{b-1}\}$$

The final roots of the tight gates in X are $\Phi^+_{W_{\langle s,t\rangle}} \cup \Phi^+_{W_{\langle t,u\rangle}}$. The final roots of the remaining tight gates are respectively:
$$\{(1,1,c_1), (c_1,c_1,c_1^2-1), (c_1,c_1,1), (c_1,c_1,1), (c_1,c_1,c_1^2-1), (1,1,c_1)\}$$
Note that these are the elementary roots with full support and that for each root there is a single pair of tight gates with the root as their final root.

A.2. **Type II.** We note that from Lemma 7.10 for types II and III there is a single elementary root of full support. The tight gates are:
$$\Gamma^0 = X \cup \{u[t,s]_{a-1}, s[t,u]_{b-1}\}$$
The final root of $u[t,s]_{a-1}$ and $s[t,u]_{b-1}$ is the root $(c_1, 1, c_2)$.

A.3. **Type III.** The tight gates are:
$$\Gamma^0 = X \cup \{utst, s[t,u]_{b-1}\}$$
The final root of $utst$ and $s[t,u]_{b-1}$ is also the root $(c_1, 1, c_2)$.

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