Bilinear maps on the ring of strictly upper triangular matrices

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Abstract. Let R be a 2-torsion free unital ring and $N_n = N_n(R)$ the ring of strictly upper triangular matrices with entries in R and center $Z = Z(N_n)$. It has been previously shown that any linear map $f : N_n \to N_n$ satisfying the condition $[f(X), X] = 0$ must be of the form $f(X) = \lambda X + \mu(X)$ for some $\lambda \in R$ and additive map μ defined on N_n. We extend these known results by providing a complete description of the bilinear maps $f : N_n \times N_n \to N_n$ satisfying the identity $[f(X, X), X] = 0$ for all $X \in N_n$.

Contents

1. Introduction

Let R be a ring with center $Z(R)$. We say that a map $f : R \to R$ is *commuting* if $[f(x), x] = 0$ for all $x \in R$ where $[a, b] = ab - ba$ denotes the standard commutator on R. Investigations into commuting maps were initiated by Posner [**25**] in 1957 when it was proven that a noncommutative prime ring cannot exhibit a nonzero commuting derivation. This theorem has since been generalized in numerous settings (see, for example, [**2, 4, 5, 6, 19, 20, 26**]).

The first general result in the study of commuting maps is due to Brešar [**5**] when it was shown that an additive commuting map f over a simple unital ring R must be of the so-called *standard form*

$$f(x) = \lambda x + \mu(x)$$

for some $\lambda \in Z(R)$ and additive $\mu : R \to Z(R)$, work which initiated the more general study of *functional identities*. Following this early finding, researchers have extensively examined the properties of commuting maps that satisfy certain conditions in various rings and algebras. The reader is referred to the survey paper by Brešar [**8**] for details on the early development of the theory of commuting maps along with an overview of classical results in the field. More information regarding the theory of functional identities can be found in [**7**] with a more detailed account provided in the book by Brešar, Chebotar, and Martindale [**9**].

Particular interest has been placed in examining the structure of commuting maps over matrix rings and algebras with notable results being established in the cases of upper and strictly upper triangular matrices ([**1, 3, 10, 13, 14, 15**]). In 2000, Beidar, Brešar, and Chebotar [**1**] proved that a linear commuting map defined on $T_r(F)$, the algebra of $r \times r$ upper triangular matrices with entries in a field F, must also be of the standard form. More precisely, given a k-linear map $f : T_n(F)^k \to T_n(F)$, $k \geq 1$, satisfying the identity $[f(X, \ldots, X), X] = 0$ for all $X \in T_n(F)$, they showed that there exist multilinear maps $\lambda_i : T_n(F)^i \to F$, $i = 0, \ldots, k$, such that

$$f(X, \ldots, X) = \sum_{i=0}^{k} \lambda_i(X, \ldots, X)X^{k-i} \tag{1.1}$$

for all $X \in T_n(F)$. Similar results in related settings have been obtained over the years, including Eremita's investigations into commuting traces over $T_n(R)$ [**12**]. A map $q : T_n(R) \to T_n(R)$ is called a *commuting trace* of a biadditive map over $T_n(R)$ if there exists a commuting biadditive map $B : T_n(R) \times T_n(R) \to T_n(R)$ such that $q(X) = B(X, X)$ for all $X \in T_n(R)$. In [**12**], Eremita proves that all commuting traces over $T_n(R)$ where R is a 2-torsion free unital ring must be of the form

$$q(X) = \lambda X^2 + \mu(X)X + \nu(X)I_n$$

where $\lambda \in Z(R)$, $\mu, \nu : T_n(R) \to Z(R)$, and I_n is the $n \times n$ identity matrix.

Extending this work to the case of strictly upper triangular matrices, denoted by $N_n(F)$, the first author has previously shown in [**3**] that linear commuting maps in this setting are almost of the standard form:

Theorem 1.1 (Theorem 1 in [**3**]). *Let $N_n(F)$ be the ring of $n \times n$ strictly upper triangular matrices, $n \geq 4$, over a field F of characteristic zero and suppose $f : N_n(F) \to N_n(F)$ is a linear commuting map. Then there exist $\lambda \in F$ and an additive map $\mu : N_n(F) \to \Omega_n$ such that*

$$f(A) = \lambda A + \mu(A)$$

for all $A \in N_n(F)$ where $\Omega_n = \{ae_{1,n-1} + be_{1,n} + ce_{2,n} : a, b, c \in F\}$ and $e_{i,j}$ denotes the standard matrix unit.

This has since been generalized by Ko and Liu [**18**] to the settings of unital and semiprime rings, giving us the following theorem.

Theorem 1.2 (Theorem 1.1 in [**18**]). *Let R be a ring with 1 and suppose $f : N_n(R) \to N_n(R)$ is an additive map satisfying $[f(X), X] = 0$ for all $X \in N_n(R)$. Then there exists $\lambda \in Z(R)$ and additive maps $\mu : N_n(R) \to Z(N_n(R))$, $\nu : N_N(R) \to \Omega$ such that*

$$f(X) = \lambda X + \mu(X) + \nu(X)$$

for all $X \in N_n(R)$ where $\nu(X) = e_{1,1}Xae_{2,n-1} + e_{2,n}aXe_{n,n}$.

In the present paper, we focus attention on bilinear maps defined over $N_n(R)$, the ring of strictly upper triangular matrices with entries in a 2-torsion free unital ring R. Our main result is the following.

Theorem 1.3. *Let $f : N_n(R) \times N_n(R) \to N_n(R)$ be a bilinear map such that*

$$[f(X, X), X] = 0$$

for all $X \in N_n(R)$. Then there exist a bilinear map $\lambda : N_n(R) \times N_n(R) \to Z(R)$ and a biadditive map $\mu : N_n(R) \times N_n(R) \to \Omega_n$ such that

$$f(X, X) = \lambda(X, X)X + \mu(X, X)$$

for all $X \in N_n(R)$, where $\Omega_n = \{ae_{1,n-1} + be_{1,n} + ce_{2,n} : a, b, c \in R\}$.

Theorem 1.3 is a clear generalization of Theorem 1.1, though the techniques we use in the present paper are notably different from those that appear in [**3**]. The proof of Theorem 1.1 given in [**3**] applies a well-known fact regarding the structure of centralizers of non-derogatory matrices over a field of characteristic 0 (see Theorem 3.2.4.2 of [**17**]). This allows one to describe the image of a particular basis for $N_n(F)$ under f, extending this result to all of $N_n(F)$ via the linearity of f. In proving Theorem 1.3, however, we proceed with an alternative method inspired by the techniques used by Eremita in [**12**]. This approach applies the process of linearization, commonly used in the study of functional identities (see, for example, Section 1.1 of [**9**]), to a decomposition of biadditive maps over $T_n(R)$ into well-chosen component functions. Using a similar decomposition, we adapt these methods to $N_n(R)$ to obtain notably different results. In particular, we would like to emphasize that the description provided in Theorem 1.3 contains no quadratic term and closely resembles the structure obtained in Theorems 1.1 and 1.2, further illustrating that the behavior of commuting maps over $N_n(R)$ is significantly different from the commuting maps over $T_n(R)$.

It is natural to ask if a more explicit description of the map μ present in Theorem 1.3 exists. In particular, it may be of note to compare this map with the maps in the conclusion of Theorem 1.2. By considering the particular case of a bilinear commuting map on $N_n(R)$ whose image is contained in Ω_n, we obtain such a description for the map μ, allowing us to refine Theorem 1.3 as follows.

Corollary 1.4. *Let $f : N_n \times N_n \to N_n$ be a bilinear map such that $[f(X, X), X] = 0$ for all $X \in N_n$. Then there exist bilinear maps $\lambda : N_n \times N_n \to R$, $\zeta : N_n \times N_n \to Z(N_n)$ and a linear map $p : N_n \to R$ such that*

$$f(X, X) = \lambda(X, X)X + x_{1,2}p(X)e_{1,n-1} + p(X)x_{n-1,n}e_{2,n} + \zeta(X)$$

for all $X \in N_n$.

The paper is structured as follows. We begin with preliminaries and initial results in Section 2. We then provide a full proof of Theorem 1.3 in Section 3. We conclude with a brief discussion about the map μ present in Theorems 1.1 and 1.3

2. Preliminaries

Throughout the remainder of this paper, we will assume $n \geq 3$ is an integer and R is a 2-torsion free unital ring unless otherwise stated. Let $e_{i,j}$ denote the $n \times n$ matrix with ij-th entry 1 and all other entries 0. Let $\bar{e}_i \in R^n$ denote the vector with i-th coordinate 1 and all other coordinates 0 and $\bar{0} \in R^n$ the zero vector. We use $N_n = N_n(R)$ to denote the ring of $n \times n$ strictly upper triangular matrices with entries in R. It is well known that N_n is a nilpotent ring in which the identity $A^n = 0_n$ holds, where we let 0_n denote the $n \times n$ zero matrix. Additionally, the center of N_n is known to be the set $Z(N_n) = \{ae_{1,n} : a \in R\}$.

We will make use of the process of linearization throughout our analysis. Roughly speaking, this process is a method through which new relations are obtained by iteratively replacing an arbitrary element satisfying a given relation by the sum of two arbitrary elements. As an illustrative example, suppose we have additive groups G, H and a bi-additive map $B : G^2 \to H$ such that $B(x, x) = 0$ for all $x \in G$. By replacing x with $x + y$, one naturally obtains $B(x,y)+B(y,x)=0$ for all $x,y\in G$. A more detailed description of this process can be found here [**16**].

We are interested in examining the structure of all bilinear maps $f : N_n \times N_n \to N_n$ satisfying $[f(X,X),X]=0$ for all $X\in N_n$. One such map is given in the following example.

EXAMPLE 2.1. Let $a,b\in R$ and define $f : N_n\times N_n\to N_n$ by

$$f(X,Y)=aX+bY+x_{1,2}y_{1,n}e_{1,n-1}+x_{1,n}y_{n-1,n}e_{2,n}.$$

f is a bilinear map satisfying

$$[f(X,X),X]=[(a+b)X,X]+(x_{1,2}x_{1,n}x_{n-1,n}-x_{1,2}x_{1,n}x_{n-1,n})e_{1,n}=0_n,$$

hence f is commuting.

While generating examples of commuting maps on N_n is a relatively simple task, it should be noted that not every map defined on N_n is necessarily commuting. This can be seen by slightly modifying Example 2.1.

EXAMPLE 2.2. Define

$$h(X,Y)=(ax_{1,2}+by_{1,2})I_n+x_{1,2}y_{1,n}e_{2,n-1}+x_{1,n}y_{n-1,n}e_{2,n}.$$

The map h is bilinear, however,

$$[h(X,X),X]=-x_{1,2}x_{1,n-1}x_{1,n}e_{1,n-1}-x_{1,2}x_{1,n}x_{n-1,n}e_{1,n}+x_{1,n-1}x_{1,n}x_{n-1,n}e_{2,n}.$$

As this expression is not identically zero, h is not commuting.

We define Ω_n to be the set $\Omega_n=\{ae_{1,n-1}+be_{1,n}+ce_{2,n} : a,b,c\in R\}$. Ω_n is essential for the conclusion of Theorem 1.1. This can be seen by taking the function f from Example 2.1 with $a=b=0$, yielding a commuting bilinear map whose image is contained in Ω.

For $n\geq 3$ we have $N_n=\begin{pmatrix}0 & R^{n-1}\\ 0 & N_{n-1}\end{pmatrix}$. Thus, we find it convenient to utilize the following notation inspired by [**12**]:

$$A_R := \begin{pmatrix}a_{1,2} & \cdots & a_{1,n}\end{pmatrix} \text{ for all } A=(a_{i,j})\in N_n;$$

$$A_N := \begin{pmatrix}0 & a_{2,3} & a_{2,4} & \cdots & a_{2,n}\\ & 0 & a_{3,4} & \cdots & a_{3,n}\\ & & \ddots & \ddots & \vdots\\ & & & 0 & a_{n-1,n}\\ & & & & 0\end{pmatrix} \text{ for all } A=(a_{i,j})\in N_n;$$

The natural projections $\pi_R : A\mapsto A_R$ and $\pi_N : A\mapsto A_N$ then yield surjective additive maps. Further, we will make use of the following embeddings:

$$\iota_R(\bar{x}) := \begin{pmatrix}0 & \bar{x}\\ 0 & 0_{n-1}\end{pmatrix} \text{ for all } \bar{x}=\begin{pmatrix}x_1 & \cdots & x_{n-1}\end{pmatrix}\in R^{n-1};$$

$$\iota_N(Y) := \begin{pmatrix}0 & \bar{0}\\ 0 & Y\end{pmatrix} \text{ for all } Y\in N_{n-1}.$$

Let G_1, G_2 be additive groups and $f : G_1 \times G_2 \to N_n$ a bilinear map. Then the following maps are all clearly bilinear:

$$
\begin{aligned}
f^{\mathcal{R}}(\bar{x}, \bar{y}) &= f(\iota_R(\bar{x}), \iota_R(\bar{y})) \\
f^{\mathcal{N}}(X, Y) &= f(\iota_N(X), \iota_N(Y)) \\
f^{\mathcal{RN}}(\bar{x}, Y) &= f(\iota_R(\bar{x}), \iota_N(Y)) + f(\iota_N(Y), \iota_R(\bar{x}))
\end{aligned}
$$

Applying the bilinearity of f we have that

$$f(X, X) = f^{\mathcal{R}}(X_R, X_R) + f^{\mathcal{RN}}(X_R, X_N) + f^{\mathcal{N}}(X_N, X_N).$$

Let $f_R(X, X) = \pi_R \circ f(X, X)$ and $f_N(X, X) = \pi_N \circ f(X, X)$. Now define

$$
\begin{aligned}
f_R^{\mathcal{R}} &= \pi_R \circ f^{\mathcal{R}}(X_R, X_R), \\
f_R^{\mathcal{N}} &= \pi_R \circ f^{\mathcal{N}}(X_N, X_N), \\
f_R^{\mathcal{RN}} &= \pi_R \circ f^{\mathcal{RN}}(X_R, X_N),
\end{aligned}
$$

and

$$f_R(X, X) = f_R^{\mathcal{R}}(X_R, X_R) + f_R^{\mathcal{N}}(X_N, X_N) + f_R^{\mathcal{RN}}(X_R, X_N). \tag{2.1}$$

Similarly, define

$$
\begin{aligned}
f_N^{\mathcal{R}} &= \pi_N \circ f^{\mathcal{R}}(X_R, X_R), \\
f_N^{\mathcal{N}} &= \pi_N \circ f^{\mathcal{N}}(X_N, X_N), \\
f_N^{\mathcal{RN}} &= \pi_N \circ f^{\mathcal{RN}}(X_R, X_N),
\end{aligned}
$$

and

$$f_N(X, X) = f_N^{\mathcal{R}}(X_R, X_R) + f_N^{\mathcal{N}}(X_N, X_N) + f_N^{\mathcal{RN}}(X_R, X_N). \tag{2.2}$$

We then have

$$f(X, X) = \begin{pmatrix} 0 & f_R(X, X) \\ 0 & f_N(X, X) \end{pmatrix} \tag{2.3}$$

We prove Theorem 1.3 by examining a bilinear commuting map f in terms of its component functions. We do so by establishing some preliminary results that describe the particular forms and properties these component functions possess, beginning with Lemma 2.3 below.

Lemma 2.3. *Fix $i \in \{1, 2, \ldots, n-1\}$. Let $f : N_n \times N_n \to R$ be a bilinear map such that*

$$x_{i,i+1} f(X, X) = 0$$

for all $X = (x_{i,j}) \in N_n$. Then $f(X, X) = 0$ for all $X \in N_n$.

Proof. A linearization of $x_{i,i+1} f(X, X) = 0$ yields

$$x_{i,i+1} f'(Y, Z) + y_{i,i+1} f'(X, Z) + z_{i,i+1} f'(X, Y) = 0 \tag{2.4}$$

for all $X, Y, Z \in N_n$ where $f'(A, B) = f(A, B) + f(B, A)$. Setting $Y = X$ and $Z = e_{i,i+1}$ in Equation (2.4) we obtain

$$-2x_{i,i+1} f'(X, e_{i,i+1}) = f'(X, X) \tag{2.5}$$

for all $X \in N_n$. Replacing X with $X + e_{i,i+1}$ in Equation (2.5) then yields

$$\begin{aligned} f'(X + e_{i,i+1}, X + e_{i,i+1}) &= -2(x_{i,i+1} + 1)f'(X + e_{i,i+1}, e_{i,i+1}) \\ &= -2(x_{i,i+1} + 1)(f'(X, e_{i,i+1}) + f'(e_{i,i+1}, e_{i,i+1})) \\ &= -2x_{i,i+1}f'(X, e_{i,i+1}) - 2f'(X, e_{i,i+1}) \end{aligned}$$

with the final equality due to the fact that $f(e_{i,i+1}, e_{i,i+1})$ must be 0 by Equation (2.3). Moreover, we have that $f'(X + e_{i,i+1}, X + e_{i,i+1}) = f'(X, X) + 2f'(X, e_{i,i+1})$, hence

$$f'(X, X) + 2f'(X, e_{i,i+1}) = -2x_{i,i+1}f'(X, e_{i,i+1}) - 2f'(X, e_{i,i+1}). \tag{2.6}$$

Combining Equations (2.5) and (2.6) we obtain

$$2f'(X, e_{i,i+1}) = -2f'(X, e_{i,i+1}), \tag{2.7}$$

indicating that $f'(X, e_{i,i+1}) = 0$. In view of Equation (2.5) we then have $f'(X, X) = 0$ and thus $f(X, X) = 0$ for all $X \in N_n$ as desired. □

Lemma 2.4. *Let $f : N_n \times N_n \to R$ be a bilinear map such that*

$$x_{1,2}x_{2,3}f(X, X) = 0$$

for all $X = (x_{i,j}) \in N_n$. Then $f(X, X) = 0$ for all $X \in N_n$.

Proof. Note that if $x_{1,2}x_{2,3}f(X, X) = 0$ for all $X \in N_n$, then

$$f(e_{1,2} + e_{2,3}, e_{1,2} + e_{2,3}) = f(e_{1,2} - e_{2,3}, e_{1,2} - e_{2,3}) = 0$$

must be the case. Define $g : N_n \times N_n \to R$ by $g(X, Y) = x_{1,2}y_{2,3}$. Then g is a bilinear map satisfying

$$g(X, X)f(X, X) = 0 \tag{2.8}$$

for all $X \in N_n$. A linearization of Equation (2.8) gives

$$g'(X, Y)f(X, X) + g(X, X)f'(X, Y) = 0 \tag{2.9}$$

for all $X, Y \in N_n$ where $f'(X, Y) = f(X, Y) + f(Y, X)$ and $g'(X, Y) = g(X, Y) + g(Y, X)$. Setting $Y = e_{2,3}$ in Equation (2.9) and applying the definition of g yields

$$x_{1,2}f(X, X) + x_{1,2}x_{2,3}f'(X, e_{2,3}) = 0 \tag{2.10}$$

for all $X \in N_n$. In particular, when $X = e_{1,2} + e_{2,3}$ we obtain

$$\begin{aligned} 0 &= f(e_{1,2} + e_{2,3}, e_{1,2} + e_{2,3}) + f'(e_{1,2} + e_{2,3}, e_{2,3}) \\ &= f'(e_{1,2}, e_{2,3}) + f'(e_{2,3}, e_{2,3}) \end{aligned} \tag{2.11}$$

Similarly, setting $X = e_{1,2} - e_{2,3}$ in Equation (2.10) yields $f'(e_{1,2}, e_{2,3}) - f'(e_{2,3}, e_{2,3}) = 0$. Combining this with Equation (2.11) we obtain $f'(e_{1,2}, e_{2,3}) = f'(e_{2,3}, e_{2,3}) = 0$. Thus, replacing X with $X + e_{2,3}$ in Equation (2.10) we obtain the following

$$\begin{aligned} 0 &= x_{1,2}f(X + e_{2,3}, X + e_{2,3}) + x_{1,2}(x_{2,3} + 1)f'(X + e_{2,3}, e_{2,3}) \\ &= x_{1,2}f(X, X) + x_{1,2}f'(X, e_{2,3}) + x_{1,2}x_{2,3}f'(X, e_{2,3}) + x_{1,2}f'(X, e_{2,3}) \\ &= x_{1,2}f'(X, e_{2,3}) + x_{1,2}f'(X, e_{2,3}) \\ &= 2x_{1,2}f'(X, e_{2,3}), \end{aligned} \tag{2.12}$$

hence $x_{1,2}f'(X, e_{2,3}) = 0$ for all $X \in N_n$. Next, replacing X with $X + e_{1,2}$ in Equation (2.12) yields

$$\begin{aligned} 0 &= 2(x_{1,2}+1)f'(X + e_{1,2}, e_{2,3}) \\ &= 2x_{1,2}f'(X, e_{2,3}) + 2f'(X, e_{2,3}) \\ &= 2f'(X, e_{2,3}). \end{aligned}$$

Therefore $f'(X, e_{2,3}) = 0$ for all $X \in N_n$. In view of Equation (2.10) this implies that $x_{1,2}f(X,X) = 0$ for all $X \in N_n$. If follows from Lemma 2.3 that $f(X,X) = 0$ for all $X \in N_n$. □

LEMMA 2.5. *Let $f : N_n \times N_n \to R^n$ be a bilinear map such that*

$$f(X,X)X = \bar{0} \tag{2.13}$$

for all $X \in N_n$. Then there exists a map $\eta : N_n \times N_n \to R$ such that $f(X,X) = \eta(X,X)\bar{e}_n$ for all $X \in N_n$.

PROOF. There exist bilinear maps $f_i : N_n \times N_n \to R$, $i = 1, \ldots, n$, such that $f(X,Y) = (f_i(X,Y))$ for all $X, Y \in N_n$. It follows from Equation (2.13) that

$$\sum_{i=1}^{k} f_i(X,X)x_{i,k+1} = 0 \tag{2.14}$$

for all $1 \leq k < n$. In particular, $f_1(X,X)x_{1,2} = 0$ for all $X \in N_n$. Applying Lemma 2.3 we have that $f_1(X,X) = 0$ for all $X \in N_n$. Proceeding by induction, fix k with $1 < k \leq n-1$ and suppose $f_j(X,X) = 0$ for all $1 \leq j \leq k-1$. Then Equation (2.14) becomes $f_k(X,X)x_{k,k+1} = 0$, which indicates that $f_k(X,X) = 0$ for all $X \in N_n$ by again applying Lemma 2.3. Therefore, $f_i(X,X) = 0$ for all $X \in N_n$, $1 \leq i < n$. Taking $\eta = f_n$ we have $f(X,X) = \eta(X,X)\bar{e}_n$ as desired. □

LEMMA 2.6. *Let $f : R^n \times R^n \to N_n$ be a bilinear map such that $f(\bar{x}, \bar{x}) \in Z(N_n)$ for all $\bar{x} \in R^n$. If*

$$\bar{x}f(\bar{x}, \bar{x}) = \bar{0} \tag{2.15}$$

for all $\bar{x} \in R^n$, then $f(\bar{x}, \bar{x}) = \bar{0}$ for all $\bar{x} \in R^n$.

PROOF. By assumption there exists a bilinear map $g : R^n \times R^n \to R$ such that $f(\bar{x}, \bar{x}) = g(\bar{x}, \bar{x})e_{1,n}$ for all $\bar{x} \in R^n$. Equation (2.15) then implies that

$$x_1 g(\bar{x}, \bar{x}) = 0 \tag{2.16}$$

for all $\bar{x} \in R^n$. As an immediate consequence, we see that $g(\bar{e}_1, \bar{e}_1) = 0$. A linearization of Equation (2.16) yields

$$x_1 g'(\bar{y}, \bar{z}) + y_1 g'(\bar{x}, \bar{z}) + z_1 g'(\bar{x}, \bar{y}) = 0 \tag{2.17}$$

for all $\bar{x}, \bar{y}, \bar{z} \in R^n$ where $g'(\bar{x}, \bar{y}) = g(\bar{x}, \bar{y}) + g(\bar{y}, \bar{x})$. Setting $\bar{y} = \bar{x}$ and $\bar{z} = \bar{e}_1$ in Equation (2.17) we obtain

$$-2x_1 g'(\bar{x}, \bar{e}_1) = g'(\bar{x}, \bar{x}). \tag{2.18}$$

Replacing $\bar{x}$ with $\bar{x} + \bar{e}_1$ in Equation (2.18) shows

$$-2(x_1+1)g'(\bar{x}, \bar{e}_1) = g'(\bar{x}, \bar{x}) + 2g'(\bar{x}, \bar{e}_1). \tag{2.19}$$

Combining Equations (2.19) and (2.18) we obtain $-2g'(\bar{x}, \bar{e}_1) = 2g'(\bar{x}, \bar{e}_1)$, implying that $g'(\bar{x}, \bar{e}_1) = 0$. Substituting this into Equation (2.18) yields $0 = g'(\bar{x}, \bar{x}) = 2g(\bar{x}, \bar{x})$, hence $g(\bar{x}, \bar{x}) = 0$ for all $\bar{x} \in R^n$. □

LEMMA 2.7. *Let $n \geq 4$, $\kappa = \{a\bar{e}_{n-2} + b\bar{e}_{n-1} : a, b \in R\} \subset R^{n-1}$ and $f : R^{n-1} \times N_{n-1} \to R^{n-1}$ be a bilinear map such that*

$$f(\bar{x}, Y)Y \in \kappa \tag{2.20}$$

for all $\bar{x} \in R^{n-1}, Y \in N_{n-1}$. Then there exist bilinear maps $g_i : R^{n-1} \times N_{n-1} \to R$, $i = 1, 2, 3$, such that

$$f(\bar{x}, Y) = \begin{pmatrix} 0 & \cdots & 0 & g_1(\bar{x}, Y) & g_2(\bar{x}, Y) & g_3(\bar{x}, Y) \end{pmatrix} \tag{2.21}$$

for all $\bar{x} \in R^{n-1}, Y \in N_{n-1}$.

PROOF. Since f is bilinear there exist bilinear maps $f_i : R^{n-1} \times N_{n-1} \to R$, $i = 1, \ldots, n-1$, such that

$$f(\bar{x}, Y) = (f_1(\bar{x}, Y), \ldots, f_{n-1}(\bar{x}, Y)). \tag{2.22}$$

For each k with $1 \leq k \leq n-4$ it follows from Equation (2.20) that

$$\sum_{i=1}^{k} f_i(\bar{x}, Y) y_{i,k+1} = 0 \tag{2.23}$$

for every $\bar{x} \in R^{n-1}$, $Y = (y_{i,j}) \in N_{n-1}$. When $k = 1$ we have $f_1(\bar{x}, Y)y_{1,2} = 0$ for all $\bar{x}, Y$. In particular, $f_1(\bar{x}, e_{1,2}) = 0$ and so

$$\begin{aligned} 0 &= f_1(\bar{x}, Y + e_{1,2})(y_{1,2} + 1) \\ &= (f_1(\bar{x}, Y) + f_1(\bar{x}, e_{1,2}))\,(y_{1,2} + 1) \\ &= f_1(\bar{x}, Y) \end{aligned}$$

for all $\bar{x} \in R^{n-1}, Y \in N_{r-1}$. Proceeding by induction, fix k with $1 \leq k \leq n-4$ and suppose $f_i(\bar{x}, Y) = 0$ for all $\bar{x}, Y$, and $1 \leq i \leq k-1$. Then Equation (2.23) becomes

$$f_k(\bar{x}, Y) y_{k,k+1} = 0. \tag{2.24}$$

This implies that $f_k(\bar{x}, e_{k,k+1}) = 0$, hence

$$\begin{aligned} 0 &= f_k(\bar{x}, Y + e_{k,k+1})(y_{k,k+1} + 1) \\ &= (f_k(\bar{x}, Y) + f_k(\bar{x}, e_{k,k+1}))\,(y_{k,k+1} + 1) \\ &= f_k(\bar{x}, Y). \end{aligned}$$

Thus, $f_k(\bar{x}, Y) = 0$ for $1 \leq k \leq n-4$. Setting $g_i = f_{n-4+i}$ for $i = 1, 2, 3$ yields the desired conclusion. □

LEMMA 2.8. *Let $f : R^{n-1} \times R^{n-1} \to R^{n-1}$ be a bilinear map such that $f(\bar{x}, \bar{x})Y \in \kappa$ for all $\bar{x} \in R^{n-1}$ and $Y \in N_{n-1}$ where $\kappa = \{a\bar{e}_{n-2} + b\bar{e}_{n-1} : a, b \in R\}$. Then there exist bilinear maps $g_i : R^{n-1} \times R^{n-1} \to R$, $i = 1, 2$, such that*

$$f(\bar{x}, \bar{x}) = \begin{pmatrix} 0 & \cdots & 0 & g_1(\bar{x}, \bar{x}) & g_2(\bar{x}, \bar{x}) \end{pmatrix} \tag{2.25}$$

for all $\bar{x} \in R^{n-1}$.

PROOF. There exist bilinear maps $f_i : R^{n-1} \times R^{n-1} \to R$, $i = 1, \dots, n-1$, such that $f(\bar{x}, \bar{x}) = (f_1(\bar{x}, \bar{x}), \dots, f_{n-1}(\bar{x}, \bar{x}))$. Taking $Y = \sum_{i=1}^{n-2} e_{i,i+1}$ we have

$$f(\bar{x}, \bar{x})Y \in \kappa \tag{2.26}$$

for every $\bar{x} \in R^{n-1}$. This implies that $f_i(\bar{x}, \bar{x}) = 0$ for each $1 \le i < n-3$. Simply set $g_i = f_{n-4+i}$ for $i = 1, 2$ to complete the proof.

□

LEMMA 2.9. *Let $f : N_n \times N_n \to \Omega'$, $n > 4$, be a bilinear map satisfying $[f(X, X), X] = 0_n$ for all $X \in N_n$ where*

$$\Omega' := \{ae_{1,n-2} + be_{1,n-1} + ce_{1,n} + de_{2,n-1} + fe_{2,n} + ge_{3,n} : a, b, c, d, f, g \in R\}.$$

Then $f(X, X) \in \Omega_n$ for all $X \in N_n$.

PROOF. Let $I = \{(1, n-2), (1, n-1), (1, n), (2, n-1), (2, n), (3, n)\}$. There exist bilinear maps $f_{i,j} : N_n \times N_n \to R$, $(i, j) \in I$, such that

$$f(X, X) = \begin{pmatrix} 0 & \cdots & 0 & f_{1,n-2}(X,X) & f_{1,n-1}(X,X) & f_{1,n}(X,X) \\ & & & 0 & f_{2,n-1}(X,X) & f_{2,n}(X,X) \\ & & & & 0 & f_{3,n}(X,X) \\ & & & \ddots & & 0 \\ & & & & & \vdots \\ & & & & & 0 \end{pmatrix}.$$

As $[f(X, X), X] = 0_n$, we have

$$x_{1,2} f_{2,n-1}(X, X) = f_{1,n-2}(X, X) x_{n-2,n-1}, \tag{2.27}$$

$$x_{2,3} f_{3,n}(X, X) = f_{2,n-1}(X, X) x_{n-1,n}, \tag{2.28}$$

and

$$f_{2,n}(X, X)x_{1,2} + f_{3,n}(X, X)x_{1,3} - x_{n-2,n} f_{1,n-2}(X, X) - x_{n-1,n} f_{1,n-1}(X, X) = 0 \tag{2.29}$$

for all $X \in N_n$. From Equation (2.27) we can find linear maps $p, p' : N_n \to R$ such that $f_{1,n-2}(X, X) = x_{1,2}p(X)$ and $f_{2,n-1}(X, X) = p'(X)x_{n-2,n-1}$, hence

$$x_{1,2}(p(X) - p'(X))x_{n-2,n-1} = 0 \tag{2.30}$$

for all $X \in N_n$. Taking B to be the bilinear map defined by $B(X, Y) = (p(X) - p'(X))y_{n-2,n-1}$, we have $x_{1,2}B(X, X) = 0$ for all $X \in N_n$. Applying Lemma 2.3 we obtain

$$0 = B(X, X) = (p(X) - p'(X))x_{n-2,n-1}. \tag{2.31}$$

Setting $X = e_{n-2,n-1}$ in the above equation, it becomes clear that $p(e_{n-2,n-1}) = p'(e_{n-2,n-1})$. Thus, replacing X with $X + e_{n-2,n-1}$ in Equation (2.31) yields

$$\begin{aligned} 0 &= (p(X + e_{n-2,n-1}) - p'(X + e_{n-2,n-1}))(x_{n-2,n-1} + 1) \\ &= (p(X) - p'(X))x_{n-2,n-1} + p(X) - p'(X) \\ &= p(X) - p'(X). \end{aligned}$$

Therefore, $p(X) = p'(X)$ for all $X \in N_n$.

Applying a similar argument to $f_{3,n}$ and $f_{2,n-1}$ using Equation (2.28), we obtain a linear map $q : N_n \to R$ such that $f_{3,n}(X,X) = q(X)x_{n-1,n}$ and $f_{2,n-1}(X,X) = x_{2,3}q(X)$. Equating our two expressions for $f_{2,n-1}$ yields

$$p(X)x_{n-2,n-1} = x_{2,3}q(X),$$

thus there must be some $a \in R$ such that $p(X) = x_{2,3}a$ and $q(X) = ax_{n-2,n-1}$. We can then conclude that

$$\begin{aligned} f_{1,n-2}(X,X) &= x_{1,2}x_{2,3}a \\ f_{2,n-1}(X,X) &= x_{2,3}ax_{n-2,n-1} \\ f_{3,n}(X,X) &= ax_{n-2,n-1}x_{n-1,n} \end{aligned}$$

for all $X \in N_n$.

Applying these expressions to Equation (2.29) yields

$$f_{2,n}(X,X)x_{1,2} + ax_{n-2,n-1}x_{n-1,n}x_{1,3} - x_{n-2,n}x_{1,2}x_{2,3}a - x_{n-1,n}f_{1,n-1}(X,X) = 0. \tag{2.32}$$

Note that $f_{1,n-1}(e_{n-1,n}, e_{n-1,n}) = 0$ must be the case.

Taking $X \in N_n$ with $x_{1,2} = 0$ we obtain

$$ax_{n-2,n-1}x_{n-1,n}x_{1,3} - x_{n-1,n}f_{1,n-1}(X,X) = 0. \tag{2.33}$$

Replacing X with $X + e_{n-1,n}$ Equation (2.33) becomes

$$\begin{aligned} 0 &= ax_{n-2,n-1}(x_{n-1,n}+1)x_{1,3} - x_{n-1,n}f_{1,n-1}(X+e_{n-1,n}, X+e_{n-1,n}) \\ &= ax_{n-2,n-1}x_{1,3} - x_{n-1,n}(f_{1,n-1}(X,e_{n-1,n}) + f_{1,n-1}(e_{n-1,n},X)). \end{aligned}$$

Taking $X \in N_n$ with $x_{n-2,n-1} = x_{1,3} = x_{n-1,n} = 1$, we obtain

$$a = f_{1,n-1}(X,e_{n-1,n}) + f_{1,n-1}(e_{n-1,n},X) \tag{2.34}$$

Replacing X with $2X$ in Equation (2.34) yields

$$\begin{aligned} a &= f_{1,n-1}(2X,e_{n-1,n}) + f_{1,n-1}(e_{n-1,n},2X) \\ &= 2(f_{1,n-1}(X,e_{n-1,n}) + f_{1,n-1}(e_{n-1,n},X)) \\ &= 2a. \end{aligned}$$

Thus, $a = 0$ must be the case. □

3. Bilinear maps on $N_n \times N_n$

We now prove Theorem 1.3.

Proof of Theorem 1.3. We proceed by induction on n. The case of $n = 3$ is trivially true, so fix $n > 3$ and suppose that any bilinear commuting map $g : N_{n-1} \times N_{n-1} \to N_{n-1}$ can be written as $g(X,X) = \lambda(X,X)X + \mu(X,X)$ for some bilinear $\lambda : N_{n-1} \times N_{n-1} \to R$ and $\mu : N_{n-1} \times N_{n-1} \to \Omega_{n-1}$.

Suppose $f : N_n \times N_n \to N_n$ is a bilinear map such that

$$[f(X,X),X] = 0_n \tag{3.1}$$

for all $X \in N_n$. Then we have that

$$[f_N(X,X),X_N] = 0_{n-1} \tag{3.2}$$

and

$$f_R(X,X)X_N = X_R f_N(X,X) \tag{3.3}$$

We obtain a description of f by first exploring the structures of $f_N = f_N^{\mathcal{R}} + f_N^{\mathcal{N}} + f_N^{\mathcal{RN}}$ and $f_R = f_R^{\mathcal{R}} + f_R^{\mathcal{N}} + f_R^{\mathcal{RN}}$.

3.1. The map $f_N^{\mathcal{N}}$.

Now, for $Y \in N_{n-1}$, it follows from Equation (3.2) that

$$0_{n-1} = [f_N(\iota_N(Y), \iota_N(Y)), Y] = [f_N^{\mathcal{N}}(Y,Y), Y],$$

hence $f_N^{\mathcal{N}} : N_{n-1} \times N_{r-1} \to N_{n-1}$ is a commuting map. Applying the inductive hypothesis, we obtain a bilinear map $\lambda : N_{n-1} \times N_{n-1} \to Z(R)$ and a biadditive map $\mu : N_{n-1} \times N_{n-1} \to \Omega$ such that

$$f_N^{\mathcal{N}}(Y,Y) = \lambda_{n-1}(Y,Y)Y + \mu_{n-1}(Y,Y) \tag{3.4}$$

for all $Y \in N_{n-1}$.

3.2. The map $f_N^{\mathcal{RN}}$.

Combining Equations (2.2) and (3.2) with the fact that $f_N^{\mathcal{N}}$ is commuting yields

$$\begin{aligned} 0_{n-1} &= [f_N^{\mathcal{N}}(X_N, X_N), X_N] + [f_N^{\mathcal{R}}(X_R, X_R), X_N] + [f_N^{\mathcal{RN}}(X_R, X_N), X_N] \\ &= [f_N^{\mathcal{R}}(X_R, X_R), X_N] + [f_N^{\mathcal{RN}}(X_R, X_N), X_N] \end{aligned} \tag{3.5}$$

for all $X \in N_n$. Let $\bar{x} \in R^{n-1}$ and $Y \in N_{n-1}$. Setting $X = \iota_R(\bar{x}) + \iota_N(Y)$ in Equation (3.5) reveals

$$0_{n-1} = [f_N^{\mathcal{R}}(\bar{x}, \bar{x}), Y] + [f_N^{\mathcal{RN}}(\bar{x}, Y), Y] \tag{3.6}$$

Similarly, setting $X = \iota_R(\bar{x}) - \iota_N(Y)$ in Equation (3.5) reveals

$$0_{n-1} = -[f_N^{\mathcal{R}}(\bar{x}, \bar{x}), Y] + [f_N^{\mathcal{RN}}(\bar{x}, Y), Y] \tag{3.7}$$

The sum of Equations (3.6) and (3.7) shows that $[f_N^{\mathcal{RN}}(\bar{x}, Y), Y] = 0_{n-1}$, thus for each $\bar{x} \in R^{n-1}$ the map $Y \mapsto f_N^{\mathcal{RN}}(\bar{x}, Y)$ is a linear commuting map from $N_{n-1} \to N_{n-1}$. Applying Theorem 1.2 we obtain $\lambda_{\bar{x}} \in Z(R)$ and additive $\mu_{\bar{x}} : N_{n-1} \to \Omega_{n-1}$ such that $f_N^{\mathcal{RN}}(\bar{x}, Y) = \lambda_{\bar{x}} Y + \mu_{\bar{x}}(Y)$ for all $Y \in N_{n-1}$. Hence, there exist maps $\lambda : R^{n-1} \to Z(R)$ and $\mu : R^{n-1} \times N_{n-1} \to \Omega_{n-1}$ such that

$$f_N^{\mathcal{RN}}(\bar{x}, Y) = \lambda(\bar{x})Y + \mu(\bar{x}, Y). \tag{3.8}$$

3.3. The map $f_N^{\mathcal{R}}$.

The difference of Equations (3.6) and (3.7) shows that $[f_N^{\mathcal{R}}(\bar{x},\bar{x}), Y] = 0_{n-1}$. Thus for each $\bar{x} \in R^{n-1}$, $f_N^{\mathcal{R}}(\bar{x},\bar{x})$ commutes with every $Y \in N_{n-1}$. As a result, $f_N^{\mathcal{R}}(\bar{x},\bar{x}) \in Z(N_{n-1})$ for every $\bar{x} \in R^{n-1}$. Setting $Y = 0_{n-1}$ in Equation (3.12) and applying Equation (2.2) we obtain

$$\bar{0} = \bar{x} f_N(\iota_R(\bar{x}), \iota_R(\bar{x})) = \bar{x} f_N^{\mathcal{R}}(\bar{x},\bar{x}). \tag{3.9}$$

As $f_N^{\mathcal{R}}(\bar{x},\bar{x}) \in Z(N_{n-1})$ for every $\bar{x} \in R^{n-1}$, Lemma 2.6 indicates that $f_N^{\mathcal{R}}(\bar{x},\bar{x}) = \bar{0}$ for all $\bar{x} \in R^{n-1}$.

We can now determine the full structure of the map f_N. Combining Equations (3.4), (3.8), (3.9), and (2.2), we conclude that

$$f_N(X,X) = \lambda_{n-1}(X_N,X_N)X_N + \mu_{n-1}(X_N,X_N) + \lambda(X_R)X_N + \mu(X_R,X_N) \tag{3.10}$$

Defining the maps $\lambda' : N_n \times N_n \to R$ and $\mu' : N_n \times N_n \to \Omega_{n-1}$ as

$$\lambda'(X,X) = \lambda_{n-1}(X_N,X_N) + \lambda(X_R)$$

and

$$\mu'(X,X) = \mu_{n-1}(X_N,X_N) + \mu(X_R,X_N),$$

Equation (3.10) becomes

$$f_N(X,X) = \lambda'(X,X)X_N + \mu'(X,X) \tag{3.11}$$

for every $X \in N_n$.

3.4. The map $f_R^{\mathcal{N}}$.

For $\bar{x} \in R^{n-1}$ and $Y \in N_{n-1}$, set $X = \iota_R(\bar{x}) + \iota_N(Y)$ in Equation (3.3) to obtain

$$f_R(\iota_R(\bar{x}) + \iota_N(Y), \iota_R(\bar{x}) + \iota_N(Y))Y = \bar{x} f_N(\iota_R(\bar{x}) + \iota_N(Y), \iota_R(\bar{x}) + \iota_N(Y)). \tag{3.12}$$

If $\bar{x} = 0$ in Equation (3.12), then applying Equation (2.1) yields

$$f_R^{\mathcal{N}}(Y,Y)Y = f_R(\iota_N(Y), \iota_N(Y))Y = \bar{0} \tag{3.13}$$

for all $Y \in N_{n-1}$. Applying Lemma 2.5 we obtain $\eta : N_{n-1} \times N_{n-1} \to R$ such that $f_R^{\mathcal{N}}(Y,Y) = \eta(Y,Y)\bar{e}_{n-1}$ for all $Y \in N_{n-1}$.

3.5. The map $f_R^{\mathcal{RN}}$.

Next, we combine Equations (3.3), (2.1), and (2.2) along with rearranging the terms to obtain

$$\bar{0} = f_R^{\mathcal{R}}(\bar{x},\bar{x})Y + f_R^{\mathcal{RN}}(\bar{x},Y)Y - \bar{x} f_N^{\mathcal{N}}(Y,Y) - \bar{x} f_N^{\mathcal{RN}}(\bar{x},Y). \tag{3.14}$$

Replacing Y with $-Y$ in the above line then yields

$$\bar{0} = -f_R^{\mathcal{R}}(\bar{x},\bar{x})Y + f_R^{\mathcal{RN}}(\bar{x},Y)Y - \bar{x} f_N^{\mathcal{N}}(Y,Y) + \bar{x} f_N^{\mathcal{RN}}(\bar{x},Y). \tag{3.15}$$

Taking the sum of Equations (3.14) and (3.15) reveals $2f_R^{\mathcal{RN}}(\bar{x},Y)Y - 2\bar{x} f_N^{\mathcal{N}}(Y,Y) = \bar{0}$, hence

$$f_R^{\mathcal{RN}}(\bar{x},Y)Y = \bar{x} f_N^{\mathcal{N}}(Y,Y). \tag{3.16}$$

Similarly, the difference of Equations (3.14) and (3.15) yields $2f_R^{\mathcal{R}}(\bar{x},\bar{x})Y - 2\bar{x}f_N^{\mathcal{RN}}(\bar{x},Y) = \bar{0}$, implying that

$$f_R^{\mathcal{R}}(\bar{x},\bar{x})Y = \bar{x}f_N^{\mathcal{RN}}(\bar{x},Y). \tag{3.17}$$

Utilizing Equation (3.4) and rearranging the terms slightly, Equation (3.16) becomes

$$\left(f_R^{\mathcal{RN}}(X_R,X_N) - \lambda_{n-1}(X_N,X_N)X_R\right)X_N = X_R\mu_{n-1}(X_N,X_N). \tag{3.18}$$

Noting that $X_R\mu_{n-1}(X_N,X_N) \in \kappa$ for all $X \in N_n$, it follows from Lemma 2.7 that there exists a bilinear map $\nu_1 : R^{n-1} \times N_{n-1} \to \kappa$ such that $f_R^{\mathcal{RN}}(X_R,X_N) - \lambda_{n-1}(X_N,X_N)X_R = \nu_1(X_R,X_N)$. Therefore,

$$f_R^{\mathcal{RN}}(X_R,X_N) = \lambda_{n-1}(X_N,X_N)X_R + \nu_1(X_R,X_N) \tag{3.19}$$

for all $X \in N_n$.

3.6. The map $f_R^{\mathcal{R}}$.

Utilizing Equation (3.8) and rearranging the terms in a similar manner, Equation (3.17) becomes

$$\left(f_R^{\mathcal{R}}(X_R,X_R) - \lambda(X_R,X_R)X_R\right)X_N = X_R\mu(X_R,X_N). \tag{3.20}$$

This time applying Lemma 2.8 we obtain

$$f_R^{\mathcal{R}}(X_R,X_R) = \lambda(X_R,X_R)X_R + \nu_2(X_R,X_N) \tag{3.21}$$

for some bilinear map $\nu_2 : R^{n-1} \times N_{n-1} \to \kappa$.

Finally, we obtain the structure for f_R. Combining Equations (2.1), (3.19), and (3.21) with the previously established fact that $f_R^{\mathcal{N}}(X_N,X_N) = 0$ for all $X \in N_n$, we have

$$f_R(X,X) = \lambda'(X_R,X_N)X_R + \nu(X_R,X_N) \tag{3.22}$$

where $\nu(X_R,X_N) = \nu_1(X_R,X_N) + \nu_2(X_R,X_N)$. Thus Equation (2.3) becomes

$$f(X,X) = \begin{pmatrix} 0 & \lambda'(X_R,X_N)X_R + \nu(X_R,X_N) \\ 0 & \lambda'(X_R,X_N)X_N + \mu'(X_R,X_N) \end{pmatrix} = \lambda'(X_R,X_N)X + \begin{pmatrix} 0 & \nu(X_R,X_N) \\ 0 & \mu'(X_R,X_N) \end{pmatrix}. \tag{3.23}$$

Defining $\mu'' : N_n \times N_n \to N_n$ by $\mu''(X,Y) = \begin{pmatrix} 0 & \nu(X_R,Y_N) \\ 0 & \mu'(X_R,Y_N) \end{pmatrix}$, we have that μ'' is a commuting bilinear map whose image is contained in Ω'. Thus, by Lemma 2.9 $\mu''(X,X) \in \Omega$ for every $X \in N_n$. Therefore, we have

$$f(X,X) = \lambda''(X,X)X + \mu''(X,X)$$

where $\lambda''(X,Y) = \lambda'(X_R,Y_N)$.

□

We believe the conclusion of Theorem 1.3 will extend to m-linear commuting maps on N_n^m for $m > 2$.

Conjecture 3.1. Let $m \geq 1$ be an integer and $f : N_n^m \to N_n$ an m-linear map such that $[f(X,\ldots,X),X] = 0$ for all $X \in N_n$. Then there exist an m-linear map $\lambda : N_n^m \to N_n$ and an m-additive map $\mu : N_n \to \Omega$ such that

$$f(X,\ldots,X) = \lambda(X,\ldots,X)X + \mu(X,\ldots,X).$$

Given the similarities between Theorem 1.3 and the results obtained by Beidar, Brešar, and Chebotar in the upper triangular case [**1**], Conjecture 3.1 seems like a natural conclusion. However, the linearization process applied throughout our proofs in the present paper leads to notably more complex equations as the number of inputs increases, suggesting an alternative approach might be necessary.

4. The map μ

We now turn our attention to the map μ present in the statement of Theorem 1.3. Suppose $f : N_n \times N_n$ is a bilinear commuting map. Applying Theorem 1.3, we obtain bilinear $\lambda : N_n \times N_n \to Z(N_n)$ and biadditive $\mu : N_n \times N_n \to \Omega$ satisfying

$$f(X,X) - \lambda(X,X)X = \mu(X,X)$$

for all $X \in N_n$, hence μ must be bilinear. Moreover, the commuting assumption on f implies that

$$\begin{aligned} 0_n &= [f(X,X), X] \\ &= [\lambda(X,X)X, X] + [\mu(X,X), X] \\ &= [\mu(X,X), X], \end{aligned}$$

meaning μ must also be commuting, naturally leading one to consider if maps of this particular form possess some common structure.

Suppose $p : N_n \to R$ is linear and $\zeta : N_n \times N_n \to Z(N_n)$ is bilinear. Then the map $h : N_n \times N_n \to N_n$ defined by

$$h(X,Y) = x_{1,2}p(X)e_{1,n-1} + p(Y)y_{n-1,n}e_{2,n} + \zeta(X,Y)$$

is bilinear and satisfies

$$\begin{aligned} [h(X,X), X] &= (x_{1,2}p(X)x_{n-1,n} - x_{1,2}p(X)x_{n-1,n})e_{1,n} \\ &= 0_n. \end{aligned}$$

We claim that μ must be of this form.

THEOREM 4.1. *Let $\mu : N_n \times N_n \to \Omega$ be a bilinear commuting map. Then there exist a linear map $p : N_n \to R$ and a bilinear map $\zeta : N_n \times N_n \to Z(N_n)$ such that*

$$\mu(X,X) = x_{1,2}p(X)e_{1,n-1} + p(X)x_{n-1,n}e_{2,n} + \zeta(X).$$

PROOF. There exist bilinear maps $\mu_1, \mu_2, \mu_3 : N_n \times N_n \to R$ such that

$$\mu(X,X) = \mu_1(X,X)e_{1,n-1} + \mu_2(X,X)e_{1,n} + \mu_3(X,X)e_{2,n}$$

for all $X \in N_n$. Since μ is commuting by assumption, it follows that

$$\mu_1(X,X)x_{n-1,n} = x_{1,2}\mu_3(X,X) \tag{4.1}$$

for all $X \in N_n$. Thus, there exist linear maps $p, p' : N_n \to R$ such that $\mu_1(X,X) = x_{1,2}p(X)$ and $\mu_3(X,X) = p'(X)x_{n-1,n}$.

Consider the linear map $q(X) = p(X) - p'(X)$. Then Equation (4.1) implies

$$x_{1,2}q(X)x_{n-1,n} = 0. \tag{4.2}$$

Clearly $q(e_{1,2}+e_{n-1,n})=0$. Therefore,

$$\begin{aligned}0 &= (x_{1,2}+1)q(X+e_{1,2}+e_{n-1,n})(x_{n-1,n}+1)\\ &= (x_{1,2}+1)q(X)(x_{n-1,n}+1)\\ &= x_{1,2}q(X)+q(X)x_{n-1,n}+q(X)\end{aligned}$$

for every $X \in N_n$. Replacing X with $X+e_{1,2}+e_{n-1,n}$ yields

$$\begin{aligned}0 &= (x_{1,2}+1)q(X+e_{1,2}+e_{n-1,n})+q(X+e_{1,2}+e_{n-1,n})(x_{n-1,n}+1)+q(X+e_{1,2}+e_{n-1,n})\\ &= x_{1,2}q(X)+q(X)+q(X)x_{n-1,n}+q(X)+q(X)\\ &= 2q(X).\end{aligned}$$

Thus, $q(X)=0$ and we conclude that $p(X)=p'(X)$ for all $X \in N_n$. Taking $\zeta=\mu_2$ we have the desired result. $\square$

Corollary 1.4 is an immediate consequence.

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