Automata for Coxeter Groups

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CONTENTS

Abstract

In this thesis, we present and compare two methods for constructing finite state automata for Coxeter groups, the Cannon and Brink-Howlett automata. We begin by introducing the algebraic and geometric viewpoints of Coxeter groups and discuss a few motivating examples. The standard geometric representation for Coxeter groups is developed, which leads to a discussion of the root system and relevant combinatorial properties. We discuss and present a proof of the key combinatorial feature leading to the finite state automata: the finiteness of a subset of the root system, the small roots.

This leads to the construction of finite state automata using the Brink-Howlett and Cannon algorithms. Automata using both algorithms are presented and compared for two affine Coxeter groups. It is shown that the Cannon automaton is minimal.

Acknowledgments

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Introduction

Coxeter Groups are abstract reflection groups. Whilst the presentation of these groups appear specialised, Coxeter groups are prevalent in many areas of mathematics. For example, they are the symmetry groups of regular polytopes and are the Weyl groups associated with Kac-Moody Lie algebras [**15**].

On one hand, Coxeter groups can be viewed in a straight forward algebraic manner via a group presentation consisting of generators and relations. On the other, they can be realised as a subgroup of linear transformations on a vector space, generated by reflections through hyperplanes with respect to a symmetric bilinear form [**12**]. Chapters 1 and 2 aim to present the connection between these two points of view.

The algebraic definition of Coxeter groups, although economical, gives notoriously little information about the nature of its elements. For example, given an arbitrary finite string of generators, is it possible to know whether the expression is "reduced"?

In 1993, members of the University of Sydney School of Mathematics and Statistics, Brigitte Brink and Professor Bob Howlett produced seminal work in the area of Coxeter groups which precisely answered the above question. They proved: *For every Coxeter group, there exists a finite state automaton which recognises the language of reduced words*. Informally, in this context, a *finite state automaton* is a finite directed graph capable of "reading" *words* (strings of generators) and giving an answer of "Yes" if and only if the group element represented by the word cannot be expressed with a shorter string of generators.

The central aim of this thesis is to thoroughly present a self-contained treatment of this result as well as give explicit examples of automaton for a few selected Coxeter groups.

Chapters 3-5 develop the preliminary notions required to construct a finite state automaton. Chapter 6 presents the Brink-Howlett algorithm and a second approach, termed the Cannon algorithm to construct finite state automaton. The Cannon algorithm is discussed in [**9**] with the genesis of the idea originating from [**5**]. We apply it in a geometric way to Coxeter groups. Examples using both approaches are presented, with calculations provided in the appendices.

CHAPTER 1

The Coxeter System

In this chapter we introduce Coxeter groups from an algebraic point of view and a geometric point of view. We give examples of this interesting class of groups and discuss the basic algebraic notions. The primary references for this chapter are [**15**] and [**17**].

We begin the algebraic point of view with the formal definition of a Coxeter system.

1.1. Definition. A *Coxeter system* is a pair (W, S) consisting of a group W and a set of generators $S \subset W$ subject only to relations of the form

$$(st)^{m(s,t)} = 1$$

where $m(s,s) = 1$ for all $s \in S$ and $m(t,s) = m(s,t) \geq 2$ for $s \neq t$. Note that $s^2 = 1$ and so $s^{-1} = s$ for all $s \in S$. We allow $m(s,t) = \infty$, in which case we consider there to be no relations between s and t.

We refer to W as the *Coxeter group* and the set S the *Coxeter generators*. The *rank* of W is $|S|$. Throughout, it will be understood that the cardinality of S is finite.

A Coxeter system can also be described with an undirected graph called a *Coxeter graph*. The Coxeter generators S form the vertex set and there is an edge between vertices s and s' if $m(s,s') \geq 3$. The edge is labeled $m(s,s')$ if $m(s,s') \geq 4$.

Geometrically, Coxeter groups can be thought of as "abstract reflection groups" or groups generated by involutions. In fact, the finite Coxeter groups are the finite reflection groups [**6**]. In view of developing finite state automata for Coxeter groups, we will be most interested in infinite Coxeter groups. In particular, we will become familiar with the "affine" Coxeter groups, which can be thought of as reflection groups induced by tesselations of Euclidean space [**17**].

1.1. Examples of finite Coxeter Groups

1.2. Example (Dihedral groups)**.** The geometric view of dihedral groups are familiar. Let α_s, α_t be two lines in $\mathbb{R}^2$ through the origin with angle $\theta = \frac{\pi}{m}$ between them. If s and t are the orthogonal reflections across α_s and α_t respectively, then st is a rotation of $\frac{2\pi}{m}$ about the origin. Thus st has order m.

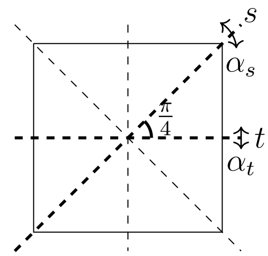
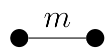


FIGURE 1. Example: The Dihedral group D_8

Hence, the dihedral groups D_{2m} have algebraic presentation:

$$D_{2m} = \langle s, t \mid s^2 = t^2 = (st)^m = 1 \rangle$$

with Coxeter graph:



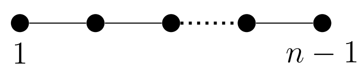
So the dihedral groups are Coxeter groups.

1.3. Remark. It will be of use later to have explicit matrices for the reflections s and t, as well as a general formula for $(st)^k$ for $k \in \mathbb{N}$ with respect to taking the lines α_s and α_t as a basis for $\mathbb{R}^2$. These are presented in Appendix C1.

1.4. Example (Symmetric groups)**.** Let S_n be the symmetric group on n letters and let s_i be the transposition $(i, i+1)$ for $1 \leq i \leq n-1$. Then S_n is generated by $S = \{ s_i \mid 1 \leq i \leq n-1\}$ and has the following presentation:

$$S_n : \langle s_1, \cdots, s_{n-1} \mid s_i^2 = 1, (s_i s_{i+1})^3 = 1, (s_i s_j)^2 = 1 \text{ for } |i-j| > 1 \rangle.$$

The symmetric groups S_n have Coxeter graph:



The relations $(s_i s_{i+1})^3 = 1$ and $(s_i s_j)^2 = 1$ for $|i-j| > 1$ are easily seen using a permutation diagram as in Figure 2. The fact that S_n is generated by s_i can be seen as follows. First note the elementary fact that every element of S_n can be written as a product of transpositions ([**19**] p16). Then let $(n\, m)$ be an arbitrary transposition with $n < m$. Then it is not difficult to see that $(n\, m)$ can be written:

$$(n\, m) = (n\, n+1) \cdots (m-2\, m-1)(m-1\, m) \cdots (n+1\, n+2)(n\, n+1)$$

Hence, $\{s_i \mid 1 \leq i \leq n-1\}$ generates S_n.

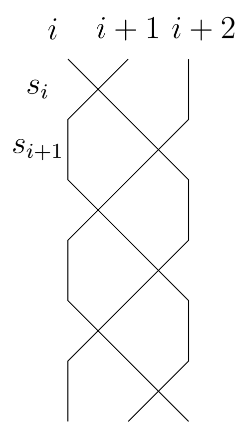


FIGURE 2. Permutation diagram for $(s_i s_{i+1})^3$

1.2. Examples of affine Coxeter Groups

In this section we give examples of two classes of affine Coxeter groups of rank 3, $\tilde{A}_2$ and $\tilde{B}_2$. The group $\tilde{A}_2$ will be a main example throughout this thesis. In Chapter 6, we present finite state automata for both of these groups.

1.5. Definition. A Coxeter group W is *affine* if it is infinite and contains a normal abelian subgroup Q, such that W/Q is a finite group.

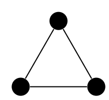
We note that the precise definition of affine Coxeter groups is not required for building automata for these groups, since the data we require is sufficiently contained in the group presentation. For our purposes, we may understand these groups less formally as those which arise from suitably defined reflections in affine space ([**3**] p8).

1.6. Example. $\tilde{A}_2$

The Coxeter group of type $\tilde{A}_2$ has the following presentation:

$$\langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (su)^3 = 1\rangle$$

with Coxeter graph



From the less formal perspective described above, $\tilde{A}_2$ acts naturally as reflections in a geometric object known as a *Coxeter complex*, part of which is illustrated in Figure 3. We briefly discuss the Coxeter complex in Chapter 6. For now, a short description should provide some intuition. Consider the triangle (or "chamber") containing the identity e below. The generator u corresponds to a reflection in the "wall" H_u bounding e, t corresponds to reflection in the H_t wall to the left of e, and s similarly acts as reflection in H_s. With rearrangement of the H_i, the Coxeter generators act in the same way on other chambers in the complex.

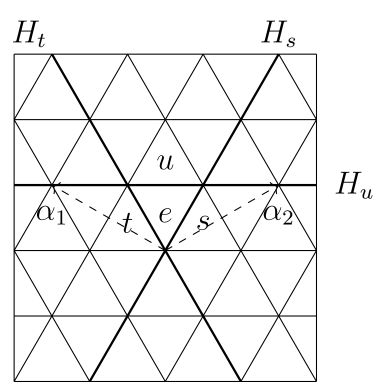


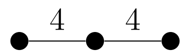
FIGURE 3. Coxeter complex for $\tilde{A}_2$

We have $\tilde{A}_2 = Q \rtimes W_0$, where $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is the normal abelian subgroup and $W_0 = \langle s, t\rangle$. Therefore $\tilde{A}_2$ is affine.

1.7. Example ($\tilde{B}_2$)**.** The Coxeter group of type $\tilde{B}_2$ has presentation:

$$\langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^4 = (tu)^4 = (su)^2 = 1 \rangle$$

and Coxeter graph



As in the example of $\tilde{A}_2$, the generators s, t, u correspond to reflections in the adjacent walls of e illustrated in Figure 4.

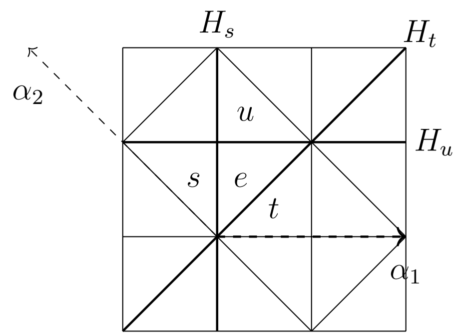
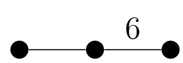


FIGURE 4. Coxeter complex for $\tilde{B}_2$

Similar to the previous example, $\tilde{B}_2 = Q \rtimes W_0$, where $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is the normal abelian subgroup and $W_0 = \langle s, t \rangle$.

There is in fact only three distinct types of affine irreducible Coxeter groups of rank 3. So we briefly mention the remaining affine Coxeter group of rank 3 for completeness. We say a Coxeter system is *irreducible* if its Coxeter diagram is connected. The notion of irreducibility will not be required for this thesis.

1.8. Example ($\tilde{G}_2$)**.** The Coxeter group $\tilde{G}_2$ has Coxeter diagram



with Coxeter complex

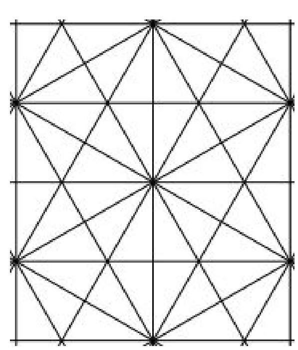
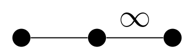


FIGURE 5. Coxeter complex for $\tilde{G}_2$

The group $\tilde{G}_2$ is affine, with $\tilde{G}_2 = Q \rtimes W_0$, where $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is the normal abelian subgroup and $W_0 = \langle t, u \rangle$.

We give one final example of a Coxeter group which is not affine, but hyperbolic. We include this example for interest and its marvelous geometric image. It suffices to understand hyperbolic Coxeter groups as reflection groups induced by tesselations of hyperbolic space.

1.9. Example. The hyperbolic Coxeter group with Coxeter diagram



Has the following geometric interpretation.

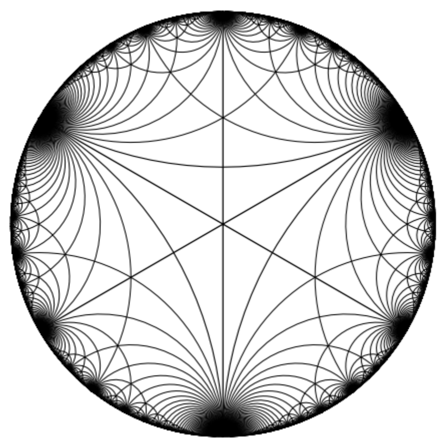


FIGURE 6. Geometric image from [**3**] page 9.

1.3. The Basics

In this section, we introduce the very basic notion of the *length* of an element w in a Coxeter system (W, S). This allows us to study the ways in which an element $w \in W$ can be written as a product of the Coxeter generators $s_i \in S$. In Chapter 3, we will see a very nice geometric characterisation of this notion. The primary reference here is [**15**].

1.10. Definition. Let (W, S) be a Coxeter system and $w \in W$. The *length* of w, $\ell(w)$ is the smallest integer k such that $w = s_1 s_2 \cdots s_k$. Equivalently,

$$\ell(w) = \min\{k \geq 0 \mid w = s_1 \cdots s_k \text{ with } s_i \in S\}.$$

We call any expression of w as a product of $\ell(w)$ elements $s_i \in S$ a *reduced expression* for w. In general, reduced expressions are not unique. For example, in the Coxeter group $\tilde{A}_2$, $sts = tst$ are both reduced expressions. We have $\ell(1) = 0$, where 1 is the identity of W, and $\ell(s) = 1$ for all $s \in S$.

1.11. Proposition. *Properties of the length function:*

(L1) $\ell(w) = \ell(w^{-1})$ *for all* $w \in W$
(L2) $\ell(w) = 1 \iff w \in S$
(L3) $\ell(ww') \leq \ell(w) + \ell(w')$ *for all* $w \in W$

(L4) $\ell(ww') \geq \ell(w) - \ell(w')$ *for all* $w \in W$
(L5) $\ell(w) - 1 \leq \ell(ws) \leq l(w) + 1$ *for* $s \in S, w \in W$

Proof. (Adapted from [**15**] p108). (L1) If $w = s_1 s_2 \cdots s_k$ is a reduced expression, then $w^{-1} = s_k \cdots s_2 s_1$ so $\ell(w^{-1}) \leq \ell(w)$. Swapping the place of w and w^{-1} yields the reverse inequality.
(L2) is obvious.
(L3) If $w = s_1 s_2 \cdots s_k$ and $w' = s'_1 s'_2 \cdots s'_l$ are reduced, then $ww' = s_1 s_2 \cdots s_k s'_1 s'_2 \cdots s'_l$ has maximum length $k + l$.
(L4) By (L3) $\ell(w) \leq \ell(ww') + \ell((w')^{-1})$ and by (L1) $\ell((w')^{-1}) = \ell(w')$
(L5) It follows directly by (L3) that $\ell(ws) \leq \ell(w) + 1$. Then by (L4) $\ell(ws) \geq \ell(w) - 1$. $\square$

CHAPTER 2

Representations of Coxeter Groups

In this chapter we build the standard geometric representation for Coxeter groups. This representation concretely concretely exhibits that a group generated by involutions and presented in terms of Coxeter relations as a subgroup of $GL(V)$ [**15**]. The primary references for this chapter are [**15**] and [**17**].

2.1. Definition. A *representation* of a group G is a vector space V over a field $\mathbb{K}$ together with a group homomorphism $\rho : G \to GL(V)$.

For any group presented with generators and relations $\langle S \mid R \rangle$, a (matrix) representation can be defined by assigning matrices $\{g_s \in GL(V) \mid s \in S\}$, such that the matrices g_s satisfy the relations of the presentation. It should be noted that additional relations beyond those specified could hold between the matrices g_s in $GL(V)$ [**11**]. We note that the representations we consider are over $\mathbb{R}$.

A representation of a group G is *faithful* if the homomorphism ρ is injective. Then G is isomorphic to the subgroup $\rho(G)$ of $GL(V)$.

2.2. Example (The sign representation)**.** Let $V = \mathbb{R}v$ be a one dimensional vector space spanned by a vector v and W a Coxeter group. Let $\rho : W \to GL(V)$ be defined by: $s \mapsto \{-1\}$ for all $s \in S$. Then ρ extends to a group homomorphism of W and the pair (V, ρ) is a representation of W called the *sign* representation of W.

Proof. We check that $s^2 = 1$ and $(st)^{m(s,t)} = 1$ for all $s, t \in S$. Clearly,

$$\rho(s)^2(v) = (-1)^2(v) = v \implies \rho(s)^2 = 1$$

and

$$(\rho(s)\rho(t))^{m(s,t)}(v) = ((-1)(-1))^{m(s,t)}(v) = v \implies (\rho(s)\rho(t))^{m(s,t)}) = 1 \qquad \square$$

Since $s(v) = -v$ for all $s \in S$, the sign representation shows that $s \neq 1$ and that $s^2 = 1$. It follows easily by induction that in fact $\ell(ws) = \ell(w) \pm 1$, strengthening (L5) of Proposition 1.11.

In the next section, we construct the standard geometric representation of W which will provides even further insight into the bridge between the combinatorial properties of W and its action as an "abstract" reflection group.

2.1. The Standard Geometric Representation

Let (W, S) be a Coxeter system and V an $\mathbb{R}$-vector space with basis $\{\alpha_s \mid s \in S\}$. Define a symmetric, bilinear form $V \times V \to \mathbb{R}$ by linearly extending:

$$\langle \alpha_s, \alpha_t \rangle = -\cos \frac{\pi}{m(s,t)}$$

Note that $\langle \alpha_s, \alpha_s \rangle = 1$ and $\langle \alpha_s, \alpha_t \rangle \leq 0$ for $s \neq t$. We define the *hyperplane orthogonal to* α_s by $H_s := \{v \in V \mid \langle v, \alpha_s \rangle = 0\}$.

For each $s \in S$, define a reflection $\sigma_s : V \to V$ by:

$$\sigma_s(\lambda) = \lambda - 2\langle \lambda, \alpha_s \rangle \alpha_s \tag{2.3}$$

Then $\sigma_s(\alpha_s) = -\alpha_s$ and σ_s fixes $\lambda \in H_s$. The following Lemma will be used often.

2.4. Lemma. *For all $w \in W$, $\lambda, \beta \in V$,*

$$\langle w(\lambda), w(\beta) \rangle = \langle \lambda, \beta \rangle$$

Proof. It suffices to show $\langle s(\lambda), s(\beta) \rangle = \langle \lambda, \beta \rangle$. By direct calculation

$$\begin{aligned} \langle s(\lambda), s(\beta) \rangle &= \langle \lambda - 2\langle \alpha_s, \lambda \rangle \alpha_s, \beta - 2\langle \alpha_s, \beta \rangle \alpha_s \rangle \\ &= \langle \lambda, \beta \rangle - 4\langle \alpha_s, \beta \rangle \langle \alpha_s, \lambda \rangle + 4\langle \alpha_s, \beta \rangle \langle \alpha_s, \lambda \rangle \\ &= \langle \lambda, \beta \rangle \end{aligned}$$

□

With the above notation, we have the following theorem:

2.5. Theorem. *The mapping $s \mapsto \sigma_s$ extends to a group homomorphism $\sigma : W \to GL(V)$.*

Proof. We verify that the relations of (W, S) hold in $GL(V)$. Direct calculation for σ_s shows that

$$\begin{aligned} (\sigma_s)^2(v) &= \sigma_s(v - 2\langle v, \alpha_s \rangle \alpha_s) \\ &= v - 2\langle v, \alpha_s \rangle \alpha_s - 2[\langle v, \alpha_s \rangle - 2\langle v, \alpha_s \rangle \langle \alpha_s, \alpha_s \rangle]\alpha_s = v \end{aligned}$$

Hence $(\sigma_s)^2 = 1$.

Next, assume that $s \neq t$ and consider the vector subspace $V_{s,t} := \mathbb{R}\alpha_s \oplus \mathbb{R}\alpha_t$. Firstly note that σ_s and σ_t preserve the subspace $V_{s,t}$. We have that $(\alpha_s) = -\alpha_s$ and $t(\alpha_t) = -\alpha_t$. Also, $\sigma_s(\alpha_t) = \alpha_t - 2\langle \alpha_t, \alpha_s \rangle \alpha_s$ and $\sigma_t(\alpha_s) = \alpha_s - 2\langle \alpha_s, \alpha_t \rangle \alpha_t$. We can therefore consider $\sigma_s\sigma_t$ as an operator on this subspace. There are two cases for $m(s,t)$:

Case (i): $m := m(s,t) < \infty$.

Claim: The bilinear form is positive definite and non-degenerate on $V_{s,t}$.

Proof. Let $\beta = a\alpha_s + b\alpha_t$. By direct calculation

$$\begin{aligned} \langle \beta, \beta \rangle &= \langle a\alpha_s + b\alpha_t, a\alpha_s + b\alpha_t \rangle \\ &= a^2 + 2ab\cos\frac{\pi}{m} + b^2 \\ &= (a - b\cos\frac{\pi}{m})^2 + b^2\sin^2\frac{\pi}{m} \end{aligned}$$

Therefore, $\langle \beta, \beta \rangle > 0$ for $\sin\frac{\pi}{m} \neq 0$ which occurs when $m < \infty$. □

We thus have that $V_{s,t}$ is the Euclidean plane and σ_s, σ_t act as orthogonal reflections about α_s and α_t respectively. Since $\langle \alpha_s, \alpha_t \rangle = -\cos\frac{\pi}{m} = \cos(\pi - \frac{\pi}{m})$, the angle between $\mathbb{R}^+\alpha_s$ and the lines $\mathbb{R}^+\alpha_t$ is $\pi - \frac{\pi}{m}$. Hence, the angle between the reflecting hyperplanes is $\frac{\pi}{m}$.

Since $\langle \cdot, \cdot \rangle$ is non-degenerate on $V_{s,t}$ it follows by Proposition C.13 that $V = V_{s,t} \oplus V_{s,t}^{\perp}$. By how σ_s and σ_t are defined, they fix $V_{s,t}^{\perp}$ point-wise, since $V_{s,t} \cap V_{s,t}^{\perp} = \{0\}$. Hence, we are in the situation of the dihedral groups seen in Chapter 1 and such a composition of reflections has order

$m(s,t)$.

Case (ii): $m := m(s,t) = \infty$.

Let $\lambda = \alpha_s + \alpha_t$. Then $\langle \lambda, \alpha_s \rangle = \langle \alpha_s, \alpha_s \rangle + \langle \alpha_t, \alpha_s \rangle = 0 = \langle \lambda, \alpha_t \rangle$. Hence σ_s and σ_t fix λ. Now

$$\begin{aligned}\sigma_s(\sigma_t(\alpha_s)) =& \sigma_s(\alpha_s - 2\langle \alpha_t, \alpha_s \rangle \alpha_s) \\ =& \sigma_s(\alpha_s + 2\alpha_t) \\ =& \alpha_s + 2\alpha_t - 2(\langle \alpha_s, \alpha_s \rangle + 2\langle \alpha_s, \alpha_t \rangle)\alpha_s \\ =& 3\alpha_s + 2\alpha_t \\ =& \alpha_s + 2\lambda\end{aligned}$$

Continuing in this way, we have $(\sigma_s\sigma_t)^k(\alpha_s) = \alpha_s + 2\lambda k$, $(k \in \mathbb{N})$. Hence, $\sigma_s\sigma_t$ has infinite order on $V_{s,t}$. $\square$

Remark. As commonly done, for the remainder of the paper we will set $w(\lambda) := \sigma_w(\lambda)$ for $\lambda \in V$ and $w \in W$, with the understanding that w acts on λ via the homomorphism σ. In the next chapter, the faithfulness of the representation will be realised.

An immediate corollary of Theorem 2.5 shows that the order of st is $m(s,t)$.

2.6. Corollary. *Let (W,S) be a Coxeter system with $s, t \in S$. Then the order of st is $m(s,t)$.*

Proof. By definition, we have ord$(st) \leq m(s,t)$. By the calculations of case (i) and (ii) in the proof of Theorem 2.5 we have ord$(st) \geq$ ord$(\sigma_s\sigma_t) = m(s,t)$. $\square$

CHAPTER 3

The Root System

Having established the standard geometric representation of (W, S), the goal of this chapter is to develop the key geometric properties of the action of W on the vector space V. In particular, we look at an important set of unit vectors, the root system. The main references for this section are [**15**] and [**14**].

3.1. Definition. The *root system* Φ of W is the following subset of unit vectors of V:

$$\Phi = \{w(\alpha_s) \mid w \in W, s \in S\}$$

Remark. By Lemma 2.4, $\langle \alpha, \alpha \rangle = 1$ for all $\alpha \in \Phi$. Thus, the roots are considered "unit vectors". However, $\langle \cdot, \cdot \rangle$ is not necessarily non-degenerate and hence $\sqrt{\langle \cdot, \cdot \rangle}$ is not a true norm.

The elements of Φ are called *roots* and $\Pi := \{\alpha_s \mid s \in S\}$ are the *simple roots*. Since Π is a basis of V, each $\alpha \in \Phi$ can be written uniquely in the form:

$$\alpha = \sum_{s \in S} c_s \alpha_s \qquad c_s \in \mathbb{R}$$

A root α is *positive* if $c_s \geq 0$ for all $s \in S$ and *negative* if $c_s < 0$ for all $s \in S$. We denote Φ^+ and Φ^- the positive and negative roots, respectively.

It is not entirely obvious that every $\alpha \in \Phi$ is either positive or negative, but it is in fact true that $\Phi = \Phi^+ \sqcup \Phi^-$. This is seen in the following proposition which gives a criterion to determine when a root is positive or negative and gives a geometric interpretation of the length function.

3.2. Proposition. *Let $w \in W$ and $s \in S$. Then*

$$\textbf{\textit{(i)}}\ \ell(ws) > \ell(w) \implies w(\alpha_s) > 0$$
$$\textbf{\textit{(ii)}}\ \ell(ws) < \ell(w) \implies w(\alpha_s) < 0$$

Proof. (Adapted from [**15**] p 111). Note that if $\ell(ws) > \ell(w) \implies w(\alpha_s) > 0$, then setting $w = \hat{w}s$ in (i), we have

$$\ell((\hat{w}s)s) > \ell(\hat{w}s) \implies \hat{w}s(\alpha_s) > 0 \implies \hat{w}(-\alpha_s) > 0 \implies \hat{w}(\alpha_s) < 0$$

And so (ii) follows from (i). We proceed to prove (i) by induction on $\ell(w)$. Suppose $\ell(ws) > \ell(w)$.

If $\ell(w) = 0$ then by Proposition 1.11, it follows that $w = 1$, so (i) holds.
If $\ell(w) > 0$, then there exists $t \in S$ such that $\ell(wt) = \ell(w) - 1$; since we can choose t to be the last factor in a reduced expression for w. By assumption, $\ell(ws) > \ell(w)$, so $s \neq t$.

Let us set $I := \{s, t\}$. Then as in Example 1.2, the subgroup W_I generated by I is dihedral. We consider the set:

$$A := \{v \in W \mid v^{-1}w \in W_I \text{ and } \ell(v) + \ell_I(v^{-1}w) = \ell(w)\}$$

Now, $w \in A$, since $w^{-1}w \in W_I$ and $\ell(w) + \ell_I(w^{-1}w) = \ell(w) + \ell_I(1) = \ell(w)$. We choose $v \in A$ such that $\ell(v)$ is as small as possible.

Write $v_I := v^{-1}w \in W_I$ so that $w = vv_I$ and $\ell(w) = \ell(v) + \ell_I(v_I)$. Recalling that we want to show $w(\alpha_s) > 0$, we now prove the following:

(1) $v(\alpha_s) > 0$ and $v(\alpha_t) > 0$
(2) $v_I(\alpha_s) = a\alpha_s + b\alpha_t$ for some $a, b \geq 0$

By proving (1) and (2), it would then follow that $w(\alpha_s) = vv_I(\alpha_s) = v(a\alpha_s + b\alpha_t) > 0$ completing the proof. We now prove (1).

We see that $wt \in A$, since $(wt)^{-1}w = t^{-1}w^{-1}w = t^{-1} \in W_I$ and $\ell(wt) + \ell_I((wt)^{-1}w) = \ell(w) - 1 + \ell_I(t^{-1}) = \ell(w)$. Since $v \in A$ and $\ell(v)$ is minimal over A, we have that $\ell(v) \leq \ell(wt) = \ell(w) - 1$.

We claim that $\ell(vs) > \ell(v)$. Suppose for a contradiction that $\ell(vs) < \ell(v)$ (i.e. $\ell(vs) = \ell(v) - 1$). Then,

$$\begin{aligned}
\ell(w) &\leq \ell(vs) + \ell((sv^{-1})w) && \text{(by Proposition 1.11)} \\
&\leq \ell(vs) + \ell_I(sv^{-1}w) \\
&= (\ell(v) - 1) + \ell_I(sv^{-1}w) \\
&\leq \ell(v) - 1 + \ell_I(v^{-1}w) + 1 && \text{(by Proposition 1.11)} \\
&= \ell(v) + \ell_I(v^{-1}w) \\
&= \ell(w) && \text{(since } v \in A\text{)}
\end{aligned}$$

Hence, $\ell(w) = \ell(vs) + \ell_I((sv^{-1})w)$ and so $vs \in A$, which contradicts that $\ell(v)$ is minimal over A. So we must have $\ell(vs) > \ell(v)$. Thus, applying the induction hypothesis to the pair (v, s), it follows that $v(\alpha_s) > 0$. Replacing s with t in the above argument, it follows that $\ell(vt) > \ell(v)$ and hence $v(\alpha_t) > 0$. This proves (1).

We now prove (2). We claim that $\ell_I(v_I s) \geq \ell_I(v_I)$. Suppose that $\ell_I(v_I s) < \ell_I(v_I)$. Then

$$\begin{aligned}
\ell(ws) &= \ell(vv^{-1}ws) \\
&\leq \ell(v) + \ell(v^{-1}ws) \\
&= \ell(v) + \ell(v_I s) \\
&\leq \ell(v) + \ell_I(v_I s) \\
&< \ell(v) + \ell_I(v_I) = \ell(w)
\end{aligned}$$

which is a contradiction, since $\ell(ws) > \ell(w)$. So $\ell_I(v_I s) \geq \ell_I(v_I)$ and any reduced expression for v_I in W_I cannot end in s and must end in t. We now consider the two possible cases for the order of st:

Case 1: $m := m(s, s') = \infty$. We saw in the proof of Theorem 2.5 that $(st)^k(\alpha_s) = \alpha_s + 2k(\alpha_s + \alpha_t)$. Hence, (2) holds in this case.

Case 2: $m := m(s,t) < \infty$
Note that we are then in the situation of the dihedral group with an angle of $\pi - \frac{\pi}{m}$ between α_s and α_t, and $\frac{\pi}{m}$ being the angle between the reflecting planes. Since $\ell_I(v_I) < m$, it follows that any $w_i \in W_I$ of length m always has a reduced expression ending in s. Since any reduced expression for v_I must end in t, there are two possibilities for a reduced expression of v_I. Either $v_I = (st)^k$ where $k < \frac{m}{2}$ or $v_I = t(st)^k$

If $v_I = (st)^k$, then by Lemma C.2 we can directly compute $(st)^k(\alpha_s)$ as follows

$$\begin{aligned}(st)^k(\alpha_s) &= \frac{1}{\sin\theta}\begin{pmatrix}\sin(2k+1)\theta & -\sin 2k\theta \\ \sin 2k\theta & -\sin(2k-1)\theta\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} \\ &= \frac{1}{\sin\theta}\begin{pmatrix}\sin(2k+1)\theta \\ \sin 2k\theta\end{pmatrix}\end{aligned}$$

For $\theta = \frac{\pi}{m}$, since $k < \frac{m}{2}$ it follows that $\frac{(2k+1)\pi}{m} < \frac{(m+1)\pi}{m}$ and $\frac{2k\pi}{m} < \pi$ and so $\sin(2k+1)\frac{\pi}{m}$ and $\sin 2k\frac{\pi}{m}$ are positive.

If $v_I = t(st)^k$, then also by Lemma C.2 with $\theta = \frac{\pi}{m}$ we have

$$\begin{aligned}t(st)^k(\alpha_s) &= \frac{1}{\sin\theta}\begin{pmatrix}1 & 0 \\ 2\cos\theta & -1\end{pmatrix}\begin{pmatrix}\sin(2k+1)\theta \\ \sin 2k\theta\end{pmatrix} \\ &= \frac{1}{\sin\theta}\begin{pmatrix}\sin(2k+1)\theta \\ 2\cos\theta - \sin 2k\theta\end{pmatrix}\end{aligned}$$

Since $2\cos\theta - \sin 2k\theta = 2\cos\theta - 2\sin k\theta\cos k\theta$ and $k < \frac{m}{2}$, it follows that $0 < \sin k\theta < 1$ and $0 < \cos k\theta < 1$. Since also $\cos k\theta \leq \cos\theta$, we conclude $2\cos\theta - \sin 2k\theta \geq 0$ and hence conclude the proof.

□

3.3. Corollary. *The homomorphism $\sigma : W \to GL(V)$ is injective.*

Proof. (Adapted from [**15**] p113) Let $w \in \text{Ker}(\sigma)$. Hence $w(v) = v$ for all $v \in V$. If $w \neq 1$, there exists $s \in S$ such that $\ell(ws) < \ell(w)$. Then by Proposition 3.2, we have $w(\alpha_s) < 0$, which is a contradiction since w fixes every $v \in V$. □

Remark. The importance of the above corollary should be emphasized. It verifies that the Coxeter group presentations are indeed 'concretely' realised as a subgroup of $GL(V)$ generated by reflections.

3.1. Roots and Reflections

Under the homomorphism $\sigma : W \to GL(V)$, each $s \in S$ acts as a reflection on V. For each root $\alpha \in \Phi$, we can specify corresponding elements in W which act as a reflection about H_α, the hyperplane orthogonal to α. Let $\alpha = w(\alpha_s)$ for some $w \in W$, $s \in S$ and consider the following:

$$wsw^{-1}(\lambda) = w[w^{-1}(\lambda) - 2\langle w^{-1}(\lambda), \alpha_s\rangle\alpha_s] \tag{3.4}$$
$$= \lambda - 2\langle w^{-1}(\lambda), \alpha_s\rangle w(\alpha_s)$$
$$= \lambda - 2\langle \lambda, w(\alpha_s)\rangle w(\alpha_s)$$
$$= \lambda - 2\langle \lambda, \alpha\rangle\alpha \tag{3.5}$$

Hence the reflection wsw^{-1} only depends on α. Proceeding, we can simply denote a reflection about $\alpha \in \Phi$ by s_α. Clearly, s_α sends $\alpha \to -\alpha$ and s_α fixes $\lambda \in H_\alpha$ since $\langle \lambda, \alpha\rangle = 0$.

We denote

$$T = \bigcup_{w \in W} wSw^{-1} = \{\text{set of reflections}, s_\alpha \mid \alpha \in \Phi\}$$

The following observations are straight forward:

3.6. Lemma. (i) *The correspondence $\alpha \to s_\alpha$ is bijective for $\alpha \in \Phi^+$*
(ii) *If $\alpha, \beta \in \Phi$ and $\beta = w(\alpha)$ for some $w \in W$ then $ws_\alpha w^{-1} = s_\beta$*

Proof.

(i) If $s_\alpha = s_\beta$, then

$$s_{\alpha(\beta)} = \beta - 2\langle \beta, \alpha\rangle\alpha = -\beta = s_{\alpha(\beta)}$$

Hence, $\langle \beta, \alpha\rangle\alpha = \beta$. Since α and β are unit vectors $\implies \alpha = \beta$.

(ii) Setting $s = s_\alpha$ in Equation (3.4) yields:

$$ws_\alpha w^{-1}(\lambda) = \lambda - 2\langle \lambda, w(\alpha)\rangle w(\alpha) = s_\beta(\lambda)$$ □

We can now extend Proposition 3.2 to arbitrary roots and reflections.

3.7. Proposition. *Let $w \in W$, $\alpha \in \Phi^+$. Then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) > 0$*

Proof. "$\Rightarrow$" We induct on $\ell(w)$. If $\ell(w) = 0$ then $w = e$ so $w(\alpha) = \alpha > 0$. Hence, assume the proposition holds for $w \in W$ such that $\ell(ws_\alpha) > \ell(w)$, with $\ell(w) > 0$.

Since $\ell(w) > 0$, there is $s \in S$ such that $\ell(sw) < \ell(w)$. Then

$$\ell((sw)s_\alpha) = \ell(s(ws_\alpha)) \geq \ell(ws_\alpha) - 1 > \ell(w) - 1 = \ell(sw)$$

So by the induction hypothesis $sw(\alpha) > 0$. Suppose $w(\alpha) < 0$. By Proposition 3.8 s sends $-\alpha_s \to \alpha_s$ and permutes the remaining negative roots, so we conclude that $w(\alpha) = -\alpha_s$. Then by Lemma 3.6, since $sw(\alpha) = \alpha_s$ this implies that $(sw)s_\alpha(sw^{-1}) = s$. So $ws_\alpha = sw$, contradicting that $\ell(ws_\alpha) > \ell(w) > \ell(sw)$. Hence, it must be that $w(\alpha) > 0$.

"$\Leftarrow$" Follows by setting $w = ws_\alpha$. □

3.2. Further Combinatorial Properties

In this section, we discuss some additional results which further highlights the relationship between the combinatorial properties of the elements $w \in W$ and its action on roots. The following properties will be regularly referenced in the subsequent material. The primary references of this section are [**15**], [**14**] and [**3**].

3.8. Proposition. *If $s \in S$, then s sends α_s to $-\alpha_s$ but permutes the remaining positive roots.*

Proof. (Adapted from [**15**] p115). Let $\alpha \in \Phi^+, \alpha \neq \alpha_s$. The root vectors are unit, so we can write:

$$\alpha = \sum_{t \in S} c_t \alpha_t$$

with non-negative coefficients and some $c_t > 0, t \neq s$. Then,

$$s(\alpha) = \sum_{t \in S} c_t \alpha_t - 2(\langle \alpha_s, \sum_{t \in S} c_t \alpha_t \rangle)\alpha_s$$

So the coefficients of α_t are unchanged and remain strictly positive. Hence, $s(\alpha_s) \notin \Phi^-$. Since $\Phi = \Phi^+ \cup \Phi^-$, this implies that $s(\alpha_s) \in \Phi^+ \setminus \{\alpha_s\}$. Therefore, $s(\Phi^+ \setminus \{\alpha_s\}) \subset \Phi^+ \setminus \{\alpha_s\}$ and applying s to both sides, it follows that $s(\Phi^+ \setminus \{\alpha_s\}) = \Phi^+ \setminus \{\alpha_s\}$. □

A subset of the positive roots which will be useful in the proceeding chapters is defined as follows.

3.9. Definition. Let $w \in W$. The *descent set* of w is

$$D(w) = \{\alpha \in \Phi^+ \ : w(\alpha) \in \Phi^-\}$$

The following proposition is quite remarkable.

3.10. Proposition. *For all* $w \in W$*,*

$$\ell(w) = |D(w)|$$

Proof. (Adapted from [**3**] p104). The result holds for $\ell(w) = 1$ by Proposition 3.8. Let us proceed by induction on length. Suppose the result holds for arbitrary $v \in W$.

Let $w = vs$, $s \in S$ with $\ell(vs) > \ell(v)$. By Proposition 3.2 then $v(\alpha_s) \in \Phi^+$. By Proposition 3.8, for $\beta \in \Phi^+$, it follows that

$$w(\beta) \in \Phi^- \iff \beta = \alpha_s \text{ or } s(\beta) \in D(v)$$

Since $\alpha_s \notin D(v)$, by the induction hypothesis we have $|D(w)| = \ell(v) + 1 = \ell(w)$. □

3.11. Proposition. $|W| < \infty \iff |\Phi| < \infty$

Proof. (Adapted from [**14**]). "$\Rightarrow$" If $|W| < \infty$ then clearly $|\Pi| < \infty$, and we have

$$|\Phi| = |\{w(\alpha_s) \mid w \in W, \alpha_s \in \Pi\}| \leq |W||\Pi| < \infty$$

"$\Leftarrow$" Now suppose $|\Phi| = m < \infty$. For each $w \in W$, define a map $\pi_w : \Phi \to \Phi$ by $v \mapsto w(v)$, $v \in \Phi$. Hence, π_w is a permutation of Φ and the mapping $w \mapsto \pi_w$ is a homomorphism $\rho : W \to \text{Sym}(\Phi)$, the symmetric group on Φ.

Now if $w \in \ker(\rho)$ then $w(v) = v$ for all $v \in \Phi$, and hence $w(\alpha_s) = \alpha_s$ for all $\alpha_s \in \Pi$. Therefore $w(v) = v$ for all $v \in V$. Since the homomorphism $\sigma : W \to GL(V)$ is injective, $w = 1$. So ρ is injective and $|W| \leq |\text{Sym}(\Phi)| = M! < \infty$. □

3.3. The Root Poset

An important component in the construction of finite state automata for Coxeter groups is a particular partial ordering of positive roots. The goal of this section is to develop the notion of the *depth* of a positive root and use this to define the partial ordering. We present the poset of positive roots for our recurring example $\tilde{A}_2$. This section largely follows [**3**] pages 109-111.

3.12. Definition. The *depth* of $\beta \in \Phi^+$ is

$$\mathrm{dp}(\beta) = min\{k : w(\beta) \in \Phi^- \text{ for some } w \in W \text{ and } \ell(w) = k\}$$

3.13. Lemma. *Let $s \in S$ and $\beta \in \Phi^+ \setminus \{\alpha_s\}$. Then,*

$$\mathrm{dp}(s(\beta)) = \begin{cases} \mathrm{dp}(\beta) - 1 & \textit{if } \langle \beta, \alpha_s \rangle > 0 \\ \mathrm{dp}(\beta) & \textit{if } \langle \beta, \alpha_s \rangle = 0 \\ \mathrm{dp}(\beta) + 1 & \textit{if } \langle \beta, \alpha_s \rangle < 0 \end{cases}$$

Proof. (Adapted from [**3**] p109). If $\langle \alpha_s, \beta \rangle = 0$ then $\alpha_s \beta = \beta - 2\langle \alpha_s, \beta \rangle \beta = \beta$. Hence, $\mathrm{dp}(s(\beta)) = \mathrm{dp}(\beta)$.

Claim. We have $\mathrm{dp}(s(\beta)) \geq \mathrm{dp}(\beta) - 1$ for $\beta \in \Phi^+ \setminus \{\alpha_s\}$.

Proof. Let $w = s_1 s_2 ... s_n$ such that $\mathrm{dp}(\beta) = \ell(w) = n$. Now if $\tilde{w}$ is such that $\tilde{w}(s(\beta)) \in \Phi^-$, then we claim that $\ell(\tilde{w}) \geq n - 1$. Suppose $\ell(\tilde{w}) < n - 1$. Then $(\tilde{w}s)(\beta)) \in \Phi^-$ and $\ell(\tilde{w}s) \leq n - 1$ contradicting the fact that $\mathrm{dp}(\beta) = n$. □

Suppose $\langle \beta, \alpha_s \rangle > 0$. By the above claim, if we can show $\mathrm{dp}(\beta) > \mathrm{dp}(s(\beta))$, then it follows that $\mathrm{dp}(\beta) - 1 \geq \mathrm{dp}(s(\beta))$ and thus $\mathrm{dp}(s(\beta)) = \mathrm{dp}(\beta) - 1$.

Hence, let $w \in W$ such that $\tilde{w}(\beta)) \in \Phi^-$ and $\mathrm{dp}(\beta) = \ell(w)$. There are two cases:

Case 1: Suppose $ws < w$ (so $\ell(ws) < \ell(w)$).
Then by the choice of w, $ws(s(\beta)) = w(\beta) \in \Phi^-$. Hence $\mathrm{dp}(s(\beta)) \leq \ell(ws) < \ell(w) = \mathrm{dp}(\beta)$.

Case 2: Suppose $ws > w$ (so $\ell(ws) > \ell(w)$). Consider the root

$$\gamma = ws(\beta) = w(\beta - 2\langle \beta, \alpha_s \rangle \alpha_s) = w(\beta) - 2\langle \beta, \alpha_s \rangle w(\alpha_s)$$

Again by choice of w, $w(\beta) \in \Phi^-$ and $\langle \beta, \alpha_s \rangle > 0$. Since $\ell(ws) > \ell(w)$, by Proposition 3.2 we have $w(\alpha_s) > 0$, so $w(\alpha_s) \in \Phi^-$. Hence $\gamma = w(\beta) - 2\langle \beta, \alpha_s \rangle w(\alpha_s)$ is the sum of two negative roots, so $\gamma \in \Phi^-$ and $\gamma \neq -\alpha_{s'}$ for $s' \in S$ (since $-\alpha_{s'}$ cannot be the sum of two negative roots).

Now let us choose $s' \in S$ such that $s'w < w$. Then by Proposition 3.8, $s'w(s(\beta)) = s'(ws(\beta)) = s'\gamma \in \Phi^-$ (since $\gamma \in \Phi^- \setminus \{\Phi^-\}$). Hence it also follows that $\mathrm{dp}(s(\beta)) \leq \ell(s'w) < \ell(w) = \mathrm{dp}(\beta)$.

Lastly, suppose $\langle \beta, \alpha_s \rangle < 0$. Then,

$$\langle s(\beta), \alpha_s \rangle = \langle \beta - 2\langle \beta, \alpha_s \rangle \alpha_s, \alpha_s \rangle = -\langle \beta, \alpha_s \rangle > 0$$

Hence by the above proof, we have $\mathrm{dp}(s(s\beta)) = \mathrm{dp}(s(\beta)) - 1$ which implies that $\mathrm{dp}(s(\beta)) = \mathrm{dp}(\beta)) + 1$. □

We use depth to define a partial ordering of the positive roots.

3.14. Definition. For $\beta, \alpha \in \Phi^+$, $\beta \leq \alpha$ if there exists $s_1, s_2, \cdots, s_k \in S$ such that:

a) $s_k s_{k-1} \cdots s_1(\beta) = \alpha$
b) $\mathrm{dp}(s_i s_{i-1} \cdots s_1(\beta)) = \mathrm{dp}(\beta) + i$ for $1 \leq i \leq k$

We now define a quantity B_s so that we have an algorithmic procedure for constructing the root poset. Let $s \in S$ and write $\beta = \sum_{s' \in S} b_{s'} \alpha_{s'}$.

Let $k_{s,s'} = 2\cos\frac{\pi}{m(s,s')}$, so that we can write:

$$s(\beta) = \beta + (\sum_{s' \in S} k_{s,s'} b_{s'})\alpha_s$$

Now define:

$$B_s = -b_s + \sum_{s' : s' - s} k_{s,s'} b_{s'}$$

Where $s' : s' - s$ are the elements $s' \in S$ such that s' is adjacent to s in the Coxeter diagram of W, and b_s is the coefficient of the s-th coordinate of β.

Claim. We have $-2\langle \alpha_s, \beta\rangle = B_s - b_s$.

Proof. On the LHS we have:

$$-2\langle \alpha_s, \beta\rangle = -2\langle \alpha_s, \sum_{s' \in S} b_{s'}\alpha_{s'}\rangle = \sum_{s' \in S} k_{s,s'} b_{s'}$$

On the RHS:

$$B_s - b_s = \sum_{s' : s' - s} b_{s'}\alpha_{s'} - 2b_s = \sum_{s' : s' - s} b_{s'}\alpha_{s'} + k_{s,s} b_s = \text{LHS}$$ ☐

By the above claim, we can now write $s(\beta) = \beta + (B_s - b_s)\alpha_s$.

3.15. Lemma. *For $s \in S$ and $\beta \in \Phi^+$, $s(\beta) > \beta$ if and only if $B_s > b_s$.*

Proof. Note that another way to state the Lemma is:

$$B_s > b_s \implies \text{dp}(s(\beta)) = \text{dp}(\beta) + 1$$
$$B_s < b_s \implies \text{dp}(s(\beta)) = \text{dp}(\beta) - 1$$

Hence by Definition 3.14, this gives us a precise criterion for whether $s(\beta)$ sits above β in the root poset.

If $B_s - b_s > 0$ then we have $\langle \alpha_s, \beta\rangle = \frac{B_s - b_s}{-2} < 0$, so by Definition 3.12 this implies that $\text{dp}(s(\beta)) = \text{dp}(\beta) + 1$. The other implication follows similarly. ☐

With the preceding information in hand, we can now succinctly describe the algorithm for generating the root poset.

3.16. Proposition. *Let (W, S) be a Coxeter system with $\Pi \subset \Phi^+$ the simple roots. A partially ordered set on Φ^+ can be constructed as follows.*

1. *Begin with the simple roots $\Pi = \{\alpha_s : s \in S\}$. Set $j = 1$. Then for each root β with depth j and each $s \in S$ such that no s-labeled edge leads down from β, compute the quantity B_s. Note that γ is also just the positive root $s(\beta)$.*
2. *If $B_s = b_s$ do nothing. If $B_s > b_s$, let γ be the vector obtained from β by replacing b_s with B_s in the s-th coordinate of β.*
3. *Then γ is a root of depth $j + 1$ and (β, γ) is an s-labeled edge.*

From (c) we denote $\beta \lhd \gamma$ and call this a *covering edge*.

3.17. Example ($\tilde{A}_2$)**.** We follow Proposition 3.16 to present a portion of the root poset for the Coxeter group $\tilde{A}_2$. Note that by Proposition 3.11, the poset is of infinite order since $\tilde{A}_2$ is infinite. In Figure 1, the reflection s is represented by the black line, t by the green line and u by the blue line.

Consider the simple root α_s. We compute the quantity B_t. Clearly, the coefficient of α_t in α_s is 0, so we have $b_t = 0$. Now

$$B_t = -2\langle \alpha_s, \alpha_t \rangle = -2(-\frac{1}{2}) = 1 > b_t.$$

Hence, we replace the coefficient of α_t, $b_t = 0$ in α_s with $B_t = 1$ to obtain $\alpha_s + \alpha_t$. So there is a t-labeled edge from α_s to $\alpha_s + \alpha_t$.

Since $\langle \alpha_s, \alpha_t \rangle = \langle \alpha_s, \alpha_u \rangle = \langle \alpha_t, \alpha_u \rangle$, we see that $u(\alpha_s) = s(\alpha_u) = \alpha_s + \alpha_u$ and $u(\alpha_t) = t(\alpha_u) = \alpha_t + \alpha_u$ are also on the poset with the corresponding edges.

Then for $\alpha_s + \alpha_t$, we only need to consider the reflection u, since we have s and t labeled edges down from the root. So

$$B_u = -2\langle \alpha_u, \alpha_s + \alpha_t \rangle = -2(-1) = 2.$$

Since the coefficient of α_u, $b_u = 0$ in $\alpha_s + \alpha_t$, we put $u(\alpha_s + \alpha_t) = \alpha_s + \alpha_t + 2\alpha_u$ on the root poset, with a u-labeled edge from $\alpha_s + \alpha_t$. The root poset for a few of the positive roots is presented below in Figure 1.

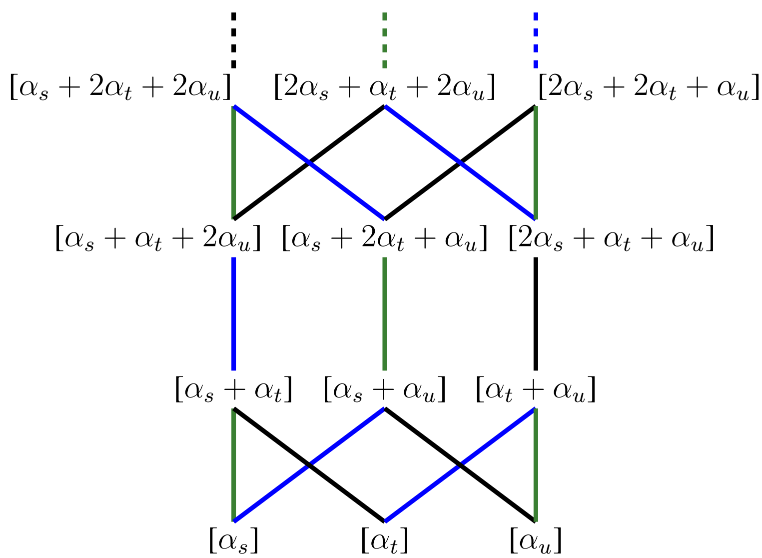


FIGURE 1. Root poset for $\tilde{A}_2$

CHAPTER 4

Subgroups of Coxeter Groups

We have so far discussed the representation of the Coxeter group W and various combinatorial properties relating to the action of W on V. In this chapter we aim to develop an understanding of the structure of subgroups of W. The main result being that every finite subgroup of a Coxeter group is conjugate of a subgroup of a finite parabolic subgroup. The primary references for this section are [**15**], [**3**] and [**14**].

4.1. Definition. Let $J \subseteq S$. We define W_J to be a *parabolic subgroup* of W generated by J.

A natural question to ask is whether W_J itself a Coxeter group. This is not immediately clear since there could be relations involving $t \in S \setminus J$ which force certain relations between $s \in J$ that are not captured with t removed from the presentation of W_J. The following proposition verifies that the parabolic subgroups are themselves Coxeter groups.

4.2. Proposition. *The parabolic subgroup $W_J \leq W$ is isomorphic to the Coxeter group with presentation:*

$$\tilde{W_J} = \langle s \in J \mid (st)^{m(s,t)} = 1 \rangle$$

Proof. (Adapted from [**14**]). Let $\varphi : \tilde{W_J} \to W_J$ be a surjective homomorphism defined by $\tilde{s} \mapsto s$. Let $\sigma, \tilde{\sigma}$ be injective representations from $W \to GL(V)$ and $\tilde{W_J} \to GL(\tilde{V})$ respectively.

The vector space $\tilde{V}$ has basis $\{\tilde{\alpha_s} \mid s \in J\}$, with bilinear form on $\tilde{V}$ defined by $\tilde{B}(\tilde{\alpha_s}, \tilde{\alpha_t}) = -\cos \frac{\pi}{m(s,t)}$ for $s, t \in J$. If V_J is the subspace of V spanned by $\{\alpha_s \mid s \in J\}$, then $\varphi : \tilde{V} \to V_J$ defined by $\tilde{\alpha_s} \mapsto \alpha_s$ is an isomorphism of vector spaces which preserves the bilinear form, since $\tilde{B}(\tilde{\alpha_s}, \tilde{\alpha_t}) = \tilde{B}(\alpha_s, \alpha_t)$ for $s, t \in J$.

Hence, let $\phi : GL(\tilde{V}) \to GL(V_J)$ be the induced isomorphism. If $s, t \in J$, then $s(\alpha_t) \in V_J$. So for all $s \in J$, we have $s(v) \in V_J$, $\forall\, v \in V_J$.

Therefore, the subspace V_J of V is W_J-invariant and there is a homomorphism $W_J \to GL(V_J)$ defined by $w \mapsto \sigma(w)|_{V_J}$. We have the following diagram:

$$\begin{tikzcd}
\tilde{W_J} \arrow[r, "\varphi"] \arrow[d, "\tilde{\sigma}"'] & W_J \arrow[d, "\sigma|_{V_J}"] \\
GL(\tilde{V}) \arrow[r, "\phi"'] & GL(V_J)
\end{tikzcd}$$

This diagram commutes since for $\lambda \in \tilde{V}$, $\tilde{\sigma_s}(\lambda) = \lambda - 2\tilde{B}(\tilde{\alpha_s}, \lambda)\tilde{\alpha_s}$ and $\phi(\tilde{\sigma_s}(\lambda)) = \lambda - 2\tilde{B}(\tilde{\alpha_s}, \lambda)\alpha_s = \lambda - 2\tilde{B}(\alpha_s, \lambda)\alpha_s$, which is the transformation of $\lambda \in V_J$ given by $\sigma(s)|_{V_J}$. Therefore, $\varphi : \tilde{W_J} \to W_J$ is an isomorphism. $\square$

The following brings to light further details regarding the internal structure of W and its subgroups.

4.3. Theorem. (i) *Let $I \subset S$. If $w = s_1 \cdots s_k$, $(s_i \in S)$ is a reduced expression for $w \in W_I$ then $s_i \in I$ for all i. Hence, the length function ℓ agrees with ℓ_I on W_I and $W_I \cap S = I$.*

(ii) *The assignment $I \to W_I$ is a bjiective, inclusion preserving map between the collection of subsets $I \subseteq S$ and the collection of subgroups $W_I \leq W$.*

(iii) *There is no $I \subset S$ which generates W. So S is a minimal generating set for W.*

Proof. (Adapted from [**15**] p113). (i) Let $w \in W_I$. We proceed by induction on $\ell(w)$. $\ell(1) = 0 = \ell_I(1)$. Now suppose $w \neq 1$ and set $s = s_k$, the last letter of w. By Proposition 3.2, since $\ell(ws) < \ell(w)$ then $w(\alpha_s) < 0$. Since $w \in W_I$, we can also write $w = t_1 \cdots t_q$ with all $t_i \in I$. Therefore, by repeated application of the formula for a reflection, we have

$$w(\alpha_s) = \alpha_s + \sum_{i=1} c_i \alpha_{t_i}$$

Since $w(\alpha_s) < 0$, it follows that $s = t_i$ for some i, which forces $s \in I$. So $ws = s_1 \cdots s_{k-1} \in W_I$. Applying this argument another $k-1$ times shows that $s_i \in I$ for all i and the last sentence of (i) immediately follows.

(ii) Let $W_I \subset W_J$. Then $I = W_I \cap S \subset W_J \cap S = J$ by (i). Hence $I \subset J \iff W_I \subset W_J$, with equality if $I = J$.

(iii) If $I \subset S$ generates W then (ii) implies $W_I = W = W_S$ and $I = S$. $\square$

The following is a rather technical proposition will be useful in later results.

4.4. Proposition. *Let $\alpha, \beta \in \Phi^+$.*

(i) *If $|\langle \alpha, \beta \rangle| < 1$, then the subgroup generated by s_α and s_β is a finite dihedral group.*

(ii) *If $\langle \alpha, \beta \rangle \leq -1$, then the subgroup generated by s_α and s_β is an infinite dihedral group. In addition, the roots $(s_\alpha s_\beta)^n(\alpha)$ for $n \in \mathbb{N}$ are positive linear combinations of α and β and are all distinct.*

Proof. (Adapted from [**3**] p107). (i) Assume that $|\langle \alpha, \beta \rangle| < 1$. Let $w \in W$ so that $w(\beta) \in \Pi$. Now $\langle \alpha, \beta \rangle = \langle w(\alpha), w(\beta) \rangle$ and by Section 3.1 we may write $s_{w(\alpha)} = w s_\alpha w^{-1}$ and $s_{w(\beta)} = w s_\beta w^{-1}$. So without loss of generality we may assume $\beta = \alpha_s$ for some $s \in S$.

Let V_0 be the subspace spanned by α and α_s and let $\gamma = \alpha - c_s \alpha_s$. Then writing $\alpha = \sum_{r \in S} c_r \alpha_r$, we can see that $\gamma \neq 0$, since $|\langle \alpha, \alpha_s \rangle < 1|$ implies that $\alpha \neq \alpha_s$. Then since $\Phi = \Phi^+ \sqcup \Phi^-$, it follows that

$$\{\lambda\gamma + \mu\alpha_s \mid \lambda, \mu \in \mathbb{R}, \lambda\mu < 0\} \cap \Phi = \emptyset \tag{4.5}$$

Now as seen in the proof of Theorem 2.5, since $|\langle \alpha, \beta \rangle| < 1$, the restriction of the bilinear form to V_0 is positive definite. Therefore, V_0 is a Euclidean plane and we are in the situation of Example 1.2. So $s_\alpha s$ acts on V_0 as a rotation through the origin of $2x$ radians, with $0 < x < \pi$ and $\cos(x) = \langle \alpha, \alpha_s \rangle$. It must follow that x is a rational multiple of π, since if it were not, the roots $(s_\alpha s)^n(\alpha)$ for $n = 0, 1, 2, \cdots$ would be dense in the unit circle, contradicting Equation (4.5). So x is of the form $\frac{p\pi}{q}$, with $p, q \in \mathbb{Z}$. Therefore, the subgroup H generated by s_α and s acts as a finite dihedral group on V_0.

It remains to show that H is isomorphic to a finite dihedral group. Since V_0 is Euclidean, $V = V_0 \oplus V_0^{\perp}$ and H fixes $V_0^{\perp}$. By Theorem 2.5, the representation of W is faithful and so it must be faithful on V_0. Therefore H is indeed isomorphic to a finite dihedral group.

(ii) Let $x = -\langle \alpha, \beta \rangle$, so that $x \geq 1$, and consider the action of $s_\alpha s_\beta$ on an arbitrary linear combination of α and β:

$$\begin{aligned} s_\alpha s_\beta(c\alpha + b\beta) &= s_\alpha(c\alpha + b\beta - 2\langle c\alpha + b\beta, \beta\rangle\beta) \\ &= s_\alpha(c\alpha + (2cx - b)\beta) = \cdots \\ &= (-c + 2x(2cx - b))\alpha + (2cx - b)\beta = \cdots \\ &= ((4x^2 - 1)c - 2xb)\alpha + (2cx - b)\beta \end{aligned}$$

Let $(s_\alpha s_\beta)^n(\alpha) = c_n\alpha + b_n\beta$, so that $(s_\alpha s_\beta)(c_n\alpha + b_n\beta) = c_{n+1}\alpha + b_{n+1}\beta$, where by the above calculation,

$$\begin{aligned} c_{n+1} &= (4x^2 - 1)c_n - 2xb_n \\ b_{n+1} &= 2xc_n - b_n \end{aligned}$$

We show by induction that $c_n > b_n \geq 0$ for all n.

For $n = 0$, $c_0 = 1$ and $b_0 = 0$, so the result is true for this base case. Hence, assume it is true for some $n > 0$.

Now let $A = 4x^2 - 2x - 1$ and $D = 2x - 1$. It follows that

$$Ac_n - Db_n = (4x^2 - 2x - 1)c_n - (2x - 1)b_n = c_{n+1} - b_{n+1}.$$

Since we assumed that $x \geq 1$ and

$$A - D = 4x^2 - 4x = (2x - 1)^2 - 1 \geq 0,$$

it follows that $A \geq D \geq 1$. Therefore, by the induction assumption, and the fact that $b_{n+1} = 2xc_n - b_n$, we have $c_{n+1} > b_{n+1} \geq 0$.

Finally, since $b_{n+1} - b_n = 2x(c_n - b_n) > 0$, we see that the positive linear combinations of α and β are all distinct. □

4.1. Finite Subgroups of Coxeter Groups

In this section we formulate the details to prove the important theorem mentioned at the beginning of the chapter; that all finite subgroups of a Coxeter group conjugate into a finite parabolic subgroup of W. This section largely follows [**14**] and the content is mainly technical in nature.

We make a preemptive note about the ingredients of the proof: the following arguments would work without the use of the dual space of V if it was the case that the bilinear form of the standard geometric representation is always non-degenerate. However, this is not always the case [**14**], which necessitates the mention of the dual of V.

Let V^* be the space of all linear functionals $V \to \mathbb{R}$. For each $\alpha \in \Pi$, we define $\delta^*_\alpha \in V^*$ by:

$$\delta^*_\alpha(\beta) = \begin{cases} 1 & \text{if } \beta = \alpha \\ 0 & \text{otherwise} \end{cases} \qquad \text{for all } \beta \in \Pi$$

Then $\{\delta^*_\alpha \mid \alpha \in \Pi\}$ is the basis of V^* dual to V.

We define a right action of W on V^* by:

$$(\alpha^* w)(v) = (\alpha^*)(wv) \qquad \text{for all } v \in V \text{and } w \in W$$

With the dual space V^* introduced, we further define two subsets of V^* as follows:

4.6. Definition. Let $\alpha^* \in V^*$. Then

$$\text{Neg}(\alpha^*) = \{v \in \Phi^+ \mid \alpha^*(v) < 0\}$$

Given Definition 4.6, we also have

$$C = \{\alpha^* \in V^* \mid \text{Neg}(\alpha^*) \cap \Pi = \emptyset\}$$

There are a few other useful characterisations of the set C. If $\alpha^* \in C$, then we have that $\alpha^*(v) \geq 0$ for all $v \in \Phi^+$, since $\alpha^*(\alpha_s) \geq 0$ for all of the simple roots. On the other hand, if $\alpha^*(v) \geq 0$ for all $v \in \Phi^+$ then it is also clear that $\alpha^*(\alpha_s) \geq 0$ for all $\alpha_s \in \Pi$. If we write

$$\alpha^* = \sum_{\alpha_s \in \Pi} \lambda_{\alpha_s} \delta^*_{\alpha_s}$$

then by the definition of the dual basis, $\alpha^*(\alpha_s) = \lambda_{\alpha_s}$ for all $\alpha_s \in \Pi$. Hence,

$$\begin{aligned} C &= \{\alpha^* \in V^* \mid \alpha^*(v) \geq 0 \text{ for all } v \in \Phi^+\} \\ &= \{\alpha^* \in V^* \mid \text{Neg}(\alpha^*) = \emptyset\} \\ &= \{\sum_{\alpha_s \in \Pi} \lambda_{\alpha_s} \delta^*_{\alpha_s} \in V^* \mid \lambda_{\alpha_s} \geq 0 \text{ for all } \alpha_s \in \Pi\} \end{aligned} \tag{4.7}$$

4.8. Proposition. *Let $\alpha^* \in C$ and $J = \{\alpha_s \in \Pi \mid \alpha^*(\alpha_s) = 0\}$. Then*

$$\{w \in W \mid \alpha^* w = \alpha^*\} = W_J = \{w \in W \mid \alpha^* w \in C\}$$

Proof. Let $\alpha^* \in C$, and write $\alpha^* = \sum_{\alpha_s \in \Pi} \lambda_{\alpha_s} \delta^*_{\alpha_s}$. Since $\lambda_{\alpha_s} = \alpha^*(\alpha_s) = 0$ for all $\alpha_s \in J$, we can in fact write

$$\alpha^* = \sum_{\alpha_s \in \Pi \setminus J} \lambda_{\alpha_s} \delta^*_{\alpha_s} \tag{4.9}$$

Now for $\alpha_s, \alpha_t, \alpha_u \in \Pi$, if $\alpha_t \neq \alpha_s$, by definition of the right action of W on V^* we have that

$$\begin{aligned} (\delta^*_{\alpha_t} s)(\alpha_u) &= \delta^*_{\alpha_t}(s(\alpha_u)) \\ &= \delta^*_{\alpha_t}(\alpha_u - s\langle \alpha_s, \alpha_u \rangle \alpha_s) \\ &= \delta^*_{\alpha_t}(\alpha_u) \end{aligned}$$

Hence if $\alpha_s \in J$ and $\alpha_t \notin J$, then it follows that $\delta^*_{\alpha_t} s = \delta^*_{\alpha_t}$. So by Equation (4.9), we have $\alpha^* s = \alpha^*$.

We can conclude that $\alpha^* w = \alpha^*$ for all $w \in W_J$, by inducting on the length of w, and hence $W_J \subseteq \{w \in W \mid \alpha^* w = \alpha^*\}$. It directly follows that $\{w \in W \mid \alpha^* w = \alpha^*\} \subseteq \{w \in W \mid w\alpha^* \in C\}$, so it only remains to prove that $\{w \in W \mid w\alpha^* \in C\} \subseteq W_J$.

Let $w \in W$ be such that $\alpha^* w = \beta^* \in C$. If $w \neq 1$, then it is possible to choose $\alpha_s \in \Pi$ so that $\ell(w^{-1}s) = \ell(w^{-1}) - 1$. If we set $w^{-1}s = w_1^{-1}$, then since

$$\begin{aligned} w_1^{-1}(\alpha_s) &= w^{-1}(s(\alpha_s)) \\ &= w^{-1}(-\alpha_s) \\ &= -w^{-1}(\alpha_s) \end{aligned}$$

and $\ell(w^{-1}s) < \ell(w^{-1})$, by Proposition 3.2 we have that $w^{-1}(\alpha_s) < 0$. Hence, $w_1^{-1}(\alpha_s) \in \Phi^+$.

Now since $\beta^* \in C$ and $w_1^{-1}(\alpha_s) \in \Phi^+$, it follows that

$$\begin{aligned} 0 \leq \alpha^*(\alpha_s) &= (\beta^* w^{-1})(\alpha_s) \\ &= (\beta^* w_1^{-1} s)(\alpha_s) \\ &= -\beta^*(w_1^{-1}(\alpha_s)) \leq 0 \end{aligned}$$

Hence, $\alpha^*(\alpha_s) = 0$ and so $\alpha_s \in J$. By the first part of the proof, we have that $\alpha^* s = \alpha^*$. Therefore, since $w = (w_1^{-1}s)^{-1} = sw_1$

$$\alpha^* w_1 = \alpha^* s w_1 = \alpha^* w = \beta^* \in C$$

and $W_J w_1 = W_J s w_1 = W_J w$.

If $w_1 \neq 1$, then repeating the argument, we can find $w_2 \in W$, where $w_2^{-1} := w_1^{-1} t$ with $\ell(w_2^{-1}) = \ell(w_1^{-1} t) = \ell(w_1^{-1}) - 1$ and $\alpha^* w_2 = \beta^* \in C$. Then again $W_J w_2 = W_J w_1 = W_J w$.

Continuing in this way, we obtain a sequence of elements $w, w_1, w_2, \cdots \in W$ such that $\ell(w_i)$ decreases at each step, until $w_j = 1$ for some j and then

$$W_J = W_J w_1 = W_J w_2 = \cdots = W_J$$

Hence $w \in W_J$ and the proof is complete. □

4.10. Proposition.

$$\{\alpha^* w \mid \alpha^* \in C \textit{ and } w \in W\} = \{\beta^* \in V^* \mid |\textit{Neg}(\beta)| < \infty\}$$

Proof. Let $\alpha^* \in C$ and $w \in W$. If $v \in \Phi^+$ then $(\alpha^* w)(v) = \alpha^*(w(v))$ is non-negative if $w(v) \in \Phi^+$. Hence

$$\begin{aligned} \text{Neg}(\alpha^* w) &= \{v \in \Phi^+ \mid \alpha^*(wv) < 0\} \\ &\subseteq \{v \in \Phi^+ \mid w(v) \in \Phi^-\} \\ &= N(w) \end{aligned}$$

which is a finite set by Proposition 3.10. Then

$$\{\alpha^* w \mid \alpha^* \in C \text{ and } w \in W\} \subseteq \{\beta^* \in V^* \mid |\text{Neg}(\beta^*)| < \infty\}$$

To obtain the reverse inclusion, let $\beta^* \in V^*$ so that $|\text{Neg}(\beta^*)| < \infty$. We will show that $\beta^* = \alpha^* w$ for some $\alpha^* \in C$ and $w \in W$.

If $\text{Neg}(\beta^*) = \emptyset$, then clearly $\beta^* \in C$ and $\alpha^* = \beta^*$ and $w = 1$ suffices. So suppose $\beta^* \notin C$ and hence $\text{Neg}(\beta^*) \cap \Pi \neq \emptyset$. Choose $\alpha_s \in \Pi$ such that $\beta^*(\alpha_s) < 0$. Now since $\beta^* s(\alpha_s) = -\beta^*(\alpha_s) > 0$, it follows that $\alpha_s \in \text{Neg}(\beta^*)$, but $\alpha_s \notin \text{Neg}(\beta^* s)$.

Now s permutes $\Phi^+ \setminus \{\alpha_s\}$, and so if $\alpha_t \in \Phi^+ \setminus \{\alpha_s\}$ then $\alpha_t \in \text{Neg}(\beta^* s)$ if and only if $(\beta^* s)(\alpha_t) = \beta^*(s(\alpha_t))$ is negative; which holds if and only if $s(\alpha_t) \in \text{Neg}(\beta^*)$. This fact along with the previous paragraph implies

$$|\text{Neg}(\beta^* s) = s(\text{Neg}(\beta^*) \setminus \{\alpha_s\})| < |\text{Neg}(\beta^*)|$$

Continuing by induction on the size of this set and keeping in mind Equation (4.7) we can deduce that $\beta^* s = \alpha^* w'$ for some $\alpha^* \in C$ and $w' \in W$. Then $\beta^* = \alpha^* w' s$. $\square$

We can now formally state and prove the theorem mentioned at the beginning of this section.

4.11. Theorem. *Let W be a Coxeter group and H a finite subgroup of W. Then wHw^{-1} is a subgroup of some finite parabolic subgroup W_J of W.*

Proof. (Adapted from [**14**]). Suppose for a contradiction that the statement of the theorem is not true. We choose a counter example W, such that the rank of W is minimal over all such groups for which the statement does not hold. Let H be a finite subgroup of W not contained in any finite parabolic subgroup of W.

It follows that W must be infinite and by Proposition 3.11, Φ is also infinite. Now let $\alpha^* = \sum_{\alpha_s \in \Pi} \lambda_{\alpha_s} \delta^*_{\alpha_s}$, where λ_{α_s} are arbitrary positive numbers. Then by our definition of the dual basis, $\alpha^*(\alpha_s) > 0$ for all $\alpha_s \in \Pi$ and hence $\alpha^* \in C$. Further, $\alpha^*(\beta) > 0$ for all $\beta \in \Phi^+$ and so $\alpha^*(v) \neq 0$ for all $v \in \Phi$.

Let $\beta^* = \sum_{h \in H} \alpha^* h$. By Proposition 4.10 $\mathrm{Neg}(\alpha^* h)$ is a finite set for each $h \in H$. Hence

$$\Omega := \bigcup_{h \in H} \mathrm{Neg}(\alpha^* h)$$

is also finite. Now if $v \in \Phi^+ \setminus \Omega$ then $v \in \Phi^+ \setminus \mathrm{Neg}(\alpha^* h)$ for all $h \in H$. Since $(\alpha^* h)(v) = \alpha^*(h(v)) \neq 0$, it follows that $(\alpha^* h)(v) > 0$ for all $h \in H$. So

$$\beta^*(v) = \sum_{h \in H} (\alpha^* h)(v) > 0$$

for all $v \in \Phi^+ \setminus \Omega$. Hence $v \notin \mathrm{Neg}(\beta^*)$ and therefore $\mathrm{Neg}(\beta^*)$ is finite. Proposition 4.10 also implies that $\beta^* = \gamma^* w$ for some $\gamma^* \in C$ and $w \in W$.

Clearly, $\Phi^+ \setminus \Omega$ is non-empty, since Φ^+ is infinite and Ω is finite, so $\beta^* \neq 0$.

For all $k \in H$ we have that

$$\beta^* k = \sum_{h \in H} \alpha^*(hk) = \sum_{g \in H} \alpha^* g = \beta^*$$

and since $\beta^* = \gamma^* w$, we have that

$$\gamma^*(wkw^{-1}) = (\gamma^* w)kw^{-1} = \beta^* k w^{-1} = \beta^* w^{-1} = \gamma^*$$

It therefore follows by Proposition 4.8 that $wkw^{-1} \in W_J$, where $J = \{\alpha_s \in \Pi \mid \gamma^*(\alpha_s) = 0\}$.

Now if $J = \Pi$ then this would mean that $\gamma^*(\alpha_s) = 0$ for all $\alpha_s \in \Pi$ and hence $\gamma^*(v) = 0$ for all $v \in V$. But then $\gamma^* = 0$, which is impossible since $\gamma^* w = \beta^* \neq 0$. Then $H_1 = wHw^{-1} \leq W_J$ and by Proposition 4.2, W_J is itself a Coxeter group which has a smaller generating set then that of W.

Since we assumed that W was the smallest example of a Coxeter group for which H is not contained in any finite parabolic subgroup of W, there must be $K \subseteq J$ such that W_K is finite and $H_1 = wHw^{-1} \leq W_K$. But then H is contained in a conjugate of the finite parabolic subgroup of W_K which is a finite parabolic subgroup of W, and hence a contradiction. □

Before we leave this chapter, we give a corollary which will be used in later chapters.

4.12. Corollary. *The set*

$$\{\langle \alpha, \alpha_s \rangle \; : \; \alpha \in \Phi^+, s \in S, |\langle \alpha, \alpha_s \rangle| < 1\}$$

is finite.

Proof. (Adapted from [**3**] p108). Let

$$\mathcal{P} = \{J \subseteq S \mid |W_J| < \infty\}$$

and

$$\mathcal{A} = \{\langle \alpha, \beta \rangle \mid \alpha, \beta \in \Phi^+, s_\alpha, s_\beta \in \bigcup_{J \in \mathcal{P}} W_J\}$$

Since $\mathcal{P}$ is finite, it is clear that $\mathcal{A}$ is finite. We show that the set in Corollary 4.12 is a subset of $\mathcal{A}$. Let $\alpha \in \Phi^+$ and $s \in S$ be such that $|\langle \alpha, \alpha_s \rangle| < 1$. Then by Proposition 4.4 the subgroup H generated by s_α and s is finite. Hence, by Theorem 4.11 there exists a $w \in W$ and $J \in \mathcal{P}$ such that $wHw^{-1} \subseteq W_J$.

Now $\langle \alpha, \alpha_s \rangle = \langle w(\alpha), w(\alpha_s) \rangle$ and the reflections about $w(\alpha), w(\alpha_s)$ are $s_{w(\alpha)} = ws_\alpha w^-1$ and $s_{w(\alpha_s)} = wsw^-1$ by Section 3.1.

Hence $\langle \alpha, \alpha_s \rangle \in \mathcal{A}$. □

CHAPTER 5

Small Roots and Root Dominance

The main result of this chapter is to describe an important subset of the positive roots of W containing the simple roots called the *small roots*, Σ. Remarkably, we will see that for all Coxeter groups, $|\Sigma| < \infty$. This fact could largely be considered the main result of this thesis and the construction of finite state automata for Coxeter groups is a corollary of this theorem. We also introduce the notion of *dominance*, which serves as a useful characterisation of small roots. The primary reference of this chapter is [**3**] pages 113-117.

5.1. Small Roots

The small roots can be inductively defined as follows:

5.1. Definition. The set of *small roots* $\Sigma \subset \Phi^+$ is constructed as follows:

a) Set $\Sigma_0 := \Pi = \{\alpha_s \mid s \in S\} \subseteq \Sigma$
b) For $\alpha \in \Sigma_{i-1}(i > 1)$ and $s \in S$: if $-1 < \langle \alpha, \alpha_s \rangle < 0$ then $s(\alpha) \in \Sigma_i$
c) $\Sigma = \cup_i \Sigma_i$

By the above construction and Equation (2.3), it follows that the roots α and $s(\alpha)$ only differ in the s-th coordinate by $-2\langle \alpha, \alpha_s \rangle$. Hence, informally, the small roots are the roots which can be reached by successive "small changes" moving up saturated chains in the root poset.

The following definition characterises the covering edges of the root poset into two kinds (see Proposition 3.16). This will be a useful distinction in the discussion of root dominance.

5.2. Definition. A covering edge $\beta \lhd \gamma$ in the root poset is *short* if $|\langle \beta, \alpha_s \rangle| < 1$, where $s \in S$ is such that $s(\beta) = \gamma$. Otherwise, the covering edge is *long*.

Since $\langle \gamma, \alpha_s \rangle = \langle s(\beta), \alpha_s \rangle = \langle \beta, \alpha_s \rangle - 2\langle \beta, \alpha_s \rangle = -\langle \beta, \alpha_s \rangle$, the following proposition regarding the relationship between small roots and short edges is a direct consequence of Definition 5.1 and Definition 5.2 above.

5.3. Proposition.

A root is small $\Longleftrightarrow$ *It is reachable from a simple root up a directed path of short edges on the root poset*

The following Lemma shows that edges down from a small root are short.

5.4. Lemma. *Let $\alpha \in \Sigma$ and $s \in S$, $\alpha \neq \alpha_s$. Then, $\langle \alpha, \alpha_s \rangle < 1$.*

Proof. We induct on $\mathrm{dp}(\alpha)$. If $\mathrm{dp}(\alpha) = 1$ then $\alpha = \alpha_r$ for some $r \in S$. Then clearly, $\langle \alpha_r, \alpha_s \rangle < 1$.

Now suppose $\mathrm{dp}(\alpha) \geq 2$. Then by our construction of Σ, there is $r \in S$ and $\beta \in \Sigma$, such that $\alpha = r(\beta)$ with $-1 < \langle \beta, \alpha_r \rangle < 0$. Consider now the possibilities for $s \in S$. If $s = r$ then

$$\langle \alpha, \alpha_s \rangle = \langle s(\beta), \alpha_s \rangle = \langle \beta, \alpha_s \rangle - 2\langle \beta, \alpha_s \rangle\langle \alpha_s, \alpha_s \rangle = -\langle \beta, \alpha_s \rangle < 1$$

Now suppose $s \neq r$. If $\beta = \alpha_s$, then

$$\langle \alpha, \alpha_s \rangle = \langle r(\alpha_s), \alpha_s \rangle = \langle \alpha_s, \alpha_s \rangle - 2(\langle \alpha_r, \alpha_s \rangle)^2 = 1 - 2(\langle \alpha_r, \alpha_s \rangle)^2 < 1.$$

Finally, if $\beta \neq \alpha_s$, then applying the induction hypothesis to β, it follows that

$$\langle \alpha, \alpha_s \rangle = \langle r(\beta), \alpha_s \rangle = \langle \beta, \alpha_s \rangle - 2\langle \alpha_r, \beta \rangle \langle \alpha_r, \alpha_s \rangle \leq \langle \beta, \alpha_s \rangle < 1. \qquad \square$$

For $\alpha \in \Sigma$, let us now define:

$$\mathcal{N}(\alpha) := \{s \in S \mid |\langle \alpha, \alpha_s \rangle| < 1\}.$$

Then we have the following:

5.5. Lemma. *Let $\beta, \alpha \in \Sigma$ be such that $\beta \lhd \alpha$ in the root poset with* $\mathrm{dp}(\beta) \geq 2$.
Then $\mathcal{N}(\alpha) \subseteq \mathcal{N}(\beta)$.

Proof. Let $s \in S \setminus \mathcal{N}(\beta)$. Since $\beta \in \Sigma$ and $s \in S$ and $\beta \neq s$, by Lemma 4.2, it follows that $\langle \beta, \alpha_s \rangle < 1$. Since $s \notin \mathcal{N}(\beta)$ it follows that $\langle \beta, \alpha_s \rangle \leq -1$.

Since $\alpha = r(\beta)$ for some $r \in S$ and $\alpha > \beta$. Applying Lemma 4.2 to α we have

$$\langle \alpha, \alpha_r \rangle < 1 \implies \langle \beta, \alpha_r \rangle - 2\langle \beta, \alpha_r \rangle \langle \alpha_r, \alpha_r \rangle < 1 \implies \langle \beta, \alpha_r \rangle > -1.$$

Then by Lemma 3.2, since $\mathrm{dp}(r(\beta)) = \mathrm{dp}(\beta) + 1$, we conclude that $-1 < \langle \beta, \alpha_r \rangle < 0$. Hence $r \neq s$ and

$$\begin{aligned}\langle \alpha, \alpha_s \rangle =& \langle r(\beta), \alpha_s \rangle \\ =& \langle \beta - 2\langle \beta, \alpha_r \rangle \alpha_r, \alpha_s \rangle \\ =& \langle \beta, \alpha_r \rangle - 2\langle \beta, \alpha_r \rangle \langle \alpha_r, \alpha_s \rangle \\ \leq& \langle \beta, \alpha_r \rangle.\end{aligned}$$

Therefore $\langle \alpha, \alpha_r \rangle \leq -1$. Hence $s \notin \mathcal{N}(\alpha)$. $\square$

We are now able to prove the remarkable main result of this thesis.

5.6. Theorem. $|\Sigma| < \infty$

Proof. (Adapted from [**3**] page 114). Suppose $|\Sigma| = \infty$. We derive a contradiction. By construction of the root poset, it is clear that at any depth there are only finitely many elements. Hence if $|\Sigma| = \infty$, there must be $\alpha \in \Sigma$ with $\mathrm{dp}(\alpha)$ arbitrarily large.

By Proposition 5.3, it follows that for each $\alpha \in \Sigma$, we have a saturated chain $\mathcal{C}_\alpha$ in the root poset, beginning at a simple root, such that each $\gamma \in \mathcal{C}_\alpha$ is in Σ. So there are saturated chains containing only small roots of arbitrary length.

Consider now the following pairs of the form (J, v): where $J \subseteq S$ and $v \in \mathbb{R}^{|J|}$ such that there exists $\gamma \in \Sigma$ with $\mathcal{N}(\gamma) = J$ and $(\langle \gamma, \alpha_s \rangle)_{s \in J} = v$. By Corollary 4.12, there are only finitely many such pairs.

Let $\mathcal{C}$ be a saturated chain in $\Sigma \setminus \Pi$, such that the length of $\mathcal{C}$ is greater than the number of pairs (J, v). We can choose a saturated segment of $\mathcal{C}$ such that: $\gamma_j \lhd \gamma_{j+1} \lhd \ldots \lhd \gamma_k$ with $\mathrm{dp}(\gamma_i) = i$ for $i = j, \ldots, k$. Such a segment can be chosen with the property that $\mathcal{N}(\gamma_j) = \mathcal{N}(\gamma_k)$ and $\langle \gamma_j, \alpha_s \rangle = \langle \gamma_k, \alpha_s \rangle \ \forall s \in \mathcal{N}(\gamma_k)$ (this also follows from Corollary 4.12).

Now let $s_i \in S$ be such that $s_i(\gamma_i) = \gamma_{i+1}$ for $i = j, \ldots, k-1$. By Lemma 5.4, edges down from a small root are short, so we have $s_{k-1} \in \mathcal{N}(\gamma_{k-1})$. Hence by Lemma 5.5 and our chosen segment of $\mathcal{C}$ it follows that

$$\mathcal{N}(\gamma_j) = \mathcal{N}(\gamma_k) \subseteq \mathcal{N}(\gamma_{k-1}) \subseteq \ldots \subseteq \mathcal{N}(\gamma_{j+1}) \subseteq \mathcal{N}(\gamma_j)$$

Thus $s_i \in \mathcal{N}(\gamma_k)$ for $i = j, ..., k-1$.

Let $\gamma_{j-1} := s_{k-1}(\gamma_j)$. We now show that $\gamma_{j-1} \lhd \gamma_j$. By the fact that $\langle \gamma_j, \alpha_s \rangle = \langle \gamma_k, \alpha_s \rangle$ for all $s \in \mathcal{N}(\gamma_k)$, we have

$$\begin{aligned}
\langle \gamma_{j-1}, \alpha_s \rangle =& \langle s_{k-1}(\gamma_j), \alpha_s \rangle \\
=& \langle \gamma_{j-1}, s_{k-1}(\alpha_s) \rangle \qquad \text{(by Lemma 2.4)} \\
=& \langle \gamma_j, \alpha_s - 2\langle \alpha_{s_{k-1}}, \alpha_s \rangle \alpha_{s_{k-1}} \rangle \\
=& \langle \gamma_k, \alpha_s - 2\langle \alpha_{s_{k-1}}, \alpha_s \rangle \alpha_{s_{k-1}} \rangle \\
=& \langle \gamma_k, s_{k-1}(\alpha_s) \rangle \\
=& \langle s_{k-1}(\gamma_k), \alpha_s \rangle \\
=& \langle \gamma_{k-1}, \alpha_s \rangle
\end{aligned}$$

Since $\gamma_j = s_{k-1}(\gamma_{j-1})$ and $\langle \gamma_{k-1}, \alpha_{s_{k-1}} \rangle = \langle \gamma_{j-1}, \alpha_{s_{k-1}} \rangle < 0$, by Lemma 3.13, we have dp$(s_{k-1}(\gamma_{j-1})) =$ dp$(\gamma_{j-1}) + 1$. Therefore by Definition 3.14, it follows that $\gamma_{j-1} \lhd \gamma_j$.

We now have a saturated chain of roots:

$$\gamma_{j-1} \lhd \gamma_j \lhd ... \lhd \gamma_{k-1}$$

such that $\langle \gamma_{j-1}, \alpha_s \rangle = \langle \gamma_{k-1}, \alpha_s \rangle$ for all $s \in \mathcal{N}(\gamma_k)$ with $s_{k-1}(\gamma_{j-1}) = \gamma_j$ and $s_i(\gamma_i) = \gamma_{i+1}$ for $i = j, ..., k-2$.

Continuing in this way, we obtain a saturated chain of small roots all the way down to a simple root γ_1:

$$\gamma_1 \lhd \gamma_2 \lhd ... \lhd \gamma_{k-1} \lhd \gamma_k$$

such that

$$\langle \gamma_1, \alpha_s \rangle = \langle \gamma_{k-j+1}, \alpha_s \rangle \tag{5.7}$$

for all $s \in \mathcal{N}(\gamma_k)$. Now let $r \in S$ be such that $\alpha_r = \gamma_1$ and consider two cases for α_r:

Case 1: $r \in \mathcal{N}(\gamma_k)$
By Equation (5.7), we have that $\langle \gamma_1, \alpha_r \rangle = \langle \gamma_{k-j+1}, \alpha_r \rangle$. Since $\alpha_r = \gamma_1$, it follows that $\langle \gamma_1, \alpha_r \rangle = 1$. But $\mathcal{N}(\gamma_{k-j+1}) = \mathcal{N}(\gamma_k)$, which implies $|\langle \gamma_{k-j+1}, \alpha_r \rangle| < 1$. This is a contradiction.

Case 2: $r \notin \mathcal{N}(\gamma_k)$
Here we have $r \neq s_i$. Let $s_i \in \{s_j, ..., s_{k-1}\}$ be such that $s_i(\gamma_{k-j}) = \gamma_{k-j+1}$. Since $\gamma_{k-j} = s_i(\gamma_{k-j+1})$ and dp$(\gamma_{k-j}) =$ dp$(\gamma_{k-j+1}) - 1$, by Lemma 3.13, this implies that $\langle \gamma_{k-j+1}, \alpha_{s_i} \rangle > 0$. But by Equation (5.7), we have

$$0 \geq \langle \gamma_1, \alpha_{s_i} \rangle = \langle \gamma_{k-j+1}, \alpha_{s_i} \rangle > 0$$

which is also a contradiction. This concludes the proof. □

5.2. Dominance

Dominance is another useful characterisation of small roots which will be purposeful in the discussion of the automaton. The primary reference for this section is [**3**] pages 116-117.

5.8. Definition. $\alpha, \beta \in \Phi^+$. α *dominates* β if

$$w(\alpha) < 0 \implies w(\beta) < 0 \text{ for all } w \in W$$

A positive root is called *humble* if it does not dominate any root but itself. By Proposition 3.8 it is clear that all simple roots are humble.

5.9. Lemma. $\alpha, \beta \in \Phi^+$. *If β dominates α, then $w(\beta)$ dominates $w(\alpha)$ for all $w \in W$ such that $w(\alpha) \in \Phi^+$.*

Proof. Suppose β dominates α. Let $w \in W$ be such that $w(\alpha) \in \Phi^+$. By dominance, $w(\beta) \in \Phi^+$. Then for any $\hat{w} \in W$ such that $\hat{w}(w(\beta)) = \hat{w}w(\beta) < 0$ it follows that $\hat{w}(w(\alpha)) = \hat{w}w(\alpha) < 0$. So $w(\beta)$ dominates $w(\alpha)$. □

5.10. Lemma. *Let (W, S) be a finite dihedral group, with $S := \{s, t\}$ and corresponding simple roots α_s and α_t. If $\beta \in \Phi^+ \setminus \{\alpha_s\}$, then there exists $w \in W$ such that $w(\alpha_s) > 0$ and $w(\beta) < 0$.*

Proof. If $\beta = \alpha_t$, then taking $w = t$, we have that $w(\alpha_s) > 0$ by Proposition 3.8 and $w(\alpha_t) = w(\beta) < 0$. Hence, suppose that $\beta \in \Phi^+ \setminus \{\alpha_s, \alpha_t\}$. Let $w_0 = stst\cdots = tstst\cdots$ be the longest element of the dihedral group. Let $w = w_0 s$. Then $\ell(w) = \ell(w_0) - 1$ and it follows that

$$\ell(ws) = \ell(w_0 ss) = \ell(w_0) = \ell(w) + 1.$$

Hence $w(\alpha_s) > 0$ by Proposition 3.8. Now by Proposition 3.10 we have that $|D(w)| = \ell(w_0) - 1$, and since $\alpha_s \notin D(w)$ it follows that $D(w) = \Phi^+ \setminus \{\alpha_s\}$. Hence $\beta \in D(w)$ and therefore $w(\beta) < 0$ as required. □

5.11. Lemma. *Let $\beta \in \Phi^+$ and $s \in S$. Then β dominates α_s if and only if $\langle \beta, \alpha_s \rangle \geq 1$.*

Proof. "$\Rightarrow$" We ignore the obvious case of $\beta = \alpha_s$ and suppose $\beta \neq \alpha_s$. Then $s(\beta) \in \Phi^+$. Suppose that β dominates α_s and $\langle \beta, \alpha_s \rangle < 1$. Now since β dominates α_s and $s_\beta(\beta) < 0$ we have that

$$s_\beta(\alpha_s) = \alpha_s - 2\langle \beta, \alpha_s \rangle \beta < 0$$

This forces $\langle \beta, \alpha_s \rangle > 0$. Note that $\langle \beta, \alpha_s \rangle \neq 0$ since the coefficient of α_s cannot be positive. Hence $|\langle \beta, \alpha_s \rangle| < 1$.

It follows by Proposition 4.4 part (i) that the subgroup generated by s_β and s is finite dihedral, so $(s_\beta s)$ acts as a rotation with finite order on $V_0 := \mathbb{R}_\beta \oplus \mathbb{R}_{\alpha_s}$. Hence by Lemma 5.10 we can find a $w \in W_{s_\beta, s}$ such that $w(\beta) < 0$ and $w(\alpha_s) > 0$ which is a contradiction.

"$\Leftarrow$" Now suppose $\langle \beta, \alpha_s \rangle \geq 1$. We have that

$$\begin{aligned}
\langle s(\beta), \beta \rangle &= \langle \beta - 2\langle \alpha_s, \beta \rangle \alpha_s, \beta \rangle \\
&= \langle \beta, \beta \rangle - 2\langle \alpha_s, \beta \rangle^2 \\
&= 1 - 2(\langle \alpha_s, \beta \rangle)^2 \leq -1
\end{aligned}$$

Then by part (ii) of Proposition 4.4 there are infinitely many positive roots of the form $\lambda\beta + \mu s(\beta)$ with $\lambda, \mu > 0$. Now let $w \in W$ is such that $w(\beta) < 0$. If $w(\alpha) > 0$ it follows that

$$w(s(\beta)) = w(\beta - 2\langle \alpha_s, \beta \rangle \alpha_s) = w(\beta) - 2\langle \alpha_s, \beta \rangle w(\alpha_s) < 0$$

Hence, $w(\beta) < 0$ and $w(s(\beta)) < 0$. But then $|N(w)| = \infty$, since all roots of the form $\lambda\beta + \mu s(\beta)$ with $\lambda, \mu > 0$ are in $N(w)$. This contradicts Proposition 3.10. Therefore we must have $w(\alpha_s) < 0$. So β dominates α_s.

□

5.12. Lemma. *Let $\beta, \alpha \in \Phi^+$ be such that $\beta \lhd \alpha$ in the root poset. Then the following holds:*

(i) *If $\beta \lhd \alpha$ is long, then α is humble.*

(ii) *If $\beta \lhd \alpha$ is short, then α is humble if and only if β is humble.*

Proof. (i) Let $s \in S$ be such that $s(\beta) = \alpha$. Suppose that the covering edge $\beta \lhd \alpha$ is long, so $|\langle \beta, \alpha_s \rangle| \geq 1$. By construction of the root poset, we have that $\langle \beta, \alpha_s \rangle < 0$ and since

$$\langle \alpha, \alpha_s \rangle = \langle s(\beta), \alpha_s \rangle = \langle \beta, \alpha_s \rangle - 2\langle \alpha_s, \beta \rangle = -\langle \alpha_s, \beta \rangle$$

It follows that $\langle, \alpha, \alpha_s \rangle \geq 1$. So by Lemma 5.11, α dominates α_s. Therefore α is not humble.

(ii) Now suppose that the covering edge $\beta \lhd \alpha$ is short, so $|\langle \beta, \alpha_s \rangle| < 1$. Hence by construction of the root poset, $-1 < \langle \beta, \alpha_s \rangle < 0$. We prove the statement by contra-positive. Suppose β is not humble. Let $\gamma \in \Phi^+ \setminus \{\beta\}$ be such that β dominates γ. By Lemma 5.11 it must be that $\gamma \neq \alpha_s$. So $s(\gamma) \in \Phi^+$ by Proposition 3.8. Then by Lemma 5.9 $\alpha = s(\beta)$ dominates $s(\gamma)$. So α is not humble. Therefore, by contra-positive, if α is humble then β is humble.

By contra-positive again, assume α is not humble. Then there is $\gamma \in \Phi^+$ such that α dominates γ. Again $\gamma \neq \alpha_s$ since $\beta \lhd \alpha$ is short. Hence $s(\gamma) \in \Phi^+$. By Lemma 5.9 again, $\beta = s(\alpha)$ dominates γ. So β is not humble. Therefore, if β is humble then α is humble. $\square$

The following theorem shows that the small roots are precisely the roots which are humble.

5.13. Theorem. *Let $\alpha \in \Phi^+$. Then $\alpha \in \Sigma$ if and only if α is humble.*

Proof. "$\Rightarrow$" Let $\alpha \in \Sigma$. Then there is a saturated chain in the root poset with a "walk" along short edges from a simple root to α. Since the small roots are humble, applying part (ii) of Lemma 5.12 implies that α is humble.

"$\Leftarrow$" Now let α_p be humble. Part (i) of Lemma 5.12 implies that the covering edge down from α_p is short. Then part (ii) of Lemma 5.12 implies that α_{p-1} is humble. Repeating this argument p-1 times implies that α_p can be reached from a simple root along a "walk" of short edges in the root poset, so $\alpha_p \in \Sigma$. $\square$

5.14. Corollary. *The small roots Σ is an order ideal in the root poset.*

Proof. Recall that an *order ideal* of a poset P, is a subset $I \subseteq P$, such that if $y \in I$ and $x < y$ then $x \in I$.

Now let $\alpha \in \Sigma$ and $\beta \in \Phi^+$ with $\beta \lhd \alpha$. Since α is humble by Theorem 5.13, it follows by Lemma 5.12 that the covering edge $\beta \lhd \alpha$ is short. Therefore β is humble and so $\beta \in \Sigma$. $\square$

CHAPTER 6

Finite State Automaton

In this final chapter, we put together the tools developed throughout the thesis to construct finite state automata for Coxeter groups. For this thesis, the automata to which we refer is always one built for the purpose of recognising the language of reduced words of a Coxeter system (W, S). We first construct the Brink-Howlett automaton and then introduce a second approach, the Cannon automaton. The primary references for this chapter are [**9**], [**3**], [**4**] and [**14**]. We begin by formalising the definition of the automaton.

6.1. Definition. A *finite state automaton* $\mathcal{A}$ is a quintuple (T, A, μ, Y, t_0), where T is a finite set, called *state set*, A a finite set called *alphabet*, $\mu : T \times A \to T$ a function, called a *transition function*, $Y \subset T$ is the set of *accept* states, and t_0 is the *initial* state.

The above definition is a general one from [**9**]. For our purposes, we may be more restrictive and specific. We consider the quintuple to be the following:

$T =$ The set of nodes of a directed graph.
$A =$ S (The Coxeter generators).
$\mu =$ The function of "walking" from node to node in the graph.
$Y =$ Subsets $\tilde{Y}$ of T, such that there is a walk in the graph containing $\tilde{Y}$.
$t_0 =$ A node of the graph designated as the "start" node.

The "operation" of the automaton works as follows. Take a string of generators (*word*) $w = s_1 s_2 \cdots s_k$ as input. Beginning at the "start" node, read the word w from the left, one generator (*letter*) at a time by following the directed edge labelled by each letter.

If there is such a path for the complete string $s_1 s_2 \cdots s_k$ we say the word is *accepted* by the automaton. Otherwise say that w is *rejected*. We denote the *reject* state of $\mathcal{A}$ simply as "No". To precisely follow Definition 6.1, there should be a node for the reject state, so that for a rejected word, there is a directed edge to this node. However in practice, this adds some clutter to the graph and so if a word is rejected, there is simply no walk in the graph which corresponds to the word.

An automaton for a Coxeter group is by no means unique, as demonstrated by the following example.

6.2. Example. Consider the infinite dihedral Coxeter group with presentation:

$$W = \langle s, t \mid s^2 = t^2 = 1 \rangle$$

Two automata recognising the reduced words of W are:

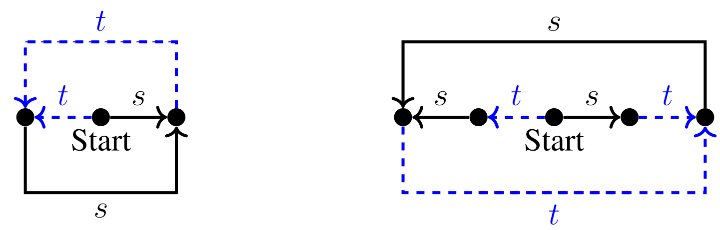


FIGURE 1. Examples of Automata for W

Since the generators s, t have no relations between them, all words of the form $ststst\cdots$ or $tststs\cdots$ are clearly reduced. In the above automatons, beginning at the start node and following the directed labels, both walks of arbitrary length exist. So words of these forms are accepted. The word stt, for example, is clearly not reduced, and it can be seen that there is no t-labeled edge emanating from the nodes reached by the walk st. So stt is rejected by both automatons.

It is not hard to see that the automaton on the left of Example 6.2 has the smallest possible number of nodes for an automaton of W. This motivates a definition.

6.3. Definition. Let W be a Coxeter group and $\mathcal{A}$ an automaton for W with corresponding state set $T_{\mathcal{A}}$. Then $\mathcal{A}$ is *minimal* if whenever $\mathcal{B}$, is also an automaton for W with corresponding state set $T_{\mathcal{B}}$, we have

$$|T_{\mathcal{A}}| \leq |T_{\mathcal{B}}|$$

6.1. The Brink-Howlett Automaton

In this section, we present the automaton described by Brink-Howlett in [**4**]. Prior to stating the algorithm, we provide a brief discussion of the idea of the construction and informally discuss why it follows by Theorem 5.6 that such an automaton exists.

Let us consider the construction of an automaton $\mathcal{A}$ for a Coxeter group W. A direct approach is to simply let $T = W$ and define a transition function by putting an s-labeled directed edge from w to ws if $\ell(ws) > \ell(w)$. However, it is clear that for an infinite Coxeter group, $\mathcal{A}$ also has infinitely many nodes. A one-to-one correspondence of nodes to elements will not work if we wish to construct a finite graph.

Another approach can be considered as follows. Recall from Definition 3.9 the *descent* set of w, $D(w)$ is the set of positive roots which become a negative root under the action of w. We let $T = D(w)$, and define a transition function $\mu : T \times S \to T$ as follows:

$$\mu(D(w), s) = \begin{cases} \text{No} & \text{if } \alpha_s \in D(w) \\ D(ws) & \text{if } \alpha_s \notin D(w) \end{cases} \tag{6.4}$$

In other words, we construct $\mathcal{A}$ by putting an s-labeled edge between $D(w)$ and $D(ws)$ whenever $\alpha_s \notin D(w)$.

To see that this constructs an automaton can be seen as follows. Recall that the cardinality of $D(w)$ is precisely $\ell(w)$ by Proposition 3.10. Hence $\ell(ws) > \ell(w)$ if and only if $|D(ws)| = |D(w)| + 1$. Now by Proposition 3.2 $\ell(ws) > \ell(w)$ if and only if $w(\alpha_s) > 0$, i.e. whenever $\alpha_s \notin D(w)$. Hence, the construction in Equation (6.4) ensures that we only have an s-labeled edge from any node if s lengthens the word.

The following proposition describes how $D(ws)$ can be computed from knowing $D(w)$.

6.5. Proposition. *Let $w \in W$ be such that $\ell(w) = k$. Suppose $D(w) = \{\beta_1, \beta_2, \cdots \beta_k\}$. If $s \in S$ is such that $\ell(ws) > \ell(w)$ then $D(ws) = \{\alpha_s\} \cup \{s(\beta_1), s(\beta_2), \cdots s(\beta_k)\}$*

Proof. Let $s \in S$ be such that $\ell(ws) > \ell(w)$. Since $w(\beta_i) < 0$ for all $\beta_i \in D(w)$, it follows that $ws(s(\beta)) = wss(\beta) = w(\beta) < 0$. By Proposition 3.8 since $s(\alpha_s) < 0$, it follows that $ws(\alpha_s) = -w(\alpha_s)$. Now $w(\alpha_s) > 0$, since if it wasn't, Proposition 3.2 implies that $\ell(ws) < \ell(w)$ which is a contradiction. Hence, by Proposition 3.10, the set $D(ws)$ is precisely $\{\alpha_s\} \cup \{s(D(w))\}$. $\square$

Proposition 6.5 provides for a method to construct walks in the automaton whereby the nodes correspond to $D(w)$ rather than w. However, we again run into the problem of finiteness. For an infinite Coxeter group, there are infinitely many $D(w)$.

The key is to consider the following: Rather than setting $T = D(w)$, we let $T = D(w) \cap \Sigma$. In other words, instead of considering all the positive roots "flipped" negative by w, we only consider the small roots which become negative when acted upon by w and "throw away" any non-small roots. We modify Equation (6.4) by replacing $D(w)$ with $D(w) \cap \Sigma$ and $D(ws)$ with $D(ws) \cap \Sigma$. By Theorem 5.6, since $|\Sigma| < \infty$, there only are finitely many such sets and hence we obtain an automaton with only finitely many nodes. This is a truly remarkable result.

We formally define the sets we require.

6.6. Definition. The *small descent set* of $w \in W$ is:

$$D_\Sigma(w) = \{\alpha \in \Sigma \mid w(\alpha) < 0\}$$

The question remains as to whether we can determine $D_\Sigma(ws)$ just by knowing $D_\Sigma(w)$. The following technical lemma and proposition shows that this is indeed possible.

6.7. Lemma. *If $\alpha \in \Sigma$, $s \in S$ and $w \in W$ is such that $s(\alpha) \in \Phi^+ \setminus \Sigma$ and $\ell(ws) > \ell(w)$. Then $ws(\alpha) > 0$.*

Proof. Since $\ell(ws) > \ell(w)$, it follows by Proposition 3.2 that $w(\alpha_s) > 0$. Therefore, $ws(\alpha_s) = -w(\alpha_s) < 0$. Now suppose that $ws(\alpha) < 0$. By Corollary 5.14, Σ is an order ideal, and since we have $\alpha \in \Sigma$ and $s(\alpha) \in \Phi^+ \setminus \Sigma$, it must be that $\langle \alpha, \alpha_s \rangle \leq -1$ (recall that if $-1 < \langle \alpha, \alpha_s \rangle < 0$ then $s(\alpha) \in \Sigma$).

Now by Proposition 4.4, the roots $(s_\alpha s)^n(\alpha)$ are non-negative distinct linear combinations of the the roots α and α_s for $n \in \mathbb{N}$. Hence, since $ws(\alpha)$ and $ws(\alpha_s)$ are negative, we have that

$$ws((s_\alpha s)^n(\alpha)) = ws(\lambda\alpha + \mu\alpha_s) = \lambda ws(\alpha) + \mu ws(\alpha_s) < 0$$

But then $|D(ws)| = \infty$, which contradicts the fact that $\ell(ws) = |D(ws)|$. Hence, we must have $ws(\alpha) > 0$. $\square$

6.8. Proposition. *$w \in W$, $s \notin D_\Sigma(w)$. Then,*

$$D_\Sigma(ws) = \{\alpha_s\} \cup \{\{s(\beta) : \beta \in D_\Sigma(w)\} \cap \Sigma\}$$

Proof. Suppose $\alpha \in D_\Sigma(ws)$. Then $\alpha \in \Sigma$ and $ws(\alpha) < 0$. So by Lemma 6.7 then $s(\alpha) \notin \Phi^+ \setminus \Sigma$. So either $s(\alpha) \in \Phi^- \implies \alpha = \alpha_s$ or $s(\alpha) \in \Sigma$. Since $w(s(\alpha)) < 0$ this implies that $s(\alpha) \in D_\Sigma(w)$. Hence, $D_\Sigma(ws) \subseteq \{\alpha_s\} \cup \{\{s(\beta) : \beta \in D_\Sigma(w)\} \cap \Sigma\}$.

Now if $\alpha \in \{\alpha_s\} \cup \{\{s(\beta) : \beta \in D_\Sigma(w)\} \cap \Sigma\}$, then either $\alpha = \alpha_s$ or $\alpha = s(\beta)$, with $\beta \in D_\Sigma(w)$. If $\alpha = \alpha_s$ then by the same argument as in the proof of Proposition 6.5 it follows that $w(s(\alpha)) = w(-\alpha_s) = -w(\alpha_s) < 0$. In the latter case, we have that $ws(\alpha) = ws(s(\beta)) = w(\beta) < 0$. Hence the reverse inclusion holds. $\square$

We give an explicit algorithm to construct the Brink-Howlett automaton.

6.9. Proposition. *(Brink-Howlett Automaton)*
Let (W, S) be a Coxeter system. Construct a finite state automaton as follows:

(1) *Using Definition 5.1 determine the small roots Σ of W.*
Let $T = \{D_\Sigma(w) \mid w \in W\}$. S will be iteratively constructed.
(2) *Initialise the algorithm by setting $D_\Sigma(e) = D \in T$.*
(3) *For each $s \in S$ such that $\alpha_s \notin D$, put an s-labeled directed edge:*

$$D \xrightarrow{s} \{\alpha_s\} \cup \{s(D) \cap \Sigma\}$$

By Proposition 6.8 $\{\alpha_s\} \cup \{s(D) \cap \Sigma\} \in T$ when $\alpha_s \notin D$.
(4) *Repeat step (3) for each $D \in S$.*

6.10. Theorem. *A word $w = s_1 \cdots s_k$ is reduced if and only if $s_1 \cdots s_k$ is a walk in the Brink-Howlett Automaton, $\mathcal{A}$.*

Proof. We prove the theorem by induction. The result is clearly true if $k = 1$. Hence assume that it is true for $k = n$. So for a reduced word $w = s_1 \cdots s_n$, $s_1 \cdots s_n$ is a walk in $\mathcal{A}$. Let $ws_{n+1} = s_1 \cdots s_n s_{n+1}$ be reduced. We want to show that $s_1 \cdots s_n s_{n+1}$ is a walk in $\mathcal{A}$. Clearly, ws_{n+1} is reduced if and only if w is reduced. By Proposition 3.2, w is reduced if and only if $w(\alpha_{n+1}) > 0$ (otherwise $\ell(ws_{n+1}) < \ell(w)$, a contradiction to ws_{n+1} being reduced).

We have the following walk in $\mathcal{A}$:

$$\bullet \xrightarrow{s_1} \bullet \dashrightarrow \bullet \xrightarrow{s_{n-1}} \bullet \xrightarrow{s_n} \bullet$$

Now by step (3) of Proposition 6.9 we only put an s_{n+1}-labeled edge out of the last node if $\alpha_{n+1} \notin D_\Sigma(w)$, i.e. $w(\alpha_{n+1}) > 0$. This concludes the proof. $\square$

6.2. The Cannon Automaton

In this section we describe another approach to constructing an automaton for Coxeter groups, termed the Cannon automaton. The Cannon automaton differs from the Brink-Howlett by being one which we apply by a geometric construction. This automaton was first introduced in [**5**] in Cannon's study of the language of geodesics of discrete groups of hyperbolic transformations ([**9**] p66). In fact, the very study of reduced words of finitely generated groups originated from [**5**].

We now introduce some definitions appropriate for this construction.

6.11. Definition. Let $w \in W$. The *cone* of w is the set:

$$C(w) = \{v \in W \mid \ell(v) = \ell(w) + \ell(w^{-1}v)\}.$$

In other words, if $v \in C(w)$, there is a reduced "walk" from the identity e to v passing through w. Note that $w \in C(w)$ for all $w \in W$.

6.12. Definition. Let $w \in W$. The *cone type* of w is the set:

$$CT(w) = w^{-1}C(w)$$

Note that by definition, the cone of w is distinct for all $w \in W$, but cone type is not. In other words, cone types are independent of representative.

6.13. Example. The infinite dihedral group in Example 6.2 has two cone types. The cone of s, are the elements which are reduced when "passing through" s, so consists of the elements $\{s, st, sts, stst, \cdots\}$. Hence, the cone type of s is $CT(s) = sC(s) = \{e, t, ts, tst, tsts, \cdots\}$. By symmetry, the cone type of t is $CT(t) = \{e, s, st, sts, stst, \cdots$. It is clear that $CT(s) = CT(ts) = CT(tst) = \cdots\}$ and $CT(t) = CT(st) = CT(sts) = \cdots$.

We denote $\mathcal{A}_C$ to be the Cannon automaton. In the language of Definition 6.1, we let our set of nodes be $T = \{CT(w) \mid w \in W\}$. Our transition function $\mu : T \times S \to T$ is defined as follows:

$$\mu(CT(w), s) = \begin{cases} \text{No} & \text{if } \ell(ws) = \ell(w) - 1 \\ CT(ws) & \text{if } \ell(ws) = \ell(w) + 1 \end{cases} \tag{6.14}$$

The transition function μ determines two things: whether ws is reduced, and if so, to which node to direct the s-labeled edge. In the Brink-Howlett automaton, we make both determinations by calculating the small descent set of ws. In this construction, we determine the length of ws and cone type geometrically.

As in the Brink-Howlett automaton, the critical question is whether the set of cone types T is finite. For a general group, the set of cone types may not be finite ([**9**] p66). The fact that $|T| < \infty$ for Coxeter groups is a corollary of the minimality of the Cannon automaton in Theorem 6.22 and the fact that the small roots are finite.

We give an explicit algorithm for the Cannon automaton.

6.15. Proposition. *(Cannon automaton) Let (W, S) be a Coxeter system. Construct an automaton as follows:*

(1) *Define $\mathcal{T}_k := \{CT(w) \mid \ell(w) = k\}$ and let $\mathcal{T} := \cup_k \mathcal{T}_k$ be the set of nodes of the graph which will be iteratively constructed. Initiate the algorithm by setting $k = 0$.*
(2) *Consider a node $CT(w) \in \mathcal{T}_k$ where $\ell(w) = k$.*
(3) *For $s \in S$ such that s such that $\ell(ws) = \ell(w) + 1$, determine $CT(ws)$.*
(4) *Put an s-label directed edge $CT(w) \to CT(ws)$.*
(5) *When all nodes $CT(w) \in \mathcal{T}_k$ have been considered, increment k by 1 and return to step 2.*

The fact that $\mathcal{A}_C$ is an automaton recognising reduced words is almost immediate by definition.

6.16. Theorem. *A word $w = s_1 \cdots s_k$ is reduced if and only if $s_1 \cdots s_k$ is a walk in the Cannon Automaton, $\mathcal{A}_C$.*

Proof. If $w = s_1 \cdots s_k$ is reduced, then $w = s_1 \cdots s_i$ is reduced for all $1 \le i \le k$. Hence, $\ell(w_i s_{i+1}) = \ell(w_i) + 1$ for all $1 \le i \le k-1$. Thus, by Equation (6.14) we have s_i-labeled edges

$$\underset{CT(e)}{\bullet} \xrightarrow{s_1} \underset{CT(s_1)}{\bullet} \dashrightarrow \bullet \xrightarrow{s_{k-1}} \bullet \xrightarrow{s_k} \underset{CT(s_1 s_2 \cdots s_k)}{\bullet}$$

Now suppose we have the above walk $s_1 \cdots s_k$ in $\mathcal{A}_C$. We induct on length. There is an s_1-labeled edge from $CT(e)$ to $CT(s_1)$ and indeed $\ell(s_1) = \ell(e) + 1$ with s_1 reduced. Hence, assume it is true for $k - 1$, i.e. there is a walk $s_1 \cdots s_{k-1}$ in $\mathcal{A}_C$ and the word $w_{k-1} = s_1 \cdots s_{k-1}$ is reduced. Since we have the walk $s_1 \cdots s_k$ in $\mathcal{A}_C$, we have an s_k-labeled edge from $CT(w_{k-1})$ to $CT(w_{k-1} s_k)$. Hence by Equation (6.14) this means that $\ell(w_{k-1} s_k) = \ell(w_{k-1}) + 1$. Therefore, $s_1 \cdots s_k$ is reduced. $\square$

We have yet to describe how one computes the cone and cone type of a word. As noted in the introduction of this section, the Cannon automaton is a definitively geometric construction. Hence, Definition 6.11 and Definition 6.12 have geometric interpretations which we will now discuss in generality. We will follow this discussion with our recurring example $\tilde{A}_2$.

To understand the geometric interpretation of cones and cone types, we very briefly introduce the notion of *chamber systems* and the *Coxeter complex*. We saw the Coxeter complex and some of the intuitive terminology in the opening examples of Chapter 1. We will now provide slightly more detail. The primary references here are [**17**], [**18**], [**2**] and [**7**].

6.17. Definition. A set $\mathcal{C}$ with a given equivalence relation $\sim_i$ is called a *chamber system* over I, if each $i \in I$ determines a partition of $\mathcal{C}$. The elements of $\mathcal{C}$ are called *chambers* and the relation $\sim_i$ is called the *i-adjacency relation*.

For a chamber system $\mathcal{C}$ over a set I, we may construct an associated simplicial complex. For our purposes, the association with a simplical complex serves to provide the geometric structure for constructing the automaton. Hence, we state this without further discussion and refer the reader to ([**18**], pp11-13) or [**7**] for further details.

6.18. Definition. Let (W, S_I) be a Coxeter system. The *Coxeter complex* $\mathcal{C}(W)$ of W is the chamber system over S. The chambers are the elements of W with $\sim_i$ adjacency relation given by $w \sim_i w$ and $w \sim_i ws_i$ for all $w \in W$ and $s_i \in S$.

Geometrically, the Coxeter group W acts on the Coxeter complex of W as reflections in its simplicial complex, $\Gamma(W)$. The chambers of the Coxeter complex are realised geometrically as simplicies in $\Gamma(W)$, which are fundamental domains for the action of W on $\Gamma(W)$. They are "glued" together so that for any vertex x, the reflections in the edges incident to x (or "walls" of the adjacent chambers) generate a finite subgroup of W.

The *cone* of a word $w = s_1 \cdots s_j$ is geometrically the intersection of *half spaces* encountered in a "walk" from the identity chamber along the path of reflecting walls $s_1 \cdots s_j$. Thus, the cone is geometrically realised as a convex polytope and hence the *cone type* $CT(w)$ is the translation of the convex polytope back to the identity chamber. (See [**15**] for further discussion).

6.19. Example ($\tilde{A}_2$)**.** We give a brief demonstration of the geometric calculation of cone types. Figure 2 illustrates the Coxeter complex. The shaded dark grey region is the identity chamber. Note that the chambers surrounding the vertex at the top right corner of the identity form the subgroup $W_{\{s,u\}}$, similarly the chambers surrounding the top left corner form the subgroup $W_{\{t,u\}}$.

The cone of s is simply given by the half space (light grey shaded region) separated by the wall H_s. The cone type of s is then the translation of this half space to the identity chamber so that the walls of the chamber of s match the walls of the chamber of e. For the simple relfections, this is in fact obtained simply by reflecting the cone in the H_s wall, giving the opposite half space. The cone types of the other simple reflections are obtained in the same way (presented in Appendix B1).

To then obtain the cone of the element su, one takes the intersection of the half spaces obtained by crossing the s-wall of e (light grey shaded region) and then the u-wall of s (darker shaded region containing the su chamber).

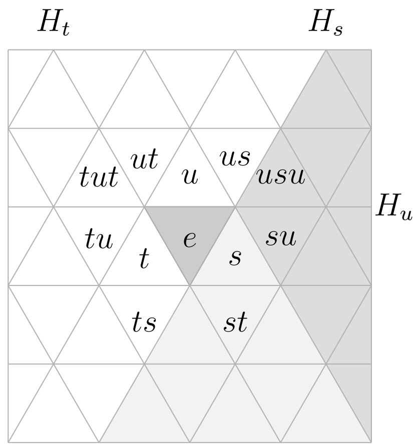


FIGURE 2. Coxeter complex for $\tilde{A}_2$

To obtain the cone type of su, one translates the cone of su to the identity chamber. Colloquially, one "picks up" the cone at su and "places it" ontop of the identity so that the walls of the chamber of su match that of the identity chamber. The cone type of su is shown in Appendix B1 along with the complete set of cone types of $\tilde{A}_2$.

6.3. Comparison of Automata

In this section we present the automatons produced by the Brink-Howlett and Cannon algorithms for the groups $\tilde{A}_2$ and $\tilde{B}_2$. The calculation of the Brink-Howlett small descent sets and Cannon cone types are presented in Appendix A and Appendix B respectively. Finally, we show that the Cannon automaton is minimal for Coxeter groups.

6.20. Example. Brink-Howlett and Cannon automaton for $\tilde{A}_2$

$\tilde{A}_2$ has presentation:

$$(p \wedge (p \implies q)) \implies q\langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (su)^3 = 1\rangle$$

The generator s is represented by the black solid line, u is represented by the blue dashed line and t is given by the green dotted line. In some overlapping areas that may cause confusion, an edge has been thickened or scattered for readability.

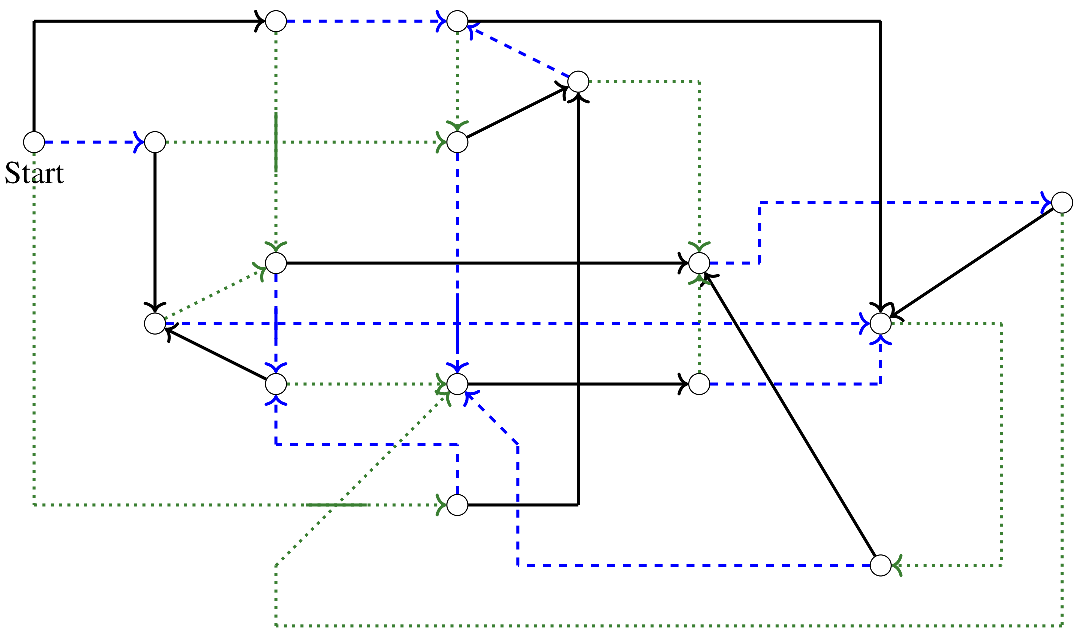


FIGURE 3. Brink-Howlett automaton for $\tilde{A}_2$

For $\tilde{A}_2$, both Brink-Howlett and Cannon algorithms produce the same number of nodes, so there is a one to one correspondence between small descent sets and cone types. It is stated in [**3**] that the Brink-Howlett automaton is minimal for the $\tilde{A}_n$ class of Coxeter groups.

6.21. Example. Brink-Howlett automaton and Cannon automaton for $\tilde{B}_2$

$\tilde{B}_2$ has the following presentation

$$\langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^4 = (tu)^4 = (su)^2 = 1\rangle$$

As above, the generator s is represented by the black solid line, u by the blue dashed line and t by the green dotted line. Figure 4 presents the automaton produced by the Brink-Howlett algorithm. This automaton has 25 nodes. The reader should take note of the node coloured red and dark blue, as well as the thickened arrows into the red node when contrasting Figure 4 with Figure 5.

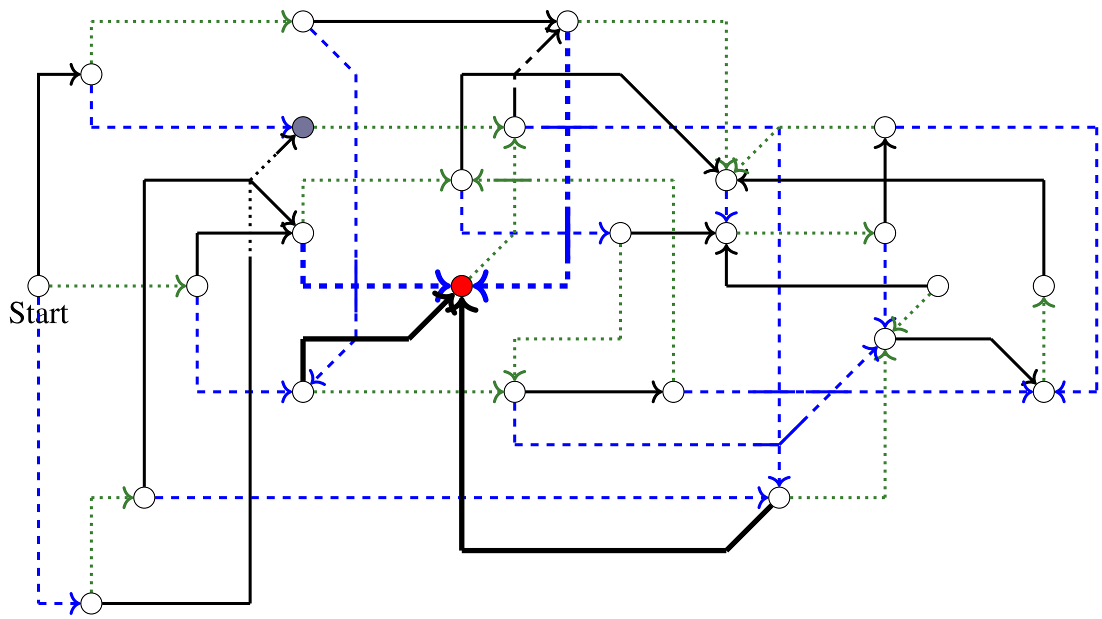


FIGURE 4. Brink-Howlett automaton for $\tilde{B}_2$

The Cannon automaton below has 24 nodes. It coincides with the Brink-Howlett automaton with the exception that the node coloured red in Figure 4 is missing below. The four thickened arrows are redirected to the node coloured dark blue.

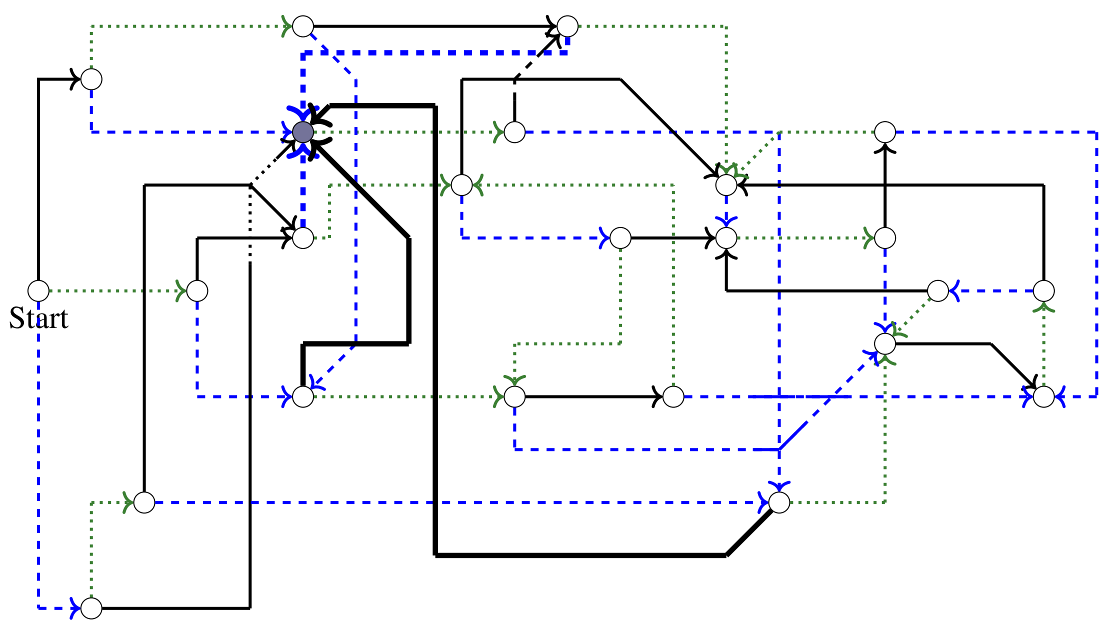


FIGURE 5. Cannon automaton for $\tilde{B}_2$

It is not hard to see that deleting the red coloured node and redirecting its arrows to the dark blue node has not resulted in any information loss. In Figure 4, both the red and dark blue nodes have a single green t-labeled path to the same end node.

For the affine Coxeter groups of rank 3: $\tilde{A}_2$ and $\tilde{B}_2$, we explicitly see that the Cannon automaton has less than or equal to the number of nodes of the Brink-Howlett automaton. In fact, we have the following theorem which has not been found throughout the course of this study in existing literature.

6.22. Theorem. *The Cannon automaton is minimal.*

Proof. (Adapted from [**16**]) Let (W, S) be a Coxeter system and $\mathcal{A}_C$ the Cannon automaton. Let $\mathcal{A}$ be an arbitrary automaton. Denote e the start node of $\mathcal{A}_C$ and e' the start node of $\mathcal{A}$.

Without loss of generality, choose a dictionary order on S which subsequently defines an order on the reduced words of W. Hence, for a node $x \in \mathcal{A}_C$, we may choose a lexicographically minimal path from e to x. Let $s_1 s_2 \cdots s_n$ be the "type" of this path. We define a function $f : T_{\mathcal{A}_C} \to T_{\mathcal{A}}$ by $f(x) = x'$, where x' is the node obtained by following the path of type $s_1 s_2 \cdots s_n$ in $\mathcal{A}$. This function is well defined since if $x = y$ for $x, y \in W$ then by the dictionary order, there is a unique lexicographically minimal path type for its reduced expression $s_1 s_2 \cdots s_n$. The path $s_1 s_2 \cdots s_n$ exists in $\mathcal{A}$, since it is an automaton recognising reduced words.

We now show that f is injective. Suppose that x and y are distinct nodes of $\mathcal{A}_C$, such that $f(x) = f(y) = z \in T_{\mathcal{A}}$. Let $s_i \cdots s_m$ and $s_j \cdots s_n$ be the lex-shortest path from e to x and e to y respectively. Since x and y are distinct nodes and they have distinct cone types. Hence, we can find a sequence of generators $t_1 \cdots t_k$ such that $s_i \cdots s_m t_1 \cdots t_k$ is reduced and $s_j \cdots s_n t_1 \cdots t_k$ is not reduced. (Note that such a sequence may not always exist if the nodes do not represent cone types, such as Brink-Howlett, for example, see the dark blue and red node of Figure 4).

By definition of $f(x)$ there is a path of type $s_i \cdots s_m$ in $\mathcal{A}$ from e' to z. Then since $s_i \cdots s_m t_1 \cdots t_k$ is a reduced expression, there is a path of type $t_1 \cdots t_k$ in $\mathcal{A}$ starting from z.

However, by definition of $f(x')$, there is also a path of type $s_j \cdots s_n$ from e' to z. But then this implies that there is a path $s_j \cdots s_n t_1 \cdots t_k$ in $\mathcal{A}$, which is a contradiction. $\square$

The fact that the Cannon automaton is finite for Coxeter groups is an immediate corollary of this fact.

6.23. Corollary. *The Cannon automaton is finite for Coxeter groups.*

Proof. Let $\mathcal{A}_C$ be the Cannon automaton and $\mathcal{A}_B$ be the Brink-Howlett automaton. Then by Theorem 6.22

$$|T_{\mathcal{A}_C}| \leq |T_{\mathcal{A}_B}| \leq |\Sigma| < \infty$$

$\square$

6.4. Final Remarks

In the course of this short study of automaton construction for Coxeter groups, we have observed that the Brink-Howlett algorithm gives an explicit and computationally straight-forward approach to constructing automata. The Cannon algorithm provides a purely geometric construction which results in minimal automata.

Automata for Coxeter groups remains an area of active research, with a recent paper [**13**] investigating an approach to automata construction distinct from the two studied here. Clearly, our work has only touched on a small subset of the area of research that was made possible by Brink and Howlett. In fact, throughout the course of this study, a number of natural questions arose which could form the basis of future work. Some of these are:

(i) Is there a smaller subset $\Sigma' \subset \Sigma$, such that an automaton can be constructed by defining a transition function $D(w) \to D(ws) \cap \Sigma'$?

(ii) In general, how many subsets $D(w) \subseteq \Sigma$ are encountered in the course of the Brink-Howlett algorithm?

(iii) What are the precise classes of Coxeter groups for which the Brink-Howlett automaton is minimal?

(iv) Applications of automata for Coxeter groups.

APPENDIX A

Brink-Howlett Small Descent Sets

In this section we present the calculation of the small descent sets in the course of constructing the Brink-Howlett automaton. The calculation first involves determination of the small roots and hence these are presented as well.

A1. Small Descent Sets for $\tilde{A}_2$

Recall that $\tilde{A}_2$ has the following presentation

$$\langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (su)^3 = 1 \rangle$$

Step 1: Determine the small roots

We use Definition 5.1 to constructively determine the small roots. Let $\Sigma_0 = \{\alpha_s, \alpha_u, \alpha_t\}$

Now, $m(s,t) = m(s,u) = m(t,u) = 3$ and since

$$\langle \alpha_s, \alpha_u \rangle = \langle \alpha_s, \alpha_t \rangle = \langle \alpha_t, \alpha_u \rangle = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

We calculate the following

$$\begin{aligned} s(\alpha_u) &= \alpha_u - 2\langle \alpha_s, \alpha_u \rangle \alpha_s = \alpha_u + \alpha_s \\ s(\alpha_t) &= \alpha_t - 2\langle \alpha_s, \alpha_t \rangle \alpha_s = \alpha_t + \alpha_s \\ t(\alpha_u) &= \alpha_u - 2\langle \alpha_t, \alpha_u \rangle \alpha_t = \alpha_u + \alpha_t \end{aligned} \tag{A.1}$$

Therefore $\Sigma_1 = \Sigma_0 \cup \{\alpha_u + \alpha_s, \alpha_t + \alpha_s, \alpha_u + \alpha_t\}$ (Note the symmetry of $s(\alpha_u) = u(\alpha_s)$ and similarly for the pairs (s,t) and (t,u)).

Now consider the bilinear form of $\alpha_s, \alpha_u, \alpha_t$ for each of the roots $\alpha_u + \alpha_s, \alpha_t + \alpha_s, \alpha_u + \alpha_t$

$$\begin{aligned}
\langle \alpha_s, \alpha_t + \alpha_s \rangle =& \langle \alpha_s, \alpha_t \rangle + \langle \alpha_s, \alpha_s \rangle = -\frac{1}{2} + 1 = \frac{1}{2} \\
\langle \alpha_s, \alpha_u + \alpha_s \rangle =& \langle \alpha_s, \alpha_u \rangle + \langle \alpha_s, \alpha_s \rangle = -\frac{1}{2} + 1 = \frac{1}{2} \\
\langle \alpha_s, \alpha_u + \alpha_t \rangle =& \langle \alpha_s, \alpha_u \rangle + \langle \alpha_s, \alpha_t \rangle = -\frac{1}{2} - \frac{1}{2} = -1
\end{aligned}$$

Since the bilinear form for each distinct pair of simple roots is $-\frac{1}{2}$, it is clear that applying α_u and α_t to each of the non-simple roots of Σ_1 yields the same results. Hence, there are no additional roots included in the small roots.

Therefore, Σ is precisely the set $\{\alpha_s, \alpha_u, \alpha_t, \alpha_u + \alpha_s, \alpha_t + \alpha_s, \alpha_u + \alpha_t\}$.

Step 2: Determine the small descent sets

The nodes of the automaton are given by the sets

$$D_\Sigma(w) = \{\alpha \in \Sigma \mid w(\alpha) < 0\}$$

To reduce the amount of proceeding computation, it will be helpful to determine the action of each of the simple reflections on the set of small roots. Note that $s(\alpha_u) = u(\alpha_s)$, $s(\alpha_t) = t(\alpha_s)$ and $t(\alpha_u) = u(\alpha_t)$ have been determined in A.1. The remaining actions are

$$\begin{aligned}
s(\alpha_u + \alpha_s) &= s(\alpha_u) + s(\alpha_s) = \alpha_u + \alpha_s - \alpha_s = \alpha_u \in \Sigma \\
t(\alpha_u + \alpha_s) &= t(\alpha_u) + t(\alpha_s) = \alpha_u + 2\alpha_t + \alpha_s \notin \Sigma \\
u(\alpha_u + \alpha_s) &= u(\alpha_u) + u(\alpha_s) = -\alpha_u + \alpha_u + \alpha_s = \alpha_s \in \Sigma \\
s(\alpha_t + \alpha_s) &= s(\alpha_t) + s(\alpha_s) = \alpha_t + \alpha_s - \alpha_s = \alpha_t \in \Sigma \\
t(\alpha_t + \alpha_s) &= t(\alpha_t) + t(\alpha_s) = -\alpha_t + \alpha_t + \alpha_s = \alpha_s \in \Sigma \\
u(\alpha_t + \alpha_s) &= u(\alpha_t) + u(\alpha_s) = 2\alpha_u + \alpha_t + \alpha_s \notin \Sigma \\
s(\alpha_u + \alpha_t) &= s(\alpha_u) + s(\alpha_t) = \alpha_u + 2\alpha_s + \alpha_t \notin \Sigma \\
u(\alpha_u + \alpha_t) &= u(\alpha_u) + u(\alpha_t) = -\alpha_u + \alpha_u + \alpha_t = \alpha_t \in \Sigma \\
t(\alpha_u + \alpha_t) &= t(\alpha_u) + t(\alpha_t) = \alpha_u + \alpha_t - \alpha_t = \alpha_u \in \Sigma
\end{aligned}$$

We begin with the identity. The start node is $D_\Sigma(e) = \{\emptyset\}$. We count the new descent sets as they are encountered using the equation numbers on the right. The next descent sets are:

$$D_\Sigma(s) = \{\alpha_s\} \cup \{s(D_\Sigma(e)) \cap \Sigma\} = \{\alpha_s\} \tag{2}$$
$$D_\Sigma(u) = \{\alpha_u\} \cup \{u(D_\Sigma(e)) \cap \Sigma\} = \{\alpha_u\} \tag{3}$$
$$D_\Sigma(t) = \{\alpha_t\} \cup \{t(D_\Sigma(e)) \cap \Sigma\} = \{\alpha_t\} \tag{4}$$

We now repeat step 2 for each of the small descent sets $D_\Sigma(s), D_\Sigma(u)$ and $D_\Sigma(t)$. We apply the reflections $j \in S$ to $D_\Sigma(w)$, where the associated simple root $\alpha_j \notin D_\Sigma(w)$ and "throw away" the roots which aren't small to obtain the next iteration of small descent sets.

$$D_\Sigma(su) = \{\alpha_u\} \cup \{u(D_\Sigma(s)) \cap \Sigma\} = \{\alpha_u, \alpha_u + \alpha_s\} \tag{5}$$
$$D_\Sigma(st) = \{\alpha_t\} \cup \{t(D_\Sigma(s)) \cap \Sigma\} = \{\alpha_t, \alpha_t + \alpha_s\} \tag{6}$$
$$D_\Sigma(us) = \{\alpha_s\} \cup \{s(D_\Sigma(u)) \cap \Sigma\} = \{\alpha_s, \alpha_u + \alpha_s\} \tag{7}$$
$$D_\Sigma(ut) = \{\alpha_t\} \cup \{t(D_\Sigma(u)) \cap \Sigma\} = \{\alpha_t, \alpha_u + \alpha_t\} \tag{8}$$
$$D_\Sigma(ts) = \{\alpha_s\} \cup \{s(D_\Sigma(t)) \cap \Sigma\} = \{\alpha_s, \alpha_t + \alpha_s\} \tag{9}$$
$$D_\Sigma(tu) = \{\alpha_u\} \cup \{u(D_\Sigma(t)) \cap \Sigma\} = \{\alpha_u, \alpha_u + \alpha_t\} \tag{10}$$

These are all distinct sets and so are distinct nodes of the automaton. We repeat step 2 with the above small descent sets:

$$D_\Sigma(sus) = \{\alpha_s\} \cup \{s(\{\alpha_u, \alpha_u + \alpha_s\}) \cap \Sigma\} = \{\alpha_s, \alpha_u, \alpha_u + \alpha_s\} \tag{11}$$
$$D_\Sigma(sut) = \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_u + \alpha_s\}) \cap \Sigma\} = \{\alpha_t, \alpha_u + \alpha_t\} = D_\Sigma(ut)$$
$$D_\Sigma(sts) = \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_t + \alpha_s\}) \cap \Sigma\} = \{\alpha_s, \alpha_t, \alpha_t + \alpha_s\} \tag{12}$$
$$D_\Sigma(stu) = \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_t + \alpha_s\}) \cap \Sigma\} = \{\alpha_u, \alpha_u + \alpha_t\} = D_\Sigma(tu)$$
$$D_\Sigma(usu) = \{\alpha_u\} \cup \{u(\{\alpha_s, \alpha_u + \alpha_s\}) \cap \Sigma\} = \{\alpha_u, \alpha_s, \alpha_u + \alpha_s\} = D_\Sigma(sus)$$
$$D_\Sigma(ust) = \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_u + \alpha_s\}) \cap \Sigma\} = \{\alpha_t, \alpha_t + \alpha_s\} = D_\Sigma(st)$$
$$D_\Sigma(utu) = \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_u, \alpha_t, \alpha_u + \alpha_t\} \tag{13}$$
$$D_\Sigma(uts) = \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_s, \alpha_t + \alpha_s\} = D_\Sigma(ts)$$
$$D_\Sigma(tsu) = \{\alpha_u\} \cup \{u(\{\alpha_s, \alpha_t + \alpha_s\}) \cap \Sigma\} = \{\alpha_u, \alpha_u + \alpha_s\} = D_\Sigma(su)$$
$$D_\Sigma(tst) = \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_t + \alpha_s\}) \cap \Sigma\} = \{\alpha_t, \alpha_s, \alpha_t + \alpha_s\} = D_\Sigma(sts)$$
$$D_\Sigma(tus) = \{\alpha_s\} \cup \{s(\{\alpha_u, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_s, \alpha_u + \alpha_s\} = D_\Sigma(us)$$
$$D_\Sigma(tut) = \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_t, \alpha_u, \alpha_u + \alpha_t\} = D_\Sigma(utu)$$

We only obtain three new descent sets. Iterating through the new sets, we obtain:

$$D_\Sigma(sust) = \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_u, \alpha_u + \alpha_s\}) \cap \Sigma\} = \{\alpha_t, \alpha_t + \alpha_s, \alpha_u + \alpha_t\} \tag{14}$$
$$D_\Sigma(stsu) = \{\alpha_u\} \cup \{u(\{\alpha_s, \alpha_t, \alpha_t + \alpha_s\}) \cap \Sigma\} = \{\alpha_u, \alpha_u + \alpha_s, \alpha_u + \alpha_t\} \tag{15}$$
$$D_\Sigma(utus) = \{\alpha_s\} \cup \{s(\{\alpha_u, \alpha_t, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_s, \alpha_u + \alpha_s, \alpha_t + \alpha_s\} \tag{16}$$

Performing one more iteration, we see that we do not obtain any new descent sets and hence come to the end of the algorithm:

$$D_\Sigma(susts) = \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_t + \alpha_s, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_s, \alpha_t, \alpha_t + \alpha_s\} = D_\Sigma(sts)$$
$$D_\Sigma(sustu) = \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_t + \alpha_s, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_u, \alpha_u + \alpha_t, \alpha_t\} = D_\Sigma(utu)$$
$$D_\Sigma(stsus) = \{\alpha_s\} \cup \{s(\{\alpha_u, \alpha_u + \alpha_s, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_s, \alpha_u + \alpha_s, \alpha_u\} = D_\Sigma(sus)$$
$$D_\Sigma(stsut) = \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_u + \alpha_s, \alpha_u + \alpha_t\}) \cap \Sigma\} = \{\alpha_t, \alpha_u + \alpha_t, \alpha_u\} = D_\Sigma(utu)$$
$$D_\Sigma(utusu) = \{\alpha_u\} \cup \{u(\{\alpha_s, \alpha_u + \alpha_s, \alpha_t + \alpha_s\}) \cap \Sigma\} = \{\alpha_u, \alpha_u + \alpha_s, \alpha_s\} = D_\Sigma(sus)$$
$$D_\Sigma(utust) = \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_u + \alpha_s, \alpha_t + \alpha_s\}) \cap \Sigma\} = \{\alpha_t, \alpha_t + \alpha_s, \alpha_s\} = D_\Sigma(sts)$$

Therefore for $\tilde{A}_2$ we have 16 small descent sets and hence 16 nodes in the automaton. We remind the reader that at each calculation, a directed edge labeled with the corresponding reflection should be drawn from the "base" node of the calculation to the one obtained. The resulting automaton is illustrated in Figure 3.

A2. Small Descent Sets for $\tilde{B}_2$

The Coxeter group $\tilde{B}_2$ has the following presentation

$$\langle s, t, u \mid s^2 = t^2 = u^2 = 1, (st)^4 = (tu)^4 = (su)^2 = 1\rangle$$

Step 1: Determine the small roots

Let $\Sigma_0 = \{\alpha_s, \alpha_u, \alpha_t\}$

$$\begin{aligned}
\langle \alpha_s, \alpha_u\rangle &= -\cos\frac{\pi}{2} = 0\\
\langle \alpha_s, \alpha_t\rangle &= -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}\\
\langle \alpha_t, \alpha_u\rangle &= -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}
\end{aligned}$$

Then

$$\begin{aligned}
s(\alpha_t) &= \alpha_t - 2\langle \alpha_s, \alpha_t\rangle\alpha_s = \alpha_t + \sqrt{2}\alpha_s && \text{(A.2)}\\
t(\alpha_s) &= \alpha_s - 2\langle \alpha_t, \alpha_s\rangle\alpha_t = \alpha_s + \sqrt{2}\alpha_t\\
t(\alpha_u) &= \alpha_u - 2\langle \alpha_t, \alpha_u\rangle\alpha_t = \alpha_u + \sqrt{2}\alpha_t\\
u(\alpha_t) &= \alpha_t - 2\langle \alpha_u, \alpha_t\rangle\alpha_u = \alpha_t + \sqrt{2}\alpha_u
\end{aligned}$$

and $\Sigma_1 = \{\alpha_s, \alpha_u, \alpha_t, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t\}$

Now

$$\begin{aligned}
\langle \alpha_s, \alpha_t + \sqrt{2}\alpha_s\rangle &= \langle \alpha_s, \alpha_t\rangle + \sqrt{2}\langle \alpha_s, \alpha_s\rangle\\
&= \frac{-\sqrt{2}}{2} + \sqrt{2} = \frac{\sqrt{2}}{2} && \text{(A.3)}\\
\langle \alpha_u, \alpha_t + \sqrt{2}\alpha_s\rangle &= \langle \alpha_u, \alpha_t\rangle + \sqrt{2}\langle \alpha_u, \alpha_s\rangle\\
&= \frac{-\sqrt{2}}{2} && \text{(A.4)}\\
\langle \alpha_t, \alpha_t + \sqrt{2}\alpha_s\rangle &= \langle \alpha_t, \alpha_t\rangle + \sqrt{2}\langle \alpha_t, \alpha_s\rangle\\
&= 1 - \sqrt{2}\frac{\sqrt{2}}{2} = 1 - 1 = 0 && \text{(A.5)}\\
\langle \alpha_s, \alpha_s + \sqrt{2}\alpha_t\rangle &= 0 \text{ (by symmetry with Equation (A.5))}
\end{aligned}$$

$$
\begin{aligned}
\langle\alpha_u, \alpha_s + \sqrt{2}\alpha_t\rangle &= \langle\alpha_u, \alpha_s\rangle + \sqrt{2}\langle\alpha_u, \alpha_t\rangle \\
&= 0 - \sqrt{2}\frac{\sqrt{2}}{2} = -1 \qquad \text{(A.6)} \\
\langle\alpha_t, \alpha_s + \sqrt{2}\alpha_t\rangle &= \frac{\sqrt{2}}{2} \text{ (by symmetry with Equation (A.3))} \\
\langle\alpha_s, \alpha_u + \sqrt{2}\alpha_t\rangle &= -1 \text{ (by symmetry with Equation (A.6))} \\
\langle\alpha_u, \alpha_u + \sqrt{2}\alpha_t\rangle &= 0 \text{ (by symmetry with Equation (A.5))} \\
\langle\alpha_t, \alpha_u + \sqrt{2}\alpha_t\rangle &= \frac{\sqrt{2}}{2} \text{ (by symmetry with Equation (A.3))} \\
\langle\alpha_s, \alpha_t + \sqrt{2}\alpha_u\rangle &= = -\frac{\sqrt{2}}{2} \text{ (by symmetry with Equation (A.4))} \\
\langle\alpha_u, \alpha_t + \sqrt{2}\alpha_u\rangle &= \frac{\sqrt{2}}{2} \text{ (by symmetry with Equation (A.3))} \\
\langle\alpha_t, \alpha_t + \sqrt{2}\alpha_u\rangle &= 0 \text{ (by symmetry with Equation (A.5))}
\end{aligned}
$$

Hence

$$
\begin{aligned}
u(\alpha_t + \sqrt{2}\alpha_s) &= u(\alpha_t) + \sqrt{2}u(\alpha_s) \\
&= \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s \in \Sigma_2 \qquad \text{(A.7)} \\
s(\alpha_t + \sqrt{2}\alpha_u) &= s(\alpha_t) + \sqrt{2}s(\alpha_u) \\
&= \alpha_t + \sqrt{2}\alpha_s + \sqrt{2}\alpha_u \\
&= u(\alpha_t + \sqrt{2}\alpha_s) \qquad \text{(A.8)}
\end{aligned}
$$

So

$$
\begin{aligned}
\Sigma_2 = \{&\alpha_s, \alpha_u, \alpha_t, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t, \\
&\alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}
\end{aligned}
$$

Finally, checking the bilinear form of each of the reflections on the new root, we see that indeed the small roots $\Sigma = \Sigma_2$

$$
\begin{aligned}
\langle\alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\rangle &= \langle\alpha_s, \alpha_t\rangle + \sqrt{2}\langle\alpha_s, \alpha_u\rangle + \sqrt{2}\langle\alpha_s, \alpha_s\rangle \\
&= -\frac{\sqrt{2}}{2} + \sqrt{2} = \frac{\sqrt{2}}{2} \\
\langle\alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\rangle &= \frac{\sqrt{2}}{2} \text{ (by symmetry with above)} \\
\langle\alpha_t, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\rangle &= \langle\alpha_t, \alpha_t\rangle + \sqrt{2}\langle\alpha_t, \alpha_u\rangle + \sqrt{2}\langle\alpha_t, \alpha_s\rangle \\
&= 1 - 1 - 1 = -1
\end{aligned}
$$

Step 2: Determine the small descent sets

We will again determine the action of the simple reflections on each of the small roots. Note that s fixes α_u and u fixes α_s since these two roots are orthogonal. The actions $s(\alpha_t), t(\alpha_s), t(\alpha_u)$ and $u(\alpha_t)$ are given in Equation (A.2). The remaining actions are

$$
\begin{aligned}
s(\alpha_t + \sqrt{2}\alpha_s) &= s(\alpha_t) + \sqrt{2}s(\alpha_s) \\
&= \alpha_t + \sqrt{2}\alpha_s - \sqrt{2}\alpha_s = \alpha_t \\
u(\alpha_t + \sqrt{2}\alpha_s) &= \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s \text{ by Equation (A.7)} \\
t(\alpha_t + \sqrt{2}\alpha_s) &= -\alpha_t + \sqrt{2}t(\alpha_s) \\
&= -\alpha_t + \sqrt{2}(\alpha_s + \sqrt{2}\alpha_t) \\
&= \alpha_t + \sqrt{2}\alpha_s \\
s(\alpha_s + \sqrt{2}\alpha_t) &= -\alpha_s + \sqrt{2}s(\alpha_t) \\
&= -\alpha_s + \sqrt{2}(\alpha_t + \sqrt{2}\alpha_s) \\
&= \alpha_s + \sqrt{2}\alpha_t \\
u(\alpha_s + \sqrt{2}\alpha_t) &= u(\alpha_s) + \sqrt{2}u(\alpha_t) \\
&= \alpha_s + \sqrt{2}(\alpha_t + \sqrt{2}\alpha_u) \notin \Sigma \\
t(\alpha_s + \sqrt{2}\alpha_t) &= t(\alpha_s) + \sqrt{2}t(\alpha_t) \\
&= \alpha_s + \sqrt{2}\alpha_t - \sqrt{2}\alpha_t \\
&= \alpha_s \\
s(\alpha_t + \sqrt{2}\alpha_u) &= \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s \text{ by Equation (A.8)} \\
u(\alpha_t + \sqrt{2}\alpha_u) &= u(\alpha_t) + \sqrt{2}u(\alpha_u) \\
&= \alpha_t + \sqrt{2}\alpha_u - \sqrt{2}\alpha_u \\
&= \alpha_t \\
t(\alpha_t + \sqrt{2}\alpha_u) &= t(\alpha_t) + \sqrt{2}t(\alpha_u) \\
&= -\alpha_t + \sqrt{2}(\alpha_u - \sqrt{2}\alpha_t) \\
&= \alpha_t + \sqrt{2}\alpha_u \\
s(\alpha_u + \sqrt{2}\alpha_t) &= \alpha_u + \sqrt{2}s(\alpha_t) \\
&= \alpha_u + \sqrt{2}(\alpha_t - \sqrt{2}\alpha_s) \notin \Sigma \\
u(\alpha_u + \sqrt{2}\alpha_t) &= -\alpha_u + \sqrt{2}u(\alpha_t) \\
&= -\alpha_u + \sqrt{2}(\alpha_t + \sqrt{2}\alpha_u) \\
&= \alpha_u + \sqrt{2}\alpha_t \\
t(\alpha_u + \sqrt{2}\alpha_t) &= t(\alpha_u) + \sqrt{2}t(\alpha_t)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_u + \sqrt{2}\alpha_t - \sqrt{2}\alpha_t \\
&= \alpha_u \\
s(\alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s) &= s(\alpha_t) + \sqrt{2}s(\alpha_u) + \sqrt{2}s(\alpha_s) \\
&= \alpha_t + \sqrt{2}\alpha_s - \sqrt{2}\alpha_u - \sqrt{2}\alpha_s \\
&= \alpha_t + \sqrt{2}\alpha_u \\
t(\alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s) &= t(\alpha_t) + \sqrt{2}t(\alpha_u) + \sqrt{2}t(\alpha_s) \\
&= -\alpha_t + \sqrt{2}(\alpha_u + \sqrt{2}\alpha_t) - \sqrt{2}(\alpha_s + \sqrt{2}\alpha_t) \\
&= \sqrt{2}\alpha_u + 3\alpha_t + \sqrt{2}\alpha_s \notin \Sigma \\
u(\alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s) &= u(\alpha_t) + \sqrt{2}u(\alpha_u) + \sqrt{2}u(\alpha_s) \\
&= \alpha_t + \sqrt{2}\alpha_u - \sqrt{2}\alpha_u + \sqrt{2}\alpha_s \\
&= \alpha_t + \sqrt{2}\alpha_s
\end{aligned}
$$

As in the calculations for $\tilde{A}_2$, we have the start node $D_\Sigma(e) = \{\emptyset\}$ with labels going to the nodes $D_\Sigma(s) = \{\alpha_s\}, D_\Sigma(u) = \{\alpha_u\}, D_\Sigma(t) = \{\alpha_t\}$. This gives us 4 descent sets. Now consider

$$
\begin{aligned}
D_\Sigma(su) &= \{\alpha_u\} \cup \{u(\alpha_s) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_s\} && (5) \\
D_\Sigma(st) &= \{\alpha_t\} \cup \{t(\alpha_s) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_s + \sqrt{2}\alpha_t\} && (6) \\
D_\Sigma(us) &= D_\Sigma(su) \\
D_\Sigma(ut) &= \{\alpha_t\} \cup \{t(\alpha_u) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_u + \sqrt{2}\alpha_t\} && (7) \\
D_\Sigma(ts) &= \{\alpha_s\} \cup \{s(\alpha_t) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_t + \sqrt{2}\alpha_s\} && (8) \\
D_\Sigma(tu) &= \{\alpha_u\} \cup \{u(\alpha_t) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_t + \sqrt{2}\alpha_u\} && (9)
\end{aligned}
$$

We have 5 new descent sets. Repeat step 2 for these sets:

$$
\begin{aligned}
D_\Sigma(sut) &= \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_s + \sqrt{2}\alpha_t\} && (10) \\
D_\Sigma(sts) &= \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_s + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t\} && (11) \\
D_\Sigma(stu) &= \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_s + \sqrt{2}\alpha_t\}) \cap \Sigma\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{\alpha_u, \alpha_t + \sqrt{2}\alpha_u\} = D_\Sigma(tu) \\
D_\Sigma(uts) &= \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_u + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_t + \sqrt{2}\alpha_s\} = D_\Sigma(ts) \\
D_\Sigma(utu) &= \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_u + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t\} \qquad (12) \\
D_\Sigma(tst) &= \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_t + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\} \qquad (13) \\
D_\Sigma(tsu) &= \{\alpha_u\} \cup \{u(\{\alpha_s, \alpha_t + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} \qquad (14) \\
D_\Sigma(tut) &= \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_t + \sqrt{2}\alpha_u\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\} \qquad (15) \\
D_\Sigma(tus) &= \{\alpha_s\} \cup \{s(\{\alpha_u, \alpha_t + \sqrt{2}\alpha_u\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} = D_\Sigma(tsu)
\end{aligned}
$$

We have 6 new descent sets. Repeat step 2:

$$
\begin{aligned}
D_\Sigma(suts) &= \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_s + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t\} = D_\Sigma(sts) \\
D_\Sigma(sutu) &= \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_s + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t\} = D_\Sigma(utu) \\
D_\Sigma(stst) &= \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t\} \qquad (16) \\
D_\Sigma(stsu) &= \{\alpha_u\} \cup \{u(\{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} = D_\Sigma(tsu) \\
D_\Sigma(utus) &= \{\alpha_s\} \cup \{s(\{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} = D_\Sigma(tsu) \\
D_\Sigma(utut) &= \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_u, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\} \qquad (17) \\
D_\Sigma(tsts) &= \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\} = D_\Sigma(stst) \\
D_\Sigma(tstu) &= \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
D_\Sigma(tsut) &= \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_s + \sqrt{2}\alpha_t\} = D_\Sigma(sut) \\
D_\Sigma(tuts) &= \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} \qquad (19) \\
D_\Sigma(tutu) &= \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_t, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t\} = D_\Sigma(utut)
\end{aligned}
$$

This yields another 4 new descent sets. Repeat step 2:

$$
\begin{aligned}
D_\Sigma(ststu) &= \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_s, \alpha_t + \sqrt{2}\alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} \qquad (20) \\
D_\Sigma(ututs) &= \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_u, \alpha_t + \sqrt{2}\alpha_u\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_u, \alpha_t + \sqrt{2}\alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} \qquad (21) \\
D_\Sigma(tstus) &= \{\alpha_s\} \cup \{s(\{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} = D_\Sigma(ststu) \\
D_\Sigma(tstut) &= \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\} = D_\Sigma(tut) \\
D_\Sigma(tutsu) &= \{\alpha_u\} \cup \{u(\{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s, \alpha_t + \sqrt{2}\alpha_s\} = D_\Sigma(ututs) \\
D_\Sigma(tutst) &= \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\} = D_\Sigma(tst)
\end{aligned}
$$

We check the two new descent sets:

$$
\begin{aligned}
D_\Sigma(ststut) &= \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_s, \alpha_t + \sqrt{2}\alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\} \qquad (22) \\
D_\Sigma(ututst) &= \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_u, \alpha_t + \sqrt{2}\alpha_s, \alpha_t + \sqrt{2}\alpha_u + +\sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\} \qquad (23)
\end{aligned}
$$

These are both new again, so repeat step 2:

$$
\begin{aligned}
D_\Sigma(ststuts) &= \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} \qquad (24) \\
D_\Sigma(ststutu) &= \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\}) \cap \Sigma\}
\end{aligned}
$$

$$
\begin{aligned}
&= \{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t, \alpha_t\} = D_\Sigma(tutu) \\
D_\Sigma(ututsts) &= \{\alpha_s\} \cup \{s(\{\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t, \alpha_t\} = D_\Sigma(stst) \\
D_\Sigma(ututstu) &= \{\alpha_u\} \cup \{u(\{\alpha_t, \alpha_s + \sqrt{2}\alpha_t, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\} \qquad (25)
\end{aligned}
$$

We have another two new sets. Perform step 2:

$$
\begin{aligned}
D_\Sigma(ststutsu) &= \{\alpha_u\} \cup \{u(\{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_u, \alpha_s, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s, \alpha_t + \sqrt{2}\alpha_s\} = D_\Sigma(ututs) \\
D_\Sigma(ststutst) &= \{\alpha_t\} \cup \{t(\{\alpha_s, \alpha_t + \sqrt{2}\alpha_s, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_s, \alpha_s + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_s\} = D_\Sigma(stst) \\
D_\Sigma(ututstus) &= \{\alpha_s\} \cup \{s(\{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_s, \alpha_u, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s, \alpha_t + \sqrt{2}\alpha_u\} = D_\Sigma(ststu) \\
D_\Sigma(ututstut) &= \{\alpha_t\} \cup \{t(\{\alpha_u, \alpha_t + \sqrt{2}\alpha_u, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u + \sqrt{2}\alpha_s\}) \cap \Sigma\} \\
&= \{\alpha_t, \alpha_u, \alpha_u + \sqrt{2}\alpha_t, \alpha_t + \sqrt{2}\alpha_u\} = D_\Sigma(tutu)
\end{aligned}
$$

This concludes the algorithm. We have obtained 25 nodes for the automaton, which is illustrated in Figure 4.

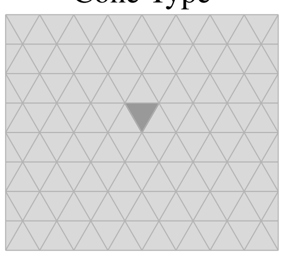
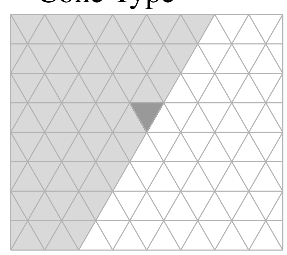
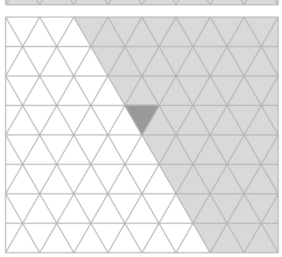
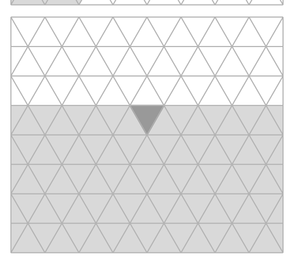
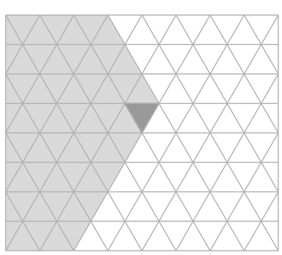
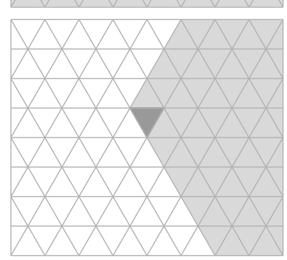
Appendix B

Cannon Cone Types calculations

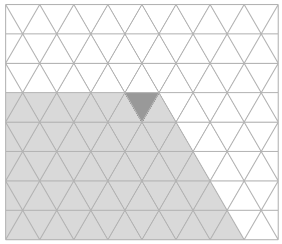
This chapter presents the various cone types produced by the Cannon algorithm for the examples given in Section 6.3. The $\sim$ symbol indicates elements of W which share the same cone type. In the same way as the Brink-Howlett algorithm, at each calculation a directed edge labeled with the corresponding reflection should be drawn from the "base" node of the calculation to the node obtained. The $\tilde{A}_2$ Cannon automaton exactly coincides with the Brink-Howlett, while the $\tilde{B}_2$ Cannon automaton is illustrated in Figure 5.

B1. Cone Types for $\tilde{A}_2$

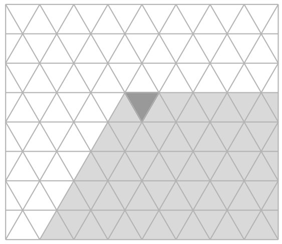
There are 16 distinct cone types for $\tilde{A}_2$.

Word	Cone Type	Word	Cone Type
1		s	
t		u	
$us \sim tus$		$ut \sim sut$	

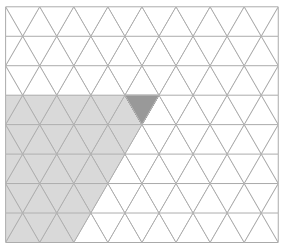
$su \sim$
tsu



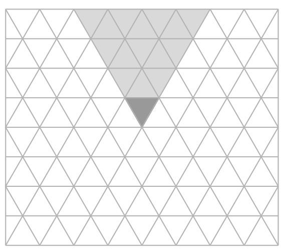
$tu \sim$
stu



$sus \sim$
$usu \sim$
$stsus \sim$
$utusu$



$sts \sim$
$tst \sim$
$susts \sim$
$utust$



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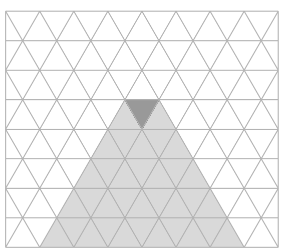
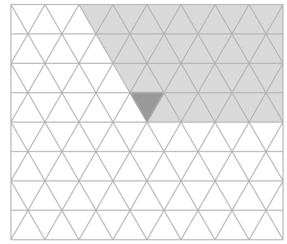
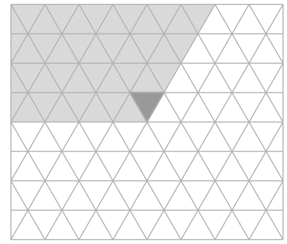


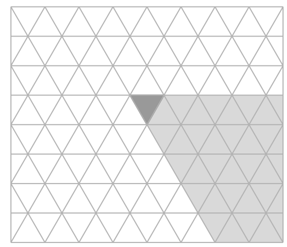
$st \sim ust$



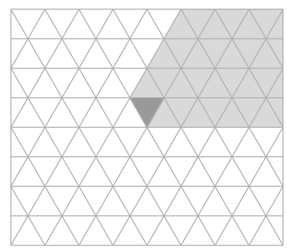
$ts \sim uts$



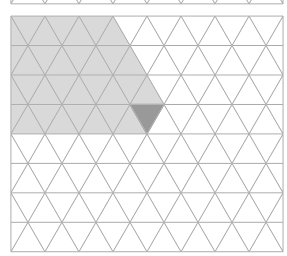
$utu \sim tut \sim$
$sustu \sim stsut$



$sust$

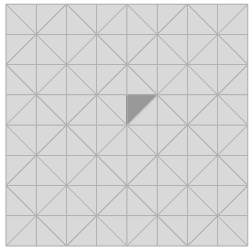
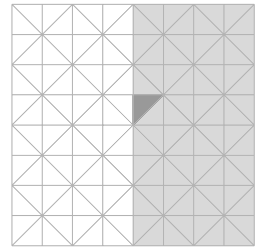
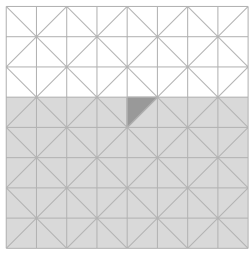
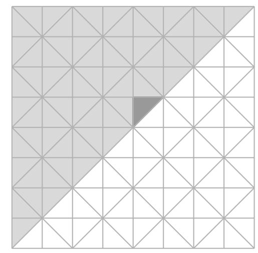
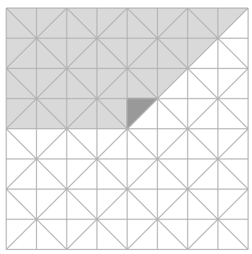
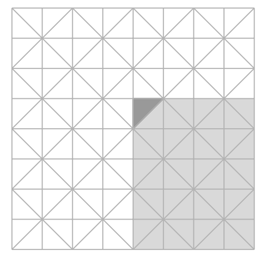
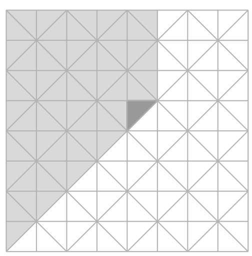
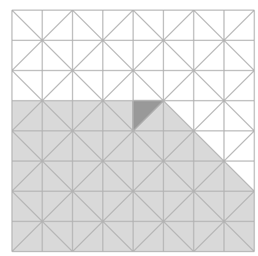
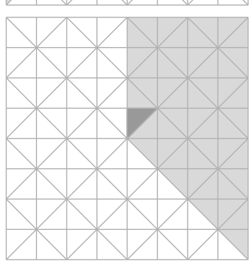
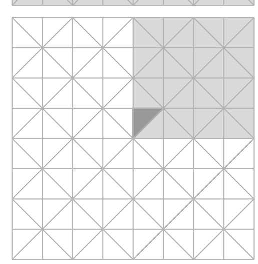


$utus$

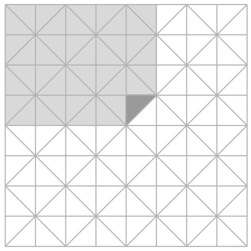


B2. Cone Types for $\tilde{B}_2$

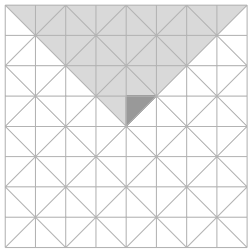
There are 24 distinct cone types for the Coxeter group of type $\tilde{B}_2$.

Word	Cone Type	Word	Cone Type
1		s	
u		t	
st		$su \sim us \sim$ $tus \sim tsu \sim$ $stsu \sim utus$	
ut		$tu \sim stu$	
$ts \sim$ uts		$sts \sim suts$	

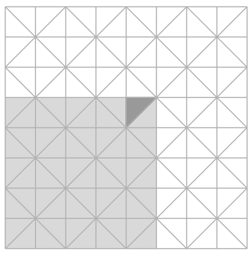
sut



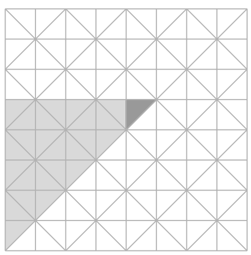
$tst \sim$
$tutst$



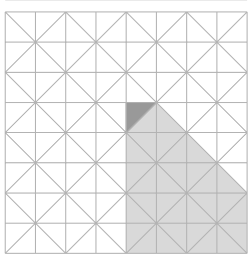
$utu \sim$
$sutu$



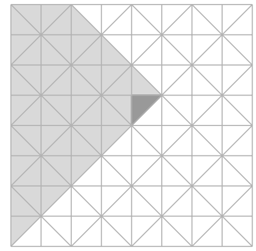
$utut \sim$
$tutu \sim$
$ststutu \sim$
$tutustut$



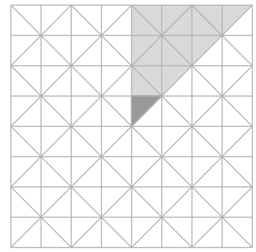
$ststu \sim$
$tstus \sim$
$tutustus$



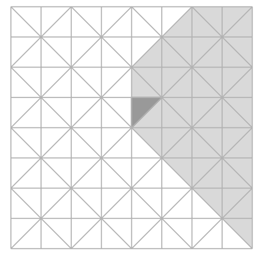
$tut \sim tstut$



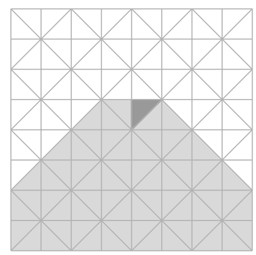
$stst \sim tsts \sim$
$tutustus \sim$
$ststutst$



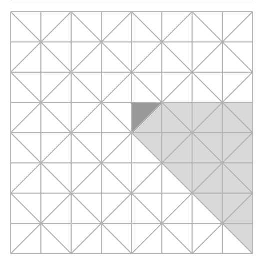
$tuts$



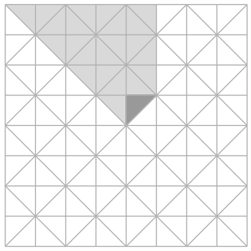
$tstu$



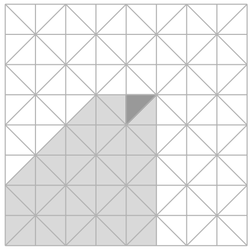
$tutus \sim$
$ututs \sim$
$tutsu \sim$
$ststutsu$



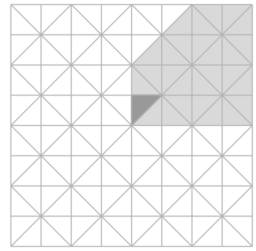
$tutust$



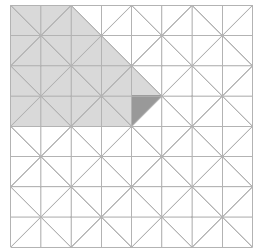
$tutustu$



$ststuts$



$ststut$



APPENDIX C

Miscellaneous Results

C1. Reflection matrices for Dihedral groups

In this section, we compute explicit matrices for the generating reflections s and t of the dihedral groups D_{2m}. The reflections s and t are with respect to the pair of basis vectors α_s and α_t such that the angle between the orthogonal hyperplanes H_s and H_t is $\frac{\pi}{m}$.

Since $s(\alpha_s) = -\alpha_s$ and $s(\alpha_t) = \alpha_t - 2\langle\alpha_t, \alpha_s\rangle\alpha_s$, we have in $GL(V)$

$$s = \begin{pmatrix} -1 & 2\cos\frac{\pi}{m} \\ 0 & 1 \end{pmatrix}$$

Where we set $m := m(s,t)$ for brevity in notation. Similarly, $t(\alpha_t) = -\alpha_t$ and $t(\alpha_s) = \alpha_s - 2\langle\alpha_s, \alpha_t\rangle\alpha_t$, so

$$t = \begin{pmatrix} 1 & 0 \\ 2\cos\frac{\pi}{m} & -1 \end{pmatrix}$$

Then

$$st = \begin{pmatrix} -1 + 4\cos^2\frac{\pi}{m} & -2\cos\frac{\pi}{m} \\ 2\cos\frac{\pi}{m} & -1 \end{pmatrix} \tag{C.1}$$

The eigenvalues for st are given by

$$\det \begin{pmatrix} \lambda + 1 - 4\cos^2\frac{\pi}{m} & -2\cos\frac{\pi}{m} \\ 2\cos\frac{\pi}{m} & \lambda + 1 \end{pmatrix}$$

$$\begin{aligned} &= (\lambda+1)(\lambda + (1 - 4\cos^2\frac{\pi}{m})) + 4\cos^2\frac{\pi}{m} \\ &= \lambda^2 + 2(1 - 2\cos^2\frac{\pi}{m})\lambda + 1 \end{aligned}$$

Using the property $2\cos^2\theta - 1 = \cos 2\theta$, we have

$$\begin{aligned} &= \lambda^2 - 2\cos\frac{2\pi}{m}\lambda + 1 \\ \implies \lambda &= \frac{1}{2}\Big(2\cos\frac{2\pi}{m} \pm \sqrt{4\cos^2\frac{\pi}{m} - 4}\Big) \\ &= \cos\frac{2\pi}{m} \pm i\sin\frac{2\pi}{m} \end{aligned}$$

$$= e^{\pm\frac{2\pi i}{m}}$$

Hence, there exists a basis of V_{st} such that st has matrix

$$\begin{pmatrix} e^{\frac{2\pi i}{m}} & 0 \\ 0 & e^{-\frac{2\pi i}{m}} \end{pmatrix}$$

Therefore, we can see explicitly that $(st)^m = 1$.

We now derive a formula for $(st)^k$, $k \in \mathbb{N}$. The calculation is a slightly long exercise in manipulating trigonometric identities. We note that the formula could also be derived through the usual method of finding invertible change of basis matrices for the diagonalised form. But that full calculation was also quite lengthy.

C.2. Lemma.

$$(st)^k = \frac{1}{\sin\theta}\begin{pmatrix} \sin(2k+1)\theta & -\sin 2k\theta \\ \sin 2k\theta & -\sin(2k-1)\theta \end{pmatrix}$$

where $\theta = \frac{\pi}{m(s,t)}$

Proof. As suggested in [**14**], we prove the Lemma by induction. Consider $k = 1$. Using the identities $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ and $\sin 3\theta = \sin\theta(2\cos 2\theta + 1)$, the $(1,1)$ entry of Equation (C.1) can be written

$$-1 + 4\cos^2\theta = -1 + 4(\frac{1}{2}(1+\cos 2\theta)) = 2\cos 2\theta + 1 = \frac{\sin 3\theta}{\sin\theta}$$

And using $\sin 2\theta = 2\sin\theta\cos\theta$, for the $(1,2)$ and $(2,1)$ entries, it follows that Equation (C.1) can be rewritten as

$$(st) = \frac{1}{\sin\theta}\begin{pmatrix} \sin 3\theta & -\sin 2\theta \\ \sin 2\theta & -\sin\theta \end{pmatrix}$$

Hence assume the claim holds for k, we show that

$$(st)^{k+1} = \frac{1}{\sin\theta}\begin{pmatrix} \sin(2k+3)\theta & -\sin(2k+2)\theta \\ \sin(2k+2)\theta & -\sin(2k+1)\theta \end{pmatrix}$$

Now

$$(st)^k(st) = \frac{1}{\sin^2\theta}\begin{pmatrix} \sin(2k+1)\theta & -\sin 2k\theta \\ \sin 2k\theta & -\sin(2k-1)\theta \end{pmatrix}\begin{pmatrix} \sin 3\theta & -\sin 2\theta \\ \sin 2\theta & -\sin\theta \end{pmatrix} \tag{C.3}$$

The $(1,1)$ entry becomes

$$\frac{1}{\sin\theta}(\sin(2k+1)\theta\sin 3\theta - \sin 2k\theta\sin 2\theta) \tag{C.4}$$

We want to show that this can be rewritten as $\sin(2k\theta + 3\theta)$. Note that

$$\sin(2k\theta + 3\theta) = \sin 2k\theta\cos 3\theta + \cos 2k\theta\sin 3\theta \tag{C.5}$$

and

$$\begin{aligned}\cos 3\theta =& \cos(2\theta+\theta)\\ &= \cos^3\theta - 3\cos\theta\sin^2\theta\end{aligned}$$

We can rewrite the numerator of Equation (C.4) as

$$\begin{aligned}&(\sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta)\sin 3\theta - \sin(2k\theta)2\sin\theta\cos\theta\\ =& \sin 2k\theta(\cos\theta\sin 3\theta - 2\sin\theta\cos\theta) + \cos 2k\theta\sin 3\theta\sin\theta\end{aligned}$$

Replacing the first of the $\sin 3\theta$ terms and using the identity $\sin 3\theta = \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$, after some rearrangement the numerator of Equation (C.4) becomes

$$\begin{aligned}\sin 2k\theta[2\cos\theta\cos^2\theta + \cos\theta\cos 2\theta - 2\cos\theta]\sin\theta\\ + \cos 2k\theta\sin 3\theta\sin\theta\end{aligned}$$

After dividing through by $\sin\theta$, we can note that if the term inside [...] is equal to $\cos 3\theta$ then by Equation (C.5) we have verified the Lemma for the $(1,1)$ entry.

Indeed we can write the term inside [...] as

$$2\cos\theta(\cos^2\theta - 1) + \cos\theta\cos 2\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta$$

Hence, the formula holds for the $(1,1)$ entry. We follow a similar procedure for the $(2,1)$ entry of Equation (C.3). The entry becomes

$$\frac{1}{\sin\theta}(\sin 2k\theta\sin 3\theta - \sin(2k-1)\theta\sin 2\theta) \tag{C.6}$$

We want to show that Equation (C.6) can be rewritten as $\sin(2k\theta + 2\theta)$. Note that

$$\sin(2k\theta + 2\theta) = \sin 2k\theta\cos 2k\theta + \cos 2k\theta\sin 2\theta \tag{C.7}$$

Using

$$\sin 3\theta = \sin(2\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin 2\theta$$

we can rewrite the numerator of Equation (C.6) as

$$\sin 2k\theta(\sin 2k\theta\cos 2k\theta + cos 2k\theta\sin 2\theta) - \sin(2k\theta - \theta)\sin 2\theta$$

$$\begin{aligned}=& \sin\theta\sin 2k\theta\cos 2\theta + 2\cos^2\theta\sin\theta\sin 2k\theta\\ &- 2\sin\theta\cos^2\theta\sin 2k\theta + \cos 2k\theta\sin\theta\sin 2\theta\end{aligned}$$

dividing above by $\sin\theta$, we can equate Equation (C.6) with Equation (C.7). Hence we verify the formula for the $(2,1)$ entry.

By the same strategy, the $(1,2)$ entry of Equation (C.3) becomes

$$\frac{1}{\sin\theta}(-\sin(2k+1)\theta\sin 2\theta+\sin 2k\theta\sin\theta) \tag{C.8}$$

And we want to show that this can be written

$$-\sin(2k\theta+\theta)=-\sin 2k\theta\cos 2\theta-\cos 2k\theta\sin 2\theta \tag{C.9}$$

Rewriting the numerator of Equation (C.8), we get

$$\begin{aligned}&-(\sin 2k\theta\cos\theta+\cos 2k\theta\sin\theta)\sin 2\theta\\ =&-\sin 2k\theta(2\cos^2\theta-1)\sin\theta-\cos 2k\theta\sin 2\theta\sin\theta\end{aligned}$$

Using that $2\cos^2\theta-1=\cos 2\theta$, and dividing by $\sin\theta$, we equate Equation (C.8) with Equation (C.9). This verifies the $(1,2)$ entry of the formula.

Finally, the $(2,2)$ entry of Equation (C.3) becomes

$$\frac{1}{\sin\theta}(-\sin 2\theta\sin 2k\theta+\sin(2k-1)\theta\sin\theta) \tag{C.10}$$

and we want to show that this can be rewritten as $-\sin(2k\theta+\theta)$. Note that

$$-\sin(2k\theta+\theta)=-\sin 2k\theta\cos\theta-\cos 2k\theta\sin\theta \tag{C.11}$$

The numerator of Equation (C.10) can be written

$$-2\sin\theta\cos\theta\sin 2k\theta+(\sin 2k\theta\cos\theta-\cos 2k\theta\sin\theta)\sin\theta \tag{C.12}$$

dividing by $\sin\theta$, we can equate Equation (C.10) with Equation (C.11). This verifies the $(2,2)$ entry. $\square$

C2. Other Propositions

C.13. Proposition. *Let V be a finite dimensional real vector space and let $\langle\cdot,\cdot\rangle$ be a symmetric bilinear form on V. Let W a subspace of V. Then $\langle\cdot,\cdot\rangle$ is non-degenerate on W if and only if $V=W\oplus W^{\perp}$.*

Proof. (Adapted from [**1**] p237) "$\Leftarrow$" is directly seen. By definition of $\langle\cdot,\cdot\rangle$ being non-degenerate, $W\cap W^{\perp}=\{0\}$, so it suffices to show that $V=W+W^{\perp}$.

Let $\{w_1,...,w_k\}$ be a basis of W and extend this to a basis of V by $\{w_1,...,w_k\}\cup\{v_1,...,v_{n-k}\}$ (so $\dim(W)=k$ and $\dim(V)=n$). Let the matrix of $\langle\cdot,\cdot\rangle$ with respect to this basis be written:

$$M=\begin{bmatrix}A & B\\ C & D\end{bmatrix}$$

where A is the $k \times k$ matrix of $\langle \cdot, \cdot \rangle$ restricted to W. We now find linearly independent vectors $\{v_1, ..., v_{n-k}\}$ so that the matrix B with entries $B_{i,j} = \langle w_i, v_j \rangle$ $1 \leq i \leq k,\ 1 \leq j \leq n-k$, is zero. Since it then follows that $\{v_1, ..., v_{n-k}\} \in W^\perp$ which implies that $V = W + W^\perp$.

Let

$$P = \begin{bmatrix} I & D \\ 0 & I \end{bmatrix}$$

be a change of basis matrix of M. We proceed to determine the upper right block D. The new basis of V corresponding to P has the form: $\{w_1, ..., w_k\} \cup \{v'_1, ..., v'_{n-k}\}$. So $\{w_1, ..., w_k\}$ is unchanged. Then the matrix M' of $\langle \cdot, \cdot \rangle$ is given by:

$$M' = P^T M P = \begin{bmatrix} I & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & D \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & AD + B \\ ... & \end{bmatrix}$$

A is invertible since $\langle \cdot, \cdot \rangle$ is non-degenerate on W, so setting $D = -A^{-1}B$ yields the required vectors $\{v_1, ..., v_{n-k}\}$. $\square$

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