A PAIR OF GARSIDE SHADOWS

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Abstract. We prove that the smallest elements of Shi parts and cone type parts exist and form Garside shadows. The latter resolves a conjecture of Parkinson and the second author as well as a conjecture of Hohlweg, Nadeau and Williams.

1. Introduction

Overview. The Shi partition and the cone type partition are examples of 'regular partitions' recently studied by Parkinson and Yau [PY22]. Regular partitions are essentially equivalent to automata recognising the language of reduced words $\mathcal{L}(W,S)$ of a Coxeter system (W,S). That is, for each regular partition $\mathscr{R}$ of W, there exists an explicitly defined automaton recognising $\mathcal{L}(W,S)$ with states being the parts of $\mathscr{R}$. Moreover, every automaton recognising $\mathcal{L}(W,S)$ arises in this way from a regular partition (see [PY22, Thm 2]).

The parts of the Shi partition are the connected components of the well-known generalised Shi arrangement, an important structure in algebraic combinatorics, geometric group theory and representation theory (see for example [DH16], [DFHM23], and the survey article [Fis19]). The cone type partition gives rise to the smallest automaton recognising $\mathcal{L}(W,S)$. Namely, it is the smallest element in the (complete) lattice of regular partitions (see [PY22, Thm 3 and Cor 4]). The Shi partition is a refinement of the cone type partition, and a critical difference to note between the two partitions is that the cone type partition does not correspond to a 'hyperplane arrangement' (see Figure 2 for the case of the Coxeter group of type $\widetilde{G}_2$).

In this article, we show that each part of the Shi partition and the cone type partition contains a smallest element. Moreover, these smallest elements form Garside shadows. We note that the results for the Shi partition were proved independently in a more general form by Dyer, Fishel, Hohlweg and Mark in [DFHM23, Thm 1.1(1)].

Terminology and notation. A *Coxeter group* W is a group generated by a finite set S subject only to relations $s^2=1$ for $s\in S$ and $(st)^{m_{st}}=1$ for $s\neq t\in S$, where $m_{st}=m_{ts}\in\{2,3,\ldots,\infty\}$. Here the convention is that $m_{st}=\infty$ means that we do not impose a relation between s and t. By X^1 we denote the *Cayley graph* of W, that is, the graph with vertex set $X^0=W$ and with edges (of length 1) joining each $g\in W$ with gs, for $s\in S$. For $g\in W$, let $\ell(g)$ denote the *word length* of g, that is, the distance in X^1 from g to id. We consider the action of W on $X^0=W$ by left multiplication. This induces an action of W on X^1.

For $r\in W$ a conjugate of an element of S, the *wall* $\mathcal{W}_r$ of r is the fixed point set of r in X^1. We call r the *reflection* in $\mathcal{W}_r$ (for fixed $\mathcal{W}_r$ such r is unique). Each

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wall $\mathcal{W}$ separates X^1 into two components, called *half-spaces*, and a geodesic edge-path in X^1 intersects $\mathcal{W}$ at most once [Ron09, Lem 2.5]. Consequently, the distance in X^1 between $g, h \in W$ is the number of walls separating g and h.

We consider the partial order $\preceq$ on W (called the 'weak order' in algebraic combinatorics), where $p \preceq g$ if p lies on a geodesic edge-path in X^1 from id to g. Equivalently, there is no wall separating p from both id and g.

Shi parts. Let $\mathcal{E}$ be the set of walls $\mathcal{W}$ such that there is no wall separating $\mathcal{W}$ from id (these walls correspond to so-called 'elementary roots'). The components of $X^1 \setminus \bigcup \mathcal{E}$ are *Shi components*. For a Shi component Y, we call $P = Y \cap X^0$ the corresponding *Shi part*.

Our first result is the following.

Theorem 1.1. *Let P be a Shi part. Then P has a smallest element with respect to $\preceq$.*

We note again that Theorem 1.1 was proved independently in a more general form by Dyer, Fishel, Hohlweg and Mark in [DFHM23, Thm 1.1(1)]). Here we give a short proof following the lines of the proof of a related result of the first author and Osajda [OP22, Thm 2.1].

In [Shi87], Shi proved Theorem 1.1 for affine W.

For $g \in W$, let $m(g)$ be the smallest element in the Shi part containing g, guaranteed by Theorem 1.1. Let $M \subset W$ be the set of elements of the form $m(g)$ for $g \in W$.

The *join* of $g, g' \in W$ is the smallest element h (if it exists) satisfying $g \preceq h$ and $g' \preceq h$. A subset $B \subseteq W$ is a *Garside shadow* if it contains S, contains $g^{-1}h$ for every $h \in B$ and $g \preceq h$, and contains the join, if it exists, of every $g, g' \in B$.

Theorem 1.2. *M is a Garside shadow.*

Theorem 1.2 was also obtained in [DFHM23, Thm 1.1(2)], where the authors showed that M is the set of so-called 'low elements' introduced in [DH16]. We give an alternative proof using 'bipodality', a notion introduced in [DH16] and rediscovered in [OP22].

Cone type parts. For each $g \in W$, let $T(g) = \{h \in W \mid \ell(gh) = \ell(g) + \ell(h)\}$. For $T \subset W$, the *cone type part* $Q(T) \subset W$ is the set of all g^{-1} with $T(g) = T$. In other words, $Q(T)$ consists of g such that T is the set of vertices on geodesic edge-paths starting at g and passing through id that appear after id, including id.

We obtain a short new proof of the following.

Theorem 1.3. [PY22, Thm 1] *Let Q be a cone type part. Then Q has a smallest element with respect to $\preceq$.*

For $g \in W$, let $\mu(g)$ be the smallest element in the cone type part containing g. Let $\Gamma \subset W$ be the set of elements of form $\mu(g)$ for $g \in W$ These elements are called the *gates* of the cone type partition in [PY22].

We also obtain the following new result, confirming in part [PY22, Conj 1].

Theorem 1.4. *For any $g, g' \in \Gamma$, if the join of g and g' exists, then it belongs to Γ.*

By [PY22, Prop 4.27(i)], this implies that Γ is a Garside shadow. Furthermore, by [PY22, Cor 4], we have that Γ is the set of states of the smallest automaton (in terms of the number of states) recognising $\mathcal{L}(W, S)$. By [HNW16, Thm 1.2], each Garside shadow B is the set of states of an automaton $\mathcal{A}_B(W,S)$ recognising $\mathcal{L}(W,S)$. Consequently, we have the following.

Corollary 1.5. *(i)* Γ *is the smallest Garside shadow.*

(ii) [HNW16, Conj 1] *The automaton* $\mathcal{A}_B(W,S)$*, where* B *is the smallest Garside shadow, is the smallest automaton recognising* $\mathcal{L}(W,S)$*.*

The paper is organised as follows. In Section 2 we discuss 'bipodality' and use it to prove Theorem 1.1 and Theorem 1.2. In Section 3 we focus on the cone type parts and give the proofs of Theorem 1.3 and Theorem 1.4.

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2. Shi parts

The following property was called *bipodality* in [DH16]. It was rediscovered in [OP22].

Definition 2.1. Let $r,q\in W$ be reflections. Distinct walls $\mathcal{W}_r,\mathcal{W}_q$ *intersect*, if $\mathcal{W}_r$ is not contained in a half-space for $\mathcal{W}_q$ (this relation is symmetric). Equivalently, $\langle r,q\rangle$ is a finite group. We say that such r,q are *sharp-angled*, if r and q do not commute and there is $g\in W$ such that both grg^{-1} and gqg^{-1} belong to S. In particular, there is a component of $X^1\setminus(\mathcal{W}_r\cup\mathcal{W}_q)$ whose intersection F with X^0 is a fundamental domain for the action of $\langle r,q\rangle$ on X^0. We call such F a *geometric fundamental domain for* $\langle r,q\rangle$.

Lemma 2.2 ([OP22, Lem 3.2], special case of [DH16, Thm 4.18])**.** *Suppose that reflections* $r,q\in W$ *are sharp-angled, and that* $g\in W$ *lies in a geometric fundamental domain for* $\langle r,q\rangle$*. Assume that there is a wall* $\mathcal{U}$ *separating* g *from* $\mathcal{W}_r$ *or from* $\mathcal{W}_q$*. Let* $\mathcal{W}'$ *be a wall distinct from* $\mathcal{W}_r,\mathcal{W}_q$ *that is the translate of* $\mathcal{W}_r$ *or* $\mathcal{W}_q$ *under an element of* $\langle r,q\rangle$*. Then there is a wall* $\mathcal{U}'$ *separating* g *from* $\mathcal{W}'$*.*

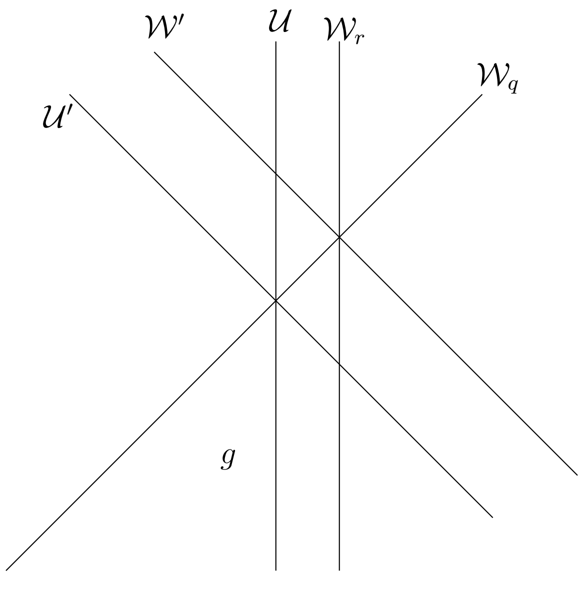

FIGURE 1. Lemma 2.2 for the case $m_{rq}=4$

The following proof is similar to that of a different result [OP22, Thm 2.1].

Proof of Theorem 1.1. Let $P = Y \cap X^0$, where Y is a Shi component. It suffices to show that for each $p_0, p_n \in P$ there is $p \in P$ satisfying $p_0 \succeq p \preceq p_n$. Let $(p_0, p_1, \ldots, p_n)$ be the vertices of a geodesic edge-path π in X^1 from p_0 to p_n, which lies in Y. Let $L = \max_{i=0}^n \ell(p_i)$.

We will now modify π and replace it by another embedded edge-path from p_0 to p_n with vertices in P, so that there is no p_i with $p_{i-1} \prec p_i \succ p_{i+1}$. Then we will be able to choose p to be the smallest p_i with respect to $\preceq$.

If $p_{i-1} \prec p_i \succ p_{i+1}$, then let $\mathcal{W}_r, \mathcal{W}_q$ be the (intersecting) walls separating p_i from p_{i-1}, p_{i+1}, respectively. Moreover, if r and q do not commute, then r, q are sharp-angled, with id in a geometric fundamental domain for $\langle r, q\rangle$. We claim that all the elements of the *residue* $R = \langle r, q\rangle(p_i)$ lie in P.

Indeed, since p_{i-1}, p_{i+1} are both in P, we have that $\mathcal{W}_r, \mathcal{W}_q \notin \mathcal{E}$. It remains to justify that each wall $\mathcal{W}' \neq \mathcal{W}_r, \mathcal{W}_q$ that is the translate of $\mathcal{W}_r$ or $\mathcal{W}_q$ under an element of $\langle r, q\rangle$ does not belong to $\mathcal{E}$. We can thus assume that r and q do not commute, since otherwise there is no such $\mathcal{W}'$. Since $\mathcal{W}_r \notin \mathcal{E}$, there is a wall $\mathcal{U}$ separating id from $\mathcal{W}_r$. By Lemma 2.2, there is a wall $\mathcal{U}'$ separating id from $\mathcal{W}'$, justifying the claim.

We now replace the subpath (p_{i-1}, p_i, p_{i+1}) of π by the second embedded edge-path with vertices in the residue R from p_{i-1} to p_{i+1}. Note that all the elements of R are $\prec p_i$, which follows from [Ron09, Thm 2.9]. Indeed, since $p_{i-1} \prec p_i \succ p_{i+1}$, the element $\mathrm{proj}_R(\mathrm{id})$ of R closest to id must be opposite to p_i, and so there is a geodesic edge-path from id to p_i through $\mathrm{proj}_R(\mathrm{id})$, and hence through any other element of R. Thus the above replacement decreases the complexity of π defined as the tuple $(n_L, \ldots, n_2, n_1)$, where n_j is the number of p_i in π with $\ell(p_i) = j$, with lexicographic order. After possibly removing a subpath, we can assume that the new edge-path is embedded. After finitely many such modifications, we obtain the desired path. $\square$

Lemma 2.3. *For $g \preceq h$, we have $m(g) \preceq m(h)$.*

Proof. Let k be the minimal number of distinct Shi components traversed by a geodesic edge-path γ from h to g. We proceed by induction on k, where for $k = 1$ we have $m(g) = m(h)$. Suppose now $k > 1$. If a neighbour f of h on γ lies in the same Shi component as h, then we can replace h by f. Thus we can assume that f lies in a different Shi component than h. Consequently, the wall $\mathcal{W}_r$ separating h from f belongs to $\mathcal{E}$. Since $g \preceq f$, by the inductive assumption we have $m(g) \preceq m(f)$. Thus it suffices to prove $m(f) \preceq m(h)$.

In the first case, where for every neighbour h' of h on a geodesic edge-path from h to id, the wall separating h from h' belongs to $\mathcal{E}$, we have $h = m(h)$ and we are done. Otherwise, let $\mathcal{W}_q$ be a wall outside $\mathcal{E}$ separating h from its neighbour $h' \prec h$. If r and q do not commute, then r, q are sharp-angled, with id in a geometric fundamental domain for $\langle r, q\rangle$. By Lemma 2.2, among the walls in $\langle r, q\rangle\{\mathcal{W}_r, \mathcal{W}_q\}$ only $\mathcal{W}_r$ belongs to $\mathcal{E}$. Let $\bar{h}, \bar{f}$ be the vertices opposite to f, h in the residue $\langle r, q\rangle h$. We have $m(\bar{h}) = m(h), m(\bar{f}) = m(f)$. Replacing h, f by $\bar{h}, \bar{f}$, and possibly repeating this procedure finitely many times, we arrive at the first case. $\square$

Lemma 2.3 has the following immediate consequence.

Corollary 2.4. *For any $g, g' \in M$, if the join of g and g' exists, then it belongs to M.*

For completeness, we include the proof of the following.

Lemma 2.5 ([DH16, Prop 4.16])**.** *For any $h \in M$ and $g \preceq h$, we have $g^{-1}h \in M$.*

Proof. For any neighbour h' of h on a geodesic edge-path from h to g, the wall $\mathcal{W}$ separating h from h' belongs to $\mathcal{E}$. Consequently, we also have $g^{-1}\mathcal{W} \in \mathcal{E}$, and so $g^{-1}h \in M$. $\square$

Also note that for each $s \in S$, we have $\mathcal{W}_s \in \mathcal{E}$ and so $m(s) = s$ implying $S \subset M$. Thus Corollary 2.4 and Lemma 2.5 imply Theorem 1.2.

3. Cone type parts

Let $T = T(g)$ for some $g \in W$. We denote by ∂T the set of walls separating adjacent vertices $h \in T$ and $h' \notin T$. In particular, the walls in ∂T separate id from g^{-1}.

We note that one of the primary differences between the cone type parts and the Shi parts is that the cone type parts do not correspond to a 'hyperplane arrangement'. See for example Figure 2.

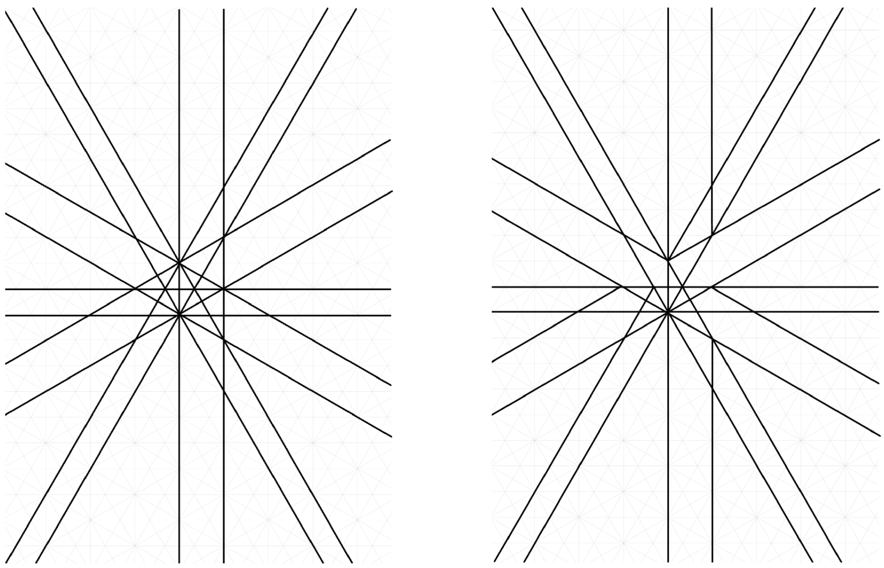

Figure 2. Shi parts and cone type parts for the Coxeter group of type $\widetilde{G}_2$

Remark 3.1. Note that for $g, g' \in Q(T)$ any geodesic edge-path from g to g' has all vertices f in $Q(T)$. Indeed, for $h \in T$, any wall separating id from f separates id from g or g' and so it does not separate id from h. Thus $h \in T(f^{-1})$ and so $T \subseteq T(f^{-1})$. Conversely, if we had $T \subsetneq T(f^{-1})$ then there would be a vertex $h \in T$ with a neighbour $h' \in T(f^{-1}) \setminus T$ separated from h by a wall $\mathcal{W}$ (in ∂T) that does not separate h from f. The wall $\mathcal{W}$ would not separate h' from g or g', contradicting $h' \notin T(g^{-1})$ or $h' \notin T(g'^{-1})$. See also [PY22, Thm 2.14] for a more general statement.

Proof of Theorem 1.3. The proof is identical to that of Theorem 1.1, with P replaced by Q. The vertices of a geodesic edge-path π in X^1 from p_0 to p_n belong to Q by Remark 3.1. We also make the following change in the proof of the claim that all the elements of $R = \langle r, q\rangle(p_i)$ lie in Q. Namely, since $T = T(p_i^{-1})$ equals $T(p_{i-1}^{-1})$, we have $\mathcal{W}_r \notin \partial T$. Analogously we obtain $\mathcal{W}_q \notin \partial T$. If r and q do not commute, we have that T is contained in a geometric fundamental domain for $\langle r, q\rangle$, and so we also have $\mathcal{W}' \notin \partial T$ for any $\mathcal{W}'$ that is a translate of $\mathcal{W}_r$ or $\mathcal{W}_q$ under an element of $\langle r, q\rangle$. This justifies the claim. □

Proof of Theorem 1.4. The proof structure is similar to that of Lemma 2.3. We need to justify that for $g \preceq h$, we have $\mu(g) \preceq \mu(h)$, where we induct on the minimal number k of distinct cone type components traversed by a geodesic edge-path γ from h to g. Suppose $k > 1$, and let $Q = Q(T)$ be the cone type component containing h. If a neighbour f of h on γ lies in Q, then we can replace h by f. Thus we can assume $f \notin Q$. Consequently, the wall $\mathcal{W}_r$ separating h from f belongs to ∂T. Since $g \preceq f$, by the inductive assumption we have $\mu(g) \preceq \mu(f)$. Thus it suffices to prove $\mu(f) \preceq \mu(h)$.

If for every neighbour h' of h on a geodesic edge-path from h to id, the wall separating h from h' belongs to ∂T, we have $h = \mu(h)$ and we are done. Otherwise, let $\mathcal{W}_q$ be a wall outside ∂T separating h from its neighbour $h' \prec h$. Let $\bar{h}, \bar{f}$ be the vertices opposite to f, h in the residue $\langle r, q\rangle h$, and let $f' = rqh$. It suffices to prove $\mu(\bar{h}) = \mu(h), \mu(\bar{f}) = \mu(f)$. To justify $\mu(\bar{h}) = \mu(h)$, or, equivalently, $\bar{h} \in Q$, it suffices to observe that among the walls in $\langle r, q\rangle\{\mathcal{W}_r, \mathcal{W}_q\}$ only $\mathcal{W}_r$ belongs to ∂T: Indeed, if r and q do not commute, then r, q are sharp-angled, with T in the geometric fundamental domain F for $\langle r, q\rangle$ containing id.

It remains to justify $\mu(\bar{f}) = \mu(f)$, or, equivalently, $T(\bar{f}^{-1}) = \widetilde{T}$ for $\widetilde{T} = T(f^{-1})$. To start with, to show $T(f'^{-1}) = \widetilde{T}$, it suffices to show that the wall $\mathcal{W} = r\mathcal{W}_q$ does not belong to $\partial\widetilde{T}$.

Otherwise, let $b \in \widetilde{T}$ be adjacent to $\mathcal{W}$. Since $\widetilde{T} \subset F \cup rF$, we have $b \in rF$. Then $rb \in F$ is adjacent to $\mathcal{W}_q$, which is outside ∂T. Consequently, $rb \notin T$. Thus there is a wall $\mathcal{W}'$ separating id from h and rb. Note that $\mathcal{W}' \neq \mathcal{W}_r$ and so $\mathcal{W}'$ separates id from f. Since id lies on a geodesic edge-path from f to b, we have that $\mathcal{W}'$ does not separate id from b. Thus $r\mathcal{W}'$ separates r and rb from f, h, b, and id, since, again, id lies on a geodesic edge-path from f to b.

Consider the distinct connected components $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ of $X^1 \setminus (\mathcal{W}_r \cup r\mathcal{W}')$ with id $\in \Lambda_1, b \in \Lambda_2, r \in \Lambda_3, rb \in \Lambda_4$. Connected components Λ_1 and Λ_3 (resp. Λ_2 and Λ_4) are *opposite* in the sense that they are separated by both $\mathcal{W}_r$ and $r\mathcal{W}'$. Since id and r are interchanged by the reflection r and they lie in the opposite connected components, we have $r\Lambda_2 \subsetneq \Lambda_1$. On the other hand, since b and rb lie in the opposite connected components, we have $r\Lambda_1 \subsetneq \Lambda_2$, which is a contradiction.

This proves that the wall $\mathcal{W}$ does not belong to $\partial\widetilde{T}$, and hence neither does any other wall in $\langle r, q\rangle\{\mathcal{W}_r, \mathcal{W}_q\}$. Consequently $T(\bar{f}^{-1}) = \widetilde{T}$, as desired. □

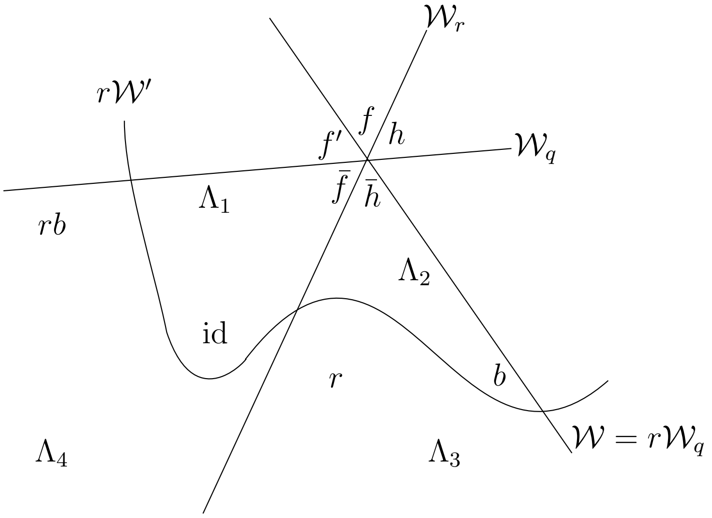

FIGURE 3. Proof of Theorem 1.4, the case of $m_{rq} = 3$

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