Mathematics for Computer Science
Lecture 2 Notes

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Lecture Notes

Logical Deductions

An inference rule is a rule for combining true propositions to form other true propositions.

Examples

- Modus Ponens: If (P and $(P \to Q)$), then Q.
- $((P \to Q) \land \neg Q) \to \neg P$
- $(\neg P \to \text{False}) \to P$

Avoid proof by intimidation: "This is obvious," "obviously," "This is simple."

Proof Outlines

Example

- Thm: $\exists n \in \mathbb{N}.\ (n \geq 10 \land isprime(n))$
- PF: We'll show that $n =$ "" works. This $n \in \mathbb{N}$ for [reasons], and n is prime and ≥ 10 because [reasons].

Abstract Version

Thm: $\exists x \in \mathbb{S}.\ P(x)$
PF: Choose $x =$ "". Then $x \in \mathbb{S}$ because [reasons], and $P(x)$ is true because [reasons].

Example

Thm: $\forall x \in \mathbb{R}.\ x^2 - 6x > -10$
PF: Suppose $x \in \mathbb{R}$. Then $(x-3)^2 \geq 0$ because all reals have non-negative squares. Equivalently, $x^2 - 6x \geq -9 > -10$ as needed. QED.

Thm: $\forall x \in \mathbb{S}.\ P(x)$
PF: Suppose x is a generic element of $\mathbb{S}$. Then $P(x)$ is true because " ".

Thm: $P \implies Q$
PF: Assume P. Then Q is true because [reasons]. *(Direct Proof)*

Thm: If n is a multiple of 10, then it is even.
PF: Assume n is a multiple of 10. Then $n = 10k$ for some $k \in \mathbb{Z}$. Thus $n = 2(5k)$, so n is even.

Proof by Contrapositive

$P \implies Q$ is equivalent to $\neg Q \implies \neg P$.
PF: By contrapositive, assume Q is false. Then P is false because [reasons].

Thm: $\forall n \in \mathbb{Z}.\ iseven(n^2) \implies iseven(n)$
PF: Suppose $n \in \mathbb{Z}$. Proof by contrapositive: assume n is odd. So $n = 2k+1$ for some $k \in \mathbb{Z}$.

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

which is odd. QED.

Proof by Contradiction

Thm: P.
PF: For the sake of contradiction, assume P is false. Then R is both true and false, which is a contradiction. So our assumption is wrong, and P is true.

Thm: $\sqrt{2} \notin \mathbb{Q}$
PF: Proof by contradiction. Assume $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = a/b$ for some integers a, b with $b \neq 0$, in lowest terms.

- $a = \sqrt{2} \cdot b$
- $a^2 = 2b^2$
- a^2 is even
- a is even by previous theorem, so $a = 2c$ for some $c \in \mathbb{Z}$

Thm: An integer n is fooish precisely when $n+1$ is barsome. $\forall n \in \mathbb{Z}.\ F(n) \iff B(n+1)$
PF: Suppose n is an integer. WTS $(F(n) \iff B(n+1))$.

- WTS $F(n) \implies B(n+1)$: Assume $F(n)$ is true. WTS $B(n+1)$.
- WTS $B(n+1) \implies F(n)$: Assume $B(n+1)$ is true. WTS $F(n)$.

Proof by Induction

Thm: $\forall n \in \mathbb{N},\ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Examples

- $n = 4$: $1 + 2 + 3 + 4 = \frac{4 \cdot 5}{2}$
- $n = 1$: $1 = \frac{1 \cdot 2}{2}$
- $n = 0$: $0 = \frac{0 \cdot 1}{2}$

If $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, then:

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2},$$

because we just added $n+1$ to both sides.

$P(n) :=$ "$1 + 2 + ... + n = \frac{n(n+1)}{2}$"
THM: $\forall n \in 0.\ P(n)$

We Proved	**We Want**
P(0)	P(0)
P(0) $\rightarrow$ P(1)	P(1)
P(1) $\rightarrow$ P(2)	P(2)
P(2) $\rightarrow$ P(3)	P(3)
P(3) $\rightarrow$ P(4)	P(4)

Principle of Induction

$LHS \implies RHS$,

(If you know P(0) $\forall n \geq 0.\ P(n) \implies P(n+1)) \implies (\forall n \geq 0.\ P(n))$

Proof by Induction

P(n):= "$\sum_{k+1}^{n} = \frac{n(n+1)}{2}$", we'll show $\forall n \geq 0$ P(n), by induction.

Base Case, WTS P(0): $0 = \frac{0*1}{2}$ – True.

Inductive Step: Assume $n \geq 0$ and assume P(n); WTS P(n+1).

In other words, assume $1 + 2 + ... + n = \frac{n(n+1)}{2}$

WTS:

$$1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2},$$