Compact = Complete + Totally Bounded

In $\mathbb{R}$ (or $\mathbb{R}^n$) a set is compact if and only if it is closed and bounded. In general metric spaces, particularly if they are "infinite dimensional", this is no longer true. However, something similar works in general: a set E in (χ, d) is compact if and only if it is complete and "totally bounded". To be totally bounded E has to satisfy the following condition: for every $\epsilon > 0$ there is a finite set of balls of radius ϵ with centers in E that cover E, meaning $E \subset \cup_{n=1}^{N} B_\epsilon(x_n)$. Remember that the condition "E is complete" will always hold when χ is complete in the metric d and E is a **closed** subset of χ.

Proof that Compact implies Complete and Totally Bounded: First I'll show that E must be complete: let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence of points in E. If E is compact, this sequence has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. So we know that $\{x_{n_k}\}_{k=1}^{\infty}$ converges to $x_\infty \in E$. But this means that the whole Cauchy sequence converges to x_∞: given $\epsilon > 0$, we know that there is an N such that $d(x_n, x_m) < \epsilon/2$ when $m, n \geq N$, and also that there is an N_1 such that $d(x_{n_k}, x_\infty) < \epsilon/2$ when $k \geq N_1$. We also know that for any k

$$d(x_n, x_\infty) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x_\infty). \tag{1}$$

Note that the definition of subsequence implies $n_k \geq k$. So we can pick k so that $k \geq N_1$ and $n_k \geq N$. Then (1) shows that $d(x_n, x_\infty) \leq \epsilon/2 + \epsilon/2 = \epsilon$, when $n \geq N$. So the Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ converges to x_∞, and E is complete. So compact implies complete.

To see that if E is compact, then E is totally bounded, suppose E is not totally bounded. Then there is an $\epsilon_0 > 0$ such that you cannot cover E with a finite number of balls of radius ϵ_0 with centers in E. So pick a ball $B_{\epsilon_0}(x_1)$ with $x_1 \in E$. Then there must be an $x_2 \in E$ that is not in $B_{\epsilon_0}(x_1)$. There is also an $x_3 \in E$ that is not in $B_{\epsilon_0}(x_1) \cup B_{\epsilon_0}(x_2)$. If you keep arguing this way, you get a sequence $\{x_n\}$ of points in E with the property that $x_n \notin \cup_{k=1}^{n-1} B_{\epsilon_0}(x_k)$. In other words each x_n is distance at least ϵ_0 away from each of its predecessors! This sequence cannot have a convergent subsequence: for any subsequence of this sequence $d(x_{n_k}, x_{n_l}) \geq \epsilon_0$ when $k \neq l$. So the subsequence cannot be a Cauchy sequence and cannot converge. This shows that E cannot be compact if it is not totally bounded. In other words compact implies totally bounded.

Proof that Totally Bounded and Complete implies Compact: For this we have to start with a sequence in E, $\{x_n\}$, and show that it has a convergent subsequence. The strategy is to use "E is totally bounded" to build a subsequence that is a Cauchy sequence, and then use "E is complete" to conclude that the subsequence converges to a point in E. You build the subsequence this way: since E is totally bounded, you can cover E with a finite number of balls of radius 1. So at least one of those balls, $B_1(y_1)$, contains an infinite number of points in the sequence $\{x_n\}$. Pick one of those points x_{n_1}. You can also cover E by a finite number of balls of radius $1/2$. In particular, these balls cover $B_1(y_1) \cap E$ and the intersection of at least one of them, call it $B_{1/2}(y_2)$ with $B_1(y_1)$ must contain an infinite number of points from $\{x_n\}$. Pick one of them x_{n_2} with $n_2 > n_1$. Continuing this way you get a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with the property that x_{n_k} belongs to a fixed ball of radius $1/j$ when $k \geq j$. In the notation I am using here that is $x_{n_k} \in B_{1/j}(y_j)$ when $k \geq j$. So for $k, l \geq j$ you have $d(x_{n_k}, x_{n_l}) \leq 2/j$. That says $\{x_{n_k}\}$ is a

Cauchy sequence, and, since E is complete it converges to a point in E. Thus we have shown that E is compact.