9/28/2016

Some Math Stuffs

Linear Algebra Review

spectrum $(A) = \{\lambda \in \mathbb{C},\ \lambda$ is the eigen-values of $A\}$

$\rho(A) = \max\{|\lambda| : \lambda \in sp(A)\}$

Rayleigh quotient: $R_A(x) = \dfrac{x^* A x}{x^* x}$

if $A = A^*$, then $\sup_{x \neq 0} R_A(x) =$ max e-value of A.

$\|A\|_2 = \rho(A^* A)^{1/2}$ $\quad A = A^* \Rightarrow \|A\|_2 = \rho(A)$

dual frame (共轭框架)

$$x = \sum_n \langle x, e_i \rangle \tilde{e}_i$$

Parseval frame $\Leftrightarrow \tilde{e}_i = \frac{1}{A} e_i$

$$x = \frac{1}{A} \sum_n \langle x, e_i \rangle e_i$$

$$\because \|x\|^2 = \frac{1}{A} \sum_n \langle x, e_i \rangle \langle e_i, x \rangle$$

$$= \frac{1}{A} \sum_n \langle x, e_i \rangle \overline{\langle x, e_i \rangle}$$

$$= \frac{1}{A} \sum_n |\langle x, e_i \rangle|^2$$

※※ Some results in Lecture 7, theorem 2.5

Frame Theory in Linear Algebra.

$\{e_i\}$ a set of vectors in Hilbert Space $\mathcal{H}$

$\{e_i\}$ must span $\mathcal{H}$, otherwise $\exists\, x \in \mathcal{H}$ such that

$x \perp e_i,\ \forall e_i \in \{e_i\}$

$\Rightarrow A\|x\|^2 \le \sum_n |\langle x, e_i \rangle|^2 = 0 \le B\|x\|^2$

eg: $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \dots \right\}$ span $\mathbb{R}^2$, however,

$$\sum_n |\langle (0,1)^T, e_k \rangle| = 0 + 0 + 1 + 2 + \dots = \infty$$

tight frame: $A = B$

parseval fram: $A = B = 1$

equal-norm fram: $\|e_i\| = c$, unit norm fram: $c = 1$

each orthonormal basis is parseval fram, However, not any Parseval fram is orthonormal basis, for example: in $\mathbb{R}^2$, $\cancel{\left\{ \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \right\}}$

$\cancel{\sum_n |\langle x, e_i \rangle|^2 = \frac{1}{2}a^2 + \frac{1}{2}b^2 +}$ $\quad \left\{ \left(\frac{\sqrt{3}}{3}, 0\right), \left(0, \frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) \right\}$

The sequence of frame coefficients is not unique compared to basis,

The frame degrade to orthonormal basis (ONB) iff it's Parseval frame + unit norm frame.

Frame operator $\mathcal{F} : \mathcal{H}_1 \to \mathcal{H}_2$

$$(\mathcal{F} x)_n = \langle x, e_n \rangle$$

$$\mathcal{F}(x) = \sum_n \langle x, e_n \rangle e_n$$

$$A\|x\|^2 \le \|Fx\|^2 \le B\|x\|^2$$

$\mathcal{F}$ is a linear operator (mapping), F is the matrix representation of $\mathcal{F}$.

$\mathcal{F}$ will map a vector in $\mathcal{H}_1$ into $\mathcal{H}_2$, to make frame operator span $\mathcal{H}$, $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2)$

Fx will get the coordinates in the Hilbert space ($\mathcal{H}_1 = \mathcal{H}_2$) with new frame $\{e_i\}$