Reading Notes 10/16/2016 (P108 Kreyszig) (P115 Luenberger)

Algebraic dual space: $X^{\#}$, space of linear functionals defined on X.

Second algebraic dual space: $X^{\#\#}$, space of linear functionals of linear functionals defined on X. (Given a point $x \in X$ fixed)

Space	General element	Value at a point
X	x	—
$X^{\#}$	f	$f(x)$
$X^{\#\#}$	g	$g(f)$

$g(f) = g_x(f) = f(x)$

Notes. We will introduce reflexivity of a normed space which is important to extend the corollary 2 of ~~Banach~~ Hahn-Banach theorem.

Canonical Mapping $C: X \to X^{\#\#}$

(canonical embedding of X into $X^{\#\#}$) $\quad x \longmapsto g$

This canonical mapping is always injective (one to one), but it may not be surjective (onto).

Proof: Consider $x \neq y$, $x, y \in X$, Suppose $Cx = Cy$.

$\because Cx = g_x, \forall x \in X \quad Cy = g_y, \forall y \in X$

$\therefore$ for all $f \in X^{\#}$, we have:

$g_x(f) = g_y(f)$, i.e. $f(x) = f(y), \forall f \in X^{\#}$

From Hahn-Banach Theorem Corollary 2, $\forall v^{\#} \in X$, there exists a $f: X \to X^{\#}$ such that $f(v) \neq 0$ which is a contradiction with $f(x-y) = 0, \forall f \in X^{\#}$. Hence $x - y = 0 \Rightarrow x = y$.

We see the canonical mapping is not always surjective mapping. If the mapping is bijective, we define space X as <u>algebraic reflexive</u>.