STAT512 Division 1 HW 4

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1. In a small scale experimental study of the relation between degree of brand liking (Y) and moisture content (X_1) and sweetness (X_2) of the product, data is in brand.csv. Sample size is 16. Use R to

a) draw a scatter plot and the correlation matrix, describe what you see.

b) fit regression model to the data without interaction, $\hat{Y} = \beta_1 X_1 + \beta_2 X_2$

c) Perform a test to see if the residuals are normal.

d) Perform BF test for accuracy of the residuals. You can make two groups ($Y \leq 81.75$, or > 81.75, where 81.75 is the average).

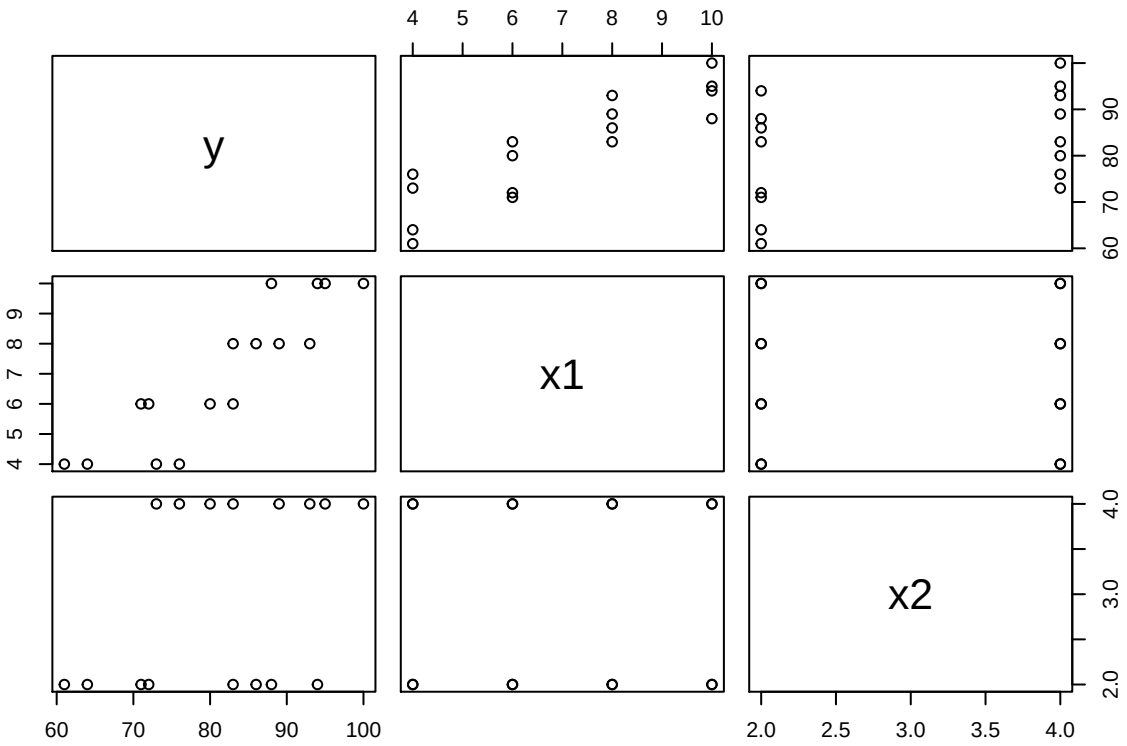
e) Perform a lack of fit test of the model use a significant level of 0.01. State H_0 and H_a, test statistics, critical value, p value and conclusion.

f) Find MSE, the variance-covariance matrix of estimators (i.e., $\Sigma_{\{\mathbf{b}\}}$), variance-covariance matrix of predictors (i.e., $\Sigma_{\{\hat{Y}_h\}}$) when $X_1 = 5, X_2 = 4$.

Solution:

a)

```
brand <- read.csv('data/brand.csv',header = TRUE)
plot(brand)
```




```
library('Hmisc')
```

```
## Loading required package: lattice
```

```
## Loading required package: survival
## Loading required package: Formula
## Loading required package: ggplot2
##
## Attaching package: 'Hmisc'
## The following objects are masked from 'package:base':
##
##     format.pval, units
```

```
rcorr(as.matrix(brand))
```

```
##       y   x1   x2
## y  1.00 0.89 0.39
## x1 0.89 1.00 0.00
## x2 0.39 0.00 1.00
##
## n= 16
##
##
## P
##    y      x1     x2
## y         0.0000 0.1304
## x1 0.0000        1.0000
## x2 0.1304 1.0000
```

The correlation matrix is given by

$$r = \begin{bmatrix} 1.00 & 0.89 & 0.39 \\ 0.89 & 1.00 & 0.00 \\ 0.39 & 0.00 & 1.00 \end{bmatrix}$$

b)

```
brand.mod <- lm(y~x1 + x2, brand)
summary(brand.mod)
```

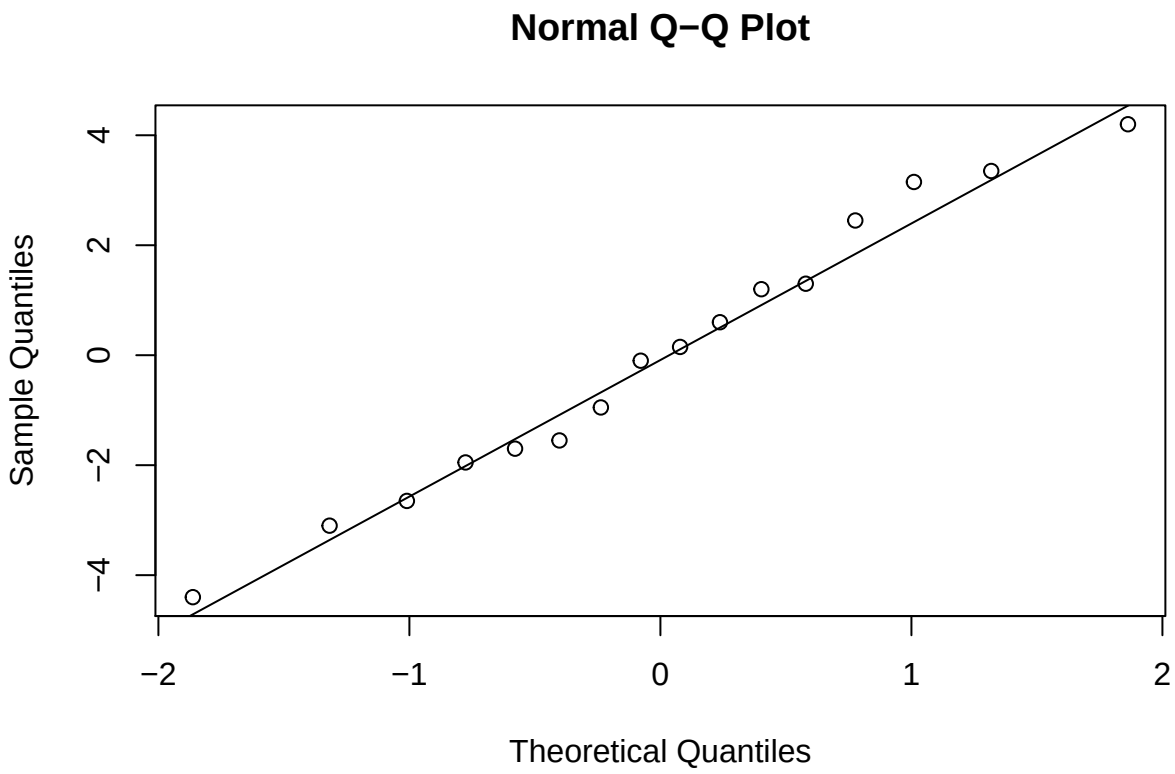
```
##
## Call:
## lm(formula = y ~ x1 + x2, data = brand)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -4.400 -1.762  0.025  1.587  4.200
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  37.6500     2.9961  12.566 1.20e-08 ***
## x1            4.4250     0.3011  14.695 1.78e-09 ***
## x2            4.3750     0.6733   6.498 2.01e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.693 on 13 degrees of freedom
## Multiple R-squared:  0.9521, Adjusted R-squared:  0.9447
## F-statistic: 129.1 on 2 and 13 DF,  p-value: 2.658e-09
```

The linear regression fit model without interaction term is given by

$$Y = 37.650 + 4.425X_1 + 4.375X_2$$

c) Let us use a probability plot and a Shapiro-Wilk test to test the normality of the error terms.

```
brand.res <- residuals(brand.mod)
qqnorm(brand.res)
qqline(brand.res)
```




```
shapiro.test(brand.res)
```

```
##
##  Shapiro-Wilk normality test
##
## data:  brand.res
## W = 0.97585, p-value = 0.9222
```

From the test results, we confirm that the residuals are normal.

d) Brown-Forsythe Test to test the constancy of residual variance is given by

```
library(ALSM)
```

```
## Loading required package: leaps
## Loading required package: SuppDists
## Loading required package: car
## Loading required package: carData
##
## Attaching package: 'ALSM'
```

```
## The following object is masked from 'package:lattice':
##
##     oneway
```

```
g <- rep(1,16)
g[brand$y <= 81.75] = 0
bftest(brand.mod,g)
```

```
##        t.value   P.Value alpha df
## [1,] 0.8629512 0.4027057  0.05 14
```

The difference is not significant, which means the variance of residuals is constant.

e) The lack of fit test is stated as follows

$$H_0 : \mathbb{E}\{Y\} = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

$$H_a : \mathbb{E}\{Y\} \neq \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

```
brand.mod.full <- lm(y~factor(x1)*factor(x2), brand)
anova(brand.mod, brand.mod.full)
```

```
## Analysis of Variance Table
##
## Model 1: y ~ x1 + x2
## Model 2: y ~ factor(x1) * factor(x2)
##   Res.Df  RSS Df Sum of Sq     F Pr(>F)
## 1     13 94.3
## 2      8 57.0  5      37.3 1.047  0.453
```

```
qf(1-0.05,5,8)
```

```
## [1] 3.687499
```

Based on the ANOVA table, $MSPE = 57.0/8 = 7.125$, $MSLF = 37.3/5 = 7.46$, the test statistic is $Ts = \frac{MSLF}{MSPE} = 1.047$, the critical value is 3.687499 at significance level $\alpha = 0.05$, p-value is 0.453, which is greater than α. Therefore, we fail to reject the null hypothesis.

f) Based on design matrix, we are able to obtain the variance-covariance matrix of estimators, variance-covariance matrixx of predictors.

```
MSE <- 94.3/13
X.design <- matrix(c(rep(1,16),brand$x1,brand$x2),nrow = 16,ncol = 3)
Normal.inv <- solve(t(X.design)%*%X.design)
Sigmab <- MSE*Normal.inv
Xh <- matrix(c(1,5,4),nrow = 3,ncol = 1)
SigmaYh <- t(Xh)%*%Sigmab%*%Xh
MSE
```

```
## [1] 7.253846
```

```
print(Sigmab)
```

```
##            [,1]          [,2]       [,3]
## [1,]  8.9766346 -6.347115e-01 -1.3600962
## [2,] -0.6347115  9.067308e-02  0.0000000
## [3,] -1.3600962  1.887513e-16  0.4533654
```

```
print(SigmaYh)
```

```
##          [,1]
## [1,] 1.269423
```

Based on the calculation, $MSE = 7.253846$.

$$\Sigma_{\{b\}} = \begin{bmatrix} 8.9766346 & -0.6347115 & -1.36009621 \\ -0.6347115 & 0.09067308 & 0.0000000 \\ -1.3600962 & 1.887513e-16 & 0.4533654 \end{bmatrix}, \quad \Sigma_{\{\hat{Y}_h\}} = 1.269423$$

2. Refer to question 1, compute the following question by hand.

a) Obtain an interval estimate of $\mathbb{E}\{Y_h\}$ (i.e., $\hat{Y}_h$) when $(X_1 = 5, X_2 = 4)$, with 99% confidence level.

b) Obtain an interval estimate of a single predictor $\hat{Y}_h\{new\}$ when $(X_1 = 5, X_2 = 4)$, with 99% confidence level.

c) Obtain an interval estimate of the average of the next two predictors, when $(X_1 = 5, X_2 = 4)$, with 90% confidence level.

d) Obtain a simultaneous estimate of the two (single) predictor $\hat{Y}_h\{new\}$ when $(X_1 = 5, X_2 = 4)$, and $(X_1 = 6, X_2 = 5)$, with a 90% confidence level.

e) Obtain a simultaneous confidence interval for all three estimators β_0, β_1 and β_2, with a 90% confidence level.

Solution:

a) The interval estimate is given by $\hat{Y}_h \pm t(1 - \alpha/2; n - p)s\{\hat{Y}_h\}$. $t(1 - 0.01/2; 13) = 3.012276$, Therefore, the interval is given by $[73.88111, 80.66889]$.

```
Xh2 <- matrix(c(1,6,5),nrow = 3,ncol = 1)
SigmaYh2 <- t(Xh2)%*%Sigmab%*%Xh2
qt(1-0.1/2,13)
```

```
## [1] 1.770933
```

```
Y <-37.650 + 4.425*5 + 4.375*4
Y2 <- 37.650 + 4.425*6 + 4.375*5
Y
```

```
## [1] 77.275
```

```
sqrt(SigmaYh + MSE)
```

```
##          [,1]
## [1,] 2.919464
```

```
Y + sqrt(SigmaYh + MSE/2)*1.770933
```

```
##          [,1]
## [1,] 81.19367
```

```
qt(1-0.1/4,13)
```

```
## [1] 2.160369
```

```
Y - sqrt(SigmaYh + MSE)*qt(1-0.1/4,13)
```

```
##          [,1]
## [1,] 70.96788
```

```
Y + sqrt(SigmaYh + MSE)*qt(1-0.1/4,13)
```

```
##          [,1]
## [1,] 83.58212
```

```
Y2 - sqrt(SigmaYh2 + MSE)*qt(1-0.1/4,13)
```

```
##           [,1]
## [1,] 79.37739
```

```
Y2 + sqrt(SigmaYh2 + MSE)*qt(1-0.1/4,13)
```

```
##           [,1]
## [1,] 92.77261
```

```
37.650 - qt(1-0.1/6,13)*sqrt(8.9766346)
```

```
## [1] 30.5205
```

```
37.650 + qt(1-0.1/6,13)*sqrt(8.9766346)
```

```
## [1] 44.7795
```

```
4.425 - qt(1-0.1/6,13)*sqrt(9.067308e-02)
```

```
## [1] 3.708458
```

```
4.425 + qt(1-0.1/6,13)*sqrt(9.067308e-02)
```

```
## [1] 5.141542
```

```
4.375 - qt(1-0.1/6,13)*sqrt(0.4533654)
```

```
## [1] 2.772763
```

```
4.375 + qt(1-0.1/6,13)*sqrt(0.4533654)
```

```
## [1] 5.977237
```

b) The confidence interval estimate for the new observation is given by $\hat{Y}_h \pm t(1-\alpha/2; n-p)s\{\hat{Y}_h\{new\}\}$ with $s\{\hat{Y}_h\{new\}\} = \sqrt{MSE + s^2\{\hat{Y}_h\}} = 2.916953$. Therefore, the numerical interval is estimated by $[68.48077, 86.06923]$.

c) Similarly, the standard deviation of prediction mean is given by $s\{predmean\} = \sqrt{MSE/m + s^2\{\hat{Y}_h\}} = 2.209455$. Therefore, the estimated confidence interval is given by $[73.35633, 81.19367]$.

d) Let us estimate with the Bonferoni simultaneous prediction, which is $\hat{Y}_h \pm Bs\{pred\}$ with $B = t(1-\alpha/2g; n-p) = t(1-0.1/4; 13) = 2.160369$. Therefore, the estimated interval is $[70.96788, 83.58212]$ and $[79.37739, 92.77261]$.

e) The Bonferroni estimated interval for coefficients can be computed, given $s\{b_0\} = \sqrt{8.9766346}, s\{b_1\} = \sqrt{9.067308e-02}, s\{b_2\} = \sqrt{0.4533654}$. Then interval is given by $b_k \pm Bs\{b_k\}$ with $B = t(1-\alpha/6, 13)$. Therefore, the interval for β_0 is $[30.5205, 44.7795]$, the interval for β_1 is $[3.708458, 5.141542]$ and the interval for β_3 is $[2.772763, 5.977237]$.

3. Refer to question 1,

a) What is the ANOVA table that decomposes the regression sum of squares into extra sums of squares associated with X_2, then with X_1, given X_2. (You may use R for this question).

b) Test whether X_1 can be dropped from the regression model given X_2 is retained. Use the partial F test with a significant level of 0.01. Define H_0 and H_a, test statistic, critical value, and state conclusion.

c) Compute R^2_{Y1}, $R^2_{Y1|2}$, $R^2_{Y2|1}$ and R^2. Explain what each coefficient measures and interpret your result.

Solution:

a) Type I ANOVA table can do that

```r
brand.mod.reverse <- lm(y~x2 + x1, brand)
brand.mod
```

```
##
## Call:
## lm(formula = y ~ x1 + x2, data = brand)
##
## Coefficients:
## (Intercept)           x1           x2
##      37.650        4.425        4.375
```

```r
brand.mod.reverse
```

```
##
## Call:
## lm(formula = y ~ x2 + x1, data = brand)
##
## Coefficients:
## (Intercept)           x2           x1
##      37.650        4.375        4.425
```

```r
anova(lm(y~x2,brand))
```

```
## Analysis of Variance Table
##
## Response: y
##           Df  Sum Sq Mean Sq F value Pr(>F)
## x2         1  306.25  306.25  2.5817 0.1304
## Residuals 14 1660.75  118.62
```

```r
anova(lm(y~x1,brand))
```

```
## Analysis of Variance Table
##
## Response: y
##           Df  Sum Sq Mean Sq F value    Pr(>F)
## x1         1 1566.45 1566.45  54.751 3.356e-06 ***
## Residuals 14  400.55   28.61
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```r
anova(brand.mod.reverse)
```

```
## Analysis of Variance Table
##
## Response: y
##           Df  Sum Sq Mean Sq F value    Pr(>F)
## x2         1  306.25  306.25  42.219 2.011e-05 ***
## x1         1 1566.45 1566.45 215.947 1.778e-09 ***
## Residuals 13   94.30    7.25
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

The extra sum of squares are given by $SSR(X_2) = 306.25$ and $SSR(X_1|X_2) = 1566.45$.

b) The Hypothesis test is drafted as follows

$$H_0 : \beta_1 = 0, \quad H_a : \beta_1 \neq 0$$

The test statistic is given by

$$F^* = \frac{MSR(X_1|X_2)}{MSE(X_1, X_2)} = \frac{1566.45/1}{94.3/13} = 215.95$$

The critical value is given by $qf(0.99; 1, 13) = 9.073806$, which is less than the test statistic. It is concluded that the Null hypothesis can be rejected, so that X_1 can be dropped.

```
qf(0.99,1,13)
```

```
## [1] 9.073806
```

c) The coefficients of partial determination are given by

```
anova(brand.mod)
```

```
## Analysis of Variance Table
##
## Response: y
##           Df  Sum Sq Mean Sq F value    Pr(>F)
## x1         1 1566.45 1566.45 215.947 1.778e-09 ***
## x2         1  306.25  306.25  42.219 2.011e-05 ***
## Residuals 13   94.30    7.25
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
library(car)
Anova(lm(y~x1+x2,brand), type='II')
```

```
## Anova Table (Type II tests)
##
## Response: y
##            Sum Sq Df F value    Pr(>F)
## x1        1566.45  1 215.947 1.778e-09 ***
## x2         306.25  1  42.219 2.011e-05 ***
## Residuals   94.30 13
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

$$R^2_{Y1} = \frac{SSR(X_1)}{SSR(X_1) + SSR(X_2|X_1) + SSE(X_1, X_2)} = \frac{1566.45}{1566.45 + 306.25 + 94.30} = 0.7964$$

$$R^2_{Y1|2} = \frac{SSR(X_1|X_2)}{SSE(X_2)} = \frac{1566.45}{94.30 + 1566.45} = 0.9432$$

$$R^2_{Y2|1} = \frac{SSR(X_2|X_1)}{SSE(X_1)} = \frac{306.25}{306.25 + 94.30} = 0.7646$$

$$R^2 = \frac{SSR(X_1, X_2)}{SST(X_1, X_2)} = \frac{1566.45 + 306.25}{1566.45 + 306.25 + 94.30} = 0.9521$$

R^2_{Y1} measures the portion of total variation of Y due to introducing predictor X_1. $R^2_{Y1|2}$ measures the relative marginal reduction in the variation in Y assocaited with X_1 when X_2 already in the model. $R^2_{Y2|1}$ measures the relative marginal reduction in the variation in Y assocaited with X_2 when X_1 already in the model. R^2 measures the proportion of variation in Y explained by the model.

4. A commercial real estate company evaluate vacancy rates, square footage, rental rates, and operating expenses for commercial properties in a large metropolitan area in order to provide clients with quantitative information upon which to make rental decisions. $N = 81$ suburban commercial properties are evaluated.

Y: rental sales

X_1: age

X_2: operating expense

X_3: vacancy rates

X_4: total square footage

According to the following ANOVA table, perform the following test, use a significant level of 0.01. Define H_0 and H_a, test statistic, critical value, and state conclusion.

Analysis of Variance Table

Response: y

	Df	Sum Sq	Mean Sq
x4	1	67.775	67.775
x1	1	42.275	42.275
x2	1	27.857	27.857
x3	1	0.420	0.420
Residuals	76	98.231	1.293

a) Whether X_3 can be dropped from the regression model given that X_1, X_2 and X_4 are retained.

b) Whether X_2 and X_3 can be dropped from the regression model given that X_1 and X_4 are retained.

c) Compute $R^2_{Y3|1,2,4}$.