

STAT545 HW 6

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1 Problem 1: Generating gamma random variables via rejection sampling

We will first generate an exponential random variable.

1. Let U be a Uniform(0,1) random variable. How will you transform this to generate an exponential with rate λ ? Use the inverse-cdf method.
2. Let Y be a random variable on $[0, 1)$, with density $p(Y = y) \propto 1/y^a$, where $a \in (0, 1)$. This is the truncated Pareto distribution. What is its normalization constant? Describe the inverse-cdf method to generate Y by transforming U .
3. Write down the density of a Gamma random variable with shape parameter α and rate parameter β . Call this $p_{\text{Gamma}}(x|\alpha, \beta)$. Note that calculating its cdf is not easy.
4. Note that the sum of two exponentials with rate 1 is distributed as Gamma(2,1). How would you use previous results to sample from a Gamma distribution with parameters (alpha_int,1), where alpha_int is an integer.

Next, we will generate a Gamma random variable, with $\alpha < 1$

5. For $\alpha < 1$, $p_{\text{Gamma}}(x|\alpha, 1) \leq C_1 x^{\alpha-1}$ for $x < 1$, and $p_{\text{Gamma}}(x) \leq C_2 \exp(-x)$ for $x \geq 1$. What are C_1 and C_2 ?
6. Use this information to write down a probability density $q(x|\alpha)$ and a scalar M such that $p_{\text{Gamma}}(x|\alpha, 1) \leq M \cdot q(x|\alpha)$.
7. Now provide pseudocode for a rejection sampler to sample from $p_{\text{Gamma}}(x|\alpha, 1)$. Write down the acceptance probability.
8. How would you use this to generate a Gamma with arbitrary shape parameter, and rate equal to one. How about arbitrary shape and rate parameters?

Solution:

1. The CDF of an exponential distribution is given by

$$F(x) = 1 - e^{-\lambda x}$$

If $P = F(X)$ is uniformly distributed, the inverse of the $F(p)$ satisfies an exponential distribution. That is,

$$F^{-1}(p) = -\frac{\log(1-p)}{\lambda} \sim \text{Expon}(\lambda)$$

where p is a random variable uniformly distributed.

2. The normalization constant for the truncated Pareto distribution is given by

$$Z_Y = \int_0^1 \frac{1}{y^a} dy = \frac{1}{1-a}$$

The PDF of Pareto distribution is given by

$$p(Y = y) = \frac{1-a}{y^a}$$

The CDF of Pareto distribution is given by

$$F(Y \leq y) = \int_0^y \frac{1-a}{t^a} dt = y^{1-a}$$

The Pareto distribution can be sampled from a uniform distribution by inverting the CDF of the Pareto distribution.

$$F^{-1}(U) = U^{a-1} \sim \text{Pareto}(a)$$

3. The density function of a Gamma distribution is given by

$$p_{\text{Gamma}}(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \beta^\alpha \exp(-\beta x)$$

with $x > 0$ and $\alpha, \beta > 0$.

4. From the stated results, it is implied that if $X_i \stackrel{\text{iid}}{\sim} \text{Expon}(\lambda)$, then $Y = X_1 + X_2 + \dots + X_k \sim \text{Gamma}(k, \lambda)$. Independently draw `alpha_int` samples from exponential distribution `Expon(1)`, add them up will give a Gamma distribution with parameter `(alpha_int, 1)`.

5. The C_1 and C_2 are given by

$$C_1 = \max_{\substack{0 < x < 1 \\ 0 < \alpha < 1}} \left\{ \frac{1}{\Gamma(\alpha)} \exp(-x) \right\} = \frac{1}{\Gamma(\alpha)}$$

$$C_2 = \max_{\substack{x \geq 1 \\ 0 < \alpha < 1}} \left\{ \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \right\} = \frac{1}{\Gamma(\alpha)}$$

6. The probability density can be given as

$$q(x|\alpha) = \frac{x^{\alpha-1}}{\frac{1}{\alpha} + e^{-1}} \quad \text{for } 0 < x < 1$$

$$q(x|\alpha) = \frac{\exp(-x)}{\frac{1}{\alpha} + e^{-1}} \quad \text{for } x \geq 1$$

The scalar coefficient is given by $M = \frac{1}{\Gamma(\alpha)} \left(\frac{1}{\alpha} + e^{-1} \right)$.

7. The pseudocode for rejection sampling is given by

```
Sample xs ~ q(x|alpha)
Sample u ~ Unif(0,1)
if (u > pGamma(xs|alpha,1)/(M*q(xs|alpha))){
  xs is rejected
} else{
  xs is accepted
}
```

The acceptance probability is given by

$$p(\text{accept}) = \int \frac{p_{\text{Gamma}}(x|\alpha, 1)}{Mq(x|\alpha)} q(x|\alpha) dx = \frac{1}{M} \int p_{\text{Gamma}}(x|\alpha, 1) dx = \frac{1}{M}$$

8. In the previous pseudocode, we see that $q(x|\alpha)$ is a combination of the truncated Pareto distribution and the exponential distribution. Therefore, we can apply the pseudocode in 7 to sample a Gamma distribution with arbitrary shape parameter and the unity rate parameter.

For a Gamma distribution with arbitrary shape parameter and rate parameter, let us consider $q(x|\alpha, \beta) = p_{\text{Gamma}}(k, \beta - 1)$, where $k = \text{Floor}(\alpha)$. The ratio of the Gamma density to the new proposal probability is

$$\frac{p_{\text{Gamma}}(x|\alpha, \beta)}{q(x|\alpha, \beta)} = \frac{\Gamma(k)\beta^\alpha}{\Gamma(\alpha)(\beta - 1)^k} x^{\alpha-k} \exp(-x)$$

This ratio obtains its maximum when $x = \alpha - k$. Therefore, setting $M = \frac{p_{\text{Gamma}}(\alpha-k|\alpha, \beta)}{p_{\text{Gamma}}(\alpha-k|k, \beta-1)}$ and proceed the rejection sampling will give samples from a Gamma distribution with arbitrary shape and rate parameter.

2 Problem 2: Exploring the space of permutations with Metropolis-Hastings

The file `message.txt` on the course webpage gives a paragraph of English encoded by a permutation code, where every symbol is mapped to a (usually) different one. The encryption key σ might thus be:

$$a \rightarrow v; b \rightarrow l; c \rightarrow n; d \rightarrow .; \dots$$

and a message like “beware of dog.” might read as “lgavfgp wp. c”. To make life simple, assume there are only 30 unique symbols, $\text{Symbol} = (\text{'a'}, \text{'b'}, \dots, \text{'z'}, \text{'.'}, \text{' '}, \text{'\n'}, \text{'\t'})$. Thus the encryption is a bijective function $\sigma: \text{Symbol} \rightarrow \text{Symbol}$. Your job is to recover the original message (or equivalently find σ).

1. How many possible functions σ are there?

Clearly, brute force search is impossible. Instead, we will fit and use a model of the English language.

Download a large corpus of English (I used ‘War and Peace’ from Project Gutenberg.) Use your favourite text-editor to cover all text to lowercase. You might also want to delete some of the clutter at the beginning and the end of the file.

2. Read the file into R using the `readChar()` command. This will return a long string, split it up into a list of characters using the `strsplit()` command. The `table()` command will then give you the frequency of each unique character. This text contains more characters than we are interested in (we only care about `Symbol`). Extract the appropriate symbols from the table, and calculate and plot a histogram of the frequency of each character in `Symbol`. What is the entropy of this distribution? (This is a very crude estimate of the entropy of the English language, if you are interested, see <http://www.math.ucsd.edu/~crypto/java/ENTROPY/>) for how Shannon got a better (model-free) estimate using human subjects.)

We will actually model English text as a first-order Markov process, characterized by a 30 X 30 transition matrix T . We will write this as $P_{Eng}(\cdot|T)$ (assume the distribution of the first symbol is uniform).

3. Given some data D , what are its sufficient statistics? These don’t have to be minimal. What is the joint probability? What is the maximum likelihood estimate of T given D ?
4. Now estimate T for the English text you download. The simplest way to do this is by understanding what `table()` with two arguments does and then running `table(txt[1:length_txt-1], txt[2:length_txt])`. Plot the estimated T using a heatmap. For which symbols does the distribution of the following symbol have the largest and smallest entropy?

We can now write a Bayesian model to estimate the permutation σ . We place a uniform prior over σ :

$$p(\sigma) \propto 1$$

Given a message X encoded with σ , we expect $\sigma^{-1}(X)$ to be English text. We can evaluate the probability it is English using our Markov model of English:

$$p(X|\sigma) = P_{Eng}(\sigma^{-1}(X)|T)$$

Solution:

1. There are 30! possibilities of σ .
- 2.

```
fileName <- 'war-and-peace.txt'
warofpeace <- readChar(fileName, file.info(fileName)$size)
wopList <- strsplit(warofpeace, '')
wopList <- wopList[[1]]
table(wopList)
```

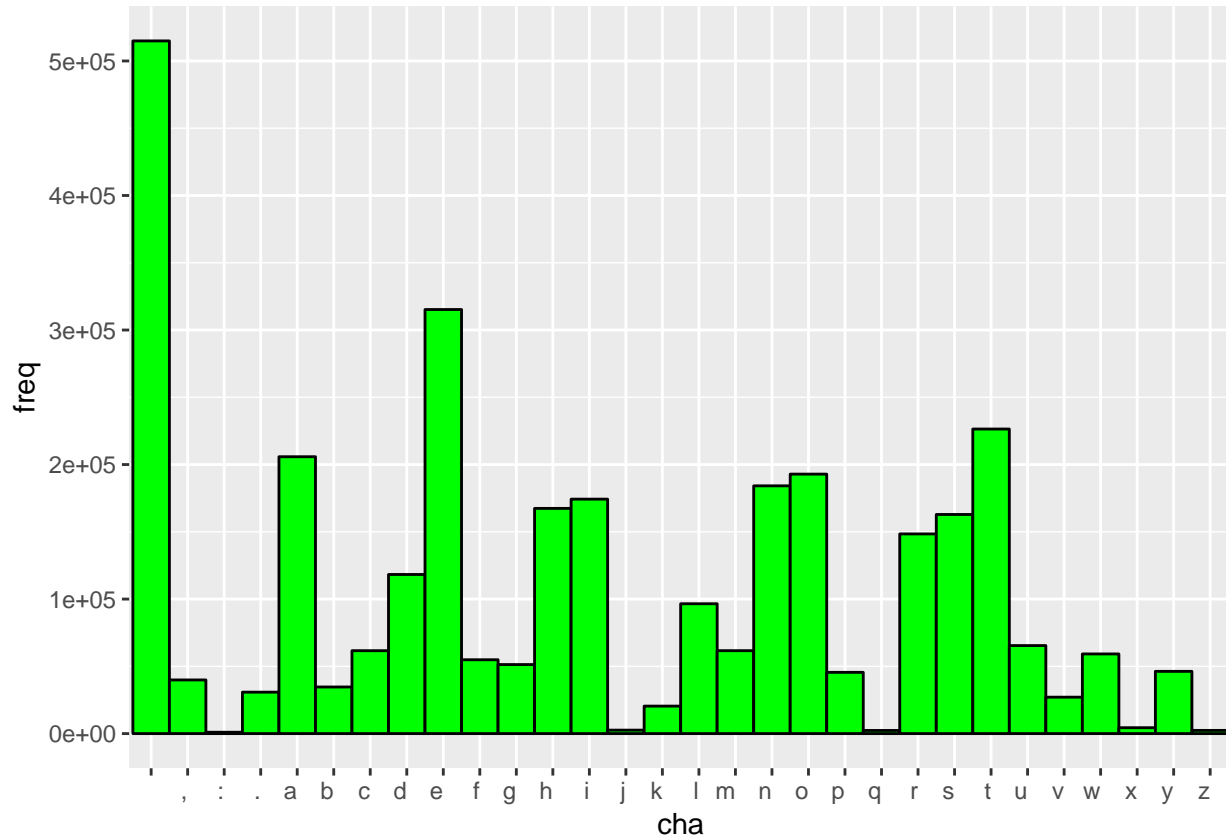
```
## wopList
##      \n      \r      -      ,      ;      :      !      ?      .
## 65007 65007 514908 6308 39891 1145 1014 3923 3137 30805
##      '      "      (      )      [      ]      @      *      /      #
## 7529 17970 670 670 1 1 2 300 29 1
##      %      =      $      0      1      2      3      4      5      6
## 1 2 2 179 392 147 61 23 55 57
## 7 8 9 a à ä b c d e
## 40 193 35 205806 4 1 34658 61623 118290 315233
## é ê f g h i j k l m
## 1 11 54896 51326 167404 174281 2574 20431 96527 61646
## n o p q r s t u v w
## 184173 192879 45533 2330 148428 162891 226406 65434 27086 59207
## x y z
## 4384 46265 2388
```

```
wopList_rev <- sub('[^a-z\\.\\,\\: ]','',wopList)
wopTable <- table(wopList_rev)
#wopTable <- wopTable[, !names(wopTable) %in% c('')]
#wopTable
names(wopTable)
```

```
## [1] "" " " ", " ":" "." "a" "b" "c" "d" "e" "f" "g" "h" "i" "j" "k" "l"
## [18] "m" "n" "o" "p" "q" "r" "s" "t" "u" "v" "w" "x" "y" "z"
```

```
wopDF <- data.frame(cha=names(wopTable),freq=as.vector(wopTable))
```

```
#wopDF
wop.df <- wopDF[-c(1),]
row.names(wop.df) <- seq(1,30)
library(ggplot2)
ggplot(data=wop.df, aes(cha, freq)) + geom_bar(stat = 'identity', width = 1, color = 'black', fill = 'g')
  theme(axis.text.x=element_text(angle = 0, hjust=1))
```



```
wop.df_freq <- wop.df$freq/sum(wop.df$freq)
entropy <- - sum(wop.df_freq * log(wop.df_freq))
entropy
```

```
## [1] 2.915402
```

The entropy of a discrete distribution is given by

$$\mathbb{H}(X) = - \sum_{k=1}^K p(X = k) \log p(X = k)$$

Hence, the entropy for this distribution is 2.915402.

3. The sufficient statistic is a function $\phi(X)$ whose value contains all the information needed to compute any estimate of the parameter. According to the factorization theorem, the joint probability is given by $p(X) = h(X)g(\theta, \phi(X))$. From the joint likelihood probability, it can be seen the maximum likelihood estimate of θ depends only on $\phi(X)$, which satisfies $\frac{\partial g(\theta, \phi(X))}{\partial \theta} = 0$.

4. The estimate of T and the heatmap of T are given as follows,

```
Trans <- table(wopList_rev[1:length(wopList_rev)-1],wopList_rev[2:length(wopList_rev)])
Trans <- Trans[2:31,2:31]
row.sums <- rowSums(Trans)
for (i in seq(1,30)){
  Trans[i,] <- Trans[i,]/row.sums[i]
}
Trans <- as.matrix(Trans)
Trans
```

```
##
```

```

##      ,      :      .      a
##      3.291472e-04 0.000000e+00 0.000000e+00 0.000000e+00 1.248467e-01
##      , 9.999422e-01 0.000000e+00 2.889004e-05 2.889004e-05 0.000000e+00
##      : 1.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
##      . 7.902559e-01 4.709823e-04 0.000000e+00 2.054529e-01 2.093254e-04
##      a 6.027072e-02 4.719524e-03 2.940513e-05 2.548445e-03 3.430599e-05
##      b 3.349310e-03 1.010568e-03 0.000000e+00 5.774672e-04 9.259687e-02
##      c 7.638302e-03 6.175648e-04 1.625171e-05 4.875512e-04 1.092927e-01
##      d 5.628628e-01 3.731171e-02 1.655924e-03 2.512375e-02 2.194544e-02
##      e 2.992882e-01 2.008926e-02 5.192381e-04 1.440721e-02 4.382961e-02
##      f 3.194330e-01 9.344213e-03 3.572234e-04 7.952922e-03 7.249756e-02
##      g 3.616191e-01 2.969496e-02 1.173234e-03 2.128004e-02 6.859374e-02
##      h 7.409393e-02 7.656996e-03 8.434141e-05 3.301364e-03 1.670803e-01
##      i 2.492327e-02 1.300495e-03 5.779979e-06 7.513973e-04 1.682552e-02
##      j 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 3.735409e-02
##      k 1.880000e-01 2.775000e-02 5.000000e-04 1.935000e-02 2.865000e-02
##      l 1.033700e-01 1.461257e-02 2.206068e-04 6.345071e-03 8.074208e-02
##      m 1.393522e-01 2.370935e-02 7.424640e-04 2.709169e-02 1.555545e-01
##      n 1.809764e-01 1.685321e-02 3.967434e-04 1.122057e-02 3.082305e-02
##      o 1.235992e-01 4.569780e-03 7.345226e-05 1.956978e-03 7.235047e-03
##      p 5.004196e-02 8.413939e-03 2.870898e-04 5.984718e-03 1.046111e-01
##      q 4.291845e-04 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
##      r 1.653364e-01 1.948384e-02 4.211602e-04 1.447134e-02 5.771966e-02
##      s 3.041701e-01 3.868409e-02 8.975575e-04 2.185966e-02 5.351607e-02
##      t 2.185623e-01 1.461397e-02 2.715510e-04 1.119695e-02 4.023933e-02
##      u 4.578964e-02 4.765577e-03 7.711289e-05 2.961135e-03 2.040407e-02
##      v 5.745559e-02 2.147483e-02 2.272469e-04 1.060486e-02 6.999205e-02
##      w 1.043213e-01 1.669552e-02 2.571179e-04 9.170538e-03 2.084712e-01
##      x 4.205056e-02 8.740846e-03 7.087172e-04 4.488542e-03 7.961257e-02
##      y 5.446197e-01 7.219189e-02 2.336232e-03 4.987047e-02 3.083364e-02
##      z 2.957330e-02 1.309675e-02 4.224757e-04 5.914660e-03 5.787917e-02
##
##      b      c      d      e      f
##      4.399404e-02 3.551655e-02 2.992300e-02 2.038165e-02 3.798319e-02
##      , 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
##      : 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
##      . 5.233136e-05 1.046627e-04 5.233136e-05 1.151290e-03 5.233136e-04
##      a 1.695206e-02 3.438930e-02 5.546788e-02 8.282446e-04 8.208933e-03
##      b 6.612000e-03 8.662008e-05 4.619738e-04 3.289253e-01 0.000000e+00
##      c 0.000000e+00 1.704804e-02 1.462654e-04 2.134987e-01 0.000000e+00
##      d 1.068338e-04 8.902817e-05 1.160927e-02 1.224939e-01 7.567394e-04
##      e 8.347245e-04 1.785785e-02 9.408726e-02 2.657645e-02 1.040119e-02
##      f 5.076333e-04 0.000000e+00 0.000000e+00 8.686170e-02 5.328270e-02
##      g 2.022817e-05 0.000000e+00 8.698115e-04 1.190023e-01 0.000000e+00
##      h 3.735120e-04 3.735120e-04 3.614632e-04 4.533170e-01 4.096583e-04
##      i 7.750952e-03 4.933212e-02 5.001416e-02 4.885239e-02 2.164602e-02
##      j 0.000000e+00 0.000000e+00 0.000000e+00 2.194553e-01 0.000000e+00
##      k 3.500000e-04 2.350000e-03 1.500000e-04 2.830000e-01 6.500000e-04
##      l 7.458610e-04 7.353559e-04 7.261114e-02 1.797210e-01 2.419321e-02
##      m 1.790163e-02 5.939712e-04 0.000000e+00 2.503424e-01 2.194394e-03
##      n 8.437781e-04 5.086697e-02 1.931805e-01 8.288583e-02 5.096196e-03
##      o 4.700944e-03 7.219307e-03 1.601784e-02 2.639035e-03 8.876180e-02
##      p 2.429221e-04 2.804646e-03 4.416766e-05 1.871384e-01 2.208383e-03
##      q 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00

```

```

## r 1.470609e-03 1.015617e-02 3.005427e-02 2.462406e-01 3.093111e-03
## s 2.705404e-03 1.417632e-02 4.010363e-04 1.117618e-01 1.909697e-03
## t 6.788774e-05 2.869389e-03 1.357755e-05 8.663381e-02 8.327563e-04
## u 1.404997e-02 3.170882e-02 2.180753e-02 2.941086e-02 5.120296e-03
## v 0.000000e+00 0.000000e+00 0.000000e+00 5.523993e-01 0.000000e+00
## w 1.028472e-04 2.742591e-04 6.290817e-03 1.329642e-01 7.542124e-04
## x 0.000000e+00 1.419797e-01 0.000000e+00 7.417907e-02 0.000000e+00
## y 3.423390e-03 3.862879e-03 3.007032e-04 5.977054e-02 2.266839e-03
## z 0.000000e+00 0.000000e+00 1.140684e-02 3.096747e-01 0.000000e+00
##
##          g          h          i          j          k
## 1.720773e-02 8.970044e-02 5.370780e-02 2.856527e-03 7.045317e-03
## , 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## : 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## . 2.616568e-04 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## a 1.724121e-02 1.538869e-03 4.297560e-02 8.233438e-04 1.260500e-02
## b 0.000000e+00 2.887336e-05 2.947970e-02 5.052838e-03 0.000000e+00
## c 0.000000e+00 1.886173e-01 4.183189e-02 0.000000e+00 4.262823e-02
## d 3.872725e-03 4.451408e-04 6.952210e-02 2.101065e-03 6.142944e-04
## e 7.085300e-03 2.346430e-03 1.139695e-02 2.793370e-04 9.004509e-04
## f 0.000000e+00 5.640370e-05 8.783936e-02 0.000000e+00 1.316086e-04
## g 9.001537e-03 1.204790e-01 5.055021e-02 0.000000e+00 0.000000e+00
## h 6.024387e-06 3.614632e-05 1.565618e-01 0.000000e+00 5.723167e-04
## i 2.481923e-02 2.889990e-05 2.658790e-03 0.000000e+00 7.554433e-03
## j 0.000000e+00 0.000000e+00 4.280156e-03 0.000000e+00 0.000000e+00
## k 6.000000e-04 4.085000e-02 1.700000e-01 0.000000e+00 0.000000e+00
## l 9.034373e-04 4.202034e-05 9.991386e-02 0.000000e+00 1.088327e-02
## m 1.649920e-05 0.000000e+00 8.355195e-02 0.000000e+00 9.899520e-05
## n 1.396425e-01 7.990746e-04 3.461725e-02 6.705521e-04 8.337198e-03
## o 4.050367e-03 1.888772e-03 1.176810e-02 6.348374e-04 1.789612e-02
## p 1.987545e-04 9.164790e-03 9.279625e-02 0.000000e+00 3.975089e-04
## q 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## r 9.638355e-03 1.456800e-03 9.215123e-02 1.380853e-05 6.365733e-03
## s 2.800889e-04 6.901008e-02 5.951252e-02 1.909697e-05 1.371162e-02
## t 4.073264e-05 3.338040e-01 6.337999e-02 0.000000e+00 4.525849e-05
## u 5.058606e-02 7.248612e-04 2.698951e-02 1.542258e-05 3.871067e-03
## v 1.136235e-04 0.000000e+00 1.841457e-01 0.000000e+00 3.029959e-04
## w 0.000000e+00 2.044087e-01 1.717719e-01 0.000000e+00 1.079895e-03
## x 0.000000e+00 1.393811e-02 1.211906e-01 0.000000e+00 0.000000e+00
## y 2.081791e-04 2.775722e-04 2.579108e-02 0.000000e+00 8.789785e-04
## z 0.000000e+00 6.506126e-02 1.035065e-01 0.000000e+00 1.689903e-03
##
##          l          m          n          o          p
## 2.350267e-02 3.458200e-02 2.599479e-02 6.197175e-02 3.188613e-02
## , 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## : 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## . 5.233136e-05 0.000000e+00 1.569941e-04 7.849704e-04 5.233136e-05
## a 6.974898e-02 2.267626e-02 2.240524e-01 2.058359e-04 2.273017e-02
## b 1.049547e-01 1.992262e-03 2.598603e-04 1.183808e-01 0.000000e+00
## c 3.256842e-02 3.250341e-05 0.000000e+00 2.085744e-01 0.000000e+00
## d 1.102169e-02 2.795484e-03 2.581817e-03 4.698907e-02 3.561127e-05
## e 3.406596e-02 2.347745e-02 8.730430e-02 5.320547e-03 1.126221e-02
## f 2.273069e-02 3.760247e-05 9.400617e-05 1.521208e-01 0.000000e+00
## g 2.993770e-02 3.641071e-04 1.873129e-02 6.290962e-02 0.000000e+00

```

```

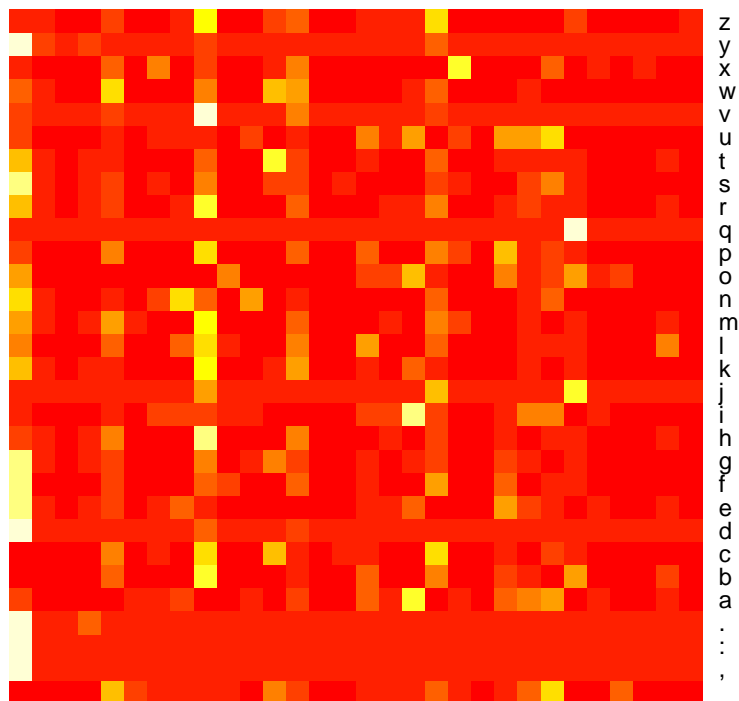
## h 9.036580e-04 2.271194e-03 6.988289e-04 8.267868e-02 2.409755e-05
## i 4.784667e-02 5.685766e-02 2.799071e-01 4.298571e-02 4.820503e-03
## j 0.000000e+00 0.000000e+00 0.000000e+00 2.785992e-01 0.000000e+00
## k 1.695000e-02 1.450000e-03 1.035500e-01 2.990000e-02 5.000000e-05
## l 1.387407e-01 4.117993e-03 1.281620e-03 8.770695e-02 3.991932e-03
## m 2.672870e-03 2.280189e-02 3.745318e-03 1.108746e-01 5.659226e-02
## n 1.175701e-02 7.711350e-04 1.089088e-02 7.085501e-02 5.308538e-04
## o 4.070304e-02 5.806401e-02 1.454722e-01 3.775446e-02 1.601784e-02
## p 9.321585e-02 5.962634e-04 1.766706e-04 1.051853e-01 5.797005e-02
## q 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## r 8.692470e-03 2.020878e-02 1.767492e-02 1.019760e-01 3.755920e-03
## s 7.645153e-03 1.073886e-02 2.794523e-03 5.729727e-02 2.061836e-02
## t 1.356850e-02 1.240083e-03 8.056012e-04 1.119967e-01 1.900857e-04
## u 9.691548e-02 2.108267e-02 1.354102e-01 2.282542e-03 4.282850e-02
## v 8.143014e-03 0.000000e+00 2.014923e-02 5.673598e-02 0.000000e+00
## w 3.411097e-03 3.428238e-05 3.808773e-02 7.878092e-02 1.714119e-05
## x 4.724781e-04 0.000000e+00 0.000000e+00 1.181195e-03 3.201039e-01
## y 4.672465e-03 3.862879e-03 1.087158e-03 1.383003e-01 1.295337e-03
## z 1.816646e-02 2.661597e-02 7.182087e-03 2.674271e-01 0.000000e+00
##
##          q          r          s          t          u
## 2.478400e-03 2.651398e-02 7.219294e-02 1.570149e-01 9.929273e-03
## , 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## : 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## . 0.000000e+00 0.000000e+00 2.616568e-04 5.233136e-05 5.233136e-05
## a 9.801711e-06 8.799486e-02 1.022613e-01 1.380130e-01 1.233545e-02
## b 0.000000e+00 5.711151e-02 1.374372e-02 7.420454e-03 1.524225e-01
## c 1.901450e-03 3.365728e-02 1.560164e-03 6.716830e-02 2.748164e-02
## d 4.006268e-04 3.184538e-02 1.981767e-02 4.184324e-04 9.819807e-03
## e 8.380109e-04 1.451961e-01 6.450712e-02 2.530464e-02 1.031904e-03
## f 0.000000e+00 1.064150e-01 2.801384e-03 4.042265e-02 3.500790e-02
## g 0.000000e+00 5.546565e-02 1.743669e-02 3.438790e-03 2.710575e-02
## h 0.000000e+00 7.723264e-03 9.398043e-04 2.527230e-02 9.639019e-03
## i 1.560594e-04 3.424638e-02 1.265295e-01 1.240210e-01 7.513973e-04
## j 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 4.603113e-01
## k 0.000000e+00 4.750000e-03 3.735000e-02 2.500000e-04 3.385000e-02
## l 1.050508e-05 4.863854e-03 1.633541e-02 1.723884e-02 1.592571e-02
## m 0.000000e+00 1.336435e-03 3.164547e-02 9.734528e-04 2.758666e-02
## n 1.352280e-03 4.135072e-04 3.449991e-02 8.937901e-02 5.353241e-03
## o 1.311647e-04 1.038877e-01 3.541973e-02 5.254984e-02 1.235519e-01
## p 0.000000e+00 1.734685e-01 2.369595e-02 4.785566e-02 2.577183e-02
## q 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00 9.995708e-01
## r 8.975545e-05 3.401041e-02 5.574504e-02 3.175962e-02 2.098897e-02
## s 1.075796e-03 1.591414e-04 6.165138e-02 1.130795e-01 2.516980e-02
## t 0.000000e+00 2.558010e-02 1.765986e-02 1.985038e-02 1.638810e-02
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## w 0.000000e+00 9.050549e-03 1.247879e-02 1.199883e-04 4.970946e-04
## x 9.449563e-04 0.000000e+00 0.000000e+00 1.112686e-01 4.016064e-03
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## z 0.000000e+00 0.000000e+00 8.449514e-04 0.000000e+00 5.534432e-02
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## b 1.328175e-03 5.774672e-04 0.000000e+00 7.362707e-02 0.000000e+00
## c 0.000000e+00 1.137619e-04 0.000000e+00 5.038029e-03 8.125853e-05
## d 2.848901e-03 3.917239e-04 0.000000e+00 1.050532e-02 1.780563e-05
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## g 0.000000e+00 3.641071e-04 0.000000e+00 1.921677e-03 4.045635e-05
## h 2.409755e-05 2.168779e-04 0.000000e+00 5.379777e-03 0.000000e+00
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## k 6.000000e-04 4.100000e-03 0.000000e+00 5.000000e-03 0.000000e+00
## l 4.443651e-03 4.905874e-03 0.000000e+00 1.051349e-01 2.626271e-04
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## u 1.233806e-03 3.084516e-05 6.631709e-04 3.238742e-04 9.454041e-03
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