Identification of Fractional-Order System Dynamics for A Rotary Inverted Pendulum

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Abstract—This report presents a novel approach to modeling the Rotary Inverted Pendulum (RIP) by incorporating fractional-order calculus into its mathematical framework. The RIP, a well-known fourth-order nonlinear dynamic system with two degrees of freedom, is traditionally described using integer-order differential equations. In this study, the conventional model is extended by replacing standard differentiators and integrators with their fractional-order equivalents, capturing system dynamics with greater flexibility. The parameters of the fractional-order model are estimated through a robust linear regression technique to ensure accuracy. A comparative evaluation reveals that the proposed fractional-order representation significantly enhances the alignment between simulated results and experimental data, providing a more precise depiction of the system's behavior. This research highlights the potential of fractional-order modeling in improving the accuracy of nonlinear dynamic systems.

Index Terms—Fractional calculus, rotary inverted pendulum (RIP), nonlinear dynamic systems, integer-order differential equations, parameter estimation coherence

1 Background and Motivation

The Rotary Inverted Pendulum (RIP) is a widely studied system in control theory, serving as a fundamental benchmark for analyzing underactuated and non-minimum phase systems. Due to its relevance, it has been extensively utilized in various engineering fields, including robotics, aerospace, and biomedical applications [1], [2], [3], [4]. The RIP system provides a practical test platform for evaluating advanced nonlinear control methodologies [5].

For centuries, integer-order calculus, independently introduced by Isaac Newton and Gottfried Leibniz in the 17th century, has been the foundation of dynamic system modeling. This well-established framework offers precise mathematical formulations and physical interpretations [6], [7], [8]. However, in recent years, fractional-order calculus has gained prominence as a more flexible alternative, particularly for capturing transient phenomena, complex switching behaviors, and intricate system dynamics that conventional integer-order models often fail to represent effectively.

The benefits of fractional-order modeling have been demonstrated across various applications. For example, fractional-order models have been shown to more accurately characterize ultra-capacitor dynamics, aligning experimental and theoretical results more effectively than integer-order approaches [9], [10], [11]. Similarly, in biomedical engineering, these models have been successfully applied to represent the dynamics of biological tissues with improved precision and compactness [12], [13], [14], [15]. Additional studies, such as [4], suggest that hydrological systems inherently exhibit fractional-order characteristics, reinforcing the natural applicability of fractional calculus in such domains. Moreover, fractional-order formulations have been explored in mechanical and electrical systems, including models for springs, dampers, capacitors, resistors, and inductors [12], [16], [17], [18]. By bridging the gap between integer-order behaviors, fractional calculus provides a more comprehensive framework for describing transitional states in dynamic systems [19], [20], [21], [22]. Despite its advantages, the physical meaning of fractional differentiation and integration remains a subject of debate, complicating efforts to establish universally accepted mathematical foundations for real-world phenomena [23], [24], [25], [26], [27].

This study introduces a fractional-order model for the RIP system, developed without imposing restrictive constraints or assumptions. The transition from the traditional integer-order formulation to a fractional-order representation is systematically carried out to maintain a direct and natural extension of the system's dynamics. Additionally, a data-driven approach is employed to identify system parameters by fitting experimental data to the fractional-order model.

The remainder of this report is structured as follows: Section II formulates the conventional integer-order equations governing the RIP system and extends them to their fractional-order counterparts. Section III presents the parameter identification process using regression-based estimation techniques. The dynamic equations are discretized, and the identification algorithms are detailed. In Section IV, a comparative analysis between the identified fractional-order model and experimental data is conducted to evaluate accuracy and performance. Finally, Section V summarizes the key findings and discusses the broader implications of fractional-order modeling for nonlinear dynamic systems.

By leveraging fractional-order techniques, this research enhances the modeling accuracy of the RIP system, demonstrating their potential for capturing complex system behaviors and advancing nonlinear system analysis methodologies.

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2 MODELLING OF THE RIP SYSTEM AND ITS FRACTIONALIZATION

A Rotary Inverted Pendulum (RIP) represents a complex electromechanical system with significant nonlinear dynamics. Fig. 1(a) illustrates the physical structure of a typical RIP system, while Fig. 1(b) depicts the coordinate system utilized for its analysis. The RIP system features two degrees of freedom: the angular displacement of the horizontal rotary arm, denoted as θ, and the angular displacement of the pendulum arm, denoted as α. The dynamic behavior of the system is governed by the nonlinear equations of motion presented in Equation (1).

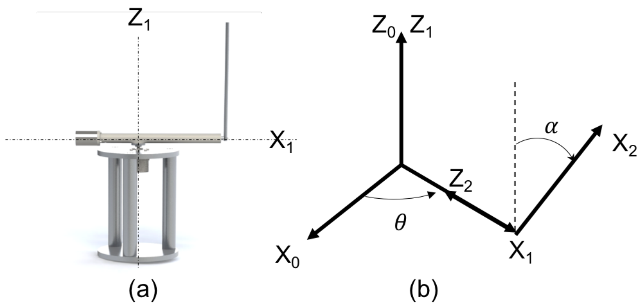

Figure 1. (a) The physical model of the RIP system. (b) The coordinate system of the RIP system.

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{\alpha} \\ \ddot{\alpha} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \frac{1}{\det(M)(C_1\tau_m + a - b)} \\ \dot{\alpha} \\ \frac{1}{\det(M)}\left(-C_3\cos\alpha\tau_m + c - d\right) \end{bmatrix} \tag{1}$$

where affiliated coefficients are summarized as follows,

$$C_1 = I_{2zz} + m_2 l_{2c}^2 \tag{2}$$

$$C_2 = I_{2yy} + m_2 l_{2c}^2 \tag{3}$$

$$C_3 = m_2 l_1 l_{2c} - l_{2xz} \tag{4}$$

$$C_4 = m_2 g l_{2c} \tag{5}$$

$$C_5 = J_r + m_1 l_{1c}^2 + I_{2xx} + m_2 l_1^2 \tag{6}$$

$$C_6 = C_2 - I_{2xx} \tag{7}$$

$$M = \begin{bmatrix} C_2\sin^2\alpha + C_5 + I_{2xx}\cos^2\alpha & C_3\cos\alpha \\ C_3\cos\alpha & C_1 \end{bmatrix} \tag{8}$$

$$a = C_1\left((-b_\theta - C_6\sin 2\alpha\dot{\alpha})\dot{\theta} + C_3\sin\alpha\dot{\alpha}^2 + \tau_{C\theta}\right) \tag{9}$$

$$b = C_3\cos\alpha\left(\frac{C_6}{2}\sin 2\alpha\dot{\theta}^2 - b_\alpha\dot{\alpha} + C_4\sin\alpha + \tau_{C\alpha}\right) \tag{10}$$

$$c = \left(C_5 + I_{2xx}\cos^2\alpha + C_2\sin^2\alpha\right)\cdot\Big(C_4\sin\alpha + \frac{C_6}{2}\sin 2\alpha\dot{\theta}^2 - b_\alpha\dot{\alpha} + \tau_{C\alpha}\Big) \tag{11}$$

$$d = C_3\cos\alpha\left((-b_\theta - C_6\sin 2\alpha\dot{\alpha})\dot{\theta} + C_3\sin\alpha\dot{\alpha}^2 + \tau_{C\theta}\right) \tag{12}$$

he system dynamics presented in Equation (1) are expressed using integer-order calculus, which is intrinsically tied to the well-established relationship between physical principles and their mathematical representations. The fractionalization of the RIP system model does not imply the introduction of new physical principles rooted in fractional calculus. Instead, it involves substituting the integer-order differentiators and integrators in the original dynamics with their fractional-order counterparts [28], [29], [30].

In this context, the integer-order derivatives, such as $\dot{\theta}$ and $\dot{\alpha}$, are replaced by the fractional derivatives $D^\lambda\theta$ and $D^\mu\alpha$, where λ and μ represent the fractional orders of the operators D^λ and D^μ, respectively. These fractional orders will be identified in Section III of this report. The resulting fractional-order representation of the RIP dynamics is formulated in Equation (13).

$$\begin{bmatrix} D^\lambda\theta \\ D^{2\lambda}\theta \\ D^\mu\alpha \\ D^{2\mu}\alpha \end{bmatrix} = \begin{bmatrix} D^\lambda\theta \\ \frac{1}{\det(M)(C_1\tau_m + a - b)} \\ D^\mu\alpha \\ \frac{1}{\det(M)}\left(-C_3\cos\alpha\tau_m + c - d\right) \end{bmatrix} \tag{13}$$

In the field of pure mathematics, numerous definitions of fractional-order differentiation exist, each offering unique properties and applications. Among the most notable are the Grunwald-Letnikov fractional derivatives, Riemann-Liouville derivatives, Caputo fractional derivatives, and Miller-Ross fractional derivatives [31], [32]. For this project, we adopt the Caputo fractional derivative, denoted as $D^\lambda := {}_a^C D_t^\lambda$.

The formal definition of the Caputo fractional derivative is provided in Equation (14), where λ is the fractional order such that $m < \lambda < m+1$, and m is a non-negative integer. This definition is particularly well-suited for systems with initial conditions expressed in terms of integer-order derivatives, making it a practical choice for the fractional modeling of dynamic systems.

$${}_a^C D_t^\lambda f(t) = \frac{1}{\Gamma(m+1-\lambda)}\int_a^t \frac{f^{(m+1)}(\tau)}{(t-\tau)^{\lambda-m}}d\tau \tag{14}$$

For the fractional-order system described in Equation (13), it is essential to identify the fractional orders that characterize the system dynamics. Specifically, this involves determining the fractional-order parameters λ and μ, which replace their integer-order counterparts in the system model presented in Equation (1). These parameters encapsulate the degree of differentiation and integration within the fractional-order framework, offering a more generalized representation of the system's behavior.

3 Discretization of Model Dynamics and Formulation of Identification Algorithms

The system dynamics in Equation (13) is an extended fractional-order state space equation, which has an equivalent form expressed in Equation (15).

$$\begin{bmatrix} D^{\lambda}x_1 \\ D^{\lambda}x_2 \\ D^{\mu}x_3 \\ D^{\mu}x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ f_1(t, x_1, x_2, x_3, x_4) \\ x_4 \\ f_2(t, x_1, x_2, x_3, x_4) \end{bmatrix} \tag{15}$$

Here, the variables are defined as $x_1 = \theta$, $x_2 = D^{\lambda}\theta$, $x_3 = \alpha$, and $x_4 = D^{\mu}\alpha$. The terms $f_1(t, x_1, x_2, x_3, x_4)$ and $f_2(t, x_1, x_2, x_3, x_4)$ correspond to the second and fourth terms on the right-hand side of the vector representation in Equation (13), respectively.

In this study, the Caputo definition of fractional-order derivatives is employed to model the system dynamics. However, under the assumption of zero initial conditions, i.e., $\theta_0 = \dot{\theta}_0 = \cdots = \theta_0^{(m+1)} = 0$ and $\alpha_0 = \dot{\alpha}_0 = \cdots = \alpha_0^{(n+1)} = 0$, where $m < \lambda < m + 1$ and $n < \mu < n + 1$, the Caputo definition becomes equivalent to the Grunwald-Letnikov definition. This equivalence allows the use of the Grunwald-Letnikov definition to discretize the reduced form of Equation (15). The mathematical formulation of the Grunwald-Letnikov fractional derivative is provided in Equation (16).

$${}_{a}^{GL}D_t^{\lambda} f(t) = \lim_{h\to 0} \frac{1}{h^{\lambda}} \sum_{k=0}^{[(t-a)/h]} (-1)^k \binom{\lambda}{k} f(t - kh) \tag{16}$$

Under zero initial conditions, both Caputo's fractional derivatives and Grunwald-Letnikov fractional derivatives can be approximated by a partial sum of the infinite series described in Equation (16). The accuracy of this approximation improves significantly as the chosen time step h becomes sufficiently small. The discrete approximation of the fractional operators is expressed in Equation (17), where a represents the initial time and T denotes the final time of interest. This approximation provides a practical approach to numerically evaluating fractional derivatives over a finite interval.

$${}_{a}^{C}D_t^{\lambda} f(t) \approx \frac{1}{h^{\lambda}} \sum_{k=0}^{[(T-a)/h]} (-1)^k \binom{\lambda}{k} f(t - kh) \tag{17}$$

The coefficients within the summation of Equation (17) are provided below and can be computed iteratively for efficient evaluation.

$$w_k^{(\lambda)} = (-1)^k \binom{\lambda}{k} \tag{18}$$

$$w_0^{(\lambda)} = 1, \quad w_{k+1}^{(\lambda)} = \left(1 - \frac{\lambda + 1}{k + 1}\right) w_k^{(\lambda)}, \quad k = 0, 1, \ldots \tag{19}$$

Substituting Equation (17) into Equation (15) gives the discrete form of the system dynamics. For example, $D^{\lambda}x_2 = f_1(t, x_1, x_2, x_3, x_4)$ is discretized as follows:

$$\frac{1}{h^{\lambda}} \sum_{k=0}^{N} w_k^{(\lambda)} x_2^{l-k} = f_1(t_k, x_1^k, x_2^k, x_3^k, x_4^k) \tag{20}$$

where $N = \text{ceiling}((t_l - a)/h) + 1$, $t_k = a + kh$, $x_1^k = x_1(t_k)$, $x_2^k = x_2(t_k)$, $x_3^k = x_3(t_k)$ and $x_4^k = x_4(t_k)$.

Equation (20) is rearranged to be computed iteratively as shown in Equation (21) and (22), i.e., the nonlinear fractional differential equation can be solved iteratively for $l = 1, 2, \ldots, N$. Similarly, the discretization of the second portion of Equation (15) is given in Equation (23) and (24).

$$x_1^l = h^{\lambda} x_2^k - \sum_{k=1}^{N} w_k^{(\lambda)} x_1^{l-k} \tag{21}$$

$$x_2^l = h^{\lambda} f_1(t_k, x_1^k, x_2^k, x_3^k, x_4^k) - \sum_{k=1}^{N} w_k^{(\lambda)} x_2^{l-k} \tag{22}$$

$$x_3^l = h^{\mu} x_4^k - \sum_{k=1}^{N} w_k^{(\mu)} x_3^{l-k} \tag{23}$$

$$x_4^l = h^{\mu} f_2(t_k, x_1^k, x_2^k, x_3^k, x_4^k) - \sum_{k=1}^{N} w_k^{(\mu)} x_4^{l-k} \tag{24}$$

The system response data is recorded by linear optical encoders. To approximate the values of two fractional orders, an optimization problem is formulated as follows:

$$\hat{\lambda}, \hat{\mu} = \arg\min_{\lambda,\mu} \mathcal{L}(\lambda, \mu) \tag{25}$$

The cost function for this optimization problem is defined as the integral of the time-weighted absolute error of the α angle. Specifically, the cost function is expressed in Equation (26), where $\alpha(t)$ represents the iteratively simulated values obtained from Equations (21) through (24), and $\tilde{\alpha}(t)$ denotes the measured real-time α angle captured by the encoders. This formulation ensures the optimization process minimizes the deviation between the simulated and measured responses over time.

$$\mathcal{L}(\lambda, \mu) = \int_0^T t|\alpha(t) - \tilde{\alpha}(t)|dt \tag{26}$$

To estimate the optimal values of λ and μ, various global optimization algorithms can be utilized. In this project, the genetic algorithm (GA) is employed. GA is a stochastic search method inspired by biological evolutionary theory, applying principles such as selection, crossover, and mutation to numerical optimization problems. The parameters of the genetic algorithm are determined empirically, and their specific values are provided in the table below. This approach ensures a robust and efficient search for the optimal fractional-order parameters.

Population size = 20	Crossover rate = 0.5
Mutation rate = 0.01	Generation number = 25

4 Identification Results and Its Evaluation

The optimal parameters are determined by solving the optimization problem, yielding $\hat{\lambda} = 0.8623$ and $\hat{\mu} = 0.9146$. The experimental data was obtained by freely releasing the pendulum arm from an initial angle of $\alpha_0 = 10°$ and recording the pendulum angles, $\alpha(t)$, at a sampling frequency of 1000 Hz. The recording spanned a duration of 20 seconds to ensure the system had fully stabilized from the initial oscillations.

The simulation results generated using the identified parameters are shown in Figure (2), illustrating the model's ability to accurately capture the system's dynamic behavior.

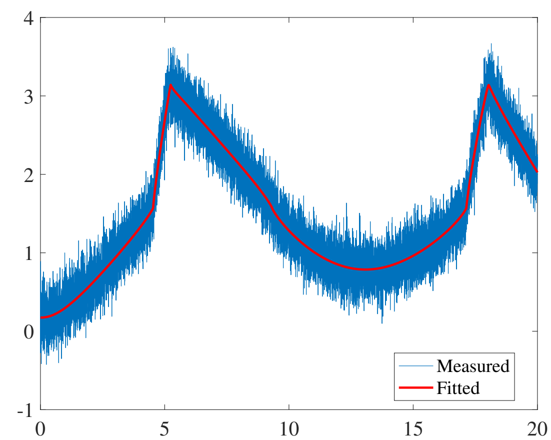

Figure 2. Measured system responses and the fitting value from the identified model.

The measurement process introduces noise into the system, which degrades the performance of the model, as depicted in Fig. 2. The residuals are defined as the differences between the measured outputs and the model's fitted output values. For the model to be valid, it is assumed that the measurement errors follow a normal distribution with constant variance. The residual plot, shown in Fig. 3, provides an approximation for verifying these assumptions of normality and constant variance. By visually inspecting the residuals, one can assess the adequacy of the model and the validity of these underlying assumptions.

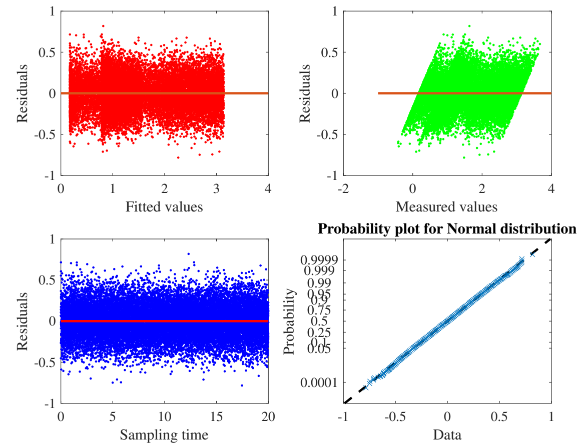

Figure 3. Residual plots and the normality probability plot for the measured residuals.

As shown in Fig. 3, the residuals from the measured data exhibit a near-perfect alignment with a normal distribution, validating the assumptions of the regression model with the identified optimal parameters.

From the previous analysis, the optimal fractional orders for the proposed rotary inverted pendulum (RIP) dynamics are determined to be $\hat{\lambda} = 0.8623$ and $\hat{\mu} = 0.9146$. The measured errors are assumed to be normally distributed, a hypothesis confirmed through residual and probability plots. However, several critical points require further discussion. The model parameters may vary if an alternative cost function is selected for the system identification process. The choice of the cost function in this study is justified by the characteristics of the measured noise data, as evidenced by the residual plots in Fig. 3. Specifically, the noise distribution is essentially normal, reinforcing the validity of our approach.

Furthermore, the model is validated through simulations incorporating different levels of noise. The model orders, λ and μ, are chosen from the entire set of real numbers. The optimal values for these orders are obtained by minimizing the proposed cost function. The structure of the model is generalized by extending the derivative and integral orders in the original rigid-body dynamic equations, which enables the more accurate representation of the system's dynamics using fractional calculus.

5 Conclusion

In this study, the fractional-order dynamic model of the rotary inverted pendulum (RIP) system is identified using experimental data. The measurements of the rotary-arm angle θ and pendulum arm angle α are assumed to be affected by white noise with constant variance. The fractional orders of the RIP dynamics, λ and μ, are estimated by solving an optimization problem, where a prescribed cost function is minimized using a genetic algorithm. The identified optimal fractional orders are then incorporated into the system model. Subsequently, the simulated output of the fitted model is compared with the experimentally measured output.

Residual and probability plots are generated following the solution of the fitted model, and the results confirm that the assumption of normally distributed measurement errors with constant variance holds true. This validation process reinforces the robustness and accuracy of the identified fractional-order model in capturing the dynamics of the RIP system.

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