Ordinary differential equations

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1 Introduction

ODE - Oridnaty differential equation. The purpose of this text is to help us solve ODEs.

An ODE of order n is defined as such

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

In most cases we would rather write the equation as such:

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

1.1 Homogeneity and Linearity of ODEs

A linear ODE is of the following is an ODE of the following format:

$$F(x, y, y', \ldots, y^{(n)}) = \sum_{i=0}^{i=n} a_i(x) y^{(i)} = b(x)$$

$\forall i(a_i(x)$ is a differentiable function)
A linear ODE is called homogeneous if $b(x) = 0$

2 Linear ODEs of first order and IVPs

Recall the form of this type of ODE is:

$$y' = p(x)y + q(x)$$

Solving $y' = q(x)$ could give us infinitly many solutions because of the integration constant. That's why we usually have these kind of problem coupled with an initial condition - $y(x_0) = y_0$. Given an ODE and an inital condition we get an IVP ir an **I**nitial **V**alue **P**roblem.

2.1 Existence and Uniqueness Theorem

Given an IVP

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} \tag{1}$$

Such that $f(x, y)$ and $f'_y(x, y)$ are continuous functions over $D \in \mathbb{R}^2$ and $(x_0, y_0) \in D$ then there exists an $\varepsilon > 0$ such that there exists only one unique solution to the problem in $[x_0 - \varepsilon, x_0 + \varepsilon]$

Accordingly there couldn't be two intersecting solutions to such an equation.

2.2 General Solution to First Order Homogeneous Linear ODEs

$$y' + p(x)y = 0$$

$$\frac{y'}{y} = -p(x) \ , \ y \neq 0^1$$

$$\int \frac{y'(x)}{y(x)} dx = \int -p(x) dx$$

$$\cdots$$

$$y(x) = k * e^{-\int p(x)dx}, \forall k \in \mathbb{R}$$

2.3 General Solution to First Order Non-Homogeneous Linear ODEs

After normalization we get:

$$y' + p(x)y = q(x)$$

Let U be a any function. We'll call U the **integration factor**.

$$Uy' + Up(x)y = Uq(x)$$

Now let's force $U' = Uq(x)$. We already know how to solve these kind of problems. Let's pronouce one such solution u. Now:

$$uy' + u'y = Uq(x)$$

$$(uy)' = z(x)$$

$$uy = \int z(x) = a(x) + C$$

$$y = \frac{a(x)}{u(x)} + \frac{C}{u(x)}$$

*Notice that the first argument is one solution to this IVP and the second is the general solution to the according Homogeneous ODE.

3 More ODEs of First Order

3.1 Separable First Order ODEs

A seperable first order ODE is of the next form:

$$y' = h(x)g(y)$$

$$\frac{y'}{g(y)} = h(x)$$

$$\frac{1}{g(y)}\frac{dy}{dx} = h(x)$$

$$\int \frac{dy}{g(y)} = \int h(x)dx$$

$$G(y(x)) = H(x) + C$$

That's an implicit solution to the ODE. Sometimes we can write it explicitly and sometimes we can't. If there exists a scalar y_0 such that $g(y_0) = 0$ then $y(x) = y_0$ is a solution - called the **singular solution**. Think why.

3.2 Homogeneous ODEs

Different then linear homogeneous ODEs. These are equations of the form:

$$y' = F(\frac{y}{x})$$

It's called that since a homogeneous function of order m is a function such that $f(tx, ty) = t^m f(x, y)$ and $F(\frac{y}{x})$ is a homogeneous function of order 0.
Define $v(x) = \frac{y(x)}{x} \Rightarrow y'(x) = v'(x)x + v(x)$ plugging that in we get

$$v'x + v = F(\frac{y}{x})$$

$$\frac{v'}{F(v) - v} = \frac{1}{x}$$

$$\int \frac{dv}{F(v) - v} = \int \frac{1}{x}dx$$

$$G(v(x)) = ln|x| + C$$

This solution is implicit, we must plug $v(x) = \frac{y(x)}{x}$ back in. Notice that if there exists a v_0 such that $F(v_0) = v_0$ then $v(x) = v_0$ is a singular solution to the separable equation. Thus $y(x) = v_0 x$ is a singular solution to the ODE.

3.3 Switching x and y

Consider the following ODE:

$$y' = \frac{y}{x + y^3}$$

That's not an ODE we have encountered so far. In analysis we talked about inverse functions and we saw that $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ and so we can instead solve

$$x' = \frac{x + y^3}{y}$$

as we would for a linear nnon-homogeneous ODE of first order.

4 Exact ODEs

Looking at equations of this form:

$$P(x,y) + Q(x,y)y' = 0$$

Or, using Leibniz notation

$$P(x,y)dx + Q(x,y) = 0$$

We'll call the equation **exact** if there exists F such that

$$\begin{cases} F'_x = P(x,y) \\ F'_y = Q(x,y) \end{cases}$$

And its solution is given implicitly with the equation $F(x,y) = c$. If we derive both sides we get

$$F'_x(x)' + F'_y(y(x))' = 0$$
$$P(x) + Q(x)y' = 0$$

Theorem - Let $Q(x,y), P(x,y)$ be partially continuously differentiable functions on a simple connected domain D.

$$\exists F : F'_x = P \wedge F'_y = Q \iff P'_y = Q'_x$$

4.1 Almost exact ODEs

If $P'_y \neq P'_x$ we can mulitply everything by an integration factor $u(x,y)$

$$u(x,y)P(x,y)dx + u(x,y)Q(x,y) = 0$$

And we want

$$u'_y P + uP'_y = u'_x + uQ'_x$$
$$u'_y P + uP'_y - u'_x Q - uQ'_x = 0$$

Which is a Partially Differentiable Equation. We can solve these under certain circumstances. If $u = u(x) \Rightarrow u(P'_y - Q'_x) = u'_x Q \Rightarrow$

$$\frac{u'_x}{u} = \frac{P'_y - Q'_x}{Q}$$

So the left fraction is dependent on x alone. A similar fraction can be generated for y.

5 Geometrical Aspects of ODEs

Let $f(x, y)$ be the function that gives the slope of a linear function that intersects with $(0, 0), (x, y)$. Now considering the ODE $y' = f(x, y)$ and recalling that a deravative of a function gives us the slope near the point of derivation we can define a graph such that each point has a vector pointing at the direction of the slope given by y'. That graph is known as This ODE's directional field.

Isoclines are defined as the points that solve $f(x, y) = C$
Nullclines are defined as the points that solve $f(x, y) = 0$

Definition - $y = \alpha(x)$ is called a
Theorem - Let