Computability Theory

1   Turing Machines

The undecidability problem: Does there exist a "definite method" that, when given any possible statement in mathematics, can decide whether that statement is true or false?

Alan Turing set out with the goal of showing that the answer to this problem is no. First, in order to prove that there is not algorithm capable of determining whether a mathematical statement is false, we need to formally define what an algorithm is, and once we have a sensible definition, it could be shown that no algorithm under this definition satisfies the undecidablity problem.

In order to mimic the ability to follow algorithms, we find useful considering the way we solve problems ourselves. We can notice, that when we are faced with a problem, in order to solve it we may need to write stuff down, we may need to read stuff, and sometimes we may need to perform certain operations with the data we gathered. These different operations can be interpreted as states of thinking, and the model we are about to define models exactly what we described so far.

Definition 1.1 (Turing machine). A Turing machine is a 7-tuple $M = (Q, \Gamma, b, \Sigma, \delta, q_0, F)$ such that:

- Q is a finite, nonempty set. The elements of Q are called states;

- $q_0 \in Q$ is called the initial state;

- $F \subseteq Q$ is a set of final states, or accepting states;

- Γ is a finite, nonempty set such that $Q \cap \Gamma = \emptyset$. The elements of Γ are called the tape alphabet symbols;

- $\Sigma \subsetneq \Gamma$ is called the set of input symbols;

- $b \in \Gamma \setminus \Sigma$ is called the blank symbol;

- $\delta \colon (Q \setminus F) \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$ is called the transition function.

In order to define a computation on a Turing machine we first have to define a configuration (a momentary state) of the machine.

Definition 1.2 (Configuration). A configuration (a momentary state) of a Turing machine M is a triple $C = (\alpha, q, i)$ such that:

- $q \in Q$ is a state of M called the current state of the computation;

- $\alpha \in \Gamma^*$ is a finite string of letters from Γ that is called the tape content;

- $i \in \mathbb{N}$ is a natural number that is called the head's position.

Additionally, the initial configuration of M on input x is the configuration $(x, q_0, 0)$.

Definition 1.3 (Finite configuration). A finite configuration is any configuration (α, q, i) such that $q \in F$.