Real Functions

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Winter 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction

1.1 Motivation

The Riemann integral we have known so far is fairly limited. For example it doesn't allow us to compute the Riemann integral of Dirichlet's function $f\colon [0,1] \to \mathbb{R}$ defined as

$$f(x) = \mathbb{1}_{\mathbb{Q}\cap[0,1]} = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & \text{otherwise} \end{cases}.$$

In his thesis Lebesgue introduced a new type of integral called a Lebesgue integral that allows us to compute integrals for functions like Dirichelt's function, and he continued to develop more concepts like measure, and almost everywhere.

1.2 Motivation for Lebesgue integral

Let $f(x) = \mathbb{1}_A$ be the function that we want $\int \mathbb{1}_A$ to be the volume of the set A.

First we would like we define what is a volume of a set. We would want to require a couple of things

(1) $\mu(A)$ is defined for all $A \subseteq \mathbb{R}^n$;

(2) $\mu([0,1]^n) = 1^n = 1$;

(3) μ to be invariant to congruations (isometries).

(4) If $\{A_i\}_{i=1}^{\infty}$ is a countable sequence of pairwise disjoint sets then

$$\mu\left(\bigcup_{i=1}^{\infty} = \sum_{i=1}^{\infty} \mu(A_i)\right).$$

Remark 1.1. Property (4) is called σ-additivity.

Theorem 1.1 (Hausdorff, 1914)**.** *There is no function that satisfies* $(1) - (4)$ *at the same time.*

We will prove this theorem later. For now we can only try to weaken the requirements. For example instead of σ-additivity we might require finite additivity.

Theorem 1.2. *There exists a function that satisfies the wanted requirements in dimenstions* 1 *and* 2 *but not in dimension* $n \geq 3$.

For example in $n = 3$ we have the Banach-Tarski paradox

Paradox 1.1 (Banach–Tarski, 1924)**.** For every $n \geq 1$ we can divide S^2 in $\mathbb{R}^n$ to a finite amount of parts such that when they are rotated and rearranged, can form a new sphere of any desired size.

Remark 1.2. The bigger sphere we would want to form, the greater is the minimal pieces we need to divide the unit sphere in order to form it.

This paradox is based on the use of the axiom of choice, but since we assume the axiom of choice in this course we still need to modify the requirements.

Instead of (4) we require subadditivity:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

This is possible but instead we would like to keep σ-additivity but instead give up on requirement (0). We will only define volume only for "nice" sets which include all of the sets we work with in analysis or geometry etc. these sets form a σ-algebra on $\mathbb{R}^n$.

2 Algebras and σ-algebras

2.1 Definitions

Definition 2.1 (Algebra)**.** An algebra on a nonempty set X is a collection $\mathcal{A}$ of subsets of X such that

(1) $\emptyset, X \in \mathcal{A}$;

(2) $\mathcal{A}$ is closed under complements;

(3) $\mathcal{A}$ is closed under finite unions and intersections.

Definition 2.2 (σ-algebra)**.** A σ-algebra on a nonemptyset X is an algebra $\mathcal{A}$ on X that is also closed under countable unions and intersections.

Remark 2.1. A σ-algebra is sometimes called a σ-field. That is why it is sometimes denoted $\mathcal{F}$.

Remark 2.2. From De-Morgan laws we know that

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$$

so it is only necessary to require closure under countable unions or countable intersections.

Remark 2.3. It is also possible to require $\mathcal{A}$ to be nonempty instead of (1). Then for $A \in \mathcal{A}$ we have

$$A \in \mathcal{A} \implies A^c \in \mathcal{A} \implies A \cup A^c = X \in \mathcal{A} \text{ and } A \cap A^c = \emptyset \in \mathcal{A}.$$

Example 2.1. $\mathcal{A} = \big\{\emptyset, X\big\}$ and $\mathcal{A} = 2^X$ are the smallest and biggest σ-algebras on X respectively.

Example 2.2. If X is not countable. Then

$$\mathcal{A} = \{E \subseteq X \colon E \text{ or } E^c \text{ are countable}\} \neq 2^X$$

is a σ-algebra.

Definition 2.3 (Cocountablility)**.** Let X be a set. Then A is called cocountable if A^c is countable.

Example 2.3 (Generated σ-algebra)**.** Let $F \subseteq 2^X$ be a family of subsets of X. The σ-algebra generated by F is defined as

$$\sigma(F) = \bigcap \Big\{\mathcal{A} \subseteq 2^X \colon \mathcal{A} \text{ is a } \sigma\text{-algebra and } F \subset \mathcal{A}\Big\}.$$

Remark 2.4. Notice that the intersection is indeed a σ-algerba.

Remark 2.5. Let $\mathcal{A}$ be a σ-algebra and $F \subseteq \mathcal{A}$. Then $\sigma(F) \subseteq A$.

Corollary 2.1. *Suppose $F_1 \subseteq \sigma(F_2)$ and $F_2 \subseteq \sigma(F_1)$. Then $\sigma(F_1) = \sigma(F_2)$.*

Definition 2.4 (Borel σ-algebra)**.** Let X be a topological space. Then we define the Borel σ-algebra as the σ-algebra generated by the open sets in X.

$$B(X) = \sigma\left(\big\{G \subset X \mid G \text{ is open}\big\}\right).$$

Remark 2.6. Recall that G denotes an open set, F a closed set, G_δ a countable intersection of open sets and F_σ a countable union of closed sets. Similarly $G_{\delta\sigma}$ is a countable union of G_δ sets etc.

Remark 2.7.
$$B(X) = \sigma\left(\{F \subset X \mid F \text{ is closed}\}\right).$$

Proposition 2.2. $B(\mathbb{R})$ *is generated by the collection of any type of interval.*

Proof. Consider the collection of open intervals in $\mathbb{R}$. Since any open set in $\mathbb{R}$ is a countable union of disjoin open intervals, it is clear that $B(\mathbb{R}) = \sigma\left((a,b)\right)$.

Consider the collection of closed intervals in $\mathbb{R}$. Since any closed interval is a countable intersection of open sets, we have that $[a,b] \subseteq \sigma\left((a,b)\right)$. Since any open interval is a countable union of closed sets, we have that $(a,b) \subseteq \sigma\left([a,b]\right)$.

The proof for other types of intervals (like $(a,b]$) is similar and thus omitted. □

2.2 The product σ-algebra

Definition 2.5 (Product σ-algebra)**.** Let $\{\mathcal{A}_i\}_i$ be a collection of σ-algebras on $\{X_i\}_i$. Then, the product σ-algebra $\otimes_{i\in I}\mathcal{A}_i$ is the σ-algebra generated by the cylindrical sets.

$$\mathcal{S} = \left\{\prod_{i\in I} U_\alpha \mid \exists j \in I \text{ s.t. } \ U_i = X_i \text{ for } i \in I \setminus \{j\} \text{ and } U_j \in \mathcal{A}_j\right\}.$$

which is the also the set that generates the product topology on the product space.

Remark 2.8. In the case of $|I| < \aleph_0$ this is equivalent to the σ-algebra generated by the open sets in $\otimes_{i\in I}\mathcal{A}_i$ with respect to the box topology.

Proposition 2.3. $B(\mathbb{R}^n) = \otimes_{i=1}^n B(\mathbb{R})$.

Proof. We need to prove that $\prod_{i=1}^n (a_i, b_i) \subseteq B(\mathbb{R}^n)$ which is clear, and also that $B(\mathbb{R}^n) \subseteq \prod_{i=1}^n (a_i, b_i)$ which is also managable because for an open set $A \subseteq \mathbb{R}^n$ we can see that it is the countable union of all the open boxes with rational sized edges contained in A around any rational point $q \in A$ (which exist because A is open). □

3 Measure

3.1 Definitions

Definition 3.1 (Measure)**.** Let $\mathcal{A}$ be a σ-algebra on X. A function $\mu\colon \mathcal{A} \to [0,\infty]$ is called a measure if

(1) $\mu(\emptyset) = 0$;

(2) Given a sequence $\{E_i\}_{i=1}^\infty \subseteq \mathcal{A}$ of disjoint sets we have
$$\mu\left(\biguplus_{i=1}^\infty E_i\right) = \sum_{i=1}^\infty \mu(E_i).$$

Remark 3.1. Property (2) is called σ-additivity. It obviouly implies finite additivity because we can choose $E_j = \emptyset$.

Definition 3.2 (Measurable space)**.** The space $(X, \mathcal{A})$ is called a measurable space. The elements of $\mathcal{A}$ are called measurable sets.

Definition 3.3 (Measure space)**.** The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

Definition 3.4 (Finite measure)**.** A finite measure is a measure μ on X such that $\mu(X) < \infty$.

Definition 3.5 (σ-finite measure)**.** A σ-finite measure is a measure μ on X such that $X = \cup_{i=1}^{\infty} E_i$ for $E_i \in \mathcal{A}$ such that $\mu(E_i) < \infty$.

Definition 3.6 (Borel measure)**.** A measure μ on a topological space X is called a Borel measure if $B(X) \subseteq \mathcal{A}$.

Example 3.1 (Delta measure)**.** Let $x_0 \in X$ and $\mathcal{A} = 2^X$. Then the delta measure is

$$\delta_{x_0}(E) = \begin{cases} 1, & x_0 \in E \\ 0, & x_0 \notin E \end{cases}.$$

Example 3.2 (Counting measure)**.** The counting measure is the measure on $\mathcal{A} = 2^X$ such that $\mu(E) = |E|$.

Example 3.3. Suppose $\aleph_0 < |X|$ and let $\mathcal{A}$ be the σ-algebra of the countable or cocountable subsets of X. Then the follwing is a measure on X,

$$\mu(E) = \begin{cases} 0, |E| \leq \aleph_0 \\ 1, |E^c| = \aleph_0 \end{cases}$$

Example 3.4. Suppose $|X| = \infty$ and $\mathcal{A} = 2^X$. Then the following is a measure on X,

$$\mu(E) = \begin{cases} 0, & |E| < \infty \\ 1, & |E| = \infty \end{cases}$$

Remark 3.2. The last example is a finite additive measure, but not σ-additive.

Proposition 3.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then*

(1) Let $E, F \in \mathcal{A}$ such that $E \subseteq F$. Then $\mu(E) \leq \mu(F)$.

(2) Let $\{E_i\}_{i=1}^{\infty} \subseteq A$. Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

(3) Let $\{E_i\}_{i=1}^{\infty} \subseteq A$ such that $E_1 \subseteq E_2 \subseteq \ldots$ then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} \mu(E_i)$$

(4) Let $\{E_i\}_{i=1}^{\infty} \subseteq A$ such that $E_1 \supseteq E_2 \supseteq \ldots$ then

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i \to \infty} \mu(E_i)$$

Proof. TO BE ADDED. □

Definition 3.7 (μ-negligible set)**.** A negligible set is a set $E \in \mathcal{A}$ such that $\mu(E) = 0$.

Definition 3.8 (Almost everywhere)**.** We say that a certain property is true almost everywhere if it is true for any $x \in X \setminus E$ such that E is μ-negligible.

Example 3.5. We have that $\sin x \neq 0$ almost everywhere because

$$|E| = \left|\left\{x \in \mathbb{R} \mid \sin x = 0\right\}\right| = \aleph_0$$

is μ-negligible for

$$\mu(E) = \begin{cases} 0, & |E| \leq \aleph_0 \\ 1, & |E^c| = \aleph_0 \end{cases}$$

Remark 3.3. A countable union of μ-negligible sets is μ-negligible from subadditivity.

Remark 3.4. A subset of a μ-negligible set is μ-negligible since measures are monotone.

3.2 Complete measure spaces

Definition 3.9 (Complete measure space)**.** A triple $(X, \mathcal{A}, \mu)$ is called a complete measure space if for any $F \in \mathcal{A}$, if $\mu(F) = 0$ then for all $E \subseteq F$ we have $E \in \mathcal{A}$.

Theorem 3.2 (Completion theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Define*

$$\begin{aligned} \mathcal{N} &= \left\{N \in \mathcal{A} \mid \mu(N) = 0\right\} \\ \overline{\mathcal{A}} &= \left\{E \cup F \mid E \in \mathcal{A} \text{ and } F \subseteq N \in \mathcal{N}\right\}. \end{aligned}$$

Then $\overline{A}$ is a σ-algebra, and there exists a unique extension $\bar{\mu}$ of μ such that $(X, \overline{\mathcal{A}}, \bar{\mu})$ is a complete measure space.

Proof. In hw □

Remark 3.5. $\overline{A}$ is the smallest σ-algebra containing $\mathcal{A}$ and all the subsets of μ-negligible sets.

Our goal is to construct a measure space $(\mathbb{R}^n, \mathcal{A}, \mu)$ such that

$$\mu\left(\prod_{i=1}^{n} [a_i, b_i]\right) = \prod_{i=1}^{n} |a_i - b_i|.$$

Recall that earlier we saw that

$$B(\mathbb{R}^n) = \bigotimes_{i=1}^{n} B(\mathbb{R}) = \sigma\left(\prod_{i=1}^{n} [a_i, b_i]\right)$$

which implies that $B(\mathbb{R}^n) \subseteq \mathcal{A}$. To define the measure for other sets we can use an outer measure

$$\mu^*(E) := \inf\left\{\sum_{i=1}^{\infty} \mu(R_i) \mid E \subseteq \bigcup_{i=1}^{n} R_i\right\}.$$

We now have two questions.

(1) Is $\mu^*(R) = \mu(R)$? Yes.

(2) Did we get a measure? No, we got an outer measure.

Remark 3.6. Defining μ^* only on "good" sets gives us a measure called Lebesgue measure. Good sets are sets for which the outer measure is equal to the inner measure.

Definition 3.10 (Outer measure)**.** Let $X \neq \emptyset$. Then $\mu^* \colon 2^X \to [0, \infty]$ is called an outer measure if

(1) $\mu^*(\emptyset) = 0$;

(2) μ^* is monotone (if $A \subset B$ then $\mu^*(A) < \mu^*(B)$);

(3) μ^* is subadditive $\left(\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu^*(E_i)\right)$.

Proposition 3.3. *Let $\mathcal{E} \subseteq 2^X$ be a family of subsets of X and $\varphi \colon \mathcal{E} \to [0, \infty]$ such that $\emptyset, X \in \mathcal{E}$ and $\varphi(\emptyset) = 0$. For all $A \subseteq X$ we define*

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(E_i) \mid E_i \in \mathcal{E} \text{ and } A \subseteq \bigcup_{i=1}^{\infty} E_i \right\}$$

then μ^* is an outer measure.

Remark 3.7. It is not promised that $\mu^*(E) = \varphi(E)$ for $E \in \mathcal{E}$.

Proof.

(1) We can cover $\emptyset$ by $E_i = \emptyset$.

(2) Let $A \subseteq B$. We can use the cover of B for A since $A \subseteq B \subseteq \bigcup E_i$ which implies $\mu^*(A) \leq \mu^*(B)$.

(3) By the definition of the infimum there exists a covering $A_i \subseteq \cup_{k=1}^{\infty} E_k^i$ such that

$$\mu\left(\bigcup_{k=1}^{\infty} E_k^i\right) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \cdot 2^{-i}$$

which implies that $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{k,i=1}^{\infty} E_k^i$ and so

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{k,i}^{\infty} \varphi(E_k^i) \leq \sum_{i=1}^{\infty} \mu(E_i) + \epsilon$$

but $\epsilon > 0$ is arbitrary which completes the proof.

□

Remark 3.8. Every outer measure can be constructed by φ in this way.

Definition 3.11 (μ^*-measurable set)**.** A set $A \subseteq X$ is called μ^*-measurable if for all $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Remark 3.9. It is clear that the $\leq$ direction is always satisfied from subadditivity, so it is only necessary to verify the direction $\geq$. In particular, it is only necessary to verify it for sets such that $\mu^*(E) < \infty$.

Let $A \subseteq E$ be such that $\mu^*(E) < \infty$. Then we need to verify that

$$\underbrace{\mu^*(A)}_{\text{outer measure}} = \underbrace{\mu^*(E) - \mu^*(E \cap A^c)}_{\text{inner measure}}.$$

3.3 Caratheodory's outer measure theorem

Theorem 3.4 (Caratheodory's outer measure theorem)**.** *Let μ^* be an outer measure on X. Then*

(1) $\mathcal{F}_{\mu^*} := \{A \subseteq X \mid A \text{ is } \mu^*\text{-measurable}\}$ *is a σ-algebra.*

(2) The triple $(X, \mathcal{F}_{\mu^*}, \mu^*|_{\mathcal{F}_{\mu^*}})$ *is a complete measure space.*

Remark 3.10. We saw in the recitations that in order to show that $\mathcal{F}$ is a σ-algebra, it suffices to prove that if $E_i \in \mathcal{F}$ are disjoint, then

$$\biguplus_{i=1}^{\infty} E_i \in \mathcal{F}.$$

Proof.

(1) Let us show that $\mathcal{F} = \mathcal{F}_{\mu^*}$ is an algebra. $\mathcal{F}$ is closed under the complement operation because by the symmetric definition, if A is μ^*-measurable so is A^c.

Let $A, B \in \mathcal{F}$ and $E \subseteq X$. Since A, B are μ^*-measurable we have that

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c).\end{aligned}$$

We also have that

$$(A \cup B) \cap E = \big((A \cap B) \cap E\big) \sqcup \big((A^c \cap B) \cap E\big) \sqcup \big((A \cap B^c) \cap E\big).$$

Thus, from subadditivity

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B))\mu^*(E \cap (A \cup B)^c)$$

which implies $A \cup B \in \mathcal{F}$. By induction we get that $\mathcal{F}$ is closed under finite unions which makes it an algebra. Moreover, let $A, B \in \mathcal{F}$ be disjoint sets. Then

$$\mu^*(\underbrace{A \sqcup B}_{E}) = \mu^*((A \sqcup B) \cap A) + \mu^*((A \sqcup B) \cap A^c) = \mu^*(A) + \mu^*(B)$$

so μ is a finite additive measure on $\mathcal{F}$.

Let $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of disjoint sets. Define

$$B_n = \biguplus_{i=1}^{n} A_i \in \mathcal{F} \quad \text{and} \quad B_\infty \biguplus_{i=1}^{\infty} A_i.$$

For all $E \subseteq X$ we have

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*((E \sqcup B_n) \cap A_n) + \mu^*((E \sqcup B_n) \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^{n} \mu^*(E \cap A_i).\end{aligned}$$

Using this and since μ^* is monotone we have

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{i=1}^{n} \mu^*(E \cap A_i) + \mu^*(E \cap B_\infty^c).$$

Taking $n \to \infty$ we have

$$\mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B_\infty^c) \geq \mu^* \left(E \cap \underbrace{\bigcup_{i=1}^{\infty} A_i}_{B_\infty} \right) + \mu^*(E \cap B_\infty^c) \geq \mu^*(E)$$

which implies that $B_\infty \in \mathcal{F}$ (σ-additive). From Remark 3.10 we have that $\mathcal{F}$ is a σ-algebra.

(2) Let $A \subseteq X$ such that $\mu^*(A) = 0$. Then we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \underbrace{\mu^*(A)}_{0} + \mu^*(E) = \mu^E$$

which implies that all the expressions must be equal so

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

which means $A \in \mathcal{F}_{\mu^*}$. Since $\mathcal{F}_{\mu^*}$ contains all μ^*-negligible sets, it is complete which also completes the proof.

□

3.3.1 Constructing Lebesgue measure on $\mathbb{R}^n$

Let

$$\mathcal{R} = \left\{ R = \prod_{i=1}^{\infty} [a_i, b_i] \mid a_i, b_i \in \mathbb{R} \right\}$$

and define

$$\varphi(R) = \mathrm{Vol}(R) = \prod_{i=1}^{\infty} |a_i - b_i|.$$

Let us define the outer Lebesgue measure for $E \subseteq \mathbb{R}^n$

$$\mu^*(E) = \inf \left\{ \sum_{i=1} \mathrm{Vol}(R_i) \mid E \subseteq \bigcup_{i=1}^{\infty} R_i \text{ s.t. } R_i \in \mathcal{R} \right\}.$$

We saw that it is an outer measure. By Theorem 3.4 the space $(\mathbb{R}^n, L(\mathbb{R}^n), \mu^*|_{L(\mathbb{R}^n)})$ is a complete measure space, such that

$\mu^*\vert_{L(\mathbb{R}^n)} = \mu$	is called Lebesgue measure	and	$L(\mathbb{R}^n) = \mathcal{F}_{\mu^*}$	is the σ-algebra of the Lebesure measurable sets in $\mathbb{R}^n$

We now need to show that

(1) For all $R \in \mathcal{R}$ we have $\mu^*(R) = \mathrm{Vol}(R)$;

(2) $R \subseteq L(\mathbb{R}^n) \iff B(\mathbb{R}^n) = \sigma(\mathcal{R}) \subseteq L(\mathbb{R}^n)$.

Remark 3.11. Recall that we defined the Cantor set as

$$C := \left\{ x \in [0,1] \mid x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ s.t. } a_n \in \{0,2\} \right\}.$$

Proposition 3.5. *The Cantor set satisfies the following properties:*

(1) It is compact;

(2) We have $m(C) = 0$;

(3) $|C| = \aleph = 2^{\aleph_0}$.

Proof.

(1) We notice that $C = \bigcap_n K_n$ such that

$$K_n := \biguplus \left\{ x_0 + [0, \frac{1}{3^n}] \mid x_0 = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ s.t. } a_n \in \{0, 2\} \right\}.$$

Since K_n is a union of a finite amount of compact sets it is compact, and since C is an intersection of a countable amount of compact sets it is compact.

(2) We have from the monotonicity of m that:

$$0 \le m(C) \le m(K_n) = \frac{2^n}{3^n} \xrightarrow{n \to \infty} 0.$$

(3) Let $x \in C$. Then it has a unique ternary expansion without the digit 1:

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \text{ s.t. } a_n \in \{0, 2\}$$

so it is clear that

$$|C| = \left| \{0, 2\}^{\mathbb{N}} \right| = 2^{\aleph_0} = \aleph = \left| [0, 1] \right|.$$

□

We can construct the almost-bijection $f\colon [0, 1] \to C$ explicitly:
Define $b_n = \frac{a_n}{2} \in \{0, 1\}$ and define

$$y = f(x) := \sum_{n=1}^{\infty} \frac{b_n}{2^n}.$$

It is clear that $f(C) = [0, 1]$ because every number in $[0, 1]$ has a binary expansion, and f is injective except for the numbers that have two binary expansions.

We notice that f is monotonic (nondecreasing). That is $x < y$ implies $f(x) \le f(y)$ and if x, y are points on the boundary of a G_n set we have $f(x) < f(y)$.

We can expand the function such that $f\colon [0, 1] \to [0, 1]$ by defining it to be constant on G_n, and this is well defined because $f(x) = f(y)$ for $x, y \in \partial G_n$. Since f is monotonic it can only have a jumping point of discontinuity, but since f is onto $[0, 1]$ it doesn't have such points and thus continuous.

The function f is called Cantor's function. It is

Proposition 3.6. *There exists a Lebesgue measurable function that is not a Borel set. That is,*

$$L(\mathbb{R}) \setminus B(\mathbb{R}) \neq \emptyset.$$

Proof. We notice that

$$|B(\mathbb{R})| = \aleph_1 = 2^{\aleph},$$

but we have that

$$|L(\mathbb{R})| \ge 2^{\aleph_1} > \aleph_1$$

because every subset of C is of measure 0, (and since the measure is complete) it is measurable, and there are $2^{|C|}$ such subsets. □

Proof. Let f be the Cantor function, and define

$$g(y) = \inf \{x \in [0,1] \mid f(x) = y\}$$

as its inverse function. The function g is strictly increasing, and thus injective, but it is not continuous. $\square$

4 Measurable functions

Definition 4.1 (Inverse image)**.** Given a map $f\colon X \to Y$. The inverse image map $f^{-1}\colon 2^Y \to 2^X$ is given by

$$f^{-1}(E) = \{x \in X \mid f(x) \in E\}, \quad E \subseteq Y.$$

Remark 4.1. The inverse image preserves taking complements, unions, and intersections of any cardinality.

From this remark we get the following proposition.

Proposition 4.1. *Let $\mathcal{A}_Y$ be a σ-algebra on Y. Then $f^{-1}(\mathcal{A}_Y) = \{f^{-1}(E) \mid E \in \mathcal{A}_Y\} \subseteq 2^X$ is a σ-algebra on X.*

Definition 4.2 (Measurable function)**.** Let $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ be measurable spaces, and let $f\colon X \to Y$ be a function. Then f is called a measurable function if for all $E \in \mathcal{A}_Y$ we have $f^{-1}(E) \in \mathcal{A}_X$. That is $f^{-1}(\mathcal{A}_Y) \subseteq \mathcal{A}_X$.

Remark 4.2. A composition of measurable functions is measurable. This easily follows from $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Lemma 4.2. *Let $\mathcal{A}_Y = \sigma(\mathcal{E})$ such that $\mathcal{E} \subseteq 2^Y$. Then $f\colon X \to Y$ is measurable if and only if for all $E \in \mathcal{E}$ we have $f^{-1}(E) \in \mathcal{A}_X$.*

Proof. Thef first direction is trivial $\square$

5 Simple functions

Definition 5.1 (Indicator function)**.** An indicator function of a set $E \subseteq X$ is defined as

$$\mathbb{1}_E(x) =_E (x) = \begin{cases} 1, & x \in E \\ 0, & \text{otherwise} \end{cases}.$$

Remark 5.1. $\mathbb{1}_E$ is measurable if and only if E is measurable.

Definition 5.2 (Simple function)**.** A function $\phi\colon (X, \mathcal{A}) \to \mathbb{R}$ is called simple if

$$\phi(X) = \sum_{j=1}^{N} c_j \cdot \mathbb{1}_{E_j}, \quad c_j \in \mathbb{R} \text{ and } E_j \in \mathcal{A}.$$

6 Integration

Set a measure space $(X, \mathcal{A}, \mu)$ and denote $\mathcal{L}^+ := \{f\colon X \to [0,\infty] \mid f \text{ is measurable}\}$.

Remark 6.1. If $\phi \in \mathcal{L}^+$ is simple, then

$$\phi = \sum_{j=1}^{N} c_j \cdot \mathbb{1}_{E_j}, \quad c_j \in [0,\infty] \text{ and } E_j \in \mathcal{A}.$$

Definition 6.1 (Integration of simple functions)**.** Let $\phi \in \mathcal{L}^+$ be simple. Then we define

$$\int \phi \,\mathrm{d}\mu = \sum_{j=1}^{N} c_j \cdot \mu(E_j) \in [0, \infty].$$

Remark 6.2. We can verify that this definition is not dependent on the representation of ϕ.

Remark 6.3. The convention is that $\underset{=c_j}{0} \cdot \underset{=\mu(E_j)}{\infty} = 0$.

Remark 6.4. We denote

$$\int \phi = \int \phi \,\mathrm{d}\mu = \int \phi(x) \,\mathrm{d}\mu(x) = \int \phi(x) mu(\mathrm{d}x)$$

and

$$\forall A \in \mathcal{A}, \quad \int_A \phi \,\mathrm{d}\mu = \int \underbrace{\phi \mathbb{1}_A}_{\text{simple function}} \mathrm{d}\mu$$

Proposition 6.1. *Let $\phi, \psi \in \mathcal{L}^+$ be simple functions. Then*

(1) For all $a \geq 0$ we have $\int a\phi = a \int \phi$.

(2) $\int(\phi + \psi) = \int \phi + \int \psi$.

(3) If $\phi \leq \psi$ then $\int \phi \leq \int \psi$.

Proof.

(1) Immediate result.

(2)

(2)

□

Definition 6.2 (Integrability)**.** Let $f \in \mathcal{L}^+$. We define

$$\int f \,\mathrm{d}\mu := \sup \left\{ \int \phi \,\mathrm{d}\mu \colon 0 \leq \phi \leq f \text{ and } \mu \text{ is simple} \right\}.$$

A function $f \in \mathcal{L}^+$ is called integrable with respect to μ if $\int f \,\mathrm{d}\mu < \infty$. We define

$$\int_A f \,\mathrm{d}\mu = \int f \cdot \mathbb{1}_A \,\mathrm{d}\mu.$$

Remark 6.5. This definition generalized the previous definition for $f = \phi$. This follows from property (3) in the previous proposition.

Proposition 6.2. *Let $\psi, \phi, \phi_1, \phi_2 \in \mathcal{L}^+$. Then*

(1) For all $a \geq 0$ we have $\int a\phi = a \int \phi$.

(2) $\int(\phi_1 + \phi_2) = \int \phi_1 + \int \phi_2$.

(3) If $0 \leq \psi \leq \phi$ then $0 \leq \int \psi \leq \int \phi$.

Theorem 6.3 (Monotone convergence theorem)**.** *If $\{f_n\} \subseteq \mathcal{L}^+$ be a pointwise monotone sequence of functions. Then*

$$\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu.$$

Remark 6.6. The function f is measurable because $f = \sup_n f_n$.

Remark 6.7. The theorem is false for the Riemann integral. Since $\mathbb{Q}$ is countable we can construct a well order on it such that $\{q_k\}_{k=1}^{\infty} = \mathbb{Q}$ and then define

$$f_n(x) = \begin{cases} 1, & x \in \bigcup_{k=1}^{n} q_k \\ 0, & \text{otherwise} \end{cases}$$

but the limit function is $f = \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ which is not Riemann integrable.

Proof. From property (3) of we have

$$\int f_1 \leq \int f_2 \leq \cdots \leq \int f_n \leq \int f$$

so the sequence $\int f_n$ converges and we have

$$\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu \leq \int f \,\mathrm{d}\mu.$$

Now set $\alpha \in (0,1)$ and let ϕ be a simple function such that $0 \leq \phi \leq f$. We define

$$A_n = \left\{x \in X \colon f_n(x) \geq \alpha \cdot \phi(x)\right\}$$

and see that they are measurable, increasing, and $\bigcup A_n = X$. Thus

$$\int f_n \geq \int f_n \mathbb{1}_A \geq \int \alpha\phi \cdot \mathbb{1}_{A_n} = \alpha \int_{A_n} \phi.$$

and then

$$\int_{A_n} \phi = \int \phi \mathbb{1}_{A_n} = \sum_{j=1}^{N} c_j \mu\left(E_j \cap A_n\right) \sum_{j=1}^{N} c_j \mu(E_j) = \int \phi.$$

Thus finally we have

$$\lim_{n\to\infty} \int f_n \geq \alpha \int \phi$$

for an arbitrary $\alpha \in (0,1)$ so we can take $\alpha \to 1$ and get

$$\lim_{n\to\infty} \int f_n \geq \int \phi.$$

We can now take $\sup_\phi$ and get that

$$\lim_{n\to\infty} \int f_n \geq \int f$$

which completes the proof. □

Corollary 6.4. *An equivalent definition to the integral of $f \in \mathcal{L}^+$ is*

$$\int f \,\mathrm{d}\mu = \lim_{n\to\infty} \int \phi_n \,\mathrm{d}\mu$$

for any sequence of positive simple functions $0 \leq \phi_n \to f$. This is because we have shown that there exists such a sequence and the limit is not dependent on the choice of sequence.