Complex Analysis - Problems

Based on lectures by

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Holomorphic functions

Exercise 1.1. Let $\Omega \subset \mathbb{C}$ be an open connected set. Let

$$\mu = \left\{(x, y) \in \mathbb{R}^2 \mid x + yi \in \Omega\right\}.$$

Let $f = u + iv$ be a holomorphic function. Suppose f has continuous partial derivatives in μ. Show that each of the following requirements imply that f is bounded.

(1) $\arg(f(z))$ is constant in Ω.

(2) $\Re(f)^2 = \Im(f)^3$ in Ω.

Solution. (1) because $\arg(f(z))$ is constant there exist $a, b \in \mathbb{R}$ different not both 0 such that

$$au(x, y) + bv(x, y) = 0.$$

From the Cauchy–Riemann equations we get that

$$\begin{cases} au_x + bv_x = 0 \\ au_y + bv_y = 0 = -av_x + bu_x \end{cases}$$

This is just like saying that

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have that

$$\det \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = -a^2 - b^2 \neq 0$$

therefore $u_x - v_x = 0$. And then from C–R we have $u_y = v_y = 0$.

(2) We have that $u^2 = v^3$. Taking the derivatives we get

$$\begin{cases} 2uu_x + 3v^2 v_x = 0 \\ 2uu_y + 3v^2 v_y = 0 \end{cases}$$

and then we have

$$2uu_y u_x = 3v^2 v_x v_x \underset{C-R}{\Longrightarrow} 2u(u_x^2 + u_y^2) = 0.$$

This implies that either $u = 0$ or $u_x = u_y = 0$ which implies from C–R that $v_x = v_y = 0$. We denote

$$\mu_0 = \left\{(x, y) \in \mu \mid u = 0\right\}$$
$$\mu \setminus \mu_0 = \left\{(x, y) \mid u \neq 0\right\}.$$

TO BE CONTINUED

Let $z = x + iy$, $\bar{z} = x - iy$. We then have $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

Definition 1.1 (Wirtinger derivative)**.** We define the Wirtinger derivative as

$$\frac{\partial}{\partial z} := \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

Exercise 1.2. Let $f = u + iv$, u, v have continuous partial derivatives. Show that

(1) If f is holomorphic, then
$$\frac{\partial f}{\partial z} = f'(z).$$

(2) f is holomorphic if and only if
$$\frac{\partial f}{\partial \bar{z}} = 0.$$

(3)

Solution. (1) We have
$$\frac{\partial f}{\partial z} := \left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}\right)(u+iv) = \frac{u_x + iv_x}{2} + \frac{1}{2}(v_y - iu_v) = f'(z)$$

(2) We have
$$\frac{\partial f}{\partial \bar{z}} := \left(\frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}\right)(u+iv) = \frac{u_x - v_y}{2} + i\frac{u_y + v_x}{2} = 0$$

We know that f is holomorphic if and only if it satisfies the C–R equations if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ which completes this part.

2 Line integrals

Exercise 2.1.

(1) Let $f(z) = \bar{z}$. Find
$$\int_\gamma f(z)\,\mathrm{d}z$$
where γ is a positively oriented circle with radius $r > 0$ around z_0.

(2) Show that in any region Ω that f is not the uniform limit of polynomials.

(3) Show that f doesn't have an antiderivative.

Solution.

(1) Define $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$. We then have
$$\int_\gamma f(z)\,\mathrm{d}z = \int_0^{2\pi} z_0 + \bar{r}e^{it}ire^{it}\,\mathrm{d}t = \int_0^{2\pi} ir\bar{z}_0e^{it} + ir^2\,\mathrm{d}t = r z_0e^{it} + ir^2t\Big|_0^{2\pi} = 2\pi ir^2.$$

(2) Assume by contradiction that Ω is a region such that f is a uniform limit of polynomials. Then for $C_r(z_0) \subseteq \Omega$ we have
$$\begin{aligned}\left|\int_{C_r(z_0)} f(z)\,\mathrm{d}z - \int_{C_r(z_0) p_n(z)\,\mathrm{d}z}\right| &= \left|\int_{C_r(z_0)} f(z) - p_n(z)\,\mathrm{d}z\right| \le 2\pi r \sum_{C_r(z_0)} |f - p_n| \\ &\le 2\pi r \sum_{\Omega} |f - p_n| \xrightarrow{n\to\infty} = 0.\end{aligned}$$

Finally we have
$$\int_\gamma f = \lim_{n\to\infty}\int_\gamma f = \lim_{n\to\infty}\int_\gamma f - p_n + \int_\gamma p_n = 0$$
where the last inequality follows from the previous calculation, and the fact that p_n has an antiderivative.

(3) Assume by contradiction that F is the antiderivative of f on Ω. Then since Ω is a region there exists some $C_r(z_0) \subseteq \Omega$. Then by a theorem

$$2\pi i r^2 = \int_{C_r(z_0)} \bar{z}\, \mathrm{d}z = 0$$

which is a contradiction.