Practice

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1 In the following sections show that $V = U \oplus W$ and find the projection on U parallel to W

1.1 $V = \mathbb{R}[x]$ with

$$W = \mathbf{Sp}\{x^2 + x + 1\}, \quad U = \{p(x) \in V \colon p(0) = 0\}$$

First we will show that $U + W = V$. Let $p \in V$ be a general polynomial:

$$p(x) = a_n x^n + \cdots + a_0 \in V$$

Now choose $w = a_0(x^2 + x + 1) \in W$ and $u = (p - w) \in U$. We know that $(p - w) \in U$ because:

$$(p - w)(0) = p(0) - w(0) = a_0 - a_0 = 0$$

Now we see that:

$$u + w = (p - w) + w = p$$

That proves that $U + W = P$. Now we will show that $U \bigcap W = \{0\}$ which will prove that $U \oplus W = V$, as we have shown in the lecture.

$$\begin{aligned} U \cap W &= \{p(x) \in V \colon p(x) \in W \wedge p(0) = 0\} \\ &= \{ax^2 + ax + a \colon a \in \mathbb{R} \wedge p(0) = 0\} \\ &= \{ax^2 + ax + a \colon a \in \mathbb{R} \wedge a = 0\} \\ &= \{0\} \end{aligned}$$

Now we will find the projection on U parallel to W. We have shown that the only way to get any specific $p \in V$ is by adding the specific:

$$u_p + w_p = (p - a_0(x^2 + x + 1)) + a_0(x^2 + x + 1)$$

So the parallel projection will be $P \colon V \to V$:

$$\begin{aligned} P(p(x)) = P(a_n x^n + \cdots + a_0) = u_p &= (p - a_0(x^2 + x + 1)) \\ &= a_n x^n + \cdots + a_3 x^3 + (a_2 - a_0)x^2 + (a_1 - a_0)x \end{aligned}$$

1.2 $V = \mathbb{R}^4$ with

$$W = \textbf{Sp}\{e_1 + e_4, e_2 + e_4\}, \quad U = \{e_1, e_2 + e_3\}$$

where $E = (e_1, ..., e_4)$ is the standard basis.

Consider the following matrix with the vectors from U and W:

$$\begin{pmatrix} — & e_1 + e_4 & — \\ — & e_2 + e_4 & — \\ — & e_1 & — \\ — & e_2 + e_3 & — \end{pmatrix}$$

By applying elementary row operations we get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Which means as we know from linear algebra 1 that $U+W = V$ and from Grassman's identity(?) we know that:

$$\underbrace{\dim(W + U)}_{4} = \underbrace{\dim(W)}_{2} + \underbrace{\dim(U)}_{2} - \dim(U \cap W)$$
$$\Rightarrow \dim(U \cap W) = 0$$
$$\Rightarrow U \cap W = \{0\}$$

Which implies that $U \oplus W = V$. Now we will find the projection on U parallel to W. For this we will need to find the unique decomposition of any $v \in V$ to vectors $u \in U$ and $w \in W$. Where for $a, b, c, d, x_1, x_2, x_3, x_4 \in \mathbb{F}$:

$$w = a(e_1 + e_4) + b(e_2 + e_4) = \begin{pmatrix} a \\ b \\ 0 \\ a + b \end{pmatrix}$$

$$u = c(e_1) + d(e_2 + e_3) = \begin{pmatrix} c \\ d \\ d \\ 0 \end{pmatrix}$$

$$u + w = \begin{pmatrix} | & | & | & | \\ e_1 + e_4 & e_2 + e_4 & e_1 & e_2 + e_3 \\ | & | & | & | \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \\ d \\ a + b \end{pmatrix} = v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

So we get that $d = x_3$

$$\begin{pmatrix} a + c \\ b + x_3 \\ x_3 \\ a + b \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Now $b = x_2 - x_3$

$$\begin{pmatrix} a+c \\ x_2 \\ x_3 \\ a+x_2-x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

So $a = x_4 - x_2 + x_3$ and we get:

$$\begin{pmatrix} x_4 - x_2 + x_3 + c \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

So $c = x_1 - x_4 + x_2 - x_3$. Finally we get that for any $v \in V$ such that:

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

We get:

$$\begin{aligned} v &= w + u = a(e_1 + e_4) + b(e_2 + e_4) + c(e_1) + d(e_2 + e_3) \\ &= (x_4 - x_2 + x_3)(e_1 + e_4) + (x_2 - x_3)(e_2 + e_4) + (x_1 - x_4 + x_2 - x_3)(e_1) + (x_3)(e_2 + e_3) \end{aligned}$$

Which means the projection on U parallel to W is $P\colon V \to V$

$$\forall \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{v} \in V\colon P(v) = (x_1 - x_4 + x_2 - x_3)(e_1) + (x_3)(e_2 + e_3) = \begin{pmatrix} x_1 - x_4 + x_2 - x_3 \\ x_3 \\ x_3 \\ 0 \end{pmatrix} = u$$

2 Prove/Disprove

2.1 The sum of projections is a projection

This is false. Let $P_1 = P_2 = \mathrm{Id}_n$ be our projections from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is clear these are projections since:

$$\mathrm{Id}_n^2 = \mathrm{Id}_n$$

But the transformation $P = P_1 + P_2$ is not a projection since:

$$P^2 = (P_1 + P_2)^2 = (2\mathrm{Id}_n)^2 = 4\mathrm{Id}_n \neq 2\mathrm{Id}_n = P$$

2.2 The composition of projections is a projection

This claim is false. Consider the following projections over $\mathbb{R}^2$:

$$P_1(x, y) = (x + y, 0) \quad \textit{and} \quad P_2(x, y) = (x, x)$$

It's easy to verify that these are indeed projections:

$$P_1^2(x, y) = P_1(x + y, 0) = (x + y, 0) = P_1(x, x)$$
$$P_2^2(x, y) = P_2(x, x) = (x, x) = P_2(x, y)$$

Yet if we consider the vector $(2, 1)$ we get:

$$(P_1 \circ P_2)(2, 1) = P_1(2, 2) = (4, 0)$$
$$(P_1 \circ P_2)^2(2, 1) = (P_1 \circ P_2)(4, 0) = P_1(4, 4) = (8, 0)$$

So:

$$(P_1 \circ P_2) \neq (P_1 \circ P_2)^2$$

Which means it's not a projection.

3 Let V be a finite-dimensional vector space, and let $P_1, ..., P_n \in$ End(V) be parallel projections. Denote $\forall i$: $R_i = \textbf{Im}P_i$

3.1 Show that tr$P_i = \dim R_i$

Since P_i is a parallel projection we know that $V = \text{Im}P_i \oplus \text{Ker}P_i$ Which means that $\text{Im}P_i \cap \text{Ker}P_i = \{0\}$. We know by a theorem we learned in class that exist:

$$B_r = \{b_1, ..., b_k\}$$

a basis for $\text{Im}P_i = R_i$. And:

$$B_k = \{r_{b+1}, ..., b_n\}$$

a basis for $\text{Ker}P_i$ such that the ordered union:

$$B = B_r \cup B_k = \{b_1, ..., b_k, b_{k+1}, ..., b_n\}$$

forms a basis for V. That means that the matrix representation of P_i by the basis B is:

$$\begin{pmatrix} | & | & | \\ [P_i(b_1)]_B & ... & [P_i(b_n)]_B \\ | & | & | \end{pmatrix}_{n\times n} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}_{n\times n}$$

So $\text{tr}([P_i]_B) = k$. And since the trace of a transformation is just the trace of its representing matrix, as shown to be a well defined trait of transformations in linear algebra 1 we conclude that:

$$\text{tr}(P_i) = \text{tr}([P_i]_B) = k = \dim \text{Im}P_i = \dim R_i$$

3.2 Let $P_1 + \cdots + P_n = \mathrm{Id}$, show that $V = \bigoplus R_i$ and infer that $\forall i \neq j\colon P_i P_j = 0$

$\underline{V = \bigoplus R_i}$ - From 3.1 we know that:

$$\dim V = \mathrm{tr}(\mathrm{Id}) = \mathrm{tr}(P_1 + \cdots + P_n) = \mathrm{tr}(P_1) + \cdots + \mathrm{tr}(P_n) = \dim R_1 + \cdots + \dim R_n$$

Now we will show that $R_1 + \cdots R_n = V$. Let $v \in V$:

$$v = \mathrm{Id}(v) = (P_1 + \cdots + P_n)(v) = P_1(v) + \cdots + P_n(v)$$

Since $\forall i\colon P_i(v) \in R_i$ we get that for any $v \in V$ exist $P_1(v) \in R_1, ..., P_n(v) \in R_n$ such that $v = P_1(v) + \cdots + P_n(v)$. So now we know that

$$V = R_1 + \cdots + R_n$$
$$\dim V = \dim R_1 + \cdots + \dim R_n$$

Denote B_{R_i} the ordered basis for R_i for any i, we get:

$$V = \mathrm{Sp}\left\{\bigcup_i B_{R_i}\right\} \qquad \Rightarrow \dim V \leq \left|\bigcup_i B_{R_i}\right|$$

$$\dim V = \sum_i |B_{R_i}| \geq \left|\bigcup_i B_{R_i}\right| \qquad \Rightarrow \left|\bigcup_i B_{R_i}\right| \leq \dim V$$

$$\Rightarrow \left|\bigcup_i B_{R_i}\right| = \dim V$$

So from:

$$\mathrm{Sp}\left\{\bigcup_i B_{R_i}\right\} = V \wedge \left|\bigcup_i B_{R_i}\right| = \dim V$$

We get that the ordered union of the ordered bases B_{R_i} form a basis of V which is equivalent as we've shown in class to saying that $V = \bigoplus R_i$

$\underline{\forall i \neq j\colon P_i P_j = 0}$ - Let $i \neq j$. Now suppose that $P_i P_j \neq 0$. that means that exists a $0 \neq v \in V$ such that $P_i P_j(v) \neq 0$, which means that $P_j(v) \notin \mathrm{Ker} P_i$. Since P_i is a projection we know that $\mathrm{Im} P_i \oplus \mathrm{Ker} P_i = V$ which means that $P_j(v) \in R_i$, but also by definition $P_j(v) \in R_j$, so:

$$\underbrace{0}_{R_1} + \cdots + \underbrace{P_j(v)}_{R_j} + \cdots + \underbrace{0}_{R_n} = P_j(v)$$

$$\underbrace{0}_{R_1} + \cdots + \underbrace{P_j(v)}_{R_i} + \cdots + \underbrace{0}_{R_n} = P_j(v)$$

but that's a contradiction to $V = \bigoplus R_i$. So $\forall i \neq j\colon P_i P_j = 0$

4 Let V be a vector space, $T, S \in \textbf{End}(V)$, and let S be diagonalizable. Prove that the eigenspaces of S are T-invariant if and only if $TS = ST$

($\Leftarrow$)
For any eignenvalue λ of S:

$$\begin{aligned} \mathrm{Ker}(S-\lambda I) &= \{s \in V | S(s) = \lambda s\} \\ \Rightarrow T(\mathrm{Ker}(S-\lambda I)) &= \{T(s) | S(s) = \lambda s\} \\ &= \{s \in V | \exists w \colon T(w) = s \wedge S(w) = \lambda w\} \end{aligned}$$

Since for $s \in T(\mathrm{Ker}(S-\lambda I))$:

$$S(s) = S(T(w)) \underset{TS=ST}{=} T(S(w)) = T(\lambda w) = \lambda T(w) = \lambda s$$

We get that $T(\mathrm{Ker}(S-\lambda I)) \subseteq \mathrm{Ker}(S-\lambda I)$ which means that all the eigenspaces of S are T-invariant.

($\Rightarrow$)
We know that S is diagnolizable so exist a base to V

$$B = (b_1, \ldots, b_n)$$

such that $[S]_B$ is a diagnonal matrix. We will show that for any $b \in B$ that $TS(b) = ST(b)$. Let $b \in B$ be an eigenvector of an eigenspace with eigenvalue λ:

$$TS(b) = T(\lambda b) = \lambda(T(b))$$

Now since $b \in V_\lambda^S$[1] is T-invariant by the assumption:

$$\lambda(T(b)) = S(T(b)) = ST(b)$$

We have shown that for any vector from the base B of V

$$TS(b) = ST(b)$$

Since B spans V and S, T are linear, we know that for any $v \in V$

$$TS(v) = ST(v)$$

Which is what we wanted to prove.

[1] λ-eigenspace of S under V not sure if this is the correct notation.

5 Let V be a vector space over a field $\mathbb{F}$, with $\dim V = n$. Let $T: V \to V$ such that any $(n-1)$-dimentional vector subspace of V is T-invariant. Prove that V is a scalar transformation.

Let $v_1 \in V$ be a vector such that $T(v_1) = v_2$ and v_2 isn't a scalar multiply of v_1. That means they are linearly independent which implies we can complete $\{v_1, v_2\}$ to a basis of V as such:

$$B = (v_1, v_2, \dots, v_n)$$

Since $\mathrm{Sp}\{v_1, v_3, \dots, v_n\}$ is a $n-1$-dimentional subspace of V, it is T-invariant, which means that:

$$T(v_1) = v_2 \in \mathrm{Sp}(v_1, v_3, \dots, v_n)$$

But that's a contradiction since if v_2 were in $\mathrm{Sp}(v_1, v_3, \dots, v_n)$ then B wouldn't be linearly independent even thought it's a basis of V. That means that for any $v \in V$ then $T(v)$ is a scalar multiple of v. Now consider the standard basis $E = (e_1, \dots, e_n)$ we know that:

$$\begin{gathered} T(e_1) = \lambda_1 e_1 \\ T(e_2) = \lambda_2 e_2 \\ \dots \\ T(e_n) = \lambda_n e_n \end{gathered}$$

We also know that $T(e_1 + \dots + e_n) = \mu \sum_{i=1}^{n} e_i$ so:

$$T(e_1 + e_2 + \dots + e_n) = T(e_1) + \dots + T(e_n) = \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{n} \mu e_i$$

Since $e_1, \dots, e_n$ are linearly independent that means that:

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = \mu$$

Finally since E is a basis, for any $v \in V$ we get that $T(v) = \mu v$. In other words that T is a scalar operator.

6 Let $T, S, Q \in \textbf{End}(v)$ such that $T = Q^{-1}SQ$. Show that $U \subseteq V$ is T-invariant $\iff Q(U)$ is S-invariant

($\Rightarrow$)
Suppose that $U \subseteq V$ is T-invariant. That means that:

$$T(U) \subseteq U$$

Now:

$$S(Q(U)) = SQ(U)$$

But we know that $T = Q^{-1}SQ \Rightarrow QT = SQ$ so:

$$S(Q(U)) = QT(U) = Q(T(U))$$

We know that $T(U) \subseteq U$ so:

$$\begin{aligned} S(Q(U)) = Q(T(U)) &\subseteq Q(U) \\ \Rightarrow S(Q(U)) &\subseteq Q(U) \end{aligned}$$

In other words - $Q(U)$ is S-invariant.

($\Leftarrow$)
Suppose that $Q(U)$ is S-invariant:

$$(*) \quad S(Q(U)) \subseteq Q(U)$$

Now:

$$T(U) = Q^{-1}SQ(U) = Q^{-1}(S(Q(U))) \underset{*}{\subseteq} Q^{-1}(Q(U)) = U$$

So:

$$T(U) \subseteq U$$

In other words U is T-invariant.

7 The one it won't be fun to typeset.

7.1 Find the Jordan normal form, a jordan basis, and the minimal polynomial of the following matrix:

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix} \in M_n(\mathbb{C})$$

First we're gonna find the characteristic polynomial of this matrix. We notice that the matrix is a blockwise triangular matrix:

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} B_{2\times 2} & 0 \\ * & C_{2\times 2} \end{pmatrix}$$

So we can solve it like we did in linear algebra 1:

$$\begin{aligned} p_A(\lambda) &= p_B(\lambda)p_C(\lambda) = ((-1-\lambda)(2-\lambda)+2)((2-\lambda)(0-\lambda)+2) \\ &= (\lambda^2-\lambda)(\lambda^2-2\lambda+1) = (\lambda(\lambda-1))(\lambda^2-2\lambda+1) = \lambda(\lambda-1)^3 \end{aligned}$$

But we can also notice that the sum of columns of these blocks is 1 so 1 is an eigenvalue of both of them, and since the sum of the eigenvalues of a matrix is equal to its trace we can find the other eigen value. We see that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 1 and $\lambda = 1$ is an eigenvalue of algebraic multiplicity 3 so the Jordan normal form will have a Jordan block $J_1(0)$ and some Jordan blocks of total size 3. Now we will find $(A - I)$ to find out how many Jordan blocks are there:

$$A - I = \begin{pmatrix} -2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 \\ -2 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$
$$\to \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we see that $(A - I) = 2$ so there are two Jordan blocks of $\lambda = 1$. That that the Jordan normal form of A must be of the form $J_2(1) \oplus J_1(1) \oplus J_1(0)$. So we want to find Jordan chains of the form:

$\lambda = 1$		$\lambda = 2$
v_2		
$\downarrow$		
v_1	v_3	v_4

We shall continue with some more calculation to find the generalized eigenspaces of A.

$$\ker(A - I) = \ker\left(\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \mathrm{Sp}\left\{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$

$$(A-I)^2 = \begin{pmatrix} -2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 \\ -2 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} -2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\ker(A-I)^2 = \ker\left(\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \mathrm{Sp}\left\{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \widetilde{V_1}$$

$$\ker(A) = \ker\left(\begin{pmatrix} -1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\right)$$

$$= \mathrm{Sp}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\} = \widetilde{V_0}$$

To test our calculations against the generalized eigenspace decomposition theorem we see that indeed:

$$V = \mathrm{Sp}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\} \oplus \mathrm{Sp}\left\{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \widetilde{V_0} \oplus \widetilde{V_1}$$

To find v_2 we would need to find a vector in $\ker(A-I)^2$ that is not in $\ker(A-I)$ for example:

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Then:

$$v_1 = (A-I)v_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

Now fo find v_3 we will just find a vector that will complement $\mathrm{Sp}\{v_1, v_2\}$ to $\widetilde{V_1}$ for example:

$$v_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

And for the last vector we can just choose any vector that is in $\widetilde{V_1}$ for example:

$$v_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

So we found all of our Jordan chains and also the Jordan basis for A:

$$B_J = \left\{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\}$$

Now we will find the minimal polynomial. To find the minimal polynomial we will see that it is excactly the product of the the polynomials of the form $p(x) = (x - \lambda)^r$ for each distinct eigenvalue λ of A and r being the size of the longest Jordan chain of its respective λ, since each vector in V can be represented as a linear combination of the Jordan base, and for any polynome that doesn't include one of these multiples of $(x - \lambda)$ we can take the top of the chain of this lambda and see that it will not be a root of the supposed polynome. Therefore:

$$m_A(x) = (x-1)^2(x-2)$$

8 The one with the polynomial operator

8.1 Let $T\colon \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ be the operator

$$T(ax^3 + bx^2 + cx + d) = 2ax^3 + (2b + 3c + d)x^2 + (2c + 3d)x + 2d$$

Does exist a basis to $\mathbb{R}_3[x]$ such that:

$$[T^2 - 4T + 4I]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We notice that:

$$[T^2 - 4T + 4I]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = J_3(0) \oplus J_1(0)$$

And since we know that the Jordan normal form of a transformation is unqiue up to order, it suffices to show that the Jordan normal form of $T^2 - 4T + 4I$ is the same or different than $J_3(0) \oplus J_1(0)$. Making some calculations we get that represented by the standard basis:

$$[T^2 - 4T + 4I]_E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underbrace{=}_{\text{denotion}} B$$

Which means that the characteristic polynomial of it is:

$$p_B(x) = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 9 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda(-\lambda^3 + 9 * 0) = \lambda^4$$

So the only eigenvalue of $T^2 - 4T + 4I$ is 0, of algebraic multiplicity 4. We know by a theorem we have proved in class that there must be at least:

$$\dim\ker(T^2 - 4T + 4I) = 3$$

Jordan blocks in $T^2 - 4T + 4I$'s Jordan normal form. This means that it can't have the Jordan normal form of $J_3(0) \oplus J_1(0)$, so we have shown that there does not exist a basis B to V such that

$$[T^2 - 4T + 4I]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

9 The one with the ranks

9.1 Let $A \in M_7(\mathbb{R})$ such that:

$$rk(A-I)^2 > rk(A-I)^3 = rk(A-I)^4$$

and $rk(A) = 3$. Calculate the Jordan normal form of A.

We know that $rk(A) = \dim \mathrm{Im}(A) = 3$ and since we also know that:

$$\underbrace{\dim \mathrm{Im}(A)}_{3} + \dim \ker(A) = \underbrace{\dim \mathbb{R}^7}_{7}$$

We know that $\dim \ker(A) = 4$ which tells us that there are 4 Jordan blocks in the Jordan normal form of A with eigenvalue 0. From similar considerations we also see that:

$$\dim \ker(A-I)^3 = 7 - rk(A-I)^3 = 7 - rk(A-I)^4 = \dim \ker(A-I)^4$$

So we know that there are:

$$\dim \ker(A-I)^4 - \dim \ker(A-I)^3 = 0$$

Jordan blocks with eigenvalue 1 of size at least 4. Also:

$$\dim \ker(A-I)^2 = 7 - rk(A-I)^2 < 7 - rk(A-I)^3 = \dim \ker(A-I)^3$$

So there is at least 1 Jordan block of size 3 in the Jordan normal form of A. Since as we have shown, there must be 4 Jordan blocks in the Jordan normal form with eigenvalue 0, and the sum of the order of the Jordan blocks must be equal to 7 the only option for the Jordan normal form of A is:

$$J_3(1) \oplus J_1(0) \oplus J_1(0) \oplus J_1(0) \oplus J_1(0)$$

10 The one with the inverses

10.1 Let $\mathbb{F}$ be a field and $0 \neq \lambda \in \mathbb{F}$. Find the Jordan normal form of $J_n(\lambda)^{-1}$. No need to explicitly compute the inverse.

We can write the Jordan block $J_n(\lambda)$ as the sum of a scalar and a nilpotent matrix like so:

$$J_n(\lambda) = \lambda I + J_n(0)$$

Now we notice that since $\lambda \neq 0$ we can multiply both sided by $\lambda^{-1}I$:

$$\lambda^{-1}IJ_n(\lambda) = \lambda^{-1}I(\lambda I + J_n(0)) = I + \lambda^{-1}J_n(0)$$

And that:

$$(I - \lambda^{-1}J_n(0))(I + \lambda^{-1}J_n(0)) = I - \lambda^{-2}J_n^2(0)$$

Now since:

$$(I + \lambda^{-2}J_n^2(0))(I - \lambda^{-2}J_n^2(0)) = I - \lambda^{-4}J_n^4(0)$$

We can keep going like:

$$(I + \lambda^{-4}J_n^4(0))(I - \lambda^{-4}J_n^4(0)) = I - \lambda^{-8}J_n^8(0)$$

So we see know that:

$$\left(\prod_{i=1}^{k}(I + \lambda^{-2^k}J_n^{2^k}(0))\right)(I - \lambda^{-1}J_n(0))(\lambda^{-1}I)J_n(\lambda) = I - \lambda^{-2^{k+1}}J_n^{2^{k+1}}(0)$$

Since $J_n(0)$ is nilpotent of order $n-1$ we can choose $k \in \mathbb{N}$ such that $2^{k+1} > n$ and then:

$$\left(\prod_{i=1}^{k}(I + \lambda^{-2^k}J_n^{2^k}(0))\right)(I - \lambda^{-1}J_n(0))(\lambda^{-1}I)J_n(\lambda) = I - \lambda^{-2^{k+1}}J_n^{2^{k+1}}(0) = I$$

From linear algebra 1 we know that a if $AB = I$ then $BA = I$ which means that we found the inverse of $J_n(\lambda)$:

$$J_n(\lambda)^{-1} = \left(\prod_{i=1}^{k}(I + \lambda^{-2^k}J_n^{2^k}(0))\right)(I - \lambda^{-1}J_n(0))(\lambda^{-1}I)$$

11 The one with the 9s

11.1 Prove that exists a matrix $A \in M_n(\mathbb{R})$ that satisfies:

$$A^9 + A^{99} = \begin{pmatrix} 2 & 99 & 999 \\ 0 & 2 & -9 \\ 0 & 0 & 2 \end{pmatrix}$$

There's no need to find one explicitly.

First we denote:

$$B = \begin{pmatrix} 2 & 99 & 999 \\ 0 & 2 & -9 \\ 0 & 0 & 2 \end{pmatrix}$$

We notice that $\lambda = 2$ is an eigenvalue of algebraic multiplicity 3 and:

$$B - 2I = \begin{pmatrix} 0 & 99 & 999 \\ 0 & 0 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

Of geometric multiplicity 1 which means that the Jordan normal form of B is $J_3(2)$ so exists an invertible P such that:

$$P^{-1}BP = J_3(2)$$

We notice that according to the combinatorial calculation we did in class we can calculate and get:

$$(J_3(2))^9 + (J_3(2))^{99} = \begin{pmatrix} 2 & 108 & 4887 \\ 0 & 2 & 108 \\ 0 & 0 & 2 \end{pmatrix}$$

This matrix is similarly similar to $J_3(2)$ which means that exists an invertible Q such that:

$$(J_3(2))^9 + (J_3(2))^{99} = Q^{-1}BQ$$

We get that:

$$Q(J_3(2))^9Q^{-1} + Q(J_3(2))^{99}Q^{-1} = B$$

And now if we denote $A = Q(J_3(2))Q^{-1}$ we get that:

$$A^9 + A^{99} = \begin{pmatrix} 2 & 99 & 999 \\ 0 & 2 & -9 \\ 0 & 0 & 2 \end{pmatrix}$$

As wanted.

12 The one with the Cauchy-Schwartz inequality

12.1 Show that for all positive $x_1, \ldots, x_n \in \mathbb{R}$:

$$n^2 \leq (x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

Let $x_1, \ldots, x_n$ be positive real numbers. Recall that the Cauchy-Shwartz inequality states that for any v, u in an inner product space, and specifically for $(\mathbb{R}^n, \langle , \rangle_{\text{std}})$ we get:

$$|\langle v, u\rangle|^2 \leq \langle v, v\rangle\langle u, u\rangle$$

Since $x_1, \ldots, x_n$ are positive we can take their roots and then for:

$$v = (\sqrt{x_1}, \ldots, \sqrt{x_n}) \quad \text{and} \quad u = (\frac{1}{\sqrt{x_1}}, \ldots, \frac{1}{\sqrt{x_n}})$$

We get:

$$|\langle v, u\rangle|^2 = |\langle(\sqrt{x_1}, \ldots, \sqrt{x_n}), (\frac{1}{\sqrt{x_1}}, \ldots, \frac{1}{\sqrt{x_n}})\rangle|^2 = |n|^2 = n^2$$

And:

$$\langle v, v\rangle\langle u, u\rangle = (x_1 + \cdots + x_n)(\frac{1}{x_1} + \cdots + \frac{1}{x_n})$$

Now substituting we get:

$$n^2 \leq (x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

Which is what we wanted to prove.

13 The one with the integral

Let $V = \mathbb{R}_2[x]$ and let:

$$\langle (p(x), q(x)) \rangle_1 = \int_0^1 p(x)q(x)\, dx$$
$$\langle (p(x), q(x)) \rangle_2 = \sum_{x \in \{-1,0,1\}} p(x)q(x)$$

Two inner products on V, and let:

$$W = \{p(x) \in V | p(x) = p(-x)\}$$

13.1 Find a basis for W and complete it to a basis for V.

We know that $W \neq V$ and $W \neq 0$ so since $x^2, 1 \in W$ and are linearly independant we get that $\dim W = 2$ and that

$$B_W = \{x^2, 1\}$$

is a basis for W. We can complete it to a basis for V as such:

$$B_V = \{x^2, 1, x\}$$

13.2 Apply the Gram-Schmidt process on V relative to each of the inner products, find $W^\perp$ and the orthogonal projection P_W on W.

According to $\langle , \rangle_1$ we get:

$$u_1' = v_1 = x^2$$
$$u_2' = v_2 - \sum_{i=1}^{1} \frac{\langle v_2, u_i' \rangle}{\langle u_i', u_i' \rangle} u_i' = v_2 - \frac{\langle v_2, u_1' \rangle}{\langle u_1', u_1' \rangle} u_1' = 1 - \frac{\langle 1, x^2 \rangle}{\langle x^2, x^2 \rangle} x^2 = 1 - \frac{\frac{1}{3}}{\frac{1}{5}} x^2 = 1 - \frac{5}{3} x^2$$
$$u_3' = v_3 - \sum_{i=1}^{2} \frac{\langle v_3, u_i' \rangle}{\langle u_i', u_i' \rangle} u_i' = x - \frac{\langle x, x^2 \rangle}{\langle x^2, x^2 \rangle} x^2 - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{4}{3} x^2 - \frac{1}{2}$$

Now to normalize the vectors:

$$u_1 = \frac{u_1'}{\|u_1'\|} = \frac{x^2}{\sqrt{\langle x^2, x^2 \rangle}} = 2x^2$$
$$u_2 = \frac{u_2'}{\|u_2'\|} = \frac{1 - \frac{5}{3}x^2}{\sqrt{\langle 1 - \frac{5}{3}x^2, 1 - \frac{5}{3}x^2 \rangle}} = \frac{3}{2} - \frac{5}{2}x^2$$
$$u_3 = \frac{u_3'}{\|u_3'\|} = \frac{x - \frac{4}{3}x^2 - \frac{1}{2}}{\sqrt{\langle x - \frac{4}{3}x^2 - \frac{1}{2}, x - \frac{4}{3}x^2 - \frac{1}{2} \rangle}} = \frac{30}{\sqrt{195}} x - \frac{40}{\sqrt{195}} x^2 - \frac{15}{\sqrt{195}}$$

Notice that when we applied Gram-Schmidt we first found an orthonormal basis for W, and since we know that $V = W \oplus W^\perp$ we get:

$$W^\perp = \mathrm{Sp}\{u_3\} = \mathrm{Sp}\left\{ \frac{30}{\sqrt{195}} x - \frac{40}{\sqrt{195}} x^2 - \frac{15}{\sqrt{195}} \right\}$$

And as we know from the lectures for all $v \in V = \mathbb{R}_2[x]$ we get:

$$\begin{aligned} P_W(v) = \sum_{i=1}^{2} \langle v(x), u_i \rangle u_i &= \langle v(x), 2x^2 \rangle 2x^2 + \langle v(x), \frac{3}{2} - \frac{5}{2}x^2 \rangle \left(\frac{3}{2} - \frac{5}{2}x^2 \right) \\ &= \left(\int_0^1 v(x) 2x^2 \, dx \right) 2x^2 + \left(\int_0^1 v(x) \left(\frac{3}{2} - \frac{5}{2}x^2 \right) dx \right) \left(\frac{3}{2} - \frac{5}{2}x^2 \right) \end{aligned}$$

According to $\langle,\rangle_2$ we get:

$$u_1' = v_1 = x^2$$

$$u_2' = v_2 - \sum_{i=1}^{1} \frac{\langle v_2, u_i'\rangle}{\langle u_i', u_i'\rangle} u_i' = v_2 - \frac{\langle v_2, u_1'\rangle}{\langle u_1', u_1'\rangle} u_1' = 1 - \frac{\langle 1, x^2\rangle}{\langle x^2, x^2\rangle} x^2 = 1 - \frac{2}{2} x^2 = 1 - x^2$$

$$u_3' = v_3 - \sum_{i=1}^{2} \frac{\langle v_3, u_i'\rangle}{\langle u_i', u_i'\rangle} u_i' = x - \frac{\langle x, x^2\rangle}{\langle x^2, x^2\rangle} x^2 - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 = x$$

Now to normalize the vectors:

$$u_1 = \frac{u_1'}{\|u_1'\|} = \frac{x^2}{\sqrt{\langle x^2, x^2\rangle}} = \frac{x^2}{\sqrt{2}}$$

$$u_2 = \frac{u_2'}{\|u_2'\|} = \frac{1 - x^2}{\sqrt{\langle 1 - x^2, 1 - x^2\rangle}} = 1 - x^2$$

$$u_3 = \frac{u_3'}{\|u_3'\|} = \frac{x}{\sqrt{\langle x, x\rangle}} = \frac{x}{\sqrt{2}}$$

Notice that when we applied Gram-Schmidt we first found an orthonormal basis for W, and since we know that $V = W \oplus W^{\perp}$ we get:

$$W^{\perp} = \mathrm{Sp}\{u_3\} = \mathrm{Sp}\left\{\frac{x}{\sqrt{2}}\right\}$$

And as we know from the lectures for all $v \in V = \mathbb{R}_2[x]$ we get:

$$\begin{aligned} P_W(v) = \sum_{i=1}^{2} \langle v, u_i\rangle u_i &= \langle v(x), \frac{x^2}{\sqrt{2}}\rangle \frac{x^2}{\sqrt{2}} + \langle v(x), 1 - x^2\rangle \left(1 - x^2\right) \\ &= \left(\sum_{x \in \{-1,0,1\}} v(x) \left(\frac{x^2}{\sqrt{2}}\right)\right) \left(\frac{x^2}{\sqrt{2}}\right) + \left(\sum_{x \in \{-1,0,1\}} v(x)(1 - x^2)\right) \left(1 - x^2\right) \\ &= (v(1) + v(-1)) \left(\frac{x^2}{2}\right) + v(0) \left(1 - x^2\right) \end{aligned}$$

13.3 Find the distance of $f(x) = x + 1$ from W according to each of the inner products.

We know that the distance of $f(x) = x+1$ from W is the distance between $x+1$ and $P_W(x+1)$ which is the point "closest" to $x+1$ on W. So first we shall calculate $P_W(x+1)$ according to each of the inner product spaces:

$$
\begin{aligned}
P_W(x+1) &= (2+0)\left(\frac{x^2}{2}\right) + 1\left(1 - x^2\right) = 1 \\
P_W(x+1) &= \left(\int_0^1 (x+1)2x^2\,dx\right)2x^2 + \left(\int_0^1 (x+1)\left(\frac{3}{2} - \frac{5}{2}x^2\right)\,dx\right)\left(\frac{3}{2} - \frac{5}{2}x^2\right) \\
&= \left(\int_0^1 2x^3 + 2x^2\,dx\right)2x^2 + \left(\int_0^1 \frac{3}{2} - \frac{5}{2}x^2 + \frac{3}{2}x - \frac{5}{2}x^3\,dx\right)\left(\frac{3}{2} - \frac{5}{2}x^2\right) \\
&= \frac{7}{3}x^2 + \frac{19(3-5x^2)}{48} = \frac{112x^2}{48} + \frac{57-95x^2}{48} = \frac{17x^2+57}{48}
\end{aligned}
$$

So now according to $\langle,\rangle_1$ we get that the distance is:

$$\sqrt{\langle x+1, 1\rangle} = \sqrt{\int_0^1 x+1\,dx} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

And now according to $\langle,\rangle_2$ we get that the distance is:

$$\sqrt{\langle x+1, \frac{17x^2+57}{48}\rangle} = \sqrt{\sum_{x=-1,0,1} \frac{17x^2+57(x+1)}{48}\,dx} = \sqrt{\frac{17+57+131}{48}} = \sqrt{\frac{205}{48}} = \frac{\sqrt{615}}{12}$$

14 The one with the contraction

Let V be a finite dimension inner product space and let $P \in \mathrm{End}(V)$ be a contraction - that is $\forall v \in V(\|Pv\| \leq \|v\|)$.

14.1 Show that P is the orthogonal projection on its own image.

We will first show that $V = \operatorname{im} P \oplus \ker P$. Since P is a projection we must have $P(v) = P^2(v)$ which implies $P(P(v) - v) = 0$ so $P(v) - v = \epsilon \in \ker P$ and then $v = P(v) + (-\epsilon)$ which shows that $V = \operatorname{im} P + \ker P$. Now let $v \in \operatorname{im} P \cap \ker P$. We get that for some $u \in V$ that $P(u) = v$ and $P^2(u) = P(v) = 0$ since $v \in \ker P$. But since $P^2(u) = P(u)$ we get $v = 0$. This shows $V = \operatorname{im} P \oplus \ker P$. We also know that $V = \operatorname{im} P \oplus \operatorname{im} P^\perp$. This shows that $\dim \operatorname{im} P^\perp = \dim \ker P$. Now we will show that $\operatorname{im} P^\perp \subseteq \ker P$. Let $v \in \operatorname{im} P^\perp$ We know that $P(v) \in \operatorname{im} P$ so:

$$\langle P(v), v \rangle = 0$$

This implies that:

$$0 = \langle P(v), v \rangle = \frac{1}{4}\left(\|P(v) + v\|^2 - \|P(v) - v\|^2 + i\|P(v) - v\|^2 - i\|P(v) + v\|^2\right)$$

This implies that:

$$\|P(v) + v\| - \|P(v) - v\| = 0$$

So using the reverse triangle identities we get:

$$0 \leq \|P(v)\| - \|v\| - \|P(v) - v\| \leq \|P(v) + v\| - \|P(v) - v\| = 0$$

So:

$$\|P(v)\| - \|v\| = \|P(v) - v\|$$

So from what we know $\|P(v)\| - \|v\|$ is a non-negative number and $\|P(v)\| \leq \|v\|$ which implies $\|P(v)\| - \|v\| = 0$ which gives:

$$\|P(v) - v\| = 0 \Rightarrow P(v) - v = 0 \Rightarrow P(v) = v$$

This shows that $v \in \operatorname{im} P$, and since $v \in \operatorname{im} P^\perp$ we know $v = 0$. But we assumed that $v \notin \ker P$ so this can't be the case, and we get a contradiction. Which means that $\operatorname{im} P^\perp \subseteq \ker P$ and we know $\dim \operatorname{im} P^\perp = \dim \ker P$ so $\operatorname{im} P^\perp = \ker P$. so P is an orthogonal projection on its own image.

15 The one with the weird inequality

Let $V = \mathbb{C}_3[x]$ with the inner product $\langle p(x), q(x)\rangle = \sum_{x=0}^{x=3} p(x)\overline{q(x)}$.

15.1 Find the minimal positive constant C such that for all $p \in V$:

$$\|p(i)\| \leq C\sqrt{\sum_{x=0}^{3} \|p(x)\|^2}$$

Notice that the following $\varphi \colon V \to \mathbb{C}$:

$$\varphi(p(x)) = p(i)$$

is a functional since for $\alpha \in \mathbb{C}$ and $p, q \in V$:

$$\varphi(\alpha p + q) = (\alpha p + q)(i) = \alpha p(i) + q(i) = \alpha\varphi(p) + \varphi(q)$$

Using riesz representation theorem we get that exists w such that:

$$\varphi(p) = p(i) = \langle p, w\rangle$$

Denote $w = a + bx + cx^2 + dx^3$, We see that for the basis $B = \{1, x, x^2, x^3\}$:

$$\begin{aligned}
1 = \varphi(1) = \langle 1, w\rangle = \overline{w(0)} + \overline{w(1)} + \overline{w(2)} + \overline{w(3)} &= w(0) + w(1) + w(2) + w(3) = 1 \\
i = \varphi(x) = \langle x, w\rangle = 0\overline{w(0)} + 1\overline{w(1)} + 2\overline{w(2)} + 3\overline{w(3)} &\Rightarrow w(1) + 2w(2) + 3w(3) = -i \\
-1 = \varphi(x^2) = \langle x^2, w\rangle = 0\overline{w(0)} + 1\overline{w(1)} + 4\overline{w(2)} + 9\overline{w(3)} &= w(1 + 4w(2) + 9w(3)) = -1 \\
-i = \varphi(x^3) = \langle x^3, w\rangle = 0\overline{w(0)} + 1\overline{w(1)} + 8\overline{w(2)} + 27\overline{w(3)} &\Rightarrow w(1) + 8w(2) + 27w(3) = i
\end{aligned}$$

Solving this system of equations gives:

$$(w(0), w(1), w(2), w(3)) = \left(\frac{5}{3}i, \frac{5-5i}{2}, -2+i, \frac{1}{2} - \frac{1}{6}i\right)$$

And now we can solve for $p(i)$ for any $p \in V$. By Cauchy-Schwartz we get:

$$\|p(i)\| = |\langle p, w\rangle| \leq \|p(x)\|\|w(x)\| = \sqrt{\sum_{x=0}^{3} \|p(x)\|^2}\sqrt{\sum_{x=0}^{3} \|w(x)\|^2}$$

And we see that:

$$\sqrt{\sum_{x=0}^{3} \|w(x)\|^2} = \frac{\sqrt{185}}{3}$$

Since we know that the CS inequality can also be an equality we get that this is the minimal constant such that the inequality is satisfied and then:

$$\boxed{C = \frac{\sqrt{185}}{3}}$$

16 The one with the invariance

Let V be a finite dimension inner product space and $T \in \mathrm{End}(V)$.

16.1 Show that $U \subseteq V$ is T-invariant iff $U^\perp$ is T^*-invariant

$\underline{U \text{ is } T\text{-invariant} \Rightarrow U^\perp \text{ is } T^*\text{-invariant:}}$
Since U is T-invariant we know that:

$$T(U) \subseteq U$$

Now suppose that $U^\perp$ is not T^*-invariant, that means that exists $u \in U^\perp$ such that $T^*(u) \notin U^\perp$, which means that:

$$\langle v, T^*(u) \rangle \neq 0$$

For some $v \in U$. This implies:

$$\langle T(v), u \rangle \neq 0$$

But since U is T-invariant we know that $T(v) \in U$, which implies that $u \notin U^\perp$ - that means that out assumption must be false so $U^\perp$ is T^*-invariant.
$\underline{U \text{ is } T\text{-invariant} \Leftarrow U^\perp \text{ is } T^*\text{-invariant:}}$
Since $U^\perp$ is T^*-invariant we know that:

$$T^*(U^\perp) \subseteq U^\perp$$

Now suppose that U is not T-invariant, that means that exists $u \in U$ such that $T(u) \notin U$, which means that:

$$\langle T(u), v \rangle \neq 0$$

For some $v \in U^\perp$. This implies:

$$\langle u, T^*(v) \rangle \neq 0$$

But since $U^\perp$ is T^*-invariant we know that $T^*(v) \in U^\perp$, which implies that $u \notin U$ - that means that out assumption must be false so U is T-invariant.

17 The one with $T*$

In the following sections find T^*

17.1 Let $(V, \langle , \rangle)$ be a finite dimension inner product space. Let $\alpha, \beta \in V$ and define $T = T_{\alpha,\beta} \in \mathrm{End}(V)$ as such:

$$T_{\alpha,\beta}(v) = \langle v, \alpha \rangle \beta$$

We see that for T defined as such:

$$\langle T(v), u \rangle = \langle \langle v, \alpha \rangle \beta, u \rangle = \langle v, \alpha \rangle \langle \beta, u \rangle = \langle v, \alpha \overline{\langle \beta, u \rangle} \rangle = \langle v, \alpha \langle u, \beta \rangle \rangle$$

And since we know that:

$$\langle T(v), u \rangle = \langle v, T^*(u) \rangle = \langle v, \alpha \langle u, \beta \rangle \rangle$$

We get that:

$$T^*(u) = \langle u, \beta \rangle \alpha$$

17.2 Let $V = (\mathrm{Mat}_n(\mathbb{F}), \langle , \rangle_{\mathrm{std}})$. Let $Q \in \mathrm{Mat}_n(\mathbb{F})$ be invertible and define $T = T_Q \in \mathrm{End}(V)$ as such:

$$T_Q(A) = QAQ^{-1}$$

We see that from properties of trace:

$$\begin{aligned}\langle T(A), B\rangle &= \langle QAQ^{-1}, B\rangle = \mathrm{tr}(QAQ^{-1}B^t) = \mathrm{tr}(B^tQAQ^{-1}) \\ &= \mathrm{tr}(Q^{-1}B^tQA) = \mathrm{tr}(AQ^{-1}B^tQ) = \langle A, (Q^{-1}B^tQ)^t\rangle\end{aligned}$$

And since we know that:

$$\langle T(A), B\rangle = \langle A, T^*(B)\rangle = \langle A, (Q^{-1}B^tQ)^t\rangle$$

We get that:

$$T^*(B) = (Q^{-1}B^tQ)^t = Q^tB(Q^{-1})^t$$

17.3 Let $Tv = J_n(\lambda)v$ for $V = \mathbb{F}_n$ with $\langle , \rangle_{\text{std}}$

We see that for T defined as such:

$$\langle T(v), u\rangle = \langle J_n(\lambda)v, u\rangle = (J_n(\lambda)v)^t u = v^t J_n(\lambda)^t u = \langle v, J_n(\lambda)^t u\rangle$$

And since we know that:

$$\langle T(v), u\rangle = \langle v, T^*(u)\rangle = \langle v, J_n(\lambda)^t u\rangle$$

We get that:

$$T^*(u) = J_n(\lambda)^t u$$

18 The one with the adjoint operator

Let $a \in \mathbb{C}, |a| \neq 1$ and let V be a finite dimension inner product space, $T \in \mathrm{End}(V)$

18.1 Show that if $T = aT^*$ then $T = 0$

We first see that T is normal since:

$$TT* = aT^*T^* = T^*aT^* = T^*T$$

This means that exists an orthonormal basis of eigenvectors of T which we shall denote $B = (v_1, \ldots, v_n)$ such that:

$$[T]_B = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

Where $\lambda_1, \ldots, \lambda_n$ denote the corresponding eigenvalues. We see that for all $1 \leq i \leq n$ that:

$$T(v_i) = \lambda_i v_i$$

But on the other hand that:

$$T(v_i) = aT^*(v_i)$$

We know from a theorem that if v_i is an eigenvector of T with eigenvalue λ_i then it is also an eigenvector of T^* with eigenvalue $\overline{\lambda_i}$ so we get:

$$\lambda_i v_i = a\overline{\lambda_i}v_i \Rightarrow \lambda_i = a\overline{\lambda_i}$$

And in particular that:

$$|\lambda_i| = |a\overline{\lambda_i}| \Rightarrow |\lambda_i| = |a||\overline{\lambda_i}|$$

But since also $|\lambda_i| = |\overline{\lambda_i}|$ we get:

$$|\lambda_i|(1 - |a|) = 0$$

And since $|a| \neq 1$ we get that $\lambda_i = 0$ which means that:

$$[T]_B = 0$$

So $T = 0$.

18.2 Show that if T is normal then $\ker T = \ker(T - aT^*)$

We can represent these tranformations and get that:

$$[T]_B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$
$$[T - aT^*)]_B = \operatorname{diag}(\lambda_1 - a\overline{\lambda_1}, \ldots, \lambda_n - a\overline{\lambda_n})$$

We know that the kernel of $v \in \ker(T)$ if and only if v is in the span of v_i with eigenvalue 0, and that $v \in \ker(T - aT^*)$ if and only if v is in the span of v_i with eigenvalue 0 but we see:

$$\lambda_i = 0 \Rightarrow \lambda_i = \overline{\lambda_i} = 0 \Rightarrow \lambda_i - a\overline{\lambda_i} = 0$$
$$\lambda_i - a\overline{\lambda_i} = 0 \Rightarrow \lambda_i = a\overline{\lambda_i} \Rightarrow |\lambda_i| = |a\overline{\lambda_i}| \Rightarrow |\lambda_i| = |a||\overline{\lambda_i}| \Rightarrow |\lambda_i|(1 - |a|) = 0 \Rightarrow \lambda_i = 0$$

Which shows that:

$$\lambda_i = 0 \iff \lambda_i - a\overline{\lambda_i} = 0$$

Which implies that the span of eignevectors from B with eigenvalue 0 in relation of T will also have eigenvalue 0 in relation to $T - aT^*$ so $\ker T = \ker(T - aT^*)$ as wanted.

19 The one with the matrix

Given:

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

19.1 Find an orthogonal matrix O and a diagonal matrix D such that $O^T A O = D$

We see that A is symmetric so it must also be normal. From the spectral theorem for normal transformations we know that exists a basis B to V such that B is an orthogonal basis in realtion to the standard inner product and also comprises of eigenvectors of A. To find that B we first will find the eigenvalues of A.

$$A = \left| \begin{pmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{pmatrix} \right| = -(\lambda+3)(\lambda+3)(\lambda-6) = 0$$

So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 6$. Now to find an orthogonal basis for $\ker(A-3)$ we do:

$$\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

And we find that:

$$\ker(A-3) = \left\{ a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

So we can choose the orthonormal basis to be:

$$B_1 = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{18}} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \right\}$$

Now for $\ker(A+6)$ we do:

$$\begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

And we find that:

$$\ker(A+6) = \left\{ a \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$$

So we can choose the orthonormal basis to be:

$$B_2 = \left\{ \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

We know that vectors of different eigenspaces are always orthogonal so we know that:

$$B = B_1 \cup B_2 = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{18}} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

And as we know from the unitary diagnolization theorem the orthogonal matrix that would diagonalize A is the matrix with these columns so:

$$O = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{1}{3} \end{pmatrix}$$

And D is just the matrix with the eigenvalues we found on the diagonal:

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

And:

$$O^T A O = D$$

20 The one with the prove disprove

Let T be an operator over a finite dimension inner product space. Prove or disprove the following:

20.1 T is unitary iff T is invertible and exists an orthonormal basis E such that $\|Te\| = 1$ for all $e \in E$

This is false. Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$:

$$T(1,0) = (1,0) \quad \text{and} \quad T(0,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

We can see that it is invertible, and exists the standard basis E which is orthonormal such that $\|T(e1)\| = \|T(e2)\| = 1$, yet if we consider $T(1,1)$ we see that:

$$\|(1,1)\| = \sqrt{2} \neq \sqrt{2+\sqrt{2}} = \left\|\left(1 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\| = \|T(1,1)\|$$

So we found a vector $v = (1,1)$ such that:

$$\|v\| \neq \|T(v)\|$$

Which means that T isn't unitary.

20.2 T is unitary iff $\|Tv\| = 1$ for all $v \in V$ such that $\|v\| = 1$

($\Rightarrow$)
Let T be unitary, then we know that for any $v \in V$ such that $\|v\| = 1$ that:

$$\|Tv\| = \|v\| = 1$$

($\Leftarrow$)
Suppose that v' is an eigenvector of T with eigenvalue λ. We can normalize v' and consider:

$$v = \frac{v'}{\|v'\|}$$

This vetcor is also an eigenvector of T with eigenvalue λ so

$$T(v) = \lambda v$$

But since $\|v\| = 1$ we also know that:

$$\|T(v)\| = \|\lambda\|\|v\| = 1 \Rightarrow \|\lambda\| = 1$$

And we know that if for any eigenvalue λ of T that $\|\lambda\| = 1$ then T is unitary. That means that we have just shown that T is unitary.

20.3 T is unitary iff for all orthonormal vectors v, u then Tv, Tu are also orthonormal

This is true. From the Gram-Schmidt theorem we know that exists $B = (v_1, \dots, v_n)$ an orthonormal basis for V, since any two vectors $u, v \in B$ are orthonormal we get that any $T(u), T(v) \in T(B)$ are also orthonormal. So the set $T(B)$ is also orthonormal. Suppose it werent linearly independent we get that exist $(a_1, \dots, a_n) \neq 0$ such that:

$$\sum_i a_i T(v_i) = 0$$

Using Parseval's identity we get that:

$$\left\| \sum_i a_i T(v_i) \right\| = \sqrt{\sum_i \|a_i\|} = \|0\| = 0$$

But this can only happen if $\forall i (a_i = 0)$ so $T(B)$ is linearly indepenedent and we got that T sends the orthonormal basis B to $T(B)$ an orthonormal basis. Let $v = \sum_i a_i v_i \in V$ we see that using Parseval's identity twice gives:

$$\|T(v)\| = \left\| T\left(\sum_i a_i v_i\right) \right\| = \left\| \sum_i a_i T(v_i) \right\| = \sqrt{\sum_i \|a_i\|} = \|v\|$$

We know that this is equivalent to T being unitary which completes the proof.

21 The one with the inequality

Let T be a operator over an inner product space V and let $TT^* = \alpha T + \beta I$ for some $\alpha, \beta \in \mathbb{R}$.

21.1 Show that $\alpha^2 + 4\beta \geq 0$

We can see that:

$$T^*T = T^{-1}(TT^*)T = T^{-1}(\alpha T + \beta I)T = \alpha T + \beta I = TT^*$$

So we get that T is normal so by the spectral theorem exists a basis of eigenvectors of T such that according to that basis T is diagonalizable. Let λ be an eigenvalue of T we get that:

$$\|\lambda\|^2 = \alpha\lambda + \beta$$

22 The one with the square root

Let T be a self-adjoint operator over a finite inner product space.

22.1 Prove that exist non-negative operators A, B such that:

$$T = A - B, \quad \sqrt{TT^*} = A + B, \quad AB = BA = 0$$

We know that if T is self-adjoint which implies it is unitary diagonalizable over $\mathbb{R}$, so exist $O \in O(n)$ and D diagonal such that:

$$O^T D O = [T]_C$$

For C the basis with the ith vector being the ith column of O. Since T is self-adjoint we know that all of eigenvalues are real. We can denote them by the entries of the main diagonal of D as such: $\lambda_i = D_{ii}$, and now we can define two matrices:

$$(A')_{ij} = \begin{cases} D_{ii} & i = j \land D_{ii} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

And:

$$(B')_{ij} = \begin{cases} -D_{ii} & i = j \land D_{ii} \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

And define the operator A, B as such:

$$A(v) = (O^T A' O)(v) \quad \text{and} \quad B(v) = (O^T B' O)(v)$$

We see that A, B are self-adjoint since $O^* = O^T$ and since all of their eigenvalues by construction are non-negative we know that they are non-negative operators. We may notice that:

$$A - B = O^T A' O - (O^T B' O) = O^T (A' - B') O = O^T D O = T$$

And also that:

$$\sqrt{TT^*} = \sqrt{O^T D D^* O} = O^T |D| O = O^T (A' + B') O = A + B$$

And since diagonal matrices commute under matrix multiplication and also $O^T = O^{-1}$ we see:

$$AB = BA = A'B' = 0$$

Since A' multiplies all the rows different than 0 in B and all the rows that are zero in a scalar. This completes the proof.

23 The one with the polynomial

Let T be a self-conjugate polynomial over the inner product space V, with eigenvalues $\lambda_1, \ldots, \lambda_n$.

23.1 For any $p(x) \in \mathbb{F}[x]$ show that the singular values of $p(T)$ are $|p(\lambda_i)|$ up to inner order.

Since $p(x)$ is a polynomial we can write:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

And:

$$\begin{aligned} p(T) &= a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0 I \\ p(T)^* &= (a_n T^n)^* + (a_{n-1}T^{n-1})^* + \cdots + (a_0 I)^* = \overline{a_n}(T^*)^n + \overline{a_{n-1}}(T^*)^{n-1} + \cdots + \overline{a_0} I \end{aligned}$$

Let λ be an eigenvalue asscociated with an eigenvector v of T. We see that:

$$\begin{aligned} p(T)(v) &= (a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0 I)(v) \\ &= a_n T(v)^n + a_{n-1}T(v)^{n-1} + \cdots + a_0 I(v) \\ &= a_n \lambda^n v + a_{n-1}\lambda^{n-1} v + \cdots + a_0 v \\ &= p(\lambda)(v) \end{aligned}$$

And:

$$\begin{aligned} p(T)^*(v) &= (\overline{a_n}(T^*)^n + \overline{a_{n-1}}(T^*)^{n-1} + \cdots + \overline{a_0} I)(v) \\ &= \overline{a_n}(T^*(v))^n + \overline{a_{n-1}}(T^*(v))^{n-1} + \cdots + \overline{a_0} v \\ &= \overline{a_n \lambda^n} v + \overline{a_{n-1}\lambda^{n-1}} v + \cdots + \overline{a_0} v \\ &= \overline{p(\lambda)} v \end{aligned}$$

So the eigenvalues of $p(T)^* p(T)$ are exactly $\overline{p(\lambda)}p(\lambda)$ which is exactly $\|p(\lambda)\|^2$. By SVD we know that the singular values of $p(T)$ are the square roots of the eigenvalues of $p(T)^* p(T)$, or in other words, the singular values of $p(T)$ are $\|p(\lambda_i)\|$ up to order.

24 The one with the operator norm

24.1 Show that $\|T^*T\|_{\text{op}} = \|T\|^2_{\text{op}}$

We know that:

$$\|T\|_{\text{op}} = \sup_{\|x\|=1} \|Tv\| = \sup_{\|x\|=1} \sqrt{\langle T(v), T(v)\rangle} = \sup_{\|x\|=1} \sqrt{\langle T^*T(v), v\rangle}$$

From this follows that:

$$\|T\|^2_{\text{op}} = \sup_{\|x\|=1} \langle T^*T(v), v\rangle$$

We may notice that $\langle T^*Tv, v\rangle$ is a non-negtive number since it's just the norm of $\langle Tv, Tv\rangle$ which means using Cauchy-Schwartz we get:

$$\|T\|^2_{\text{op}} = \sup_{\|x\|=1} \langle T^*T(v), v\rangle \leq \|T^*T\|\|x\| = \|T^*T\|$$

So we got that $\|T\|^2_{\text{op}} \leq \|T^*T\|$. To prove the other direction we recall that we saw in the rehearsal that T and T^* have the same singular values and in particular that:

$$\|T\|_{\text{op}} = \|T^*\|_{\text{op}}$$

So using this and properties of the norm we get:

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

From this and the other inequality we get:

$$\boxed{\|T^*T\|_{\text{op}} = \|T\|^2_{\text{op}}}$$

25 The one with the reflexive bilinear form

Let f be a reflexive bilinear form over a finite dimension V.

25.1 Show that if $\text{rank} f = r$ then exist $\phi_1, \tau_1, \ldots, \phi_r, \tau_r \in V^*$ such that:

$$f(x, y) = \phi_1(x)\tau_1(y) + \cdots + \phi_r(x)\tau_r(y)$$

We know that $\text{rank} f = r$ so if we denote $A = [f]_B$ we get that $\dim \ker(A) = n - r$. Denote the basis for the kernel at $B_k = \{e_{n-r+1}, \ldots, e_n\}$ and complete it to a basis for V as such $B = \{e_1, \ldots, e_n\}$ Now for each $u, v \in \text{span}\{e_1, \ldots, e_r\}$ we can denote:

$$u = \sum_{i=1}^{n} \alpha_i e_i$$
$$v = \sum_{i=1}^{n} \beta_i e_i$$

And now for any u, v we see:

$$f(u, v) = f\left(\sum_{i=1}^{n} \alpha_i e_i, v\right) = \sum_{i=1}^{n} \alpha_i f(e_i, v) = \sum_{i=1}^{r} \alpha_i f(e_i, v)$$

The last equality is true since we know that:

$$f(v, e_i) = [v]_B A [e_i]_B = [v]_B 0 = 0$$

And since f is reflexive we get $f(e_i, v) = 0$ as well. Let $\phi_i \in V^*$ where $1 \leq r \leq n$ be defined as:

$$\phi_i\left(\sum_j \alpha_j e_j\right) = \alpha_i$$

And:

$$\tau_i(v) = f(e_i, v)$$

These are trivially linear functionals. From the above calculations we see that:

$$f(u, v) = \sum_{i=1}^{r} \phi_i(u)\tau_i(v)$$

Which is what we wanted to prove.

26 The one where we show some things are unique

Let V be a finite dimension inner product space over $\mathbb{R}$, f be a bilinear form over V.

26.1 Show that exists a unique $T \in \mathrm{End}(V)$ such that:

$$f(u,v) = \langle u, T(v) \rangle\,, \quad \forall u, v \in V$$

We know by Gram-Schmidt that V has an orthonormal basis B which implies:

$$\langle v, u \rangle = \langle [v]_B, [u]_B \rangle_{\mathrm{std}}$$

So we need to show that exists a unique $T \in \mathrm{End}(T)$ such that:

$$f(u,v) = \langle [u]_B, [T(v)]_B \rangle_{\mathrm{std}}\,, \quad \forall u, v \in V$$

Let:

$$[T(v)]_B = [f]_B [v]_B \in \mathrm{End}(T)$$

We see that:

$$f(u,v) = [u]_B^* [f]_B [v]_B = [u]_B^* [T(v)]_B = \langle [u]_B, [T(v)]_B \rangle_{\mathrm{std}}$$

This shows that exists a T as wanted, we will now show it's unique. Let $S \neq T$ such that:

$$\langle u, T(v) \rangle = \langle u, S(v) \rangle$$

From this follows that:

$$\begin{aligned} \langle u, T(v) \rangle - \langle u, S(v) \rangle &= 0 \\ \Rightarrow \langle u, T(v) - S(v) \rangle &= 0 \\ \Rightarrow \langle u, (T-S)(v) \rangle &= 0 \end{aligned}$$

Since $T \neq S$ exists v' such that $(T-S)(v') \neq 0$ and for all $u \in V$ and specifivally for $T(v')$ we get:

$$\left\langle T(v'), (T-S)(v) \right\rangle = \left\langle T(v'), T(v') \right\rangle = \|T(v')\|^2 = 0$$

But since $T(v') \neq 0$ this can't be. This implies that T is indeed unique.

27 The one with the inner product

Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be symmetric and also satisfy:

$$(A^2 - 5A + 7I)^3 = I$$

27.1 Show that:

$$f(x, y) = x^T Ay$$

is an inner product on $\mathbb{R}^2$

To show that this is an inner product on $\mathbb{R}^2$ we need to show that f is positive-definite. Since A is symmetric and real it is self conjugate. By a theorem from class we know that if it is self conjugate and all of its eigenvalues are positive then A is positive definite and then f is an inner product. Let λ be an eigen value of A with a corresponding eigenvector v_λ such that:

$$Av_\lambda = \lambda v_\lambda$$

Since A satisfies the above equality we see that:

$$v_\lambda = Iv_\lambda = (A^2-5A+7I)^3 v_\lambda = (A^2-5A+7I)^2(\lambda^2 v_\lambda - 5\lambda v_\lambda + 7v_\lambda) = (A^2-5A+7I)^2(\lambda^2-5\lambda+7)v_\lambda$$

Consider the real polynimial $g(x) = x^2 - 5x + 7$. We see that its discriminant is $\sqrt{25 - 28}$ which means it doens't have any roots. Since the coefficient of x^2 is positive that means that $g(x) > 0$ for any real x and specifically that $g(\lambda) > 0$ which gives:

$$(A^2 - 5A + 7I)^3 v_\lambda = (A^2 - 5A + 7I)^2 g(\lambda)v_\lambda = (A^2 - 5A + 7I)g(\lambda)g(\lambda)v_\lambda = g(\lambda)g(\lambda)g(\lambda)v_\lambda$$

This implies that $1 = g(\lambda)^3$. The only real solution to that equation is $g(\lambda) = 1$, considering the equation $g(x) = 1$ we see:

$$g(x) = 1 \Rightarrow x^2 - 5x + 7 - 1 = 0 \Rightarrow (x - 2)(x - 3) = 0$$

So $\lambda = 2$ or $\lambda = 3$. This implies that all the eigenvalues of A are positive and as we said that implies that f is an inner product and completes the proof.

28 The one with equivalence

28.1 How many bilinear forms are there over $\mathbb{R}^2$ for which exists $0 \neq x \in \mathbb{R}^2$ such that $f(x,x) > 0$ up to isomorphism?

Let B be a bilinear form and E be a basis for $\mathbb{R}^2$. We know that each bilinear form defines a quadratic form q. We also know that any quadratic form can be represented by a symmetric matrix S_q. Since S_q is symmteric we can use Sylvester's law of inertia and get that each S_q is uniquely congruent to a matrix of the form:

$$I_{n_+} \oplus -I_{n_-} \oplus O_{n_0}$$

We now need to consider all the options that are not negative semi-definite so there would be an $x \neq 0$ such that $f(x,x) > 0$. Since we are talking about a 2×2 matrix here there are only 5 such options:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So there are exactly 5 bilinear forms over $\mathbb{R}^2$ for which exists $x \neq 0$ such that $f(x,x) > 0$ up to isomorphism.

29 The one with the two

Let f be a symmetric bilinear form over a real finite-dimension vector space V.

29.1 Prove that if $W \subseteq V$ is a subspace such that $f|_W$ is positive definite, then $\dim W \leq n_+(f)$

Denote $\dim(W) = k$ and let $B_W = (v_1, \dots, v_k)$ be a basis for W, and $B_v = (v_1, \dots, v_k, \dots v_n)$ be a basis for V. We know that f is a symmteric bilinear form, which implies that $[f]_B$ is symmetric. So by Sylvester's law of inertia we get that exists a diagonal matrix D and an invertible matrix S such that $[f]_B$ is congruent to D and:

$$S^T [f]_B S = D$$

We also know by the orthogonal diagonalization theorem for real symmetric matrices that exists $O \in O(n)$ such that:

$$O^T [f]_B O = D'$$

Where D is diagonal with the eigenvalues of $[f]_B$ on its diagonal. Since we know that $f|_W$ is positive definite that means that all of its eigenvalues are positive and moreover that $D'_{11}, \dots, D'_{kk}$ are the eigenvalues of W and thus positive. Since the positive values on the diagonal corresponds to $n_+(D')$ we get that $n_+(D') \geq \dim W$ and since Sylvester's character and the rank don't change between congruent matrices [2] we get that $n_+(f) \geq \dim W$ too, which is exactly what we wanted to prove.

[2] Notice that D and D' are congruent because congruency is an equivalence relation

29.2 Let $B = (b_1, \ldots, b_n)$ be a Sylvester basis such that:

$$[f]_B = I_{n_+} \oplus (-I_{n_-}) \oplus O_{n_0}$$

Does it necessarily follow that $W \subseteq \mathrm{sp}\{b_1, \ldots, b_{n_+}\}$

No. Let $V = \mathbb{R}^2$ and $E = \{e_1, 2e_2\}$ be a basis to $\mathbb{R}^2$ such that e_1, e_2 are the vectors from the standard basis and:

$$[f]_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We see that E is a Sylvester's basis but if we consider:

$$W = \mathrm{sp}\{(1,1)\}$$

Then W is indeed a linear subspace of V and if we let $w = (a, a) \in W$ we see that:

$$\langle [f]_B[w]_B, [w]_B \rangle = \begin{pmatrix} 2a & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2a \\ a \end{pmatrix} = 3a^2$$

And of course $f|_B$ is also symmetric so by a theorem it is positive definite, yet as we can easily see $W \nsubseteq \mathrm{sp}\{e_1\}$

30 The one where we prove... or disprove?

Let A be a symmetric real matrix of order $n \times n$ over V.

30.1 A is non-negative iff $\Delta_i(A) \geq 0$ for all $i = 1, \ldots, n$. Consider both directions

($\Leftarrow$)
This is false because we can look at the matrix over $\mathbb{R}$:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

We see that:

$$\Delta_1(A) = 0 \quad \text{and} \quad \Delta_2(A) = \det(A) = 0$$

But still we see that is is symmetric and it has a negative eigenvalue.
($\Rightarrow$)
Assume that A is non-negative. This clearly implies that any principle minor corresponding to $\Delta_i(A)$ is also non-negative, which means that all of its eigenvalues are non-negative. Since the determinant of any principle minor is the product of its eigenvalues we get that for all $i = 1, \ldots, n$ that $\Delta_i(A) \geq 0$ which is what we wanted to prove.

31 Past Tests

Find the Jordan normal form for the following matrix:

$$A := \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & -2 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in M_4(\mathbb{R})$$

And find an invertible P such that $P^{-1}AP$ is in Jordan normal form
First we will start by finding the eigenvalues of A by using the charecteristic polynomial:

$$\det(A - \lambda I) = (\lambda + 1)^4$$

So the eigenvalues are $\lambda = -1$ with an algebraic multiplicity of 4 and geometric multiplicity of 2. This implies that there are 2 Jordan blocks with eigenvalues of -1 and their sum is 4 which implies that:

$$\text{JordanNormalForm}(A) = J_2(-1) \oplus J_2(-1)$$

To find P we need to find the generalized eigenspaces of A in relation to $\lambda = -1$. We see that:

$$\ker(A+I) = \ker \begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{Sp}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$\ker(A+I)^2 = \ker \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathbb{R}^4$$

Now to find the Jordan basis we just need to choose v_2, v_4 to be linearly independent vectors in $\ker(A+I)^2$ that are not in $\ker(A+I)$. We can choose:

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

And now the Jordan base will be:

$$B_J = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

And since these are the columns of P we know that:

$$P = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Prove that the matrices A^{-1} and A^3 are similar.
We see that:

$$A^{-1} = PJ^{-1}P^{-1}$$
$$A^3 = PJ^{-3}P^{-1}$$

Whis means it is sufficient to prove that A^{-1} and A^3 are similar. Indeed we see that:

$$A^{-1} = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad A^3 = \begin{pmatrix} -1 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Both these matrices have the same eigenvalues with the same algebraic and geometric multiplicity so their Jordan normal form is identical which means they are similar.

Let V be an inner product space over $\mathbb{C}$ and let $T\colon V \to V$ a linear transformation such that:

$$(7I - T)T^* = 10I$$

Show that T is self-adjoint.
We get that:

$$T^* = \frac{TT^* + 10I}{7}$$

And now:

$$T = \frac{T^*T + 10I}{7}$$

Since we know that TT^* is always self adjoint we get that $T = T^*$ so T is self-adjoint.
Let λ be an eigenvalue of T. Show that $2 \leq \lambda \leq 5$
We know that all the eigenvalues of T are real since it is a normal operator and by the spectral theorem for normal operators exists a basis B of orthonormal eigenvectors of T to V. By that basis we get:

$$[T]_B = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

We can also notice that since the basis is made up by orthonormal vectors:

$$\text{diag}\{\overline{\lambda_1}, \dots, \overline{\lambda_n}\} = [T]_B^* = [T^*]_B = [T]_B = \text{diag}\{\lambda_1, \dots, \lambda_n\}$$

Which means that for any $1 \leq i \leq n$ we get $\lambda_i = \overline{\lambda_i}$ or in other words that $\lambda_i \in \mathbb{R}$. Suppose λ was an eigenvalue of $T$for $v \in V$. We get that:

$$\begin{aligned}(7I - T)T^*(v) &= 10I(v)\\ \lambda(7I - T)v &= 10v\\ \lambda(7v - \lambda v) &= 10v\\ 7\lambda v - \lambda^2 v - 10v &= 0\\ 7\lambda - \lambda^2 - 10 &= 0\\ (\lambda - 2)(\lambda - 5) &= 0\end{aligned}$$

This means that $2 \leq \lambda \leq 5$ as wanted.
Let $v \in V$ such that $\|v\| = 1$. Prove that $2 \leq \|T(v)\| \leq 5$.
It might be hard considering the norm only, but luckily we can use the inner product and see that:

$$\|T(v)\|^2 = \langle T(v), T(v)\rangle = \left\langle \sum_i \langle v, e_i\rangle Te_i, \sum_i \langle v, e_i\rangle Te_i \right\rangle = \sum_i \| \langle v, e_i\rangle \|^2 \|\lambda_i\|^2 \leq 25\|v\| = 25$$

We get that $\|T(v)\| \leq 5$ and similarly we can ge that $2 \leq \|T(v)\|$ and finish the proof.

Let A_1, A_2 be two real symmetrical and invertible matrics of order 2. Show that if they are not congruent exist D_1, D_2 diagonal and P invetible matrix such that:

$$P^T A_1 P = D_1$$
$$P^T A_2 P = D_2$$

Since A_1 and A_2 are real matrices of order 2 invetible and symmetrical, by Sylvester's law of inertia at least one of them must be congruent to $\pm I$. Which means that exists P invertible such that:

$$P^T A_1 P = \pm I \quad \text{and} \quad P^T A_2 P = B$$

We know that B is symmetrical and invertible so exists an orthogonal matrix Q that diagonlaizes it, and then we get:

$$(PQ)^t A_1 (PQ) = Q^T \pm IQ = \pm I = D_1$$
$$(PQ)^t A_2 (PQ) = Q^T BQ = D_2$$

Which is what we wanted to prove.

Define the following vector space over $\mathbb{R}$

$$V = \{A \in M_2(\mathbb{C}) | A = A^*\}$$

And also define $q\colon V \to \mathbb{R}$ as $A \mapsto 2\det(A)$. Show that q is a quadratic form and find its signature.

We can notice that:

$$V \left\{ \begin{pmatrix} a & b+ci \\ b-ci & a \end{pmatrix} \middle| a, b, c \in \mathbb{R} \right\}$$

Which means that:

$$q(A) = 2(a^2 - b^2 - c^2)$$

Notice that even though it may seem as if we are mapping a 2×2 matrix to a real number, we are basically sending the vector (a, b, c) from $\mathbb{R}^3$ to $2(a^2 - b^2 - c^2)$ and so the symmetrical bilinear form that defines q according to the basis:

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\}$$

is the matrix:

$$[f]_B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

This shows that q is a quadratic form and we can see that the signature of it is just $(1, 2)$ with rank 3.

Let V be a vector space over $\mathbb{C}$ and let $D, T \in \mathrm{End}(v)$ such that $TD = DT$ and D is diagonalizable. Prove that the eigenspaces of D are T-invariant.
Let λ be an eigenvalue of D. The corresponding eigenspace is:

$$V_\lambda = \{v \in V \mid D(v) = \lambda v\}$$

To show this space is T-invariant by definition we can consider $v \in V_\lambda$ and indeed:

$$D(T(v)) = T(D(v)) = T(\lambda v) = \lambda T(v)$$

Which implies that $T(v) \in V_\lambda$ as wanted.
Show that exists a basis B of V such that $[D]_B$ is diagonal and $[T]_B$ is in Jordan normal form.
In the previous exercise we have shown that the eigenspaces of D are T-invariant, and since we know they also span V we get that:

$$V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_k}$$

Is a direct product decomposition of T. From Jordan's theorem for every V_{λ_i} exists a basis B_i that changes T to a normal form, so we get that the ordered union $B = \bigcup B_i$ is a basis such that $[T]_B$ is in Jordan normal form and since it is made of eigenvectors of D it of course diagonalizes D as well.

For $\mathbb{R}^3$ coupled with the standard inner product we denote the norm and metric it induces as $|\cdot|$ and $d(\cdot,\cdot)$ correspondingly. Let:

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Find the value of $\max\{d(Av, v) \mid v \in \mathbb{R}^3 \wedge \|v\| = 1\}$
This is a clearly a question about SVD decomposition. We see that:

$$\max\{d(Av, v) \mid v \in \mathbb{R}^3 \wedge \|v\| = 1\} = \max\{\|(A - I)(v)\| \mid v \in \mathbb{R}^3 \wedge \|v\| = 1\}$$

We denote:

$$B = \begin{pmatrix} 3 & 3 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To find it's singular values we calculate:

$$B^T B = \begin{pmatrix} 13 & 5 & 0 \\ 5 & 13 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a block matrix so we can easily find that its eigenvalues are

$$(\lambda_1, \lambda_2, \lambda_3) = (18, 8, 1)$$

Thus the singular values of B are:

$$(\sigma_1, \sigma_2, \sigma_3) = (\sqrt{18}, \sqrt{8}, \sqrt{1})$$

So finally:

$$\max\{d(Av, v) \mid v \in \mathbb{R}^3 \wedge \|v\| = 1\} = \sqrt{18}$$

Find a unit vector $v \in V$ that gives that maximum
This is just the unit eigenvector corresponding to the eigenvalue 18 of $B^T B$. We can easily find one:

$$\tilde{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

And now normalizing it we get:

$$v = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Let $A \in U_n$. Prove that $|\mathbf{tr}(A)| = n$ if and only if A is a scalar matrix

We know that A is unitary so it is also normal and from the spectral theorem for normal matrices it is diagonalizable. Since it is unitary we also know that all of its eigenvalues are on the unit circle which means that:

$$|\mathrm{tr}(A)| = \left|\sum_i \lambda_i\right| \leq \sum_i |\lambda_i| = n$$

We know that the equality holds if and only if all the vectors are in the same direction. Thus $|\mathrm{tr}(A)| = n$ if and only if A is a scalar matrix.

Let $B \in O_n$. Prove that $|\mathbf{tr}(A)| = n$ if and only if A is $\pm I$

Since $A \in O_n$ it is also in U_n which implies that $|\mathrm{tr}(A)| = n$ if and only if A is a scalar matrix. We can see that:

$$n = |\mathrm{tr}(A)| = |\mathrm{tr}(cI)| = |c * n| = |n||c|$$

This implies that $|\mathrm{tr}(A)| = n$ if and only if A is $\pm I$.

Let V be a finite dimension vector space over $\mathbb{R}$ and let $\varphi, \phi\colon V \to \mathbb{R}$ be linear functionals. Define:

$$q(v) = \varphi(v)\phi(v)$$

Prove that $q(v)$ is a quadratic form on V.
We know that:

$$f(v,u) = \varphi(v)\phi(u)$$

Is a bilinear form, and since the quadratic form it induces is q we get that q is a bilinear form by definition. Usually when working with bilinear and quadratic forms it's convenient to find the symmetric bilinear form that induces q and it is:

$$g(v,u) = \frac{1}{2}\left(\varphi(v)\phi(u) + \varphi(u)\phi(v)\right)$$

Show that $n_+ \leq 1$ and also that $n_- \leq 1$
Denote $\dim(V) = n$. We know that φ, ϕ are linear functionals and thus,

$$\text{null}\varphi \geq n-1$$
$$\text{null}\phi \geq n-1$$

The intuition is that we want to eventually get to a matrix that has a block of 0_{n-2} so we can consider $K = \ker\varphi \cup \ker\phi$. We know that $\dim K \geq n-2$. Complete it with some space W to V as such:

$$V = W \oplus K$$

We can now make a basis for V as the union of the basis for W and the basis for K denoted $B = B_W \cup B_K$. The representing matrix of g with this basis will be:

$$B = [g|_{W\times W}]_{B_W} \oplus 0_{n-2}$$

This tells us that $n_- + n_+ \leq 2$. Suppose $n_+ = 2$ this would mean that $\dim W = 2$ and that for any $0 \neq w \in V - K$ that $q(w) > 0$ but that can't be the case since $\dim\ker\phi \geq n-1$ so we can choose a vector that is not in W such that $q(w) = 0$ in contradiction. Thus $n_-, n_+ \leq 1$ as wanted.
Find examples for functionals that gives all 4 options $(0,0), (1,0), (0,1), (1,1)$
We can choose the functionals for (n_+, n_-) such that:

$$\varphi(a,b) = n_+a + n_-b$$
$$\phi(a,b) = n_+a - n_-b$$

Let V be a vector space over $\mathbb{C}$ and $T \in \mathrm{End}(V)$. Prove or disprove that for any $0 \leq k \leq \dim V$ exists a subspace U that is T-invariant of dimension k.
We notice that the trivial subspace $\{0\}$ is T-invariant. Assuming we have invariant subspaces of dimension $m < k \leq n$ we will prove an invariant subspace of size k exists. Let U be an invariant subspace of size $k-1$. Complete U to V with W such that:

$$V = U \oplus W$$

If exists $w \in W$ such that $T(w) \in U$ than if we add w to U we get an invariant subspace of dimension k as wanted. Otherwise we have:

$$T(W) \subseteq W$$

We see now that $V = U \oplus W$ is a direct decomposition of T. Since we are working over $\mathbb{C}$ this means that exists at least one eigenvector in W which we denote w'. Since it is a eigenvector we have:

$$T(\mathrm{Sp}\{w'\}) \subseteq \mathrm{Sp}\{w'\}$$

And now if we add w' to U we get an invariant subspace of dimension k as wanted which completes the proof.
Let V be a vector space over $\mathbb{C}$ and $T \in \mathrm{End}(V)$. Prove or disprove that for any $0 \leq k \leq \dim V$ exists at most one subspace U that is T-invariant of dimension k. Take:

$$V = \mathbb{C}^2 \quad T = \mathrm{Id}$$

And the subspaces will be:

$$A_1 = \mathrm{Sp}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$
$$A_2 = \mathrm{Sp}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

It is clear that:

$$T(A_1) \subseteq A_1$$
$$T(A_2) \subseteq A_2$$

Which means we found two different T-invariant subspaces of dimension 1.

Let $V = \mathbb{C}_4[x]$. Define:

$$(T(p))(x) = p''(x^2)$$

Find the Jordan normal form of T and a Jordan basis.
We see that:

$$T(ax^4 + bx^3 + cx^2 + dx + e) = 12x^2 + 6bx + 2c$$

So the representing matrix of T under the standard basis is:

$$A = [T]_E = \begin{pmatrix} 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

To find its Jordan normal form we can first find its eigenvalues. We see that it is a lower triangular matrix and so its eigenvalues are $\lambda_1 = 0$ with algebraic multiplicity

$\lambda_1 = 0$	$\lambda_2 = 12$	
4	1	algebraic multiplicity
2	1	geometric multiplicity

Now to find the Jordan chains of λ_1 we get:

$$\ker(A - 12I) = \ker \begin{pmatrix} 12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} = \mathrm{Sp}\{e_4, e_5\}$$

$$\ker(A - 12I)^2 = \ker \begin{pmatrix} 144 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 0 \end{pmatrix} = \mathrm{Sp}\{e_4, e_5, e_3\}$$

$$\ker(A - 12I)^3 = \ker \begin{pmatrix} 1728 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{Sp}\{e_4, e_5, e_3, e_2\}$$

This means that there is at least one block of size 3 with eigenvalue λ_1. So considering the multiplicities from earlier we get that the Jordan normal form of A up to order is:

$$A = J_1(12) \oplus J_3(0) \oplus J_1(0)$$

Which means that the chain of λ_1 of length 3 is:

$$v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = Av_4 = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = Av_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 12 \end{pmatrix}$$

And the Jordan basis is:

$$B_J = \{e_1, 12e_5, 6e_3, e_2, e_4\}$$

Calculate $T^{100}(x^4 + x^3 + x^2 + x + 1)$
We can see that:

$$T^{100}(x^4 + x^3 + x^2 + x + 1) = T^{100}(x^3 + x^2 + x + 1) + T^{100}(x^4) = 0 + 12^{100}x^4 = 12^{100}x^4$$

Notice that there is also an easier way to solve the earlier problem by noticing that:

$$T(x^4) = 12x^4$$

So x^4 is an eigenvector with the eigenvalue 12, and we can also see that:

$$\begin{aligned}&T(x^3) = 6x^2 \quad T(6x^2) = 12 \quad T(12) = 0\\&T(x) = 0\end{aligned}$$

Which means that $(12, 6x^2, x^3), (x)$ are Jordan chains with eigenvalue 0 and then we get the same Jordan normal form and the same basis.

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{R}$. Let U_1, U_2 be two non-trivial subspaces of V. Define:

$$\angle(U_1, U_2) = \min\{\angle(u_1, u_2) \mid u_1, u_2 \neq 0 \wedge (u_1, u_2) \in U_1 \times U_2\}$$

Show that if P_1, P_2 are the orthogonal projections on U_1, U_2 then the first singular value of $P_1 \circ P_2$ is $\cos(\angle(U_1, U_2))$.
First we recall that the biggest singular value of $P_2 \circ P_1$ is:

$$\sigma_1 = \max\{\|P_2 \circ P_1(v)\| \mid \|v\| = 1\}$$

Let u_1, u_2 be the vectors where we attain the minimum for $\angle(U_1, U_2)$, without loss of generality we can assume they are unit vectors. Now by calculation we get:

$\cos(\angle(U_1, U_2)) = \cos(\angle(u_1, u_2)) = \cos(\angle(u_1, u_2))\|u_1\|\|u_2\| = \langle u_1, u_2 \rangle =_{(*)} \langle u_1, P_1(u_2) \rangle \leq_{\mathrm{CS}} \|u_1\|\|P_1(u_2)\| = \|P$

The equality $(*)$ is from the fact that $u_2 = P_1(u_2) + u^\perp$ where $u^\perp \in U_1^\perp$.

Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space of dimension 5 over $\mathbb{R}$. Let T be an invertible linear operator such that:

$$T^* = 3I - 2T^{-1}$$

And also $\mathbf{tr}(T) = 9$. Show that T is diagonalizble and find its diagonal form. First we can notice that T and $T*$ commute because T commutes with I and with T^-1 so T is normal. From the spectral theorem it follows that it has an orthonormal basis of eigenvectors such that T is diagonalizble and all of its eigenvalues are real. Let v be an eigenvector of T with eigenvalue λ. We know that v is also an eigenvalue of T^* and T^{-1} with eigenvalues $\overline{\lambda}$ and λ^{-1} accordingly. It follows that:

$$\overline{\lambda} = 3\lambda - 2\lambda^{-1}$$

Solving this equation we get:

$$\begin{aligned}
\overline{\lambda} &= 3 - 2\lambda^{-1} \\
\|\lambda\|^2 &= 3\lambda - 2 \\
0 &= \|\lambda\|^2 - 3\lambda + 2 \\
0 &= \lambda^2 - 3\lambda + 2, \quad \lambda \in \mathbb{R} \\
0 &= (\lambda - 1)(\lambda - 2)
\end{aligned}$$

So the possible eigenvalues of T are $\lambda_1 = 1$ and $\lambda_2 = 2$ and we only need find their multiplicities. Denote their multiplicities r_1, r_2 and we get:

$$\begin{cases} r_1 + 2r_2 = 9 \\ r_1 + r_2 = \dim V = 5 \end{cases}$$

The only solution in natrual numbers is $(r_1, r_2) = (1, 4)$ so the diagonal form of T is $\mathrm{diag}(1, 2, 2, 2, 2)$.

Let $A \in O(n)$. Prove that:

$$q_A(v) = \|v\|^2 \cos(\angle(v, Av))$$

Is a quadratic form.
We define the function:

$$f(u, v) = \langle u, Av \rangle = \|u\| \|Av\| \cos(\angle(u, Av))$$

It is clear that this is a bilinear form and we have:

$$f(v, v) = \|v\|^2 \cos(\angle(v, Av))$$

So $q_A(v)$ is indeed a quadratic form.
Now suppose that $n = 2$. For any $\theta \in [0, 2\pi]$ find the signature of the quadratic form q_A for the rotation matrix:

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

If the matrix is the rotation matrix in angle θ we get that:

$$q_A(v) = \|v\|^2 \cos(\angle(v, Av)) = \|v\|^2 \cos(\theta)$$

This means that if $x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ then $q_A(v) > 0$ for all $0 \neq v \in V$ so $(n_0, n_+, n_-) = (0, 2, 0)$ and similarly if $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ then $(n_0, n_+, n_-) = (0, 0, 2)$ and if $\theta \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ then $(n_0, n_+, n_-) = (2, 0, 0)$.
For θ values such that q_A is positive definite, find an orthonormal basis for the inner product that q_A is its sqaured norm.
We have seen that $q_A(v) = \cos(\theta)\|v\|^2$ so a symmetric bilinear form the induces it is $g(u, v) = \cos(\theta)\|v\|\|u\|$. In

Let $T \in End(V)$ where V is a complex vector space of finite dimension. Show that there is a finite amount of T-invariant subspaces iff $p_T(x) = m_T(X)$