Topology

Based on lectures by
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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction

Before getting into the main part of the course, we can first look at topology from the viewpoint it was considered hundreds of years ago.

Definition 1.1 (Geodesic triangle)**.** A geodesic triangle is the area enscribed inside 3 points on a sphere.

Theorem 1.1 (Girard's theorem)**.** *Let T be a geodesic triangle. Denote its angles θ_1, θ_2, θ_3. Then we have $\theta_1 + \theta_2 + \theta_3 > 180$ and*

$$\text{Area}(T) = \theta_1 + \theta_2 + \theta_3 - \pi.$$

Proof. Geometric. □

Theorem 1.2 (Euler's theorem)**.** *Let P be a convex polyhydron. Denote E the number of edges in P, V the number of vertices in P, and F the number of faces of P. Then*

$$V - E + F = 2.$$

Proof. Begin by ensphering the polyhydron in the unit sphere. Put a flashlight inside the polyhydron such that the faces of P cast shadows of geodesic polygons on the sphere.

Now we trianglize all the geodesic polygons. We have

$$\boxed{2E = 3F}$$

Denote the trianlges $\sigma_1, \ldots, \sigma_F$. From Theorem 1.1 we have that

$$\sum_{i=1}^{F} \frac{\theta_1^i + \theta_2^i + \theta_3^i}{V} - \pi = 4\pi.$$

From this we get

$$2\pi V - \pi F = 4\pi \implies \boxed{2V - F = 4}$$

From these equations we can deduce the desired relation

$$\boxed{V - E + F = 2}$$

which completes the proof. □

Example 1.1. box with a hold in the middle.

2 Quotient spaces and complexes

2.1 Quotient spaces

Let (X, τ) be a topological space, and let $\sim$ be an equivalence relation on X. Denote:

$$[x] := \left\{ y \in X \mid y \sim x \right\}.$$

Also define the function

$$\begin{array}{cccc} \pi & : & X & \longrightarrow & X/_{\sim} \\ & & x & \longmapsto & [x] \end{array}.$$

Definition 2.1 (Quotient topology)**.** We define on $X/_{\sim}$ the quotient topology $\tau_{\sim}$ by $U \in \tau_{\sim}$ if and only if $\pi^{-1}(U) \in \tau$.

In other words, the quotient topolgoy is the topology generated by π.

Proposition 2.1 (Universal property of quotient spaces)**.** *Let X, Y be topological spaces, let $\sim$ be an equivalence relation on X, and $f: X \to Y$ be a function such that for every $x \sim x' \in X$ we have $f(x) = f(x')$. Then there exists $f': X/_{\sim} \to Y$ such that $f' \circ \pi = f$ where π is the quotient projection. Moreover, f' is continuous if and only if f is continuous. In a diagram it looks like this:*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X/_{\sim} \arrow[r, dashed, "f'"'] & Y \end{tikzcd}$$

Proof. The first part of the proof is clear, so we will focus on the equivalence of continuity between f and f'. It is clear that π is continuous so when f' is continuous we also have that $f = f' \circ \pi$ is continuous.

Next suppose f is continuous. To be added ☐

Example 2.1. Let $X = [0, 1]$ with the topology induced by the standard topology on $\mathbb{R}$. Let $\sim$ be the equivalence relation generated by $0 \sim 1$. Thus,

$$X/_{\sim} \cong S^1 = \left\{ z \in \mathbb{C}^2 : |z| = 1 \right\}.$$

We define $f: X \to S^1$ by $f(t) = e^{2\pi i t}$. Since f is continuous and $f(0) = f(1)$, from the universal property of quotient spaces there exists a continuous function $f': X/_{\sim} \to S^1$ defined by $f'([x]) = f(x)$. It is clear that f is one to one and onto. To show that it is a homeomorphism we can use the following lemma

Lemma 2.2. *Let X, Y be topological spaces. Suppose that X is compact, Y is Hausdorff. Then every continuous function from X to Y is a closed transformation. In particular, if f is one to one and onto, it is a homeomorphism.*

Proof. Let C be a closed set in X. Since X is compact C is compact. Since f is continuous $f(C)$ is compact in the Hausdorff space Y and thus closed which completes the proof. ☐

Remark 2.1. Recall that the last part of the lemma is true because an open bijection is a homeomorphism.

Example 2.2. We will show that

$$X = {\mathbb{C} \setminus \{0\}} \big/ {x \sim \lambda x} \quad \forall 0\lambda \in \mathbb{R}$$

is homeomorphic to the unit sphere S^1. First define the function

$$\begin{array}{cccc} f & : & \mathbb{C} \setminus \{0\} & \longrightarrow & S^1 \\ & & z & \longmapsto & \left(\frac{z}{|z|}\right)^2 \end{array}$$

The function f is continuous and satisfies $f(x) = f(\lambda x)$ for all $0 \neq \lambda \in \mathbb{R}$. Thus, from the universal property of quotient spaces there exists a continuous function $f' \colon X \to S^1$. It is clear that f is one to one and onto. We have that $X = \pi(S^1)$ because every element in $C \setminus \{0\}$ is equivalent to some element in S^1, and since π is continuous and S^1 is compact, it follows that X is also compact. It is clear that S^1 is Hausdorff, and thus from the previous lemma we have that f' is a homeomorphism between X and S^1.

From now on we denote

$$\begin{aligned} D^n &:= \left\{x \in \mathbb{R}^n \colon \|x\| \leq 1\right\}; \\ S^n &:= \partial D^{n+1} = \left\{x \in \mathbb{R}^{n+1} \colon \|x\| = 1\right\}; \\ T^n &:= (S^1)^n. \end{aligned}$$

We can see that

$$D^0 = \{0\} \text{ and } \partial D^0 = \emptyset \text{ and } S^0 = \{-1, 1\}.$$

Example 2.3. We have that ${D^n} \big/ {S^{n-1}} \cong S^n$. The equivalence relation here is $x \sim y$ if and only if $x, y \in S^{n-1}$.

Example 2.4. We have that ${\mathbb{R}} \big/ {\mathbb{Z}} \cong S^1$ where $x \sim x + k$ for all $x \in \mathbb{R}$, $k \in \mathbb{Z}$.

Example 2.5. For all $x \in \mathbb{R}^n \setminus \{0\}$ and $\lambda \in (0, \infty)$ we have:

$$S^n \cong {\mathbb{R}^n \setminus \{0\}} \big/ {x \sim \lambda x}.$$

2.2 Pastings

The following lemma will allow us to discuss more pastings of spaces.

Lemma 2.3. *Let X be a topological space, $\sim$ an equivalence relation on X, let π the quotient projection, and f a continuous function such that*

(1) f is constant on the fibers of π.

(2) f' is one to one and onto.

(3) f is closed, open, or (${X} \big/ {\sim}$ is compact and Y is Hausdorff).

then f' is a homeomorphism.

Example 2.6. Let $X = [0, 1] \cup [2, 3]$ and $\sim := 1 \sim 2$. Then ${X} \big/ {\sim} = [0, 2]$. We can see this by defining the function $f \colon X \to [0, 2]$ as such

$$x \mapsto \begin{cases} x, & 0 \leq x \leq 1x - 1, \\ 2 \leq x \leq 3 \end{cases}.$$

Example 2.7. Define

$$\mathbb{R}_n^+ = \{(x_1, \dots, x_n) \mid x_i \geq 0\}$$
$$\mathbb{R}_n^- = \{(x_1, \dots, x_n) \mid x_i \leq 0\}$$

Define $X = \mathbb{R}_n^+ \cup \mathbb{R}_n^+$ (note that we treat this as a dijoint union). We define the equivalence relation $(x_1^+, \dots, x_{n-1}^+, 0) \sim (x_1^-, \dots, x_{n-1}^-, 0)$ and then we have $\mathbb{R}^n \cong X/\sim$.

Example 2.8. Let $X = I \times I$ where $I = [0, 1]$. We define the equivalence relation as such $(s, 0) \sim (s, 1)$ for all $0 \leq s \leq 1$. Then $X/\sim$ is homeomorphic to a cylinder. We can see this by defining the function

$$\begin{array}{cccc} f & : & I \times I & \longrightarrow & S \times I \\ & & (s,t) & \longmapsto & (\cos 2\pi t, \sin 2\pi t, s) \end{array}$$

Example 2.9. Let $X = I \times I$. Define the equivalence relation $(s, 0) \sim (1 - s, 1)$. The space $X/\sim$ is homeomorphic to a Mobius strip.

Example 2.10. Let $X = I \times I$. Define the equivalence relation $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, t)$. The space $X/\sim$ is homeomorphic to a torus (specifically T^2).

2.3 CW complexes

2.3.1 Disjoint union

Definition 2.2 (Disjoint union topology)**.** Let $\{X_\alpha\}_\alpha$ be a collection of topological spaces. We define the dijoint union topology in the following manner. We say that $U \subset \coprod_\alpha X_\alpha$ is open if and only if $U \cap X_\alpha$ is open in X_α.

Example 2.11. A disjoint union of points is an open set if every X_α is endowed with the discrete topology.

Example 2.12. We have that $D^1 \coprod D^1 \cong [0, 1] \cup [2, 3]$ with the standard topology.

Proposition 2.4 (Universal property of disjoint union)**.** *Let $\{X_\alpha\}_\alpha, Y$ be topological spaces, let $f_\alpha \colon X_\alpha \to Y$ be continuous functions. Then exists a unique continuous function $f \colon \coprod_\alpha X_\alpha \to Y$ such that $f_\alpha = f \circ i_\alpha$ where $i_\alpha \colon X_\alpha \to \coprod_\alpha X_\alpha$ is the injection map. In a diagram, it looks like this:*

$$\begin{tikzcd} \coprod_\alpha X_\alpha \arrow[dr, dashed, "\exists! f"] & \\ X_\alpha \arrow[u, "i_\alpha"] \arrow[r, "f_i"'] & Y \end{tikzcd}$$

2.3.2 CW complexes

Definition 2.3 (CW complex)**.** A CW complex is a topological space X with subspaces

$$\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X = \bigcup_n X^n$$

such that the spaces X^n are constructed inductively in the following way. Define

$$X^{-1} := \emptyset.$$

For $n \geq 1$ assume that X^{n-1} is defined. Let $\{D_\alpha^n\}_\alpha$ be a collection of n-dimensional discs, let $\{f_\alpha \colon \partial D_\alpha^n \to X^{n-1}\}_\alpha$ be a collection of continuous functions. Define:

$$X^n := \left(X^{n-1} \amalg \coprod_\alpha D_\alpha^n\right)/\sim$$

where $\sim$ is generated by $x \sim f_\alpha(x)$ for all α and for all $x \in \partial D^n$. We endow the space $X = \cup_n X^n$ with the topology such that $U \subset X$ open if and only if $U \cap X^n$ is open for every n. The space X is called a CW complex. The X^n subspace is called the n-skeleton of the complex. The pairs (D^n_α, f_α) are called the n-cells of the complex.

Remark 2.2. In most examples, the construction process of the complex is finite. In this case there exists n such that $X^n = X$. We call n the dimension of the complex. In particular, the n-skeleton X^n is a CW complex of dimension n (at most).

Example 2.13. A 0-dimensional complex X is a discrete set. The 0-cells are called the vertices of X.

Example 2.14. A 1-dimensional complex X is a topological graph. The 1-cells are called the edges of X. Each edge connects to vertices in its extremes.

Remark 2.3. Notice that a topological graph is not a simple graph — each edge can connect a vertice to itself, and more than a single edge can connect the same vertices.

Example 2.15. We describe a CW complex for the n-dimensional sphere S^n. Let $X^0 = \{*\}$ be with a single vertice, and a single n-cell D^n_α. Since there are no cells of dimensions $0 < i < n$ we get $X^{n-1} = \{*\}$. The pasting map of the cell D^n_α is the constant map $F_\alpha \colon \partial D^n_\alpha \to X^{n-1} = \{*\}$, and indeed we have $S^n = X^n = \{*\} \coprod D^n_\alpha /\!\sim$ where $x \sim *$ for all $x \in \partial D^n_\alpha$.

Example 2.16. Using the construction from the previous example, we can construct a CW complex for the torus $T^2 = S^1 \times S^1$ in the following way. Let v be a vertice, a, b be edges, and s be 2-cell. We connect the 2-cell by TO BE CONTINUED

In general, we only require the pasting maps of CW complexes to be continuous, but in practice we will only consider very nice pasting maps. For example pasting faces of polygons.

Definition 2.4 (Polygonal complex)**.** A polygonal complex is a CW complex of dimension 2 such that the pasting maps of the 2-cells identify the 2-cell as a (regular) polygon, and paste each of its edges to an edge in X^1.