Complex Analysis - Problems

Based on lectures by

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Winter 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Holomorphic functions

Exercise 1.1. Let $\Omega \subset \mathbb{C}$ be an open connected set. Let

$$\mu = \left\{(x, y) \in \mathbb{R}^2 \mid x + yi \in \Omega\right\}.$$

Let $f = u + iv$ be a holomorphic function. Suppose f has continuous partial derivatives in μ. Show that each of the following requirements imply that f is bounded.

(1) $\arg(f(z))$ is constant in Ω.

(2) $\Re(f)^2 = \Im(f)^3$ in Ω.

Solution. (1) because $\arg(f(z))$ is constant there exist $a, b \in \mathbb{R}$ different not both 0 such that

$$au(x, y) + bv(x, y) = 0.$$

From the Cauchy–Riemann equations we get that

$$\begin{cases} au_x + bv_x = 0 \\ au_y + bv_y = 0 = -av_x + bu_x \end{cases}$$

This is just like saying that

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We have that

$$\det \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = -a^2 - b^2 \neq 0$$

therefore $u_x - v_x = 0$. And then from C–R we have $u_y = v_y = 0$.

(2) We have that $u^2 = v^3$. Taking the derivatives we get

$$\begin{cases} 2uu_x + 3v^2 v_x = 0 \\ 2uu_y + 3v^2 v_y = 0 \end{cases}$$

and then we have

$$2uu_y u_x = 3v^2 v_x v_x \underset{C-R}{\Longrightarrow} 2u(u_x^2 + u_y^2) = 0.$$

This implies that either $u = 0$ or $u_x = u_y = 0$ which implies from C–R that $v_x = v_y = 0$. We denote

$$\mu_0 = \left\{(x, y) \in \mu \mid u = 0\right\}$$
$$\mu \setminus \mu_0 = \left\{(x, y) \mid u \neq 0\right\}.$$

TO BE CONTINUED

Let $z = x + iy$, $\bar{z} = x - iy$. We then have $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

Definition 1.1 (Wirtinger derivative)**.** We define the Wirtinger derivative as

$$\frac{\partial}{\partial z} := \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y} \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

Exercise 1.2. Let $f = u + iv$, u, v have continuous partial derivatives. Show that

(1) If f is holomorphic, then

$$\frac{\partial f}{\partial z} = f'(z).$$

(2) f is holomorphic if and only if

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

(3)

Solution. (1) We have

$$\frac{\partial f}{\partial z} := \left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}\right)(u+iv) = \frac{u_x + iv_x}{2} + \frac{1}{2}(v_y - iu_v) = f'(z)$$

(2) We have

$$\frac{\partial f}{\partial \bar{z}} := \left(\frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}\right)(u+iv) = \frac{u_x - v_y}{2} + i\frac{u_y + v_x}{2} = 0$$

We know that f is holomorphic if and only if it satisfies the C–R equations if and only if $\frac{\partial f}{\partial \bar{z}} = 0$ which completes this part.

Remark 1.1.

$$f\bar{z}$$