1 Metric Spaces

Exercise 1.1. Let $\mathbb{R}^n$ be the Euclidean space with the Euclidean metric.

(1) Show that any set in $\mathbb{R}^n$ is open if and only if it is a union of open balls.

(2) Show that any set in $\mathbb{R}^n$ is open if and only if it is a countable union of open balls.

Solution.

(1) Suppose that $A \subset \mathbb{R}^n$ is a union of open balls, then since each open ball is an open set, and the set of open sets is closed under unions, the set A is open.

Next suppose that $A \subset \mathbb{R}^n$ is open. Since A is open, for each $a \in A$ there exists r_a such that

$$(*)\, B(a, r_a) \subset A.$$

Consider the set

$$B := \bigcup_{a \in A} B(a, r_a).$$

Let $a \in A$. Then there exists $B(a, r_a)$ such that $a \in B(a, r_a) \subset B$. Then $a \in B$. Then there exists $B(a, r_a)$ such that $a \in B(a, r_a) \underset{(*)}{\subset} A$. Therefore $A = B$ which means that A is a union of open balls which completes the proof.

(2) Let A be a countable union of open balls. It is clear from (1) that it is open. Next suppose that A is open. Denote

$$\mathcal{Q} := \left\{ x \in A \mid x \in \mathbb{Q}^n \right\}$$

We have that

$$|\mathcal{Q}| \leq |\mathbb{Q}^n| = \aleph_0$$

so $\mathcal{Q}$ is countable. For each $x \in \mathcal{Q}$ denote

$$\mathcal{B}_x := \left\{ B(x, q) \mid q \in \mathbb{Q} \text{ and } B(x, q) \subset A \right\}.$$

It is clear that $\mathcal{B}_x$ is countable. Therefore

$$B := \bigcup_{\substack{x \in \mathcal{Q} \\ B(x,q) \in \mathcal{B}_x}} B(x, q)$$

is a countable union of open balls.

Let $x \in B$. Then there exists $x' \in \mathcal{Q}$ and $B(x', q) \in B_{x'}$ such that $x \in B(x, q) \subset A$. Let $x \in A$. Since A is open, there exists $\delta > 0$ such that $B(x, \delta) \subset A$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ there exists $q \in \mathbb{Q}^n$ such that $d(q, x) < \delta/2$, which implies that $q \in B(x, \delta/2) \subset A$ so we also have $q \in \mathcal{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we can choose $d(x, q) < \epsilon < \delta/2$ such that $\epsilon \in \mathbb{Q}$. We notice that $B(q, \epsilon) \subset B(x, \delta) \subset A$ and that $x \in B(q, \epsilon) \in B$. Therefore $A = B$ which implies that A is a countable union of open balls which completes the proof.