Complex Analysis

Based on lectures by
Notes taken by yehelip

Winter 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction

1.1 Complex numbers and the complex plane

1.1.1 Preliminaries

Definition 1.1 (Complex number)**.** A complex number is an expression of the form $x+yi$ such that $x, y \in \mathbb{R}$ and i is a 'imaginary number' not in $\mathbb{R}$. We denote

$$\Re(z) := x \quad \text{and} \quad \Im(z) := y.$$

If $\Im(z) = 0$ then z is said to be a real number, and if $\Re(z) = 0$ then it is said to be purely imaginary.

The set of all complex numbers is denoted as $\mathbb{C}$ and it can be made into a field with the following operations.

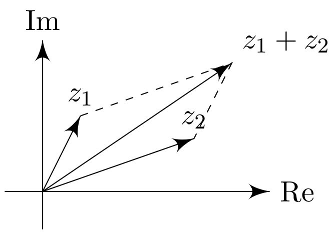
$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \quad \text{and} \quad z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The field $\mathbb{C}$ is called the complex plane.

Note that $i^2 = -1$. Also note that $T(x + yi) = (x, y)$ is a bijection between $\mathbb{C}$ and $\mathbb{R}$ and moreover, we have that T is additive. That is

$$T(z_1 + z_2) = T(z_1) + T(z_2)$$

which gives complex addition a geometric meaning.



The absolute value of a complex number $x + yi = z \in \mathbb{C}$ is defined by

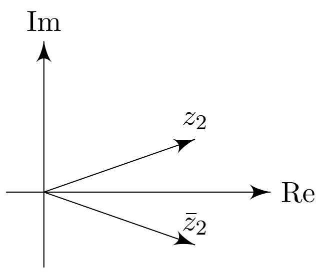
$$|z| = \sqrt{x^2 + y^2}.$$

Note that $|z| = \|(x, y)\| = \|T(z)\|$ where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^2$.

This implies that $|z - w|$ should be considered the distance between natural numbers z, w. Because we have that $|z| = \|T(z)\|$ we also have that the triangle inequality holds:

$$|z + w| \leq |z| + |w| \text{ for all } z, w \in \mathbb{C}.$$

Definition 1.2 (Complex conjugate)**.** The complex conjugate of $x+yi = z \in C$ is the complex number $x - yi$. The complex conjugate of z is denoted $\bar{z}$.



It is easy to verify that

$$\Re(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \Re(z) = \frac{z - \bar{z}}{2i} \quad \text{and} \quad |z|^2 = z\bar{z}.$$

Given θ we can denote $e^{i\theta} = \cos\theta + i\sin\theta$, and then describe any complex number $z \in \mathbb{C}$ as $re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and $r > 0$. We get that $|z| = |re^{i\theta}| = r$. We also have that θ describes the angle of z with the x-axis and it is usually denoted $\theta = \arg(z)$.

1.1.2 Convergence

Definition 1.3 (Convergence)**.** We say that the sequence $\{z_n\}_{n\geq 1} \subset \mathbb{C}$ converges to some $z_0 \in \mathbb{C}$ if $|z - z_0| \xrightarrow{n\to\infty} 0$. In this case, we call z_0 the limit of the sequence of $\{z_n\}_{n\geq 1}$.

Remark 1.1. It is easy to verify that the limit is unique, and that $z_n \xrightarrow{n\to\infty} z$ if and only if $T(z_n) \xrightarrow{n\to\infty} T(z)$ in the Euclidean metric.

Definition 1.4 (Cauchy sequence)**.** A sequence $\{z_n\}_{n\geq 1}$ is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists $N > 1$ such that for all $n, m > N$ we have that $|z_n - z_m| < \epsilon$.

Proposition 1.1. *The complex plane $\mathbb{C}$ is complete. That is, every Cauchy sequence converges in $\mathbb{C}$.*

Proof. The proof follows immediately from the known fact that $\mathbb{R}$ is complete and the previous remark. $\square$

1.1.3 Sets in the complex plane

Definition 1.5 (Open disc)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$D_r(z_0) := \left\{z \in \mathbb{C} \colon |z - z_0| < r\right\}.$$

We call $D_r(z_0)$ the open disc at center z_0 with radius r.

Definition 1.6 (Closed disc)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$\overline{D}_r(z_0) := \left\{z \in \mathbb{C} \colon |z - z_0| \leq r\right\}.$$

We call $\overline{D}_r(z_0)$ the closed disc at center z_0 with radius r.

Definition 1.7 (Circle)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$C_r(z_0) := \left\{z \in \mathbb{C} \colon |z - z_0| = r\right\}.$$

We call $C_r(z_0)$ the circle at center z_0 with radius r.

Definition 1.8 (Interior point)**.** Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if exists $r > 0$ such that $D_r(z) \subset \Omega$.

Definition 1.9 (Interior of a set)**.** Given $\Omega \subset \mathbb{C}$, we say that the interior of Ω is the collection of all interior points of Ω. We denote the interior as $\mathrm{Int}(\Omega)$.

Definition 1.10 (Open set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω is an open set if $\mathrm{Int}(\Omega) = \Omega$.

Definition 1.11 (Closed set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω a closed set if $\Omega^c := \mathbb{C} \setminus \Omega$ is open.

Definition 1.12 (Limit point)**.** Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if there exists a sequence z_n such that $z_n \neq z$ for all $n > 1$ and $z_n \xrightarrow{n\to\infty} z$.

Proposition 1.2. *Let $\Omega \subset \mathbb{C}$ be given. Then Ω is closed if and only if it contains all of its limit points.*

Proof. Clear. □

Definition 1.13 (Closure)**.** Let $\Omega \subset \mathbb{C}$ be given. The closure of Ω, denoted $\overline{\Omega}$, is defined as

$$\overline{\Omega} = \Omega \cup \left\{ z \in \mathbb{C} \mid x \text{ is a limit point of } \Omega \right\}.$$

Remark 1.2. Note that Ω is closed if and only if $\overline{\Omega} = \Omega$.

Definition 1.14 (Boundary)**.** The boundary of $\Omega \subset \mathbb{C}$ is denoted by $\partial\Omega$ and defined by $\partial\Omega := \Omega \setminus \mathrm{Int}(\Omega)$.

Definition 1.15 (Diameter)**.** Given $\Omega \subset \mathbb{C}$, we define the diameter of Ω as

$$\mathrm{diam}(\Omega) := \sup \left\{ |z - w| \colon z, w \in \Omega \right\}.$$

Definition 1.16 (Bounded set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω is bounded if $\mathrm{diam}(\Omega) < \infty$.

Remark 1.3. It is clear that a set $\Omega \subset \mathbb{C}$ is bounded if and only if there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $\Omega \subset D_r(z_0)$.

Definition 1.17 (Compact set)**.** A subset Ω of $\mathbb{C}$ is said to be compact if it is closed and bounded.

Theorem 1.3. (Bolzano–Weierstrass theorem). *A subset Ω in $\mathbb{C}$ is compact if and only if every seqeuence $\{z_n\}_{n\geq 1}$ has a subsequence $\left\{z_{n_k}\right\}$ such that $z_{n_k} \xrightarrow{k\to\infty} z$ for some $z \in \mathbb{C}$.*

Theorem 1.4. (Cantor's intersection lemma). *Let $\Omega_1, \Omega_2, \ldots$ be nonempty compact subsets of $\mathbb{C}$. Suppose that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 1$, and that $\mathrm{diam}(\Omega_n) \xrightarrow{n\to\infty} 0$. Then $\cap_{n\geq 1}\Omega_n = \{z\}$ for some $z \in \mathbb{C}$.*

Proof. Choose $z_n \in \Omega_n$ for all $n \geq 1$. Because $\mathrm{diam}\,\Omega \xrightarrow{n\to\infty}= 0$ we have that $\{z_n\}_{n\geq 1}$ is a Cauchy sequence and therefore it converges to some $z \in \mathbb{C}$. Because Ω_n is compact for every $n \geq 1$ we get that $z \in \cap_{n\geq 1}\Omega_n$. This means that $\cap_{n\geq 1}\Omega_n \neq \emptyset$.

Let $z, w \in \Omega$. Because $\mathrm{diam}\,\Omega \xrightarrow{n\to\infty} 0$ we have that $|z - w| \leq 0$ and thus $z = w$ which implies that $\cap_{n\geq 1}\Omega_n = \{z\}$ which completes the proof. □

Definition 1.18 (Connected open set)**.** A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty open subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

Definition 1.19 (Region)**.** A connected open set in $\mathbb{C}$ will be called a region.

Definition 1.20 (Connected closed set)**.** A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty closed subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

Remark 1.4. It can be shown that Ω is connected if and only if for any $z, w \in \Omega$ there exists a curve $\gamma \colon [0, 1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1)$. This implies that open and closed discs, as well as circles, are connected.

1.1.4 Continuous functions

Definition 1.21 (Continuous function)**.** Let Ω be a nonempty subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is continuous at a point $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that $|f(z) - f(z_0)| < \epsilon$ for all $z \in \Omega$ with $|z - z_0| < \delta$. We say that f is continuous on Ω if it is continuous at every $z_0 \in \Omega$.

Remark 1.5. It is easy to verify that the functions $\Im$, $\Re$, $|\cdot|$, and $\theta \mapsto e^{i\theta}$ are all continuous.

Proposition 1.5. *The composition of continuous functions is continuous.*

Definition 1.22 (Bounded function)**.** Let Ω be a nonempty subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is bounded if there exists $M > 0$ so that $|f(z)| < M$ for all $z \in \Omega$. We say that f attains a maximum if there exists $z_M \in \Omega$ such that $f(z) \leq f(z_M)$ for all $z \in \Omega$. We define when f attains a minimum similarly.

Proposition 1.6. *Let Ω be a nonempty compact subset of $\mathbb{C}$, and let $f\colon \Omega \to \mathbb{C}$ be continuous. Then f is bounded, and it attains its maximum and minimum on Ω.*

1.2 Holomorphic functions

Definition 1.23 (Holomorphic function)**.** Let Ω be a nonempty open subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is holomorphic at a point $z \in \Omega$ if the following limit exists

$$f'(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

The number $f'(z)$ is called the derivative of f at z. It is said that f is holomorphic if it is holomorphic at every $z \in \Omega$. Given a closed subset $C \subset \Omega$, we say that f is holomorphic on C if there exists $C \subset \Omega' \subset \Omega$ so that Ω' is open and f is holomorphic on Ω'.

Definition 1.24 (Entire function)**.** We say that $f\colon \mathbb{C} \to \mathbb{C}$ is entire if it is holomorphic on $\mathbb{C}$.

Remark 1.6. Note that h is a complex number and can approach 0 from any direction.

Remark 1.7. It is also useful to notice that $f\colon \Omega \to \mathbb{C}$ is holomorphic at $z \in \Omega$ if and only if there exist $a \in \mathbb{C}$, $r > 0$ with $D_r(z) \subset \Omega$, and a function $\psi\colon D_r(0) \to \mathbb{C}$ with $\lim_{h\to 0} \psi(h) = 0$, so that

$$f(z+h) = f(z) + ah + h\psi(h) \text{ for all } h \in D_r(0).$$

From this formulation is it clear that f is continuous at z whenever f is holomorphic at z.

Example 1.1. It follows directly from the definition that the function $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$ with $f'(z) = -1/z^2$. For all $0 \neq z \in \mathbb{C}$ we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{z+h} - \frac{1}{z}\right) = \lim_{h\to 0} \frac{-1}{z(z+h)} = \frac{-1}{z^2}.$$

Example 1.2. The function $f(z) = \bar{z}$ is not holomorphic. For any $z \in \mathbb{C}$ and $r \in \mathbb{R}$ we have that

$$\frac{f(z+t) - f(z)}{t} = 1 \quad \text{and} \quad \frac{f(z+ti) - f(z)}{ti} = -1$$

Proposition 1.7. *Let $\Omega \subset \mathbb{C}$ be open and let $f, g\colon \Omega \to \mathbb{C}$ be holomorphic at $z \in \Omega$. Then*

(1) $f + g$ is holomorphic at z with $(f+g)'(z) = f'(z) + g'(z)$.

(2) fg is holomorphic at z with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

Proof. We will only prove (2) because the proof of (1) is much simpler. Because f and g are holomorphic at z, they are also continuous there. Thus,

$$\begin{aligned}\lim_{h\to 0}\frac{(fg)(z+h)-(fg)(z)}{h} &= \lim_{h\to 0}\left(\frac{f(z+h)-f(z)}{h}g(z+h)+f(z)\frac{g(z+h)-g(z)}{h}\right)\\ &= f'(z)g(z)+f(z)g'(z),\end{aligned}$$

which completes the proof. □

Corollary 1.8. *It's quite easy to prove that constant function of the form $f(z)=c$ for some $c\in\mathbb{C}$ and $f(z)=z$ are holomorphic. It follows immediately from Proposition 1.7 that all polynomials, functions of the form $p(z)=\sum_{k=0}^{n}a_kz^k$ are entire, with $p'(z)=\sum_{k=1}^{n}ka_kz^{k-1}$ for all $z\in\mathbb{C}$.*

Proposition 1.9. *A composition of holomorphic functions at z is holomorphic at z, with $(g\circ f)'(z)=g'(f(z))f'(z)$.*

Corollary 1.10. *Let $\Omega\subset\mathbb{C}$ be open and let $f,g\colon\Omega\to\mathbb{C}$ be holomorphic at $z\in\Omega$. Suppose also that $g(z)\neq 0$. Then f/g is holomorphic at z with*

$$(f/g)'(z)=\frac{f'(z)g(z)+f(z)g'(z)}{g(z)^2}.$$

Proof. Let $h\colon\mathbb{C}\setminus\{0\}\to\mathbb{C}$ be with $h(z)=1/z$. We now have that

$$\begin{aligned}(f/g)'(z)&=(f\cdot(h\circ g))'(z)=f'(z)(h\circ g)(z)+f(z)(h\circ g)'(z)\\ &=f'(z)/g(z)+f(z)h'(g(z))g'(z)=f'(z)/g(z)-f(z)g(z)^{-2}g'(z).\end{aligned}$$

□

Recall that $T\colon\mathbb{C}\to\mathbb{R}^2$ is the operator $T(x+yi)=(x,y)$.

Proposition 1.11. *Let $\Omega\subset\mathbb{C}$ be open, let $f\colon\Omega\to\mathbb{C}$, let $u,v\colon T(\Omega)\to\mathbb{R}$ be with $f(x+yi)=u(x,y)+iv(x,y)$ for $x+iy\in\Omega$, and let $F\colon T(\Omega)\to\mathbb{R}^2$ be with $F(x,y)=(u(x,y),v(x,y))$ for $(x,y)\in T(\Omega)$. Fix $x_0+iy_0=z_0\in\Omega$, write $p=(x_0,y_0)$, and suppose that f is holomorphic at z_0. Then,*

(1) the partial derivatives of u and v exist at p, and

$$f'(z_0)=\frac{\partial u}{\partial x}(p)+i\frac{\partial v}{\partial x}(p)=\frac{\partial v}{\partial y}(p)-i\frac{\partial u}{\partial y}(p);$$

(2) The Cauchy–Riemann equations are satisfied:

$$\frac{\partial u}{\partial x}(p)=\frac{\partial v}{\partial y}(p)\ \text{and}\ \frac{\partial u}{\partial y}(p)=-\frac{\partial v}{\partial x}(p).$$

(3) F is differentiable at p with,

$$dF_p=\begin{pmatrix}\frac{\partial u}{\partial x}(p) & -\frac{\partial v}{\partial x}(p)\\ \frac{\partial v}{\partial x}(p) & \frac{\partial u}{\partial x}(p)\end{pmatrix}=\begin{pmatrix}\frac{\partial v}{\partial y}(p) & \frac{\partial u}{\partial y}(p)\\ -\frac{\partial u}{\partial y}(p) & \frac{\partial v}{\partial y}(p)\end{pmatrix}.$$

Remark 1.8. Note that $u=\Re\circ f\circ T^{-1}$, $v=\Im\circ f\circ T^{-1}$ and $F=T\circ f\circ T^{-1}$. Thus, F is the map corresponding to f under the identification of $\mathbb{C}$ with $\mathbb{R}^2$ via T.

Remark 1.9. Note that from (3) we have

$$\det(dF_p) = \left(\tfrac{\partial u}{\partial x}(p)\right)^2 + \left(\tfrac{\partial v}{\partial x}(p)\right)^2.$$

From this and from (1), it follows that $\det(dF_p) > 0$ whenever $f'(z_0) \neq 0$. Moreover, we have that $\sqrt{\det(dF_p)} \cdot dF_p$ is an orthogonal matrix.

We now prove Proposition 1.11.

Proof.

(1) We can first let $t \to 0$ in $\mathbb{R}$ and see that

$$\begin{aligned} f'(z_0) &= \lim_{t\to 0} \frac{f(z+t) - f(z)}{h} \\ &= \lim_{t\to 0} \frac{u(x_0+t, y_0) - u(x_0, y_0) + iv(x_0+t, y_0) - iv(x_0, y_0)}{t} \\ &= \frac{\partial u}{\partial x}(p) + i\frac{\partial v}{\partial x}(p). \end{aligned}$$

Similarly,

$$\begin{aligned} f'(z_0) &= \lim_{t\to 0} \frac{f(z+it) - f(z)}{h} \\ &= \lim_{t\to 0} \frac{u(x_0, y_0+t) - u(x_0, y_0) + iv(x_0, y_0+t) - iv(x_0, y_0)}{t} \\ &= \frac{\partial v}{\partial y}(p) - i\frac{\partial u}{\partial y}(p) \end{aligned}$$

which completes the proof of (1). From the equation

$$\frac{\partial u}{\partial x}(p) + i\frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i\frac{\partial u}{\partial y}(p)$$

(2) This is an immediate result of (1).

(3)

□

The following proposition is a kind of converse to the previous proposition.

Proposition 1.12. *Let $\Omega \subset \mathbb{C}$ be open, let $f\colon \Omega \to \mathbb{C}$, and let u and v be as in Proposition 1.11. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p := (x_0, y_0)$, ans suppose that u and v are differentiable at p, that is $\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p)$ and $\frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$. Then f is holomorphic at z_0.*

Proof. To be added. □

1.3 Power series

Definition 1.25 (Power series)**.** A power series centered at $z_0 \in \mathbb{C}$ is an expression of the form $\sum_0^\infty a_n(z - z_0)^n$, where $\{a_n\}_{n\geq 0} \subset \mathbb{C}$. Given $z \in \mathbb{C}$, we say that the power series converges at z if the limit $\lim_{N\to\infty} \sum_{n=0}^N a_n(z - z_0)^n$ exists in $\mathbb{C}$. If this limit does not exist, we say that the series diverges at z.

Definition 1.26 (Absolute convergence)**.** Given a power series $\sum_{n=0}^\infty a_n(z - z_0)$, we say that it converges absolutely at $z \in \mathbb{C}$ if $\sum_{n=0}^\infty |a_n| \cdot \left|(z - z_0)\right| < \infty$.

Proposition 1.13. *If a power series converges absolutely at z then it also converges at z. This follows from the completeness of $\mathbb{C}$.*

In the following proposition we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proposition 1.14 (Hadamard's theorem)**.** *Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series, and let $0 \leq \mathbb{R} \leq \infty$ be given by*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Then for $z \in \mathbb{C}$ the series converges absolutely if $|z-z_0| < R$, and the series diverges if $|z-z_0| > R$.

Remark 1.10. The number R is called the radius of convergence of the power series, and the region $\left\{z \in \mathbb{C} \colon |z-z_0| < R\right\}$ is called the disc of convergence.

We now proceed to prove Proposition 1.14

Proof. Set $L := 1/R$. Suppose first that $0 < R \leq \infty$, so that $0 \leq L < \infty$. Let $z \in \mathbb{C}$ be such that $|z-z_0| < R$, then there exists $L < M < \infty$ so that $M|z-z_0| < 1$. By the definition of L (the limsup) there exists $N \geq 1$ so that $|a_n|^{\frac{1}{n}} < M$ forall $n > N$. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n| \cdot |z-z_0|^n &= \sum_{n=0}^{N-1} |a_n| \cdot |z-z_0|^n + \sum_{n=N}^{\infty} \left(|a_n|^{\frac{1}{n}} \cdot |z-z_0|\right)^n \\ &\leq \sum_{n=0}^{N-1} |a_n| \cdot |z-z_0|^n + \sum_{n=N}^{\infty} \big(M|z-z_0|\big)^n < \infty.\end{aligned}$$

Suppose next that $0 \leq R < \infty$, so that $0 < L\infty$. Let $z \in \mathbb{C}$ be such that $|z-z_0| > R$, then similarly there exists $0 < M < L$ so that $M|z-z_0| > 1$. Then, for every $N \geq 1$ there exists $n \geq N$ so that $|a_n|^{\frac{1}{n}} > M$. For such n we have

$$\begin{aligned}\left|\sum_{k=0}^{n} a_k(z-z_0)^k - \sum_{k=0}^{n-1} a_k(z-z_0)^k\right| &= |a_n| \cdot |z-z_0|^n \\ &= \left(|a_n|^{\frac{1}{n}} \cdot |z-z_0|\right)^n > \big(M|z-z_0|\big)^n > 1,\end{aligned}$$

which shows that the partial sums do not form a Cauchy sequence. Thus the series diverges at z, which completes the proof. $\square$

Example 1.3. Considert the power series $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Because we have

$$\sqrt[n]{(2n)!} \geq \sqrt[n]{n^n} = n$$

we also have for every $n \geq 1$,

$$\left(\frac{1}{(2n)!}\right)^{\frac{1}{2n}} \leq \frac{1}{n^{\frac{1}{2}}} \quad \text{and} \quad \left(\frac{1}{(2n+1)!}\right)^{\frac{1}{2n+1}} \leq \frac{1}{n^{\frac{1}{2}}}.$$

Since $n^{-\frac{1}{2}} \xrightarrow{n\to\infty} = 0$ we get that the radius of convergence is ∞ for the series. The map $z \mapsto e^z$ is called the exponantial function. We also have that

$$e^z e^w = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) + \left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

Example 1.4. Consider the power series $f(z) := \sum_{n=0}^{\infty} z^n$. Since $1^{\frac{1}{n}} = 1$ we get that the radius of convergence in this case is 1. Thus f defined a function from $D_1(0)$ to $\mathbb{C}$. Moreover, since we have

$$(1-z)\sum_{n=0}^{N} z^n = 1 - z^{N+1},$$

we get for $z \in D_1(0)$ that

$$f(z) = \lim_{N\to\infty} \sum_{n=0}^{N} z^n = \lim_{N\to\infty} \frac{1-z^{N+1}}{1-z} = \frac{1}{1-z}.$$

Proposition 1.15. *Let $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series, and let R be the radius of convergence of f. Then,*

(1) R is also the radius of convergence of $\sum_{n=0}^{\infty} na_n(z-z_0)^{n-1}$;

(2) suppose $R > 0$, then f is holomorphic in its disc of convergence with

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}.$$

Proof.
Since $n^{1/n} \xrightarrow{n\to\infty} 1$ and $\frac{n-1}{n} \xrightarrow{n\to\infty} 1$, we have

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |na_n|^{\frac{1}{n-1}}$$

which gives the first part of the proposition.

(2) By the chain rule, the derivative of $f(z)$ wouldn't change for any $z_0 \in \mathbb{C}$ so we may choose $z_0 = 0$ and then $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose $R > 0$, let $0 < r < R$, fix $w \in D_r(0)$, and define

$$g(z) := \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}.$$

Since

$$\sum_{n=1}^{\infty} n|a_n|r^{n-1} \leq \sum_{n=1}^{\infty} n|a_n||z|^{n-1} < \infty$$

there exists $N \geq 1$ so that

$$\sum_{n=N+1}^{\infty} n|a_n|r^{n-1} < \frac{\epsilon}{3}.$$

For $z \in D_r(0)$ set

$$S(z) = \sum_{n=0}^{N} a_n z^n \text{ and } E(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Since $w \in D_r(0)$ and S is holomorphic at w (since it is a polynomial), there exists $\delta > 0$ so that $D_\delta(w) \subset D_r(0)$ and

$$\left|\frac{f(w+h)-f(w)}{h} - g(w)\right| \leq \left|\frac{S(w+h)-S(w)}{h} - S'(w)\right| + \left|S'(w) - g(w)\right| + \left|\frac{E(w+h)-E(w)}{h}\right|.$$

Recall that

$$S'(w) = \sum_{n=1}^{N} na_n z^{n-1}.$$

Since $w \in B_r(0)$ we get

$$\left|S'(w) - g(w)\right| = \left|\sum_{n=N+1}^{\infty} na_n w^{n-1}\right| \leq \left|\sum_{n=N+1}^{\infty} n|a_n|r^{n-1}\right| < \frac{\epsilon}{3}.$$

Notice that for each $n \geq 1$ and $a, b \in \mathbb{C}$,

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}) = (a-b)\sum_{k=0}^{n-1} a^{n-k-1}b^k.$$

Using this fact, and since $w, w+h \in D_r(0)$,

$$\left|(w+h)^n - w^n\right| = \left|h\sum_{k=0}^{n-1}(w+h)^{n-k-1}w^k\right| \leq |h|nr^{n-1}.$$

We now have that

$$\begin{aligned}
\left|\frac{E(w+h) - E(w)}{h}\right| &= \frac{1}{|h|}\left|\sum_{n=N+1}^{\infty} a_n((w+h)^n - w^n)\right| \\
&= \frac{1}{|h|}\sum_{n=N+1}^{\infty} |a_n| \cdot |(w+h)^n - w^n| \\
&= \sum_{n=N+1}^{\infty} |a_n|nr^{n-1} < \frac{\epsilon}{3}.
\end{aligned}$$

It now follows that

$$\left|\frac{f(w+h) - f(w)}{h} - g(w)\right| < \epsilon \text{ for } 0 \neq h \in D_\delta(0).$$

This shows that f is holomorphic at w with $f'(w) = g(w)$, which completes the proof. □

Corollary 1.16. *A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.*

Example 1.5. The function e^z is entire with

$$(e^z)' = \sum_{n=1}^{\infty} n\frac{z^{n-1}}{n!} = e^z \text{ for all } z \in \mathbb{C}.$$

Example 1.6. The standard trigonometric functions are given by

$$\cos z := \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} \text{ and } \sin z := \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}.$$

It is easy to verify that in both cases the radius of convergence is ∞. We also see that

$$(\cos z)' = \sum_{n=1}^{\infty}(-1)^n\frac{z^{2n-1}}{(2n-1)!} = -\sin z$$

and

$$(\sin z)' = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = \cos z.$$

From these equalities it is easy to check that $\sin z$ and $\cos z$ agree with their respective real versions. Moreover, a simple calculation gives the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \text{ and } \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

which are called Euler formulas. By adding these identites we get

$$e^{iz} = \cos z + i \sin z.$$

It follows that for all $x, y \in \mathbb{R}$ we have

$$e^{x+yi} = e^x e^{yi} = e^x(\cos y + i \sin y).$$

Definition 1.27 (Analytic function)**.** Let $\Omega \subset \mathbb{C}$ be open. A function $f\colon \Omega \to \mathbb{C}$ is said to be analytic at $z_0 \in \Omega$ if there exists a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, with radius of convergence $R > 0$, such that for some $0 < r < R$ with $D_r(z_0) \subset \Omega$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \text{ for } z \in D_r(z_0).$$

If f has a power series expansion at every point in Ω, we say that f is analytic on Ω.

Remark 1.11. From Proposition 1.15 we get that an analytic function on Ω is also holomorphic there. A deep theorem theorem we shall prove later is that every holomorphic function is analytic. This is why the terms holomorphic and analytic are often used interchangeably.

1.4 Integration along paths

Definition 1.28 (Path)**.** A path (in $\mathbb{C}$) is a continuous function $\gamma\colon [a, b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$ with $a < b$.

Definition 1.29 (Closed path)**.** A closed path is a path such that $\gamma(a) = \gamma(b)$.

Definition 1.30 (Simple path)**.** A path is called simple if γ is injective, unless the path is closed, in which case we only require $\gamma|_{(a,b)}$ to be injective.

Remark 1.12. We sgakk write γ^* for the image of γ, i.e. $\gamma^* := \gamma([a, b])$. Note that γ^* is a compact subset of $\mathbb{C}$.

Definition 1.31 (Differentiable path)**.** Let $\gamma\colon [a, b] \to \mathbb{C}$ be a path. We say that γ is differentiable at $t \in [a, b]$ if the following limit exists

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t + h) - \gamma(t)}{h},$$

where at the endpoints a, b the limit is one-sided.

Definition 1.32 (Smooth path)**.** Let $\gamma\colon [a, b] \to \mathbb{C}$ be a path. We say that γ is smooth if $\gamma'(t)$ exists at all $t \in [a, b]$, and the map $\gamma'\colon [a, b] \to \mathbb{C}$ is continuous.

Definition 1.33 (Piecewise-smooth path)**.** Let $\gamma\colon [a, b] \to \mathbb{C}$ be a path. We say that γ is piecewise-smooth if there exist points $a = a_0 < a_1 < \cdots < a_n = b$ so that for each $0 \le k < n$ the restriction of γ to $[a_k, a_{k+1}]$ is smooth.

Remark 1.13. Note that a piecewise-smooth path is continuous and not just piecewise continuous. Also, from now on every time we say 'path' we mean 'piecewise-smooth path'.

Definition 1.34 (Integral)**.** Given a continuous function $f\colon [a,b] \to \mathbb{C}$ we define

$$\int_a^b f(t)\,\mathrm{d}t := \int_a^b \Re(f(t))\,\mathrm{d}t + \int_a^b \Im(f(t))\,\mathrm{d}t.$$

Definition 1.35 (Integral along path)**.** Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path, and let $a = a_0 < a_1 < \cdots < a_n = b$ be such that γ is smooth on $[a_k, a_{k+1}]$ for each $0 \le k < n$. Given a continuous $f\colon \gamma^* \to \mathbb{C}$, we define the integral of f along γ as follows

$$\int_\gamma f(z)\,\mathrm{d}z := \sum_{k=0}^{n-1} \int_{a_k}^{a^{k+1}} f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Remark 1.14. Notice that the integrand $f(\gamma(t))\gamma'(t)$ is well defined and continuous at each $t \in [a,b] \setminus \{a_1, \ldots, a_{n-1}\}$. This we can also write,

$$\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Definition 1.36 (Length of a path)**.** Let $\gamma\colon [a,b] \to \mathbb{C}$ be a path, and let $a = a_0 < a_1 < \cdots < a_n = b$ be such that γ is smooth on $[a_k, a_{k+1}]$ for each $0 \le k < n$. The length of γ is defined as follows,

$$\operatorname{length}(\gamma) := \int_a^b |\gamma'(t)|\,\mathrm{d}t.$$

Example 1.7 (Length of a circle)**.** Let $z_0 \in C$ and $r > 0$ be given. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be with $\gamma(\theta) = z_0 + re^{i\theta}$ for $\theta \in [0, 2\pi]$. It is clear that γ is a smooth, closed and simple path. Let $f\colon C_r(z_0) \to \mathbb{C}$ be a continuous. Using a simple substitution we see that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{C_r(z_0)} f(z)\,\mathrm{d}z = \int_0^{2\pi} f(z_0 + re^{i\theta}) ire^{i\theta}\,\mathrm{d}\theta.$$

Also,

$$\operatorname{length}(\gamma) = \operatorname{length}(C_r(z_0)) = \int_0^{2\pi} |ire^{i\theta}|\,\mathrm{d}\theta = \int_0^{2\pi} |r|\,\mathrm{d}\theta = 2\pi r.$$

Remark 1.15. The curve γ in this example is called a positively oriented circle with center z_0 and radius r. That is because when travelling along the curve, the interior of the circle would be on the left, and the exterior on the right. If we chose $\gamma(\theta) = z_0 + r^{e^{-i\theta}}$ then it would be the other way around, so we would call it a negatively oriented circle. Every simple closed curve has such an orientation by a heavy theorem called the Jordan curve theorem which we will not prove in these notes.

Example 1.8. Let $\alpha, \beta \in \mathbb{C}$ be given. Let $\gamma\colon [0,1] \to \mathbb{C}$ be such that $\gamma(t) = t\beta + (1-t)\alpha$ for $t \in [0,1]$. Then γ is a smooth path, and it is simple if and only if $\alpha \neq \beta$. We denote the image of γ by $[\alpha, \beta]$. Let $f\colon [\alpha, \beta] \to \mathbb{C}$. We have

$$\int_{[\alpha,\beta]} f(z)\,\mathrm{d}z = \int_0^1 f(t\beta + (1-t)\alpha)(\beta - \alpha)\,\mathrm{d}t.$$

Also,

$$\operatorname{length}(\gamma) = \int_0^1 |\beta - \alpha|\,\mathrm{d}t = |\beta - \alpha|.$$

Example 1.9. Let $\gamma\colon[\alpha,\beta]\to\mathbb{C}$ be a path, and let $\gamma^-\colon[\alpha,\beta]$ be such that $\gamma^-(t)=\gamma(\alpha+\beta-t)$. It is clear that γ^- is also a path. We call γ^- the path opposite to γ. Given a continuous $f\colon\gamma^*\to\mathbb{C}$,

$$\int_{\gamma^-} f(z)\,\mathrm{d}z=\int_a^b f(\gamma^-(t))(\gamma^-)'(t)\,\mathrm{d}t=-\int_a^b f(\gamma(a+b-t))\gamma'(b+a-t)\,\mathrm{d}t.$$

Thus by the substitution $s=b+a-t$,

$$\int_{\gamma^-} f(z)\,\mathrm{d}z=-\int_a^b f(\gamma(s))\gamma'(s)\,\mathrm{d}t=-\int_\gamma f(z)\,\mathrm{d}z.$$

Moreover, by a similar substitution,

$$\mathrm{length}(\gamma^-)=\int_a^b|\gamma'(b+a-t)|\,\mathrm{d}t=\int_a^b|\gamma'(s)|\,\mathrm{d}t=\mathrm{length}(\gamma).$$

Remark 1.16. It is worth pointing out that if $w,\eta\in\mathbb{C}$ and γ is the oriented interval from w to η, then γ^- is the oriented interval from η to w.

Proposition 1.17. *For continuous $f,g\colon[a,b]\to\mathbb{C}$ and $\alpha\in\mathbb{C}$,*

$$\alpha\int_a^b f(t)\,\mathrm{d}t=\int_a^b \alpha f(t)\,\mathrm{d}t \ \textit{and}\ \int_a^b(f+g)(t)\,\mathrm{d}t=\int_a^b f(t)\,\mathrm{d}t+\int_a^b g(t)\,\mathrm{d}t.$$

Moreover,

$$\left|\int_a^b f(t)\,\mathrm{d}t\right|\le\int_a^b\big|f(t)\big|\,\mathrm{d}t.$$

Proof. The first two parts of the proposition are rather simple so we will only prove the final part. We may assume that $\int_a^b f(t)\,\mathrm{d}t\neq 0$. Set $\theta:=\arg(\int_a^b f(t)\,\mathrm{d}t)$. Then,

$$\left|\int_a^b f(t)\,\mathrm{d}t\right|=e^{-i\theta}\int_a^b f(t)\,\mathrm{d}t=\int_a^b e^{-i\theta}f(t)\,\mathrm{d}t.$$

Since the last expression is real we can use properties of the Riemann integral and the fact that $|e^{-i\theta}|=1$ to get

$$\left|\int_a^b f(t)\,\mathrm{d}t\right|=\int_a^b\Re\left(e^{-i\theta}f(t)\right)\mathrm{d}t\le\int_a^b\left|e^{-i\theta}f(t)\right|\mathrm{d}t=\int_a^b\big|f(t)\big|\,\mathrm{d}t,$$

which completes the proof of the proposition. □

Proposition 1.18. *Let $\gamma\colon[a,b]\to\mathbb{C}$ be a path. Then,*

(1) for every continuous $f,g\colon\gamma^*\to\mathbb{C}$ and $\alpha,\beta\in\mathbb{C}$

$$\int_\gamma \alpha f(z)+\beta g(z)\,\mathrm{d}z=\alpha\int_\gamma f(z)\,\mathrm{d}z+\beta\int_\gamma g(z)\,\mathrm{d}z.$$

(2) for every continuous $f\colon\gamma^*\to\mathbb{C}$

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right|\le\|f\|_{\gamma^*}\cdot\mathrm{length}(\gamma).$$

Proof. The first property follows from the previous proposition, and its proof is omitted. Given a continuous $f\colon \gamma^* \to \mathbb{C}$

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t\right| \le \int_a^b \left|f(\gamma(t))\gamma'(t)\right|\mathrm{d}t \le \|f\|_{\gamma^*}\int_a^b \left|\gamma'(t)\right|\mathrm{d}t = \|f\|_{\gamma^*}\cdot \mathrm{length}(\gamma),$$

which completes the proof. □

Definition 1.37 (Equivalence of paths)**.** Two paths $\gamma_1\colon [a,b]\to\mathbb{C}$ and $\gamma_2\colon [c,d]\to\mathbb{C}$ are said to be equivalent if there exists a continuously differentiable bijection $\varphi\colon [a,b]\to[c,d]$ so that $\varphi'(t)>0$ and $\gamma_1(t)=\gamma_2(\varphi(t))$ for all $t\in[a,b]$.

Proposition 1.19. *Let $\gamma_1\colon [a,b]\to\mathbb{C}$ and $\gamma_2\colon [c,d]\to\mathbb{C}$ be equivalent paths. Then* $\mathrm{length}(\gamma_1) = \mathrm{length}(\gamma_2)$*, and $\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$ for every continuous function $f\colon \gamma_1^*\to\mathbb{C}$.*

Proof. Since γ_1 and γ_2 are equivalent, there exists a continuously differentiable bijection φ from $[a,b]$ to $[c,d]$ so that $\varphi'(t)>0$ and $\gamma_1(t)=\gamma_2(\varphi(t))$ for all $t\in[a,b]$. Thus,

$$\mathrm{length}\gamma_1 = \int_a^b |\gamma_1'(t)|\,\mathrm{d}t = \int_a^b |\gamma_2'(\varphi(t))\varphi'(t)|\,\mathrm{d}t = \int_a^b |\gamma_2'(s)|\,\mathrm{d}s = \mathrm{length}\gamma_2.$$

Similarly,

$$\begin{aligned}\int_{\gamma_1} f(z)\,\mathrm{d}z &= \int_a^b f(\gamma_1(t))\gamma_1'(t)\,\mathrm{d}t = \int_a^b f(\gamma_2(\varphi(t)))\gamma_2'(\varphi(t))\varphi'(t)\,\mathrm{d}t \\ &= \int_c^d f(\gamma_2(s))\gamma_2'(s)\,\mathrm{d}s = \int_{\gamma_2} f(z)\,\mathrm{d}z,\end{aligned}$$

which completes the proof. □

Definition 1.38 (Primitive function)**.** Let Ω be a nonempty open subset of $\mathbb{C}$ and let $f\colon\Omega\to\mathbb{C}$. A primitive for f on Ω is a function $F\colon\Omega\to\mathbb{C}$ that is holomorphic on Ω with $F'(z)=f(z)$ for all $z\in\Omega$.

The following are the complex version of the chain rule and fundamental theorem of calculus.

Lemma 1.20. *Let $\emptyset\ne\Omega\subset\mathbb{C}$ be open, let $F\colon\Omega\to\mathbb{C}$ be holomorphic on Ω, and let $\gamma\colon [a,b]\to\Omega$ be a smooth path. Then $F\circ\gamma$ is also a smooth path with $(F\circ\gamma)'(t)=F'(\gamma(t))\gamma'(t)$ for all $t\in[a,b]$.*

Proposition 1.21. *Let $\emptyset\ne\Omega\subset\mathbb{C}$ be open, let $f\colon\Omega\to\mathbb{C}$ be continuous, let $F\colon\Omega\to\mathbb{C}$ be a primitive for f on Ω, and let $\gamma\colon[a,b]\to\Omega$ be a path. Then,*

$$\int_\gamma f(z)\,\mathrm{d}z = F\big(\gamma(b)\big) - F\big(\gamma(a)\big).$$

Corollary 1.22. *Let $\emptyset\ne\Omega\subset\mathbb{C}$ be open, let $\gamma\colon[a,b]\to\Omega$ be a closed path, and let $f\colon\Omega\to\mathbb{C}$ be continuous. Suppose that f has a primitive on Ω, then $\int_\gamma f(z)\,\mathrm{d}z=0$.*

Proof. Since γ is closed we have $\gamma(a)=\gamma(b)$. This together with the previous proposition completes the proof. □

Corollary 1.23. *Let $\gamma\colon[a,b]\to\mathbb{C}$ be a closed path. Then $\int_\gamma p(z)\,\mathrm{d}z=0$ for every polynomial $p\colon\mathbb{C}\to\mathbb{C}$.*

This corollary follows immediately from the fact that $z\mapsto z^{n+1}/(n+1)$ is a primitive for $z\mapsto z^n$ and previous propositions. However, Corollary 1.22 can also shine when trying to prove a function has no primitive.

Example 1.10. Let $f\colon \mathbb{C}\setminus\{0\}\to\mathbb{C}$ be defined with $f(z)=1/z$. We have that

$$\int_{C_1(0)} f(z)\,\mathrm{d}z = \int_0^{2\pi} f(e^{i\theta})ie^{i\theta}\,\mathrm{d}\theta = \int_0^{2\pi} i\,\mathrm{d}\theta = 2\pi i.$$

From Corollary 1.22 we get that f does not have a primitive on $\mathbb{C}\setminus\{0\}$.

Corollary 1.24. *Let $\Omega\subset\mathbb{C}$ be a region, and let $f\colon\Omega\to\mathbb{C}$ be holomorphic. Suppose that $f'(z)=0$ for all $z\in\Omega$, then f is constant.*

Proof.

Lemma 1.25. *Let $\Omega\subset\mathbb{C}$ be a region. Then for every $z,w\in\Omega$ there exists a path $\gamma\colon[0,1]\to\Omega$ with $\gamma(0)=z$ and $\gamma(1)=w$.*

This means that a region is also path connected, and we will not prove this here although the proof is quite standard.

Let $z,w\in\Omega$ be given. Then there exists a path $\gamma\colon[0,1]\to\Omega$ with $\gamma(0)=z$ and $\gamma(1)=w$. Now since $f'(z)=0$ on Ω, from Corollary 1.22 we can conclude that

$$0=\int_\gamma f'(z)\,\mathrm{d}z = f(\gamma(1))-f(\gamma(0)) = f(w)-f(z) \implies f(w)=f(z).$$

This implies that f is constant and completes the proof. □

1.5 The index of a point with respect to a path