Analysis 2

1 Proper Integrals

Definition 1.1 (Riemann integrability)**.** A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be Riemann integrable on the interval $[a, b]$ if there exists $I \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for any partition $X = (x_0, x_1, \ldots, x_n)$ with $\lambda(X) < \delta$, any sequence $(c_1, c_2, \ldots, c_n)$ such that $c_i \in [x_{i-1}, x_i]$ for all $1 \le i \le n$ also satisfies:

$$\left| \sum_{i=1}^{n} f(c_i)\Delta X_i - I \right| < \varepsilon.$$

We also denote $I = \int_a^b f(x)\,\mathrm{d}x$.

Proposition 1.1. *Suppose f is a Riemann integrable function, then f^2 is also Riemann integrable.*

Proof. We can notice that

$$\begin{aligned}
&U(f,P) - D(f,P) < \varepsilon \\
\Rightarrow\, &U(f,P) < D(f,P) + \varepsilon \\
\Rightarrow\, &U(f,P)^2 < D(f,P)^2 + 2\varepsilon D(f,P) + \varepsilon^2 \\
\Rightarrow\, &U(f,P)^2 - D(f,P)^2 < \varepsilon(2D(f,P) + \varepsilon)
\end{aligned}$$

Since f is integrable we know $2D(f,P) + \varepsilon$ is bounded. Denote the bound M. Let $\varepsilon > 0$. Choose the δ that matches $\varepsilon_\delta = \min(\frac{\varepsilon}{2M+1}, 1)$ under f's integrability. We get:

$$\begin{aligned}
&U(f,P) - D(f,P) < \varepsilon_\delta \\
\Rightarrow\, &U(f,P)^2 - D(f,P)^2 < \varepsilon_\delta(2D(f,P) + \varepsilon_\delta) \\
\Rightarrow\, &U(f,P)^2 - D(f,P)^2 < \frac{\varepsilon}{2M+1}(2M+1) = \varepsilon \\
\Rightarrow\, &U(f^2,P) - D(f^2,P) < \varepsilon
\end{aligned}$$

This shows that f^2 is Darboux integrable, which implies it is Riemann integrable, which completes the proof. □

Definition 1.2. If f is continuous, then f is integrable.

Proof. Let f be continuous on $[a, b]$. By the Cantor–Heine theorem it is uniformally continuous. We have

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)\Delta X_i <^{*} \sum_{i=1}^{n} \varepsilon \Delta X_i = \varepsilon(b - a)$$

$(*)$ This is becuase by definition we have that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$. This implies that $d(f(x), f(y)) < \varepsilon$ and that delta is exactly what we wanted, which completes the proof. □

Proposition 1.2 (Intermediate Value Theorem for Integrals)**.** *Let f be a continuous funtion on $[a, b]$ then exists $x_0 \in [a, b]$ such that:*

$$\int_a^b f(x)\,\mathrm{d}x = f(x_0)(b - a).$$

Proof. Since f is continuous it is Riemann integrable. From Weierstrass's theorem f has a minimum and maximum in $[a, b]$ which we will denote m and M. We know have

$$m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a).$$

From this follows that

$$m \leq \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a} \leq M.$$

Denote $c := \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}$. By the intermediate value theorem we know that exists $x_0 \in (a, b)$ such that $f(x_0) = c$ and thus:

$$\int_a^b f(x)\,\mathrm{d}x = f(x_0)(b-a)$$

$\square$

1.1 The fundamental theorem of calculus

Theorem 1.3. (Fundamental theorem of calculus, part one). *Let f be a continuous real-valued function defined on a closed interval $[a,b]$. Let F be the function defined as:*

$$F(x) = \int_a^x f(t)\,dt$$

for all $x \in [a,b]$. Then F is uniformly continuous on $[a,b]$ and differentiable on the open interval (a,b), and

$$F'(x) = f(x) \quad \forall x \in (a,b).$$

Theorem 1.4. (Fundamental theorem of calculus, part two). *Under the conditions of part one, if f is Riemann integrable on $[a,b]$. Then:*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

Proof. **Part One** For any $x_1, x_1 + \Delta x \in [a,b]$ we get:

$$F(x_1 + \Delta x) - F(x_1) = \int_a^{x_1+\Delta x} f(t)\,dt - \int_a^{x_1} f(t)\,dt = \int_{x_1}^{x_1+\Delta x} f(t)\,dt$$

According to the mean value theorem for integration we get that for $c \in [x_1, x_1 + \Delta x]$:

$$\int_{x_1}^{x_1+\Delta x} f(t)\,dt = f(c)\Delta x$$

$$\lim_{\Delta x \to 0} \frac{F(x_1 + \Delta x) - F(x_1)}{\Delta x} = \lim_{\Delta x \to 0} f(c) = f(x_1)$$

And thus from the squeeze theorem and f's continuity we get $F'(x_1) = f(x_1)$. □

Proof. **Part Two** Let $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a,b]$ such that $(x_0, x_n) = (a,b)$. Then we have:

$$\begin{aligned} F(b) - F(a) &= F(x_n) + [F(x_{n-1}) - F(x_{n-1})] + \cdots + [F(x_1) - F(x_1)] - F(x_0) \\ &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \cdots + [F(x_1) - F(x_0)] \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]. \end{aligned}$$

Because F is continuous on $[a,b]$ and differentiable on (a,b), we can use Lagrange's theorem on $[x_i, x_{i-1}]$. Thus, there exists $c_i \in [x_i, x_{i-1}]$ such that

$$\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} \left[F'(c_i)(x_i - x_{i-1})\right].$$

According to part one we get that $F'(c_i) = f(c_i)$ and so

$$F(b) - F(a) = \sum_{i=1}^{n} [f(c_i)(\Delta x_i)]$$

$$\Longleftrightarrow$$

$$\lim_{\|\Delta x_i\| \to 0} (F(b) - F(a)) = \lim_{\|\Delta x_i\| \to 0} \left(\sum_{i=1}^{n} [f(c_i)(\Delta x_i)] \right)$$

$$\Longleftrightarrow$$

$$F(b) - F(a) = \int_a^b f(x)\,\mathrm{d}x.$$

□

1.2 Length of a curve

Using integrals, we can actually find a formula for the length of a continuous graph. Approximating the length of a graph using the pythagorian theorem for partition $X = (x_0, x_1, \ldots, x_n)$ we get:

$$\sum_{i=1}^{n} \sqrt{d(x_{i-1}, x_i)^2 + d(f(x_{i-1}), f(x)_i)^2}.$$

Assuming f is continuous on $[a, b]$ and differentiable on (a, b) from Lagrange's theorem we get:

$$\begin{aligned}
\sum_{i=1}^{n} \sqrt{d(x_{i-1}, x_i)^2 + d(f(x_{i-1}), f(x)_i)^2} &= \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f'(c_i)(x_i - x_{i-1}))^2} \\
&= \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 (1 + (f'(c_i))^2)} \\
&= \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \Delta X_i.
\end{aligned}$$

We can see that this summation is matching the integral

$$\lim_{\|\Delta x_i\| \to 0} \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \Delta X_i = \int_a^b \sqrt{1 + (f'(x))^2} \, \mathrm{d}x$$

The result of this integral will give us the length of any continuous graph.

1.3 The limit comparison test

Proposition 1.5 (Limit comparison test)**.** *Let f, g be two integrable positive functions on $[a, M]$ for any $M \in \mathbb{R}$. Suppose that*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c.$$

Then

- *If $c \in (0, \infty)$ then either both series converge or both series diverge.*
- *If $c = 0$ the converganve of f implies the convergance of g.*
- *If $c = \infty$ the converganve of g implies the convergance of f.*

Proof. Assume $c \in (0, \infty)$. Let $\varepsilon > 0$ we know that exists $x_0 \in \mathbb{R}$ such that for all $x_0 < x$ we have:

$$g(x)(c - \varepsilon) < f(x) < g(x)(c + \varepsilon).$$

If $g(x)$ converges then $f(x)$ converges by the squeeze theorem. Similarly if g diverges we know that

$$g(x)(c - \varepsilon) < f(x),$$

So from a certain point onwards f will meet the requirments of the direct comparison test and will diverge which completes the proof. $\square$

Here is some practise of the limit comparison test.

Exercise 1.1. Find the value of

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right).$$

Solution. By direct calculation we have:

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} \frac{1}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f(\frac{k}{n}) \frac{1}{n} \\
&= \int_0^1 \frac{1}{1+x}
\end{aligned}$$

Exercise 1.2. Check if the following integral converges or diverges:

$$\int_{\frac{1}{2}}^{1} \frac{1}{x\sqrt{1-x}} \, \mathrm{d}x.$$

Solution. This function seems to behave like $\frac{1}{\sqrt{1-x}}$ near 1 so let's compare them using the limit comparison test:

$$\lim_{x\to 1^-} \frac{\frac{1}{x\sqrt{1-x}}}{\frac{1}{\sqrt{1-x}}} = \lim_{x\to 1^-} \frac{1}{x} = 1.$$

By the limit comparison test we get that the integral converges.

2 Improper Integrals

Definition 2.1 (Improper integral)**.** Let $f\colon [a,\infty) \to \mathbb{R}$ be a function that is integrable on $[a, M]$ for all $M > a$. Define the improper integral

$$\int_a^\infty f(x)\,\mathrm{d}x := \lim_{M\to\infty} \int_a^M f(x)\,\mathrm{d}x.$$

If the limit exists we say that the integral converges.

Definition 2.2 (Absolute convergence)**.** An improper integral of a function f is said to converge absolutely if the integral of the absolute value of the integrand is finite—that is, if $\int_a^\infty |f(x)|\,\mathrm{d}x = L$ for some finite $L \in \mathbb{R}$.

Remark 2.1. An improper integral of a function f that converges, but does not converge absolutely, is said to converge conditionally.

Proposition 2.1. *Let $\int_a^\infty f(x)\,\mathrm{d}x$ be an improper integral that converge absolutely, then it also converges. In other words:*

$$\int_a^\infty |f(x)|\,\mathrm{d}x \quad \text{converges} \qquad \Longrightarrow \int_a^\infty f(x)\,\mathrm{d}x \quad \text{converges.}$$

Proof. Suppose f converges absolutely. Consider:

$$f^+ = \begin{cases} f(x), & f(x) > 0 \\ 0, & f(x) \le 0 \end{cases}$$

$$f^- = \begin{cases} -f(x), & f(x) < 0 \\ 0, & f(x) \ge 0 \end{cases}$$

We know that $|f| = f^+ + f^-$ convereges and so by the direct comparison test we get that $\int_a^\infty f^+$ and $\int_a^\infty f^-$ converge. Since $f = f^+ - f^-$ we also get that $\int_a^\infty f$ converges as well which completes the proof. □

Proposition 2.2 (Dirichlet's Test)**.** *If a_n is a monotonic sequence and $\lim_{n\to\infty} a_n = 0$ and $\sum b_n$ is bounded by M then*

$$\sum_{n=1}^\infty a_n b_n \quad \text{converges.}$$

Proof. Denote $B_n = \sum_{k=1}^n b_n$ and by summation by parts we see that

$$\sum_{k=1}^n a_n b_n = a_n B_n - \sum_{k=1}^{n-1} (a_{k+1} - a_k) B_k.$$

Since a_n converges to 0 and B_n is bounded $\lim_{n\to\infty} a_n B_n$ exists. WLOG assume a_n is increasing, we can also see that

$$\sum_{k=1}^{n-1} |(a_{k+1} - a_k)B_k| \le \sum_{k=1}^{n-1} M\,|(a_{k+1} - a_k)| \le M \sum_{k=1}^{n-1} (a_{k+1} - a_k)$$

And

$$M \sum_{k=1}^{n-1} (a_{k+1} - a_k) = M(a_n - a_1)$$

Which converges to $-Ma_1$ since $\lim_{n\to\infty} a_n = 0$. That means that this sequence is bounded. Which means that $\sum_{k=1}^{n-1} |(a_{k+1} - a_k)B_k|$ is also bounded. It is also monotonic which means it converges. And if $\sum_{k=1}^{n-1} (a_{k+1} - a_k)B_k$ converges absolutly it also converges conditionally which means $\sum_{k=1}^{n-1} (a_{k+1} - a_k)B_k$ converges. And by additivity of limits we know $\lim_{n\to\infty} \sum_{k=1}^n a_n b_n$ exists so $\sum_{n=1}^\infty a_n b_n$ converges. □

Proposition 2.3 (Abel's test)**.** *Suppose $\sum_1^\infty a_n$ converges, and b_n is monotone and bounded. Then $\sum_1^\infty a_n b_n$ also converges.*

Proof. We know b_n is monotone and bounded so it has a limit $\lim_{n\to\infty} b_n = b$ This implies $\lim_{n\to\infty} b_n - b = 0$. Since $b_n - b$ is also monotonic we know by Dirichle's test that $\sum_1^\infty a_n(b_n - b)$ converges. And by homogenity of series we know that $\sum_{n=1}^\infty a_n b$ converges as well. That means $\sum_{n=1}^\infty (a_n b) + a_n(b_n - b)$ converges. So $\sum_{n=1}^\infty (a_n b) + a_n(b_n - b) = \sum_{n=1}^\infty a_n b_n$ converges. □

Proposition 2.4 (Root test)**.** *Suppose that we have the series $\sum_{n=1} \infty a_n$. Define*

$$L := \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Then for,

- $L < 1$ *the series is absolutely convergent (thus also convergent);*
- $L > 1$ *the series is divergent;*
- $L = 1$ *the series may be divergent, conditionally convergent, or absolutely convergent.*

Proposition 2.5 (Ratio test)**.** *Suppose that we have the series $\sum_{n=1} \infty a_n$. Define*

$$L := \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

Then for,

- $L < 1$ *the series is absolutely convergent (thus also convergent);*
- $L > 1$ *the series is divergent;*
- $L = 1$ *the series may be divergent, conditionally convergent, or absolutely convergent.*

In case the ratio test doesn't work, there does exist a stronger test which we can use.

Proposition 2.6 (Rabbe's test)**.** *Let a_n be a positive sequence, then if*

$$\lim_{n\to\infty} \left(n \left(1 - \frac{a_n + 1}{a_n} \right) \right) = q,$$

then for,

- $q > 1$ *the series is convergent;*
- $q < 1$ *the series is divergent;*
- $q = 1$ *the series may be divergent, conditionally convergent, or absolutely convergent.*

Proposition 2.7 (Integral test for series)**.** *Let $f(x)$ be a positive monotone decreasing function on $[1, \infty]$. Define $a_n = f(n)$. Then*

$$\sum_{n=1}^{\infty} a_n \quad \textit{converges} \quad \Longleftrightarrow \int_1^{\infty} f(x)dx \quad \textit{converges.}$$

Proof. Suppose the series converges then we get:

$$\left|\sum_{n=1}^{\infty} a_n\right| < M,$$

but we also know that:

$$0 \le \sum_{n=2}^{\infty} a_n \le \int_1^{N+1} f(x)dx \le \sum_{n=1}^{\infty} a_n.$$

This means that the integral is increasing and bounded so it's converging. Suppose the integral was converging, to prove the series is also converging we could show similarly it's "bounded" by the integral's bound. $\square$

Proposition 2.8 (Cauchy condensation test)**.** *Let a_n be a non-increasing sequence of non-negative number. Then*

$$\sum_{n=1}^{\infty} f(n) \leq \sum_{n=0}^{\infty} 2^n f(2^n) \leq 2\sum_{n=1}^{\infty} f(n).$$

Proof. This follows from a simple rearrangement of the series:

$$\begin{aligned}
\sum_{n=1}^{\infty} f(n) &= f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) + \cdots \\
&= f(1) + (f(2) + f(3)) + (f(4) + f(5) + f(6) + f(7)) + \cdots \\
&\leq f(1) + (f(2) + f(2)) + (f(4) + f(4) + f(4) + f(4)) + \cdots \\
&= \sum_{n=0}^{\infty} 2^n f(2^n) \\
&= f(1) + (f(2) + f(2)) + (f(4) + f(4) + f(4) + f(4)) + \cdots \\
&\leq (f(1) + f(1)) + (f(2) + f(2)) + (f(3) + f(3)) + (f(4) + f(4)) \cdots \\
&= 2\sum_{n=1}^{\infty} f(n).
\end{aligned}$$

□

Proposition 2.9 (Leibniz's Test)**.** *Let a_n be a monotone decreasing positive sequence such that*

$$\lim_{n\to\infty} a_n = 0.$$

Then

$$\sum_{n=1}^{\infty}(-1)^{\infty}a_n \qquad \textit{converges.}$$

Proof. Since a_n is monotonically decreasing then we can say that

$$S_{2(m+1)} = S_{2m} + (a_{2m+1} - a_{2m}) \geq S_{2m}$$
$$S_{2(m+1)+1} = S_{2m+1} - (a_{2m+2} + a_{2m+3}) \leq S_{2m+1}$$

Or in other words S_{2m} monotonically increases and S_{2m+1} monotonically decreases. But we also know that

$$S_{2m+1} - S_{2m} = a_{2m+1} \geq 0$$

And that means that

$$a_1 - a_2 = S_2 \leq S_{2m} \leq S_{2m+1} \leq S_1 = a_1$$

In other words our monotone sequences are bounded and so they converge. Recall as $m \to \infty$

$$S_{2m+1} - S_{2m} = a_{2m+1} \to 0$$

So by Cantor's lemma S_{2m+1}, S_{2m} converge to the same limit L. Moreover,

$$S_{2m} \leq L \leq S_{2m+1}$$

And also

$$|S_k - L| \leq a_{k+1}$$

□

Definition 2.3 (Permutation)**.** A permutation is defined as a bijection from a set to itself.

Definition 2.4 (Absolute convergence)**.** A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n| = L$ for some finite $L \in \mathbb{R}$.

Remark 2.2. A series that converges, but does not converges absolutely, is said to be conditionally convergent.

Remark 2.3. If a series converges absolutely, it converges.

Theorem 2.10. (Riemann series theorem). *Suppose that $(a_1, a_2, a_3, \ldots)$ is a sequence of real numbers, and that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Let $M \in \mathbb{R} \cup \{\infty, -\infty\}$. Then there exists a permutation σ such that:*

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = M.$$

There also exists a bijection π such that $\sum_{n=1}^{\infty} a_{\pi(n)}$ diverges.

Remark 2.4. If $\sum_{n=1}^{\infty} a_n$ is converges absolutely to $M \in \mathbb{R} \cup \{\infty, -\infty\}$, then for any bijection σ, the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ will also converge to M.

Remark 2.5. If a series is converging then putting parentheses will not change the value to which it converges.

3 Function Sequences

Definition 3.1 (Pointwise convergece)**.** A function sequence $\{f_n\}_{n=1}^{\infty}$ is said to converge pointwise to a given function f, if for all $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} f_n(x) = f(x).$$

The function f is said to be the pointwise limit of $\{f_n\}_{n=1}^{\infty}$.

Definition 3.2 (Uniform convergece)**.** A function sequence $\{f_n\}_{n=1}^{\infty}$ is said to converge uniformly to a given function f, if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ and $x \in \mathbb{R}$ we have

$$|f_n(x) - f(x)| < \epsilon.$$

Equivalently, the sequence $\{f_n\}_{n=1}^{\infty}$ uniformly converges to f if

$$\lim_{n\to\infty} \sup\{\, |f_n(x) - f(x)| : x \in \mathbb{R}\} = 0.$$

Theorem 3.1. (Dini's theorem). *Let $f_n(x) \to f(x)$ converge pointwise in $D = [a,b]$. If for all $x \in D$ the sequence $f_n(x)$ is monotonic, and f, $\{f_n\}_{n\geq 1}$ are continuous, then $f_n(x)$ converges to $f(x)$ uniformally.*

Here are some important examples to why all the conditions in Dini's theorem are necessary.

Example 3.1. Consider the function sequence $f_n = x^n$ on the interval $(0,1)$. The seuqence converges pointwise to the constant function $f(x) = 0$. For all $x \in (0,1)$ the sequence $f_n(x)$ is monotonic. The function f, $\{f_n\}_{n\geq 1}$ are continuous. But since the interval is open and not closed, the conditions of Dini's theorem are not fulfilled, and indeed the function sequence does not converge uniformly on $(0,1)$.

Example 3.2. The function sequence $f_n = \frac{nx}{1+n^2x^2}$ satisfies all the conditions exepct that it is not pointwise monotonic, and it does not converge unifromly on $\mathbb{R}$.

3.1 Properties of Uniformally Converging Function Sequences

Proposition 3.2. *Suppose f_n converges to f uniformally, and that f_n is continuous for any naturaln. Then f is continuous.*

Proposition 3.3. *Suppose f_n converges to f uniformally on $[a,b]$, and that f_n is integrable for any natural n. Then f is integrable, and we have*

$$\int_a^b \int f_n \xrightarrow{n\to\infty} \int_a^b \int f.$$

Proposition 3.4. *Suppose f_n is continuous differentiable on I for any natural n such that:*

- *f_n' converges uniformly on I;*
- *There exists $x_0 \in I$ such that $f_n(x_0)$ converges;*

then f_n uniformally converge on I to the function f and $f_n' \to f'$.

Proposition 3.5 (Weierstrass M-test)**.** *Let $\sum_{n=1}^{\infty} f_n(x)$ be a function series. Suppose exists a sequence M_n such that:*

- *For any natural n we have $|f_n(x)| \leq M_n$;*
- *The series $\sum_{n=1}^{\infty} M_n$ converges.*

Then $\sum_{n=1}^{\infty} f_n(x)$ *converges uniformally.*

Proof. Since M_n converges we can use Cauchy's criterion for the convegence of a series get that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ and $p \in \mathbb{N}$ we have

$$\left|\sum_{k=1}^{n+p} M_k - \sum_{k=1}^{n} M_k\right| < \varepsilon.$$

Since $0 \leq M_n$ we get

$$\sum_{k=n+1}^{n+p} M_k < \varepsilon,$$

and finally, we have

$$\left|\sum_{k=n+1}^{n+p} f_k(x)\right| \leq \sum_{k=n+1}^{n+p} |f_k(x)| \leq \sum_{k=n+1}^{n+p} M_n < \varepsilon.$$

which completes the proof. □

4 Power Series Theorems

Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for some x_0, then it absolutely converges for any x such that $|x| < |x_0|$. Since the power series converges $\lim_{n\to\infty} a_n x_0^n = 0$ and so the sequence is bounded and we denote that bound M.

$$0 \le |a_n x^n| = |a_n x_0^n| \left|\frac{x^n}{x_0^n}\right| < M \left|\frac{x}{x_0}\right|^n$$

And this sequence's sum is a geometric series so it converges and so does $\sum_{n=1}^{\infty} |a_n x^n|$. We also know that $|a_n x^n| < |a_n x_0^n|$ for all $n \in \mathbb{N}$ so according to Weierstrass's M test $\sum_{n=1}^{\infty} |a_n x^n|$ uniformally converges. Let

$$X = \left\{ x \in \mathbb{R} \mid \sum_{n=1}^{\infty} a_n x^n \text{ converges} \right\}$$

We claim that exists $R = \sup X$ - the radius of convergance - and that the series converges if $|x| < |R|$ and diverges fo $|x| > |R|$. For any $x > R$ the series diverges by definition of R. If $x < -R$ we know that exists x_1 such that $R < x_1 < |x|$ such that the series converges, in contradiction to R's defintion. if $|x| < |R|$ than there exists x_2 such that $|x| < |x_2| < |R|$ for which the series converges and then it converges for x as well.

Exercise 4.1. We know the series converges uniformally for any close intervel properly inside $[-R, R]$. If it converges uniformally on $[0, R]$ then it is converging in R as well.

Exercise 4.2. Let a function series converge uniformally to f. Prove f is continuous on (a, b).

5 Cauchy–Hadamard + D'alembert Theorem

Proposition 5.1 (Cauchy–Hadamard). *Let $\sum_{n=0}^{\infty} a_n x^n$ be a series and let R be the radius of convergence of the series. Then:*

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Proof. Let

$$\begin{aligned}
&\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \\
&\Rightarrow \sqrt[n]{|a_n x^n|} = \sqrt[n]{|a_n|}|x| \to L|x| \\
&\Rightarrow \begin{cases} |x| < \frac{1}{L} = R & \text{The series converges} \\ |x| > \frac{1}{L} = R & \text{The series diverges} \end{cases}
\end{aligned}$$

This proves the series converges or diverges absolutely according to the root test. If it converges absolutly this implies it converges in the usual sense. Suppose it diverges - by the root test we know that if the series diverges the partial sums don't converge to 0 so the series must diverge as well. □

Proposition 5.2 (D'alembert). *Let $\sum_{n=0}^{\infty} a_n x^n$ be a series and let R be the radius of convergence of the series. Then:*

$$R = \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}.$$

Proof. Let

$$\begin{aligned}
&\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \\
&\Rightarrow \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|a_{n+1}|}{|a_n|}|x| \to L|x| \\
&\Rightarrow \begin{cases} |x| < \frac{1}{L} = R & \text{The series converges} \\ |x| > \frac{1}{L} = R & \text{The series diverges} \end{cases}
\end{aligned}$$

If the series converges absolutly we can be sure yet again that it converges. If it diverges - than by the quotient test the partial sums diverge and so the series must also diverge, and the series will diverge as we claimed. □

6 A Note on the Taylor Series

If f is smooth on $(-R, R)$ then f can be the limit of a power series if and only if:

$$\forall x \in (-R, R)$$
$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \sum_{n=N+1}^{\infty} a_n x^n = 0$$

This is because the following are equivalent:

$$\begin{aligned}
&f \text{ can be the limit of a power series} \\
&\lim_{n\to\infty} S_n(x) = f(x), \forall x \in (-R, R) \\
&\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} f(x) - S_n(x) = 0, \quad \forall x \in (-R, R)
\end{aligned}$$

7 Continuous Partial Derivatives imply Differentiability

7.1 Semi-Proof

We want to find the tangetial plane to f for (x_0, y_0) assuming that the partial derivatives are continuous at that point. Let's denote

$$z_o = f(x_0, y_0) \quad \text{and} \quad A = \frac{\partial f}{\partial x}(x_0, y_0) \quad \text{and} \quad B = \frac{\partial f}{\partial y}(x_0, y_0)$$

Now the tangential lines that intersect at z_0 and are parallel to the axes (and in turn are perpendicular to one another) are (since the derivatives are continuous)

$$\begin{aligned} z &= B(y - y_0) + z_0 \\ z &= A(x - x_0) + z_0 \end{aligned}$$

Their directional vectors are in turn

$$\begin{aligned} \vec{\beta} &= (0, 1, B) \\ \vec{\alpha} &= (1, 0, A) \end{aligned}$$

And the normal vector to their spanning plane is

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & B \\ 1 & 0 & A \end{vmatrix} = (A, B, -1)$$

And so the plane equation is

$$A(x - x_0) + B(y - y_0) - (z - z_0) = 0$$

Which means that

$$z = z_0 + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0).$$

We have shown that if continuous partial derivatives exists at (x_0, y_0) then f has a tangential plane at (x_0, y_0) which is equivalent to being differentiable at (x_0, y_0)

7.2 Note on Diffferentiability

We say that f is differentiable at (x_0, y_0) if exist A, B such that

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = Ah + Bk + \alpha(h, k)\sqrt{h^2 + k^2} = Ah + Bj + \alpha(h, k)h + \beta(h, k)k$$

where $\lim_{(h,k)\to(0,0)} \alpha(h, k) = 0$ and $\lim_{(h,k)\to(0,0)} \beta(h, k) = 0$. This is equivalent to

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)}{\sqrt{(y - y_0)^2 + (y - y_0)^2}} = 0$$

8 Leibniz integral rule

Theorem 8.1. (Leibniz integral rule). *Let $f(x,y)$ be continuous on a rectangle $[a,b]\times[c,d]$ and suppose $\frac{\partial f}{\partial y}(x,y)$ exists and is continuous on $[a,b]\times[c,d]$. Define $F(y)=\int_a^b f(x,y)\,\mathrm{d}x$. Then F is differentiable on $[c,d]$ and*

$$F'(y)=\frac{d}{dy}\int_a^b f(x,y)\,\mathrm{d}x=\int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

In order to prove this theorem, we first need to prove a lemma.

Lemma 8.2. *Let $f(x,y)$ be continuous on $[a,b]\times[c,d]$. Then $F(y)=\int_a^b f(x,y)\,\mathrm{d}x$ is uniforamlly contiuous on $[c,d]$.*

Proof. We know f is contiuous on a compact space so it is uniformally continuous there. This means that for any $\varepsilon>0$ there exists $\delta>0$ such that for any $p_1=(x_1,y_2)$, $p_2=(x_2,y_2)$ such that $d(p_1,p_2)<\delta$ we have

$$|f(p_1)-f(p_2)|<\varepsilon.$$

Now consider $y_1,y_2\in[c,d]$ such that $d(y_1,y_2)<\delta$. We know that for all $x\in[a,b]$ that $d((x,y_1),(x,y_2))>\delta$ and then we can see that

$$\begin{aligned}|F(y_1)-F(y_2)|=\left|\int_a^b f(x,y_1)\,\mathrm{d}x-\int_a^b f(x,y_2)\,\mathrm{d}x\right|&=\left|\int_a^b (f(x,y_1)-f(x,y_2))\,\mathrm{d}x\right|\\ &\le\int_a^b |(f(x,y_1)-f(x,y_2))|\,\mathrm{d}x<\varepsilon(b-a)\end{aligned}$$

□

We can now go back to prove Theorem 8.1.

Proof. Now denote $G(y):=\int_a^b \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x$, by the previous lemma G is continuous. Thus,

$$\Delta F=F(y+\Delta y)-F(y)=\int_a^b f(x,y+\Delta y)\,\mathrm{d}x-\int_a^b f(x,y)\,\mathrm{d}x=\int_a^b (f(x,y+\Delta y)-f(x,y))\,\mathrm{d}x.$$

We know by Lagrange's theorem that exists $t\in(0,1)$ such that

$$\begin{aligned}&\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=\frac{\partial f}{\partial y}(x,y+\Delta y)\\ &\Rightarrow\int_a^b (f(x,y+\Delta y)-f(x,y))\,\mathrm{d}x=\int_a^b\left(\frac{\partial f}{\partial y}(x,y+t\Delta y)\Delta y\right)\mathrm{d}x\\ &\Rightarrow\frac{\Delta F}{\Delta y}=\int_a^b\left(\frac{\partial f}{\partial y}(x,y+t\Delta y)\right)\mathrm{d}x={}^2G(y+t\Delta y)\to G(y)\end{aligned}$$

which completes the proof. □

We can generalize this rule further to get the generalized Leibniz integral rule.

Theorem 8.3. (Generalized Leibniz integral theorem). *Let $f(x,y)$ be a continuously differentiable function on $[a,b]\times[c,d]$. Suppose that $\frac{\partial f}{\partial y}(x,y)$ exists and is continuous on $[a,b]\times[c,d]$, and that $\alpha(y)$, $\beta(y)$ are differentiable on $[c,d]$. Define $F(y) := \int_{\alpha(y)}^{\beta(y)} f(x,y)\,\mathrm{d}x$. Then F is differentiable on $[c,d]$ and*

$$F'(y) = \int_{\alpha(y)}^{\beta(y)} \frac{\partial f}{\partial y}(x,y)\,dx + f(\beta(y),y)\beta'(y) - f(\alpha(y),y)\alpha'(y).$$

Proof. Denote $\Phi(s,t,y) := \int_s^t f(x,y)\,\mathrm{d}x$. Then

$$F(y) = \Phi(\alpha(y),\beta(y),y) = \int_s^t f(x,y)\,\mathrm{d}x.$$

And now

$$F'(y) = \frac{\partial\Phi}{\partial s}\frac{ds}{dy} + \frac{\partial\Phi}{\partial t}\frac{dt}{dy} + \frac{\partial\Phi}{\partial y}\frac{dy}{dy}.$$

By Theorem 8.1 and the fundamental theorem we get

$$F'(y) = -f(\alpha(y),y)\alpha'(y) + f(\beta(y),y)\beta'(y) + \int_{\alpha(y)}^{\beta(y)} \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x,$$

which completes the proof. $\square$

9 Fubini's Theorem

Theorem 9.1 (Fubini's theorem)**.** *Let $f(x,y)$ be a continuous function on $[a,b]\times[c,d]$. Then*

$$\int_a^b \left(\int_c^d f(x,y)\,dy\right)\,\mathrm{d}x = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy.$$

Proof. Denote

$$\varphi(t) := \int_c^t \left(\int_a^b f(x,y)\,dx\right)\,dy$$
$$\Psi(t) := \int_a^b \left(\int_c^t f(x,y)\,dy\right)\,\mathrm{d}x.$$

Since $f(x,y)$ is continuous we get that $F(y) = \int_a^b f(x,y)\,\mathrm{d}x$ is also continuous. Thus, by the fundemetal theorem,

$$\varphi'(t) = \frac{d}{dt}\int_c^t F(y)\,dy = F(t) = \int_a^b f(x,t)\,\mathrm{d}x.$$

Denote $G(x,t) := \int_c^t f(x,y)\,dy$. Then, by the fundemental theorem again we get

$$\frac{\partial G}{\partial t} = f(x,t).$$

And thus by the generalized Leibniz Integral Rule

$$\Psi'(t) = \frac{d}{dt}\int_a^b G(x,t)\,\mathrm{d}x = \int_a^b f(x,t)\,\mathrm{d}x.$$

We concluded that φ, Ψ have the same derivative. That means that they at most differ by a constant L. However, since $\varphi(c) = \Psi(c) = 0$ we get that $L = 0$ and thus

$$\varphi = \Psi,$$

and more specifically $\varphi(d) = \Psi(d)$. This means that

$$\int_a^b \left(\int_c^d f(x,y)\,dy\right)\,\mathrm{d}x = \int_c^d \left(\int_a^b f(x,y)\,dx\right)\,dy$$

which completes the proof. □

10 The Chain Rule

Proposition 10.1 (The chain rule). *Let $f(x, y)$ have continuous partial derivatives on domain D. Let $x(t)$, $y(t)$ be differentiable on the interval I such that for all $t \in I$ we have $(x(t), y(t) \in D)$. Denote $F(t) := f(x(t), y(t))$. Then,*

$$F'(t) = \frac{\partial f}{\partial x}\bigg|_{(x(t),y(t))} \frac{dx}{dt} + \frac{\partial f}{\partial y}\bigg|_{(x(t),y(t))} \frac{dy}{dt}$$

Proof.

$$\frac{dF}{dt} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}$$

Denote

$$\begin{cases} \Delta x = x(t+\Delta t) - x(t) \\ \Delta y = y(t+\Delta t) - y(t) \end{cases}$$

$$= \lim_{\Delta t \to 0} \frac{f(x(t)+\Delta x, y(t)+\Delta y) - f(x(t), y(t))}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}.$$

Since f is differentiable

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \alpha(\Delta x, \Delta y)\Delta x + \beta(\Delta x, \Delta y)\Delta y$$

where $\alpha, \beta \to 0$ so:

$$F'(t) = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \alpha(\Delta x, \Delta y)\Delta x + \beta(\Delta x, \Delta y)\Delta y}{\Delta t}.$$

$$= \frac{\partial f}{\partial x}\bigg|_{(x(t),y(t))} \frac{dx}{dt} + \frac{\partial f}{\partial y}\bigg|_{(x(t),y(t))} \frac{dy}{dt}$$

which completes the proof. □

Corollary 10.2. *suppose $F(u, v) = f(x(u, v), y(u, v))$ then we see that*

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad \textit{and} \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

11 Substitution For Multiple Variables

Let f be integrable over Domain D. Let $x(u,v)$ and $y(u,v)$ be in C_1[1], and let them define an invertible transformation $\varphi : D \to E$ where D is defined on an xy plane and E on an uv plane. Now suppose

$$\mathbb{J} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \neq 0 \quad \forall (u,v) \in E$$

($*$) It can be equal to 0 in the domain if the measure of the set of those points is 0.

Then

$$\iint_D f(x,y)\,dx\,dy = \iint_E f(x(u,v),y(u,v))|\mathbb{J}|\,du\,dv.$$

[1]continuously differentiable

Exercise 11.1 (the Gaussian integral)**.** Calculate the value of the following integral:

$$\iint_{-\infty}^{\infty} e^{-x^2}.$$

Solution. First notice that

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy.$$

Now consider the integral in polar coordinates.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy &= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2} r\,d\theta\,dr = \int_{0}^{\infty}\int_{0}^{2\pi} re^{-r^2}\,d\theta\,dr \\
&= \int_{0}^{\infty} \theta r e^{-r^2}\Big|_{\theta=0}^{\theta=2\pi} dr = 2\pi\int_{0}^{\infty} re^{-r^2}\,dr \\
&= 2\pi\int_{0}^{\infty} re^{-r^2}\,dr.
\end{aligned}$$

Finally we get

$$\begin{aligned}
2\pi\int_{0}^{\infty} re^{-r^2}\,dr &= 2\pi\lim_{M\to\infty} -\frac{1}{2}e^{-r^2}\Big|_{0}^{M} \\
&= 2\pi\lim_{M\to\infty} -\frac{1}{2}(e^{-M^2} - e^{-0^2}) = 2\pi(-\frac{1}{2}(0-1)) = \pi.
\end{aligned}$$

Going back to the original integral we get

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2} = \sqrt{\pi}.$$