

# Complex Analysis

Based on lectures by  
Notes taken by yehelip

Winter 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

## Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Complex numbers and the complex plane . . . . .	3
1.1.1	Preliminaries . . . . .	3
1.1.2	Convergence . . . . .	4
1.1.3	Sets in the complex plane . . . . .	4
1.1.4	Continuous functions . . . . .	5
1.2	Holomorphic functions . . . . .	6
1.3	Power series . . . . .	8

# 1 Introduction

## 1.1 Complex numbers and the complex plane

### 1.1.1 Preliminaries

**Definition 1.1** (Complex number). A complex number is an expression of the form  $x + yi$  such that  $x, y \in \mathbb{R}$  and  $i$  is a ‘imaginary number’ not in  $\mathbb{R}$ . We denote

$$\Re(z) := x \quad \text{and} \quad \Im(z) := y.$$

If  $\Im(z) = 0$  then  $z$  is said to be a real number, and if  $\Re(z) = 0$  then it is said to be purely imaginary.

The set of all complex numbers is denoted as  $\mathbb{C}$  and it can be made into a field with the following operations.

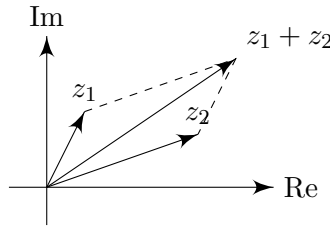
$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \quad \text{and} \quad z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The field  $\mathbb{C}$  is called the complex plane.

Note that  $i^2 = -1$ . Also note that  $T(x + yi) = (x, y)$  is a bijection between  $\mathbb{C}$  and  $\mathbb{R}^2$  and moreover, we have that  $T$  is additive. That is

$$T(z_1 + z_2) = T(z_1) + T(z_2)$$

which gives complex addition a geometric meaning.



The absolute value of a complex number  $x + yi = z \in \mathbb{C}$  is defined by

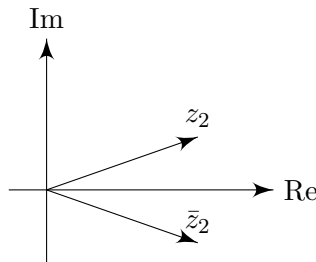
$$|z| = \sqrt{x^2 + y^2}.$$

Note that  $|z| = \|(x, y)\| = \|T(z)\|$  where  $\|\cdot\|$  is the standard Euclidean norm on  $\mathbb{R}^2$ .

This implies that  $|z - w|$  should be considered the distance between natural numbers  $z, w$ . Because we have that  $|z| = \|T(z)\|$  we also have that the triangle inequality holds:

$$|z + w| \leq |z| + |w| \quad \text{for all } z, w \in \mathbb{C}.$$

**Definition 1.2** (Complex conjugate). The complex conjugate of  $x + yi = z \in \mathbb{C}$  is the complex number  $x - yi$ . The complex conjugate of  $z$  is denoted  $\bar{z}$ .



It is easy to verify that

$$\Re(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \Im(z) = \frac{z - \bar{z}}{2i} \quad \text{and} \quad |z|^2 = z\bar{z}.$$

Given  $\theta$  we can denote  $e^{i\theta} = \cos \theta + i \sin \theta$ , and then describe any complex number  $z \in \mathbb{C}$  as  $re^{i\theta}$  for some  $\theta \in [0, 2\pi)$  and  $r > 0$ . We get that  $|z| = |re^{i\theta}| = r$ . We also have that  $\theta$  describes the angle of  $z$  with the  $x$ -axis and it is usually denoted  $\theta = \arg(z)$ .

### 1.1.2 Convergence

**Definition 1.3** (Convergence). We say that the sequence  $\{z_n\}_{n \geq 1} \subset \mathbb{C}$  converges to some  $z_0 \in \mathbb{C}$  if  $|z - z_0| \xrightarrow{n \rightarrow \infty} 0$ . In this case, we call  $z_0$  the limit of the sequence of  $\{z_n\}_{n \geq 1}$ .

**Remark 1.1.** It is easy to verify that the limit is unique, and that  $z_n \xrightarrow{n \rightarrow \infty} z$  if and only if  $T(z_n) \xrightarrow{n \rightarrow \infty} T(z)$  in the Euclidean metric.

**Definition 1.4** (Cauchy sequence). A sequence  $\{z_n\}_{n \geq 1}$  is said to be a Cauchy sequence if for all  $\epsilon > 0$  there exists  $N > 1$  such that for all  $n, m > N$  we have that  $|z_n - z_m| < \epsilon$ .

**Proposition 1.1.** *The complex plane  $\mathbb{C}$  is complete. That is, every Cauchy sequence converges in  $\mathbb{C}$ .*

*Proof.* The proof follows immediately from the known fact that  $\mathbb{R}$  is complete and the previous remark.  $\square$

### 1.1.3 Sets in the complex plane

**Definition 1.5** (Open disc). For  $z_0 \in \mathbb{C}$  and  $r > 0$  we set

$$D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

We call  $D_r(z_0)$  the open disc at center  $z_0$  with radius  $r$ .

**Definition 1.6** (Closed disc). For  $z_0 \in \mathbb{C}$  and  $r > 0$  we set

$$\overline{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

We call  $\overline{D}_r(z_0)$  the closed disc at center  $z_0$  with radius  $r$ .

**Definition 1.7** (Circle). For  $z_0 \in \mathbb{C}$  and  $r > 0$  we set

$$C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}.$$

We call  $C_r(z_0)$  the circle at center  $z_0$  with radius  $r$ .

**Definition 1.8** (Interior point). Given  $\Omega \subset \mathbb{C}$ , we say that  $z \in \Omega$  is an interior point of  $\Omega$  if exists  $r > 0$  such that  $D_r(z) \subset \Omega$ .

**Definition 1.9** (Interior of a set). Given  $\Omega \subset \mathbb{C}$ , we say that the interior of  $\Omega$  is the collection of all interior points of  $\Omega$ . We denote the interior as  $\text{Int}(\Omega)$ .

**Definition 1.10** (Open set). Given  $\Omega \subset \mathbb{C}$ , we say that  $\Omega$  is an open set if  $\text{Int}(\Omega) = \Omega$ .

**Definition 1.11** (Closed set). Given  $\Omega \subset \mathbb{C}$ , we say that  $\Omega$  a closed set if  $\Omega^c := \mathbb{C} \setminus \Omega$  is open.

**Definition 1.12** (Limit point). Given  $\Omega \subset \mathbb{C}$ , we say that  $z \in \Omega$  is an interior point of  $\Omega$  if there exists a sequence  $z_n$  such that  $z_n \neq z$  for all  $n > 1$  and  $z_n \xrightarrow{n \rightarrow \infty} z$ .

**Proposition 1.2.** *Let  $\Omega \subset \mathbb{C}$  be given. Then  $\Omega$  is closed if and only if it contains all of its limit points.*

*Proof.* Clear. □

**Definition 1.13** (Closure). Let  $\Omega \subset \mathbb{C}$  be given. The closure of  $\Omega$ , denoted  $\overline{\Omega}$ , is defined as

$$\overline{\Omega} = \Omega \cup \{z \in \mathbb{C} \mid z \text{ is a limit point of } \Omega\}.$$

**Remark 1.2.** Note that  $\Omega$  is closed if and only if  $\overline{\Omega} = \Omega$ .

**Definition 1.14** (Boundary). The boundary of  $\Omega \subset \mathbb{C}$  is denoted by  $\partial\Omega$  and defined by  $\partial\Omega := \overline{\Omega} \setminus \text{Int}(\Omega)$ .

**Definition 1.15** (Diameter). Given  $\Omega \subset \mathbb{C}$ , we define the diameter of  $\Omega$  as

$$\text{diam}(\Omega) := \sup \{|z - w| : z, w \in \Omega\}.$$

**Definition 1.16** (Bounded set). Given  $\Omega \subset \mathbb{C}$ , we say that  $\Omega$  is bounded if  $\text{diam}(\Omega) < \infty$ .

**Remark 1.3.** It is clear that a set  $\Omega \subset \mathbb{C}$  is bounded if and only if there exists  $z_0 \in \mathbb{C}$  and  $r > 0$  such that  $\Omega \subset D_r(z_0)$ .

**Definition 1.17** (Compact set). A subset  $\Omega$  of  $\mathbb{C}$  is said to be compact if it is closed and bounded.

**Theorem 1.3. (Bolzano–Weierstrass theorem).** *A subset  $\Omega$  in  $\mathbb{C}$  is compact if and only if every sequence  $\{z_n\}_{n \geq 1}$  has a subsequence  $\{z_{n_k}\}$  such that  $z_{n_k} \xrightarrow{k \rightarrow \infty} z$  for some  $z \in \mathbb{C}$ .*

**Theorem 1.4. (Cantor’s intersection lemma).** *Let  $\Omega_1, \Omega_2, \dots$  be nonempty compact subsets of  $\mathbb{C}$ . Suppose that  $\Omega_{n+1} \subset \Omega_n$  for all  $n \geq 1$ , and that  $\text{diam}(\Omega_n) \xrightarrow{n \rightarrow \infty} 0$ . Then  $\bigcap_{n \geq 1} \Omega_n = \{z\}$  for some  $z \in \mathbb{C}$ .*

*Proof.* Choose  $z_n \in \Omega_n$  for all  $n \geq 1$ . Because  $\text{diam} \Omega_n \xrightarrow{n \rightarrow \infty} 0$  we have that  $\{z_n\}_{n \geq 1}$  is a Cauchy sequence and therefore it converges to some  $z \in \mathbb{C}$ . Because  $\Omega_n$  is compact for every  $n \geq 1$  we get that  $z \in \bigcap_{n \geq 1} \Omega_n$ . This means that  $\bigcap_{n \geq 1} \Omega_n \neq \emptyset$ .

Let  $z, w \in \bigcap_{n \geq 1} \Omega_n$ . Because  $\text{diam} \Omega_n \xrightarrow{n \rightarrow \infty} 0$  we have that  $|z - w| \leq 0$  and thus  $z = w$  which implies that  $\bigcap_{n \geq 1} \Omega_n = \{z\}$  which completes the proof. □

**Definition 1.18** (Connected open set). A nonempty open set  $\Omega \subset \mathbb{C}$  is said to be connected if it does not contain disjoint nonempty open subsets  $\Omega_1, \Omega_2$  such that  $\Omega = \Omega_1 \cup \Omega_2$ . A connected open set in  $\mathbb{C}$  will be called a region.

**Definition 1.19** (Connected closed set). A nonempty open set  $\Omega \subset \mathbb{C}$  is said to be connected if it does not contain disjoint nonempty closed subsets  $\Omega_1, \Omega_2$  such that  $\Omega = \Omega_1 \cup \Omega_2$ .

**Remark 1.4.** It can be shown that  $\Omega$  is connected if and only if for any  $z, w \in \Omega$  there exists a curve  $\gamma: [0, 1] \rightarrow \Omega$  such that  $\gamma(0) = z$  and  $\gamma(1) = w$ . This implies that open and closed discs, as well as circles, are connected.

#### 1.1.4 Continuous functions

**Definition 1.20** (Continuous function). Let  $\Omega$  be a nonempty subset of  $\mathbb{C}$  and let  $f: \Omega \rightarrow \mathbb{C}$  be given. We say that  $f$  is continuous at a point  $z_0 \in \Omega$  if for all  $\epsilon > 0$  there exists  $\delta > 0$  so that  $|f(z) - f(z_0)| < \epsilon$  for all  $z \in \Omega$  with  $|z - z_0| < \delta$ . We say that  $f$  is continuous on  $\Omega$  if it is continuous at every  $z_0 \in \Omega$ .

**Remark 1.5.** It is easy to verify that the functions  $\Im$ ,  $\Re$ ,  $|\cdot|$ , and  $\theta \mapsto e^{i\theta}$  are all continuous.

**Proposition 1.5.** *The composition of continuous functions is continuous.*

**Definition 1.21** (Bounded function). Let  $\Omega$  be a nonempty subset of  $\mathbb{C}$  and let  $f: \Omega \rightarrow \mathbb{C}$  be given. We say that  $f$  is bounded if there exists  $M > 0$  so that  $|f(z)| < M$  for all  $z \in \Omega$ . We say that  $f$  attains a maximum if there exists  $z_M \in \Omega$  such that  $f(z) \leq f(z_M)$  for all  $z \in \Omega$ . We define when  $f$  attains a minimum similarly.

**Proposition 1.6.** *Let  $\Omega$  be a nonempty compact subset of  $\mathbb{C}$ , and let  $f: \Omega \rightarrow \mathbb{C}$  be continuous. Then  $f$  is bounded, and it attains its maximum and minimum on  $\Omega$ .*

## 1.2 Holomorphic functions

**Definition 1.22** (Holomorphic function). Let  $\Omega$  be a nonempty open subset of  $\mathbb{C}$  and let  $f: \Omega \rightarrow \mathbb{C}$  be given. We say that  $f$  is holomorphic at a point  $z \in \Omega$  if the following limit exists

$$f'(z) := \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}.$$

The number  $f'(z)$  is called the derivative of  $f$  at  $z$ . It is said that  $f$  is holomorphic if it is holomorphic at every  $z \in \Omega$ . Given a closed subset  $C \subset \Omega$ , we say that  $f$  is holomorphic on  $C$  if there exists  $C \subset \Omega' \subset \Omega$  so that  $\Omega'$  is open and  $f$  is holomorphic on  $\Omega'$ .

**Definition 1.23** (Entire function). We say that  $f: \mathbb{C} \rightarrow \mathbb{C}$  is entire if it is holomorphic on  $\mathbb{C}$ .

**Remark 1.6.** Note that  $h$  is a complex number and can approach 0 from any direction.

**Remark 1.7.** It is also useful to notice that  $f: \Omega \rightarrow \mathbb{C}$  is holomorphic at  $z \in \Omega$  if and only if there exist  $a \in \mathbb{C}$ ,  $r > 0$  with  $D_r(z) \subset \Omega$ , and a function  $\psi: D_r(0) \rightarrow \mathbb{C}$  with  $\lim_{h \rightarrow 0} \psi(h) = 0$ , so that

$$f(z+h) = f(z) + ah + h\psi(h) \text{ for all } h \in D_r(0).$$

From this formulation it is clear that  $f$  is continuous at  $z$  whenever  $f$  is holomorphic at  $z$ .

**Example 1.1.** It follows directly from the definition that the function  $1/z$  is holomorphic on  $\mathbb{C} \setminus \{0\}$  with  $f'(z) = -1/z^2$ . For all  $0 \neq z \in \mathbb{C}$  we have

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{1}{z+h} - \frac{1}{z} \right) = \lim_{h \rightarrow 0} \frac{-1}{z(z+h)} = \frac{-1}{z^2}.$$

**Example 1.2.** The function  $f(z) = \bar{z}$  is not holomorphic. For any  $z \in \mathbb{C}$  and  $r \in \mathbb{R}$  we have that

$$\frac{f(z+t) - f(z)}{t} = 1 \quad \text{and} \quad \frac{f(z+ti) - f(z)}{ti} = -1$$

**Proposition 1.7.** *Let  $\Omega \subset \mathbb{C}$  be open and let  $f, g: \Omega \rightarrow \mathbb{C}$  be holomorphic at  $z \in \Omega$ . Then*

- (1)  $f + g$  is holomorphic at  $z$  with  $(f + g)'(z) = f'(z) + g'(z)$ .
- (2)  $fg$  is holomorphic at  $z$  with  $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$ .

*Proof.* We will only prove (2) because the proof of (1) is much simpler. Because  $f$  and  $g$  are holomorphic at  $z$ , they are also continuous there. Thus,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(fg)(z+h) - (fg)(z)}{h} &= \lim_{h \rightarrow 0} \left( \frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right) \\ &= f'(z)g(z) + f(z)g'(z), \end{aligned}$$

which completes the proof. □

**Corollary 1.8.** *It's quite easy to prove that constant function of the form  $f(z) = c$  for some  $c \in \mathbb{C}$  and  $f(z) = z$  are holomorphic. It follows immediately from Proposition 1.7 that all polynomials, functions of the form  $p(z) = \sum_{k=0}^n a_k z^k$  are entire, with  $p'(z) = \sum_{k=1}^n k a_k z^{k-1}$  for all  $z \in \mathbb{C}$ .*

**Proposition 1.9.** *A composition of holomorphic functions at  $z$  is holomorphic at  $z$ , with  $(g \circ f)'(z) = g'(f(z))f'(z)$ .*

**Corollary 1.10.** *Let  $\Omega \subset \mathbb{C}$  be open and let  $f, g: \Omega \rightarrow \mathbb{C}$  be holomorphic at  $z \in \Omega$ . Suppose also that  $g(z) \neq 0$ . Then  $f/g$  is holomorphic at  $z$  with*

$$(f/g)'(z) = \frac{f'(z)g(z) + f(z)g'(z)}{g(z)^2}.$$

*Proof.* Let  $h: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$  be with  $h(z) = 1/z$ . We now have that

$$\begin{aligned} (f/g)'(z) &= (f \cdot (h \circ g))'(z) = f'(z)(h \circ g)(z) + f(z)(h \circ g)'(z) \\ &= f'(z)/g(z) + f(z)h'(g(z))g'(z) = f'(z)/g(z) - f(z)g(z)^{-2}g'(z). \end{aligned}$$

□

Recall that  $T: \mathbb{C} \rightarrow \mathbb{R}^2$  is the operator  $T(x + yi) = (x, y)$ .

**Proposition 1.11.** *Let  $\Omega \subset \mathbb{C}$  be open, let  $f: \Omega \rightarrow \mathbb{C}$ , let  $u, v: T(\Omega) \rightarrow \mathbb{R}$  be with  $f(x + yi) = u(x, y) + iv(x, y)$  for  $x + iy \in \Omega$ , and let  $F: T(\Omega) \rightarrow \mathbb{R}^2$  be with  $F(x, y) = (u(x, y), v(x, y))$  for  $(x, y) \in T(\Omega)$ . Fix  $x_0 + iy_0 = z_0 \in \Omega$ , write  $p = (x_0, y_0)$ , and suppose that  $f$  is holomorphic at  $z_0$ . Then,*

(1) *the partial derivatives of  $u$  and  $v$  exist at  $p$ , and*

$$f'(z_0) = \frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i \frac{\partial u}{\partial y}(p);$$

(2) *The Cauchy–Riemann equations are satisfied:*

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p) \text{ and } \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p).$$

(3)  *$F$  is differentiable at  $p$  with,*

$$dF_p = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & -\frac{\partial v}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial u}{\partial x}(p) \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y}(p) & \frac{\partial u}{\partial y}(p) \\ -\frac{\partial u}{\partial y}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}.$$

**Remark 1.8.** Note that  $u = \Re \circ f \circ T^{-1}$ ,  $v = \Im \circ f \circ T^{-1}$  and  $F = T \circ f \circ T^{-1}$ . Thus,  $F$  is the map corresponding to  $f$  under the identification of  $\mathbb{C}$  with  $\mathbb{R}^2$  via  $T$ .

**Remark 1.9.** Note that from (3) we have

$$\det(dF_p) = \left( \frac{\partial u}{\partial x}(p) \right)^2 + \left( \frac{\partial v}{\partial x}(p) \right)^2.$$

From this and from (1), it follows that  $\det(dF_p) > 0$  whenever  $f'(z_0) \neq 0$ . Moreover, we have that  $\sqrt{\det(dF_p)} \cdot dF_p$  is an orthogonal matrix.

We now prove Proposition 1.11.

*Proof.* To be added.

□

The following proposition is a kind of converse to the previous proposition.

**Proposition 1.12.** *Let  $\Omega \subset \mathbb{C}$  be open, let  $f: \Omega \rightarrow \mathbb{C}$ , and let  $u$  and  $v$  be as in Proposition 1.11. Fix  $x_0 + iy_0 = z_0 \in \Omega$ , write  $p := (x_0, y_0)$ , and suppose that  $u$  and  $v$  are differentiable at  $p$ , that is  $\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p)$  and  $\frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$ . Then  $f$  is holomorphic at  $z_0$ .*

*Proof.* To be added.

□

### 1.3 Power series

**Definition 1.24** (Power series). A power series centered at  $z_0 \in \mathbb{C}$  is an expression of the form  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ , where  $\{a_n\}_{n \geq 0} \subset \mathbb{C}$ . Given  $z \in \mathbb{C}$ , we say that the power series converges at  $z$  if the limit  $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n(z - z_0)^n$  exists in  $\mathbb{C}$ . If this limit does not exist, we say that the series diverges at  $z$ .

**Definition 1.25** (Absolute convergence). Given a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)$ , we say that it converges absolutely at  $z \in \mathbb{C}$  if  $\sum_{n=0}^{\infty} |a_n| \cdot |(z - z_0)| < \infty$ .

**Proposition 1.13.** *If a power series converges absolutely at  $z$  then it also converges at  $z$ . This follows from the completeness of  $\mathbb{C}$ .*

In the following proposition we use the convention that  $\frac{1}{0} = \infty$  and  $\frac{1}{\infty} = 0$ .

**Proposition 1.14** (Hadamard's theorem). *Let  $\sum_{n=0}^{\infty} a_n(z - z_0)$  be a power series, and let  $0 \leq R \leq \infty$  be given by*

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}.$$

*Then for  $z \in \mathbb{C}$  the series converges absolutely if  $|z - z_0| < R$ , and the series diverges if  $|z - z_0| > R$ .*

**Remark 1.10.** The number  $R$  is called the radius of convergence of the power series, and the region  $\{z \in \mathbb{C} : |z - z_0| < R\}$  is called the disc of convergence.

We now proceed to prove Proposition 1.14

*Proof.* Set  $L := 1/R$ . Suppose first that  $0 < R \leq \infty$ , so that  $0 \leq L < \infty$ . Let  $z \in \mathbb{C}$  be such that  $|z - z_0| < R$ , then there exists  $L < M < \infty$  so that  $M|z - z_0| < 1$ . By the definition of  $L$  (the limsup) there exists  $N \geq 1$  so that  $|a_n|^{\frac{1}{n}} < M$  for all  $n > N$ . Thus

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| \cdot |z - z_0|^n &= \sum_{n=0}^{N-1} |a_n| \cdot |z - z_0|^n + \sum_{n=N}^{\infty} \left( |a_n|^{\frac{1}{n}} \cdot |z - z_0| \right)^n \\ &\leq \sum_{n=0}^{N-1} |a_n| \cdot |z - z_0|^n + \sum_{n=N}^{\infty} (M|z - z_0|)^n < \infty. \end{aligned}$$

Suppose next that  $0 \leq R < \infty$ , so that  $0 < L = \infty$ . Let  $z \in \mathbb{C}$  be such that  $|z - z_0| > R$ , then similarly there exists  $0 < M < L$  so that  $M|z - z_0| > 1$ . Then, for every  $N \geq 1$  there exists  $n \geq N$  so that  $|a_n|^{\frac{1}{n}} > M$ . For such  $n$  we have

$$\begin{aligned} \left| \sum_{k=0}^n a_k(z - z_0)^k - \sum_{k=0}^{n-1} a_k(z - z_0)^k \right| &= |a_n| \cdot |z - z_0|^n \\ &= \left( |a_n|^{\frac{1}{n}} \cdot |z - z_0| \right)^n > (M|z - z_0|)^n > 1, \end{aligned}$$

which shows that the partial sums do not form a Cauchy sequence. Thus the series diverges at  $z$ , which completes the proof.  $\square$

**Example 1.3.** Consider the power series  $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ . Because we have

$$\sqrt[n]{(2n)!} \geq \sqrt[n]{n^n} = n$$

we also have for every  $n \geq 1$ ,

$$\left( \frac{1}{(2n)!} \right)^{\frac{1}{2n}} \leq \frac{1}{n^{\frac{1}{2}}} \quad \text{and} \quad \left( \frac{1}{(2n+1)!} \right)^{\frac{1}{2n+1}} \leq \frac{1}{n^{\frac{1}{2}}}.$$



Since  $n^{-\frac{1}{2}} \xrightarrow{n \rightarrow \infty} 0$  we get that the radius of convergence is  $\infty$  for the series. The map  $z \mapsto e^z$  is called the exponential function. We also have that

$$e^z e^w = \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{w^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

**Example 1.4.** Consider the power series  $f(z) := \sum_{n=0}^{\infty} z^n$ . Since  $1^{\frac{1}{n}} = 1$  we get that the radius of convergence in this case is 1. Thus  $f$  defined a function from  $D_1(0)$  to  $\mathbb{C}$ . Moreover, since we have

$$(1-z) \sum_{n=0}^N z^n = 1 - z^{N+1},$$

we get for  $z \in D_1(0)$  that

$$f(z) = \lim_{N \rightarrow \infty} \sum_{n=0}^N z^n = \lim_{N \rightarrow \infty} \frac{1 - z^{N+1}}{1 - z} = \frac{1}{1 - z}.$$

**Proposition 1.15.** *Let*