Riemann Surfaces

Based on lectures by

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction

Definition 1.1 (Riemann surface)**.** A Riemann surface is a 1-dimensional complex manifold.

Definition 1.2 (Riemann surface)**.** A Riemann surface is a topological space X together with open subsets $\{U_k\}_{k\in I}$ of X with $\cup_{k\in I} U_k = X$ together with maps $f_i : U_i \to \mathbb{C}$ such that

(1) Each f_i is a homeomorphism onto its image.

(2) If $U_i \cap U_j \neq \emptyset$ then $f_i \circ f_j^{-1} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ are *biholomorphic.*

Remark 1.1. A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at p if $f'(p) = \lim_{z\to p} \frac{f(z)-f(p)}{z-p}$ exists.

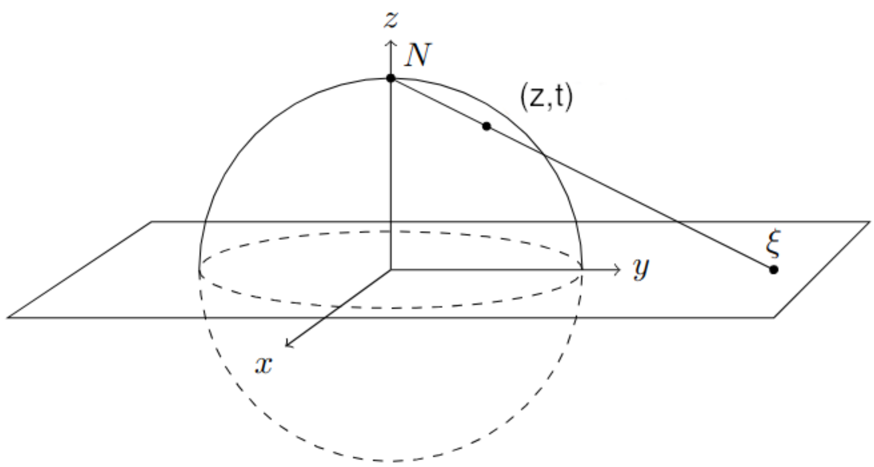
Definition 1.3 (Biholomorphism)**.** A function $f : \mathbb{C} \to \mathbb{C}$ is called biholomorphic if it has an inverse and both f and f' are holomorphic.

Definition 1.4 (Atlas)**.** The $\{(U_i, f_i)\}_{i\in I}$ are called an atlas of the Riemann surface.

Definition 1.5 (Chart)**.** Each individual (U_i, f_i) is called a chart of the Riemann surface.

Example 1.1. Let $U \subset \mathbb{C}$. Then U can take an atlas with one chart which is the identity map.

Example 1.2 (Riemann sphere)**.** Let $X = \{(z, t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + t^2 = \mathbb{R}\}$. We identify $\mathbb{C}$ with the xy plane. Denote N and S the north and south poles of the sphere accordingly. We define $\pi_N : \mathbb{C} \to S$ such that π_N sends each point (z, t) on the sphere to its stereographic projection from N onto the plane (point ξ) as can be seen in the figure below:



We can similarly define π_S and verify that the images of the projections are $X\setminus\{N\}$ and $X\setminus\{S\}$ accordingly.

Now X is a Riemann surface with an atlas consisting of $\pi_S : X\setminus\{S\} \to \mathbb{C}$ and $\pi_N : X\setminus\{N\} \to \mathbb{C}$. We denote the Riemann sphere as $\hat{\mathbb{C}}$.

Definition 1.6 (Biholomorphism of Riemann surfaces)**.** Let $(X, (U_i, f_i))$, $(Y, (W_i, g_i))$ be two Riemann surfaces. A biholomorphism between them is a homeomorphism $X \xrightarrow{\phi} Y$ such that $g_i \circ \phi \circ f_i^{-1}$ are biholomorphisms on their domains of definition.

A main problem in Riemann surfaces was classifying certain types of Riemann surfaces up to biholomorphisms.

Theorem 1.1. (Riemann mapping theorem). *Any two proper open simply connected subsets of $\mathbb{C}$ are biholomorphic.*

A generalization of the Riemann mapping theorem is the uniformization theorem proved by Kobe in 1907.

Theorem 1.2. (Uniformization theorem). *Any simply connected Riemann surface is biholomorphic to one of the following:*

(1) $\mathbb{C}$

(2) $\hat{\mathbb{C}}$

(3) $\mathbb{H} = \left\{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\right\}$

We will give a proof for this theorem in the end of the class.

A natural question that arises is what about non simply connected Riemann surfaces?

Theorem 1.3. (Uniformization theorem, part II). *Any connected Riemann surface is biholomorphic either to* $\hat{\mathbb{C}}$ *or to a quotient of* $\mathbb{C}$ *or* $\mathbb{H}$ *by a properly discontinuous torsion-free subgroup of biholomorphisms.*

Remark 1.2. Biholomorphisms of $U = \mathbb{C}$ or $\mathbb{H}$ (or any subset of $\mathbb{C}$) forms a group under composition. We denote that group by $\operatorname{Bih}(U)$.

Definition 1.7 (Properly discontinuous group)**.** A countable subgroup of $\operatorname{Bih}(U)$ is said to be properly discontinuous if for all compact $K \subseteq U$, the set $\left\{g \in G \colon gK \cap K \neq \emptyset\right\}$ is finite.

Definition 1.8 (torsion-free action)**.** $G \subseteq \operatorname{Bih}(U)$ is torsion-free if $gp = p$ for some $p \in U$ implies g is the identity.

Remark 1.3. Notice that multiplication in gp is the group action of g on the set U. That us $gp = g(p)$.

We can know define the quotient space ${}^{U}/_{G}$ where $p \sim q$ if there exists $g \in G$ such that $gp = q$.

Introduce a topology on ${}^{U}/_{G}$ which is the coarsest topology such that the canonical projections $U \to {}^{U}/_{G}$ are continuous.

Under the assumptions that G is properly discontinuous and torsion-free, ${}^{U}/_{G}$ is a Riemann surface with the following charts. By assumptions on G, we can find for any $p \in U$ a neighbourhood W of $p \in U$ such that $\pi \colon U \to {}^{U}/_{G}$ is a homeomorphism onto its image when restricted to W.

So, restrictions of π to these neighbourhoods W provide an atlas.

Definition 1.9 (Free action)**.** Let G

2 Introduction to Teichmuller spaces

Let S be a topological space. Then

$$\text{Modulispace}(S) = \{\text{Riemann surfaces homeomorphic to } S \text{ up to biholomorphisms}\}$$

Definition 2.1 (Teichmuller space)**.** We consider pairs (X, f) where X is a Riemann space and $f \to X$ is a homeomorphism. Then

$$\text{Teich}(S) = \big\{(X, f) \colon f \colon S \to X\big\} /_{\sim}$$

where $(X_1, f_1) \sim (X_2, f_2)$ if a biholomorphism $X_1 \xrightarrow{\phi} X_2$ homotopic to $f_2 \circ f_1^{-1}$. In other words commutes up to homotopy.

Remark 2.1. The pair (X, f) is called a *marking* for S.

Definition 2.2 (Homotopy)**.** We say that two continuous functions f, g from a topological space X to Y are homotopic if there exists a continuous function $H \colon X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Definition 2.3 (Mapping class group)**.** First we define:

$$\text{Homeo}^+(S) = \{\text{Orientation preserving homeomorphisms } S \to S\}$$

$$\text{Homeo}^0 \triangleleft \text{Homeo}^+(S) = \text{the homeo. homotopic to the identity.}$$

And now we define

$$\text{MCG}(S) = \text{Homeo}^+(S) \Big/ \text{Homeo}^0(S) = \frac{\text{Diff}^+(S)}{\text{Diff}^0(S)}.$$

We have that acts on $\text{Teich}(S)$ by $\varphi \in \text{Homeo}^+(S), [\varphi] \in \text{MCG}(S)$ as such

$$[\varphi][(X, f)] = [(X, f \circ e^{-1})].$$

Now $\mathcal{M}(S) = \text{Teich}(S) \Big/ \text{MCG}(S)$. Our goal is to determine $\mathcal{M}(S)$ and $\text{MCG}(S)$ when S is a torus.

Theorem 2.1. *Assume S is a torus $T \cong \mathbb{R}^2 / \mathbb{Z}^2$. Then $\text{MCG}(S) = \text{SL}_2(\mathbb{Z})$ acting linearly on the torus. $\mathcal{M}(S)$ can be identified with $\mathbb{H} \Big/ \text{SL}_2(\mathbb{Z})$ where the action is by Mobius transformations.*

Proof. Any Riemann surfaces homeomorphic to S has to form $\mathbb{C} / \Lambda$ where $\Lambda \subseteq \text{Bih}(\mathbb{C})$ is properly disc, free, and $\Lambda\langle z \to z_{\tau 1}, z \to z + \tau_2, \rangle where \tau_1, \tau_2 \notin \mathbb{R}$. We have to determine when different $\mathbb{C} / \Lambda$ are biholomorphic. We can also write $\mathbb{C}\Lambda$ where $\Lambda = \langle \tau_1, \tau_2 \rangle \subseteq \mathbb{C}$. If $\Lambda_1 = \langle \tau_1, \tau_2 \rangle$, $\Lambda_2 = \langle c\tau_1, c\tau_2 \rangle$ for $c \in \mathbb{C}$. Then $\mathbb{C} / \Lambda_1 \to \mathbb{C} / \Lambda_2$ is given by $[z] \to [cz]$.

For any $\tau_1, \tau_2 \in \mathbb{C}$ there exists $c \in \mathbb{C}^\times$ such that (up to change in order)

This tells us that any Riemann torus is biholomorphic to $\mathbb{C} \Big/ \langle XX, XX \rangle$ where $\tau \in \mathbb{H}$.

So we have a surjection $\mathbb{H} \to \mathcal{M}(S)$. □

Theorem 2.2. $\mathbb{C} \Big/ \langle 1, \tau_1 \rangle$ *is biholomorphic to* $\mathbb{C} \Big/ \langle 1, \tau_2 \rangle$ *iff exists $A \in \text{SL}_2(\mathbb{Z})$ such that writing τ_1, τ_2 as elements of $\mathbb{R}^2$, we have $A\tau_1 = \tau_2$.*

Proof. Suppose there exists a biholomorphism $f \colon \mathbb{C} \Big/ \langle 1, \tau_1 \rangle \to \mathbb{C} \Big/ \langle 1, \tau_2 \rangle$. Let $\bar{f} \colon \mathbb{C} \to \mathbb{C}$ be a lift of f. This means $f(g + x) - f(x) \in \langle 1, \tau_1, \rangle whenever g \in \langle 1, \tau_1 \rangle$.

$$f \colon \mathbb{C} \Big/ \langle 1, \tau_1 \rangle \to \mathbb{C} \Big/ \langle 1, \tau_2 \rangle bih.$$
$$\bar{f} \colon \mathbb{C} \to \mathbb{C} lift.$$

By post composing with a biholomorphism of $\mathbb{C}$ we can assume $\bar{f}(0) = 0$. We know that $\bar{f}(\tau_2)$ and $\bar{f}(1)$ are equivalent mod $\langle 1, \tau_1 \rangle$. □