Introduction to Metric and Topological Spaces

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1 Metric Spaces

First we will begin with metric spaces.

Definition 1.1. Let X be a non-empty set. A metric on X is a function $d\colon X \times X \to [0, \infty]$ such that for all $x, y, z \in X$,

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality);

The pair (X, d) is said to be a ***metric space****.*

Example 1.1. Let X be a non-empty set. Let $d\colon X \times X \to [0, \infty)$ be the function such that for $x, y \in X$,

$$d(x, y) := \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

The function d is a metric and it is called **the discrete metric** on X.

Example 1.2. Let $X = \mathbb{R}^n$ and define the function:

$$d(x - y) := |x - y|$$

Where $|\cdot|\colon \mathbb{R} \to \mathbb{R}$ is the Eclidean norm function. Then the pair (X, d) forms a metric space.

Example 1.3. Let (X, N) be an arbitrary normed space and define the function:

$$d(x - y) := N(x - y)$$

Then the pair (X, d) forms a metric space.

Example 1.4. The pair $(C([0, 1]), d)$ such that $C([0, 1])$ is the space of all continuous functions on $[0, 1]$ paired with the metric:

$$d(f, g) = \int_0^1 |f(x) - g(x)|\, dx$$

Is also a metric space.

Example 1.5. The pair $(C([0, 1]), d)$ paired with the supremum metric:

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

Is also also metric space.

Example 1.6. Let Λ be a nonempty set which will represent an alphabet. The set $\Lambda^{\mathbb{N}}$ represents all the sequences over that alphabet. The pair $(\Lambda^{\mathbb{N}}, d)$ with the metric d defined on two sequences $\omega = (\omega_n)_{n=1}^{\infty}, \eta = (\eta_n)_{n=1}^{\infty}$ as:

$$d(\omega, \eta) = \begin{cases} 2^{-\min\{n \geq 0 | \omega_n \neq \eta_n\}} & \omega \neq \eta \\ 0 & \omega = \eta \end{cases}$$

2 Compactness

Let X be a fixed topological space.

Definition 2.1. A class $\mathcal{U} := \{U_i\}_{i\in I}$ of open subsets of a X is said to be an *open cover of* X if $X = \bigcup_{i\in I} U_i$. A subclass of $\mathcal{U}$ is said to be a subcover of $\mathcal{U}$ if it is in itself an open cover of X.

Definition 2.2. The space X is said to be compact if every open cover of X has a finite subcover.

Definition 2.3. A subset Y of X is said to be compact if for every family of open sets $\{U_i\}_{i\in I}$ such that $Y \subset \bigcup_{i\in I} U_i$ exists a finite index set $I_0 \subset I$ such that $Y \subset \bigcup_{i\in I_0} U_i$.

Remark 2.1. It follows easily from the definition of the subspace topology that a nonempty subset Y of X is compact if and only if Y is a compact space when equipped with the subspace topology.

Proposition 2.1. *Suppose that X is compact and let $F \subset X$ be closed. Then F is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of F. Since F is closed we know that $X \setminus F \cup \{U_i\}_{i\in I}$ is an open cover of X. Since X is compact exists a finite index set $I_0 \subset I$ such that $X \setminus F \cup \{U_i\}_{i\in I_0}$ is a finite open cover of X. It is clear that $F \subset \{U_i\}_{i\in I_0}$ which completes the proof. □

Proposition 2.2. *Suppose X is compact, let Y be a topological space, and let $f\colon X \to Y$ be continuous. Then $f(X)$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $f(X)$. Since f is continuous $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is compact exists a finite index set $I_0 \subset I$ such that $\{f^{-1}(U_i)\}_{i\in I_0}$ is an open cover of X. We now have:

$$f(X) = f(\cup_{i\in I_0} f^{-1}(U_i)) = \cup_{i\in I_0} f(f^{-1}(U_i)) = \cup_{i\in I_0} U_i$$

Which completes the proof. □

Here are some more equivalent forms of compactness that are ofter easier to apply.

Proposition 2.3. *The space X is compact if and only if for every class $\{F_i\}_{i\in I}$ of closed subsets of X with $\cap_{i\in I} F_i = \emptyset$ there exists a finite $I_0 \subset I$ with $\cap_{i\in I_0} F_i = \emptyset$.*

Proof. Assume X is compact. Let $\{F_i\}_{i\in I}$ be a family of closed subsets of X with $\cap_{i\in I} F_i = \emptyset$ then we have $\cap_{i\in I} X \setminus F_i = X$ which is a cover of X thus exists a finite $I_0 \subset I$ with $\cap_{i\in I_0} X \setminus F_i = X$ being a finite subcover of X. This implies that $\cap_{i\in I_0} F_i = \emptyset$ which completes the proof. The proof of the other direction is similar and thus omitted. □

Definition 2.4. Let S be a nonempty set. A class of subsets $\{E_i\}_{i\in I}$ of S is said to have the **finite intersection property** if $\cap_{i\in I_0} E_i \neq \emptyset$ for every finite $I_0 \subset I$.

Proposition 2.4. *The space X is compact if and only if every class of closed subsets of X with the finite intersection property has nonempty intersection.*

Proposition 2.5. *Let B be an open base for X. Suppose that every open cover $\{B_i\}_{i\in I} \subset B$ of X has a finite subcover. Then X is compact.*

Definition 2.5. A family $\mathcal{B}$ of closed subsets of X is called a **closed base** for X if the collection

$$\{X \setminus B \colon B \in \mathcal{B}\}$$

is an open base for X. Similarly, a family $\mathcal{S}$ of closed subsets of X is called a **closed subbase** for X if the collection $\{X \setminus S \colon S \in \mathcal{S}\}$ is an open subbase for X.

Remark 2.2. Note that if $\mathcal{S}$ is a closed subbase for X then the set $\mathcal{B}$ of all finite unions of elements of $\mathcal{S}$ forms a closed base for X. This is so since, by definition, the set of all finite intersections of an open subbase forms an open base. We call $\mathcal{B}$ the closed base generated by $\mathcal{S}$.

Proposition 2.6. *Let $\mathcal{B}$ be a closed base for X. Suppose that for every $\{B_i\}_{i\in I} \subset \mathcal{B}$ with the finite intersection property we have $\cap_{i\in I} B_i = \emptyset$. Then X is compact.*

3 The Alexander Subbase Theorem

Theorem 3.1. *Let $\mathcal{S}$ be an open subbase for X. Suppose that every open cover $\{S_i\}_{i \in I} \subset \mathcal{S}$ of X has a finite subcover. Then X is compact.*

The thoerem also has a second form:

Theorem 3.2. *Let $\mathcal{S}$ be a closed subbase for X. Suppose that $\cap_{i \in I} S_i = \emptyset$ for every $\{S_i\} i \in I \subset \mathcal{S}$ with the finite intersection property. Then X is compact.*

The proof of this theorem is concerned with Zorn's lemma and will be omitted for now.

4 Boundedness

Definition 4.1. Let X be a metric space. We say that $A \subset X$ is **bounded** if exists $r > 0$ and $x \in X$ such that $A \subset B(x, r)$.

Note that it is easy to see that $A \subset X$ is bounded if and only if it has a finite diameter.

Lemma 4.1. *Let $\mathcal{S}$ be an open subbase for a topological space X. If $Y \subset X$ is a subset of X equiped with the subspace topology induced by X then $\{S \cap Y \mid S \in \mathcal{S}\}$ is an open subbase for Y.*

Proof. Let U be a nonempty subset of Y and let $y \in U$. There exists W an open set in X such that $W \cap Y = U$. Because $\mathcal{S}$ is a subbase for X exists $S_1, \ldots, S_n \in \mathcal{S}$ such that $y \in \cap_{i=1}^n S_i \subset W$ and thus because $y \in Y$:

$$y \in \cap_{i=1}^n S_i \cap Y \subset W \cap Y = U$$

Because $S_i \cap Y$ are all open in Y we have that indeed $\{S \cap Y \mid S \in \mathcal{S}\}$ is an open subbase as wanted. □

We will now prove the Heine-Borel theorem in $\mathbb{R}$.

Theorem 4.2. *Every closed and bounded set in $\mathbb{R}$ is compact.*

Proof. Let A be a closed and bounded set in $\mathbb{R}$. Because A is bounded we know that exist real numbers $a, b \in \mathbb{R}$ such that $a < b$ and also $A \subset [a, b]$. If we equip $[a, b]$ with the subspace topology induced on it by $\mathbb{R}$ it is not hard to see that A is closed in $[a, b]$ and thus it suffices to verify that $[a, b]$ is compact in $\mathbb{R}$. It's easy to check that the set:

$$\{(-\infty, c) \mid c \in \mathbb{R}\} \cup \{(d, \infty) \mid d \in \mathbb{R}\}$$

Is an open subbase to $\mathbb{R}$. From the lemma we have that the set:

$$S = \{[a, c) \mid a < c \leq b\} \cup \{(d, b] \mid a < d \leq b\}$$

Is an open subbase for $[a, b]$. Let $\mathcal{U} \subset S$ be an open cover of $[a, b]$, by Alexander's subbase theorem it suffices to show that $\mathcal{S}$ has a finite subcover. Since $\mathcal{U} \subset \mathcal{S}$ there exist index sets I, J such that:

$$\mathcal{U} = \{[a, c_i) \mid i \in I\} \cup \{(d_j, b] \mid j \in J\}$$

We have that $a \in [a, b]$ and $\mathcal{U}$ a cover of $[a, b]$ which means that $I \neq \emptyset$. Denote $s = \sup\{c_i\}_{i \in I}$, if we have $s \leq d_j$ for all $j \in J$ we have $s \notin \cup\mathcal{U}$ which is a contradiction. Otherwise exists $j_0 \in J$ such that $d_{j_0} < s$ and then by definition exists $i_0 \in I$ such that $d_{j_0} < c_{i_0} < s$ and then we have that $\{[a, c_{i_0}), (d_{j_0}, b]\}$ is a finite subcover of $[a, b]$ which completes the proof. □

5 Tychonoff's theorem

Theorem 5.1. *Let $\{X_i\}_{i\in I}$ be a nonempty family of compact topological spaces. Equip $\prod_{i\in I} X_i$ with the product topology. Then $\prod_{i\in I} X_i$ is compact.*

Proof. Set:

$$\mathcal{S} = \left\{ \prod_{i\in I} F_i \mid \exists i_0 \in I \text{ s.t. } (\forall i \in I \setminus \{i_0\})(F_i = X_i) \wedge F_{i_0} \text{ is closed in } X_{i_0} \right\}$$

This is the standard closed subbase for $\prod_{i\in I} X_i$. Let $\{S_j\}_{j\in J} \subset \mathcal{S}$ be with the finite intersection property. By Alexander's subbase theorem it suffices to prove that $\cap_{j\in J}\{S_j\} \neq \emptyset$ to conclude that $\prod_{i\in I} X_i$ is compact. NOT COMPLETED □

We can now prove a couple lemmas and then show that the Heine-Borel theorem is applicable on $\mathbb{R}^d$ as well for $d \in \mathbb{N}$.

Definition 5.1. A topological space X is called **locally compact** if for any $x \in X$ exists a neighbourhood $U \subset X$ of x so that $\overline{U}$ is compact.

As an immediate result we get that for each $d \geq 1$ that $\mathbb{R}^d$ is locally compact.

Definition 5.2. The metric space X is said to be **sequentially compact** if every sequence in X has a convergent subsequence.

Definition 5.3. The metric space X is said to have the **Bolzano–Weierstrass property** if every infinite subset of X has a limit point in X.

It is important to note that in metric spaces, sequential compactness and the Bolzano–Weierstrass property are both equivalent to compactness. We will omit the proofs because there's not enough time. Here are some more definitions without motivation, and a lemma without a proof.

Definition 5.4. Let $\{U_i\}_{i\in I}$ be an open cover of X. A real number $\delta > 0$ is said to be a **Lebesgue number** for $\{U_i\}_{i\in I}$ if for all nonempty $A \subset X$ with $\operatorname{diam}(A) < \delta$ there exists $i \in I$ so that $A \subset U_i$.

Lemma 5.2. *(Lebesgue's covering lemma). Suppose that X is sequentially compact. Let $\{U_i\}_{i\in I}$ be an open cover of X. Then $\{U_i\}_{i\in I}$ has a Lebesgue number.*

Definition 5.5. Let $\epsilon > 0$ be given. A nonempty subset A of X is said to be an **ϵ-net** if A is finite and $X = \cup_{a\in A} B(a, \epsilon)$.

Definition 5.6. We say that X is **totally bounded** if it has an ϵ-net for all $\epsilon > 0$.

It is clear that a totally bounded space is also bounded. Using Lebesgue's lemma we can also prove the following proposition:

Proposition 5.3. *Suppose that a metric space X is compact. Let (Y, d_Y) be a metric space, and let $f\colon X \to Y$ be continuous. Then f is uniformly continuous.*

Proof. Let $\epsilon > 0$. Since f is continuous the set $f^{-1}(B(f(x), \epsilon/2))$ is open for any $x \in X$ and thus the set:

$$\mathcal{U} := \{f^{-1}(B(f(x), \epsilon/2))\}_{x\in X}$$

Is an open cover for X. Because X is a compact metric space it is also sequencially compact, and thus from Lebesgue's lemma we have that exists a Lebesgue number $\rho > 0$ for $\mathcal{U}$. Now let $x_1, x_2 \in X$ such that $d(x_1, x_2) < \rho$, by definition exists $x \in X$ such that $x_1, x_2 \in f^{-1}(B(f(x), \epsilon/2))$, thus:

$$d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), f(x)) + d_Y(f(x), f(x_2)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

There is also a connection between compactness and total boundness as we see in the following proposition.

Proposition 5.4. *The metric space X is compact if and only if it is complete and totally bounded.*

The proof will be omitted for now.

Corollary 5.5. *Suppose that X is complete and let A be a nonempty closed subset of X. Then A is compact if and only if it is totally bounded.*

6 The Arzelà–Ascoli theorem

First we define a new structure. Let K be a field and A a vector space. Let $|\cdot|\colon A \times A \to A$ be a binary operation. Then A is called an **algebra** if for each $x, y, z \in V$ the following identites hold:

- Left distributiviy: $(x + y) \cdot z = x \cdot z + y \cdot z$.
- Right distributiviy: $z \cdot (x + y) = z \cdot x + z \cdot y$.
- Compatibility with scalars: $(ax) \cdot (by) = (ab)(x \cdot y)$.

These identites actually just imply that the operation is bilinear. An algebra over K is sometimes called a K-algebra and K is called the base field of A. Notice that we didn't require the operation to be associative or commutative, although some authors use the term "algebra" to refer to an associative algebra.

Definition 6.1. Given K-algebras A, B then a homomorphism of K-algebras is a K-linear map $f\colon A \to B$ such that $f(xy) = f(x)f(y)$ for all $x, y \in A$. If A and B are unital then the morphism $f(1_A) = 1_B$ is called the unital homomorphism. The space of all K-algebra homomorphisms between A and B is usually written as $\mathrm{Hom}_{K\text{-alg}}(A, B)$. A K-algebra isomorphism is a bijective K-algebra homomorphism.

A subalgebra of a K-algebra A is a linear subspace of A such that all products and sums of the subspace are themselves elements of the subspace. For examples $\mathbb{R}$ with complex addition and multiplication as a subspace of the $\mathbb{R}$-algebra $\mathbb{C}$ is an example of a subalgebra.

Similarly to rings, algebras also have a concept of ideals. A left ideal L of a K-algebra A, is a linear subspace of A such that for any $x, y \in L$, $c \in K$, $z \in A$ the following three identities are satisfied:

- L is closed under addition: $x + y \in L$
- L is closed under scalar multiplication: $cx \in L$
- L is closed under vector multiplication from the left by arbitrary elements: $z \cdot x \in L$

We can similarly define a right ideal. An ideal that is both a left and a right ideal is called a two-sided ideal or simply an ideal. Notice that every ideal is a subalgebra and that in a commutative algebra any ideal is a two-sided ideal. Also notice that in contrast to an ideal of rings, here we also have a the requirement for closure under scalar multiplication and not just being a subgroup of addition. If the algebra is unital then the third requirement implies the second one.

You can also talk about extension of scalars but I don't know what that is yet.

Let (X, d) be a fixed compact metric space. Denote $C(X)$ the algebra of all continuous functions $f\colon X \to \mathbb{R}$ and $C_b(X)$ the subalgebra of all the boundded functions in $C(X)$. Because X is compact we know that the image $f(X)$ of any $f \in C(X)$ is compact and in particular bounded and thus $C_b(X) = C(X)$. This means we can set the norm $|\cdot|_\infty$ on $C(X)$. We can thus consider $C(X)$ as a metric space with the metric induced on it by $|\cdot|_\infty$. We will soon establish a useful characterisation of the compact sets in $C(X)$.

Definition 6.2. A subset $F \subset C(X)$ is called **equicontinuous** if for any $\varepsilon > 0$ exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for any $f \in F$ and $x, y \in X$ with $d(x, y) < \delta$.

Theorem 6.1. *(Arzelà–Ascoli theorem). Let F be a nonempty closed subset of $C(X)$. Then F is compact if and only if it is bounded and equicontinuous.*

Remark 6.1. It is easy to see that F is bounded if and only if there exists $M > 1$ so that $|f(x)| \leq M$ for all $f \in F$ and $x \in X$.

7 Seperation

Let X be a fixed topological space.

Definition 7.1. We say that X is a T_1**-space** if and only if for every $x_1, x_2 \in X$ exist neighbourhoods U_1 of x_1 and U_2 of x_2 such that $x_1 \notin U_2$ and $x_2 \notin U_1$.

We can also verify that if X is a T_1-space then every topological subspace of X is also a T_1-space.

Proposition 7.1. *The space X is a T_1**-space** if and only if $\{x\}$ is closed in X for every $x \in X$.*

Proof. Suppose that X is a T_1-space. Let $x \in X$. For every $y \in X \setminus \{x\}$ exists a neighbourhood $U_y \subset X \setminus \{x\}$ the union of which gives $X \setminus \{x\}$ and then $\{x\}$ is closed as wanted. Now assume that $\{x\}$ is closed for every $x \in X$. For two points $x_1, x_2 \in X$ the sets $\{x_1\}, \{x_2\}$ are closed and thus we have $U_1 := X \setminus \{x_1\}$ neighbourhood of x_1 and $U_2 := X \setminus \{x_2\}$ neighbourhood of x_2 such that $x_1 \notin U_2$ and $x_2 \notin U_1$. $\square$

Definition 7.2. We say that X is a **Hausdorff space** if for all distinct $x_1, x_2 \in X$ there exist open sets $U1, U2 \subset X$ with $x_1 \in U_1, x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

We can verify that every Hausdorff space is a T_1-space and that if X if a Hausdorff is a topological space then every subspace of X is also a Hausdorff space.

Proposition 7.2. *Let $\{X_i\}_{i\in I}$ be a nonempty family of Hausdorff spaces. Then the product space $\prod_{i\in I} X_i$ is also a Hausdorff space.*

Proof. Let $\{x_i\}_{i\in I}, \{y_i\}_{i\in I}$ be distinct points in $\prod_{i\in I} X_i$. Therefore exists $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$. Because X_{i_0} is a Hausdorff space there exist open sets $U_x, U_y \subset X_{i_0}$ with $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$. We know that the projection $\pi_{i_0} \colon \prod_{i\in I} X_i \to X_{i_0}$ is continuous and thus $\pi_{i_0}^{-1}(U_x)$ and $\pi_{i_0}^{-1}(U_y)$ are two open and disjoint sets of $\prod_{i\in I} X_i$ such that $\{x_i\}_{i\in I} \in \pi_{i_0}^{-1}(U_x)$ and $\{y_i\}_{i\in I} \in \pi_{i_0}^{-1}(U_y)$ as wanted. This shows that $\prod_{i\in I} X_i$ is a Hausdorff space which completes the proof. $\square$

The following proposition is one of the most important properties of Hausdorff spaces.

Proposition 7.3. *Suppose that X is a Hausdorff space. Let K be a compact subset of X with $K \neq X$, and let $x \in X \setminus K$. Then there exist open sets $U, V \subset X$ so that $x \in U, K \subset V$ and $U \cap V = \emptyset$.*

Proof. First we may suppose that $K \neq \emptyset$ otherwise we could choose $U = X$ and $V = \emptyset$. Since X is Hausdorff for every $y \in K$ exist $U_y, V_y \subset X$ disjoint open sets such that $x \in U_y$ and $y \in V_y$. We have $K \subset \cup_{y\in Y} V_y$ but since K is compact exist $y_1, \ldots, y_n$ such that $K \subset \cup_{i=1}^n V_{y_i}$. We now define:

$$V := \cup_{i=1}^n V_{y_i}$$
$$U := \cap_{i=1}^n U_{y_i}$$

It is clear that both sets are open, and that $x \in U$ and $K \subset V$ and for every $i \in [n]$ we also see that:

$$Y_{y_i} \cap U \subset V_{y_i} \cap U_{y_i} = \emptyset$$

Which means that $U \cap V = \emptyset$ as wanted which completes the proof. $\square$

Corollary 7.4. *Suppose that X is a Hausdorff space. Then every compact subset of X is closed.*

Proof. Let $K \subset X$ be compact. We may clearly assume that $K \neq X$. Given $x \in X \subset K$, it follows from the previous proposition that there exists a neighbourhood U of x which is contained in $X \setminus K$. This shows that $X \setminus K$ is a union of open sets, and so it is itself open. Thus K is closed, which completes the proof. □

One particularly useful result of this corollary is the following proposition:

Proposition 7.5. *Suppose that X is a Hausdorff space, let Y be a compact topological space, and let $f: Y \to X$ be a continuous bijection. Then f is a homeomorphism.*

Proof. All that's left to show is that f is an open map. Let $U \subset Y$ be open. It follows that $Y \setminus U$ is closed in a compact space and thus compact. Since f is continuous $f(Y \setminus U)$ is compact. From the previous corollary $f(Y \setminus U)$ is closed. Since f is a bijection we also have $f(Y \setminus U) = X \setminus f(U)$. This implies that U is open, so f is an open map and the proof is complete. □

8 Completely regular spaces and normal spaces.

Definition 8.1. We say that $C_b(X)$ separates points if for every distinct $x, y \in X$ there exists $f \in C_b(X)$ with $f(x) \neq f(y)$.