Complex Analysis

Based on lectures by

Notes taken by yehelip

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction

1.1 Complex numbers and the complex plane

1.1.1 Preliminaries

Definition 1.1 (Complex number)**.** A complex number is an expression of the form $x+yi$ such that $x, y \in \mathbb{R}$ and i is a 'imaginary number' not in $\mathbb{R}$. We denote

$$\Re(z) := x \quad \text{and} \quad \Im(z) := y.$$

If $\Im(z) = 0$ then z is said to be a real number, and if $\Re(z) = 0$ then it is said to be purely imaginary.

The set of all complex numbers is denoted as $\mathbb{C}$ and it can be made into a field with the following operations.

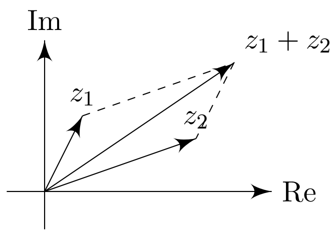
$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \quad \text{and} \quad z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The field $\mathbb{C}$ is called the complex plane.

Note that $i^2 = -1$. Also note that $T(x + yi) = (x, y)$ is a bijection between $\mathbb{C}$ and $\mathbb{R}$ and moreover, we have that T is additive. That is

$$T(z_1 + z_2) = T(z_1) + T(z_2)$$

which gives complex addition a geometric meaning.



The absolute value of a complex number $x + yi = z \in \mathbb{C}$ is defined by

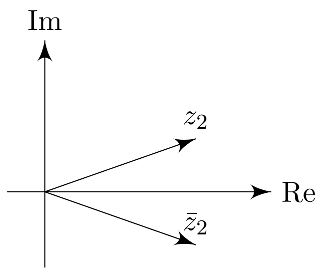
$$|z| = \sqrt{x^2 + y^2}.$$

Note that $|z| = \left\|(x, y)\right\| = \left\|T(z)\right\|$ where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^2$.

This implies that $|z - w|$ should be considered the distance between natural numbers z, w. Because we have that $|z| = \left\|T(z)\right\|$ we also have that the triangle inequality holds:

$$|z + w| \le |z| + |w| \text{ for all } z, w \in \mathbb{C}.$$

Definition 1.2 (Complex conjugate)**.** The complex conjugate of $x+yi = z \in C$ is the complex number $x - yi$. The complex conjugate of z is denoted $\bar{z}$.



It is easy to verify that

$$\Re(z) = \frac{z+\bar{z}}{2} \quad \text{and} \quad \Re(z) = \frac{z-\bar{z}}{2i} \quad \text{and} \quad |z|^2 = z\bar{z}.$$

Given θ we can denote $e^{i\theta} = \cos\theta + i\sin\theta$, and then describe any complex number $z \in \mathbb{C}$ as $re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and $r > 0$. We get that $|z| = |re^{i\theta}| = r$. We also have that θ describes the angle of z with the x-axis and it is usually denoted $\theta = \arg(z)$.

1.1.2 Convergence

Definition 1.3 (Convergence)**.** We say that the sequence $\{z_n\}_{n\geq 1} \subset \mathbb{C}$ converges to some $z_0 \in \mathbb{C}$ if $|z - z_0| \xrightarrow{n\to\infty} 0$. In this case, we call z_0 the limit of the sequence of $\{z_n\}_{n\geq 1}$.

Remark 1.1. It is easy to verify that the limit is unique, and that $z_n \xrightarrow{n\to\infty} z$ if and only if $T(z_n) \xrightarrow{n\to\infty} T(z)$ in the Euclidean metric.

Definition 1.4 (Cauchy sequence)**.** A sequence $\{z_n\}_{n\geq 1}$ is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists $N > 1$ such that for all $n, m > N$ we have that $|z_n - z_m| < \epsilon$.

Proposition 1.1. *The complex plane $\mathbb{C}$ is complete. That is, every Cauchy sequence converges in $\mathbb{C}$.*

Proof. The proof follows immediately from the known fact that $\mathbb{R}$ is complete and the previous remark. □

1.1.3 Sets in the complex plane

Definition 1.5 (Open disc)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$D_r(z_0) := \left\{ z \in \mathbb{C} \colon |z - z_0| < r \right\}.$$

We call $D_r(z_0)$ the open disc at center z_0 with radius r.

Definition 1.6 (Closed disc)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$\overline{D}_r(z_0) := \left\{ z \in \mathbb{C} \colon |z - z_0| \leq r \right\}.$$

We call $\overline{D}_r(z_0)$ the closed disc at center z_0 with radius r.

Definition 1.7 (Circle)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$C_r(z_0) := \left\{ z \in \mathbb{C} \colon |z - z_0| = r \right\}.$$

We call $C_r(z_0)$ the circle at center z_0 with radius r.

Definition 1.8 (Interior point)**.** Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if exists $r > 0$ such that $D_r(z) \subset \Omega$.

Definition 1.9 (Interior of a set)**.** Given $\Omega \subset \mathbb{C}$, we say that the interior of Ω is the collection of all interior points of Ω. We denote the interior as $\mathrm{Int}(\Omega)$.

Definition 1.10 (Open set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω is an open set if $\mathrm{Int}(\Omega) = \Omega$.

Definition 1.11 (Closed set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω a closed set if $\Omega^c := \mathbb{C} \setminus \Omega$ is open.

Definition 1.12 (Limit point)**.** Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if there exists a sequence z_n such that $z_n \neq z$ for all $n > 1$ and $z_n \xrightarrow{n\to\infty} z$.

Proposition 1.2. *Let $\Omega \subset \mathbb{C}$ be given. Then Ω is closed if and only if it contains all of its limit points.*

Proof. Clear. □

Definition 1.13 (Closure)**.** Let $\Omega \subset \mathbb{C}$ be given. The closure of Ω, denoted $\overline{\Omega}$, is defined as

$$\overline{\Omega} = \Omega \cup \left\{ z \in \mathbb{C} \mid x \text{ is a limit point of } \Omega \right\}.$$

Remark 1.2. Note that Ω is closed if and only if $\overline{\Omega} = \Omega$.

Definition 1.14 (Boundary)**.** The boundary of $\Omega \subset \mathbb{C}$ is denoted by $\partial\Omega$ and defined by $\partial\Omega := \Omega \setminus \mathrm{Int}(\Omega)$.

Definition 1.15 (Diameter)**.** Given $\Omega \subset \mathbb{C}$, we define the diameter of Ω as

$$\operatorname{diam}(\Omega) := \sup \left\{ |z - w| \colon z, w \in \Omega \right\}.$$

Definition 1.16 (Bounded set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω is bounded if $\operatorname{diam}(\Omega) < \infty$.

Remark 1.3. It is clear that a set $\Omega \subset \mathbb{C}$ is bounded if and only if there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $\Omega \subset D_r(z_0)$.

Definition 1.17 (Compact set)**.** A subset Ω of $\mathbb{C}$ is said to be compact if it is closed and bounded.

Theorem 1.3. (Bolzano–Weierstrass theorem). *A subset Ω in $\mathbb{C}$ is compact if and only if every seqeuence $\{z_n\}_{n\geq 1}$ has a subsequence $\left\{z_{n_k}\right\}$ such that $z_{n_k} \xrightarrow{k\to\infty} z$ for some $z \in \mathbb{C}$.*

Theorem 1.4. (Cantor's intersection lemma). *Let $\Omega_1, \Omega_2, \ldots$ be nonempty compact subsets of $\mathbb{C}$. Suppose that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 1$, and that* $\operatorname{diam}(\Omega_n) \xrightarrow{n\to\infty} 0$. *Then $\cap_{n\geq 1}\Omega_n = \{z\}$ for some $z \in \mathbb{C}$.*

Proof. Choose $z_n \in \Omega_n$ for all $n \geq 1$. Because $\operatorname{diam}\Omega \xrightarrow{n\to\infty}= 0$ we have that $\{z_n\}_{n\geq 1}$ is a Cauchy sequence and therefore it converges to some $z \in \mathbb{C}$. Because Ω_n is compact for every $n \geq 1$ we get that $z \in \cap_{n\geq 1}\Omega_n$. This means that $\cap_{n\geq 1}\Omega_n \neq \emptyset$.

Let $z, w \in \Omega$. Because $\operatorname{diam}\Omega \xrightarrow{n\to\infty} 0$ we have that $|z - w| \leq 0$ and thus $z = w$ which implies that $\cap_{n\geq 1}\Omega_n = \{z\}$ which completes the proof. □

Definition 1.18 (Connected open set)**.** A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty open subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. A connected open set in $\mathbb{C}$ will be called a region.

Definition 1.19 (Connected closed set)**.** A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty closed subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

Remark 1.4. It can be shown that Ω is connected if and only if for any $z, w \in \Omega$ there exists a curve $\gamma\colon [0,1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1)$. This implies that open and closed discs, as well as circles, are connected.

1.1.4 Continuous functions

Definition 1.20 (Continuous function)**.** Let Ω be a nonempty subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is continuous at a point $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that $|f(z) - f(z_0)| < \epsilon$ for all $z \in \Omega$ with $|z - z_0| < \delta$. We say that f is continuous on Ω if it is continuous at every $z_0 \in \Omega$.

Remark 1.5. It is easy to verify that the functions $\Im$, $\Re$, $|\cdot|$, and $\theta \mapsto e^{i\theta}$ are all continuous.

Proposition 1.5. *The composition of continuous functions is continuous.*

Definition 1.21 (Bounded function)**.** Let Ω be a nonempty subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is bounded if there exists $M > 0$ so that $|f(z)| < M$ for all $z \in \Omega$. We say that f attains a maximum if there exists $z_M \in \Omega$ such that $f(z) \leq f(z_M)$ for all $z \in \Omega$. We define when f attains a minimum similarly.

Proposition 1.6. *Let Ω be a nonempty compact subset of $\mathbb{C}$, and let $f\colon \Omega \to \mathbb{C}$ be continuous. Then f is bounded, and it attains its maximum and minimum on Ω.*

1.2 Holomorphic functions

Definition 1.22 (Holomorphic function)**.** Let Ω be a nonempty open subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is holomorphic at a point $z \in \Omega$ if the following limit exists

$$f'(z) := \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}.$$

The number $f'(z)$ is called the derivative of f at z. It is said that f is holomorphic if it is holomorphic at every $z \in \Omega$. Given a closed subset $C \subset \Omega$, we say that f is holomorphic on C if there exists $C \subset \Omega' \subset \Omega$ so that Ω' is open and f is holomorphic on Ω'.

Definition 1.23 (Entire function)**.** We say that $f\colon \mathbb{C} \to \mathbb{C}$ is entire if it is holomorphic on $\mathbb{C}$.

Remark 1.6. Note that h is a complex number and can approach 0 from any direction.

Remark 1.7. It is also useful to notice that $f\colon \Omega \to \mathbb{C}$ is holomorphic at $z \in \Omega$ if and only if there exist $a \in \mathbb{C}$, $r > 0$ with $D_r(z) \subset \Omega$, and a function $\psi\colon D_r(0) \to \mathbb{C}$ with $\lim_{h\to 0} \psi(h) = 0$, so that

$$f(z+h) = f(z) + ah + h\psi(h) \text{ for all } h \in D_r(0).$$

From this formulation is it clear that f is continuous at z whenever f is holomorphic at z.

Example 1.1. It follows directly from the definition that the function $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$ with $f'(z) = -1/z^2$. For all $0 \neq z \in \mathbb{C}$ we have

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{z+h} - \frac{1}{z}\right) = \lim_{h\to 0} \frac{-1}{z(z+h)} = \frac{-1}{z^2}.$$

Example 1.2. The function $f(z) = \bar{z}$ is not holomorphic. For any $z \in \mathbb{C}$ and $r \in \mathbb{R}$ we have that

$$\frac{f(z+t) - f(z)}{t} = 1 \quad \text{and} \quad \frac{f(z+ti) - f(z)}{ti} = -1$$

Proposition 1.7. *Let $\Omega \subset \mathbb{C}$ be open and let $f, g\colon \Omega \to \mathbb{C}$ be holomorphic at $z \in \Omega$. Then*

(1) $f + g$ is holomorphic at z with $(f+g)'(z) = f'(z) + g'(z)$.

(2) fg is holomorphic at z with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

Proof. We will only prove (2) because the proof of (1) is much simpler. Because f and g are holomorphic at z, they are also continuous there. Thus,

$$\begin{aligned}\lim_{h\to 0} \frac{(fg)(z+h) - (fg)(z)}{h} &= \lim_{h\to 0}\left(\frac{f(z+h) - f(z)}{h} g(z+h) + f(z)\frac{g(z+h) - g(z)}{h}\right) \\ &= f'(z)g(z) + f(z)g'(z),\end{aligned}$$

which completes the proof. ∎

Corollary 1.8. *It's quite easy to prove that constant function of the form $f(z) = c$ for some $c \in \mathbb{C}$ and $f(z) = z$ are holomorphic. It follows immediately from Proposition 1.7 that all polynomials, functions of the form $p(z) = \sum_{k=0}^{n} a_k z^k$ are entire, with $p'(z) = \sum_{k=1}^{n} k a_k z^{k-1}$ for all $z \in \mathbb{C}$.*

Proposition 1.9. *A composition of holomorphic functions at z is holomorphic at z, with $(g \circ f)'(z) = g'(f(z))f'(z)$.*

Corollary 1.10. *Let $\Omega \subset \mathbb{C}$ be open and let $f, g \colon \Omega \to \mathbb{C}$ be holomorphic at $z \in \Omega$. Suppose also that $g(z) \neq 0$. Then f/g is holomorphic at z with*

$$(f/g)'(z) = \frac{f'(z)g(z) + f(z)g'(z)}{g(z)^2}.$$

Proof. Let $h \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be with $h(z) = 1/z$. We now have that

$$\begin{aligned}(f/g)'(z) &= (f \cdot (h \circ g))'(z) = f'(z)(h \circ g)(z) + f(z)(h \circ g)'(z) \\ &= f'(z)/g(z) + f(z)h'(g(z))g'(z) = f'(z)/g(z) - f(z)g(z)^{-2}g'(z).\end{aligned}$$

□

Recall that $T \colon \mathbb{C} \to \mathbb{R}^2$ is the operator $T(x + yi) = (x, y)$.

Proposition 1.11. *Let $\Omega \subset \mathbb{C}$ be open, let $f \colon \Omega \to \mathbb{C}$, let $u, v \colon T(\Omega) \to \mathbb{R}$ be with $f(x + yi) = u(x, y) + iv(x, y)$ for $x + iy \in \Omega$, and let $F \colon T(\Omega) \to \mathbb{R}^2$ be with $F(x, y) = (u(x, y), v(x, y))$ for $(x, y) \in T(\Omega)$. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p = (x_0, y_0)$, and suppose that f is holomorphic at z_0. Then,*

(1) the partial derivatives of u and v exist at p, and

$$f'(z_0) = \frac{\partial u}{\partial x}(p) + i\frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i\frac{\partial u}{\partial y}(p);$$

(2) The Cauchy–Riemann equations are satisfied:

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p) \text{ and } \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p).$$

(3) F is differentiable at p with,

$$dF_p = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & -\frac{\partial v}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial u}{\partial x}(p) \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y}(p) & \frac{\partial u}{\partial y}(p) \\ -\frac{\partial u}{\partial y}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}.$$

Remark 1.8. Note that $u = \Re \circ f \circ T^{-1}$, $v = \Im \circ f \circ T^{-1}$ and $F = T \circ f \circ T^{-1}$. Thus, F is the map corresponding to f under the identification of $\mathbb{C}$ with $\mathbb{R}^2$ via T.

Remark 1.9. Note that from (3) we have

$$\det(dF_p) = \left(\tfrac{\partial u}{\partial x}(p)\right)^2 + \left(\tfrac{\partial v}{\partial x}(p)\right)^2.$$

From this and from (1), it follows that $\det(dF_p) > 0$ whenever $f'(z_0) \neq 0$. Moreover, we have that $\sqrt{\det(dF_p)} \cdot dF_p$ is an orthogonal matrix.

We now prove Proposition 1.11.

Proof. For (1) we can first let $t \to 0$ in $\mathbb{R}$ and see that

$$\begin{aligned} f'(z_0) &= \lim_{t\to} \frac{f(z+t)-f(z)}{h} \\ &= \lim_{t\to 0} \frac{u(x_0+t,y_0)-u(x_0,y_0)+iv(x_0+t,y_0)-iv(x_0,y_0)}{t} \\ &= \tfrac{\partial u}{\partial x}(p) + i\tfrac{\partial v}{\partial x}(p). \end{aligned}$$

Similarly,

$$\begin{aligned} f'(z_0) &= \lim_{t\to} \frac{f(z+it)-f(z)}{h} \\ &= \lim_{t\to 0} \frac{u(x_0,y_0+t)-u(x_0,y_0)+iv(x_0,y_0+t)-iv(x_0,y_0)}{t} \\ &= \tfrac{\partial v}{\partial y}(p) - i\tfrac{\partial u}{\partial y}(p) \end{aligned}$$

which completes the proof of (1). From the equation

$$\frac{\partial u}{\partial x}(p) + i\frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i\frac{\partial u}{\partial y}(p)$$

we immediately get (2). □

The following proposition is a kind of converse to the previous proposition.

Proposition 1.12. *Let $\Omega \subset \mathbb{C}$ be open, let $f\colon \Omega \to \mathbb{C}$, and let u and v be as in Proposition 1.11. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p := (x_0, y_0)$, ans suppose that u and v are differentiable at p, that is $\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p)$ and $\frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$. Then f is holomorphic at z_0.*

Proof. To be added. □

1.3 Power series

Definition 1.24 (Power series)**.** A power series centered at $z_0 \in \mathbb{C}$ is an expression of the form $\sum_0^\infty a_n(z-z_0)^n$, where $\{a_n\}_{n\geq 0} \subset \mathbb{C}$. Given $z \in \mathbb{C}$, we say that the power series converges at z if the limit $\lim_{N\to\infty} \sum_{n=0}^N a_n(z-z_0)^n$ exists in $\mathbb{C}$. If this limit does not exist, we say that the series diverges at z.

Definition 1.25 (Absolute convergence)**.** Given a power series $\sum_{n=0}^\infty a_n(z-z_0)$, we say that it converges absolutely at $z \in \mathbb{C}$ if $\sum_{n=0}^\infty |a_n| \cdot \big|(z-z_0)\big| < \infty$.

Proposition 1.13. *If a power series converges absolutely at z then it also converges at z. This follows from the completeness of $\mathbb{C}$.*

In the following proposition we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

Proposition 1.14 (Hadamard's theorem)**.** *Let $\sum_{n=0}^\infty a_n(z-z_0)$ be a power series, and let $0 \leq \mathbb{R} \leq \infty$ be given by*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

Then for $z \in \mathbb{C}$ the series converges absolutely if $|z-z_0| < R$, and the series diverges if $|z-z_0| > R$.

Remark 1.10. The number R is called the radius of convergence of the power series, and the region $\big\{z \in \mathbb{C}\colon |z-z_0| < R\big\}$ is called the disc of convergence.

We now proceed to prove Proposition 1.14

Proof. Set $L := 1/R$. Suppose first that $0 < R \leq \infty$, so that $0 \leq L < \infty$. Let $z \in \mathbb{C}$ be such that $|z - z_0| < R$, then there exists $L < M < \infty$ so that $M|z - z_0| < 1$. By the definition of L (the limsup) there exists $N \geq 1$ so that $|a_n|^{\frac{1}{n}} < M$ forall $n > N$. Thus

$$\begin{aligned}\sum_{n=0}^{\infty} |a_n| \cdot |z - z_0|^n &= \sum_{n=0}^{N-1} |a_n| \cdot |z - z_0|^n + \sum_{n=N}^{\infty} \Big(|a_n|^{\frac{1}{n}} \cdot |z - z_0| \Big)^n \\ &\leq \sum_{n=0}^{N-1} |a_n| \cdot |z - z_0|^n + \sum_{n=N}^{\infty} \big(M|z - z_0| \big)^n < \infty.\end{aligned}$$

Suppose next that $0 \leq R < \infty$, so that $0 < L\infty$. Let $z \in \mathbb{C}$ be such that $|z - z_0| > R$, then similarly there exists $0 < M < L$ so that $M|z - z_0| > 1$. Then, for every $N \geq 1$ there exists $n \geq N$ so that $|a_n|^{\frac{1}{n}} > M$. For such n we have

$$\begin{aligned}\left| \sum_{k=0}^{n} a_k (z - z_0)^k - \sum_{k=0}^{n-1} a_k (z - z_0)^k \right| &= |a_n| \cdot |z - z_0|^n \\ &= \Big(|a_n|^{\frac{1}{n}} \cdot |z - z_0| \Big)^n > \big(M|z - z_0| \big)^n > 1,\end{aligned}$$

which shows that the partial sums do not form a Cauchy sequence. Thus the series diverges at z, which completes the proof. $\square$

Example 1.3. Considert the power series $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Because we have

$$\sqrt[n]{(2n)!} \geq \sqrt[n]{n^n} = n$$

we also have for every $n \geq 1$,

$$\left(\frac{1}{(2n)!} \right)^{\frac{1}{2n}} \leq \frac{1}{n^{\frac{1}{2}}} \quad \text{and} \quad \left(\frac{1}{(2n+1)!} \right)^{\frac{1}{2n+1}} \leq \frac{1}{n^{\frac{1}{2}}}.$$

Since $n^{-\frac{1}{2}} \xrightarrow{n \to \infty} = 0$ we get that the radius of convergence is ∞ for the series. The map $z \mapsto e^z$ is called the exponantial function. We also have that

$$e^z e^w = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) + \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

Example 1.4. Consider the power series $f(z) := \sum_{n=0}^{\infty} z^n$. Since $1^{\frac{1}{n}} = 1$ we get that the radius of convergence in this case is 1. Thus f defined a function from $D_1(0)$ to $\mathbb{C}$. Moreover, since we have

$$(1 - z) \sum_{n=0}^{N} z^n = 1 - z^{N+1},$$

we get for $z \in D_1(0)$ that

$$f(z) = \lim_{N \to \infty} \sum_{n=0}^{N} z^n = \lim_{N \to \infty} \frac{1 - z^{N+1}}{1 - z} = \frac{1}{1 - z}.$$

Proposition 1.15. *Let*