Analysis 3 - Problems

Based on lectures by
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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Topology in Euclidean spaces

Exercise 1.1. Prove that a continuous function $f\colon A \to B$ has a maximum in a compact space

Solution. By the completeness axiom for the real numbers we know that the set $f(A)$ has a supremum which we will denote S. By the definition of the supremum it is possible to construct a sequence that converges to it which we shall denote $f(x^n)$. We don't know whether x^n converges or not but we know it has a subsequence that converges so:

$$\lim_{k\to\infty} x^{n_k} = x \quad \text{and} \quad \lim_{k\to\infty} f(x^{n_k}) = S$$

Since f is close we get $x \in A$ and since it is continuous $f(x) = S$ which shows that it is continuous and also has a maximum.

Exercise 1.2. Prove that a set E is closed if and only if it's complement E^c is open

Solution. First suppose that E is closed, and E^c is not open. Then exists $x \notin E$ such that for all $r > 0$ we get $B_r(x) \cap E \neq \emptyset$ which means we can construct a sequence in E that converges to x but $x \notin E$ in contradiction to the assumption that E is close.

Suppose next that E^c is open and E is not closed, then exists a sequence $(x^n)_{n=1}^{\infty}$ that converges to some $x \in E^c$ which means that for every $r > 0$ that $B_r(x) \cap E \neq \emptyset$ in contradiction to E^c being open.