

Complex Analysis

Based on lectures by
Notes taken by yehelip

Winter 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1	Introduction	3
1.1	Complex numbers and the complex plane	3
1.1.1	Preliminaries	3
1.1.2	Convergence	4
1.1.3	Sets in the complex plane	4
1.1.4	Continuous functions	5
1.2	Holomorphic functions	6
1.3	Power series	8

1 Introduction

1.1 Complex numbers and the complex plane

1.1.1 Preliminaries

Definition 1.1 (Complex number). A complex number is an expression of the form $x + yi$ such that $x, y \in \mathbb{R}$ and i is a ‘imaginary number’ not in \mathbb{R} . We denote

$$\Re(z) := x \quad \text{and} \quad \Im(z) := y.$$

If $\Im(z) = 0$ then z is said to be a real number, and if $\Re(z) = 0$ then it is said to be purely imaginary.

The set of all complex numbers is denoted as \mathbb{C} and it can be made into a field with the following operations.

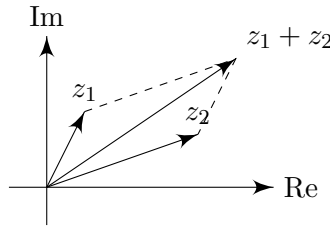
$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \quad \text{and} \quad z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The field \mathbb{C} is called the complex plane.

Note that $i^2 = -1$. Also note that $T(x + yi) = (x, y)$ is a bijection between \mathbb{C} and \mathbb{R}^2 and moreover, we have that T is additive. That is

$$T(z_1 + z_2) = T(z_1) + T(z_2)$$

which gives complex addition a geometric meaning.



The absolute value of a complex number $x + yi = z \in \mathbb{C}$ is defined by

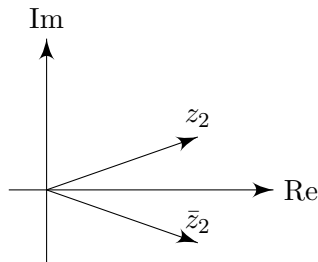
$$|z| = \sqrt{x^2 + y^2}.$$

Note that $|z| = \|(x, y)\| = \|T(z)\|$ where $\|\cdot\|$ is the standard Euclidean norm on \mathbb{R}^2 .

This implies that $|z - w|$ should be considered the distance between natural numbers z, w . Because we have that $|z| = \|T(z)\|$ we also have that the triangle inequality holds:

$$|z + w| \leq |z| + |w| \quad \text{for all } z, w \in \mathbb{C}.$$

Definition 1.2 (Complex conjugate). The complex conjugate of $x + yi = z \in \mathbb{C}$ is the complex number $x - yi$. The complex conjugate of z is denoted \bar{z} .



It is easy to verify that

$$\Re(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \Im(z) = \frac{z - \bar{z}}{2i} \quad \text{and} \quad |z|^2 = z\bar{z}.$$

Given θ we can denote $e^{i\theta} = \cos \theta + i \sin \theta$, and then describe any complex number $z \in \mathbb{C}$ as $re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and $r > 0$. We get that $|z| = |re^{i\theta}| = r$. We also have that θ describes the angle of z with the x -axis and it is usually denoted $\theta = \arg(z)$.

1.1.2 Convergence

Definition 1.3 (Convergence). We say that the sequence $\{z_n\}_{n \geq 1} \subset \mathbb{C}$ converges to some $z_0 \in \mathbb{C}$ if $|z - z_0| \xrightarrow{n \rightarrow \infty} 0$. In this case, we call z_0 the limit of the sequence of $\{z_n\}_{n \geq 1}$.

Remark 1.1. It is easy to verify that the limit is unique, and that $z_n \xrightarrow{n \rightarrow \infty} z$ if and only if $T(z_n) \xrightarrow{n \rightarrow \infty} T(z)$ in the Euclidean metric.

Definition 1.4 (Cauchy sequence). A sequence $\{z_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists $N > 1$ such that for all $n, m > N$ we have that $|z_n - z_m| < \epsilon$.

Proposition 1.1. *The complex plane \mathbb{C} is complete. That is, every Cauchy sequence converges in \mathbb{C} .*

Proof. The proof follows immediately from the known fact that \mathbb{R} is complete and the previous remark. \square

1.1.3 Sets in the complex plane

Definition 1.5 (Open disc). For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}.$$

We call $D_r(z_0)$ the open disc at center z_0 with radius r .

Definition 1.6 (Closed disc). For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$\overline{D}_r(z_0) := \{z \in \mathbb{C} : |z - z_0| \leq r\}.$$

We call $\overline{D}_r(z_0)$ the closed disc at center z_0 with radius r .

Definition 1.7 (Circle). For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}.$$

We call $C_r(z_0)$ the circle at center z_0 with radius r .

Definition 1.8 (Interior point). Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if exists $r > 0$ such that $D_r(z) \subset \Omega$.

Definition 1.9 (Interior of a set). Given $\Omega \subset \mathbb{C}$, we say that the interior of Ω is the collection of all interior points of Ω . We denote the interior as $\text{Int}(\Omega)$.

Definition 1.10 (Open set). Given $\Omega \subset \mathbb{C}$, we say that Ω is an open set if $\text{Int}(\Omega) = \Omega$.

Definition 1.11 (Closed set). Given $\Omega \subset \mathbb{C}$, we say that Ω a closed set if $\Omega^c := \mathbb{C} \setminus \Omega$ is open.

Definition 1.12 (Limit point). Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if there exists a sequence z_n such that $z_n \neq z$ for all $n > 1$ and $z_n \xrightarrow{n \rightarrow \infty} z$.

Proposition 1.2. *Let $\Omega \subset \mathbb{C}$ be given. Then Ω is closed if and only if it contains all of its limit points.*

Proof. Clear. □

Definition 1.13 (Closure). Let $\Omega \subset \mathbb{C}$ be given. The closure of Ω , denoted $\overline{\Omega}$, is defined as

$$\overline{\Omega} = \Omega \cup \{z \in \mathbb{C} \mid z \text{ is a limit point of } \Omega\}.$$

Remark 1.2. Note that Ω is closed if and only if $\overline{\Omega} = \Omega$.

Definition 1.14 (Boundary). The boundary of $\Omega \subset \mathbb{C}$ is denoted by $\partial\Omega$ and defined by $\partial\Omega := \overline{\Omega} \setminus \text{Int}(\Omega)$.

Definition 1.15 (Diameter). Given $\Omega \subset \mathbb{C}$, we define the diameter of Ω as

$$\text{diam}(\Omega) := \sup \{|z - w| : z, w \in \Omega\}.$$

Definition 1.16 (Bounded set). Given $\Omega \subset \mathbb{C}$, we say that Ω is bounded if $\text{diam}(\Omega) < \infty$.

Remark 1.3. It is clear that a set $\Omega \subset \mathbb{C}$ is bounded if and only if there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $\Omega \subset D_r(z_0)$.

Definition 1.17 (Compact set). A subset Ω of \mathbb{C} is said to be compact if it is closed and bounded.

Theorem 1.3. (Bolzano–Weierstrass theorem). *A subset Ω in \mathbb{C} is compact if and only if every sequence $\{z_n\}_{n \geq 1}$ has a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \xrightarrow{k \rightarrow \infty} z$ for some $z \in \mathbb{C}$.*

Theorem 1.4. (Cantor’s intersection lemma). *Let $\Omega_1, \Omega_2, \dots$ be nonempty compact subsets of \mathbb{C} . Suppose that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 1$, and that $\text{diam}(\Omega_n) \xrightarrow{n \rightarrow \infty} 0$. Then $\bigcap_{n \geq 1} \Omega_n = \{z\}$ for some $z \in \mathbb{C}$.*

Proof. Choose $z_n \in \Omega_n$ for all $n \geq 1$. Because $\text{diam} \Omega_n \xrightarrow{n \rightarrow \infty} 0$ we have that $\{z_n\}_{n \geq 1}$ is a Cauchy sequence and therefore it converges to some $z \in \mathbb{C}$. Because Ω_n is compact for every $n \geq 1$ we get that $z \in \bigcap_{n \geq 1} \Omega_n$. This means that $\bigcap_{n \geq 1} \Omega_n \neq \emptyset$.

Let $z, w \in \bigcap_{n \geq 1} \Omega_n$. Because $\text{diam} \Omega_n \xrightarrow{n \rightarrow \infty} 0$ we have that $|z - w| \leq 0$ and thus $z = w$ which implies that $\bigcap_{n \geq 1} \Omega_n = \{z\}$ which completes the proof. □

Definition 1.18 (Connected open set). A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty open subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. A connected open set in \mathbb{C} will be called a region.

Definition 1.19 (Connected closed set). A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty closed subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

Remark 1.4. It can be shown that Ω is connected if and only if for any $z, w \in \Omega$ there exists a curve $\gamma: [0, 1] \rightarrow \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$. This implies that open and closed discs, as well as circles, are connected.

1.1.4 Continuous functions

Definition 1.20 (Continuous function). Let Ω be a nonempty subset of \mathbb{C} and let $f: \Omega \rightarrow \mathbb{C}$ be given. We say that f is continuous at a point $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that $|f(z) - f(z_0)| < \epsilon$ for all $z \in \Omega$ with $|z - z_0| < \delta$. We say that f is continuous on Ω if it is continuous at every $z_0 \in \Omega$.

Remark 1.5. It is easy to verify that the functions \Im , \Re , $|\cdot|$, and $\theta \mapsto e^{i\theta}$ are all continuous.

Proposition 1.5. *The composition of continuous functions is continuous.*

Definition 1.21 (Bounded function). Let Ω be a nonempty subset of \mathbb{C} and let $f: \Omega \rightarrow \mathbb{C}$ be given. We say that f is bounded if there exists $M > 0$ so that $|f(z)| < M$ for all $z \in \Omega$. We say that f attains a maximum if there exists $z_M \in \Omega$ such that $f(z) \leq f(z_M)$ for all $z \in \Omega$. We define when f attains a minimum similarly.

Proposition 1.6. *Let Ω be a nonempty compact subset of \mathbb{C} , and let $f: \Omega \rightarrow \mathbb{C}$ be continuous. Then f is bounded, and it attains its maximum and minimum on Ω .*

1.2 Holomorphic functions

Definition 1.22 (Holomorphic function). Let Ω be a nonempty open subset of \mathbb{C} and let $f: \Omega \rightarrow \mathbb{C}$ be given. We say that f is holomorphic at a point $z \in \Omega$ if the following limit exists

$$f'(z) := \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}.$$

The number $f'(z)$ is called the derivative of f at z . It is said that f is holomorphic if it is holomorphic at every $z \in \Omega$. Given a closed subset $C \subset \Omega$, we say that f is holomorphic on C if there exists $C \subset \Omega' \subset \Omega$ so that Ω' is open and f is holomorphic on Ω' .

Definition 1.23 (Entire function). We say that $f: \mathbb{C} \rightarrow \mathbb{C}$ is entire if it is holomorphic on \mathbb{C} .

Remark 1.6. Note that h is a complex number and can approach 0 from any direction.

Remark 1.7. It is also useful to notice that $f: \Omega \rightarrow \mathbb{C}$ is holomorphic at $z \in \Omega$ if and only if there exist $a \in \mathbb{C}$, $r > 0$ with $D_r(z) \subset \Omega$, and a function $\psi: D_r(0) \rightarrow \mathbb{C}$ with $\lim_{h \rightarrow 0} \psi(h) = 0$, so that

$$f(z+h) = f(z) + ah + h\psi(h) \text{ for all } h \in D_r(0).$$

From this formulation it is clear that f is continuous at z whenever f is holomorphic at z .

Example 1.1. It follows directly from the definition that the function $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$ with $f'(z) = -1/z^2$. For all $0 \neq z \in \mathbb{C}$ we have

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{z+h} - \frac{1}{z} \right) = \lim_{h \rightarrow 0} \frac{-1}{z(z+h)} = \frac{-1}{z^2}.$$

Example 1.2. The function $f(z) = \bar{z}$ is not holomorphic. For any $z \in \mathbb{C}$ and $r \in \mathbb{R}$ we have that

$$\frac{f(z+t) - f(z)}{t} = 1 \quad \text{and} \quad \frac{f(z+ti) - f(z)}{ti} = -1$$

Proposition 1.7. *Let $\Omega \subset \mathbb{C}$ be open and let $f, g: \Omega \rightarrow \mathbb{C}$ be holomorphic at $z \in \Omega$. Then*

- (1) $f + g$ is holomorphic at z with $(f + g)'(z) = f'(z) + g'(z)$.
- (2) fg is holomorphic at z with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

Proof. We will only prove (2) because the proof of (1) is much simpler. Because f and g are holomorphic at z , they are also continuous there. Thus,

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(fg)(z+h) - (fg)(z)}{h} &= \lim_{h \rightarrow 0} \left(\frac{f(z+h) - f(z)}{h} g(z+h) + f(z) \frac{g(z+h) - g(z)}{h} \right) \\ &= f'(z)g(z) + f(z)g'(z), \end{aligned}$$

which completes the proof. □

Corollary 1.8. *It's quite easy to prove that constant function of the form $f(z) = c$ for some $c \in \mathbb{C}$ and $f(z) = z$ are holomorphic. It follows immediately from Proposition 1.7 that all polynomials, functions of the form $p(z) = \sum_{k=0}^n a_k z^k$ are entire, with $p'(z) = \sum_{k=1}^n k a_k z^{k-1}$ for all $z \in \mathbb{C}$.*

Proposition 1.9. *A composition of holomorphic functions at z is holomorphic at z , with $(g \circ f)'(z) = g'(f(z))f'(z)$.*

Corollary 1.10. *Let $\Omega \subset \mathbb{C}$ be open and let $f, g: \Omega \rightarrow \mathbb{C}$ be holomorphic at $z \in \Omega$. Suppose also that $g(z) \neq 0$. Then f/g is holomorphic at z with*

$$(f/g)'(z) = \frac{f'(z)g(z) + f(z)g'(z)}{g(z)^2}.$$

Proof. Let $h: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ be with $h(z) = 1/z$. We now have that

$$\begin{aligned} (f/g)'(z) &= (f \cdot (h \circ g))'(z) = f'(z)(h \circ g)(z) + f(z)(h \circ g)'(z) \\ &= f'(z)/g(z) + f(z)h'(g(z))g'(z) = f'(z)/g(z) - f(z)g(z)^{-2}g'(z). \end{aligned}$$

□

Recall that $T: \mathbb{C} \rightarrow \mathbb{R}^2$ is the operator $T(x + yi) = (x, y)$.

Proposition 1.11. *Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$, let $u, v: T(\Omega) \rightarrow \mathbb{R}$ be with $f(x + yi) = u(x, y) + iv(x, y)$ for $x + iy \in \Omega$, and let $F: T(\Omega) \rightarrow \mathbb{R}^2$ be with $F(x, y) = (u(x, y), v(x, y))$ for $(x, y) \in T(\Omega)$. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p = (x_0, y_0)$, and suppose that f is holomorphic at z_0 . Then,*

(1) *the partial derivatives of u and v exist at p , and*

$$f'(z_0) = \frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i \frac{\partial u}{\partial y}(p);$$

(2) *The Cauchy–Riemann equations are satisfied:*

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p) \text{ and } \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p).$$

(3) *F is differentiable at p with,*

$$dF_p = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & -\frac{\partial v}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial u}{\partial x}(p) \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y}(p) & \frac{\partial u}{\partial y}(p) \\ -\frac{\partial u}{\partial y}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}.$$

Remark 1.8. Note that $u = \Re \circ f \circ T^{-1}$, $v = \Im \circ f \circ T^{-1}$ and $F = T \circ f \circ T^{-1}$. Thus, F is the map corresponding to f under the identification of \mathbb{C} with \mathbb{R}^2 via T .

Remark 1.9. Note that from (3) we have

$$\det(dF_p) = \left(\frac{\partial u}{\partial x}(p) \right)^2 + \left(\frac{\partial v}{\partial x}(p) \right)^2.$$

From this and from (1), it follows that $\det(dF_p) > 0$ whenever $f'(z_0) \neq 0$. Moreover, we have that $\sqrt{\det(dF_p)} \cdot dF_p$ is an orthogonal matrix.

We now prove Proposition 1.11.

Proof. To be added.

□

The following proposition is a kind of converse to the previous proposition.

Proposition 1.12. *Let $\Omega \subset \mathbb{C}$ be open, let $f: \Omega \rightarrow \mathbb{C}$, and let u and v be as in Proposition 1.11. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p := (x_0, y_0)$, and suppose that u and v are differentiable at p , that is $\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p)$ and $\frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$. Then f is holomorphic at z_0 .*

Proof. To be added.

□

1.3 Power series

Definition 1.24 (Power series). A power series centered at $z_0 \in \mathbb{C}$ is an expression of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, where $\{a_n\}_{n \geq 0} \subset \mathbb{C}$. Given $z \in \mathbb{C}$, we say that the power series converges at z if the limit $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n(z - z_0)^n$ exists in \mathbb{C} . If this limit does not exist, we say that the series diverges at z .

Definition 1.25 (Absolute convergence). Given a power series $\sum_{n=0}^{\infty} a_n(z - z_0)$, we say that it converges absolutely at $z \in \mathbb{C}$ if $\sum_{n=0}^{\infty} |a_n| \cdot |(z - z_0)| < \infty$.

Proposition 1.13. *If a power series converges absolutely at z then it also converges at z . This follows from the completeness of \mathbb{C} .*

Proposition 1.14 (Hadamard's formula). *Let $\sum_{n=0}^{\infty} a_n(z - z_0)$ be a power series, and let $0 \leq R \leq \infty$ be given by*

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}.$$

Then for $z \in \mathbb{C}$ the series converges absolutely if $|z - z_0| < R$, and the series diverges if $|z - z_0| > R$.