Complex Analysis

Based on lectures by
Notes taken by yehelip

Winter 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction

1.1 Complex numbers

Definition 1.1 (Complex number)**.** A complex number is an expression of the form $x + yi$ such that $x, y \in \mathbb{R}$ and i is a 'imaginary number' not in $\mathbb{R}$. We denote

$$\Re(z) := x \quad \text{and} \quad \Im(z) := y.$$

If $\Im(z) = 0$ then z is said to be a real number, and if $\Re(z) = 0$ then it is said to be purely imaginary.

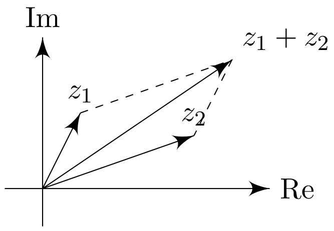
The set of all complex numbers is denoted as $\mathbb{C}$ and it can be made into a field with the following operations.

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \quad \text{and} \quad z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

Note that $i^2 = -1$. Also note that $T(x + yi) = (x, y)$ is a bijection between $\mathbb{C}$ and $\mathbb{R}$ and moreover, we have that T is additive. That is

$$T(z_1 + z_2) = T(z_1) + T(z_2)$$

which gives complex addition a geometric meaning.



The absolute value of a complex number $x + yi = z \in \mathbb{C}$ is defined by

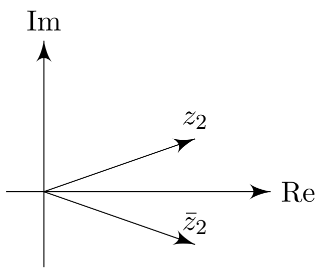
$$|z| = \sqrt{x^2 + y^2}.$$

Note that $|z| = \|(x, y)\| = \|T(z)\|$ where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^2$.

This implies that $|z - w|$ should be considered the distance between natural numbers z, w. Because we have that $|z| = \|T(z)\|$ we also have that the triangle inequality holds:

$$|z + w| \leq |z| + |w| \text{ for all } z, w \in \mathbb{C}.$$

Definition 1.2 (Complex conjugate)**.** The complex conjugate of $x + yi = z \in C$ is the complex number $x - yi$. The complex conjugate of z is denoted $\bar{z}$.



It is easy to verify that

$$\Re(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \Re(z) = \frac{z - \bar{z}}{2i} \quad \text{and} \quad |z|^2 = z\bar{z}.$$

Given θ we can denote $e^{i\theta} = \cos\theta + i \sin\theta$, and then describe any complex number $z \in \mathbb{C}$ as $re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and $r > 0$. We get that $|z| = |re^{i\theta}| = r$. We also have that θ describes the angle of z with the x-axis and it is usually denoted $\theta = \arg(z)$.

1.2 Convergence

Definition 1.3 (Convergence)**.** We say that the sequence $\{z_n\}_{n\geq 1} \subset \mathbb{C}$ converges to some $z_0 \in \mathbb{C}$ if $|z - z_0| \xrightarrow{n\to\infty} 0$. In this case, we call z_0 the limit of the sequence of $\{z_n\}_{n\geq 1}$.

Remark 1.1. It is easy to verify that the limit is unique, and that $z_n \xrightarrow{n\to\infty} z$ if and only if $T(z_n) \xrightarrow{n\to\infty} T(z)$ in the Euclidean metric.

Definition 1.4 (Cauchy sequence)**.** A sequence $\{z_n\}_{n\geq 1}$ is said to be a Cauchy sequence if for all > 0 there exists $N > 1$ such that for all $n, m > N$ we have that $|z_n - z_m| < \epsilon$.

Proposition 1.1. *The complex plane $\mathbb{C}$ is complete. That is, every Cauchy sequence converges in $\mathbb{C}$.*

Proof. The proof follows immediately from the known fact that $\mathbb{R}$ is complete and the previous remark. □

1.3 Sets in the complex plane

Definition 1.5 (Open disc)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$D_r(z_0) := \left\{z \in \mathbb{C} \colon |z - z_0| < r\right\}.$$

We call $D_r(z_0)$ the open disc at center z_0 with radius r.

Definition 1.6 (Closed disc)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$\overline{D}_r(z_0) := \left\{z \in \mathbb{C} \colon |z - z_0| \leq r\right\}.$$

We call $\overline{D}_r(z_0)$ the closed disc at center z_0 with radius r.

Definition 1.7 (Circle)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$C_r(z_0) := \left\{z \in \mathbb{C} \colon |z - z_0| = r\right\}.$$

We call $C_r(z_0)$ the circle at center z_0 with radius r.

Definition 1.8 (Interior point)**.** Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if exists $r > 0$ such that $D_r(z) \subset \Omega$.

Definition 1.9 (Interior of a set)**.** Given $\Omega \subset \mathbb{C}$, we say that the interior of Ω is the collection of all interior points of Ω. We denote the interior as $\mathrm{Int}(\Omega)$.

Definition 1.10 (Open set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω is an open set if $\mathrm{Int}(\Omega) = \Omega$.

Definition 1.11 (Closed set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω a closed set if $\Omega^c := \mathbb{C} \setminus \Omega$ is open.

Definition 1.12 (Limit point)**.** Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of Ω if there exists a sequence z_n such that $z_n \neq z$ for all $n > 1$ and $z_n \xrightarrow{n\to\infty} z$.

Proposition 1.2. *Let $\Omega \subset \mathbb{C}$ be given. Then Ω is closed if and only if it contains all of its limit points.*

Proof. Clear. □

Definition 1.13 (Closure)**.** Let $\Omega \subset \mathbb{C}$ be given. The closure of Ω, denoted $\overline{\Omega}$, is defined as

$$\overline{\Omega} = \Omega \cup \left\{z \in \mathbb{C} \mid x \text{ is a limit point of } \Omega\right\}.$$

Remark 1.2. Note that Ω is closed if and only if $\overline{\Omega} = \Omega$.

Definition 1.14 (Boundary)**.** The boundary of $\Omega \subset \mathbb{C}$ is denoted by $\partial\Omega$ and defined by $\partial\Omega := \Omega \setminus \mathrm{Int}(\Omega)$.

Definition 1.15 (Diameter)**.** Given $\Omega \subset \mathbb{C}$, we define the diameter of Ω as

$$\mathrm{diam}(\Omega) := \sup\left\{|z-w|\colon z, w \in \Omega\right\}.$$

Definition 1.16 (Bounded set)**.** Given $\Omega \subset \mathbb{C}$, we say that Ω is bounded if $\mathrm{diam}(\Omega) < \infty$.

Remark 1.3. It is clear that a set $\Omega\mathbb{C}$ is bounded if and only if there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $\Omega D_r(z_0)$.

Definition 1.17 (Compact set)**.** A subset Ω of $\mathbb{C}$ is said to be compact if it is closed and bounded.

Theorem 1.3. (Bolzano–Weierstrass theorem). *A subset Ω in $\mathbb{C}$ is compact if and only if every seqeuence $\{z_n\}_{n\geq 1}$ has a subsequence $\left\{z_{n_k}\right\}$ such that $z_{n_k} \xrightarrow{k\to\infty} z$ for some $z \in \mathbb{C}$.*

Theorem 1.4. (Cantor's intersection lemma). *Let $\Omega_1, \Omega_2, \ldots$ be nonempty compact subsets of $\mathbb{C}$. Suppose that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 1$, and that $\mathrm{diam}(\Omega_n) \xrightarrow{n\to\infty} 0$. Then $\cap_{n\geq 1}\Omega_n = \{z\}$ for some $z \in \mathbb{C}$.*

Proof. Choose $z_n \in \Omega_n$ for all $n \geq 1$. Because $\mathrm{diam}\,\Omega \xrightarrow{n\to\infty}= 0$ we have that $\{z_n\}_{n\geq 1}$ is a Cauchy sequence and therefore it converges to some $z \in \mathbb{C}$. Because Ω_n is compact for every $n \geq 1$ we get that $z \in \cap_{n\geq 1}\Omega_n$. This means that $\cap_{n\geq 1}\Omega_n \neq \emptyset$.

Let $z, w \in \Omega$. Because $\mathrm{diam}\,\Omega \xrightarrow{n\to\infty} 0$ we have that $|z-w| \leq 0$ and thus $z = w$ which implies that $\cap_{n\geq 1}\Omega_n = \{z\}$ which completes the proof. ☐

Definition 1.18 (Connected open set)**.** A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty open subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. A connected open set in $\mathbb{C}$ will be called a region.

Definition 1.19 (Connected closed set)**.** A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty closed subsets Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$.

Remark 1.4. It can be shown that Ω is connected if and only if for any $z, w \in \Omega$ there exists a curve $\gamma\colon [0,1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1)$. This implies that open and closed discs, as well as circles, are connected.

1.4 Continuous functions

Definition 1.20 (Continuous function)**.** Let Ω be a nonempty subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is continuous at a point $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that $|f(z) - f(z_0)| < \epsilon$ for all $z \in \Omega$ with $|z - z_0| < \delta$. We say that f is continuous on Ω if it is continuous at every $z_0 \in \Omega$.

Remark 1.5. It is easy to verify that the functions $\Im$, $\Re$, $|\cdot|$, and $\theta \mapsto e^{i\theta}$ are all continuous.

Proposition 1.5. *The composition of continuous functions is continuous.*

Definition 1.21 (Bounded function)**.** Let Ω be a nonempty subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is bounded if there exists $M > 0$ so that $|f(z)| < M$ for all $z \in \Omega$. We say that f attains a maximum if there exists $z_M \in \Omega$ such that $f(z) \leq f(z_M)$ for all $z \in \Omega$. We define when f attains a minimum similarly.

Proposition 1.6. *Let Ω be a nonempty compact subset of $\mathbb{C}$, and let $f\colon \Omega \to \mathbb{C}$ be continuous. Then f is bounded, and it attains its maximum and minimum on Ω.*

1.5 Holomorphic functions

Definition 1.22 (Holomorphic function)**.** Let Ω be a nonempty open subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that f is holomorphic at a point $z \in \Omega$ if the following limit exists

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

The number $f'(z)$ is called the derivative of f at z.