Riemann Surfaces

Based on lectures by
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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction

Definition 1.1 (Riemann surface)**.** A Riemann surface is a 1-dimensional complex manifold.

Definition 1.2 (Riemann surface)**.** A Riemann surface is a topological space X together with open subsets $\{U_k\}_{k\in I}$ of X with $\cup_{k\in I} U_k = X$ together with maps $f_i\colon U_i \to \mathbb{C}$ such that

(1) Each f_i is a homeomorphism onto its image.

(2) If $U_i \cap U_j \neq \emptyset$ then $f_i \circ f_j^{-1}\colon f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ are *biholomorphic.*

Remark 1.1. A function $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic at p if $f'(p) = \lim_{z\to p} \frac{f(z)-f(p)}{z-p}$ exists.

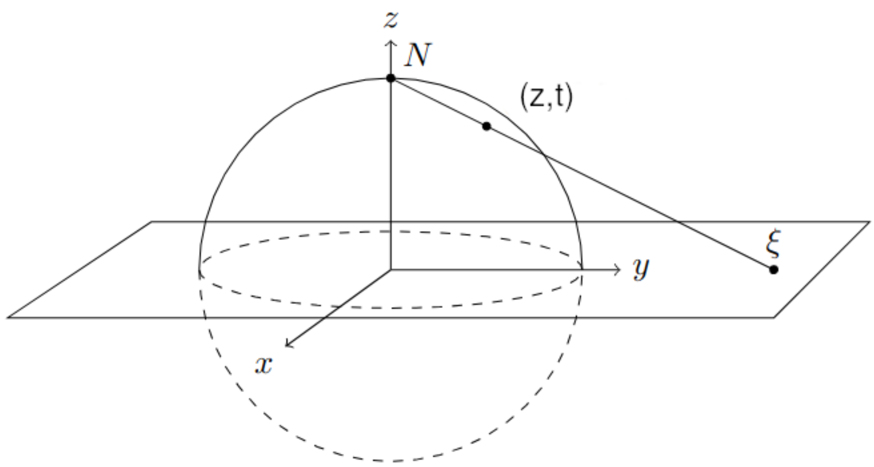
Definition 1.3 (Biholomorphism)**.** A function $f\colon \mathbb{C} \to \mathbb{C}$ is called biholomorphic if it has an inverse and both f and f' are holomorphic.

Definition 1.4 (Atlas)**.** The $\{(U_i, f_i)\}_{i\in I}$ are called an atlas of the Riemann surface.

Definition 1.5 (Chart)**.** Each individual (U_i, f_i) is called a chart of the Riemann surface.

Example 1.1. Let $U \subset \mathbb{C}$. Then U can take an atlas with one chart which is the identity map.

Example 1.2 (Riemann sphere)**.** Let $X = \{(z,t) \in \mathbb{C} \times \mathbb{R}\colon |z|^2 + t^2 = \mathbb{R}\}$. We identify $\mathbb{C}$ with the xy plane. Denote N and S the north and south poles of the sphere accordingly. We define $\pi_N\colon \mathbb{C} \to S$ such that π_N sends each point (z,t) on the sphere to its stereographic projection from N onto the plane (point ξ) as can be seen in the figure below:



We can similarly define π_S and verify that the images of the projections are $X\setminus\{N\}$ and $X\setminus\{S\}$ accordingly.

Now X is a Riemann surface with an atlas consisting of $\pi_S\colon X\setminus\{S\} \to \mathbb{C}$ and $\pi_N\colon X\setminus\{N\} \to \mathbb{C}$. We denote the Riemann sphere as $\hat{\mathbb{C}}$.

Definition 1.6 (Biholomorphism of Riemann surfaces)**.** Let $(X, (U_i, f_i))$, $(Y, (W_i, g_i))$ be two Riemann surfaces. A biholomorphism between them is a homeomorphism $X \xrightarrow{\phi} Y$ such that $g_i \circ \phi \circ f_i^{-1}$ are biholomorphisms on their domains of definition.

A main problem in Riemann surfaces was classifying certain types of Riemann surfaces up to biholomorphisms.

Theorem 1.1. (Riemann mapping theorem). *Any two proper open simply connected subsets of $\mathbb{C}$ are biholomorphic.*

A generalization of the Riemann mapping theorem is the uniformization theorem proved by Kobe in 1907.

Theorem 1.2. (Uniformization theorem). *Any simply connected Riemann surface is biholomorphic to one of the following:*

(1) $\mathbb{C}$

(2) $\hat{\mathbb{C}}$

(3) $\mathbb{H} = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$

We will give a proof for this theorem in the end of the class.

A natural question that arises is what about non simply connected Riemann surfaces?

Theorem 1.3. (Uniformization theorem, part II). *Any connected Riemann surface is biholomorphic either to* $\hat{\mathbb{C}}$ *or to a quotient of* $\mathbb{C}$ *or* $\mathbb{H}$ *by a properly discontinuous torsion-free subgroup of biholomorphisms.*

Remark 1.2. Biholomorphisms of $U = \mathbb{C}$ or $\mathbb{H}$ (or any subset of $\mathbb{C}$) forms a group under composition. We denote that group by $\mathrm{Bih}(U)$.

Definition 1.7 (Properly discontinuous group)**.** A countable subgroup of $\mathrm{Bih}(U)$ is said to be properly discontinuous if for all compact $K \subseteq U$, the set $\{g \in G \colon gK \cap K \neq \emptyset\}$ is finite.

Definition 1.8 (torsion-free action)**.** $G \subseteq \mathrm{Bih}(U)$ is torsion-free if $gp = p$ for some $p \in U$ implies g is the identity.

Remark 1.3. Notice that multiplication in gp is the group action of g on the set U. That us $gp = g(p)$.

We can know define the quotient space ${}^{U}/_{G}$ where $p \sim q$ if there exists $g \in G$ such that $gp = q$.

Introduce a topology on ${}^{U}/_{G}$ which is the coarsest topology such that the canonical projections $U \to {}^{U}/_{G}$ are continuous.

Under the assumptions that G is properly discontinuous and torsion-free, ${}^{U}/_{G}$ is a Riemann surface with the following charts. By assumptions on G, we can find for any $p \in U$ a neighbourhood W of $p \in U$ such that $\pi \colon U \to {}^{U}/_{G}$ is a homeomorphism onto its image when restricted to W.

So, restrictions of π to these neighbourhoods W provide an atlas.

Definition 1.9 (Free action)**.** Let G

2 Introduction to Teichmuller spaces

Let S be a topological space. Then

$$\text{Modulispace}(S) = \{\text{Riemann surfaces homeomorphic to } S \text{ up to biholomorphisms}\}$$

Definition 2.1 (Teichmuller space)**.** We consider pairs (X, f) where X is a Riemann space and $f\colon S \to X$ is a homeomorphism. Then

$$\text{Teich}(S) = \left\{(X, f)\colon f\colon S \to X\right\} /\sim$$

where $(X_1, f_1) \sim (X_2, f_2)$ if there exists a biholomorphism $X_1 \xrightarrow{\phi} X_2$ such that $f_2 \circ f_1^{-1}$ is homotopic to ϕ. In other words commutes up to homotopy.

Remark 2.1. The pair (X, f) is called a *marking* for S.

Definition 2.2 (Homotopy)**.** We say that two continuous functions f, g from a topological space X to Y are homotopic if there exists a continuous function $H\colon X \times [0,1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Definition 2.3 (Mapping class group)**.** First we define:

$$\begin{aligned}
&\text{Homeo}^+(S) = \{\text{Orientation preserving homeomorphisms } S \to S\}\\
&\text{Homeo}^0 \triangleleft \text{Homeo}^+(S) = \{\text{the homeo. homotopic to the identity.}\}
\end{aligned}$$

And now we define

$$\text{MCG}(S) = \text{Homeo}^+(S) \Big/ \text{Homeo}^0(S) = \frac{\text{Diff}^+(S)}{\text{Diff}^0(S)}.$$

We have that MCG acts on $\text{Teich}(S)$ by $\varphi \in \text{Homeo}^+(S), [\varphi] \in \text{MCG}(S)$ as such

$$[\varphi][(X, f)] = [(X, f \circ e^{-1})].$$

Now $\mathcal{M}(S) = \text{Teich}(S) \Big/ \text{MCG}(S)$. Our goal is to determine $\mathcal{M}(S)$ and $\text{MCG}(S)$ when S is a torus.

Theorem 2.1. *Assume S is a torus $T \cong \mathbb{R}^2 / \mathbb{Z}^2$. Then $\text{MCG}(S) = \text{SL}_2(\mathbb{Z})$ acting linearly on the torus. $\mathcal{M}(S)$ can be identified with $\mathbb{H} \Big/ \text{SL}_2(\mathbb{Z})$ where the action is by Mobius transformations.*

Proof. Any Riemann surfaces homeomorphic to S has to form $\mathbb{C}/\Lambda$ where $\Lambda \subseteq \text{Bih}(\mathbb{C})$ is properly disc, free, and $\Lambda\langle z \to z + \tau_1, z \to z + \tau_2\rangle$ where $\tau_1, \tau_2 \notin \mathbb{R}$. We have to determine when different $\mathbb{C}/\Lambda$ are biholomorphic. We can also write $\mathbb{C}/\Lambda$ where $\Lambda = \langle \tau_1, \tau_2\rangle \subseteq \mathbb{C}$. If $\Lambda_1 = \langle \tau_1, \tau_2\rangle$, $\Lambda_2 = \langle c\tau_1, c\tau_2\rangle$ for $c \in \mathbb{C}^*$. Then $\mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ is given by $[z] \to [cz]$.

For any $\tau_1, \tau_2 \in \mathbb{C}$ there exists $c \in \mathbb{C}^\times$ such that (up to change in order)

This tells us that any Riemann torus is biholomorphic to $\mathbb{C} \Big/ \langle XX, XX\rangle$ where $\tau \in \mathbb{H}$.

So we have a surjection $\mathbb{H} \to \mathcal{M}(S)$. $\square$

Theorem 2.2. *$\mathbb{C} \Big/ \langle 1, \tau_1\rangle$ is biholomorphic to $\mathbb{C} \Big/ \langle 1, \tau_2\rangle$ iff exists $A \in \text{SL}_2(\mathbb{Z})$ such that writing τ_1, τ_2 as elements of $\mathbb{R}^2$, we have $A\tau_1 = \tau_2$.*

Proof. Suppose there exists a biholomorphism $f\colon \mathbb{C}\big/\langle 1,\tau_1\rangle \to \mathbb{C}\big/\langle 1,\tau_2\rangle$. Let $\bar{f}\colon \mathbb{C}\to\mathbb{C}$ be a lift of f. This means $f(g+x)-f(x)\in\langle 1,\tau_1\rangle$ whenever $g\in\langle 1,\tau_1\rangle$.

$$\begin{aligned} &f\colon \mathbb{C}\big/\langle 1,\tau_1\rangle \to \mathbb{C}\big/\langle 1,\tau_2\rangle bih. \\ &\bar{f}\colon \mathbb{C}\to\mathbb{C}lift. \end{aligned}$$

By post composing with a biholomorphism of $\mathbb{C}$ we can assume $\bar{f}(0)=0$. We know that $\bar{f}(\tau_2)$ and $\bar{f}(1)$ are equivalent mod $\langle 1,\tau_1\rangle$. □

Remark 2.2. Let S be a Riemann surface. Recall that $\mathcal{M}(S)$ is the moduli space of S, which is the space of Riemann surfaces homeo to S up to biholomorphism.

Recall that we define

$$\mathrm{MCG}(S)=\mathrm{Homeo}^+(S)\big/\mathrm{Homeo}^0(S)$$

And the Teichmuller space of S

$$\mathrm{Teich}(S)=\left\{(X,f)\colon f\colon S\to X\right\}/\sim$$

where $(X_1,f_1)\sim(X_2,f_2)$ if there exists a biholomorphism $X_1\xrightarrow{\phi}X_2$ such that $f_2\circ f_1^{-1}$ is homotopic to ϕ.

Our goal today is to show that for $T=\text{torus}=\mathbb{R}^2/\mathbb{Z}^2$.

- $\mathrm{MCG}(S)\cong\mathrm{SL}_2(\mathbb{Z})$.
- $\mathcal{M}(S)\cong\mathbb{H}\big/\mathrm{SL}_2(\mathbb{Z})$.
- $\mathrm{Teich}(S)\cong\mathbb{H}$.

Notice that because of the relation

$$\mathrm{Teich}(S)\big/\mathrm{MCG}(S)=\mathcal{M}(S)$$

we only need to prove two of these propositions.

First let's prove that $\mathcal{M}(S)\cong\mathbb{H}\big/\mathrm{SL}_2(\mathbb{Z})$. Any Riemann surface is homeomorphic to T (a complex torus) is biholomorphic to $\mathbb{C}\big/\langle\tau_1,\tau_2\rangle$.

Recall that if $c\in\mathbb{C}^\times$ then $\mathbb{C}\big/\langle\tau_1,\tau_2\rangle$ is biholomorphic to $\mathbb{C}\big/\langle c\tau_1,c\tau_2\rangle$ via the map

$$f(z+\langle\tau_1,\tau_2\rangle)=cz+\langle c\tau_1,c\tau_2\rangle.$$

Up to changing the order of τ_1, τ_2 we can find $c\in\mathbb{C}\times$ such that $c\tau_1=1$ and $c_2\in\mathbb{H}$.

Our goal now is to determine when there exists a biholomorphism

$$\mathbb{C}\big/\langle 1,\tau_1\rangle\mapsto\mathbb{C}\big/\langle 1,\tau_2\rangle,\quad \tau_1,\tau_2\in\mathbb{H}$$

Denote $\Lambda_\tau=\langle 1,\tau\rangle$, when $\tau\in\mathbb{H}$.

Theorem 2.3. *There exists a biholomorphism*

$$\mathbb{C}\big/\langle 1,\tau_1\rangle\mapsto\mathbb{C}\big/\langle 1,\tau_2\rangle,\quad \tau_1,\tau_2\in\mathbb{H}$$

iff there exists

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})$$

with $\tau_2=\frac{a\tau_1+b}{c\tau_2+d}$. *This will imply that*

$\mathcal{M}(S) \cong \mathbb{H} \big/ \mathrm{SL}_2(\mathbb{Z})$.

Proof. Suppose there exists a biholomorphism

$$f\colon \mathbb{C} \big/ \langle 1, \tau_1 \rangle \to \mathbb{C} \big/ \langle 1, \tau_2 \rangle .$$

Lift it up to $\bar{f}\colon \mathbb{C} \to \mathbb{C}$. Then $\bar{f}(g+x) - \bar{f}(x) \in \Lambda_{\tau_1}$ whenever $x \in \mathbb{C}$, $g \in \Lambda_{\tau_2}$. We can replace $\bar{f}$ with $\bar{f} - \bar{f}(0)$. We can assume that $\bar{f}(0) = 0$ (and is still a lift off).

Remark 2.3. Since $\bar{f}$ is a lift off of f we have $\bar{f}(\Lambda_{\tau_2}) = \Lambda_{\tau_1}$.

We know that $\bar{f}$ has form $\bar{f}(z) = az + b$ such that $a \in \mathbb{C}^\times$, $b \in \mathbb{C}$. As $\bar{f}(0) = 0$ we have $\bar{f}(z) = az$. Also, $\{0, 1, \tau_2\} \subseteq \Lambda_{\tau_2}$ so $0 = \bar{f}(0), \bar{f}(1) = a, \bar{f}(\tau_2) = a\tau_2$ are in Λ_{τ_1} so we can write $a = \bar{f}(1) =$ We now have

$$\tau_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and we still need to show it is an element of $\mathrm{SL}_2(\mathbb{Z})$. $\square$

3 Isotopy and Homotopy

Let X be a metric space and $F_0, F_1 \colon X \to X$ homeomorphisms. Then we say F_0, F_1 are homotopic if there exists a family F_t for $t \in [0,1]$ of continuous maps such that $t \mapsto F_t(x)$ is continuous for all $x \in X$ (we don't require F_0, F_1 to be homeomorphisms).

The maps F_0, F_1 are said to be isotopic if F_t are required to be homeomorphisms.

For a surface S, we defined $\mathrm{MCG}(S)$

Theorem 3.1 (Baire, Epstein)**.** *If S is a finite type surface (e.g. closed surface of XXX) then two homeomorphisms $F_0, F_1 \colon X \to X$ are homotopic iff they are isotopic.*

Last time we have shown that if $T = \mathbb{R}^2/\mathbb{Z}^2$ is a torus, then $\mathrm{MCG}(S) = \mathrm{SL}_2(\mathbb{Z})$. For any $A \in \mathrm{SL}_2(\mathbb{Z})$ we obtained a homeomorphism

$$\begin{array}{rcccc} \psi_A & : & T & \longrightarrow & T \\ & & [x] & \longmapsto & [Ax] \end{array}$$

We showed that any orientation preserving homeomorphism $\phi \colon T \to T$ is homotopic to ψ_A for some $A \in \mathrm{SL}_2(\mathbb{Z})$.

This gives a map

$$\begin{array}{rcccc} \Phi & : & \mathrm{SL}_2(\mathbb{Z}) & \longrightarrow & \mathrm{MCG}(T) \\ & & [A] & \longmapsto & [\psi_A] \end{array}$$

which we know is surjective. Why is Φ injective?

We want to show that for $A \in \mathrm{SL}_2(\mathbb{Z})$ and $A \neq I$ that ψ_A is not homotopic to the identity map.

We will see this by showing that ψ_A acts nontrivially on the fundamental group $\Pi_1(T)$.

Definition 3.1 (Loop)**.** A loop based at p is a continuous function $\gamma \colon [0,1] \to X$ with $f(0) = f(1) = 0$.

Definition 3.2 (Homotopy)**.** We say that two continuous functions f, g from a topological space X to Y are homotopic if there exists a continuous function $H \colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.

Definition 3.3 (Fundamental group)**.** The fundamental group of a topological space X is the space of all loops in X based at a point $p \in X$ up to homotopy.

A homeomorphism $\psi \colon X \to X$ acts on $\Pi_1(X)$ by

$$\psi[\gamma] = [\psi \circ \gamma].$$

The group operation is concatenation of loops.

Invesrion is changing the direction of the loop.

Exercise 3.1 (Fundamental group of the torus)**.** The fundamental group of T is $\Pi_1(T) \cong \mathbb{Z}^2$ is generated by

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \quad \text{and} \quad \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].$$

This means any loop is homotopic to $[\begin{pmatrix} a \\ b \end{pmatrix}]$ by doing $[\begin{pmatrix} 1 \\ 0 \end{pmatrix}]$ a times and $[\begin{pmatrix} 0 \\ 1 \end{pmatrix}]$ b times.

For $A \in \mathrm{SL}_2(\mathbb{Z})$ and $\psi_A\colon T \to T$ action on $\Pi_1(T,(0,0))$ (fundamental group of the torus based in $(0,0)$) defined by

$$(x,y) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}.$$

If ψ_A was homotopic to the identity it owuld act trivially on $\Pi_1(T,(0,0))$ which means that

$$[0,1] = [A(1,0)] = [(b,d)] \text{ and } [(1,0)] = [A(1,0)] = [(a,c)]$$

so $(a,c) = (1,0)$ and $(b,d) = (0,1)$ so A is the identity matrix.

4 Hyperbolic geometry

Recall that $\mathrm{Bih}(\mathbb{H})$ is the set of Mobius transformations with a representing matrix in $\mathrm{SL}_2(\mathbb{Z})$.

But $z \mapsto \frac{1}{z}$ does not present Euclidean metric on $\mathbb{H} = \{z \colon \Im(z) > 0\}$.

We will introduce a metric $\rho = \rho_{hyp}$ on $\mathbb{H}$ s.t.

$$\mathrm{Bih}(\mathbb{H}) = \text{Orientation preserving isometrics of } (\mathbb{H}, \rho).$$

Definition 4.1 (Length)**.** Let $\gamma \colon [0,1] \to \mathbb{H}$ be a pointwise continuous smooth path between a and b.

The hyperbolic length of γ is

$$L_\rho(\gamma) := \int_0^1$$

Recall that every element in every $\mathrm{Mob}(\hat{\mathbb{C}})$ is conjugate to either $z \mapsto z+1$ or $z \mapsto \Lambda z$ for $\Lambda \in \mathbb{C} \setminus \{0,1\}$.

Recall that $T_\lambda(z) = \Lambda z$ and $T_{\Lambda^{-1}}(z) = \Lambda^{-1} z$ are conjugate via the transform

$$S(z) = \frac{1}{z}.$$

Indeed this can be verified.

So in the above classification we can assume $|\Lambda| \geq 1$.

Corollary 4.1. *A complete set of nonindentity conjugacy class representations in* $\mathrm{Mob}(\mathbb{C})$ *is* $z \mapsto z+1$ *or* $z \mapsto \Lambda z$ *for* $\{\Lambda, 1/\Lambda\}$ *distinct unordered pairs.*

Proof. We know that $z \mapsto z+1$ cannot be conjugate to $z \mapsto \Lambda z$ which has two fixed points in $\hat{\mathbb{C}}$.

We need to show that for different $\Lambda \neq 1$ and $|\Lambda| \geq 1$, $z \mapsto \Lambda z$ are not conjugate.

Recall that in general if $\psi \in \mathrm{Mob}(\hat{\mathbb{C}})$ then

$$\psi = \psi_A(z) = \frac{az+b}{cz+d} \text{ s.t. } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

There are two matrices representing ψ which are A and $-A$.

Since $\mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Mob}(\hat{\mathbb{C}})$ defined as $[A] \mapsto \psi_A$ is an isomorphism, we know that ψ_A and ψ_B are conjugate in $\mathrm{Mob}(\hat{C})$ if and only if A is conjugate to $\pm B$ in $\mathrm{SL}_2(\mathbb{C})$.

Now, if A is conjugate to B then A, B have the same eigenvalues.

Coming back to $z \mapsto \Lambda z$.

If $\Lambda_1, \Lambda_2 \in \mathbb{C} \setminus 0, 1$ such that $|\Lambda_1|, |\Lambda_2| \geq 1$. $z \mapsto \Lambda z$ is represented by

$$\begin{pmatrix} \Lambda & 0 \\ 0 & 1/\Lambda \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

which has eigenvalues $(1, 1/\Lambda)$ for distinct $|\Lambda| \geq 1$ so they can't be conjugate. $\square$

4.1 Qualitative properties of Möbius transformations

The map $z \mapsto z+1$ preserves the unique line in $\mathbb{H}$ which converges to ∞ (which is the fixed point of $z \mapsto z+1$) in both directions. In other words, all the horizontal lines.

The map $z \mapsto \Lambda z$ preserves the geodesic in $\mathbb{H}$ which converges to the two fixed points $z \to \Lambda z$ (namely 0, ∞).