Some Calculus Proofs

Yeheli Fomberg

Contents

1 Getting Used to Integrals

1.1 Suppose f is Riemann integrable then f^2 is also Riemann integrable

First recall the definition of a Riemann integrable function. A function f is Riemann integrable on $[a, b]$ iff

$\exists I \in \mathbb{R} : \forall \varepsilon > 0$ exists $\delta > 0$ such that for any partition $X = (x_0, x_1, \ldots, x_n)$: $\lambda(X) < \delta$ any sequence $(c_1, c_2, \ldots, c_n)$ that satisfies $c_i \in [x_{i-1}, x_i]$ also satisfies :

$$|\sum_{i=1}^{n} f(c_i)\Delta X_i - I| < \varepsilon$$

And we denote $I = \int_a^b f(x)dx$
Now consider f^2 :

$$\begin{aligned} U(f, P) - D(f, P) &< \varepsilon \\ \Rightarrow U(f, P)^2 &< D(f, P)^2 + 2\varepsilon D(f, P) + \varepsilon^2 \\ \Rightarrow U(f, P)^2 - D(f, P)^2 &< \varepsilon(2D(f, P) + \varepsilon) \end{aligned}$$

Since $2D(f, P) + \varepsilon$ is bounded by M for $\varepsilon < 1$ we can choose $\varepsilon = \min(\frac{\varepsilon}{M}, 1)$ and for that ε a matching δ that satisfies the condition.

$$\begin{aligned} U(f, P)^2 - D(f, P)^2 &< \varepsilon \\ \Rightarrow U(f^2, P)^2 - D(f^2, P) &< \varepsilon \end{aligned}$$

And thus we finished.

1.2 If f is Continuous Then f is Integrable

Let f be continuous on $[a, b]^*$. By Cantor-Heine it is uniformally continuous.

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)\Delta X_i <^* \sum_{i=1}^{n} \varepsilon \Delta X_i = \varepsilon(b - a)$$

(*) This is becuase by definition

$$\forall \varepsilon > 0 : \exists \delta > 0 (d(x, y) < \delta \to d(f(x), f(y)) < \varepsilon)$$

And that delta is exactly what we wanted.

*Think about the case (a, b)

2 Intermediate Value Theorem for Integrals

Let f be a continuous funtion on $[a, b]$ then

$$\exists c \in [a, b] \text{ such that } \int_a^b f(x)dx = f(c)(b-a)$$

f is continuous so it is Riemann integrable. We know that by Weierstrass that since f is continuous it has a minimum and maximum which we'll denote m, M

$$m(b-a) < \int_a^b f(x)dx < M(b-a)$$

$$\Rightarrow m < \frac{\int_a^b f(x)dx}{b-a} < M$$

Let $c = \frac{\int_a^b f(x)dx}{b-a}$. By the intermediate value theorem we know that this c exists and then

$$\int_a^b f(x)dx = f(c)(b-a)$$

3 Fundamental Theorem of Calculus

Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Let F be the function defined, for all x in $[a, b]$, by:

$$F(x) = \int_a^x f(t)dt$$

Then F is uniformly continuous on $[a, b]$ and differentiable on the open interval (a, b), and

$$F'(x) = f(x) \quad \forall x \in (a, b)$$

If f is Riemann integrable on $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

3.1 Part One

For any $x_1, x_1 + \Delta x \in [a, b]$ we get

$$F(x_1 + \Delta x) - F(x_1) = \int_a^{x_1+\Delta x} f(t)\,dt - \int_a^{x_1} f(t)\,dt = \int_{x_1}^{x_1+\Delta x}$$

According to the mean value theorem for integration we get for $c \in [x_1, x_1 + \Delta x]$

$$\int_{x_1}^{x_1+\Delta x} f(d)\,dt = f(c)\Delta x$$

And so

$$\lim_{\Delta x \to 0} \frac{F(x_1 + \Delta x) - F(x_1)}{\Delta x} = \lim_{\Delta x \to 0} f(c) = f(x_1)$$

And thus according to the squeeze theorem and f's continuity $F'(x_1) = f(x_1)$

3.2 Part Two

Let $P = (x_0, x_1, \ldots, x_n)$ a partition of $[a, b]$ such that $(x_0, x_n) = (a, b)$

$$\begin{aligned}
F(b) - F(a) &= F(x_n) + [F(x_{n-1}) + F(x_{n-1})] + \ldots + F(x_n) + [F(x_1) + F(x_1)] - F(x_0) \\
&= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)] \\
&= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]
\end{aligned}$$

Since F is continuous on $[a, b]$ and differentiable on (a, b) according to the mean value theorem on $[x_i, x_{i-1}]$ where $c_i \in [x_i, x_{i-1}]$ we get

$$\sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] = \sum_{i=1}^{n} [F'(c_i)(x_i - x_{i-1})]$$

According to part one we get that $F'(c_i) = f(c_i)$ and so

$$F(b) - F(a) = \sum_{i=1}^{n} [f(c_i)(\Delta x_i)]$$

$$\Longleftrightarrow$$

$$\lim_{||\Delta x_i|| \to 0} F(b) - F(a) = \lim_{||\Delta x_i|| \to 0} \sum_{i=1}^{n} [f(c_i)(\Delta x_i)]$$

$$\Longleftrightarrow$$

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

4 Find Formula For Length of Continuous Graph

Approximating the length of a graph using the pythagorian theorem for partition $X = (x_0, x_1, \ldots, x_n)$ we get

$$\sum_{i=1}^{n} \sqrt{d(x_{i-1}, x_i)^2 + d(f(x_{i-1}), f(x)_i)^2}$$

And assuming f is continuous on $[a, b]$ and differentiable on (a, b) by Lagrange we get

$$\begin{aligned}
\sum_{i=1}^{n} \sqrt{d(x_{i-1}, x_i)^2 + d(f(x_{i-1}), f(x)_i)^2} &= \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f'(c_i)(x_i - x_{i-1}))^2} \\
&= \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 (1 + (f'(c_i))^2)} \\
&= \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \Delta X_i
\end{aligned}$$

We can see that this summation is matching an integral

$$\lim_{||\Delta x_i|| \to 0} \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \Delta X_i = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

And so the result of this integral will give us the length of any continuous graph

5 The Limit Comparison Test

Let f, g be two integrable non-positive functions on $[a, M]$ for any M. Now suppose

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$$

If $c \in (0, \infty)$ then either both series converge or both series diverge. And if $c = 0$ or $c = \infty$ you could say something as well(think what).

5.1 Proof

Let $\varepsilon > 0$ we know that $\exists x_0 \in \mathbb{R} : \forall x_0 < x$

$$g(x)(c - \varepsilon) < f(x) < g(x)(c + \varepsilon)$$

Then if $g(x)$ converges then $f(x)$ converges by the squeeze theorem. Similarly if g diverges we know that

$$g(x)(c - \varepsilon) < f(x)$$

So from a certain point onwards f will meet the requirments of the direct comparison test and so will diverge.

6 Some Practise

6.1 Limits With Series: Find $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right)$

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right) &= \sum_{k=1}^{n}\frac{1}{n+k}\\ &= \sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\frac{1}{n}\\ &= \sum_{k=1}^{n}f(\frac{k}{n})\frac{1}{n}\\ &= \int_0^1\frac{1}{1+x}\end{aligned}$$

6.2 Check convergion: $\int_{\frac{1}{2}}^{1}\frac{1}{x\sqrt{1-x}}$

This seems to behave a lot like $\frac{1}{\sqrt{1-x}}$ so let's compare them using the limit comparison test

$$\lim_{x\to 1^-}\frac{\frac{1}{x\sqrt{1-x}}}{\frac{1}{\sqrt{1-x}}}=\lim_{x\to 1^-}\frac{1}{x}=0$$

So by the limit comparison test we know that the integral converges.

7 Absolute Convergance Implies Conditional Convergance

$$\int_a^\infty |f(x)|\,dx \quad \text{converges} \quad \Rightarrow \int_a^\infty f(x)\,dx \quad \text{converges}$$

The first condition is called **"absolute convergence"**. We will prove it implies the second condition **"conditional convergance"**. Suppose f converges absolutely. Consider:

$$f^+ = \begin{cases} f(x) & f(x) > 0 \\ 0 & f(x) \leq 0 \end{cases}$$

$$f^- = \begin{cases} -f(x) & f(x) < 0 \\ 0 & f(x) \geq 0 \end{cases}$$

We know that $|f| = f^+ + f^-$ convereges and so by the direct comparison test we get that $\int_a^\infty f^+, \int_a^\infty f^-$ converge and since $f = f^+ - f^-$ we also get that $\int_a^\infty f$ as well.

8 Convergence Tests

8.1 Dirichlet's Test

If a_n is a monotonic and $\lim_{n\to\infty} a_n = 0$ and $\sum b_n$ is bounded then

$$\sum_{n=1}^{\infty} a_n b_n \text{ converges}$$

8.2 Abel's Test

Suppose $\sum a_n$ is a convergent series, and b_n is monotone and bounded. Then $\sum a_n b_n$ also converges.

8.3 Root And Quotient Test

Just like in Calc1. Suppose the limit of

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q > 1$$

Then eventually it'll be greater than q^n and by the comparison test it diverges. If $q < 1$ then eventually it'll be smaller than q^n. Same idea for the root test.

9 Rabbe's Test

In case the quotient test doesn't work - let a_n be a positive sequence then if

$$\lim_{n\to\infty} \left(n \left(1 - \frac{a_n + 1}{a_n} \right) \right) = q$$

then if(and it is different from the root/quotient test)

$$\begin{cases} q > 1 & \text{the series converges} \\ q < 1 & \text{the series diverges} \\ q = 1 & \text{we must check using a better Rabbe...} \end{cases}$$

10 Integral Test for Series

Let $f(x)$ be a positive monotone decreasing function on $[1, \infty]$. Define $a_n = f(n)$ then:

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \Longleftrightarrow \int_1^{\infty} f(x)dx \quad \text{converges}$$

That is true since suppose the series converges then we get:

$$\sum_{n=1}^{\infty} a_n < M$$

But we also know that:

$$0 \leq \sum_{n=2}^{\infty} a_n \leq \int_1^{N+1} f(x)dx \leq \sum_{n=1}^{\infty} a_n$$

That means thatthe integral is increasing and bounded so it's converging. Suppose the integral was converging to prove the series is also converging we could show it's "bounded" by the integral's bound.

11 Cauchy Condensation Test

Let a_n be a non-increasing sequence of non-negative number.

$$\sum_{n=1}^{\infty} f(n) \leq \sum_{n=0}^{\infty} 2^n f(2^n) \leq 2 \sum_{n=1}^{\infty} f(n)$$

This is becuase of simple rearrangement of the numbers:

$$\begin{aligned}
\sum_{n=1}^{\infty} f(n) &= f(1) + f(2) + f(3) + f(4) + \dots \\
&= f(1) + (f(2) + f(3)) + (f(4) + f(5) + f(6) + f(7)) \dots \\
&\leq f(1) + (f(2) + f(2)) + (f(4) + f(4) + f(4) + f(4)) \dots \\
&= \sum_{n=0}^{\infty} 2^n f(2^n) \\
&= f(1) + (f(2) + f(2)) + (f(4) + f(4) + f(4) + f(4)) \\
&\leq (f(1) + f(1)) + (f(2) + f(2)) + (f(3) + f(3)) + (f(4) + f(4)) \dots \\
&= 2 \sum_{n=1}^{\infty} f(n)
\end{aligned}$$

12 Leibniz's Test

Let a_n be a monotone decreasing positive sequence such that

$$\lim_{n\to\infty} a_n = 0$$

Then

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n \text{ converges}$$

Since a_n is monotonically decreasing then we can say that

$$S_{2(m+1)} = S_{2m} + a_{2m+1} - a_{2m}$$

$$S_{2(m+1)+1} = S_{2m+1} - a_{2m+2} + a_{2m+3}$$

Or in other words S_{2m} monotonically increases and S_{2m+1} monotonically decreases. But we also know that

$$S_{2m+1} - S_{2m} = a_{2m+1} \geq 0$$

And that means that

$$a_1 - a_2 = S_2 \leq S_{2m} \leq S_{2m+1} \leq S_1 = a_1$$

In other words our monotone sequences are bounded and so they converge. Recall as $m \to \infty$

$$S_{2m+1} - S_{2m} = a_{2m+1} \to 0$$

And so they must converge to the same limit L. Moreover

$$S_{2m} \leq L \leq S_{2m+1}$$

And also

$$|S_k - L| \leq a_{k+1}$$

13 Riemann Series Theorem

Suppose that $(a_1, a_2, a_3, \ldots)$ is a sequence of real numbers, and that $\sum_{n=1}^{\infty} a_n$ is conditionally convergent. Let M be a real number. Then there exists a permutation σ such that:

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = M$$

This is also the case for $M = \pm\infty$
If a_n is absolutely converging then rearrangement of the elements is possible.
If a series is converging then by creating a new sequence we can show putting parentheses is valid.

13.1 Bonus: Dini's Theorem

Let $f_n(x) \to f(x)$ converge pointwise in $D = [a, b]$ and $\forall x \in D(f_n(x)$ is monotonic) and f, f_n are continuous then $f_n(x) \to f(x)$ converges uniformally.

14 Properties of Uniformally Converging Function Sequences

14.1 Continuity

Suppose $f_n \to f$ converges uniformally, and f_n is continuous for any $n \in \mathbb{N}$ then f is continuous. The proof is based on the triangle inequality.

14.2 Integrability

Suppose $f_n \to f$ converges uniformally on $[a, b]$, and f_n is integrable for any $n \in \mathbb{N}$ then f is integrable and as $n \to \infty$

$$\int_a^b \int f_n \to \int_a^b \int f$$

14.3 Differentiability

Suppose $f_n \in C^1$ on I such that:

- f_n' uniformally converges on I
- $\exists x_0 : f_n(x_0)$ converges

Then f_n uniformally converge on I to f and

$$f_n' \to f'$$

15 Weierstrass M-Test

Let $\sum_{n=1}^{\infty} f_n(x)$ be a function series. Suppose $\exists M_n$ such that

- $\forall n \in \mathbb{N}(|f_n(x)| \leq M_n)$
- $\sum_{n=1}^{\infty} M_n$ converges

Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformally.

15.1 Proof

Since M_n converges we can use an quivalent definition for the convegence of the series and so

$$\forall \varepsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{such that} \quad (n > N) \wedge (p \in \mathbb{N}) \rightarrow \left(\left| \sum_{k=1}^{n+p} M_k(x) - \sum_{k=1}^{n} M_k(x) \right| < \varepsilon \right)$$

And since $0 \leq M_n$ that implies

$$\sum_{k=n+1}^{n+p} M_k(x) < \varepsilon$$

And so we get that:

$$\left| \sum_{k=n+1}^{n+p} f_k(x) \right| \leq \sum_{k=n+1}^{n+p} |f_k(x)| \leq \sum_{k=n+1}^{n+p} M_n < \varepsilon$$

16 Power Series Theorems

Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for some x_0, then it absolutely converges for any x such that $|x| < |x_0|$.

Since the power series converges $\lim_{n\to\infty} a_n x_0^n = 0$ and so the sequence is bounded and we denote that bound M.

$$0 \le |a_n x^n| = |a_n x_0^n| \left|\frac{x^n}{x_0^n}\right| < M \left|\frac{x}{x_0}\right|^n$$

And this sequence's sum is a geometric series so it converges and so does $\sum_{n=1}^{\infty} |a_n x^n|$. We also know that $|a_n x^n| < |a_n x_0^n|$ for all $n \in \mathbb{N}$ so according to Weierstrass $\sum_{n=1}^{\infty} |a_n x^n|$ uniformally converges.

Let

$$X = \{x \in \mathbb{R} : \sum_{n=1}^{\infty} a_n x^n \text{ converges}\}$$

We claim $R = \sup X$. For any $x > R$ the series cannot converge. And if $x < -R$ we know that exists $R < x_0 < |x|$ for which the series converges, but that's not a contradiction. if $|x_0| < |R|$ than there exists an $|x_0| < |x| < |R|$ for which the series converges and then it converges for x_0 as well.

Now we know the series converges uniformally for any close intervel inside $[-R, R]$ and if we know it CU in $[0, R]$ then it is converging in R as well but this proof is not easy and takes time so we won't learn it.

Let a function series converge to f. Why is f continuous on (a, b)?

17 Cauchy–Hadamard + D'alembert Theorem

Let $\sum_{n=0}^{\infty} a_n x^n$ be a series and let R be the radius of convergence of the series - that is to say the series converges for any $x \in (-R, R)$ then:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|}$$

17.1 Proof the 1st

Let

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

$$\Rightarrow \sqrt[n]{|a_n x^n|} = \sqrt[n]{|a_n|}|x| \to L|x|$$

$$\Rightarrow \begin{cases} |x| < \frac{1}{L} = R & \text{The series converges} \\ |x| > \frac{1}{L} = R & \text{The series diverges} \end{cases}$$

This proves the series converges/diverges absolutely according to the root test. If it converges we know that the original series converges. Suppose it diverges - by the root test we know that if the series diverges the |partial sums| don't converge to zero and thus the original partial sums don't either, and the series diverges.

17.2 Proof the 2nd

Let

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$$

$$\Rightarrow \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|a_{n+1}|}{|a_n|}|x| \to L|x|$$

$$\Rightarrow \begin{cases} |x| < \frac{1}{L} = R & \text{The series converges} \\ |x| > \frac{1}{L} = R & \text{The series diverges} \end{cases}$$

If the series converges absolutly we can be sure yet again that it converges. If it diverges - than by the quotient test the |partial sums| diverge and so the parial sums of the series must also diverge and the series will diverge as we claimed.

18 A Note on the Taylor Series

If f is smooth on $(-R, R)$ then f can be the limit of a power series iff

$$\forall x \in (-R, R)$$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \sum_{n=N+1}^{\infty} a_n x^n = 0$$

That is because

$$f \text{ can be the limit of a power series}$$

$$\iff$$

$$\lim_{n\to\infty} S_n(x) = f(x), \forall x \in (-R, R)$$

$$\iff$$

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} f(x) - S_n(x) = 0, \forall x \in (-R, R)$$

19 Continuous Partial Derivatives imply Differentiability

19.1 Semi-Proof

We want to find the tangetial plane to f for (x_0, y_0) assuming that the partial derivatives are continuous at that point. Let's denote

$$z_o = f(x_0, y_0) \quad \text{and} \quad A = \frac{\partial f}{\partial x}(x_0, y_0) \quad \text{and} \quad B = \frac{\partial f}{\partial y}(x_0, y_0)$$

Now the tangential lines that intersect at z_0 and are parallel to the axes (and in turn are perpendicular to one another) are

$$z = B(y - y_0) + z_0 \tag{1}$$
$$z = A(x - x_0) + z_0 \tag{2}$$

Their directional vectors are in turn

$$\vec{\beta} = (0, 1, B) \tag{3}$$
$$\vec{\alpha} = (1, 0, A) \tag{4}$$

And the normal to their spanning plane is

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & B \\ 1 & 0 & A \end{vmatrix} = (A, B, -1)$$

And so the plane equation is

$$A(x - x_0) + B(y - y_0) - (z - z_0) = 0$$
$$\Rightarrow z = z_0 + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

We have shown that if continuous partial derivatives exists at (x_0, y_0) then f has a tangential plane at (x_0, y_0) which is equivalent to being differentiable at (x_0, y_0)

19.2 Note on Diffferentiability

We say that f is differentiable at (x_0, y_0) if exist A, B such that

$$f(x_0+h, y_0+k) - f(x_0, y_0) = Ah + Bk + \alpha(h, k)\sqrt{h^2 + k^2} = Ah + Bj + \alpha(h, k)h + \beta(h, k)k$$

and $\lim_{(h,k)\to(0,0)} \alpha(h, k)$ and $\lim_{(h,k)\to(0,0)} \beta(h, k)$ That's equivalent to

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x_0, y_0) - \frac{\partial f}{\partial y}(x_0, y_0)}{\sqrt{(y - y_0)^2 + (y - y_0)^2}} = 0$$

20 Leibniz integral rule

Let $f(x, y)$ be continuous on a rectangle $[a, b] \times [c, d]$ and suppose $\frac{\partial f}{\partial y}(x, y)$ exists and is continuous on $[a, b] \times [c, d]$. Define $F(y) = \int_a^b f(x, y)\, dx$ then F is differentiable on $[c, d]$ and

$$F'(y) = \frac{d}{dy} \int_a^b f(x, y)\, dx = \int_a^b \frac{\partial f}{\partial y}(x, y)\, dx$$

20.1 Lemma

Lemma: if $f(x, y)$ is continuous on $[a, b] \times [c, d]$ then $F(y) = \int_a^b f(x, y)\, dx$ is uniforamlly contiuous on $[c, d]$.

We know f is contiuous on a compact space so it is uniformally continuous there:

$\forall \varepsilon > 0\, \exists \delta > 0\, (\forall (p_1 = (x_1, y_1), p_2 = (x_2, y_2)) : d(p_1, p_2) < \delta \to |f(p_1) - f(p_2)| < \varepsilon)$

Now consider $y_1, y_2 \in [c, d]$. such that $d(y_1, y_2) < \delta$ we know that $\forall x \in [a, b]$ that $d((x, y_1), (x, y_2)) > \delta$ and then we can see that

$$|F(y_1) - F(y_2)| = \left| \int_a^b f(x, y_1)\, dx - \int_a^b f(x, y_2)\, dx \right| = \left| \int_a^b (f(x, y_1) - f(x, y_2))\, dx \right|$$

$$\leq \int_a^b |(f(x, y_1) - f(x, y_2))|\, dx < \varepsilon(b - a)$$

20.2 The Rule

Now denote $G(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\, dx$, by the lemma G is continuous.

$$\Delta F = F(y + \Delta y) - F(y) = \int_a^b f(x, y + \Delta y)\, dx - \int_a^b f(x, y)\, dx = \int_a^b (f(x, y + \Delta y) - f(x, y))\, dx$$

We know by the Lagrange theorem that $\exists t \in (0, 1)$ such that

$$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{\partial f}{\partial y}(x, y + \Delta y)$$

$$\Rightarrow \int_a^b (f(x, y + \Delta y) - f(x, y))\, dx = \int_a^b \left(\frac{\partial f}{\partial y}(x, y + t\Delta y)\Delta y \right) dx$$

$$\Rightarrow \frac{\Delta F}{\Delta y} = \int_a^b \left(\frac{\partial f}{\partial y}(x, y + t\Delta y) \right) dx = {}^{\dagger} G(y + t\Delta y) \to G(y)$$

† Since G is continuous as $\Delta y \to 0$

20.3 Generalization

Let $f(x,y)$ be continuously differentiable on a rectangle $[a,b]\times[c,d]$ and suppose $\frac{\partial f}{\partial y}(x,y)$ exists and is continuous on $[a,b]\times[c,d]$, and $\alpha(y),\beta(y)$ are differentiable on $[c,d]$. Define $F(y)=\int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx$ then F is differentiable on $[c,d]$ and

$$F'(y)=\int_{\alpha(y)}^{\beta(y)} \frac{\partial f}{\partial y}(x,y)\,dx + f(\beta(y),y)\beta'(y) - f(\alpha(y),y)\alpha'(y)$$

Denote $\Phi(s,t,y)=\int_s^t f(x,y)\,dx$ then:

$$F(y)=\Phi(\alpha(y),\beta(y),y)=\int_s^t f(x,y)\,dx$$

And now

$$F'(y)=\frac{\partial\Phi}{\partial s}\frac{ds}{dy}+\frac{\partial\Phi}{\partial t}\frac{dt}{dy}+\frac{\partial\Phi}{\partial y}\frac{dy}{dy}$$

So by the rule we proved earlier and the fundamental theorem

$$F'(y)=-f(\alpha(y),y)\alpha'(y)+f(\beta(y),y)\beta'(y)+\int_{\alpha(y)}^{\beta(y)}\frac{\partial f}{\partial y}(x,y)\,dx$$

21 Fubini's Theorem

Let $f(x,y)$ be continuous on rectangle $[a,b] \times [c,d]$ then

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

21.1 Proof

Denote

$$\begin{cases} \varphi(t) = \int_c^t \left(\int_a^b f(x,y)\,dx \right) dy \\ \Psi(t) = \int_a^b \left(\int_c^t f(x,y)\,dy \right) dx \end{cases}$$

Since f is continuous we know that $F(y) = \int_a^b f(x,y)\,dx$ is continuous and so by the fundemetal theorem:

$$\varphi'(t) = \frac{d}{dt} \int_c^t F(y)\,dy = F(t) = \int_a^b f(x,t)\,dx$$

Denote $G(x,t) = \int_c^t f(x,y)\,dy$. Then by the fundemental theorem we get

$$\frac{\partial G}{\partial t} = f(x,t)$$

And thus by the Leibniz Integral Rule

$$\Psi'(t) = \frac{d}{dt} \int_a^b G(x,t)\,dx = \int_a^b f(x,t)\,dx$$

We concluded that φ, Ψ have the same derivative. That means that

$$\varphi = \Psi + const.$$

We know that $\varphi(c) = \Psi(c) = 0$ and so $const. = 0$ and so

$$\varphi = \Psi$$

and specifically

$$\varphi(d) = \Psi(d)$$

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

22 The Chain Rule

Let $f(x,y)$ have continuous partial derivatives on domain D. Let $x(t), y(t)$ be differentiable on Interval I such that $\forall t \in I : (x(t), y(t) \in D)$ and denote $F(t) = f(x(t), y(t))$ then

$$F'(t) = \left.\frac{\partial f}{\partial x}\right|_{(x(t),y(t))} \frac{dx}{dt} + \left.\frac{\partial f}{\partial y}\right|_{(x(t),y(t))} \frac{dy}{dt}$$

22.1 Proof

$$\frac{dF}{dt} = \lim_{\Delta t \to 0} \frac{F(t+\Delta t) - F(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t}$$

Denote

$$\begin{cases} \Delta x = x(t+\Delta t) - x(t) \\ \Delta y = y(t+\Delta t) - y(t) \end{cases}$$

$$= \lim_{\Delta t \to 0} \frac{f(x(t)+\Delta x, y(t)+\Delta y) - f(x(t), y(t))}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}$$

Since f is differentiable

$$\Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \alpha(\Delta x, \Delta y)\Delta x + \beta(\Delta x, \Delta y)\Delta y$$

Where $\alpha, \beta \to 0$ So:

$$F'(t) = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \alpha(\Delta x, \Delta y)\Delta x + \beta(\Delta x, \Delta y)\Delta y}{\Delta t}$$

$$= \left.\frac{\partial f}{\partial x}\right|_{(x(t),y(t))} \frac{dx}{dt} + \left.\frac{\partial f}{\partial y}\right|_{(x(t),y(t))} \frac{dy}{dt}$$

22.2 Corollary

suppose $F(u,v) = f(x(u,v), y(u,v))$ then we see that

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

23 Substitution For Multiple Variables

Let f be integrable over Domain D. Let $x(u,v)$ and $y(u,v)$ be in C_1[†] and let them define an invertible transformation $\varphi : D \to E$ where D is defined on an xy plane and E on an uv plane. Now suppose

$$\mathbb{J} = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \neq 0 \quad \forall (u,v) \in E$$

*It can be equal to 0 in the domain if the measure of the set of those points is 0.

Then

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_E f(x(u,v),y(u,v))|\mathbb{J}|\,du\,dv$$

[†]continuously differentiable

24 Calculate $\iint_{-\infty}^{\infty} e^{-x^2}$

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Now consider the integral in polar coordinates.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy &= \int_0^{\infty}\int_0^{2\pi} e^{-r^2} r^{\dagger}\,d\theta\,dr = \int_0^{\infty}\int_0^{2\pi} re^{-r^2}\,d\theta\,dr \\
&= \int_0^{\infty} \theta re^{-r^2}\Big|_{\theta=0}^{\theta=2\pi}\,dr = 2\pi\int_0^{\infty} re^{-r^2}\,dr \\
&= 2\pi\int_0^{\infty} re^{-r^2}\,dr
\end{aligned}$$

And

$$\begin{aligned}
2\pi\int_0^{\infty} re^{-r^2}\,dr &= 2\pi \lim_{M\to\infty} -\frac{1}{2}e^{-r^2}\Big|_0^M \\
&= 2\pi \lim_{M\to\infty} -\frac{1}{2}(e^{-M^2} - e^{-0^2}) = 2\pi(-\frac{1}{2}(0-1)) = \pi \\
&\Rightarrow \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi \\
&\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}
\end{aligned}$$