Practice

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1 In the following sections show that $V = U \oplus W$ and find the projection on U parallel to W

1.1 $V = \mathbb{R}[x]$ with

$$W = \mathbf{Sp}\{x^2 + x + 1\}, \quad U = \{p(x) \in V : p(0) = 0\}$$

First we will show that $U + W = V$. Let $p \in V$ be a general polynomial:

$$p(x) = a_n x^n + \cdots + a_0 \in V$$

Now choose $w = a_0(x^2 + x + 1) \in W$ and $u = (p - w) \in U$. We know that $(p - w) \in U$ because:

$$(p - w)(0) = p(0) - w(0) = a_0 - a_0 = 0$$

Now we see that:

$$u + w = (p - w) + w = p$$

That proves that $U + W = P$. Now we will show that $U \bigcap W = \{0\}$ which will prove that $U \oplus W = V$, as we have shown in the lecture.

$$\begin{aligned} U \cap W &= \{p(x) \in V : p(x) \in W \wedge p(0) = 0\} \\ &= \{ax^2 + ax + a : a \in \mathbb{R} \wedge p(0) = 0\} \\ &= \{ax^2 + ax + a : a \in \mathbb{R} \wedge a = 0\} \\ &= \{0\} \end{aligned}$$

Now we will find the projection on U parallel to W. We have shown that the only way to get any specific $p \in V$ is by adding the specific:

$$u_p + w_p = (p - a_0(x^2 + x + 1)) + a_0(x^2 + x + 1)$$

So the parallel projection will be $P : V \to V$:

$$\begin{aligned} P(p(x)) = P(a_n x^n + \cdots + a_0) = u_p &= (p - a_0(x^2 + x + 1)) \\ &= a_n x^n + \cdots + a_3 x^3 + (a_2 - a_0)x^2 + (a_1 - a_0)x \end{aligned}$$

1.2 $V = \mathbb{R}^4$ with

$$W = \mathbf{Sp}\{e_1 + e_4, e_2 + e_4\}, \quad U = \{e_1, e_2 + e_3\}$$

where $E = (e_1, ..., e_4)$ is the standard basis.

Consider the following matrix with the vectors from U and W:

$$\begin{pmatrix} — & e_1 + e_4 & — \\ — & e_2 + e_4 & — \\ — & e_1 & — \\ — & e_2 + e_3 & — \end{pmatrix}$$

By applying elementary row operations we get:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Which means as we know from linear algebra 1 that $U+W = V$ and from Grassman's identity(?) we know that:

$$\underbrace{\dim(W + U)}_{4} = \underbrace{\dim(W)}_{2} + \underbrace{\dim(U)}_{2} - \dim(U \cap W)$$
$$\Rightarrow \dim(U \cap W) = 0$$
$$\Rightarrow U \cap W = \{0\}$$

Which implies that $U \oplus W = V$. Now we will find the projection on U parallel to W. For this we will need to find the unique decomposition of any $v \in V$ to vectors $u \in U$ and $w \in W$. Where for $a, b, c, d, x_1, x_2, x_3, x_4 \in \mathbb{F}$:

$$w = a(e_1 + e_4) + b(e_2 + e_4) = \begin{pmatrix} a \\ b \\ 0 \\ a+b \end{pmatrix}$$

$$u = c(e_1) + d(e_2 + e_3) = \begin{pmatrix} c \\ d \\ d \\ 0 \end{pmatrix}$$

$$u + w = \begin{pmatrix} | & | & | & | \\ e_1 + e_4 & e_2 + e_4 & e_1 & e_2 + e_3 \\ | & | & | & | \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \\ d \\ a+b \end{pmatrix} = v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

So we get that $d = x_3$

$$\begin{pmatrix} a+c \\ b+x_3 \\ x_3 \\ a+b \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Now $b = x_2 - x_3$

$$\begin{pmatrix} a+c \\ x_2 \\ x_3 \\ a + x_2 - x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

So $a = x_4 - x_2 + x_3$ and we get:

$$\begin{pmatrix} x_4 - x_2 + x_3 + c \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

So $c = x_1 - x_4 + x_2 - x_3$. Finally we get that for any $v \in V$ such that:

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

We get:

$$\begin{aligned} v &= w + u = a(e_1 + e_4) + b(e_2 + e_4) + c(e_1) + d(e_2 + e_3) \\ &= (x_4 - x_2 + x_3)(e_1 + e_4) + (x_2 - x_3)(e_2 + e_4) + (x_1 - x_4 + x_2 - x_3)(e_1) + (x_3)(e_2 + e_3) \end{aligned}$$

Which means the projection on U parallel to W is $P\colon V \to V$

$$\forall \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{v} \in V\colon P(v) = (x_1 - x_4 + x_2 - x_3)(e_1) + (x_3)(e_2 + e_3) = \begin{pmatrix} x_1 - x_4 + x_2 - x_3 \\ x_3 \\ x_3 \\ 0 \end{pmatrix} = u$$

2 Prove/Disprove

2.1 The sum of projections is a projection

This is false. Let $P_1 = P_2 = \mathrm{Id}_n$ be our projections from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is clear these are projections since:

$$\mathrm{Id}_n^2 = \mathrm{Id}_n$$

But the transformation $P = P_1 + P_2$ is not a projection since:

$$P^2 = (P_1 + P_2)^2 = (2\mathrm{Id}_n)^2 = 4\mathrm{Id}_n \neq 2\mathrm{Id}_n = P$$

2.2 The composition of projections is a projection

This claim is false. Consider the following projections over $\mathbb{R}^2$:

$$P_1(x, y) = (x + y, 0) \quad \textit{and} \quad P_2(x, y) = (x, x)$$

It's easy to verify that these are indeed projections:

$$P_1^2(x, y) = P_1(x + y, 0) = (x + y, 0) = P_1(x, x)$$
$$P_2^2(x, y) = P_2(x, x) = (x, x) = P_2(x, y)$$

Yet if we consider the vector $(2, 1)$ we get:

$$(P_1 \circ P_2)(2, 1) = P_1(2, 2) = (4, 0)$$
$$(P_1 \circ P_2)^2(2, 1) = (P_1 \circ P_2)(4, 0) = P_1(4, 4) = (8, 0)$$

So:

$$(P_1 \circ P_2) \neq (P_1 \circ P_2)^2$$

Which means it's not a projection.

3 Let V be a finite-dimensional vector space, and let $P_1, ..., P_n \in$ End(V) be parallel projections. Denote $\forall i$: $R_i = \mathbf{Im} P_i$

3.1 Show that $\mathbf{tr} P_i = \dim R_i$

Since P_i is a parallel projection we know that $V = \text{Im}P_i \oplus \text{Ker}P_i$ Which means that $\text{Im}P_i \cap \text{Ker}P_i = \{0\}$. We know by a theorem we learned in class that exist:

$$B_r = \{b_1, ..., b_k\}$$

a basis for $\text{Im}P_i = R_i$. And:

$$B_k = \{r_{b+1}, ..., b_n\}$$

a basis for $\text{Ker}P_i$ such that the ordered union:

$$B = B_r \cup B_k = \{b_1, ..., b_k, b_{k+1}, ..., b_n\}$$

forms a basis for V. That means that the matrix representation of P_i by the basis B is:

$$\begin{pmatrix} | & | & | \\ [P_i(b_1)]_B & ... & [P_i(b_n)]_B \\ | & | & | \end{pmatrix}_{n\times n} = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}_{n\times n}$$

So $\text{tr}([P_i]_B) = k$. And since the trace of a transformation is just the trace of its representing matrix, as shown to be a well defined trait of transformations in linear algebra 1 we conclude that:

$$\text{tr}(P_i) = \text{tr}([P_i]_B) = k = \dim \text{Im}P_i = \dim R_i$$

3.2 Let $P_1 + \cdots + P_n = \mathrm{Id}$, show that $V = \bigoplus R_i$ and infer that $\forall i \neq j \colon P_iP_j = 0$

$\underline{V = \bigoplus R_i}$ - From 3.1 we know that:

$$\dim V = \mathrm{tr}(\mathrm{Id}) = \mathrm{tr}(P_1 + \cdots + P_n) = \mathrm{tr}(P_1) + \cdots + \mathrm{tr}(P_n) = \dim R_1 + \cdots + \dim R_n$$

Now we will show that $R_1 + \cdots R_n = V$. Let $v \in V$:

$$v = \mathrm{Id}(v) = (P_1 + \cdots + P_n)(v) = P_1(v) + \cdots + P_n(v)$$

Since $\forall i \colon P_i(v) \in R_i$ we get that for any $v \in V$ exist $P_1(v) \in R_1, ..., P_n(v) \in R_n$ such that $v = P_1(v) + \cdots + P_n(v)$. So now we know that

$$\begin{gathered} V = R_1 + \cdots + R_n \\ \dim V = \dim R_1 + \cdots + \dim R_n \end{gathered}$$

Denote B_{R_i} the ordered basis for R_i for any i, we get:

$$\begin{aligned} V &= \mathrm{Sp}\left\{\bigcup_i B_{R_i}\right\} && \Rightarrow \dim V \leq \left|\bigcup_i B_{R_i}\right| \\ \dim V &= \sum_i |B_{R_i}| \geq \left|\bigcup_i B_{R_i}\right| && \Rightarrow \left|\bigcup_i B_{R_i}\right| \leq \dim V \\ &\Rightarrow \left|\bigcup_i B_{R_i}\right| = \dim V \end{aligned}$$

So from:

$$\mathrm{Sp}\left\{\bigcup_i B_{R_i}\right\} = V \wedge \left|\bigcup_i B_{R_i}\right| = \dim V$$

We get that the ordered union of the ordered bases B_{R_i} form a basis of V which is equivalent as we've shown in class to saying that $V = \bigoplus R_i$

$\underline{\forall i \neq j \colon P_iP_j = 0}$ - Let $i \neq j$. Now suppose that $P_iP_j \neq 0$. that means that exists a $0 \neq v \in V$ such that $P_iP_j(v) \neq 0$, which means that $P_j(v) \notin \mathrm{Ker}P_i$. Since P_i is a projection we know that $\mathrm{Im}P_i \oplus \mathrm{Ker}P_i = V$ which means that $P_j(v) \in R_i$, but also by definition $P_j(v) \in R_j$, so:

$$\begin{gathered} \underbrace{0}_{R_1} + \cdots + \underbrace{P_j(v)}_{R_j} + \cdots + \underbrace{0}_{R_n} = P_j(v) \\ \underbrace{0}_{R_1} + \cdots + \underbrace{P_j(v)}_{R_i} + \cdots + \underbrace{0}_{R_n} = P_j(v) \end{gathered}$$

but that's a contradiction to $V = \bigoplus R_i$. So $\forall i \neq j \colon P_iP_j = 0$

4 Let V be a vector space, $T, S \in \mathbf{End}(V)$, and let S be diagonalizable. Prove:

$$\textbf{the eigenspaces of } S \textbf{ are } T\textbf{-invariant} \iff TS = ST$$

$(\Leftarrow)$
For any eignenvalue λ of S:

$$\begin{aligned} \operatorname{Ker}(S - \lambda I) &= \{s \in V | S(s) = \lambda s\} \\ \Rightarrow T(\operatorname{Ker}(S - \lambda I)) &= \{T(s) | S(s) = \lambda s\} \\ &= \{s \in V | \exists w \colon T(w) = s \wedge S(w) = \lambda w\} \end{aligned}$$

Since for $s \in T(\operatorname{Ker}(S - \lambda I))$:

$$S(s) = S(T(w)) \underset{TS=ST}{=} T(S(w)) = T(\lambda w) = \lambda T(w) = \lambda s$$

We get that $T(\operatorname{Ker}(S - \lambda I)) \subseteq \operatorname{Ker}(S - \lambda I)$ which means that all the eigenspaces of S are T-invariant.

$(\Rightarrow)$
We know that S is diagnolizable so exist a base to V

$$B = (b_1, \dots, b_n)$$

such that $[S]_B$ is a diagnonal matrix. We will show that for any $b \in B$ that $TS(b) = ST(b)$. Let $b \in B$ be an eigenvector of an eigenspace with eigenvalue λ:

$$TS(b) = T(\lambda b) = \lambda(T(b))$$

Now since $b \in V_\lambda^S$[1] is T-invariant by the assumption:

$$\lambda(T(b)) = S(T(b)) = ST(b)$$

We have shown that for any vector from the base B of V

$$TS(b) = ST(b)$$

Since B spans V and S, T are linear, we know that for any $v \in V$

$$TS(v) = ST(v)$$

Which is what we wanted to prove.

[1] λ-eigenspace of S under V not sure if this is the correct notation.

5 Let V be a vector space over a field $\mathbb{F}$, with $\dim V = n$. Let $T\colon V \to V$ such that any $(n-1)$-dimentional vector subspace of V is T-invariant. Prove that V is a scalar transformation.

Let $v_1 \in V$ be a vector such that $T(v_1) = v_2$ and v_2 isn't a scalar multiply of v_1. That means they are linearly independent which implies we can complete $\{v_1, v_2\}$ to a basis of V as such:

$$B = (v_1, v_2, \dots, v_n)$$

Since $\mathrm{Sp}\{v_1, v_3, \dots, v_n\}$ is a $n-1$-dimentional subspace of V, it is T-invariant, which means that:

$$T(v_1) = v_2 \in \mathrm{Sp}(v_1, v_3, \dots, v_n)$$

But that's a contradiction since if v_2 were in $\mathrm{Sp}(v_1, v_3, \dots, v_n)$ then B wouldn't be linearly independent even thought it's a basis of V. That means that for any $v \in V$ then $T(v)$ is a scalar multiple of v. Now consider the standard basis $E = (e_1, \dots, e_n)$ we know that:

$$\begin{gathered} T(e_1) = \lambda_1 e_1 \\ T(e_2) = \lambda_2 e_2 \\ \dots \\ T(e_n) = \lambda_n e_n \end{gathered}$$

We also know that $T(e_1 + \dots + e_n) = \mu \sum_{i=1}^{n} e_i$ so:

$$T(e_1 + e_2 + \dots + e_n) = T(e_1) + \dots + T(e_n) = \sum_{i=1}^{n} \lambda_i e_i = \sum_{i=1}^{n} \mu e_i$$

Since $e_1, \dots, e_n$ are linearly independent that means that:

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = \mu$$

Finally since E is a basis, for any $v \in V$ we get that $T(v) = \mu v$. In other words that T is a scalar operator.

6 Let $T, S, Q \in \textbf{End}(v)$ such that $T = Q^{-1}SQ$. Show that $U \subseteq V$ is T-invariant $\iff$ $Q(U)$ is S-invariant

<u>($\Rightarrow$)</u> Suppose that $U \subseteq V$ is T-invariant. That means that:

$$T(U) \subseteq U$$

Now:

$$S(Q(U)) = SQ(U)$$

But we know that $T = Q^{-1}SQ \Rightarrow QT = SQ$ so:

$$S(Q(U)) = QT(U) = Q(T(U))$$

We know that $T(U) \subseteq U$ so:

$$\begin{aligned} S(Q(U)) &= Q(T(U)) \subseteq Q(U) \\ &\Rightarrow S(Q(U)) \subseteq Q(U) \end{aligned}$$

In other words - $Q(U)$ is S-invariant.

<u>($\Leftarrow$)</u> Suppose that $Q(U)$ is S-invariant:

$$(*) \quad S(Q(U)) \subseteq Q(U)$$

Now:

$$T(U) = Q^{-1}SQ(U) = Q^{-1}(S(Q(U))) \underset{*}{\subseteq} Q^{-1}(Q(U)) = U$$

So:

$$T(U) \subseteq U$$

In other words U is T-invariant.

7 The one it won't be fun to typeset.

7.1 Find the Jordan normal form, a jordan basis, and the minimal polynomial of the following matrix:

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix} \in M_n(\mathbb{C})$$

First we're gonna find the characteristic polynomial of this matrix. We notice that the matrix is a blockwise triangular matrix:

$$A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} B_{2\times 2} & 0 \\ * & C_{2\times 2} \end{pmatrix}$$

So we can solve it like we did in linear algebra 1:

$$\begin{aligned} p_A(\lambda) = p_B(\lambda)p_C(\lambda) &= ((-1-\lambda)(2-\lambda)+2)((2-\lambda)(0-\lambda)+2) \\ &= (\lambda^2-\lambda)(\lambda^2-2\lambda+1) = (\lambda(\lambda-1))(\lambda^2-2\lambda+1) = \lambda(\lambda-1)^3 \end{aligned}$$

But we can also notice that the sum of columns of these blocks is 1 so 1 is an eigenvalue of both of them, and since the sum of the eigenvalues of a matrix is equal to its trace we can find the other eigen value. We see that $\lambda = 0$ is an eigenvalue of algebraic multiplicity 1 and $\lambda = 1$ is an eigenvalue of algebraic multiplicity 3 so the Jordan normal form will have a Jordan block $J_1(0)$ and some Jordan blocks of total size 3. Now we will find $(A - I)$ to find out how many Jordan blocks are there:

$$A - I = \begin{pmatrix} -2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 \\ -2 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$
$$\to \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So we see that $(A - I) = 2$ so there are two Jordan blocks of $\lambda = 1$. That that the Jordan normal form of A must be of the form $J_2(1) \oplus J_1(1) \oplus J_1(0)$. So we want to find Jordan chains of the form:

$$\begin{array}{cc|c} \lambda = 1 & & \lambda = 2 \\ \hline v_2 & & \\ \downarrow & & \\ v_1 & v_3 & v_4 \end{array}$$

We shall continue with some more calculation to find the generalized eigenspaces of A.

$$\ker(A - I) = \ker\left(\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \mathrm{Sp}\left\{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$

$$(A-I)^2 = \begin{pmatrix} -2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 1 \\ -2 & -1 & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\ker(A-I)^2 = \ker\left(\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \mathrm{Sp}\left\{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \widetilde{V_1}$$

$$\ker(A) = \ker\left(\begin{pmatrix} -1 & -1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 2 & 2 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\right) = \ker\left(\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}\right)$$

$$= \mathrm{Sp}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\} = \widetilde{V_0}$$

To test our calculations against the generalized eigenspace decomposition theorem we see that indeed:

$$V = \mathrm{Sp}\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\right\} \oplus \mathrm{Sp}\left\{\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \widetilde{V_0} \oplus \widetilde{V_1}$$

To find v_2 we would need to find a vector in $\ker(A-I)^2$ that is not in $\ker(A-I)$ for example:

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Then:

$$v_1 = (A-I)v_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

Now fo find v_3 we will just find a vector that will complement $\mathrm{Sp}\{v_1, v_2\}$ to $\widetilde{V_1}$ for example:

$$v_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$$

And for the last vector we can just choose any vector that is in $\widetilde{V_1}$ for example:

$$v_4 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

So we found all of our Jordan chains and also the Jordan basis for A:

$$B_J = \{\begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\}$$

Now we will find the minimal polynomial. To find the minimal polynomial we will see that it is excactly the product of the the polynomials of the form $p(x) = (x - \lambda)^r$ for each distinct eigenvalue λ of A and r being the size of the longest Jordan chain of its respective λ, since each vector in V can be represented as a linear combination of the Jordan base, and for any polynome that doesn't include one of these multiples of $(x - \lambda)$ we can take the top of the chain of this lambda and see that it will not be a root of the supposed polynome. Therefore:

$$m_A(x) = (x - 1)^2(x - 2)$$

8 The one with the polynomial operator

8.1 Let $T\colon \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ be the operator

$$T(ax^3 + bx^2 + cx + d) = 2ax^3 + (2b + 3c + d)x^2 + (2c + 3d)x + 2d$$

Does exist a basis to $\mathbb{R}_3[x]$ such that:

$$[T^2 - 4T + 4I]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We notice that:

$$[T^2 - 4T + 4I]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = J_3(0) \oplus J_1(0)$$

And since we know that the Jordan normal form of a transformation is unqiue up to order, it suffices to show that the Jordan normal form of $T^2 - 4T + 4I$ is the same or different than $J_3(0) \oplus J_1(0)$. Making some calculations we get that represented by the standard basis:

$$[T^2 - 4T + 4I]_E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \underbrace{=}_{\text{denotion}} B$$

Which means that the characteristic polynomial of it is:

$$p_B(x) = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 9 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda(-\lambda^3 + 9 * 0) = \lambda^4$$

So the only eigenvalue of $T^2 - 4T + 4I$ is 0, of algebraic multiplicity 4. We know by a theorem we have proved in class that there must be at least:

$$\dim\ker(T^2 - 4T + 4I) = 3$$

Jordan blocks in $T^2 - 4T + 4I$'s Jordan normal form. This means that it can't have the Jordan normal form of $J_3(0) \oplus J_1(0)$, so we have shown that there does not exist a basis B to V such that

$$[T^2 - 4T + 4I]_B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

9 The one with the ranks

9.1 Let $A \in M_7(\mathbb{R})$ such that:

$$rk(A-I)^2 > rk(A-I)^3 = rk(A-I)^4$$

and $rk(A) = 3$. Calculate the Jordan normal form of A.

We know that $rk(A) = \dim \operatorname{Im}(A) = 3$ and since we also know that:

$$\underbrace{\dim \operatorname{Im}(A)}_{3} + \dim \ker(A) = \underbrace{\dim \mathbb{R}^7}_{7}$$

We know that $\dim \ker(A) = 4$ which tells us that there are 4 Jordan blocks in the Jordan normal form of A with eigenvalue 0. From similar considerations we also see that:

$$\dim \ker(A-I)^3 = 7 - rk(A-I)^3 = 7 - rk(A-I)^4 = \dim \ker(A-I)^4$$

So we know that there are:

$$\dim \ker(A-I)^4 - \dim \ker(A-I)^3 = 0$$

Jordan blocks with eigenvalue 1 of size at least 4. Also:

$$\dim \ker(A-I)^2 = 7 - rk(A-I)^2 < 7 - rk(A-I)^3 = \dim \ker(A-I)^3$$

So there is at least 1 Jordan block of size 3 in the Jordan normal form of A. Since as we have shown, there must be 4 Jordan blocks in the Jordan normal form with eigenvalue 0, and the sum of the order of the Jordan blocks must be equal to 7 the only option for the Jordan normal form of A is:

$$J_3(1) \oplus J_1(0) \oplus J_1(0) \oplus J_1(0) \oplus J_1(0)$$

10 The one with the inverses

10.1 Let $\mathbb{F}$ be a field and $0 \neq \lambda \in \mathbb{F}$. Find the Jordan normal form of $J_n(\lambda)^{-1}$. No need to explicitly compute the inverse.

We can write the Jordan block $J_n(\lambda)$ as the sum of a scalar and a nilpotent matrix like so:

$$J_n(\lambda) = \lambda I + J_n(0)$$

Now we notice that since $\lambda \neq 0$ we can multiply both sided by $\lambda^{-1} I$:

$$\lambda^{-1} I J_n(\lambda) = \lambda^{-1} I(\lambda I + J_n(0)) = I + \lambda^{-1} J_n(0)$$

And that:

$$(I - \lambda^{-1} J_n(0))(I + \lambda^{-1} J_n(0)) = I - \lambda^{-2} J_n^2(0)$$

Now since:

$$(I + \lambda^{-2} J_n^2(0))(I - \lambda^{-2} J_n^2(0)) = I - \lambda^{-4} J_n^4(0)$$

We can keep going like:

$$(I + \lambda^{-4} J_n^4(0))(I - \lambda^{-4} J_n^4(0)) = I - \lambda^{-8} J_n^8(0)$$

So we see know that:

$$\left(\prod_{i=1}^{k} (I + \lambda^{-2^k} J_n^{2k}(0)) \right) (I - \lambda^{-1} J_n(0))(\lambda^{-1} I) J_n(\lambda) = I - \lambda^{-2^{k+1}} J_n^{2^{k+1}}(0)$$

Since $J_n(0)$ is nilpotent of order $n-1$ we can choose $k \in \mathbb{N}$ such that $2^{k+1} > n$ and then:

$$\left(\prod_{i=1}^{k} (I + \lambda^{-2^k} J_n^{2k}(0)) \right) (I - \lambda^{-1} J_n(0))(\lambda^{-1} I) J_n(\lambda) = I - \lambda^{-2^{k+1}} J_n^{2^{k+1}}(0) = I$$

From linear algebra 1 we know that a if $AB = I$ then $BA = I$ which means that we found the inverse of $J_n(\lambda)$:

$$J_n(\lambda)^{-1} = \left(\prod_{i=1}^{k} (I + \lambda^{-2^k} J_n^{2k}(0)) \right) (I - \lambda^{-1} J_n(0))(\lambda^{-1} I)$$

10.2 Find a sufficient and necessary condition that a real matrix has to meet to be similar to its inverse.

11 The one with the 9s

11.1 Prove that exists a matrix $A \in M_n(\mathbb{R})$ that satisfies:

$$A^9 + A^{99} = \begin{pmatrix} 2 & 99 & 999 \\ 0 & 2 & -9 \\ 0 & 0 & 2 \end{pmatrix}$$

There's no need to find one explicitly.

Since the matrix we get by the calculation is of order 3 we know that A is also of order 3. 1

Consider $A = J_3(\lambda)$, by a theorem we proved in class we can see that for the polynome $f(x) = x^9 + x^{99}$:

$$f(A) = \begin{pmatrix} f(\lambda) & f'(\lambda) & \frac{1}{2}f''(\lambda) \\ 0 & f(\lambda) & f'(\lambda) \\ 0 & 0 & f(\lambda) \end{pmatrix}$$

12 The one with high powers

12.1 Find all the matrices $A \in M_4(\mathbb{C})$ that satisfy $A^4 - 2A^2 + 1 = 0$ up to similarity.

13 The one with invariant subspaces

13.1 Compute the invariant subspaces of a jordan block $J_n(\lambda)$. Use what we saw in the rehearsal about the invariant subspaces of $J_n(0)$.

13.2 **Let $T \in End(V)$ where V is a complex vector space of finite dimension. Show that there is a finite amount of T-invariant subspaces iff $p_T(x) = m_T(X)$**

14 The one with the Cauchy-Schwartz inequality

14.1 Show that for all positive $x_1, \dots, x_n \in \mathbb{R}$:

$$n^2 \leq (x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

Let $x_1, \dots, x_n$ be positive real numbers. Recall that the Cauchy-Shwartz inequality states that for any v, u in an inner product space, and specifically for $(\mathbb{R}^n, \langle , \rangle_{\text{std}})$ we get:

$$|\langle v, u\rangle|^2 \leq \langle v, v\rangle\langle u, u\rangle$$

Since $x_1, \dots, x_n$ are positive we can take their roots and then for:

$$v = (\sqrt{x_1}, \dots, \sqrt{x_n}) \quad \text{and} \quad u = (\frac{1}{\sqrt{x_1}}, \dots, \frac{1}{\sqrt{x_n}})$$

We get:

$$|\langle v, u\rangle|^2 = |\langle(\sqrt{x_1}, \dots, \sqrt{x_n}), (\frac{1}{\sqrt{x_1}}, \dots, \frac{1}{\sqrt{x_n}})\rangle|^2 = |n|^2 = n^2$$

And:

$$\langle v, v\rangle\langle u, u\rangle = (x_1 + \cdots + x_n)(\frac{1}{x_1} + \cdots + \frac{1}{x_n})$$

Now substituting we get:

$$n^2 \leq (x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)$$

Which is what we wanted to prove.

15 The one with the integral

Let $V = \mathbb{R}_2[x]$ and let:

$$\langle(p(x), q(x))\rangle_1 = \int_0^1 p(x)q(x)\,dx$$
$$\langle(p(x), q(x))\rangle_2 = \sum_{x\in\{-1,0,1\}} p(x)q(x)$$

Two inner products on V, and let:

$$W = \{p(x) \in V | p(x) = p(-x)\}$$

15.1 Find a basis for W and complete it to a basis for V.

We know that $W \neq V$ and $W \neq 0$ so since $x^2, 1 \in W$ and are linearly independant we get that $\dim W = 2$ and that

$$B_W = \{x^2, 1\}$$

is a basis for W. We can complete it to a basis for V as such:

$$B_V = \{x^2, 1, x\}$$

15.2 Apply the Gram-Schmidt process on V relative to each of the inner products, find $W^{\perp}$ and the orthogonal projection P_W on W.

According to $\langle , \rangle_1$ we get:

$$u_1' = v_1 = x^2$$
$$u_2' = v_2 - \sum_{i=1}^{1} \frac{\langle v_2, u_i' \rangle}{\langle u_i', u_i' \rangle} u_i' = v_2 - \frac{\langle v_2, u_1' \rangle}{\langle u_1', u_1' \rangle} u_1' = 1 - \frac{\langle 1, x^2 \rangle}{\langle x^2, x^2 \rangle} x^2 = 1 - \frac{\frac{1}{3}}{\frac{1}{5}} x^2 = 1 - \frac{5}{3} x^2$$
$$u_3' = v_3 - \sum_{i=1}^{2} \frac{\langle v_3, u_i' \rangle}{\langle u_i', u_i' \rangle} u_i' = x - \frac{\langle x, x^2 \rangle}{\langle x^2, x^2 \rangle} x^2 - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x - \frac{4}{3} x^2 - \frac{1}{2}$$

Now to normalize the vectors:

$$u_1 = \frac{u_1'}{\|u_1'\|} = \frac{x^2}{\sqrt{\langle x^2, x^2 \rangle}} = 2x^2$$
$$u_2 = \frac{u_2'}{\|u_2'\|} = \frac{1 - \frac{5}{3}x^2}{\sqrt{\langle 1 - \frac{5}{3}x^2, 1 - \frac{5}{3}x^2 \rangle}} = \frac{3}{2} - \frac{5}{2} x^2$$
$$u_3 = \frac{u_3'}{\|u_3'\|} = \frac{x - \frac{4}{3}x^2 - \frac{1}{2}}{\sqrt{\langle x - \frac{4}{3}x^2 - \frac{1}{2}, x - \frac{4}{3}x^2 - \frac{1}{2} \rangle}} = \frac{30}{\sqrt{195}} x - \frac{40}{\sqrt{195}} x^2 - \frac{15}{\sqrt{195}}$$

Notice that when we applied Gram-Schmidt we first found an orthonormal basis for W, and since we know that $V = W \oplus W^{\perp}$ we get:

$$W^{\perp} = \operatorname{Sp}\{u_3\} = \operatorname{Sp}\left\{ \frac{30}{\sqrt{195}} x - \frac{40}{\sqrt{195}} x^2 - \frac{15}{\sqrt{195}} \right\}$$

And as we know from the lectures for all $v \in V = \mathbb{R}_2[x]$ we get:

$$\begin{aligned} P_W(v) = \sum_{i=1}^{2} \langle v(x), u_i \rangle u_i &= \langle v(x), 2x^2 \rangle 2x^2 + \langle v(x), \frac{3}{2} - \frac{5}{2} x^2 \rangle \left(\frac{3}{2} - \frac{5}{2} x^2 \right) \\ &= \left(\int_0^1 v(x) 2x^2 \, dx \right) 2x^2 + \left(\int_0^1 v(x) \left(\frac{3}{2} - \frac{5}{2} x^2 \right) dx \right) \left(\frac{3}{2} - \frac{5}{2} x^2 \right) \end{aligned}$$

According to $\langle , \rangle_2$ we get:

$$
\begin{aligned}
&u_1' = v_1 = x^2 \\
&u_2' = v_2 - \sum_{i=1}^{1} \frac{\langle v_2, u_i' \rangle}{\langle u_i', u_i' \rangle} u_i' = v_2 - \frac{\langle v_2, u_1' \rangle}{\langle u_1', u_1' \rangle} u_1' = 1 - \frac{\langle 1, x^2 \rangle}{\langle x^2, x^2 \rangle} x^2 = 1 - \frac{2}{2} x^2 = 1 - x^2 \\
&u_3' = v_3 - \sum_{i=1}^{2} \frac{\langle v_3, u_i' \rangle}{\langle u_i', u_i' \rangle} u_i' = x - \frac{\langle x, x^2 \rangle}{\langle x^2, x^2 \rangle} x^2 - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} 1 = x
\end{aligned}
$$

Now to normalize the vectors:

$$
\begin{aligned}
u_1 &= \frac{u_1'}{\|u_1'\|} = \frac{x^2}{\sqrt{\langle x^2, x^2 \rangle}} = \frac{x^2}{\sqrt{2}} \\
u_2 &= \frac{u_2'}{\|u_2'\|} = \frac{1 - x^2}{\sqrt{\langle 1 - x^2, 1 - x^2 \rangle}} = 1 - x^2 \\
u_3 &= \frac{u_3'}{\|u_3'\|} = \frac{x}{\sqrt{\langle x, x \rangle}} = \frac{x}{\sqrt{2}}
\end{aligned}
$$

Notice that when we applied Gram-Schmidt we first found an orthonormal basis for W, and since we know that $V = W \oplus W^{\perp}$ we get:

$$
W^{\perp} = \mathrm{Sp}\{u_3\} = \mathrm{Sp}\left\{ \frac{x}{\sqrt{2}} \right\}
$$

And as we know from the lectures for all $v \in V = \mathbb{R}_2[x]$ we get:

$$
\begin{aligned}
P_W(v) = \sum_{i=1}^{2} \langle v, u_i \rangle u_i &= \langle v(x), \frac{x^2}{\sqrt{2}} \rangle \frac{x^2}{\sqrt{2}} + \langle v(x), 1 - x^2 \rangle \left(1 - x^2\right) \\
&= \left(\sum_{x \in \{-1,0,1\}} v(x) \left(\frac{x^2}{\sqrt{2}} \right) \right) \left(\frac{x^2}{\sqrt{2}} \right) + \left(\sum_{x \in \{-1,0,1\}} v(x)(1 - x^2) \right) \left(1 - x^2\right) \\
&= (v(1) + v(-1)) \left(\frac{x^2}{2} \right) + v(0) \left(1 - x^2\right)
\end{aligned}
$$

15.3 Find the distance of $f(x) = x + 1$ from W according to each of the inner products.

We know that the distance of $f(x) = x+1$ from W is the distance between $x+1$ and $P_W(x+1)$ which is the point "closest" to $x+1$ on W. So first we shall calculate $P_W(x+1)$ according to each of the inner product spaces:

$$\begin{aligned}
P_W(x+1) &= (2+0)\left(\frac{x^2}{2}\right) + 1\left(1-x^2\right) = 1 \\
P_W(x+1) &= \left(\int_0^1 (x+1)2x^2\,dx\right)2x^2 + \left(\int_0^1 (x+1)\left(\frac{3}{2}-\frac{5}{2}x^2\right)\,dx\right)\left(\frac{3}{2}-\frac{5}{2}x^2\right) \\
&= \left(\int_0^1 2x^3+2x^2\,dx\right)2x^2 + \left(\int_0^1 \frac{3}{2}-\frac{5}{2}x^2+\frac{3}{2}x-\frac{5}{2}x^3\,dx\right)\left(\frac{3}{2}-\frac{5}{2}x^2\right) \\
&= \frac{7}{3}x^2 + \frac{19(3-5x^2)}{48} = \frac{112x^2}{48} + \frac{57-95x^2}{48} = \frac{17x^2+57}{48}
\end{aligned}$$

So now according to $\langle,\rangle_1$ we get that the distance is:

$$\sqrt{\langle x+1, 1\rangle} = \sqrt{\int_0^1 x+1\,dx} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

And now according to $\langle,\rangle_2$ we get that the distance is:

$$\sqrt{\langle x+1, \frac{17x^2+57}{48}\rangle} = \sqrt{\sum_{x=-1,0,1} \frac{17x^2+57(x+1)}{48}\,dx} = \sqrt{\frac{17+57+131}{48}} = \sqrt{\frac{205}{48}} = \frac{\sqrt{615}}{12}$$

16 The one with the contraction

Let V be a finite dimension inner product space and let $P \in \mathrm{End}(V)$ be a contraction - that is $\forall v \in V(\|Pv\| \leq \|v\|)$.

16.1 Show that P is the orthogonal projection on its own image.

We will first show that $V = \operatorname{im} P \oplus \ker P$. Since P is a projection we must have $P(v) = P^2(v)$ which implies $P(P(v) - v) = 0$ so $P(v) - v = \epsilon \in \ker P$ and then $v = P(v) + (-\epsilon)$ which shows that $V = \operatorname{im} P + \ker P$. Now let $v \in \operatorname{im} P \cap \ker P$. We get that for some $u \in V$ that $P(u) = v$ and $P^2(u) = P(v) = 0$ since $v \in \ker P$. But since $P^2(u) = P(u)$ we get $v = 0$. This shows $V = \operatorname{im} P \oplus \ker P$. We also know that $V = \operatorname{im} P \oplus \operatorname{im} P^{\perp}$. This shows that $\dim \operatorname{im} P^{\perp} = \dim \ker P$. Now we will show that $\operatorname{im} P^{\perp} \subseteq \ker P$. Let $v \in \operatorname{im} P^{\perp}$ We know that $P(v) \in \operatorname{im} P$ so:

$$\langle P(v), v \rangle = 0$$

This implies that:

$$0 = \langle P(v), v \rangle = \frac{1}{4}\left(\|P(v) + v\|^2 - \|P(v) - v\|^2 + i\|P(v) - v\|^2 - i\|P(v) + v\|^2\right)$$

This implies that:

$$\|P(v) + v\| - \|P(v) - v\| = 0$$

So using the reverse triangle identities we get:

$$0 \leq \|P(v)\| - \|v\| - \|P(v) - v\| \leq \|P(v) + v\| - \|P(v) - v\| = 0$$

So:

$$\|P(v)\| - \|v\| = \|P(v) - v\|$$

So from what we know $\|P(v)\| - \|v\|$ is a non-negative number and $\|P(v)\| \leq \|v\|$ which implies $\|P(v)\| - \|v\| = 0$ which gives:

$$\|P(v) - v\| = 0 \Rightarrow P(v) - v = 0 \Rightarrow P(v) = v$$

This shows that $v \in \operatorname{im} P$, and since $v \in \operatorname{im} P^{\perp}$ we know $v = 0$. But we assumed that $v \notin \ker P$ so this can't be the case, and we get a contradiction. Which means that $\operatorname{im} P^{\perp} \subseteq \ker P$ and we know $\dim \operatorname{im} P^{\perp} = \dim \ker P$ so $\operatorname{im} P^{\perp} = \ker P$. so P is an orthogonal projection on its own image.

17 The one with the weird inequality

Let $V = \mathbb{C}_3[x]$ with the inner product $\langle p(x), q(x)\rangle = \sum_{x=0}^{x=3} p(x)\overline{q(x)}$.

17.1 Find the minimal positive constant C such that for all $p \in V$:

$$\|p(i)\| \leq C\sqrt{\sum_{x=0}^{3} \|p(x)\|^2}$$

Notice that the following $\varphi\colon V \to \mathbb{C}$:

$$\varphi(p(x)) = p(i)$$

is a functional since for $\alpha \in \mathbb{C}$ and $p, q \in V$:

$$\varphi(\alpha p + q) = (\alpha p + q)(i) = \alpha p(i) + q(i) = \alpha\varphi(p) + \varphi(q)$$

Using riesz representation theorem we get that exists w such that:

$$\varphi(p) = p(i) = \langle p, w\rangle$$

Denote $w = a + bx + cx^2 + dx^3$, We see that for the basis $B = \{1, x, x^2, x^3\}$:

$$\begin{aligned}
1 = \varphi(1) = \langle 1, w\rangle &= \overline{w(0)} + \overline{w(1)} + \overline{w(2)} + \overline{w(3)} = w(0) + w(1) + w(2) + w(3) = 1 \\
i = \varphi(x) = \langle x, w\rangle &= 0\overline{w(0)} + 1\overline{w(1)} + 2\overline{w(2)} + 3\overline{w(3)} \Rightarrow w(1) + 2w(2) + 3w(3) = -i \\
-1 = \varphi(x^2) = \langle x^2, w\rangle &= 0\overline{w(0)} + 1\overline{w(1)} + 4\overline{w(2)} + 9\overline{w(3)} = w(1 + 4w(2) + 9w(3)) = -1 \\
-i = \varphi(x^3) = \langle x^3, w\rangle &= 0\overline{w(0)} + 1\overline{w(1)} + 8\overline{w(2)} + 27\overline{w(3)} \Rightarrow w(1) + 8w(2) + 27w(3) = i
\end{aligned}$$

Solving this system of equations gives:

$$(w(0), w(1), w(2), w(3)) = \left(\frac{5}{3}i, \frac{5-5i}{2}, -2+i, \frac{1}{2} - \frac{1}{6}i\right)$$

And now we can solve for $p(i)$ for any $p \in V$. By Cauchy-Schwartz we get:

$$\|p(i)\| = |\langle p, w\rangle| \leq \|p(x)\|\|w(x)\| = \sqrt{\sum_{x=0}^{3}\|p(x)\|^2}\sqrt{\sum_{x=0}^{3}\|w(x)\|^2}$$

And we see that:

$$\sqrt{\sum_{x=0}^{3}\|w(x)\|^2} = \frac{\sqrt{185}}{3}$$

Since we know that the CS inequality can also be an equality we get that this is the minimal constant such that the inequality is satisfied and then:

$$\boxed{C = \frac{\sqrt{185}}{3}}$$

18 The one with the invariance

Let V be a finite dimension inner product space and $T \in \text{End}(V)$.

18.1 Show that $U \subseteq V$ is T-invariant iff $U^\perp$ is T^*-invariant

<u>U is T-invariant $\Rightarrow U^\perp$ is T^*-invariant:</u>
Since U is T-invariant we know that:

$$T(U) \subseteq U$$

Now suppose that $U^\perp$ is not T^*-invariant, that means that exists $u \in U^\perp$ such that $T^*(u) \notin U^\perp$, which means that:

$$\langle v, T^*(u) \rangle \neq 0$$

For some $v \in U$. This implies:

$$\langle T(v), u \rangle \neq 0$$

But since U is T-invariant we know that $T(v) \in U$, which implies that $u \notin U^\perp$ - that means that out assumption must be false so $U^\perp$ is T^*-invariant.
<u>U is T-invariant $\Leftarrow U^\perp$ is T^*-invariant:</u>
Since $U^\perp$ is T^*-invariant we know that:

$$T^*(U^\perp) \subseteq U^\perp$$

Now suppose that U is not T-invariant, that means that exists $u \in U$ such that $T(u) \notin U$, which means that:

$$\langle T(u), v \rangle \neq 0$$

For some $v \in U^\perp$. This implies:

$$\langle u, T^*(v) \rangle \neq 0$$

But since $U^\perp$ is T^*-invariant we know that $T^*(v) \in U^\perp$, which implies that $u \notin U$ - that means that out assumption must be false so U is T-invariant.

19 The one with $T*$

In the following sections find T^*

19.1 Let $(V, \langle , \rangle)$ be a finite dimension inner product space. Let $\alpha, \beta \in V$ and define $T = T_{\alpha,\beta} \in \mathrm{End}(V)$ as such:

$$T_{\alpha,\beta}(v) = \langle v, \alpha \rangle \beta$$

We see that for T defined as such:

$$\langle T(v), u \rangle = \langle \langle v, \alpha \rangle \beta, u \rangle = \langle v, \alpha \rangle \langle \beta, u \rangle = \langle v, \alpha \overline{\langle \beta, u \rangle} \rangle = \langle v, \alpha \langle u, \beta \rangle \rangle$$

And since we know that:

$$\langle T(v), u \rangle = \langle v, T^*(u) \rangle = \langle v, \alpha \langle u, \beta \rangle \rangle$$

We get that:

$$T^*(u) = \langle u, \beta \rangle \alpha$$

19.2 Let $V = (\mathrm{Mat}_n(\mathbb{F}), \langle , \rangle_{\mathrm{std}})$. Let $Q \in \mathrm{Mat}_n(\mathbb{F})$ be invertible and define $T = T_Q \in \mathrm{End}(V)$ as such:

$$T_Q(A) = QAQ^{-1}$$

We see that from properties of trace:

$$\begin{aligned}\langle T(A), B\rangle &= \langle QAQ^{-1}, B\rangle = \mathrm{tr}(QAQ^{-1}B^t) = \mathrm{tr}(B^tQAQ^{-1}) \\ &= \mathrm{tr}(Q^{-1}B^tQA) = \mathrm{tr}(AQ^{-1}B^tQ) = \langle A, (Q^{-1}B^tQ)^t\rangle\end{aligned}$$

And since we know that:

$$\langle T(A), B\rangle = \langle A, T^*(B)\rangle = \langle A, (Q^{-1}B^tQ)^t\rangle$$

We get that:

$$T^*(B) = (Q^{-1}B^tQ)^t = Q^tB(Q^{-1})^t$$

19.3 Let $Tv = J_n(\lambda)v$ for $V = \mathbb{F}_n$ with $\langle , \rangle_{\text{std}}$

We see that for T defined as such:

$$\langle T(v), u\rangle = \langle J_n(\lambda)v, u\rangle = (J_n(\lambda)v)^t u = v^t J_n(\lambda)^t u = \langle v, J_n(\lambda)^t u\rangle$$

And since we know that:

$$\langle T(v), u\rangle = \langle v, T^*(u)\rangle = \langle v, J_n(\lambda)^t u\rangle$$

We get that:

$$T^*(u) = J_n(\lambda)^t u$$

20 The one with the adjoint operator

Let $a \in \mathbb{C}, |a| \neq 1$ and let V be a finite dimension inner product space, $T \in \text{End}(V)$

20.1 Show that if $T = aT^*$ then $T = 0$

We first see that T is normal since:

$$TT* = aT^*T^* = T^*aT^* = T^*T$$

This means that exists an orthonormal basis of eigenvectors of T which we shall denote $B = (v_1, \ldots, v_n)$ such that:

$$[T]_B = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

Where $\lambda_1, \ldots, \lambda_n$ denote the corresponding eigenvalues. We see that for all $1 \leq i \leq n$ that:

$$T(v_i) = \lambda_i v_i$$

But on the other hand that:

$$T(v_i) = aT^*(v_i)$$

We know from a theorem that if v_i is an eigenvector of T with eigenvalue λ_i then it is also an eigenvector of T^* with eigenvalue $\overline{\lambda_i}$ so we get:

$$\lambda_i v_i = a\overline{\lambda_i} v_i \Rightarrow \lambda_i = a\overline{\lambda_i}$$

And in particular that:

$$|\lambda_i| = |a\overline{\lambda_i}| \Rightarrow |\lambda_i| = |a||\overline{\lambda_i}|$$

But since also $|\lambda_i| = |\overline{\lambda_i}|$ we get:

$$|\lambda_i|(1 - |a|) = 0$$

And since $|a| \neq 1$ we get that $\lambda_i = 0$ which means that:

$$[T]_B = 0$$

So $T = 0$.

20.2 Show that if T is normal then $\ker T = \ker(T - aT^*)$

We can represent these tranformations and get that:

$$[T]_B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$
$$[T - aT^*)]_B = \operatorname{diag}(\lambda_1 - a\overline{\lambda_1}, \ldots, \lambda_n - a\overline{\lambda_n})$$

We know that the kernel of $v \in \ker(T)$ if and only if v is in the span of v_i with eigenvalue 0, and that $v \in \ker(T - aT^*)$ if and only if v is in the span of v_i with eigenvalue 0 but we see:

$$\lambda_i = 0 \Rightarrow \lambda_i = \overline{\lambda_i} = 0 \Rightarrow \lambda_i - a\overline{\lambda_i} = 0$$
$$\lambda_i - a\overline{\lambda_i} = 0 \Rightarrow \lambda_i = a\overline{\lambda_i} \Rightarrow |\lambda_i| = |a\overline{\lambda_i}| \Rightarrow |\lambda_i| = |a||\overline{\lambda_i}| \Rightarrow |\lambda_i|(1 - |a|) = 0 \Rightarrow \lambda_i = 0$$

Which shows that:

$$\lambda_i = 0 \iff \lambda_i - a\overline{\lambda_i} = 0$$

Which implies that the span of eignevectors from B with eigenvalue 0 in relation of T will also have eigenvalue 0 in relation to $T - aT^*$ so $\ker T = \ker(T - aT^*)$ as wanted.

21 The one with the matrix

Given:

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -4 & 1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

21.1 Find an orthogonal matrix O and a diagonal matrix D such that $O^T AO = D$

We see that A is symmetric so it must also be normal. From the spectral theorem for normal transformations we know that exists a basis B to V such that B is an orthogonal basis in realtion to the standard inner product and also comprises of eigenvectors of A. To find that B we first will find the eigenvalues of A.

$$A = \left| \begin{pmatrix} 1-\lambda & -4 & 2 \\ -4 & 1-\lambda & -2 \\ 2 & -2 & -2-\lambda \end{pmatrix} \right| = -(\lambda+3)(\lambda+3)(\lambda-6) = 0$$

So the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 6$. Now to find an orthogonal basis for $\ker(A-3)$ we do:

$$\begin{pmatrix} 4 & -4 & 2 \\ -4 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

And we find that:

$$\ker(A-3) = \left\{ a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$$

So we can choose the orthonormal basis to be:

$$B_1 = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{18}} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \right\}$$

Now for $\ker(A+6)$ we do:

$$\begin{pmatrix} -5 & -4 & 2 \\ -4 & -5 & -2 \\ 2 & -2 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

And we find that:

$$\ker(A+6) = \left\{ a \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$$

So we can choose the orthonormal basis to be:

$$B_2 = \left\{ \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

We know that vectors of different eigenspaces are always orthogonal so we know that:

$$B = B_1 \cup B_2 = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{18}} \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

And as we know from the unitary diagnolization theorem the orthogonal matrix that would diagonalize A is the matrix with these columns so:

$$O = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{1}{3} \end{pmatrix}$$

And D is just the matrix with the eigenvalues we found on the diagonal:

$$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

And:

$$O^T A O = D$$

22 The one with the prove disprove

Let T be an operator over a finite dimension inner product space. Prove or disprove the following:

22.1 T is unitary iff T is invertible and exists an orthonormal basis E such that $\|Te\| = 1$ for all $e \in E$

This is false. Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$:

$$T(1,0) = (1,0) \quad \text{and} \quad T(0,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

We can see that it is invertible, and exists the standard basis E which is orthonormal such that $\|T(e1)\| = \|T(e2)\| = 1$, yet if we consider $T(1,1)$ we see that:

$$\|(1,1)\| = \sqrt{2} \neq \sqrt{2+\sqrt{2}} = \left\| \left(1 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \right\| = \|T(1,1)\|$$

So we found a vector $v = (1,1)$ such that:

$$\|v\| \neq \|T(v)\|$$

Which means that T isn't unitary.

22.2 T is unitary iff $\|Tv\| = 1$ for all $v \in V$ such that $\|v\| = 1$

($\Rightarrow$)
Let T be unitary, then we know that for any $v \in V$ such that $\|v\| = 1$ that:

$$\|Tv\| = \|v\| = 1$$

($\Leftarrow$)
Suppose that v' is an eigenvector of T with eigenvalue λ. We can normalize v' and consider:

$$v = \frac{v'}{\|v'\|}$$

This vetcor is also an eigenvector of T with eigenvalue λ so

$$T(v) = \lambda v$$

But since $\|v\| = 1$ we also know that:

$$\|T(v)\| = \|\lambda\|\|v\| = 1 \Rightarrow \|\lambda\| = 1$$

And we know that if for any eigenvalue λ of T that $\|\lambda\| = 1$ then T is unitary. That means that we have just shown that T is unitary.

22.3 T is unitary iff for all orthonormal vectors v, u then Tv, Tu are also orthonormal

This is true. From the Gram-Schmidt theorem we know that exists $B = (v_1, \ldots, v_n)$ an orthonormal basis for V, since any two vectors $u, v \in B$ are orthonormal we get that any $T(u), T(v) \in T(B)$ are also orthonormal. So the set $T(B)$ is also orthonormal. Suppose it werent linearly independent we get that exist $(a_1, \ldots, a_n) \neq 0$ such that:

$$\sum_i a_i T(v_i) = 0$$

Using Parseval's identity we get that:

$$\left\| \sum_i a_i T(v_i) \right\| = \sqrt{\sum_i \|a_i\|} = \|0\| = 0$$

But this can only happen if $\forall i(a_i = 0)$ so $T(B)$ is linearly indepenedent and we got that T sends the orthonormal basis B to $T(B)$ an orthonormal basis. Let $v = \sum_i a_i v_i \in V$ we see that using Parseval's identity twice gives:

$$\|T(v)\| = \left\| T\left(\sum_i a_i v_i\right) \right\| = \left\| \sum_i a_i T(v_i) \right\| = \sqrt{\sum_i \|a_i\|} = \|v\|$$

We know that this is equivalent to T being unitary which completes the proof.

23 The one with the inequality

Let T be a operator over an inner product space V and let $TT^* = \alpha T + \beta I$ for some $\alpha, \beta \in \mathbb{R}$.

23.1 Show that $\alpha^2 + 4\beta \geq 0$

<u>case $a = 0$</u>

$$TT^* = \beta I$$

So

$$\beta T^{-1} = T^*$$

This implies???

<u>case $a \neq 0$</u>

We know that TT^* is self-adjoint, and since $\alpha, \beta \in \mathbb{R}$ we get that:

$$\alpha T + \beta I = (TT^*) = (TT^*)^* = (\alpha T + \beta I)^* = \alpha T^* + \beta I$$

Because $a \neq 0$ we get:

$$T = T^*$$

Which means that:

$$T^2 = \alpha T + \beta I$$
$$\Rightarrow p(T) = T^2 - \alpha T - \beta I = 0$$

This implies???

24 The one with the square root

Let T be a self-adjoint operator over a finite inner product space.

24.1 Prove that exist non-negative operators A, B such that:

$$T = A - B, \quad \sqrt{TT^*} = A + B, \quad AB = BA = 0$$

We know that if T is self-adjoint which implies it is unitary diagonalizable over $\mathbb{R}$, so exist $O \in O(n)$ and D diagonal such that:

$$O^T D O = [T]_C$$

For C the basis with the ith vector being the ith column of O. Since T is self-adjoint we know that all of eigenvalues are real. We can denote them by the entries of the main diagonal of D as such: $\lambda_i = D_{ii}$, and now we can define two matrices:

$$(A')_{ij} = \begin{cases} D_{ii} & i = j \wedge D_{ii} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

And:

$$(B')_{ij} = \begin{cases} -D_{ii} & i = j \wedge D_{ii} \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

And define the operator A, B as such:

$$A(v) = (O^T A' O)(v) \quad \text{and} \quad B(v) = (O^T B' O)(v)$$

We see that A, B are self-adjoint since $O^* = O^T$ and since all of their eigenvalues by construction are non-negative we know that they are non-negative operators. We may notice that:

$$A - B = O^T A' O - (O^T B' O) = O^T (A' - B') O = O^T D O = T$$

And also that:

$$\sqrt{TT^*} = \sqrt{O^T D D^* O} = O^T |D| O = O^T (A' + B') O = A + B$$

And since diagonal matrices commute under matrix multiplication and also $O^T = O^{-1}$ we see:

$$AB = BA = A'B' = 0$$

Since A' multiplies all the rows different than 0 in B and all the rows that are zero in a scalar. This completes the proof.

25 The one with the polynomial

Let T be a self-conjugate polynomial over the inner product space V, with eigenvalues $\lambda_1, \ldots, \lambda_n$.

25.1 For any $p(x) \in \mathbb{F}[x]$ show that the singular values of $p(T)$ are $|p(\lambda_i)|$ up to inner order.

Since $p(x)$ is a polynomial we can write:

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

And:

$$p(T) = a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0 I$$
$$p(T)^* = (a_n T^n)^* + (a_{n-1}T^{n-1})^* + \cdots + (a_0 I)^* = \overline{a_n}(T^*)^n + \overline{a_{n-1}}(T^*)^{n-1} + \cdots + \overline{a_0} I$$

Let λ be an eigenvalue asscociated with an eigenvector v of T. We see that:

$$\begin{aligned}
p(T)(v) &= (a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0 I)(v) \\
&= a_n T(v)^n + a_{n-1}T(v)^{n-1} + \cdots + a_0 I(v) \\
&= a_n \lambda^n v + a_{n-1}\lambda^{n-1} v + \cdots + a_0 v \\
&= p(\lambda)(v)
\end{aligned}$$

And:

$$\begin{aligned}
p(T)^*(v) &= (\overline{a_n}(T^*)^n + \overline{a_{n-1}}(T^*)^{n-1} + \cdots + \overline{a_0} I)(v) \\
&= \overline{a_n}(T^*(v))^n + \overline{a_{n-1}}(T^*(v))^{n-1} + \cdots + \overline{a_0} v \\
&= \overline{a_n \lambda^n} v + \overline{a_{n-1}\lambda^{n-1}} v + \cdots + \overline{a_0} v \\
&= \overline{p(\lambda)} v
\end{aligned}$$

So the eigenvalues of $p(T)^* p(T)$ are exactly $\overline{p(\lambda)}p(\lambda)$ which is exactly $\|p(\lambda)\|^2$. By SVD we know that the singular values of $p(T)$ are the square roots of the eigenvalues of $p(T)^* p(T)$, or in other words, the singular values of $p(T)$ are $\|p(\lambda_i)\|$ up to order.

26 The one with the operator norm

26.1 Show that $\|T^*T\|_{\mathbf{op}} = \|T\|_{\mathbf{op}}^2$

We know that:

$$\|T\|_{\text{op}} = \sup_{\|x\|=1} \|Tv\| = \sup_{\|x\|=1} \sqrt{\langle T(v), T(v)\rangle} = \sup_{\|x\|=1} \sqrt{\langle T^*T(v), v\rangle}$$

From this follows that:

$$\|T\|_{\text{op}}^2 = \sup_{\|x\|=1} \langle T^*T(v), v\rangle$$

We may notice that $\langle T^*Tv, v\rangle$ is a non-negtive number since it's just the norm of $\langle Tv, Tv\rangle$ which means using Cauchy-Schwartz we get:

$$\|T\|_{\text{op}}^2 = \sup_{\|x\|=1} \langle T^*T(v), v\rangle \leq \|T^*T\|\|x\| = \|T^*T\|$$

So we got that $\|T\|_{\text{op}}^2 \leq \|T^*T\|$. To prove the other direction we recall that we saw in the rehearsal that T and T^* have the same singular values and in particular that:

$$\|T\|_{\text{op}} = \|T^*\|_{\text{op}}$$

So using this and properties of the norm we get:

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

From this and the other inequality we get:

$$\boxed{\|T^*T\|_{\text{op}} = \|T\|_{\text{op}}^2}$$

27 The one with the reflexive bilinear form

Let f be a reflexive bilinear form over a finite dimension V.

27.1 Show that if $\text{rank} f = r$ then exist $\phi_1, \tau_1, \dots, \phi_r, \tau_r \in V^*$ such that:

$$f(x, y) = \phi_1(x)\tau_1(y) + \cdots + \phi_r(x)\tau_r(y)$$

We know that $\text{rank} f = r$ so if we denote $A = [f]_B$ we get that $\dim\ker(A) = n - r$. Denote the basis for the kernel at $B_k = \{e_{n-r+1}, \dots, e_n\}$ and complete it to a basis for V as such $B = \{e_1, \dots, e_n\}$ Now for each $u, v \in \text{span}\{e_1, \dots, e_r\}$ we can denote:

$$u = \sum_{i=1}^{n} \alpha_i e_i$$
$$v = \sum_{i=1}^{n} \beta_i e_i$$

And now for any u, v we see:

$$f(u, v) = f\left(\sum_{i=1}^{n} \alpha_i e_i, v\right) = \sum_{i=1}^{n} \alpha_i f(e_i, v) = \sum_{i=1}^{r} \alpha_i f(e_i, v)$$

The last equality is true since we know that:

$$f(v, e_i) = [v]_B A[e_i]_B = [v]_B 0 = 0$$

And since f is reflexive we get $f(e_i, v) = 0$ as well. Let $\phi_i \in V^*$ where $1 \le r \le n$ be defined as:

$$\phi_i\left(\sum_j \alpha_j e_j\right) = \alpha_i$$

And:

$$\tau_i(v) = f(e_i, v)$$

These are trivially linear functionals. From the above calculations we see that:

$$f(u, v) = \sum_{i=1}^{r} \phi_i(u)\tau_i(v)$$

Which is what we wanted to prove.

28 The one where we show some things are unique

Let V be a finite dimension inner product space over $\mathbb{R}$, f be a bilinear form over V.

28.1 Show that exists a unique $T \in \text{End}(V)$ such that:

$$f(u,v) = \langle u, T(v) \rangle, \quad \forall u, v \in V$$

We know by Gram-Schmidt that V has an orthonormal basis B which implies:

$$\langle v, u \rangle = \langle [v]_B, [u]_B \rangle_{\text{std}}$$

So we need to show that exists a unique $T \in \text{End}(T)$ such that:

$$f(u,v) = \langle [u]_B, [T(v)]_B \rangle_{\text{std}}, \quad \forall u, v \in V$$

Let:

$$[T(v)]_B = [f]_B [v]_B \in \text{End}(T)$$

We see that:

$$f(u,v) = [u]_B^* [f]_B [v]_B = [u]_B^* [T(v)]_B = \langle [u]_B, [T(v)]_B \rangle_{\text{std}}$$

This shows that exists a T as wanted, we will now show it's unique. Let $S \neq T$ such that:

$$\langle u, T(v) \rangle = \langle u, S(v) \rangle$$

From this follows that:

$$\begin{aligned} \langle u, T(v) \rangle - \langle u, S(v) \rangle &= 0 \\ \Rightarrow \langle u, T(v) - S(v) \rangle &= 0 \\ \Rightarrow \langle u, (T-S)(v) \rangle &= 0 \end{aligned}$$

Since $T \neq S$ exists v' such that $(T-S)(v') \neq 0$ and for all $u \in V$ and specifivally for $T(v')$ we get:

$$\langle T(v'), (T-S)(v) \rangle = \langle T(v'), T(v') \rangle = \|T(v')\|^2 = 0$$

But since $T(v') \neq 0$ this can't be. This implies that T is indeed unique.

29 The one with the inner product

Let $A \in \mathrm{Mat}_n(\mathbb{R})$ be symmetric and also satisfy:

$$(A^2 - 5A + 7I)^3 = I$$

29.1 Show that:

$$f(x, y) = x^T Ay$$

is an inner product on $\mathbb{R}^2$

To show that this is an inner product on $\mathbb{R}^2$ we need to show that f is positive-definite. Since A is symmetric and real it is self conjugate. By a theorem from class we know that if it is self conjugate and all of its eigenvalues are positive then A is positive definite and then f is an inner product. Let λ be an eigen value of A with a corresponding eigenvector v_λ such that:

$$Av_\lambda = \lambda v_\lambda$$

Since A satisfies the above equality we see that:

$$v_\lambda = Iv_\lambda = (A^2-5A+7I)^3 v_\lambda = (A^2-5A+7I)^2(\lambda^2 v_\lambda - 5\lambda v_\lambda + 7v_\lambda) = (A^2-5A+7I)^2(\lambda^2-5\lambda+7)v_\lambda$$

Consider the real polynimial $g(x) = x^2 - 5x + 7$. We see that its discriminant is $\sqrt{25 - 28}$ which means it doens't have any roots. Since the coefficient of x^2 is positive that means that $g(x) > 0$ for any real x and specifically that $g(\lambda) > 0$ which gives:

$$(A^2 - 5A + 7I)^3 v_\lambda = (A^2 - 5A + 7I)^2 g(\lambda)v_\lambda = (A^2 - 5A + 7I)g(\lambda)g(\lambda)v_\lambda = g(\lambda)g(\lambda)g(\lambda)v_\lambda$$

This implies that $1 = g(\lambda)^3$. The only real solution to that equation is $g(\lambda) = 1$, considering the equation $g(x) = 1$ we see:

$$g(x) = 1 \Rightarrow x^2 - 5x + 7 - 1 = 0 \Rightarrow (x - 2)(x - 3) = 0$$

So $\lambda = 2$ or $\lambda = 3$. This implies that all the eigenvalues of A are positive and as we said that implies that f is an inner product and completes the proof.

30 The one with equivalence

30.1 How many bilinear forms are there over $\mathbb{R}^2$ for which exists $0 \neq x \in \mathbb{R}^2$ such that $f(x, x) > 0$ up to isomorphism?

Let B be a bilinear form and E be a basis for $\mathbb{R}^2$. We know that each bilinear form defines a quadratic form q. We also know that any quadratic form can be represented by a symmetric matrix S_q. Since S_q is symmteric we can use Sylvester's law of inertia and get that each S_q is uniquely congruent to a matrix of the form:

$$I_{n_+} \oplus -I_{n_-} \oplus O_{n_0}$$

We now need to consider all the options that are not negative semi-definite so there would be an $x \neq 0$ such that $f(x, x) > 0$. Since we are talking about a 2×2 matrix here there are only 5 such options:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

So there are exactly 5 bilinear forms over $\mathbb{R}^2$ for which exists $x \neq 0$ such that $f(x, x) > 0$ up to isomorphism.

31 The one with the two

Let f be a symmetric bilinear form over a real finite-dimension vector space V.

31.1 Prove that if $W \subseteq V$ is a subspace such that $f|_W$ is positive definite, then $\dim W \leq n_+(f)$

Denote $\dim(W) = k$ and let $B_W = (v_1, \ldots, v_k)$ be a basis for W, and $B_v = (v_1, \ldots, v_k, \ldots v_n)$ be a basis for V. We know that f is a symmteric bilinear form, which implies that $[f]_B$ is symmetric. So by Sylvester's law of inertia we get that exists a diagonal matrix D and an invertible matrix S such that $[f]_B$ is congruent to D and:

$$S^T[f]_B S = D$$

We also know by the orthogonal diagonalization theorem for real symmetric matrices that exists $O \in O(n)$ such that:

$$O^T[f]_B O = D'$$

Where D is diagonal with the eigenvalues of $[f]_B$ on its diagonal. Since we know that $f|_W$ is positive definite that means that all of its eigenvalues are positive and moreover that $D'_{11}, \ldots, D'_{kk}$ are the eigenvalues of W and thus positive. Since the positive values on the diagonal corresponds to $n_+(D')$ we get that $n_+(D') \geq \dim W$ and since Sylvester's character and the rank don't change between congruent matrices [2] we get that $n_+(f) \geq \dim W$ too, which is exactly what we wanted to prove.

[2]Notice that D and D' are congruent because congruency is an equivalence relation

31.2 Let $B = (b_1, \ldots, b_n)$ be a Sylvester basis such that:

$$[f]_B = I_{n_+} \oplus (-I_{n_-}) \oplus O_{n_0}$$

Does it necessarily follow that $W \subseteq \text{sp}\{b_1, \ldots, b_{n_+}\}$

No. Let $V = \mathbb{R}^2$ and $E = \{e_1, 2e_2\}$ be a basis to $\mathbb{R}^2$ such that e_1, e_2 are the vectors from the standard basis and:

$$[f]_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We see that E is a Sylvester's basis but if we consider:

$$W = \text{sp}\{(1,1)\}$$

Then W is indeed a linear subspace of V and if we let $w = (a, a) \in W$ we see that:

$$\langle [f]_B [w]_B, [w]_B \rangle = \begin{pmatrix} 2a & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2a \\ a \end{pmatrix} = 3a^2$$

And of course $f|_B$ is also symmetric so by a theorem it is positive definite, yet as we can easily see $W \not\subseteq \text{sp}\{e_1\}$

32 The one where we prove... or disprove?

Let A be a symmetric real matrix of order $n \times n$ over V.

32.1 A is non-negative iff $\Delta_i(A) \geq 0$ for all $i = 1, \ldots, n$. Consider both directions

($\Leftarrow$)
This is false because we can look at the matrix over $\mathbb{R}$:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

We see that:

$$\Delta_1(A) = 0 \quad \text{and} \quad \Delta_2(A) = \det(A) = 0$$

But still we see that is is symmetric and it has a negative eigenvalue.
($\Rightarrow$)
Assume that A is non-negative. This clearly implies that any principle minor corresponding to $\Delta_i(A)$ is also non-negative, which means that all of its eigenvalues are non-negative. Since the determinant of any principle minor is the product of its eigenvalues we get that for all $i = 1, \ldots, n$ that $\Delta_i(A) \geq 0$ which is what we wanted to prove.