

Real Functions

Based on lectures by Emanuel Milman

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

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1 Introduction

1.1 Motivation

The Riemann integral we have known so far is fairly limited. For example it doesn't allow us to compute the Riemann integral of Dirichlet's function $f: [0, 1] \rightarrow \mathbb{R}$ defined as

$$f(x) = \mathbb{1}_{\mathbb{Q} \cap [0,1]} = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{otherwise} \end{cases}.$$

In his thesis Lebesgue introduced a new type of integral called a Lebesgue integral that allows us to compute integrals for functions like Dirichlet's function, and he continued to develop more concepts like measure, and almost everywhere.

1.2 Motivation for Lebesgue integral

Let $f(x) = \mathbb{1}_A$ be the function that we want $\int \mathbb{1}_A$ to be the volume of the set A .

First we would like we define what is a volume of a set. We would want to require a couple of things

- (1) $\mu(A)$ is defined for all $A \subseteq \mathbb{R}^n$;
- (2) $\mu([0, 1]^n) = 1^n = 1$;
- (3) μ to be invariant to congruations (isometries).
- (4) If $\{A_i\}_{i=1}^\infty$ is a countable sequence of pairwise disjoint sets then

$$\mu \left(\bigcup_{i=1}^\infty A_i \right) = \sum_{i=1}^\infty \mu(A_i).$$

Remark 1.1. Property (4) is called σ -additivity.

Theorem 1.1 (Hausdorff, 1914). *There is no function that satisfies (1) – (4) at the same time.*

We will prove this theorem later. For now we can only try to weaken the requirements. For example instead of σ -additivity we might require finite additivity.

Theorem 1.2. *There exists a function that satisfies the wanted requirements in dimensions 1 and 2 but not in dimension $n \geq 3$.*

For example in $n = 3$ we have the Banach-Tarski paradox

Paradox 1.1 (Banach–Tarski, 1924). For every $n \geq 1$ we can divide S^2 in \mathbb{R}^n to a finite amount of parts such that when they are rotated and rearranged, can form a new sphere of any desired size.

Remark 1.2. The bigger sphere we would want to form, the greater is the minimal pieces we need to divide the unit sphere in order to form it.

This paradox is based on the use of the axiom of choice, but since we assume the axiom of choice in this course we still need to modify the requirements.

Instead of (4) we require subadditivity:

$$\mu \left(\bigcup_{i=1}^\infty A_i \right) \leq \sum_{i=1}^\infty \mu(A_i).$$

This is possible but instead we would like to keep σ -additivity but instead give up on requirement (0). We will only define volume only for “nice” sets which include all of the sets we work with in analysis or geometry etc. these sets form a σ -algebra on \mathbb{R}^n .

2 Algebras and σ -algebras

2.1 Definitions

Definition 2.1 (Algebra). An algebra on a nonempty set X is a collection \mathcal{A} of subsets of X such that

- (1) $\emptyset, X \in \mathcal{A}$;
- (2) \mathcal{A} is closed under complement;
- (3) \mathcal{A} is closed under finite unions and intersections.

Definition 2.2 (σ -algebra). A σ -algebra on a nonempty set X is an algebra \mathcal{A} on X that is also closed under countable unions and intersections.

Remark 2.1. A σ -algebra is sometimes called a σ -field. That is why it is sometimes denoted \mathcal{F} .

Remark 2.2. From De-Morgan laws we know that

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c$$

so it is only necessary to require closure under countable unions or countable intersections.

Remark 2.3. It is also possible to require \mathcal{A} to be nonempty. Then for $A \in \mathcal{A}$ we have

$$A \in \mathcal{A} \implies A^c \in \mathcal{A} \implies A \cup A^c = X \in \mathcal{A} \text{ and } A \cap A^c = \emptyset \in \mathcal{A}.$$

Example 2.1. $\mathcal{A} = 2^X$ and $\{\emptyset, X\}$ are the smallest and biggest σ -algebras.

Example 2.2. If X is not countable. Then

$$\mathcal{A} = \{E \subseteq X : E \text{ or } E^c \text{ are countable}\} \neq 2^X$$

is a σ -algebra.

Definition 2.3 (Cocountability). Let X be a set. Then A is called cocountable if A^c is countable.

Example 2.3 (Generated σ -algebra). Let $F \subseteq 2^X$ be a family of subsets of X . The σ -algebra generated by F is defined as

$$\sigma(F) = \bigcap \left\{ \mathcal{A} \subseteq 2^X : \mathcal{A} \text{ is a } \sigma\text{-algebra and } F \subset \mathcal{A} \right\}$$

Remark 2.4. Notice that the intersection is indeed a σ -algebra.

Remark 2.5. Let \mathcal{A} be a σ -algebra and $F \subseteq \mathcal{A}$. Then $\sigma(F) \subseteq \mathcal{A}$.

Corollary 2.1. Suppose $F_1 \subseteq \sigma(F_2)$ and $F_2 \subseteq \sigma(F_1)$. Then $\sigma(F_1) = \sigma(F_2)$.

Definition 2.4 (Generated σ -algebra). Let X be a topological space. Then we define the Borel σ -algebra as the σ -algebra generated by the open sets in X .

$$B(X) = \sigma \left(\{G \subset X \mid G \text{ is open}\} \right).$$

Remark 2.6. Recall that G denotes an open set, F a closed set, G_δ a countable intersection of open sets and F_σ a countable union of closed sets. Similarly $G_{\delta\sigma}$ is a countable union of G_δ sets etc.

Remark 2.7.

$$B(X) = \sigma \left(\{F \subset X \mid F \text{ is closed}\} \right).$$

Proposition 2.2. $B(\mathbb{R})$ is generated by the collection of any type of interval.

Proof. Consider the collection of open intervals in \mathbb{R} . Since any open set in \mathbb{R} is a countable union of disjoint open intervals, it is clear that $B(\mathbb{R}) = \sigma((a, b))$.

Consider the collection of closed intervals in \mathbb{R} . Since any closed interval is a countable intersection of open sets, we have that $[a, b] \subseteq \sigma((a, b))$. Since any open interval is a countable union of closed sets, we have that $(a, b) \subseteq \sigma([a, b])$.

The proof for other types of intervals (like $(a, b]$) is similar and thus omitted. \square

2.2 The product σ -algebra

Definition 2.5 (Product σ -algebra). Let $\{\mathcal{A}_i\}_i$ be a collection of σ -algebras on $\{X_i\}_i$. Then, the product σ -algebra $\otimes_{i \in I} \mathcal{A}_i$ is the σ -algebra generated by the cylindrical sets.

$$\mathcal{S} = \left\{ \prod_{i \in I} U_i \mid \exists j \in I \text{ s.t. } U_i = X_i \text{ for } i \in I \setminus \{j\} \text{ and } U_j \in \mathcal{A}_j \right\}.$$

which is also the set that generated the product topology on the product space.

Remark 2.8. In the case of $|I| < \aleph_0$ this is equivalent to the σ -algebra generated by the set of the box topology.

Proposition 2.3. $B(\mathbb{R}^n) = \otimes_{i=1}^n B(\mathbb{R})$.

Proof. We need to prove that $\prod_{i=1}^n (a_i, b_i) \subseteq B(\mathbb{R}^n)$ which is clear, and also that $B(\mathbb{R}^n) \subseteq \prod_{i=1}^n (a_i, b_i)$ which is also manageable because for an open set $A \subseteq \mathbb{R}^n$ we can see that it is the countable union of all the open boxes with rational sized edges contained in A around any rational point $q \in A$ (which exist because A is open). \square

3 Measure

3.1 Definitions

Definition 3.1 (Measure). Let \mathcal{A} be a σ -algebra on X . A function $\mu: \mathcal{A} \rightarrow [0, \infty]$ is called a measure if

- (1) $\mu(\emptyset) = 0$;
- (2) Given a sequence $\{E_i\}_{i=1}^\infty \subseteq \mathcal{A}$ of disjoint sets we have

$$\mu \left(\biguplus_{i=1}^\infty E_i \right) = \sum_{i=1}^\infty \mu(E_i).$$

Remark 3.1. Property (2) is called σ -additivity. This obviously implies finite additivity because we can choose $E_j = \emptyset$.

Definition 3.2 (Measurable space). The space (X, \mathcal{A}) is called a measurable space. The elements of \mathcal{A} are called measurable sets.

Definition 3.3 (Measure space). The triple (X, \mathcal{A}, μ) is called a measure space.

Definition 3.4 (Finite measure). A finite measure is a measure μ on X such that $\mu(X) < \infty$.

Definition 3.5 (σ -finite measure). A σ -finite measure is a measure μ on X such that $X = \bigcup_{i=1}^{\infty} E_i$ for $E_i \in \mathcal{A}$ such that $\mu(E_i) < \infty$.

Definition 3.6 (Borel measure). A measure μ on a topological space X is called a Borel measure if $B(X) \subseteq \mathcal{A}$.

Example 3.1 (Delta measure). Let $x_0 \in X$ and $\mathcal{A} = 2^X$. Then the delta measure is

$$\delta_{x_0}(E) = \begin{cases} 1, & x_0 \in E \\ 0, & x_0 \notin E \end{cases}.$$

Example 3.2 (Counting measure). The counting measure is the measure on $\mathcal{A} = 2^X$ such that $\mu(E) = |E|$.

Example 3.3. Suppose $\aleph_0 < |X|$ and let \mathcal{A} be the σ -algebra of the countable or cocountable subsets of X . Then the following is a measure on X ,

$$\mu(E) = \begin{cases} 0, & |E| \leq \aleph_0 \\ 1, & |E^c| = \aleph_0 \end{cases}$$

Example 3.4. Suppose $|X| = \infty$ and $\mathcal{A} = 2^X$. Then the following is a measure on X ,

$$\mu(E) = \begin{cases} 0, & |E| < \infty \\ 1, & |E| = \infty \end{cases}$$

Remark 3.2. The last example is a finite additive measure, but not σ -additive.

Proposition 3.1. Let (X, \mathcal{A}, μ) be a measure space. Then

- (1) Let $E, F \in \mathcal{A}$ such that $E \subseteq F$. Then $\mu(E) \leq \mu(F)$.
- (2) Let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$. Then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

- (3) Let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ such that $E_1 \subseteq E_2 \subseteq \dots$ then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i \rightarrow \infty} \mu(E_i)$$

- (4) Let $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ such that $E_1 \supseteq E_2 \supseteq \dots$ then

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i \rightarrow \infty} \mu(E_i)$$

Proof. TO BE ADDED. □

Definition 3.7 (μ -negligible set). A negligible set is a set $E \in \mathcal{A}$ such that $\mu(E) = 0$.

Definition 3.8 (Almost everywhere). We say that a certain property is true almost everywhere if it is true for any $x \in X \setminus E$ such that E is μ -negligible.

Example 3.5. We have that $\sin x \neq 0$ almost everywhere because

$$|E| = \left| \{x \in \mathbb{R} \mid \sin x = 0\} \right| = \aleph_0$$

is μ -negligible for

$$\mu(E) = \begin{cases} 0, & |E| \leq \aleph_0 \\ 1, & |E^c| = \aleph_0 \end{cases}$$

Remark 3.3. A countable union of μ -negligible sets is μ -negligible from subadditivity.

Remark 3.4. A subset of a μ -negligible set is μ -negligible since measures are monotone.

3.2 Complete measure spaces

Definition 3.9 (Complete measure space). A triple (X, \mathcal{A}, μ) is called a complete measure space if for any $F \in \mathcal{A}$, if $\mu(F) = 0$ then for all $E \subseteq F$ we have $E \in \mathcal{A}$.

Theorem 3.2 (Completion theorem). Let (X, \mathcal{A}, μ) be a measure space. Define

$$\begin{aligned} \mathcal{N} &= \{N \in \mathcal{A} \mid \mu(N) = 0\} \\ \overline{\mathcal{A}} &= \{E \cup F \mid E \in \mathcal{A} \text{ and } F \subseteq N \in \mathcal{N}\}. \end{aligned}$$

Then $\overline{\mathcal{A}}$ is a σ -algebra, and there exists a unique extension $\bar{\mu}$ of μ such that $(X, \overline{\mathcal{A}}, \bar{\mu})$ is a complete measure space.

Proof. In hw □

Remark 3.5. $\overline{\mathcal{A}}$ is the smallest σ -algebra containing \mathcal{A} and all the subsets of μ -negligible sets.

Our goal is to construct a measure space $(\mathbb{R}^n, \mathcal{A}, \mu)$ such that

$$\mu \left(\prod_{i=1}^n [a_i, b_i] \right) = \prod_{i=1}^n |a_i - b_i|.$$

Recall that earlier we saw that

$$B(\mathbb{R}^n) = \bigotimes_{i=1}^n B(\mathbb{R}) = \sigma \left(\prod_{i=1}^n [a_i, b_i] \right)$$

which implies that $B(\mathbb{R}^n) \subseteq \mathcal{A}$. To define the measure for other sets we can use an outer measure

$$\mu^*(E) := \inf \left\{ \sum_{i=1}^{\infty} \mu(R_i) \mid E \subseteq \bigcup_{i=1}^{\infty} R_i \right\}.$$

We now have two questions.

- (1) Is $\mu^*(R) = \mu(R)$? Yes.
- (2) Did we get a measure? No, we got an outer measure.

Remark 3.6. Defining μ^* only on “good” sets gives us a measure called Lebesgue measure. Good sets are sets for which the outer measure is equal to the inner measure.

Definition 3.10 (Outer measure). Let $X \neq \emptyset$. Then $\mu^*: 2^X \rightarrow [0, \infty]$ is called an outer measure if

- (1) $\mu^*(\emptyset) = 0$;
- (2) μ^* is monotone (if $A \subset B$ then $\mu(A) < \mu(B)$);
- (3) μ is subadditive $\left(\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)\right)$.

Proposition 3.3. Let $\mathcal{E} \subseteq 2^X$ be a family of subsets of X and $\varphi: \mathcal{E} \rightarrow [0, \infty]$ such that $\emptyset, X \in \mathcal{E}$ and $\varphi(\emptyset) = 0$. For all $A \subseteq X$ we define

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(E_i) \mid E_i \in \mathcal{E} \text{ and } A \subseteq \bigcup_{i=1}^{\infty} E_i \right\}$$

then μ^* is an outer measure.

Remark 3.7. It is not promised that $\mu^*(E) = \varphi(E)$ for $E \in \mathcal{E}$.

Proof.

- (1) We can cover \emptyset by $E_i = \emptyset$.
- (2) Let $A \subseteq B$. We can use the cover of B for A since $A \subseteq B \subseteq \bigcup E_i$ which implies $\mu^*(A) \leq \mu^*(B)$.
- (3) By the definition of the infimum there exists a covering $A_i \subseteq \bigcup_{k=1}^{\infty} E_k^i$ such that

$$\mu \left(\bigcup_{k=1}^{\infty} E_k^i \right) \leq \sum_{i=1}^{\infty} \mu^*(A_i) + \epsilon \cdot 2^{-i}$$

which implies that $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{k,i=1}^{\infty} E_k^i$ and so

$$\mu \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{k,i} \varphi(E_k^i) \leq \sum_{i=1}^{\infty} \mu(E_i) + \epsilon$$

but $\epsilon > 0$ is arbitrary which completes the proof. □

Remark 3.8. Every outer measure can be constructed by φ in this way.

Definition 3.11 (μ^* -measurable set). A set $A \subseteq X$ is called μ^* -measurable if for all $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Remark 3.9. It is clear that the \leq direction is always satisfied from subadditivity, so it is only necessary to verify the direction \geq . In particular, it is only necessary to verify it for sets such that $\mu^*(E) < \infty$.

Let $A \subseteq E$ be such that $\mu^*(E) < \infty$. Then we need to verify that

$$\underbrace{\mu^*(A)}_{\text{outer measure}} = \underbrace{\mu^*(E) - \mu^*(E \cap A^c)}_{\text{inner measure}}.$$

3.3 Caratheodory's outer measure theorem

Theorem 3.4 (Caratheodory's outer measure theorem). *Let μ^* be an outer measure on X . Then*

- (1) $\mathcal{F}_{\mu^*} := \{A \subseteq X \mid A \text{ is } \mu^*\text{-measurable}\}$ is a σ -algebra.
- (2) The triple $(X, \mathcal{F}_{\mu^*}, \mu^*|_{\mathcal{F}_{\mu^*}})$ is a complete measure space.

Remark 3.10. We saw in the recitations that in order to show that \mathcal{F} is a σ -algebra, it suffices to prove that if $E_i \in \mathcal{F}$ are disjoint, then

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}.$$

Proof.

- (1) Let us show that $\mathcal{F} = \mathcal{F}_{\mu^*}$ is an algebra. \mathcal{F} is closed under the complement operation because by the symmetric definition, if A is μ^* -measurable so is A^c .

Let $A, B \in \mathcal{F}$ and $E \subseteq X$. Since A, B are μ^* -measurable we have that

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c). \end{aligned}$$

We also have that

$$(A \cup B) \cap E = ((A \cap B) \cap E) \sqcup ((A^c \cap B) \cap E) \sqcup ((A \cap B^c) \cap E).$$

Thus, from subadditivity

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$$

which implies $A \cup B \in \mathcal{F}$. By induction we get that \mathcal{F} is closed under finite unions which makes it an algebra. Moreover, let $A, B \in \mathcal{F}$ be disjoint sets. Then

$$\mu^*\left(\underbrace{A \sqcup B}_E\right) = \mu^*((A \sqcup B) \cap A) + \mu^*((A \sqcup B) \cap A^c) = \mu^*(A) + \mu^*(B)$$

so μ is a finite additive measure on \mathcal{F} .

Let $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of disjoint sets. Define

$$B_n = \bigcup_{i=1}^n A_i \in \mathcal{F} \quad \text{and} \quad B_{\infty} = \bigcup_{i=1}^{\infty} A_i.$$

For all $E \subseteq X$ we have

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*((E \sqcup B_n) \cap A_n) + \mu^*((E \sqcup B_n) \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) \\ &= \sum_{i=1}^n \mu^*(E \cap A_i). \end{aligned}$$

Using this and since μ^* is monotone we have

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{i=1}^n \mu^*(E \cap A_i) + \mu^*(E \cap B_{\infty}^c).$$

Taking $n \rightarrow \infty$ we have

$$\mu^*(E) \geq \sum_{i=1}^{\infty} \mu^*(E \cap A_i) + \mu^*(E \cap B_{\infty}^c) \geq \mu^*\left(E \cap \underbrace{\bigcup_{i=1}^{\infty} A_i}_{B_{\infty}}\right) + \mu^*(E \cap B_{\infty}^c) \geq \mu^*(E)$$

which implies that $B_{\infty} \in \mathcal{F}$ (σ -additive). From Remark 3.10 we have that \mathcal{F} is a σ -algebra.

(2) Let $A \subseteq X$ such that $\mu^*(A) = 0$. Then we have

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \underbrace{\mu^*(A)}_0 + \mu^*(E) = \mu^*(E)$$

which implies that all the expressions must be equal so

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

which means $A \in \mathcal{F}_{\mu^*}$. Since \mathcal{F}_{μ^*} contains all μ^* -negligible sets, it is complete which also completes the proof. □

3.3.1 Constructing Lebesgue measure on \mathbb{R}^n

Let

$$\mathcal{R} = \left\{ R = \prod_{i=1}^{\infty} [a_i, b_i] \mid a_i, b_i \in \mathbb{R} \right\}$$

and define

$$\varphi(R) = \text{Vol}(R) = \prod_{i=1}^{\infty} |a_i - b_i|.$$

Let us define the outer Lebesgue measure for $E \subseteq \mathbb{R}^n$

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \text{Vol}(R_i) \mid E \subseteq \bigcup_{i=1}^{\infty} R_i \text{ s.t. } R_i \in \mathcal{R} \right\}.$$

We saw that it is an outer measure. By Theorem 3.4 the space $(\mathbb{R}^n, L(\mathbb{R}^n), \mu^*|_{L(\mathbb{R}^n)})$ is a complete measure space, such that

$$\mu^*|_{L(\mathbb{R}^n)} = \mu \quad \begin{array}{l} \text{is called} \\ \text{Lebesgue measure} \end{array} \quad \text{and} \quad L(\mathbb{R}^n) = \mathcal{F}_{\mu^*} \quad \begin{array}{l} \text{is the } \sigma\text{-algebra} \\ \text{of the Lebesgue} \\ \text{measurable sets in } \mathbb{R}^n \end{array}$$

We now need to show that

- (1) For all $R \in \mathcal{R}$ we have $\mu^*(R) = \text{Vol}(R)$;
- (2) $R \subseteq L(\mathbb{R}^n) \iff B(\mathbb{R}^n) = \sigma(\mathcal{R}) \subseteq L(\mathbb{R}^n)$.