Practice

Yeheli Fomberg

326269651

1 Basic Integrals

The integral is:

$$\int \frac{x^4}{(x-1)(x-2)}\,dx$$

To calculate the integral we shall use polynomial division and get that:

$$\frac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \frac{15x-14}{(x-1)(x-2)}$$

Applying linearity:

$$\int \frac{x^4}{(x-1)(x-2)}\,dx = \int x^2 + 3x + 7 + \frac{15x-14}{(x-1)(x-2)}\,dx = \int x^2 + 3x + 7\,dx + \int \frac{15x-14}{(x-1)(x-2)}\,dx$$

The first integral is:

$$\int x^2 + 3x + 7\,dx = \frac{x^3}{3} + 3\frac{x^2}{2} + 7x + C$$

And the second one is:

$$\int \frac{15x-14}{(x-1)(x-2)}\,dx$$

Which we can solve with partial fraction decomposition that gives:

$$\int \frac{15x-14}{(x-1)(x-2)}\,dx = \int \frac{16}{x-2}\,dx + \int \frac{-1}{x-1}\,dx = 16\ln|x-2| - \ln|x-1| + C$$

finnaly:

$$\frac{x^4}{(x-1)(x-2)} = \frac{x^3}{3} + 3\frac{x^2}{2} + 7x + 16\ln|x-2| - \ln|x-1| + C$$

Now the integral is:

$$\int \frac{x+1}{(x^2+4)^2}\,dx$$

We shall use linearity and see that:

$$\int \frac{x+1}{(x^2+4)^2}\,dx = \int \frac{x}{(x^2+4)^2}\,dx + \int \frac{1}{(x^2+4)^2}\,dx$$

We will of course use substitution with $u = x^2 + 4$ and get:

$$\int \frac{x}{(x+4)^2}\,dx = \int \frac{1}{2u^2}\,du = (-2u)^{-1} + C = -(2x^2+8)^{-1} + C$$

For the second intergal:

$$\int \frac{1}{(x^2+4)^2}\,dx$$

We would like to get to something of the form:

$$\int \frac{1}{(u^2+1)^n}\,dx$$

Because we have a solution for those integrals. We see that:

$$\int \frac{1}{(x^2+4)^2}\,dx = \int \frac{1}{(4(\left(\frac{x}{2}\right)^2+1))^2}\,dx$$

From here we can get with substitution of $u = \frac{x}{2}$ to the above form and solve with the formula, or solve directly with integration by parts.

Now the integral is:

$$\int \frac{x\cos(\sqrt{x^2-1})}{\sqrt{x^2-1}}\,dx$$

We can solve this by substituting $u = \sqrt{x^2-1}$ and get:

$$\int \frac{x\cos(\sqrt{x^2-1})}{\sqrt{x^2-1}}\,dx = \int \cos(u)\,du$$

And from here it's trivial.

Now we will solve a trigonometric integral:

$$\int \frac{1+\sin(x)}{1+\cos(x)}\,dx$$

If we use the trigonometric substitution $u = \tan(\frac{x}{2})$ we get the following:

$$dx = \frac{2du}{1+u^2} \qquad \sin x = \frac{2u}{1+u^2} \qquad \cos x = \frac{1-u^2}{1+u^2}$$

From there it's like the previous integrals.

Now to practice integration by parts:

$$\int \sqrt{x} \ln x \, dx$$

We will denote:

$$f'(x) = \sqrt{x}$$
$$g(x) = \ln x$$

And then:

$$f(x) = \frac{2}{3}x^{\frac{3}{2}}$$
$$g'(x) = \frac{1}{x}$$

Substituting in the formula gives:

$$\int \sqrt{x} \ln x \, dx = \frac{2}{3}x^{\frac{3}{2}} \ln x - \int \frac{2}{3}x^{\frac{1}{2}} \, dx$$

And from here it's like the previous ones.

The integral now will involve a slight algebraic manipulation.

$$\int x \ln(x + x^{-1})\, dx$$

In this case we want to get rid of the x^{-1} inside the ln function. We can do that thanks to logarithmic arithmetics.

$$\int x \ln(x + x^{-1})\, dx = \int x \ln(x^{-1}(x^2 + 1))\, dx = \int x \ln(x^{-1}) + x \ln(x^2 + 1)\, dx$$

Some more manipulation gives:

$$\int x \ln(x^{-1}) + x \ln(x^2 + 1)\, dx = \int x \ln(x^2 + 1)\, dx - \int x \ln(x)\, dx$$

We may notice after some more pratice that the first part can be done easily with substitution, and the second part can be done with integration by parts.

2 Basic Theorems

One

In this question we are asked to prove that if a function f has piecewise antiderivatives, it has an antiderivative over all of the real line. This can be shown with tools from real analysis one if we understand that the antiderivative if a family of functions that differ by a constant "+ C". We can then set the constants accordingly and the result will follow. The function will look like:

$$f(x) = \begin{cases} F_1(x) \\ F_2(x) + F_1(x') - F_2(x') \end{cases} + C$$

If x' is the point connecting the two intervals.

Now we will look at this disasterous looking integral:

$$\int \frac{g(f^{-1}(x))}{f'(f^{-1}(x))}\,dx$$

Now it may not be obvioius at first, but of course, there is a trick. This time we may recall from real analysis one that:

$$\frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))}$$

Eureka! This looks exactly like we wanted it to look. If we substitute $u = f^{-1}(x)$ we get:

$$\int \frac{g(f^{-1}(x))}{f'(f^{-1}(x))}\,dx = \int g(u)\,du$$

And if we know $G(x)$ this becomes trivial. Notice we can you this to solve the third integral from **??**.

The next problem is rather long, so we may write it in bold font
Given a polynomial $P(x)$ **with** $\deg(P) = n$ **and also given all his derivatives** $P', P'', ..., P^{(n)}$**. Use** $k, n, P, P', P'', ..., P^{(n)}$ **to find the antiderivative of the function**

$$f(x) = P(x)e^{kx}$$

First we apply integration by parts:

$$\int P(x)e^{kx}\,dx = P(x)\frac{e^{kx}}{k} - \int P'(x)\frac{e^{kx}}{k}\,dx = \frac{P(x)e^{kx}}{k} - \frac{1}{k}\int P'(x)e^{kx}\,dx$$

We can see a pattern here. We will prove by induction that for a polynomial of degree n:

$$\int P(x)e^{kx}\,dx = \sum_{i=0}^{n}\left(\frac{1}{k}\left(\frac{-1}{k}\right)^{i} P^{(i)}(x)e^{kx}\right) + C$$

If $\deg(P) = 0$ we see that indeed:

$$\int P(x)e^{kx}\,dx = P(x)\frac{e^{kx}}{k} - \int \underbrace{P'(x)}_{0}\frac{e^{kx}}{k}\,dx = \frac{P(x)e^{kx}}{k} + C$$
$$= \sum_{i=0}^{0}\left(\frac{1}{k}\left(\frac{-1}{k}\right)^{i} P^{(i)}(x)e^{kx}\right) + C$$

Now we can assume the equation is true for any polynomial of $\deg(P) = d < n$ and prove for a polynomial with $\deg(P) = n$. Using integration by parts we get:

$$\int P(x)e^{kx}\,dx = P(x)\frac{e^{kx}}{k} - \int P'(x)\frac{e^{kx}}{k}\,dx = \frac{P(x)e^{kx}}{k} - \frac{1}{k}\int P'(x)e^{kx}\,dx$$

Since $\deg(P') = n - 1 < n$ so we can plug in the sum:

$$\frac{P(x)e^{kx}}{k} - \frac{1}{k}\int P'(x)e^{kx}\,dx = \frac{P(x)e^{kx}}{k} - \sum_{i=0}^{n-1}\left(\frac{1}{k}\left(\frac{-1}{k}\right)^{i} P^{(i+1)}(x)e^{kx}\right) + C$$
$$= \sum_{i=0}^{n}\left(\frac{1}{k}\left(\frac{-1}{k}\right)^{i} P^{(i)}(x)e^{kx}\right) + C$$

Which shows that for any polynomial of $\deg(P) = n$:

$$\int f(x)\,dx = \int P(x)e^{kx}\,dx = \sum_{i=0}^{n}\left(\frac{1}{k}\left(\frac{-1}{k}\right)^{i} P^{(i)}(x)e^{kx}\right) + C$$

3 Definition of the definite integral

In this section there will be some exercises about the definition of the definite integral. We shall start with proving that $f(x) = x^3$ is Darboux integrable on $[0, n]$ and infer the value of the integral. Recall the definition for a Darboux integrable function. We need to show that for all $\varepsilon > 0$ exists $\delta > 0$ such that for any partition of the interval P that satisfies $\lambda(P) < \delta$ we get that:

$$U(f,P) - L(f,P) < \varepsilon$$

An equivalent definition is that for any $\varepsilon > 0$ exists a partition such that:

$$U(f,P) - L(f,P) < \varepsilon$$

Think about the definition of the infimum and maximum and try to prove that. We can now look at a very natural partition of $[0, n]$ by fractions:

$$P_k = \left(0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{nk}{k}\right)$$

Then, since f is strictly monotonous we get that:

$$U(f, P_k) = \sum_{i}^{nk} M_i \frac{1}{k} = \frac{1}{k^4} \sum_{i}^{nk} i^3$$

Remember that we know that:

$$\sum_{i=1}^{n} i^3 = \left[\frac{i(i+1)}{2}\right]^2$$

Substituting we will get that:

$$\inf U(f, P_k) = \sup L(f, P_k) = \frac{n^4}{4}$$

So according to the alternative definition for Darboux integrable we get that the function is integrable and the integral is:

$$\int_0^n x^3 \, dx = \frac{n^4}{4}$$

The next exercise is to show the following two intresting theorems for odd and even functions accordingly:

$$\int_{-a}^{a} f(x)\,dx = 0$$
$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{2a} f(x)\,dx$$

This can be done by choosing any partition sequence with $\lambda(P_k)$ converging to zero, then inside the definition we can use the properties of odd or even functions, and get what we want.

This question is a bit tricky. We need to show that if f is Darboux integrable on $[a, b]$ then f^2 is also Darboux integrable on $[a, b]$. As we don't know any tricks yet, we need to show this by definition and hope things will work. Since we are working with differences of squares we may use:

$$|f^2(x) - f^2(y)| = |f(x) + f(y)||f(x) - f(y)|$$

Hey wait a minute. We do know that if f is integrable it is bounded. If we denote that bound with B we get:

$$|f^2(x) - f^2(y)| \leq 2B|f(x) - f(y)|$$

It's a bit gentle but from here we can get that:

$$U(f^2, P) - L(f^2, P) = \sum_{i=1}^{n} [M_i^2 - m_i^2](\Delta x_i) \leq 2B \sum_{i=1}^{n} [M_i - m_i](\Delta x_i) \leq 2B\varepsilon$$

The next question is pretty cool. We can define:

$$f(x) = \frac{\sin(x)\ln(x)}{x} \quad a_n = \int_n^{n+1} f(x)\,dx$$

And now we need to show the sequence is well defined and find the limit of the sequence. To show that it's well defined in this context means to show it is integrable. We just proved that if f is integrable then f^2 is also integrable. Applying the following trick:

$$2fg = (f+g)^2 - f^2 - g^2$$

We can see that fg is also integrable. We also know that a continuous function is integrable, together this shows that the sequence is well defined. To calculate the limit of the series we can notice that:

$$\lim_{x\to\infty} f(x) = 0$$

With some manipulation it can be shown by definition that the limit of the sequence is also zero.

This is a more geometrical exercise. We define the function:

$$f(x) = \sqrt{1 - x^2}$$

The geometrical way to think of this function relates to the unit circle. Try to think about it yourself and find:

$$\int_0^1 f(x)\, dx$$

After you've done that you can try to find:

$$\lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} \sqrt{n^2 - k^2}$$

Note that this is a Riemann sum.

Now we practise more definite integrals. The first is:

$$\int_0^{\pi} e^{\cos^2(x)} \sin(2x)\, dx$$

We can solve this by first using the identity $\sin(2x) = \sin(x)\cos(x)$ and get the integral:

$$\int_0^{\pi} e^{\cos^2(x)} \sin(x)\cos(x)\, dx$$

Now we see it help to substitute $u = \cos(x)$ and we get:

$$\int_1^{-1} -e^{u^2} u\, dx$$

We can continue from here but we can also see that since the integrand is an odd function the result of the integral is:

$$\int_0^{\pi} e^{\cos^2(x)} \sin(2x)\, dx = 0$$

The following integral is meant for you to practice a basic property of definite integrals:

$$\int_0^2 \min\{x, x^2\}\, dx$$

Try it yourself!

When we look at the following integral:

$$\int_0^1 e^x \cos(2x)\, dx$$

We can see it makes sense to try integration by parts on it, and we can even assume that if we apply it two times we could solve the integral rather easily. See for yourself.

In the following integral:

$$\int_0^{\frac{1}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx$$

Did you notice it? If we substitute $u = 1 - x^2$ we would get a similar expression to $\frac{x}{\sqrt{x}}$ and we can deal with these things quite easily by now.

Another one:

$$\int_1^{2\pi} \frac{\ln(x)}{x} \cos(\ln(x))\, dx$$

Now this might be a bit more tricky, but since the derivative of $\ln x$ is just $\frac{1}{x}$ we can see why we have motivation to apply substitution with $u = \ln(x)$. After that we get another simpler integral which we can already solve.

This will be the last one:

$$\int_{1}^{2e} |\ln(x) - 1| \, dx$$

Here we can notice that the integrand is just:

$$|\ln(x) - 1| = \begin{cases} 1 - \ln(x) & x \in [1, e] \\ \ln(x) - 1 & x \in [e, 2e] \end{cases}$$

Applying linearty we get a rather simple integral.

4 Basic Indefinite Integral Theorems

Let $f\colon [a,b] \to \mathbb{R}$ be integrable. Show that exists $c \in [a,b]$ such that:

$$\int_a^c f(x)\,dx = \int_c^b f(x)\,dx$$

Of course we need to define a function here. We will define the function:

$$g(x) = \int_a^x f(x)\,dx$$

We know that this function is continuous from the fundamental theorem of calculus, and we can even show easily that it is Lipshcitz continuous. We also know that:

$$g(a) = \int_a^a f(x)\,dx = 0$$

So according to the mean value theorem exists c such that:

$$g(c) = \frac{1}{2}g(b)$$

Or in other words:

$$\int_a^c f(x)\,dx = \int_c^b f(x)\,dx$$

This is the exercises:

$$\lim_{x\to 0} \frac{\int_0^{x^2} \arctan t \, dt}{\int_0^x t^2 \sin t \, dt}$$

And:

$$\lim_{n\to\infty} \left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2} \right)$$

Don't forget that:

$$\int \frac{1}{1+x^2} \, dx = \arctan(x) + C$$

Good luck!

Here are some integrals to prove:

$$\int_{\frac{1}{2}}^{2} \frac{\ln x}{x^2+1}\,dx = 0$$

This can simply be done by divide and conquer. We will divide as such:

$$\int_{\frac{1}{2}}^{1} \frac{\ln x}{x^2+1}\,dx = -\int_{1}^{2} \frac{\ln x}{x^2+1}\,dx$$

And conquer by substituting $u = \ln(x)$.

The following is to show:

$$\int_{2+\sin(x)}^{3^x} \ln(t)\, dt$$

Is monotonically increasing on $[1, \infty)$. This can be done by using the linearity as well.

The next statements are to prove, or disprove. The first being Let $f\colon [a,\infty) \to \mathbb{R}$ is monotone, decreasing, and non-negative, if $\int_a^\infty f(x)\,dx$ converges, then $\lim_{x\to\infty} f(x) = 0$. The solution will be on the next page.

Since the function is monotonic, decreasing and non-negative, if it's not converging to zero we will get from the comparison test

$$\int_a^\infty \epsilon \, dx \leq \int_a^\infty f(x) \, dx$$

For some $\epsilon > 0$. This is the contrapositive of the statement so we are done.

Let $f\colon [a,\infty) \to \mathbb{R}$ is continuous, and non-negative, if $\int_a^\infty f(x)\,dx$ converges, then $\lim_{x\to\infty} f(x) = 0$. The solution will be on the next page.

This is false because we can choose the tent function.

Let $f\colon [a,\infty) \to \mathbb{R}$ is continuous, if $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty f^2(x)\,dx$ converges. The solution will be on the next page. This is a relatively hard question. Here is a hint [1]

[1] Try to make a rectangles function

This is false. We can choose a function similar to the tent function but with trapezoids. The intuition is to take rectangles with an area of $\frac{1}{n^3} * n$, this way the area will converge but if we square the function the integral will be the harmonic sum which we know diverges.

5 Improper integrals

In this section we have to check whether the improper integrals converge or diverge. Starting with this one:

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan(x)}\, dx$$

We will use the substitution $t = \sqrt{(}\tan(x))$ and after some manipulation that:

$$dx = \frac{2t}{1+t^4} dt$$

This implies that:

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan(x)}\, dx = \int_0^{\infty} t \frac{2t}{1+t^4}\, dt$$

Which converges by the limit comparison test with $\int_0^{\infty} \frac{1}{t^2}\, dt$. Notice we don't have an issue near zero.

Now the integral is:

$$\int_{-\infty}^{\infty} \frac{dx}{x^3 - 1}$$

We can write:

$$\int_{-\infty}^{\infty} \frac{dx}{x^3 - 1} = \int_{-\infty}^{1} \frac{dx}{x^3 - 1} + \int_{1}^{\infty} \frac{dx}{x^3 - 1}$$

The integrals converge near negative and positive infinities for example by the limit comparison test with $\int_1^\infty \frac{1}{x^3}\,dx$ and $\int_{-\infty}^1 \frac{1}{x^3}\,dx$. We do know that:

$$\frac{1}{x^3 - 1} = \frac{1}{(x-1)(x^2+x+1)}$$

So there is another problematic point which is $x = 1$. We can see that:

$$\lim_{x \to 1} \frac{\frac{1}{(x-1)(x^2+x+1)}}{\frac{1}{(x-1)}} = \frac{1}{3}$$

This means that the integral converges iff:

$$\int_0^1 \frac{1}{x-1}\,dx$$

Converges, but we know it doesn't, so the original integral also diverges.

The next integral is kind of annoying but we shall deal with it:

$$\int_2^\infty \frac{1}{x^p(\ln x)^q}\,dx$$

We can try to substitute $t = \ln(x)$ and get:

$$\int_2^\infty \frac{1}{x^p(\ln x)^q}\,dx = \int_{\ln 2}^\infty \frac{1}{e^{(p-1)t}t^q\,dt}$$

And now if $p < 1$ we can see the integral diverges. If $p > 1$ then we can see it converges. If $p = 1$ then we get that:

$$\int_2^\infty \frac{1}{x^p(\ln x)^q\,dx} = \int_2^\infty \frac{1}{t^q}\,dt$$

Which we know converges for $q > 1$.

And now for something completely different:

$$\int_2^\infty x^p (\sin x)^q \, dx$$

We see that $x > 2$ so we know that $x^p > 1$ for any p and x over the integration interval. We also know that exists an $\epsilon > 0$ such that for any interval of the form $[\frac{\pi}{2} + 2\pi k - \epsilon, \frac{\pi}{2} + 2\pi k - \epsilon]$ we get $\sin(x) > \frac{1}{2}$ which means that:

$$\int_{\frac{\pi}{2}+2\pi k-\epsilon}^{\frac{\pi}{2}+2\pi k-\epsilon} x^p (\sin x)^q \, dx > 2\epsilon \frac{1}{2} = \epsilon$$

So by Cauchy's integral test we know this integral divergs.

And now:

$$\int_0^\infty x^p \sin(e^x)\, dx$$

Ok, we would much rather work with $\sin(u)$ here so let's substitute $u = e^x$. We get:

$$\int_1^\infty \frac{\ln^p(u)}{u} \sin(u)\, du$$

We might think of using integration by parts now, but it'd be much better to use another test we know, Dirichlet's test. If we choose:

$$f(x) = \sin(u)$$
$$g(x) = \frac{\ln^p(u)}{u}$$

We can see that:

$$\int_1^T \sin(x)\, dx = \cos(1) - \cos(T) \leq 2$$

We see that g is differentiable on $[1, \infty)$ and we get:

$$g'(t) = \frac{p \ln^{p-1}(t)}{t^2} - \frac{\ln^p(t)}{t^2}$$

So:

$$|g'(t)| \leq \frac{p \ln^{p-1}(t)}{t^2} + \frac{\ln^p(t)}{t^2}$$

We need to show the convergance of:

$$\int_1^\infty |g'(t)|\, dt$$

From the direct comparison test we can just check that:

$$\int_1^\infty \frac{p \ln^{p-1}(t)}{t^2} + \frac{\ln^p(t)}{t^2}\, dt = p \int_1^\infty \frac{\ln^{p-1}(t)}{t^2}\, dt + \int_1^\infty \frac{\ln^p(t)}{t^2}\, dt$$

Which... wait... these integrals just converges from that previous exercise! great, we've shown this integral converges.

Now we calculate the following limit:

$$\lim_{n\to\infty}\int_0^\infty e^{-nx^2}\,dx$$

From intuition we would want to say the limit is zero. The intuition is correct and so we'll prove it. Considering the integral:

$$\int_0^\infty e^{-nx^2}\,dx$$

We see that the integrand acts quite differently when x is lower than 1 near to 1 or greater than 1. We will accomodate by choosing $0 < \epsilon < 1$ and we may also consider $\frac{\epsilon}{3}$ because eventually we will show the three parts are smaller than one epsilon:

$$\int_0^\infty e^{-nx^2}\,dx = \int_0^{\frac{\epsilon}{3}} e^{-nx^2}\,dx + \int_{\frac{\epsilon}{3}}^1 e^{-nx^2}\,dx + \int_1^\infty e^{-nx^2}\,dx$$

We see that for the first integral we see:

$$\int_0^{\frac{\epsilon}{3}} e^{-nx^2}\,dx \leq \int_0^{\frac{\epsilon}{3}} 1\,dx = \frac{\epsilon}{3}$$

For the second integral we see:

$$\int_{\frac{\epsilon}{3}}^1 e^{-nx^2}\,dx$$

We get a maximum at $x = \frac{\epsilon}{3}$ which means we can choose an n large enough such that the bound will be $\frac{\epsilon}{3}$ and then we will get:

$$\int_{\frac{\epsilon}{3}}^1 e^{-nx^2}\,dx \leq \left(1 - \frac{\epsilon}{3}\right)\frac{\epsilon}{3} \leq \frac{\epsilon}{3}$$

For the last integral we see:

$$\int_1^\infty e^{-nx^2}\,dx$$

Ok now this might take some time to get used to, but hopefully by now you may thing to multiply thei ntegrand by x. Why? Because that an integral we can calculate using integration by parts. We get:

$$\int_1^\infty e^{-nx^2}\,dx \leq \int_1^\infty xe^{-nx^2}\,dx = \frac{1}{2n}$$

Again, we can see that for a sufficiently large n this will be less than $\frac{\epsilon}{3}$ and thus we get:

$$\int_0^\infty e^{-nx^2}\,dx \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

6 Series

First, as we do, let's figure out if some series converge or not.

$$\sum_{n=2}^{\infty} \frac{\cos(\pi n)}{\ln n}$$

Here we can see that the series is equivalent to:

$$\sum_{n=2}^{\infty} (-1)^n \ln(n)$$

We know that $\ln(n)$ is monotonically decreasing and $\lim_{n\to\infty} \ln(n) = 0$ so by Leibniz's test the sum converges.

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

When we see factorials we need to thing of something like the quotient test. Indeed:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}$$

So by D'Alembert's test the series converges.

$$\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2-9}}$$

We see that this series is positive and we can compare it with the harmonic sum. We get:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2-9}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-9}} = 1$$

And since the harmonic sum does not converge this series diverges as well.

$$\sum_{n=2}^{\infty} \frac{1}{(\ln(n))^n}$$

Here the root test is needed. We get:

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{(\ln(n))^n}\right|} = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$$

Since $0 < 1$ this series absolutely converges.

$$\sum_{n=2}^{\infty} \ln\left(1+\frac{(-1)^n}{n}\right)$$

We can notice the following pattern:

$$\ln\left(1+\frac{(-1)^n}{n}\right) = \begin{cases} \ln\left(\frac{n+1}{n}\right) & n \text{ is even} \\ \ln\left(\frac{n-1}{n}\right) & n \text{ is odd} \end{cases} = \begin{cases} \ln(n+1)-\ln(n) & n \text{ is even} \\ \ln(n-1)-\ln(n) & n \text{ is odd} \end{cases}$$

We see that if we look on the odd subsequence of the partial sum [2] we get:

$$\sum_{n=2}^{2k+1} \ln\left(1+\frac{(-1)^n}{n}\right) = 0 \xrightarrow{k\to\infty} 0$$

And if we look on the even partial sum we get:

$$\sum_{n=2}^{2k} \ln\left(1+\frac{(-1)^n}{n}\right) = \ln\left(1+\frac{(-1)^{2k}}{2k}\right) \xrightarrow{k\to\infty} \ln(1) = 0$$

And as we know that means the series converges, moreover, it converges to 0, and since all the elements are positive the convergance is absolute.

[2] k must be a natural number strictly greater than 0

$$\sum_{n=2}^{\infty}\left(\frac{n-1}{n+1}\right)^{n(n-1)}$$

We need to use the root test and we get:

$$\begin{aligned}
\lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{n-1}{n+1}\right)^{n(n-1)}\right|} &= \lim_{n\to\infty}\left(\frac{n-1}{n+1}\right)^{n-1} \\
&= \lim_{n\to\infty}\left(1-\frac{2}{n+1}\right)^{n-1} \\
&= \lim_{n\to\infty}\left(1-\frac{2}{n+1}\right)^{n+1}\left(1-\frac{2}{n+1}\right)^{-2} \\
&= \lim_{n\to\infty}\underbrace{\left(1-\frac{2}{n+1}\right)^{n+1}}_{\xrightarrow{n\to\infty}e^{-2}}\underbrace{\left(1-\frac{2}{n+1}\right)^{-2}}_{\xrightarrow{n\to\infty}1} \\
&= e^{-2}
\end{aligned}$$

Since $e^{-2} < 1$ by the root test the series converges, moreover, it converges absolutely.

For what values of $p, q \in \mathbb{R}$ does the following series converge?

$$\sum_{n=2}^{\infty} \frac{1}{n^p (\ln n)^q}$$

We can solve this using the integral test and the relavent previous integral we did.

Calculate the following series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)}$$

We may get something more familiar if we get rid of this $(-1)^n$ part:

$$\begin{aligned}
\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)} &= \sum_{n=1}^{\infty} \left(\frac{2(2n-1)+1}{(2n-1)(2n)} - \frac{2(2n)+1}{2n(2n+1)} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{4n-1}{2n(2n-1)} - \frac{4n+1}{2n(2n+1)} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2n} \left(\frac{4n-1}{2n-1} - \frac{4n+1}{2n+1} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2n} \frac{4n}{(2n+1)(2n-1)} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n-1)} \\
&= 2 \sum_{n=1}^{\infty} \left(\frac{\frac{1}{2}}{2n-1} - \frac{\frac{1}{2}}{2n+1} \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)
\end{aligned}$$

Looking at the partial sums we see that:

$$\sum_{n=1}^{k} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = 1 - \frac{1}{2k+1}$$

So:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n+1}{n(n+1)} = \lim_{k \to \infty} \sum_{n=1}^{k} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) = \lim_{k \to \infty} \left(1 - \frac{1}{2k+1} \right) = 1$$

7 Series Theorems

If $\lim_{n\to\infty} a_n = 0$ then exists a subsequnce a_{n_k} such that $\sum_{k=1}^{\infty} a_{n_k}$ converges. The idea here is to take a subseuqence such that:

$$a_{n_k} \leq 2^{-k}$$

That way using the comparison test we can see that $\sum_{k=1}^{\infty} a_{n_k}$ converges.

Given the positive series $\sum a_n, \sum b_n$ show that if for any natural $n \in \mathbb{N}$ that:

$$\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}$$

Then if $\sum b_n$ converges then $\sum_{a_n}$ converges and that if $\sum a_n$ diverges then $\sum_{a_n}$ diverges. The idea here is to notice:

$$a_1 \frac{b_1}{a_1} \leq b_1$$

And then by induction it can be shown that:

$$a_n \frac{b_1}{a_1} \leq b_n$$

And we can solve using the direct comparison test.

Let f be a function differentiable at $x = 0$ such that $f(0) = 0$ and $f'(0) \neq 0$ and let a_n be a positive sequence that converges to 0. Show that:

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \iff \sum_{n=1}^{\infty} f(a_n) \quad \text{converges}$$

We can look at the definition of the derivative with sequences and get:

$$\lim_{n\to\infty} \frac{f(a_n) - f(0)}{a_n - 0} = \lim_{n\to\infty} \frac{f(a_n)}{a_n} \neq 0$$

Since a_n is positive we get that $f(a_n)$ doesn't change its sign. Since we have shown:

$$\lim_{n\to\infty} \frac{f(a_n)}{a_n} = L \neq 0$$

We are also showing that:

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \iff \sum_{n=1}^{\infty} f(a_n) \quad \text{converges}$$

Let a_n be a positive series. Show that if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \sqrt{a_n a_{n-1}}$ converges, and that if a_n is monotonic then the other direction is also true.

For the forward direction we can notice that by the AM-GM inequality:

$$\sqrt{a_n a_{n-1}} \leq \frac{a_n + a_{n-1}}{2}$$

We can also notice that:

$$\sum_{n=1}^{\infty} a_n = \frac{a_1}{2} + \sum_{n=2}^{\infty} \frac{a_{n-1} + a_n}{2}$$

Since $\sum a_n$ converges we know that $\sum_{n=1}^{\infty} \frac{a_{n-1}+a_n}{2}$ must also converge. Now by the direct comparison test since as we remember:

$$0 \leq \sqrt{a_n a_{n-1}} \leq \frac{a_n + a_{n-1}}{2}$$

We get that $\sum_{n=1}^{\infty} \sqrt{a_n a_{n-1}}$ converges. Now to prove the other direction. We know that a_n is monotonic and it can't be increasing because it must also converge to zero. This implies:

$$a_n \leq \sqrt{a_{n-1} a_n}$$

And so by the direct comparison test we get that $\sum_{n=1}^{\infty} a_n$ converges

Let $\sum_{n=1}^{\infty} a_n$ be a conditionally converging series. Show that for all natural $k > 1$ that the series $\sum_{n=1}^{\infty} a_n n^k$ diverges. Consider the series:

$$\sum_{n=1}^{\infty} a_n n^2$$

Suppose it were converging, that means that the limit of the inside seuquence is:

$$\lim_{n\to\infty} a_n n^2 = 0$$

But this means that:

$$\lim_{n\to\infty} |a_n n^2| = \lim_{n\to\infty} |a_n| n^2 = 0$$

So for some natural N for any $n > N$ we get that:

$$|a_n| \leq \frac{1}{n^2}$$

And by the comparison test we get that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. Since that's a contradiction we get that the above series is not converging and by the direct compparison test the result follows.

8 Uniform Convergence

Ah, uniform convergence, Show that each of the following function sequences uniformly converge on I and not uniformly converges on J:

$$
\begin{aligned}
f_n(x) &= \sin(x^n(1-x)) \\
I &= [0,1] \\
J &= [0,2]
\end{aligned}
$$

We can see that for any $x \in I$ the function goes to 0 and since all the function and the zero function is continuous and I is closed we get uniform convergence. To show the convergence on J is not uniform we will try to use some theorems. Specifically we know that if it were converging uniformly to f then since f_n are continuous f is continuous and there is a $\delta > 0$ such that for all x in $(1, 1+\delta)$ we get $|f(x)| \leq \frac{1}{4}$ yet we get that:

$$(1+\delta)^n(1-(1+\delta)) = -\delta(1+\delta)^n$$

So if we choose n large enough we get that:

$$-\delta(1+\delta)^n \leq -\frac{\pi}{2}$$

And from the middle value theorem we can choose x_n such that:

$$(x_n)^n(1-(x_n)) = -\frac{\pi}{2}$$

And then:

$$f_n(x_n) = -1$$

But that's a contradiction because:

$$\sup_{x\in J} |f_n(x) - f(x)| \geq \sup_{x\in J} |f_n(x_n) - f(x_n)| > -\frac{3}{4}$$

Because $|f(x)| \leq \frac{1}{4}$ which completes the proof.

$$
\begin{aligned}
f_n(x) &= x\arctan(nx) - \frac{1}{2n}\ln(n^2x^2+1) \\
I &= [1,2] \\
J &= [1,\infty)
\end{aligned}
$$

We can notice that:

$$f_n'(x) = \arctan(nx)$$

All of f_n' are continuous and also they converge pointwise to $g(x) = \frac{\pi}{2}$ and monotonically. By Dini's theorem they converge uniformly to g. Now if we check convergence for $x = 1$ in respect to $f_n(x)$ we get that:

$$\lim_{n\to\infty}\left(x\arctan(nx) - \frac{1}{2n}\ln(n^2x^2+1)\right) = \frac{\pi}{2}$$

Since the sequence of f_n' converges uniformly and f converges at $x = 1$ by a theorem f_n uniformly converges. Looking at J we can notice that the first argument is "stronger" than the second one and thus exists a sequence of a_n such that:

$$f_n(a_n) > \pi$$

And then we get:

$$\sup_{x\in J}|f_n(x) - f(x)| \geq |f_n(a_n) - f(a_n)| > \frac{\pi}{2}$$

And thus the convergence is not uniform.

9 Uniform Convergence Theorems

Let $I \subseteq \mathbb{R}$ be an interval or ray and let $f_n \colon I \to \mathbb{R}$ be a function sequence that converges pointwise to $f \colon I \to \mathbb{R}$.

Show that if the convergence is uniform and each f_n is uniformly continuous on I then f is also uniformly continuous on I

We know that each f_n is uniformly continuous so for all $\epsilon > 0$ and for all n exists a delta such that if $|x - y| < \delta$ then:

$$|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$$

We also know from pointwise convergence that exists N large enough such that:

$$|f(x) - f_n(x)| < \frac{\varepsilon}{3}$$
$$|f_n(y) - f(y)| < \frac{\varepsilon}{3}$$

Thus for any $\varepsilon > 0$ we can choose $n > N$ and then for all $|x - y| < \delta$ we get:

$$|f(x) - f(y)| < |f_n(x) - f(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

Show that if the convergence is not uniform, the uniform continuity of f_n does not necessarily imply that f is uniformly continuous. Consider the following function sequence:

$$f_n(x) = x^n$$

We know that it converges pointwise to:

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

It is clear that for all $n \in \mathbb{N}$ that $f_n(x)$ is uniformly continuous on $[0,1]$ but the function $f(x)$ is not even continous, thus we have shown that the uniform continuity of f_n does not necessarily imply that f is uniformly continuous.

10 Limits with uniform convergence theorems

$$\lim_{n \to \infty} \int_0^1 \arctan\left(\frac{x}{n}\right) \, dx$$

We know that the function sequence $g_n(x) = \frac{x}{n}$ converges to 0 and since:

$$\lim_{x \to 0} \arctan(x) = 0$$

We get that:

$$\lim_{n \to \infty} \arctan\left(\frac{x}{n}\right) = 0$$

Since the function sequence converges pointwise, to $f(x) = 0$ is monotonic, and all the functions are continuous on a closed interval, we know by Dini's theorem that the convergence is uniform. This means that:

$$\lim_{n \to \infty} \int_0^1 \arctan\left(\frac{x}{n}\right) \, dx = \int_0^1 f(x) \, dx = \int_0^1 0 \, dx = 0$$

Now we need to find:

$$\lim_{n\to\infty}\int_0^1 \sin(x^n)\,dx$$

We know that the function sequence:

$$f_n(x) = \sin(x^n)$$

Is monotonic and converges pointwise to $f(x) = 0$ on the interval $[0, 1-\epsilon]$ which means by Dini's theorem that the convergence is uniform. We also know that:

$$\int_{1-\epsilon}^1 \sin(x^n)\,dx \leq \int_{1-\epsilon}^1 1\,dx = \epsilon$$

We get:

$$\lim_{n\to\infty}\int_0^1 \sin(x^n)\,dx = \lim_{n\to\infty}\int_{1-\epsilon}^1 \sin(x^n)\,dx \leq \epsilon$$

Which completes the proof.

11 Function series

We are now going to practice Weierstrass's M-test now.

$$\sum_{n=1}^{\infty} \frac{n\sin(nx)}{2^n} \quad \text{on} \quad I = [0, \pi]$$

First we need to find the right bounds for each term in the series:

$$\left| \frac{n\sin(nx)}{2^n} \right| \leq \frac{n}{2^n} = M_n$$

Now all that's left is to notice that using the quotient test on M_n gives:

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$

This means that the series:

$$\sum_{n=1}^{\infty} M_n$$

Converges which completes the proof.

$$\sum_{n=1}^{\infty}\left(\frac{\ln x}{x}\right)^n \quad \text{on} \quad I=[1,\infty]$$

Here we shall also use Weierstrass's M-test. We know that the limit of the function:

$$f(x)=\frac{\ln x}{x}$$

Exists since it's a continuous function and converges to zero at infinity. We can mark it a and we can show it's smaller than 1 because we know that on the interval:

$$\ln x \leq x$$

This implies that:

$$\left(\frac{\ln x}{x}\right)^n \leq a^n = M_n$$

And since $a<1$ we get that the series:

$$\sum_{n=1}^{\infty} M_n$$

Converges which completes the proof.

12 Convergence radiuses

To calculate convergence radiuses of power series we can use Chauchy-Hadamard theorem, which states that for a power series:

$$\sum_{n=1}^{\infty} a_n x^n$$

The convergence radius will be:

$$R = \limsup_{n\to\infty} \frac{1}{|a_n|^{\frac{1}{n}}}$$

Or using D'Alembert test:

$$R = \frac{1}{\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|}$$

Using that we can determine the radius of convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+2^n}}{1+3^n} x^n$$

using D'Alembert's theorem and get that since:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{\sqrt{(n+1)^2+2^{n+1}}}{1+3^{n+1}}}{\frac{\sqrt{n^2+2^n}}{1+3^n}} \right| &= \lim_{n\to\infty} \left| \frac{\sqrt{(n+1)^2+2^{n+1}}(1+3^n)}{(1+3^{n+1})\sqrt{n^2+2^n}} \right| \\
&= \lim_{n\to\infty} \frac{\sqrt{(n+1)^2+2^{n+1}}(1+3^n)}{(1+3^{n+1})\sqrt{n^2+2^n}} \\
&= \lim_{n\to\infty} \sqrt{\frac{(n+1)^2+2^{n+1}}{n^2+2^n}} \frac{1+3^n}{1+3^{n+1}} \\
&= \sqrt{2}\frac{1}{3} \\
&= \frac{\sqrt{2}}{3}
\end{aligned}$$

The radius of convergence is $R = \frac{3}{\sqrt{2}}$

$$\sum_{n=1}^{\infty} \frac{e^n n!}{n^n} x^n$$

We see that:

$$\lim_{n\to\infty} \sqrt[n]{\frac{e^n n!}{n^n}} = \lim_{n\to\infty} e \sqrt[n]{\frac{n!}{n^n}} = e\frac{1}{e} = 1$$

So by Chauchy-Hadamard the radius of convergence is $R = 1$

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(1+\frac{1}{n}\right)^{n^2} x^{2n+1}$$

For this series we need to notice that the coefficients for x^n are actually:

$$a_n = \begin{cases} 0 & n \text{ is even or } n = 1 \\ \frac{1}{(\frac{n-1}{2})}\left(1+\frac{1}{(\frac{n-1}{2})}\right)^{(\frac{n-1}{2})^2} & \text{otherwise} \end{cases}$$

We see that we get:

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\frac{1}{(\frac{n-1}{2})}\left(1+\frac{1}{(\frac{n-1}{2})}\right)^{(\frac{n-1}{2})^2}} &= \lim_{n\to\infty} \sqrt[n]{\frac{2}{n-1}}\left(1+\frac{2}{n-1}\right)^{(\frac{n^2-2n+1}{4n})} \\
&= \lim_{n\to\infty} \sqrt[n]{\frac{2}{n-1}}\left(1+\frac{2}{n-1}\right)^{\frac{n-1}{4}} \\
&= \lim_{n\to\infty} \sqrt[n]{\frac{2}{n-1}}\sqrt{\left(1+\frac{2}{n-1}\right)^{\frac{n-1}{2}}} \\
&= 1 * \sqrt{e} \\
&= \sqrt{e}
\end{aligned}$$

So by Cauchy-d'Alembert we get that the radius of convergence is $R = \frac{1}{\sqrt{e}}$

13 Power Seires Theorems

Give example to a power series $\sum_{n=0}^{\infty} a_n x^n$ with convergence radius $R = 1$ such that $\sum_{n=0}^{\infty} a_n x^n$ does not converge at $x = 1$ but $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ does converge at $x = 1$

Consider the following power series:

$$\sum_{n=0}^{\infty} (-1)^n x^n$$

This is actually just the alternating series of the power series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Which we know has a radius of convergence equal to 1. By Leibniz's test for alternating series we can see that for $|x| < 1$ the series converges, and we see that for $|x| > 1$ the series diverges, that means that its radius of convergence is also $R = 1$. Checking for convergence at $x = 1$ we get:

$$\sum_{n=0}^{\infty} (-1)^n 1^n = \sum_{n=0}^{\infty} (-1)^n$$

Which we know does not converge. But the series other series at $x = 1$ gives:

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

Which is the alternating harmonic series which we know does converge.

Give example to a power series $\sum_{n=0}^{\infty} a_n x^n$ with convergence radius 1, such that $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 1$ but $\sum_{n=0}^{\infty} n a_n x^{n-1}$ does not converge at $x = 1$
Let us define:

$$a_n = \frac{(-1)^n}{n+1}$$

We can use D'Alembert's test and see that:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{n+1+1}}{\frac{(-1)^n}{n+1}} \right| = \lim_{n\to\infty} \frac{n+2}{n+1} = 1$$

So we know it has a radius of convergence $R = 1$. By the previous exercise we know that its power series converges at $x = 1$ yet we see that at $x = 1$ we get:

$$\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n \frac{(-1)^n}{n+1} = \sum_{n=0}^{\infty} \left((-1)^n - \frac{(-1)^n}{n+1} \right)$$

Which does not converge since it doesn't satisfy Cauchy's criteria.

Even though more beatiful and elegant solutions exists to the pervious section, we shall now expand some functions as power series around $x = 0$ and find the radius of convergence.

$$f_1(x) = \frac{1}{1 + ax}, \quad a > 0$$

First we shall calculate all the derivatives of f_1 as such:

$$\begin{aligned} f_1^{(0)}(x) &= \frac{1}{1 + ax} \\ f_1^{(1)}(x) &= -\frac{a}{(ax+1)^2} \\ f_1^{(2)}(x) &= \frac{2a^2}{(ax+1)^3} \\ &\cdots \\ f_1^{(n)}(x) &= (-1)^n \frac{n! a^n}{(ax+1)^{n+1}} \end{aligned}$$

This means that if exists a power series that converges to f_1 around $x = 0$ its coefficients must be:

$$a_n = \frac{(-1)^n \dfrac{n! a^n}{(a0+1)^{n+1}}}{n!} = (-1)^n a^n$$

We can see that:

$$\lim_{N\to\infty} R_N(x) = \lim_{N\to\infty} \sum_{n=N}^{\infty} (-1)^n a^n x^n = \lim_{N\to\infty} \sum_{n=N}^{\infty} (-1)^n (ax)^n$$

Which is an alternating geometric sum so it converges if and only if:

$$-1 < ax < 1 \iff x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$$

And it converges to:

$$\frac{(-1)^N (ax)^N}{1 - ax}$$

Now we know:

$$\lim_{N\to\infty} R_N(x) = \lim_{N\to\infty} \frac{(-1)^N (ax)^N}{1 - ax} = \lim_{N\to\infty} \frac{(-ax)^N}{1 - ax}$$

Which converges to 0 since $x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$. We have shown that the original series can be expressed as a power series iff $\lim_{N\to\infty} R_n(x) = 0$ and this happens iff

$$x \in \left(-\frac{1}{a}, \frac{1}{a}\right)$$

Finally we can say that f_1 can be expressed as a power series around 0 with radius of convergence $R = 1$ and the expression would be:

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n}{n!} x^n$$

The previous way to expand the function was not very easy, but in fact there are easier ways. For example let's look at the function:

$$f_2(x) = \frac{1}{3x^2 - 2x - 1}$$

This function looks a tad weird, but we can see it decomposes to:

$$f_2(x) = \frac{1}{3x^2 - 2x - 1} = \frac{1}{(3x+1)(x-1)}$$

From here using partial fraction decomposition we get:

$$f_2(x) = \frac{1}{4}\frac{1}{x-1} - \frac{3}{4}\frac{1}{3x+1} = -\frac{1}{4}\frac{1}{1-x} - \frac{3}{4}\frac{1}{1-(-3x)}$$

And now we can use:

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

And get that:

$$f_2(x) = -\frac{1}{4}\sum_{n=1}^{\infty} x^n - \frac{3}{4}\sum_{n=1}^{\infty} (-3x)^n = \sum_{n=1}^{\infty} -\frac{1}{4}(1 + (-3)^{n+1})x^n$$

Notice that to add the series we had to assume they both converge. We see that the radius of convergence for the series is:

$$\limsup_{n\to\infty} \frac{1}{\left|-\frac{1}{4}(1 + (-3)^{n+1})\right|^{\frac{1}{n}}} = \frac{1}{3}$$

And in that radius indeed both of the series we added together converge.

$$f_3(x) = \int_0^x \frac{t - \sin t}{t^2}\, dt$$

Ok now this looks kind of scary because it's a weird integral, but if we look first on the integrand we can see that finding it's power series is not that hard.

$$x - \sin(x) = x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = \left(\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots\right)$$

And then since dividing by x doesn't change the radius of convergence we get:

$$\frac{x - \sin(x)}{x^2} = \left(\frac{x}{3!} - \frac{x^3}{5!} + \cdots\right)$$

Now if we integrate both sides we get:

$$\int_0^x \frac{t - \sin(t)}{t^2}\, dt = \left(\frac{x^2}{2 * 3!} - \frac{x^4}{4 * 5!} + \cdots\right)$$

In other words we get:

$$f_3(x) = \sum_{n=1}^{\infty} \frac{1}{2n(2n+1)!} x^{2n}$$

The last function is a very famous function:

$$f_4(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}t^2}\, dt$$

This is known as the error function. We see we can solve this question in a very similar method. First we take the series expansion of the exponent function:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Now with a simple substitution we get:

$$e^{-\frac{1}{2}x^2} = 1 + -\frac{1}{2}x^2 + \frac{(-\frac{1}{2}x^2)^2}{2!} + \frac{(-\frac{1}{2}x^2)^3}{3!} + \cdots$$

Performing integration by parts we get:

$$\int_0^x e^{-\frac{1}{2}t^2}\, dt = x - \frac{x^3}{3*2} + \frac{x^5}{5*2^2*2!} - \frac{x^7}{7*2^3*3!} + \cdots$$

Finnaly we get that:

$$f_4(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{1}{2}t^2}\, dt = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{(-1)^n}{(2n+1)2^n n!} x^{2n+1}$$

14 Series Calculations

Find the radius of convergence and the explicit function for the following series:

$$1 + 3x + 5x^2 + 7x^3 + \cdots$$

We notice that this is the series:

$$\sum_{n=0}^{\infty} (2n+1)x^n = 2\sum_{n=0}^{\infty} nx^n + \sum_{n=0}^{\infty} x^n$$

The latter sum is known to be the geometric sum:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

And for the first one we need to work a bit. We see that we may want to intergate at sum point so we can calculate:

$$f(x) = \sum_{n=0}^{\infty} nx^n$$
$$f(x) - 1 = \sum_{n=1}^{\infty} nx^n$$
$$\frac{f(x)}{x} - \frac{1}{x} = \sum_{n=1}^{\infty} nx^{n-1}$$

Now if we integrate we get:

$$\int_0^x \frac{f(t)}{t} - \frac{1}{t}\, dt = \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

This implies that:

$$\frac{f(x)}{x} - \frac{1}{x} = \left[\frac{1}{1-x}\right]' = \frac{1}{(1-x)^2}$$

Or in other words:

$$f(x) = \frac{x}{(1-x)^2} + 1$$

Finally:

$$\sum_{n=0}^{\infty} (2n+1)x^n = 2\left(\frac{x}{(1-x)^2} + 1\right) + \frac{1}{1-x} = \frac{x+1}{(1-x)^2} + 1$$

With a radius of convergence of ∞

Describe the following series as an explicit function:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos^{2n}(x)}{n}$$

We see that this series is very similar to a power series. We can substitute $t = \cos(x)$ and get:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos^{2n}(x)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{n}$$

This is a known series for $\ln(1 - t)$ so the explicit function is:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos^{2n}(x)}{n} = \ln(1 - \cos^2(x))$$

Calculate:

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!4^n}$$

We can see that this is also very similar to a power series:

$$\cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

We know that the radius of convergence is all of ∞ so we can substitute $x = \frac{1}{2}$ and get:

$$\cos\left(\frac{1}{2}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!4^n}$$

15 Multivariable Calculus

This section is about multivariable calculus. All the definitions are very similar, so we can just jump straight into it. Let:

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Do $f_x(0,0)$ or $f_y(0,0)$ exist?
We see that:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{(h^2+0^2)\sin\left(\dfrac{1}{\sqrt{h^2+0^2}}\right) - 0}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\dfrac{1}{\sqrt{h^2}}\right)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{|h|}\right) = 0$$

And that:

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{(0^2+h^2)\sin\left(\dfrac{1}{\sqrt{0^2+h^2}}\right) - 0}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\dfrac{1}{\sqrt{h^2}}\right)}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{|h|}\right) = 0$$

So both the partial derivatives exist.

Is f differentiable at $(0,0)$?

We know that $f(x,y)$ is differentiable at $(0,0)$ if and only if the following is equal to zero:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \frac{\partial f}{\partial x}(x-0) - \frac{\partial f}{\partial y}(y-0)}{\sqrt{(x-0)^2 + (y-0)^2}} &= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} \\
&= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)
\end{aligned}$$

And in polar coordinates we get:

$$\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) = \lim_{r\to 0} r\sin\left(\frac{1}{r}\right) = 0$$

Which means the function is differetiable at $(x,y) = (0,0)$.

Is f_x continuous at $(0,0)$?
We saw that $f_x(0,0) = 0$ and we see that at any point different from $(0,0)$ we get:

$$f_x(x,y) = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}$$

And now:

$$\lim_{(x,y)\to(0,0)}\left(2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}\right) = -\lim_{(x,y)\to(0,0)}\left(\frac{x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}\right)$$

Writing the function in polar form gives:

$$\lim_{(x,y)\to(0,0)}\left(\frac{x\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)}{\sqrt{x^2+y^2}}\right) = \lim_{(x,y)\to(0,0)}\left(\frac{r\cos(\theta)\cos(r^{-1})}{r}\right) = \lim_{(x,y)\to(0,0)}\cos(\theta)\cos(r^{-1})$$

Since the limit does not exist we know that f_x is not continuous at $(x,y) = (0,0)$.

Let:

$$f(x) = \begin{cases} \frac{x^2y-3xy^2}{(2x^2+y^2)^\alpha} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

For which values of α is f continuous?

We see that $f(0,0) = 0$ and also that in polar coordinates:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y - 3xy^2}{(2x^2+y^2)^\alpha} = \lim_{r\to 0} \frac{(r\cos(\theta))^2(r\sin(\theta)) - 3(r\cos(\theta))(r\sin(\theta))^2}{(2(r\cos(\theta))^2 + (r\sin(\theta))^2)^\alpha} = \lim_{r\to 0} \frac{r^3\alpha(\theta)}{(r^2\beta(\theta))^\alpha}$$
$$= \lim_{r\to 0} \frac{r^3}{r^{2\alpha}}\gamma(\theta)$$

Such that $\alpha(\theta)$, $\beta(\theta)$ are functions bounded by θ and:

$$\gamma(\theta) = \frac{\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta)}{(1+\cos^2(\theta))^\alpha} = \frac{\cos(\theta)\sin(\theta)(\cos(\theta) - 3\sin(\theta))}{(1+\cos^2(\theta))^\alpha}$$

Since $(1+\cos^2(\theta))^\alpha$ is bounded as such:

$$0 < 2^{-\alpha} \leq (1+\cos^2(\theta))^\alpha \leq 2^\alpha$$

We get that $\gamma(\theta)$ is a bounded function and thus f converges to 0 if and only if $r^{3-2\alpha}$ is converging to 0. That means that f is continuous iff:

$$0 < 3 - 2\alpha \Rightarrow \boxed{\alpha < 1.5}$$

For which values of α is f differentiable?

We see that the function is trivially differetiable everywhere except at $(0,0)$ as composition of diffrentiable functions. We will check differentiablity at the point $(0,0)$ directly. First we will calculate the partial derivatives:

$$f_x(x,y) = \lim_{h \to 0} \frac{\frac{0}{2h^{2\alpha}} - 0}{h} = 0$$

$$f_y(x,y) = \lim_{h \to 0} \frac{\frac{0}{h^{2\alpha}} - 0}{h} = 0$$

We see that f is differentiable iff the following limit is 0:

$$\lim_{(x,y)\to(0,0)} \frac{\frac{x^2y - 3xy^2}{(2x^2+y^2)^\alpha} - 0 - f_x(0,0)(x-0) - f_y(0,0)(y-0)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^2y-3xy^2}{(2x^2+y^2)^\alpha}}{\sqrt{x^2+y^2}}$$

Moving to polar form we get:

$$\lim_{r\to 0} \frac{\frac{(r\cos(\theta))^2(r\sin(\theta)) - 3(r\cos(\theta))(r\sin(\theta))^2}{(2(r\cos(\theta))^2 + (r\sin(\theta))^2)^\alpha}}{r} = \lim_{r \to 0} \frac{r^2}{r^{2\alpha}}\gamma(\theta)$$

As we saw $\gamma(\theta)$ is bounded and not always zero and thus the limit is 0 iff:

$$0 < 2 - 2\alpha \Rightarrow \boxed{\alpha < 1}$$

For which values of α does f have a directional derivative on the direction $u = (1, 2)$?
Using the polar form of the function we see that the function will have a directional derivative if and only if:

$$\lim_{r\to 0} \frac{\frac{(r\cos(\theta))^2(r\sin(\theta)) - 3(r\cos(\theta))(r\sin(\theta))^2}{(2(r\cos(\theta))^2 + (r\sin(\theta))^2)^\alpha} - 0}{r}$$

exists when $\theta = \arctan(2)$. We see that it equals to:

$$\lim_{r\to 0} \frac{r^2}{r^{2\alpha}}\gamma(\arctan(2)) = \gamma(\arctan(2)) \lim_{r\to 0} \frac{r^2}{r^{2\alpha}}$$

Which exists iff:

$$0 \leq 2 - 2\alpha \Rightarrow \boxed{\alpha \leq 1}$$

Raise an example of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that f has a partial derivative on any direction at $(0,0)$ **but is not differentiable at** $(0,0)$

Consider the following function:

$$f(x) = \begin{cases} \frac{x^2y-3xy^2}{2x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

This is exactly the same function from the previous question except we set $\alpha = 1$. As we saw in previously this function is not differentiable at $(0,0)$ but we see that the derivative on any direction $u = (u_1, u_2)$ exists whenever the following limit exists with $c = \arctan(\frac{u_2}{u_1})$ for $\alpha = 1$, and indeed:

$$\lim_{r\to 0} \frac{r^2}{r^2}\gamma(c) = \gamma(c) \lim_{r\to 0} \frac{r^2}{r^2} = \gamma(c)$$

We see that the limit exists for any direction so $f(x,y)$ has a directional derivative at any direction at $(0,0)$ but is not differetiable there. Another good exmaple is:

$$f(x) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and $v = (v_1, v_2) \neq (0,0)$ and $0 < a$. Show that:

$$\frac{\partial f}{\partial (av)}(x,y) = a\frac{\partial f}{\partial v}(x,y)$$

We see that by definition:

$$\frac{\partial f}{\partial (av)}(x,y) = \lim_{h\to 0} \frac{f(x+hav_1, y+hav_2) - f(x,y)}{h}$$

Denote $c = ha$ and we get:

$$\frac{\partial f}{\partial (av)}(x,y) = \lim_{\frac{c}{a}\to 0} a\frac{f(x+cv_1, y+cv_2) - f(x,y)}{c}$$

That is the value such that for every $\varepsilon > 0$ exists δ' such that for all $0 < \frac{c}{a} < \delta'$ we get:

$$\left| a\frac{f(x+cv_1, y+cv_2) - f(x,y)}{c} - \frac{\partial f}{\partial (av)}(x,y) \right| < \varepsilon$$

But for that same value we can choose for any epsilon a delta $0 < \delta = \min(\delta', \frac{\delta'}{a})$ since $a > 0$ and get that for all $0 < c < \delta$ that:

$$\left| a\frac{f(x+cv_1, y+cv_2) - f(x,y)}{c} - \frac{\partial f}{\partial (av)}(x,y) \right| < \varepsilon$$

Which means that:

$$\lim_{\frac{c}{a}\to 0} a\frac{f(x+cv_1, y+cv_2) - f(x,y)}{c} = a\lim_{c\to 0} \frac{f(x+cv_1, y+cv_2) - f(x,y)}{c} = a\frac{\partial f}{\partial v}(x,y)$$

This shows that indeed:

$$\boxed{\frac{\partial f}{\partial (av)}(x,y) = a\frac{\partial f}{\partial v}(x,y)}$$

Let $g(h)\colon \mathbb{R} \to \mathbb{R}$ such that:

$$\lim_{h\to 0} g(h) = 0$$

And let $f\colon \mathbb{R}^2 \to \mathbb{R}$ have a directional derivative on any direction at (x_0, y_0). Assume that for any $v = (v_1, v_2) \neq (0,0)$ that:

$$0 \leq \left| \frac{f(x_0 + hv_1, y_0 + hv_2) - f(x_0, y_0)}{h} - f_x(x_0, y_0)v_1 - f_y(x_0, y_0)v_2 \right| \leq g(h)$$

Show that f is differentiable at (x_0, y_0)
First we see from the squeeze theorem that:

$$\lim_{h\to 0} \left(\frac{f(x_0 + hv_1, y_0 + hv_2) - f(x_0, y_0)}{h} - f_x(x_0, y_0)v_1 - f_y(x_0, y_0)v_2 \right) = 0$$

And since all directional derivatives exist we know that:

$$\lim_{h\to 0} \frac{f(x_0 + hv_1, y_0 + hv_2) - f(x_0, y_0)}{h}$$

Exists. From arithmetic of limits we get that:

$$\lim_{h\to 0} \frac{f(x_0 + hv_1, y_0 + hv_2) - f(x_0, y_0)}{h} = f_x(x_0, y_0)v_1 - f_y(x_0, y_0)v_2$$

Denote $hv_1 = \Delta x$ and $hv_2 = \Delta hv_2$ and we get that:

$$\lim_{h\to 0} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{h} = f_x(x_0, y_0)\frac{\Delta x}{h} - f_y(x_0, y_0)v_2\frac{\Delta y}{h}$$

We see know that for some function $\alpha(\Delta x, \Delta y) \xrightarrow{h\to 0} 0$, which is equivalent to $\alpha(\Delta x, \Delta y) \xrightarrow{(\Delta x, \Delta y)\to(0,0)} 0$ that:

$$\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{h} = f_x(x_0, y_0)\frac{\Delta x}{h} - f_y(x_0, y_0)v_2\frac{\Delta y}{h} + \alpha(\Delta x, \Delta y)$$

Then:

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)v_2\Delta y + \alpha(\Delta x, \Delta y)h$$

But we also see that:

$$|h||v| = \sqrt{h^2(v_1^2 + v_2^2)} = \sqrt{\Delta x^2 + \Delta y^2}$$

Which gives:

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)v_2\Delta y + \frac{\alpha(\Delta x, \Delta y)}{v}\sqrt{\Delta x^2 + \Delta y^2}$$

And that is exactly the definition of f being differentiable at (x_0, y_0).

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a differential function at $(0,0)$.
Show that:

$$g(x,y) = f(x,y) - f(y,x)$$

is differentiable at $(0,0)$
We need to show that:

$$\lim_{(x,y)\to(0,0)} \frac{g(x,y) - g(0,0) - \frac{\partial g}{\partial x}(0,0)(x) - \frac{\partial g}{\partial y}(0,0)(y)}{\sqrt{x^2+y^2}} = 0$$

Or in another words that:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(y,x) - \frac{\partial f(x,y)-f(y,x)}{\partial x}(0,0)(x) - \frac{\partial f(x,y)-f(y,x)}{\partial y}(0,0)(y)}{\sqrt{x^2+y^2}} = 0$$

We can use derivation arithmetic to see that the expression is just:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(y,x) - \frac{\partial f(x,y)}{\partial x}(0,0)(x) + \frac{\partial f(y,x)}{\partial x}(0,0)(x) - \frac{\partial f(x,y)}{\partial y}(0,0)(y) + \frac{\partial f(y,x)}{\partial y}(0,0)(y)}{\sqrt{x^2+y^2}}$$

Reordering gives:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - \frac{\partial f(x,y)}{\partial x}(0,0)(x) - \frac{\partial f(x,y)}{\partial y}(0,0)(y)}{\sqrt{x^2+y^2}} \\ - \frac{f(y,x) - f(0,0) - \frac{\partial f(y,x)}{\partial x}(0,0)(x) - \frac{\partial f(y,x)}{\partial y}(0,0)(y)}{\sqrt{x^2+y^2}}$$

The first summand goes to 0 since $f(x,y)$ is differentiable at zero. If we show that $f(y,x)$ is also differentiable at zero that would mean the other part will also go to 0 which will complete the proof. We see that $f(y,x)$ is just an isometry of $f(x,y)$ since it is a reflection on the axis $y = x$. That means that if property P_{xy} is true for $f(x,y)$ then the property P_{yx} is true for $f(y,x)$. In particular if P_{00} is being differentible at $(0,0)$ then we know that $f(y,x)$ is also differetiable at $(0,0)$ which completes the proof.

Let:

$$h(x,y) = f(|x|, y^2)$$

show that:

$$h \textbf{ is differentiable at } (0,0) \textbf{ if and only if } f_x(0,0) = 0$$

Let $f_x(0,0) = a \neq 0$ we get that:

$$\lim_{x\to 0^+} \frac{h(x,0)-h(0,0)}{x} = \lim_{x\to 0^+} \frac{f(|x|,0)-f(0,0)}{x} = f_x(0,0)$$

And

$$\lim_{x\to 0^-} \frac{h(x,0)-h(0,0)}{x} = \lim_{x\to 0^-} \frac{f(|x|,0)-f(0,0)}{-|x|} = -f_x(0,0)$$

Since $f_x(0,0) \neq 0$ these are two different numbers, so the directional derivative in the direction of $u = (1,0)$ does not exist which implies that h is not differentiable. The other direction can be proved using the definition of differentiability and manipulation the limits.

16 The Chain Rule and Integrals Depending on a Parameter

Let $f\colon [a,\infty) \to \mathbb{R}$ continuous and define:

$$I_n(x) = \int_a^x (x-t)^{n-1} f(t)\,dt$$

For every natural n and any $x > a$ prove that:

$$\left(\frac{d^n}{dx^n} I_n\right)(x) = (n-1)! f(x)$$

When we see integral with parameters the intuition should be to start thinking about functions in multiple variables so we may define:

$$f(x,y) = (x-y)^{n-1} f(y)$$

We can see that this function is defined and is continuous on any rectangle of the form $[a,b] \times [c,d]$ such that $a < b$ and also $a < c < d$ and we see that the function is also continuously differentiable in respect to x and we get:

$$\frac{\partial^m}{\partial x^m} f(x,y) = (n-1)(n-2)\dots(n-m)(x-y)^{n-m-1} f(y)$$

Using Leibniz's integral rule we get that:

$$\frac{d}{dx} I_n(x) = \frac{d}{dx} \int_a^x (x-y)^{n-1} f(y)\,dy = (n-1) \int_a^x (x-y)^{n-2} f(y)\,dy$$

Similarly for any $m < n-1$ we get that:

$$\frac{d^m}{dx^m} I_n(x) = \frac{d^m}{dx^m} \int_a^x (x-y)^{n-1} f(y)\,dy = (n-1)(n-2)\dots(n-m) \int_a^x (x-y)^{n-m-1} f(y)\,dy$$

And specifically for $m = n-1$ we get:

$$\frac{d^{n-1}}{dx^{n-1}} I_n(x) = \frac{d^{n-1}}{dx^{n-1}} \int_a^x (x-y)^{n-1} f(y)\,dy = (n-1)! \int_a^x f(y)\,dy$$

And now from the fundamental theorem of calculus we get that:

$$\frac{d^n}{dx^n} I_n(x) = \frac{d}{dx} (n-1)! \int_a^x f(y)\,dy = (n-1)! f(x)$$

Like we wanted.

Let $f(x,y)$ be a function with continuous partial derivatives in D, and let $p=(x_1,y_2)$ and $q=(x_2,y_2)$ be two points in D such that:

$$\{(tx_1+(1-t)x_2, ty_1+(1-t)y_2)\colon t\in[0,1]\}\subseteq D$$

Meaning the straight line connecting the points p,q is inside D and we can denote it $[p,q]$. Show that exists a point $w=(x',y')\in[p,q]$ such that:

$$f(q)-f(p)=\frac{\partial}{\partial x}(w)(x_1-x_2)+\frac{\partial}{\partial y}(w)(y_1-y_2)$$

We can notice that this question seems just like Lagrange's theorem. We may define the function

$$\gamma(t)=(tx_1+(1-t)x_2, ty_1+(1-t)y_2),\quad t\in[0,1]$$

We may notice that this function is differentiable and thus if we consider the composition $F:=f\circ\gamma$ we can see that it is differentiable on $[0,1]$ so using Lagrange's theorem and the chain rule we get that exists $c\in[0,1]$ such that:

$$F(1)-F(0)=F'(c)$$

We can substitute:

$$F(0)=f(p)$$
$$F(1)=f(q)$$

And we also get from the chain rule:

$$F'(c)=\frac{\partial f}{\partial x}(\gamma(c))(\gamma_1)'(c)+\frac{\partial f}{\partial y}(\gamma(c))(\gamma_2)'(c)=\frac{\partial f}{\partial x}(\gamma(c))(x_1-x_2)+\frac{\partial f}{\partial y}(\gamma(c))(y_1-y_2)$$

So now if we just define $w=\gamma(c)$ we get:

$$f(q)-f(p)=\frac{\partial}{\partial x}(w)(x_1-x_2)+\frac{\partial}{\partial y}(w)(y_1-y_2)$$

17 Leibniz's Rule

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$$

We can consider the function:

$$F(y) = \int_0^y \frac{\ln(1+xy)}{1+x^2}\,dx$$

The integrand is the function:

$$f(x,y) = \frac{\ln(1+xy)}{1+x^2}$$

Which is continuous on $[0,2]\times[0,2]$ and we see that:

$$\frac{\partial f}{\partial y} = \frac{x}{(x^2+1)(xy+1)}$$

Which is continuous on $[0,2]\times[0,2]$ as well. To use Leibniz's rule we also need to calculate:

$$\begin{aligned}\alpha(y) &= 0\\ \alpha'(y) &= 0\\ \beta(y) &= y\\ \beta'(y) &= 1\end{aligned}$$

Now using Leibniz's rule we get:

$$\frac{d}{dy}F(y) = \int_0^y \frac{x}{(x^2+1)(xy+1)}\,dx + f(\beta(y),y) = \int_0^y \frac{x}{(x^2+1)(xy+1)}\,dx + \frac{\ln(1+y^2)}{1+y^2}$$

After performing partial fraction decomposition we get that:

$$F'(y) = \frac{\ln(1+y^2)}{1+y^2} + \frac{1}{1+y^2}\int_0^y \frac{y+x}{1+x^2}\,dx - \frac{1}{1+y^2}\int_0^y \frac{y}{1+xy}\,dx$$

And now these are things we already know how to solve.

$$\int_0^y \frac{y+x}{1+x^2}\,dx = y\arctan(x)\Big|_0^y + \frac{1}{2}\ln(1+x^2)\Big|_0^y$$

And:

$$\int_0^y \frac{y}{1+xy}\,dx = y\ln(1+xy)\Big|_0^y$$

Finally we get:

$$F'(y) = \frac{y}{1+y^2}\arctan(y) + \frac{1}{2}\frac{1}{1+y^2}\ln(1+y^2) = \left[\frac{1}{2}\arctan(y)\ln(1+y^2)\right]'$$

We get that:

$$F(y) = \frac{1}{2}\arctan(y)\ln(1+y^2) + C$$

But we know that $F(0)=0$ so we get that $C=0$ and then we end up with:

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = F(1) = \frac{1}{2}\arctan(1)\ln(2) = \frac{\pi}{8}\ln(2)$$

$$\int_0^1 \frac{dx}{(x^2+a^2)^3}$$

We can easily solve the integral if $a = 0$ so from now on we shall assume that $a >$. We are advised to look at the function:

$$F(y) = \int_0^1 \frac{dx}{x^2+y^2}$$

We see that the integrand is:

$$f(x,y) = \frac{1}{x^2+y^2}$$

This function is continuous and differentiable for any $(x,y) \neq (0,0)$ as we can see and it's derivatives are:

$$\frac{\partial f}{\partial y}(x,y) = -\frac{2y}{\left(y^2+x^2\right)^2}$$
$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{8y^2}{\left(y^2+x^2\right)^3} - \frac{2}{\left(y^2+x^2\right)^2}$$

These are continuous also for any $(x,y) \neq (0,0)$, using Leibniz's rule twice we get on the rectangle $[0,1] \times [\frac{a}{2}, 2a]$ that:

$$F'(y) = \int_0^1 -\frac{2y}{\left(y^2+x^2\right)^2}\,dx \quad \text{and} \quad F''(y) = \int_0^1 \left(\frac{8y^2}{\left(y^2+x^2\right)^3} - \frac{2}{\left(y^2+x^2\right)^2}\right) dx$$

We notice that the second derivative can be also written as:

$$F''(y) = \frac{F'(y)}{y} + 8y^2 \int_0^1 \frac{dx}{(x^2+y^2)^3}$$

So we want to find:

$$\frac{1}{8a^2}\left(F''(a) - \frac{F'(a)}{a}\right)$$

We actually can find $F(y)$ directly using substitution and find out that:

$$F(y) = \frac{\arctan(y^{-1})}{y}$$

We can now calculate the derivatives directly:

$$F'(y) = \frac{-1}{1+y^2}$$
$$F''(y) = \frac{-2y}{(1+y^2)^2}$$

Finally we can substitute everything and get that:

$$\int_0^1 \frac{dx}{(x^2+a^2)^3} = \frac{1}{8a^2}\left(F''(a) - \frac{F'(a)}{a}\right) = \frac{a^2-1}{8a^3(1+a^2)^2}$$

$$\frac{d}{dx}\left(\int_0^1 \frac{\sin(xy)}{y}\,dx\right)$$

Here again we obviously need to use Leibniz's rule. We can define:

$$f(x,y) = \frac{\sin(xy)}{y}$$

This function is continuous at any point except where $y = 0$ so using Lebiniz's rule we get:

$$\frac{d}{dx}\left(\int_0^1 \frac{\sin(xy)}{y}\,dx\right) = \int_0^1 \frac{\partial}{\partial x}\left(\frac{\sin(xy)}{y}\right)\,dx = \int_0^1 \cos(xy)\,dx = -\frac{\sin(x)}{x}$$

Writing again we got that:

$$\frac{d}{dx}\left(\int_0^1 \frac{\sin(xy)}{y}\,dx\right) = -\frac{\sin(x)}{x}$$

We so still need to show that we have used Leibniz's rule correctly and indeed by definition we know that:

$$\int_0^1 \frac{\sin(xy)}{y}\,dx = \lim_{\epsilon\to 0^+}\int_\epsilon^1 \frac{\sin(xy)}{y}\,dx$$

So we know we could use Leibniz's rule on the rectangles $[\epsilon, 1] \times [0, 1]$ and now we know that our result holds.

A function $f(x,y)$ is called homogeneous of degree $0 \leq m$ if:

$$f(tx,ty) = t^m f(x,y)$$

For every $t > 0$ and every $(x,y) \in \mathbb{R}^2$.

A homogeneous function of degree m has a limit at the origin if and only if it is constant

By definition we get that:

$$f(tx,ty) = t^0 f(x,y) = f(x,y)$$

For every $t > 0$, if the function is constant then it has a limit at the origin which is the constant. For the other direction if we assume the function has a limit at the origin we can write:

$$\lim_{(x,y)\to(0,0)} f(x,y) = L$$

Using the homogeneity we get that for any $t > 0$:

$$f(x,y) = f\left(\frac{x}{t}, \frac{y}{t}\right)$$

Which means that:

$$f(x,y) = \lim_{t\to\infty} f\left(\frac{x}{t}, \frac{y}{t}\right) = L$$

Which means the function is constant.

Show that a homogeneous function of degree $0 \leq m$ with continuous dervatives satisfies:

$$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = mf(x,y)$$

For every $(x,y) \neq (0,0)$

Suppose we look at (x_0, y_0) since we see we need to do something with the chain rule later we define a new function:

$$\gamma(t) = (tx_0, ty_0)$$

Since f is homogeneous of degree m:

$$f \circ \gamma(t) = f(tx_0, ty_0) = t^m f(x_0, y_0)$$

We will now derive both sides while using the chain rule and we get:

$$\frac{d}{dt} f \circ \gamma(t) = x_0 \frac{\partial f}{\partial x}(x_0, y_0) + y_0 \frac{\partial f}{\partial y}(x_0, y_0)$$
$$\frac{d}{dt} t^m f(x_0, y_0) = mt^{m-1} f(x_0, y_0)$$

Since this is true for any $t > 0$ if we substitute $t = 1$ we get:

$$x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y) = mf(x,y)$$

18 Leibniz's Theorem and Fubini's Theorem

$$\int_0^{\frac{\pi}{2}} (a^2 \cos^2(x) + b^2 \sin^2(x))\, dx$$

We are advised to consider the following function:

$$F(y) = \int_0^{\frac{\pi}{2}} (y^2 \cos^2(x) + b^2 \sin^2(x))\, dx$$

Ok this can be solved but not now.

Some more integrals...

Show that for any $p > 1$ and for any $m \in \mathbb{N}$ that:

$$\int_0^1 x^p (\ln x)^m \, dx = \frac{(-1)^m m!}{(p+1)^{m+1}}$$

We may try to solve this intergal by applying integration by parts multiple times, but there is a more elegant way to solve this by noticing:

$$a^t = e^{t\ln(a)} \quad \text{and} \quad (a^t)' = \ln(a)e^{t\ln(a)} = \ln(a)a^t$$

We can see that:

$$\frac{d^m}{dt^m}a^t = \ln^m(a)a^t$$

So if we define $f(x,y) = x^y$ we get:

$$\frac{d^m}{dy^m}f(x,y) = x^y(\ln x)^m$$

Which means that our integrand is continuously differentiable for $x > 0$ and $y > 1$ and now we can define:

$$G_m(y) = \int_0^1 x^y(\ln x)^m \, dx = \int_0^1 \frac{d^m}{dy^m}f(x,y) \, dx$$

And:

$$F(y) = \int_0^1 x^y \, dx$$

Using Leibniz's rule on the rectangle $R = [0,1] \times [1,p]$ on $F(y)$ we get:

$$F^{(m)}(y) = G_m(y)$$

So we want to find $F^{(m)}(p)$ but we know that:

$$F(y) = \int_0^1 x^y \, dx = \left.\frac{x^{y+1}}{y+1}\right|_0^1 = \frac{1}{1+y}$$

And we can also calculate the mth derivative of F manually:

$$\begin{gathered} F'(y) = -\frac{1}{(y+1)^2} \\ F''(y) = \frac{2}{(y+1)^3} \\ \cdots \\ F^{(m)}(y) = \frac{(-1)^m m!}{(y+1)^{m+1}} \end{gathered}$$

So the result is:

$$\int_0^1 x^p(\ln x)^m \, dx = \frac{(-1)^m m!}{(p+1)^{m+1}}$$

As wanted.

19 Test 1

Check convergence for the following series and integral:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\sin\left(\frac{1}{n}\right)\right)^{\frac{2}{3}}$$

The intuition here is not very clear at first. We may want to use the comparison test somehow and we will. We notice a couple of things. First that from some index we will get that:

$$0<\frac{1}{n}<\sin\left(\frac{1}{n}\right)$$

Which means that the series is positive from some place. And also from L'Hôpital's rule we can see that:

$$\lim_{x\to 0^+}\frac{x-\sin(x)}{x^3}=\lim_{x\to 0^+}\frac{1-\cos(x)}{3x^2}=\lim_{x\to 0^+}\frac{\sin(x)}{6x}=\frac{1}{6}$$

And more precisely we get that:

$$\lim_{n\to\infty}\frac{\left(\frac{1}{n}-\sin\left(\frac{1}{n}\right)\right)^{\frac{2}{3}}}{1/n^2}=\left(\frac{1}{6}\right)^{\frac{2}{3}}$$

This means that by the limit comparison test our series converges because we know that the series:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Converges.

$$\int_1^\infty \ln\left(1+\frac{\sin x}{x}\right)\,dx$$

We are advised to first show that the improper integral:

$$\int_1^\infty \left(\ln\left(1+\frac{\sin x}{x}\right)-\frac{\sin x}{x}\right)\,dx$$

Absolutely converges. We know that near $x = 0$ we have:

$$\ln(1+x) = x - \frac{x^2}{2} + o(x^2)$$

Now we can use a similar technique to the one from the previous question and consider:

$$\lim_{x\to\infty} \frac{\ln(1+x)-x}{x^2} = -\frac{1}{2}$$

Now we get that:

$$\lim_{x\to\infty} \frac{\ln(1+\frac{\sin x}{x})-\frac{\sin x}{x}}{(\frac{\sin x}{x})^2} = -\frac{1}{2}$$

We know that the nominator is negative for a small enough delta and thus we can you the limit comparison test with:

$$\int_1^\infty \left(\frac{\sin x}{x}\right)^2\,dx$$

and know that:

$$\int_1^\infty \left(\ln\left(1+\frac{\sin x}{x}\right)-\frac{\sin x}{x}\right)\,dx$$

Absolutely converges. Because from Dirichlet's test we know that:

$$\int_1^\infty \frac{\sin x}{x}\,dx$$

Converges we can use improper integral arithmetic to get that:

$$\int_1^\infty \ln\left(1+\frac{\sin x}{x}\right)\,dx$$

Converges.

Prove or disprove the following claim:
Let $\sum a_n$ converge and b_n be bounded then $\sum a_n b_n$ also converges.
This is not true because we can choose:

$$a_n = \frac{(-1)^n}{n}$$
$$b_n = (-1)^n$$

It is clear that the conditions are satisfied and we see that:

$$\sum a_n b_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

Which does not converge.

If $f(x, y)$ is defined around the origin and has partial derivatives in the origin then it is continuous there
This is false. We can consider the following counter example:

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

The partial derivatives exist because by definition:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = 0$$

Similarly the other partial derivative exists but we get that:

$$\lim_{t \to 0} f(t, t) = \frac{1}{2}$$

So the function is not continuous at the origin.

Let $f(x)$ have a derivative $f'(0) = a \neq 0$ and $f(0) = 0$ Show that any positive sequence a_n that converges to 0 satisfies:

$$\sum_{n=1}^{\infty} a_n \quad \textbf{Converges} \iff \sum_{n=1}^{\infty} f(a_n) \quad \textbf{Converges}$$

Let a_n converge to 0, we know that since:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = a$$

That:

$$\lim_{n \to \infty} \frac{f(a_n)}{a_n} = a$$

Suppose that $a > 0$ we get that by the limit comparison that that from some point $f(a_n) > 0$ and we know $a_n > 0$ so we get that:

$$\sum_{n=1}^{\infty} a_n \quad \text{Converges} \iff \sum_{n=1}^{\infty} f(a_n) \quad \text{Converges}$$

If $a < 0$ we can change f with $-f$ to get the same result.

Give an example for a function $f(x)$ such that $f(0) = f'(0) = 0$ and $f''(0) \neq 0$ and a positive sequence a_n such that:

$$\sum_{n=1}^{\infty} a_n \quad \textbf{Diverges and} \quad \sum_{n=1}^{\infty} f(a_n) \quad \textbf{Converges}$$

We can choose:

$$f(x) = x^2$$
$$a_n = \frac{1}{n}$$

We get that:

$$f(0) = f'(0) = 0 \quad \text{and} \quad f''(0) = 2 \neq 0$$

And of course:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{Diverges and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{Converges}$$

Prove the following series converges for any $x \in [0, \infty)$ and find the sum:

$$\sum_{n=0}^{\infty} \frac{x}{(1+x)^n}$$

For $x = 0$ we get that the sum is 0 and otherwise we get a geometric series with a factor of $\frac{1}{1+x}$ with a sum of:

$$\frac{a_1}{1-q} = \frac{x}{1-\frac{1}{1+x}} = 1+x$$

Which means that:

$$s(x) = \begin{cases} 0 & x = 0 \\ 1+x & x \in (0, \infty) \end{cases}$$

Show that the series does not uniformly converge on $[0, 1]$

There are two ways to show this. First, to consider uniform convergence we want to consider the series as a sequence of functions and we denote:

$$s_m(x) = \sum_{n=0}^{m} \frac{x}{(1+x)^n}$$

And now we can see that:

$$s(x) - s_m(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{(1+x)^m} & x \in (0, \infty) \end{cases}$$

In this case:

$$\sup_{x\in[0,1]} s(x) - s_m(x) = \sup_{x\in(0,1]} \frac{1}{(1+x)^m} = 1 \neq 0$$

Which means the convergence is not uniform. The other way to solve this is by considering that all the functions $s_m(x)$ are continuous which means that if the convergence were uniform then $s(x)$ must also be continuous, but since it is not continuous we can say that the convergence was also not uniform.

Prove that the series:

$$\sum_{n=0}^{\infty} \frac{x^2}{(1+x)^n}$$

Converges uniformly on $[0,1]$

This can be done in two ways as well. It would be much better for us by now to think first of all about Dini's theorem. The function series is non-negative, all the functions are continuous and the point-wise limit of the function is:

$$s(x) = \begin{cases} 0 & x = 0 \\ x(1+x) & x \in (0,\infty) \end{cases}$$

Which is continuous. By Dini's theorem, the convergence is uniform. The other way would be to calculate directly of course like we did earlier but there is no need to show that calculation or do that at all if you were comfortable with the previous direct calculation.

Let $0 < a < b$ be constant and define $f(x,y) = x^y$ for any $(x,y) \in [0,1] \times [a,b]$ prove by calculation that for any $x \in [0,1]$:

$$\int_a^b x^y \, dy = I(x) = \begin{cases} 0 & x = 0 \\ b - a & x = 1 \\ \frac{x^b - x^a}{\ln x} & x \in (0,1) \end{cases}$$

And explain why $I(x)$ is continuous on $[0,1]$

It may be hard to notice but the question actually states that we should solve this by calculation, which may seem tiring but we can see that in this case it's quite the opposite. When working with exponentials it's always a good idea to consider the exponent function e^x and this indeed solves:

$$\int_a^b x^y \, dy = \int_a^b e^{y \ln x} \, dy = \frac{e^{y \ln x}}{\ln x}\bigg|_{y=a}^{y=b} = \frac{e^{b \ln x}}{\ln x} - \frac{e^{a \ln x}}{\ln x} = \frac{x^b - x^a}{\ln x}$$

We can also notice that the edge cases are trivial:

$$\int_a^b 0^y \, dx = 0$$

$$\int_a^b 1^y \, dx = b - a$$

If we want to show that $I(x)$ is continuous we can use the theorem from class that if $f(x,y)$ is continuous then:

$$\int_a^b x^y \, dy = I(x)$$

Is also continuous. We can also calculate directly but that doesn't yield much merit here.

Use the previous exercise and Fubini's theorem to find:

$$\int_0^1 I(x)\,dx$$

Fubini's theorem states that if $f(x, y)$ is continuous on the relevant rectangle in this case $[0, 1] \times [a, b]$ which as we can know, it is, then change the order of the integrals and get:

$$\int_0^1 I(x)\,dx = \int_0^1 \int_a^b x^y\,dy\,dx = \int_a^b \int_0^1 x^y\,dx\,dy = \int_a^b \frac{x^{y+1}}{y+1}\bigg|_{x=0}^{x=1} dy = \int_a^b \frac{dy}{y+1} =$$

Finally:

$$\int_a^b \frac{dy}{y+1} = \ln\left(\frac{b+1}{a+1}\right)$$

20 More practice

Saving for volume integrals, the Jacobian matrix and some more fun stuff we are mostly done with this course's matrial, so we are going to solve miscellaneous exercises without consideration for their order. We can start with a straight forward one using the chain rule.

Define the function:

$$f(3u^2+v,-6u+v^3)=e^{u-v}$$

We are asked to calculate the equation of the tangential plane at the point $(4,-5)$, and also the directional derivative there with the direction $u=(3,-4)$. To calculate the partial derivative at the point $(x,y)=(4,-5)$ we can also calculate the partial derivatives of the function:

$$f(3u^2+v,-6u+v^3)=e^{u-v}$$

At $(1,1)$ and we can do that using the chain rule. To help us we can denote:

$$\begin{aligned}
&f(x(u,v),y(u,v))=e^{u-v}\\
&x(u,v)=3u^2+v\\
&y(u,v)=-6u+v^3
\end{aligned}$$

And the chain rule gives:

$$\frac{\partial f}{\partial u}(1,1)=\frac{\partial f}{\partial x}(4,-5)\frac{\partial x}{\partial u}(1,1)+\frac{\partial f}{\partial y}(4,-5)\frac{\partial y}{\partial u}(1,1)=\frac{\partial f}{\partial x}(4,-5)(6)+\frac{\partial f}{\partial y}(4,-5)(-6)$$

$$\frac{\partial f}{\partial v}(1,1)=\frac{\partial f}{\partial x}(4,-5)\frac{\partial x}{\partial v}(1,1)+\frac{\partial f}{\partial y}(4,-5)\frac{\partial y}{\partial v}(1,1)=\frac{\partial f}{\partial x}(4,-5)(1)+\frac{\partial f}{\partial y}(4,-5)(3)$$

But we also get that the partial derivatives are equal to:

$$\begin{aligned}
\frac{\partial e^{u-v}}{\partial u}(1,1)&=1\\
\frac{\partial e^{u-v}}{\partial v}(1,1)&=-1
\end{aligned}$$

Which means we now have a system of equations:

$$\begin{cases}6\frac{\partial f}{\partial x}(4,-5)-6\frac{\partial f}{\partial y}(4,-5) & =1\\ \frac{\partial f}{\partial x}(4,-5)+3\frac{\partial f}{\partial y}(4,-5) & =-1\end{cases}$$

Solving that we can find that:

$$\nabla f(4,-5)=\left(-\frac{1}{8},-\frac{7}{24}\right)$$

This solves the first part of the question, and to solve the second part we can just use calculate:

$$\nabla f(4,-5)\cdot\frac{u}{\|u\|}$$

Prove the following inequality:

$$\left(\int_0^x e^{-t^2}\,dt\right)^2 + \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\,dt = \frac{\pi}{4}$$

We can see that this is actually just the parameter integral:

$$F(x) = \left(\int_0^x e^{-t^2}\,dt\right)^2 + \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\,dt$$

In this case we can prove this by showing that the derivative of this function in respect to x is 0. Using Leibniz's rule we get that the derivative is:

$$2e^{-x^2}\int_0^x e^{-t^2}\,dt + \int_0^1 -2xe^{-x^2(t^2+1)}\,dt$$

Notice, how we used the fundamental theorem of calculus here. Now applying variable substitution on the second integral $u = xt$ gives:

$$2e^{-x^2}\int_0^x e^{-t^2}\,dt - 2e^{-x^2}\int_0^x e^{-u^2}\,du = 0$$

So the original function is constant. Now if we calculate $F(0)$ we see:

$$F(0) = \left(\int_0^0 e^{-t^2}\,dt\right)^2 + \int_0^1 \frac{e^{-0^2(t^2+1)}}{t^2+1}\,dt = \int_0^1 \frac{1}{t^2+1}\,dt = \arctan(x)|_0^1 = \frac{\pi}{4}$$

Which completes the proof.

Find the value of the integral for any natural $n \in \mathbb{N}$:

$$\int_0^1 x^n \ln(x)\, dx$$

We did a similar question to this one in the pase and indeed we can see that if we define:

$$f(x,t) = x^t$$

The partial derivative is:

$$\frac{\partial f}{\partial t} = x^t \ln(x)$$

And then:

$$\int_0^1 x^n \ln(x)\, dx = \int_0^1 \frac{\partial}{\partial t} f(x,t)\, dx = \frac{d}{dt} \int_0^1 x^t\, dx = \frac{d}{dt} \frac{1}{t+1} = -\frac{1}{(1+t)^2}$$

And we are done.

Find the value of the integral:

$$\int_0^1 \int_0^1 e^{x\sqrt{y}}\,dy\,dx$$

This isn't going to be easy to integrate first by y but since the integrand is continuous on $[0,1] \times [0,1]$ we can use Fubini's theorem:

$$\int_0^1 \int_0^1 e^{x\sqrt{y}}\,dy\,dx = \int_0^1 \int_0^1 e^{x\sqrt{y}}\,dx\,dy = \int_0^1 \frac{e^{\sqrt{y}} - 1}{\sqrt{y}}\,dy$$

Now using substitution $u = \sqrt{y}$ we get:

$$\int_0^1 2(e^u - 1)\,du = 2(e-2)$$

We need to find the convergence or divergence of the following improper integral:

$$\int_{2}^{\infty} e^{-(\ln x)^{\frac{1}{2}}}\, dx$$

We mya try to do integral substitution $u = \ln x$ and get:

$$\int_{\ln 2}^{\infty} e^{u-\sqrt{u}}\, du$$

But we can notice:

$$\lim_{u\to\infty} e^{u-\sqrt{u}} = \infty$$

Which means that the integral diverges.

Find the anti-derivative of the following function:

$$f(x) = \frac{x - \ln(1+x)}{x(1+x)\ln(1+x)}$$

We might try to think that using the series expansion would be a good idea but since we have division here we probably need to think about something else. Well, first of all by the linearity of the integral:

$$\int f(x)\,dx = \int \frac{1}{(1+x)\ln(1+x)}\,dx - \int \frac{1}{x(1+x)}\,dx$$

And calculating them seperately:

$$\int \frac{1}{x(1+x)}\,dx = \ln|x| - \ln|1+x| + C$$

And using the substitution $u = \ln(1+x)$ we get:

$$\int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln(1+x)| + C$$

So finally:

$$\int f(x)\,dx = \ln|\ln(1+x)| - \ln|x| - \ln|1+x| + C$$

Disprove the following claim: "If f is bounded on $[0,1]$ and the limit:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$

exists, then f is integrable on $[0,1]$"
This is not true because we can define the function:

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

Because then we know that the function is not integrable on $[0,1]$ but still the limit:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 0 = 0$$

exists.

Disprove the following claim: "If the series

$$\sum_{n=1}^{\infty} a_n$$

converges then the series

$$\sum_{n=1}^{\infty} a_n^4$$

Also converges"

This is not true because we can consider the sequence:

$$a_n = \frac{(-1)^n}{\sqrt[4]{n}}$$

The series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$$

Converges because the sequence $\frac{1}{\sqrt[4]{n}}$ is positive and monotonically decreasing so by Leibniz's test it converges.
But if we consider:

$$\sum_{n=1}^{\infty} a_n^4 = \sum_{n=1}^{\infty} \frac{1}{n}$$

It is the harmonic sum and thus does not converge.

Prove that for any $a > 0$ that the function series:

$$\sum_{n=1}^{\infty} \frac{x}{x^2+n^2}$$

Converges uniformly on $[-a, a]$
We can do that using Weirstrass's M-test. We can see that:

$$\frac{x}{x^2+n^2} \leq \frac{a}{n^2}$$

And the series:

$$\sum_{n=1}^{\infty} \frac{a}{n^2}$$

Converges. Thus we can deduce unifrom convergence on $[-a, a]$ as wanted. We can see that it is does not uniformly converge on $\mathbb{R}$ because if it did then for every $\varepsilon > 0$ we would get that exists a natural N such that for every $m > n > N$ we would get:

$$\sup_{x \in \mathbb{R}} \left| \sum_{j=n+1}^{2n} \frac{x}{x^2+j^2} \right| < \varepsilon$$

But this is not the case since if we choose $x = n$ we would get:

$$\lim_{n\to\infty} \sum_{j=n+1}^{2n} \frac{n}{n^2+j^2} = \lim_{n\to\infty} \sum_{j=n+1}^{2n} \frac{1}{n} \frac{1}{1+(\frac{j}{n})^2} = \int_1^2 \frac{1}{1+x^2}\, dx = \arctan(2) - \arctan(1) > 0$$

Which ends the proof.

Check convergence for the series:

$$\sum_{n=2}^{\infty} \frac{(-1)^{\lfloor n/2 \rfloor}}{n \ln n}$$

We can check that this series convergence according to Dirichlet's test. We know that the partial sums:

$$\sum_{n=2}^{m} (-1)^{\lfloor n/2 \rfloor}$$

Is bounded, and we know that:

$$\frac{1}{n \ln n}$$

Is monotonic and converges to 0 which means that our series converges.

Check for convergence for the following improper integral:

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{\pi}{2}-x}}{\ln(1+\cos x)}\,dx$$

We can see that the problamatic point is $x=\frac{\pi}{2}$ and so we need to calculate the limit of the function there. We will use L'Hôpital's rule:

$$\lim_{x\to\frac{\pi}{2}^+} \frac{\sqrt{\frac{\pi}{2}-x}}{\ln(1+\cos x)} = \lim_{x\to\frac{\pi}{2}^+} \frac{1+\cos(x)}{2\sin(x)\sqrt{\frac{\pi}{2}-x}} = \infty$$

Find the following integral

$$\int x \arctan(x)\, dx$$

We can do that by using integration by parts because we want to get rid of the arctan part:

$$\int x \arctan(x)\, dx = \frac{x^2}{2} \arctan(x) - \int \frac{x^2}{2(1+x^2)}\, dx$$

Now we need to solve the more simple second integral:

$$\int \frac{x^2}{2(1+x^2)}\, dx = \frac{1}{2} \int \frac{x^2}{(1+x^2)}\, dx$$

And we may want to use integration by parts, but that is a grave mistake:

$$\int \frac{x^2}{(1+x^2)}\, dx = \int 1 - \frac{1}{(1+x^2)}\, dx = x - \arctan(x) + C$$

Now combining all the results we get:

$$\int x \arctan(x)\, dx = \frac{x^2}{2} \arctan(x) - \frac{x - \arctan(x)}{2} + C$$

Or if we choose to simplify we get:

$$\int x \arctan(x)\, dx = \frac{(x^2+1) \arctan(x) - x}{2} + C$$