

Riemann Surfaces

Based on lectures by
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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1	Introduction	3
2	Introduction to Teichmuller spaces	5

1 Introduction

Definition 1.1 (Riemann surface). A Riemann surface is a 1-dimensional complex manifold.

Definition 1.2 (Riemann surface). A Riemann surface is a topological space X together with open subsets $\{U_k\}_{k \in I}$ of X with $\cup_{k \in I} U_k = X$ together with maps $f_i: U_i \rightarrow \mathbb{C}$ such that

- (1) Each f_i is a homeomorphism onto its image.
- (2) If $U_i \cap U_j \neq \emptyset$ then $f_i \circ f_j^{-1}: f_j(U_i \cap U_j) \rightarrow f_i(U_i \cap U_j)$ are *biholomorphic*.

Remark 1.1. A function $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic at p if $f'(p) = \lim_{z \rightarrow p} \frac{f(z) - f(p)}{z - p}$ exists.

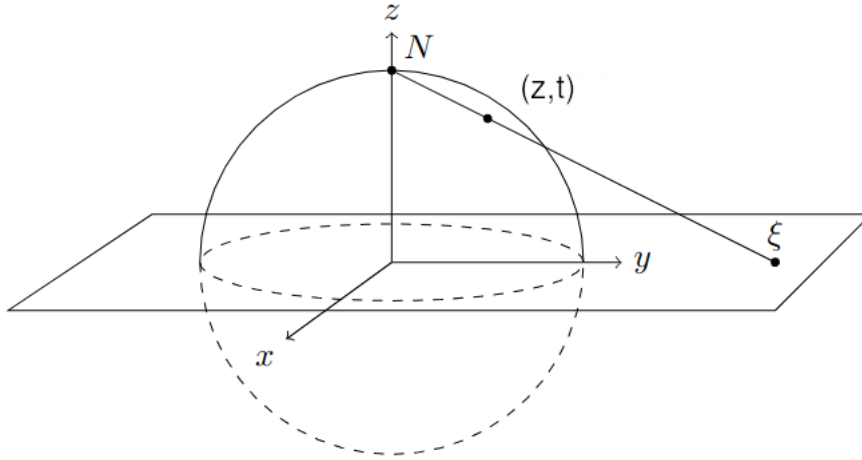
Definition 1.3 (Biholomorphism). A function $f: \mathbb{C} \rightarrow \mathbb{C}$ is called biholomorphic if it has an inverse and both f and f' are holomorphic.

Definition 1.4 (Atlas). The $\{(U_i, f_i)\}_{i \in I}$ are called an atlas of the Riemann surface.

Definition 1.5 (Chart). Each individual (U_i, f_i) is called a chart of the Riemann surface.

Example 1.1. Let $U \subset \mathbb{C}$. Then U can take an atlas with one chart which is the identity map.

Example 1.2 (Riemann sphere). Let $X = \{(z, t) \in \mathbb{C} \times \mathbb{R} : |z|^2 + t^2 = \mathbb{R}\}$. We identify \mathbb{C} with the xy plane. Denote N and S the north and south poles of the sphere accordingly. We define $\pi_N: \mathbb{C} \rightarrow S$ such that π_N sends each point (z, t) on the sphere to its stereographic projection from N onto the plane (point ξ) as can be seen in the figure below:



We can similarly define π_S and verify that the images of the projections are $X \setminus \{N\}$ and $X \setminus \{S\}$ accordingly.

Now X is a Riemann surface with an atlas consisting of $\pi_S: X \setminus \{S\} \rightarrow \mathbb{C}$ and $\pi_N: X \setminus \{N\} \rightarrow \mathbb{C}$. We denote the Riemann sphere as $\hat{\mathbb{C}}$.

Definition 1.6 (Biholomorphism of Riemann surfaces). Let $(X, (U_i, f_i))$, $(Y, (W_i, g_i))$ be two Riemann surfaces. A biholomorphism between them is a homeomorphism $X \xrightarrow{\phi} Y$ such that $g_i \circ \phi \circ f_i^{-1}$ are biholomorphisms on their domains of definition.

A main problem in Riemann surfaces was classifying certain types of Riemann surfaces up to biholomorphisms.

Theorem 1.1. (Riemann mapping theorem). Any two proper open simply connected subsets of \mathbb{C} are biholomorphic.

A generalization of the Riemann mapping theorem is the uniformization theorem proved by Koebe in 1907.

Theorem 1.2. (Uniformization theorem). *Any simply connected Riemann surface is biholomorphic to one of the following:*

(1) \mathbb{C}

(2) $\hat{\mathbb{C}}$

(3) $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$

We will give a proof for this theorem in the end of the class.

A natural question that arises is what about non simply connected Riemann surfaces?

Theorem 1.3. (Uniformization theorem, part II). *Any connected Riemann surface is biholomorphic either to $\hat{\mathbb{C}}$ or to a quotient of \mathbb{C} or \mathbb{H} by a properly discontinuous torsion-free subgroup of biholomorphisms.*

Remark 1.2. Biholomorphisms of $U = \mathbb{C}$ or \mathbb{H} (or any subset of \mathbb{C}) forms a group under composition. We denote that group by $\text{Bih}(U)$.

Definition 1.7 (Properly discontinuous group). A countable subgroup of $\text{Bih}(U)$ is said to be properly discontinuous if for all compact $K \subseteq U$, the set $\{g \in G : gK \cap K \neq \emptyset\}$ is finite.

Definition 1.8 (torsion-free action). $G \subseteq \text{Bih}(U)$ is torsion-free if $gp = p$ for some $p \in U$ implies g is the identity.

Remark 1.3. Notice that multiplication in gp is the group action of g on the set U . That is $gp = g(p)$.

We can now define the quotient space U/G where $p \sim q$ if there exists $g \in G$ such that $gp = q$.

Introduce a topology on U/G which is the coarsest topology such that the canonical projections $U \rightarrow U/G$ are continuous.

Under the assumptions that G is properly discontinuous and torsion-free, U/G is a Riemann surface with the following charts. By assumptions on G , we can find for any $p \in U$ a neighbourhood W of $p \in U$ such that $\pi : U \rightarrow U/G$ is a homeomorphism onto its image when restricted to W .

So, restrictions of π to these neighbourhoods W provide an atlas.

Definition 1.9 (Free action). Let G

2 Introduction to Teichmuller spaces

Let S be a topological space. Then

$$\text{Modulispace}(S) = \{\text{Riemann surfaces homeomorphic to } S \text{ up to biholomorphisms}\}$$

Definition 2.1 (Teichmuller space). We consider pairs (X, f) where X is a Riemann space and $f \rightarrow X$ is a homeomorphism. Then

$$\text{Teich}(S) = \{(X, f) : f : S \rightarrow X\} / \sim$$

where $(X_1, f_1) \sim (X_2, f_2)$ if there exists a biholomorphism $X_1 \xrightarrow{\phi} X_2$ such that $f_2 \circ f_1^{-1}$ is homotopic to ϕ . In other words commutes up to homotopy.

Remark 2.1. The pair (X, f) is called a *marking* for S .

Definition 2.2 (Homotopy). We say that two continuous functions f, g from a topological space X to Y are homotopic if there exists a continuous function $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Definition 2.3 (Mapping class group). First we define:

$$\begin{aligned} \text{Homeo}^+(S) &= \{\text{Orientation preserving homeomorphisms } S \rightarrow S\} \\ \text{Homeo}^0 \triangleleft \text{Homeo}^+(S) &= \{\text{the homeo. homotopic to the identity.}\} \end{aligned}$$

And now we define

$$\text{MCG}(S) = \text{Homeo}^+(S) / \text{Homeo}^0(S) = \frac{\text{Diff}^+(S)}{\text{Diff}^0(S)}.$$

We have that φ acts on $\text{Teich}(S)$ by $\varphi \in \text{Homeo}^+(S)$, $[\varphi] \in \text{MCG}(S)$ as such

$$[\varphi][(X, f)] = [(X, f \circ e^{-1})].$$

Now $\mathcal{M}(S) = \text{Teich}(S) / \text{MCG}(S)$. Our goal is to determine $\mathcal{M}(S)$ and $\text{MCG}(S)$ when S is a torus.

Theorem 2.1. Assume S is a torus $T \cong \mathbb{R}^2 / \mathbb{Z}^2$. Then $\text{MCG}(S) = \text{SL}_2(\mathbb{Z})$ acting linearly on the torus. $\mathcal{M}(S)$ can be identified with $\mathbb{H} / \text{SL}_2(\mathbb{Z})$ where the action is by Mobius transformations.

Proof. Any Riemann surfaces homeomorphic to S has to form \mathbb{C} / Λ where $\Lambda \subseteq \text{Bih}(\mathbb{C})$ is properly disc, free, and $\Lambda \langle z \rightarrow z\tau_1, z \rightarrow z + \tau_2 \rangle$ where $\tau_1, \tau_2 \notin \mathbb{R}$. We have to determine when different \mathbb{C} / Λ are biholomorphic. We can also write \mathbb{C} / Λ where $\Lambda = \langle \tau_1, \tau_2 \rangle \subseteq \mathbb{C}$. If $\Lambda_1 = \langle \tau_1, \tau_2 \rangle$, $\Lambda_2 = \langle c\tau_1, c\tau_2 \rangle$ for $c \in \mathbb{C}$. Then $\mathbb{C} / \Lambda_1 \rightarrow \mathbb{C} / \Lambda_2$ is given by $[z] \rightarrow [cz]$.

For any $\tau_1, \tau_2 \in \mathbb{C}$ there exists $c \in \mathbb{C}^\times$ such that (up to change in order)

This tells us that any Riemann torus is biholomorphic to $\mathbb{C} / \langle X, X \rangle$ where $\tau \in \mathbb{H}$.

So we have a surjection $\mathbb{H} \rightarrow \mathcal{M}(S)$. □

Theorem 2.2. $\mathbb{C} / \langle 1, \tau_1 \rangle$ is biholomorphic to $\mathbb{C} / \langle 1, \tau_2 \rangle$ iff exists $A \in \text{SL}_2(\mathbb{Z})$ such that writing τ_1, τ_2 as elements of \mathbb{R}^2 , we have $A\tau_1 = \tau_2$.

Proof. Suppose there exists a biholomorphism $f: \mathbb{C} / \langle 1, \tau_1 \rangle \rightarrow \mathbb{C} / \langle 1, \tau_2 \rangle$. Let $\bar{f}: \mathbb{C} \rightarrow \mathbb{C}$ be a lift of f . This means $f(g + x) - f(x) \in \langle 1, \tau_1 \rangle$ whenever $g \in \langle 1, \tau_1 \rangle$.

$$\begin{aligned} f: \mathbb{C} / \langle 1, \tau_1 \rangle &\rightarrow \mathbb{C} / \langle 1, \tau_2 \rangle \text{ bih.} \\ \bar{f}: \mathbb{C} &\rightarrow \mathbb{C} \text{ lift.} \end{aligned}$$

By post composing with a biholomorphism of \mathbb{C} we can assume $\bar{f}(0) = 0$. We know that $\bar{f}(\tau_2)$ and $\bar{f}(1)$ are equivalent mod $\langle 1, \tau_1 \rangle$. \square

Remark 2.2. Let S be a Riemann surface. Recall that $\mathcal{M}(S)$ is the moduli space of S , which is the space of Riemann surfaces homeo to S up to biholomorphism.

Recall that we define

$$\text{MCG}(S) = \text{Homeo}^+(S) / \text{Homeo}^0(S)$$

And the Teichmüller space of S

$$\text{Teich}(S) = \{(X, f): f: S \rightarrow X\} / \sim$$

where $(X_1, f_1) \sim (X_2, f_2)$ if there exists a biholomorphism $X_1 \xrightarrow{\phi} X_2$ such that $f_2 \circ f_1^{-1}$ is homotopic to ϕ .

Our goal today is to show that for $T = \text{torus} = \mathbb{R}^2 / \mathbb{Z}^2$.

- $\text{MCG}(S) \cong \text{SL}_2(\mathbb{Z})$.
- $\mathcal{M}(S) \cong \mathbb{H} / \text{SL}_2(\mathbb{Z})$.
- $\text{Teich}(S) \cong \mathbb{H}$.

Notice that because of the relation

$$\text{Teich}(S) / \text{MCG}(S) = \mathcal{M}(S)$$

we only need to prove two of these propositions.

First let's prove that $\mathcal{M}(S) \cong \mathbb{H} / \text{SL}_2(\mathbb{Z})$. Any Riemann surface is homeomorphic to T (a complex torus) is biholomorphic to $\mathbb{C} / \langle \tau_1, \tau_2 \rangle$.

Recall that if $c \in \mathbb{C}^\times$ then $\mathbb{C} / \langle \tau_1, \tau_2 \rangle$ is biholomorphic to $\mathbb{C} / \langle c\tau_1, c\tau_2 \rangle$ via the map

$$f(z + \langle \tau_1, \tau_2 \rangle) = cz + \langle c\tau_1, c\tau_2 \rangle.$$

Up to changing the order of τ_1, τ_2 we can find $c \in \mathbb{C}^\times$ such that $c\tau_1 = 1$ and $c_2 \in \mathbb{H}$.

Our goal now is to determine when there exists a biholomorphism

$$\mathbb{C} / \langle 1, \tau_1 \rangle \mapsto \mathbb{C} / \langle 1, \tau_2 \rangle, \quad \tau_1, \tau_2 \in \mathbb{H}$$

Denote $\Lambda_\tau = \langle 1, \tau \rangle$, when $\tau \in \mathbb{H}$.

Theorem 2.3. *There exists a biholomorphism*

$$\mathbb{C} / \langle 1, \tau_1 \rangle \mapsto \mathbb{C} / \langle 1, \tau_2 \rangle, \quad \tau_1, \tau_2 \in \mathbb{H}$$

iff there exists

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$$

with $\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}$. This will imply that

$$\mathcal{M}(S) \cong \mathbb{H} / \mathrm{SL}_2(\mathbb{Z}).$$

Proof. Suppose there exists a biholomorphism

$$f: \mathbb{C} / \langle 1, \tau_1 \rangle \rightarrow \mathbb{C} / \langle 1, \tau_2 \rangle.$$

Lift it up to $\bar{f}: \mathbb{C} \rightarrow \mathbb{C}$. Then $\bar{f}(g+x) - \bar{f}(x) \in \Lambda_{\tau_1}$ whenever $x \in \mathbb{C}$, $g \in \Lambda_{\tau_2}$. We can replace \bar{f} with $\bar{f} - \bar{f}(0)$. We can assume that $\bar{f}(0) = 0$ (and is still a lift off).

Remark 2.3. Since \bar{f} is a lift off of f we have $\bar{f}(\Lambda_{\tau_2}) = \Lambda_{\tau_1}$.

We know that \bar{f} has form $\bar{f}(z) = az + b$ such that $a \in \mathbb{C}^\times$, $b \in \mathbb{C}$. As $\bar{f}(0) = 0$ we have $\bar{f}(z) = az$. Also, $\{0, 1, \tau_2\} \subseteq \Lambda_{\tau_2}$ so $0 = \bar{f}(0)$, $\bar{f}(1) = a$, $\bar{f}(\tau_2) = a\tau_2$ are in Λ_{τ_1} so we can write $a = \bar{f}(1)$. We now have

$$\tau_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and we still need to show it is an element of $\mathrm{SL}_2(\mathbb{Z})$. □