Introrduction to Metric and Topological Spaces

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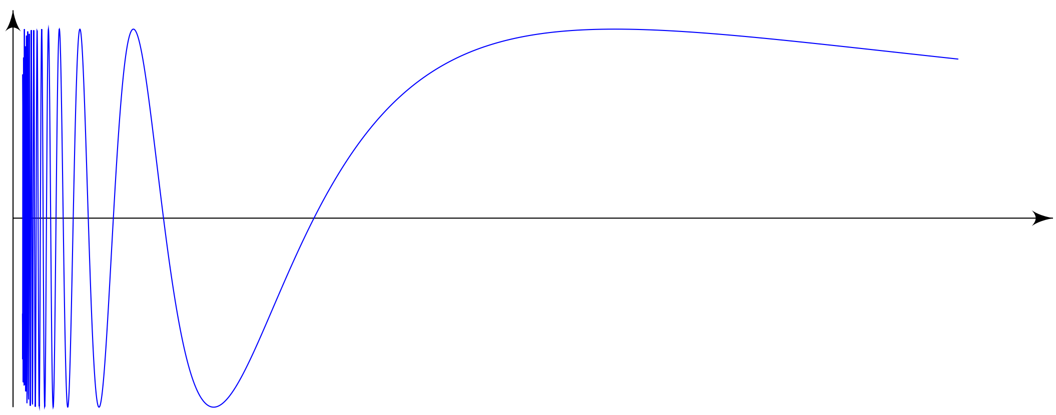
These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.



Contents

1 Metric Spaces

First we will begin with metric spaces.

Definition 1.1 (Metric space)**.** Let X be a non-empty set. A metric on X is a function $d\colon X \times X \to [0, \infty]$ such that for all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ (symmetry);

(3) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality);

The pair (X, d) is said to be a metric space.

1.1 Standard examples

Example 1.1. Let X be a non-empty set. Let $d\colon X \times X \to [0, \infty)$ be the function such that for $x, y \in X$,

$$d(x, y) := \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}.$$

The function d is a metric and it is called the discrete metric on X.

Example 1.2. Let $X = \mathbb{R}^n$ and define the function:

$$d(x - y) := \left| x - y \right|,$$

where $\left| \cdot \right| : \mathbb{R} \to \mathbb{R}$ is the Eclidean norm function. Then the pair (X, d) forms a metric space.

Example 1.3. Let (X, N) be an arbitrary normed space and define the function:

$$d(x - y) := N(x - y).$$

Then the pair (X, d) forms a metric space.

1.1.1 The L^p metric

Example 1.4. The pair $(C\left([0, 1]\right), d)$ such that $C([0, 1])$ is the space of all continuous functions on $[0, 1]$ paired with the metric:

$$d(f, g) = \int_0^1 \left| f(x) - g(x) \right| \mathrm{d}x$$

is also a metric space. This metric is called the L^1 metric.

Remark 1.1. In general, the p-metrics are induced by the p-norms, defined on $C\left([0, 1]\right)$ for every $1 \leq p < \infty$ as such:

$$d(f, g) = \int_0^1 \left| f(x) - g(x) \right|^p \mathrm{d}x.$$

Similarly we can define the L^∞ space on $C\left([0, 1]\right)$ as in the following example.

Example 1.5. The pair $(C([0, 1]), d)$ paired with the supremum metric:

$$d(f, g) = \sup_{x \in [0,1]} \left| f(x) - g(x) \right|$$

is also metric space. This metric is called the L^∞ metric.

Example 1.6. Let Λ be a nonempty set which will represent an alphabet. The set $\Lambda^{\mathbb{N}}$ represents all the sequences over that alphabet. The pair $(\Lambda^{\mathbb{N}}, d)$ with the metric d defined on two sequences $\omega = (\omega_n)_{n=1}^{\infty}, \eta = (\eta_n)_{n=1}^{\infty}$ as:

$$d(\omega, \eta) = \begin{cases} 2^{-\min\{n \geq 0 | \omega_n \neq \eta_n\}} & \omega \neq \eta \\ 0 & \omega = \eta \end{cases}$$

is also a metric space.

Example 1.7. Another simple way to construct a metric space is by constructing it from another space. Let (X, d) be a metric space and let $Y \subset X$. The pair (Y, d_Y) where d_Y is the metric d constrained to Y is also a metric space, and it is called a metric subspace of X.

1.2 Open sets

Definition 1.2 (Open ball)**.** Let (X, d) be a metric space. For $x \in X$ and $r > 0$ write:

$$B(x, r) := \left\{ y \in X \mid d(x, y) < r \right\}.$$

The set $B(x, r)$ is called the open ball in X with center x and radius r.

Definition 1.3 (Open subset)**.** A subset U of a metric space X is said to be open if for every $x \in X$ exists $r > 0$ such that $B(x, r) \subset X$.

Proposition 1.1. *Every open ball in X is an open subset of X.*

Proof. Let $B(x, r_x)$ be an open ball in X. Let $y \in B(x, r_x)$. Then for $r_y = r_x - d(x, y)$ we have that for every $z \in B(y, r_y)$ that

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + (r_x - d(x, y)) = r_x,$$

which means that $B(y, r_y) \subset B(x, r_x)$ which completes the proof. □

Proposition 1.2 (Properties of open subsets)**.** *The following properties are always satisfied:*

(1) $\emptyset$ and X are open;

(2) A union of open sets remains open;

(3) A finite intersection of open sets remains open;

These are the basic properties of open subsets, they can be verified directly from the definitions.

Proposition 1.3. *A subset U of X is open if and only if it is a countable union of open balls.*

Proof. Let U be a countable union of open balls, since every open ball is open, and a contable union of open subsets remains open, we get that U is open in X.

Let U be an open subset of X. Then for every $x \in X$ exists $r_x > 0$ such that $B(x, r_x) \subset U$. We have that

$$\bigcup_{x \in U} B(x, r_x) = U,$$

which completes the proof. □

Theorem 1.4. *Every nonempty open subset of $\mathbb{R}$ is a countable union of disjoint open intervals.*

Before we prove the following theorem, we need to prove a lemma.

Lemma 1.5. *Let $\{I_\alpha\}_{\alpha\in A}$ be a family of open intervals of $\mathbb{R}$. Suppose that $\cap_{\alpha\in A} I_\alpha \neq \emptyset$, then $\cap_{\alpha\in A} I_\alpha$ is an open interval.*

Proof. Let $x \in \cap_{\alpha\in A} I_\alpha$. For every I_α exists a_α, b_α such that $I_\alpha = (a_\alpha, b_\alpha)$. Denote

$$a = \inf_{\alpha\in A} a_\alpha \quad \text{and} \quad b = \sup_{\alpha\in A} b_\alpha.$$

We now clearly have that $\cup_{\alpha\in A} A_\alpha = (a, b)$, which completes the proof. □

We will now prove Theorem 1.4.

Proof. Let $U \subset \mathbb{R}$ be open and nonempty. For any $x \in U$ let I_x be the union of all open intervals $I \subset U$ with $x \in I$. From the previous lemma we have that I_x is an open interval for all $x \in U$. Consider the set

$$\mathcal{E} := \{I_x\}\, x \in U.$$

It is clear that $\cup_{I\in\mathcal{E}} I = U$. Notice that $I_x \neq I_y$ if and only if $I_x \cap I_y = \emptyset$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for every $I \in \mathcal{E}$ exists $q_I \in \mathbb{Q}$ such that $q \in I$. Since all the elements in $\mathcal{E}$ are disjoint, we have that

$$|\mathcal{E}| = \left|\{q_I \mid I \in \mathcal{E}\}\right| \leq |\mathbb{Q}| = |\mathbb{N}|\,,$$

which completes the proof. □

1.3 Sequences

Definition 1.4 (Convergence)**.** Let $\{x_n\}_{n\geq 1}$ be a sequence in X, and let $x \in X$. We say that $\{x_n\}_{n\geq 1}$ converges to x and write

$$x_n \xrightarrow{n\to\infty} x \quad \text{or} \quad \lim_{n\to\infty}$$

if for all $\epsilon > 0$ there exists $N \geq 1$ such that $d(x_n, x) \leq \epsilon$ for all $n \geq N$.

Proposition 1.6 (Uniqueness of the limit)**.** *Let $\{x_n\}_{n\geq 1}$ be a sequence in X and $x, x' \in X$ such that $\{x_n\}_{n\geq 1} \xrightarrow{n\to\infty} x$ and $\{x_n\}_{n\geq 1} \xrightarrow{n\to\infty} x'$. Then $x = x'$.*

Proof. For all $n \geq 1$ we have that

$$d(x, x') \leq d(x, x_n) + d(x_n, x') < \epsilon.$$

This shows that $d(x, x') \leq 0$ and thus $d(x, x') = 0$ and $x = x'$ as wanted. □

1.4 Closed sets

Definition 1.5 (Closed subset)**.** We say that $F \subset X$ is a closed subset of X if for every sequence $\{x_n\}_{n\geq 1} \subset F$ and $x \in X$ such that $\{x_n\}_{n\geq 1} \xrightarrow{n\to\infty} x$, we have that $x \in F$.

Proposition 1.7. *Let F be a subset of X. Then F is closed if and only if $X \setminus F$ is open.*

Proof. Suppose first that F is not closed. Then exists $\{x_n\}_{n\geq 1}$ and $x \in X$ such that $\{x_n\}_{n\geq 1}$ converges to x and $x \in X \setminus F$. Let $r > 0$, then we know that exists $N \geq 1$ such that for all $n > N$ we have

$$d(x_n, x) < r \quad \text{and} \quad x_n \in X$$

which shows that $X \setminus F$ is not open.

Suppose next that $X \setminus F$ is not open. Then exists a sequence $x \in X$ such that for all $\frac{1}{n} > 0$ exists $x_n \in F$ such that $d(x, x_n) \leq 1/n$. It follows that

$$\{x_n\}_{n\geq 1} \xrightarrow{n\to\infty} x \quad \text{and} \quad x \in X \setminus F$$

which shows that F is not closed which completes the proof. □

Proposition 1.8 (Properties of closed subsets)**.** *The following properties are always satisfied:*

(1) $\emptyset$ and X are closed;

(2) An intersection of closed sets remains closed;

(3) A finite union of closed sets remains closed;

These are the basic properties of closed subsets, they can be verified directly from the definitions.

Definition 1.6 (Closed ball)**.** Let (X, d) be a metric space. Let $x \in X$ and $r > 0$. We define

$$\overline{B}(x, r) := \left\{ y \in X \mid d(x, y) \leq B \right\}.$$

The set $\overline{B}(x, r)$ is called the closed ball in x with center x and radius r.

Proposition 1.9. *The set $\overline{B}(x, r)$ is a closed subset of X for all $x \in X$ and $r > 0$.*

Proof. It suffices to show that $X \setminus \overline{B}(x, r)$ is open. □

1.4.1 The middle third Cantor set

Example 1.8 (The middle third Cantor set)**.** Set $C_0 := [0, 1]$. Let C_1 be the set obtained by deleting the middle third of C_0, that is $C_1 := \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. We can continue this process infinitely many times:

$$\begin{aligned} C_0 &:= [0, 1] \\ C_1 &:= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\ C_2 &:= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\ &\vdots \end{aligned}$$

Since for every $n \in \mathbb{N}$ the set $\mathbb{N}$ is a finite union of closed sets, we have that C_n are closed for all $n \in \mathbb{N}$. It then follows that the set $C := \cap_{n \in \mathbb{N}} C_n$ is also closed. The set C is called the middle third Cantor set.

1.5 Functions

Definition 1.7 (Continuity)**.** Let $f \colon X \to Y$ be a function between two metric spaces. We say that f is continuous at $x \in X$ if for every $\{x_n\}_{n \geq 1}$

$$\{x_n\}_{n \geq 1} \xrightarrow{n \to \infty} x \implies \{f(x_n)\}_{n \geq 1} \xrightarrow{n \to \infty} f(x).$$

We say that f is continuous if it is continuous at x for all $x \in X$.

Proposition 1.10. *Let $f \colon X \to Y$ and $x \in X$ be given. Then f is continuous at x if and only if for every $\epsilon > 0$ there exists $\delta > 0$ so that $f(B(x, \delta)) \subset B(f(x), \epsilon)$.*

Proof. To be added. □

Proposition 1.11. *A mapping $f \colon X \to Y$ is continuous if and only if $f^{-1}(U)$ is open for every open $U \subset Y$.*

Proof. To be added. □

2 Topological Spaces

Definition 2.1 (Topological space)**.** Let X be a nonempty set. A collection $\tau \subset P(X)$ is said to be a topology on X if it satisfies the following properties,

(1) $X, \emptyset \in \tau$;

(2) Any union of sets in τ is a set in τ;

(3) Any finite intersection of sets in τ is a set in τ;

The pair (X, τ) is said to be a topological space and $U \in \tau$ an open set of (X, τ). An element $x \in X$ is said to be a point of (X, τ).

2.1 Standard examples

Example 2.1 (Topology induced by metric)**.** Every metric spaces can induce a topological space. Let (X, d) be a metric space. Define

$$\tau := \left\{ U \subset X \mid \forall x \in U \quad \exists \epsilon > 0 \text{ s.t. } B(x, \epsilon) \subset U \right\}.$$

It can be verified that (X, τ) is a topological space.

Definition 2.2 (Metrizable space)**.** We say that a topological space (X, τ) is metrizable, if exists a metric d on X, such that the topology that d induces on X is equal to τ.

Example 2.2 (Discrete topology)**.** Let X be a nonempty set and let $\tau := P(X)$. The topology τ is called the discrete topology and the space (X, τ) is called the discrete space. Is it metrizable?

Example 2.3 (Trivial topology)**.** Let X be a nonempty set and let $\tau := \{\emptyset, X\}$. The topology τ is called the trivial topology. Is it metrizable when $|X| = 1$? Is it metrizable when $|X| > 1$?

Example 2.4 (Finite complement topology)**.** Let X be any infinite set and let

$$\tau := \left\{ A \subset X \mid \left| X \setminus A \right| < \infty \right\} \cup \{\emptyset\}.$$

The topology τ is called the finite complement topology. Is it metrizable?

2.2 Continuous functions

Definition 2.3 (Continuity)**.** A mapping $f \colon X \to Y$ is said to be continuous if for every open set $U \subset Y$ the set $f^{-1}(U)$ is open.

Notice that the definition for continuity in topological spaces is equivalent to the defintion we gave in metric spaces, under the topology induced by the metric.

Definition 2.4 (Neighbourhood)**.** Given $x \in X$, an open $U \subset X$ containing x is said to be a neighbourhood of x.

Definition 2.5 (Continuiuty at a point)**.** A mapping $f \colon X \to Y$ is said to be continuous at x if for every neighbourhood U of $f(x)$ there exists a neighbourhood V of x such that $f(V) \subset U$.

Definition 2.6 (Open map)**.** A mapping $f \colon X \to Y$ is said to be open if $f(U)$ is open in Y for every open $U \subset X$.

2.3 Homeomorphisms

Definition 2.7 (Homeomorphism)**.** A mapping $f\colon X \to Y$ is said to be a homeomorphism if it is injective, surjective, continuous and open. If there exists such an f, then we say that X and Y are homeomorphic.

Proposition 2.1. *Let $f\colon X \to Y$ be continuous. It follows that f is a homoemorphism if and only if it has a continuous inverse.*

Remark 2.1. We say that a property P is a *topological property* if for every two homeomorphic spaces X and Y, then P holds for X if and only if it holds for P. The branch that deals with topological properties is called topology.

2.4 The subspace topology

Definition 2.8 (Subspace topology)**.** Let (X, τ_X) be a topological space and let $\emptyset \neq Y \subset X$. Define

$$\tau_Y = \left\{U \cap Y \mid U \in \tau_X\right\}$$

We call τ_Y the subspace topology, induced by τ_X on Y.

2.4.1 Characteristic property of the subspace topology

Theorem 2.2. (Characteristic property of the subspace topology). *Let (X, τ_X) be a topological space, let $\emptyset \neq Y \subset X$, and write τ_Y for the subspace topology on Y. Then τ_Y is the unique topology on Y which satisfies the following property. Let Z be a topological space and let $f\colon Z \to X$ be with $f(Z) \subset Y$. Then f is continuous as a map into (X, τ_X) if and only if it is continuous as a map into (Y, τ_Y).*

Throughout this section let X be a fixed topological space.

2.5 Closed sets

Definition 2.9 (Closed set)**.** A subset F of X is said to be closed if $F^c = X \setminus F$ is open.

Proposition 2.3 (Properties of closed sets)**.** *The following properties are always satisfied:*

(1) X, $\emptyset$ are closed;

(2) Any intersection of closed sets is closed;

(3) Any finite union of closed sets is closed;

Definition 2.10 (Closure)**.** Given $A \subset X$ we denote $\overline{A}$ to be the intersection of all $F \subset X$ such that $A \subset F$ and F is closed. We call $\overline{A}$ the closure of A.

Remark 2.2. We can also define the closure of A in an alternate way:

$$\overline{A} = \left\{x \in X \mid A \cap U \neq \emptyset \text{ for each neighbourhood } U \text{ of } x\right\}.$$

You may try to prove that both definitions are equivalent.

Definition 2.11 (Dense subset)**.** A subset A of X is said to be dense in X if $\overline{A} = X$.

Using the second definition of closure we get that A is dense in X if and only if $A \cap U \neq \emptyset$ for every nonempty $U \subset X$.

Definition 2.12 (Seperability)**.** We say that X is seperable if it has a countable dense subset.

Example 2.5. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$. Since $\mathbb{Q}$ is countable, it follows that $\mathbb{R}$ is seperable.

2.6 Isolated points and limit points

Definition 2.13 (Isolated point)**.** Let $A \subset X$. We say that x is an isolated point of A if exists U open in X such that $X \cap U = \{x\}$.

This is exactly the same as saying that x is an isolated point if and only if the singleton $\{x\}$ is open in the subspace topology.

Definition 2.14 (Limit point)**.** Let $A \subset X$. We say that x is a limit point of A if for every neighbourhood U of x there exists $a \in U \cap A$ with $a \neq x$. The set of all limit points of A is called the derived set of A and is denoted by $D(A)$.

Example 2.6. Consider the set $A = \left\{\frac{1}{n} \mid n \in \mathbb{Z}_+\right\} \cup \{0\}$ as a subset of $\mathbb{R}$. Then 0 is a limit point of A, and every other point in A is an isolated point.

Proposition 2.4. *Let $A \subset X$ be given, then*

1. $\overline{A} = A \cup D(A)$.

2. *A is closed if and only if $D(A) \subset A$.*

Proof. Let $x \in X$. Suppose that $x \notin \overline{A}$. It follows that exists a neighbourhood U of x such that $U \cap A = \emptyset$. This implies that $x \notin D(A)$ and thus $x \notin A \cup D(A)$.

Now suppose that $x \notin A \cup D(A)$. Since $x \notin D(A)$ exists a neighbourhood U of x such that $U \cap A \setminus \{x\} = \emptyset$. Since $x \notin A$ we have $A = A \setminus \{x\}$ and thus $U \cap A = \emptyset$. This shows that $x \notin \overline{A}$ which completes the proof of the first part.

For the second part, suppose that $D(A) \subset A$. This implies that $\overline{A} = A \cup D(A) = A$ which means that A is closed.

Next suppose that A is closed. This implies that $\overline{A} = A$ and thus $A = A \cup D(A)$ which implies that $D(A) \subset A$ and completes the proof of the second part. □

Corollary 2.5. *Let $A \subset X$ be closed and write $I(A)$ for the set of all isolated points of A. Then A is the disjoint union of $D(A)$ and $I(A)$.*

Proof. If A is closed then $A = \overline{A}$ and then

$$A = \overline{A} = A \cup D(A) = \big(A \setminus D(A)\big) \cup D(A) = I(A) \cup D(A).$$

The last equality follows directly from the definitions and the fact that $I(A)$ and $D(A)$ are disjoint too. □

2.7 The interior and boundary of a topological space

Definition 2.15 (Interior)**.** Let A be a subset of X. The interior of A is denoted by $\operatorname{Int}(A)$ and is defined to be the union of all open subsets U of X so that $U \subset A$. A point $x \in \operatorname{Int}(A)$ is said to be an interior point of A of a topological space.

Proposition 2.6. *Let $A \subset X$ be given, then*

1. $\operatorname{Int} A = X \setminus \overline{(X \setminus A)}$.

2. $\operatorname{Int}(A)$ *is open and contained in A.*

3. *A is open if and only if $A = \operatorname{Int}(A)$.*

Example 2.7. Considering $[a, b]$ as a subset of $\mathbb{R}$ we have $\operatorname{Int}([0, 1]) = (0, 1)$.

Proposition 2.7. *For $A \subset X$ we have*

(1) $\mathrm{Int}(A) = X \setminus \overline{X \setminus (X \setminus A)}$;

(2) $\mathrm{Int}(A)$ *is open and contained in* A*;*

(3) A *is open if and only if* $A = \mathrm{Int}(A)$*;*

Definition 2.16 (Boundary)**.** Let $A \subset X$ be given. A point $x \in X$ is said to be a boundary point of A if for every neighbourhood U of x we have $U \cap A \neq \emptyset$ and $U \cap (X \setminus A) \neq \emptyset$. The set of all boundary points of A is called the boundary of A and is denoted by ∂A.

Example 2.8. Considering $[a, b)$ as a subset of $\mathbb{R}$ we have $\partial A = \{a, b\}$.

Proposition 2.8. *For* $A \subset X$ *we have* $\partial A = \overline{A} \cap \overline{X \setminus A}$ *and in particular* ∂A *is closed.*

Proof. The equality follows directly from Definition 2.16 and Remark 2.2. We have that ∂A is closed as it it an intersection of two closed sets. □

Proposition 2.9. *Let* $A \subset X$ *be given, then* $\overline{A}$ *is the disjoint union of* $\mathrm{Int}(A)$ *and* ∂A*.*

2.8 Open bases

Definition 2.17 (Open base)**.** A family $\mathcal{B}$ of subsets of X is said to be an open base for X if for each open $U \subset X$ and $x \in U$ exists $B \in \mathcal{B}$ such that $x \in B \subset U$.

Remark 2.3. It is easy to see that a family $\mathcal{B}$ of open subsets of X is an open base for X if and only if each open $U \subset X$ is a union of elements of $\mathcal{B}$.

Proposition 2.10. *Let* Y *be a topological space, let* $\mathcal{B}$ *be an open base for* Y*, and let* $f\colon X \to Y$*. Suppose that* $f^{-1}(B)$ *is open for all* $B \in \mathcal{B}$*, then* f *is continuous.*

2.8.1 Countability

Definition 2.18 (Second countability)**.** We say that X is second countable, or that it satisfies the second axiom of countability, if it has a countable open base.

Proposition 2.11. *Suppose that* X *is second countable, then* X *is separable.*

Proof. Let $\mathcal{B}$ be a countable open base for X. We have that $\mathcal{B}\setminus\{\emptyset\}$ is also an open base. Choose an arbitrary $x_B \in B$ for each $B \in \mathcal{B}$. Since $\mathcal{B}$ is countable $\{x_B\}_{B\in\mathcal{B}}$ is also countable. Let U be open in X. By defintion of an open base exists $B \in \mathcal{B}$ such that $x_B \in B$ and $B \subset U$ so $x_B \in U$. This shows that $\{x_B\}_{B\in\mathcal{B}}$ is dense in X and thus X is seperable. □

Remark 2.4. In metric spaces, the property of being seperable and second countable is equivalent. If we denote (X, d) the metric space and A the countable dense set, then

$$\mathcal{B} = \big\{B(a, q) \mid a \in A \text{ and } q \in \mathbb{Q} \cap (0, \infty)\big\}$$

will form the desired countable open base for X.

Example 2.9. A classic example of a topological space that is seperable but not second countable is the Sorgenfrey line, also known as the lower limit topology, which we will discuss later.

Theorem 2.12. (Lindelöf's Theorem). *Suppose that* X *is second countable. Let* $\{U_i\}_{i\in I}$ *be a family of open subsets of* X*. Then there exists a countable* $I_0 \subset I$ *so that* $\cup_{i\in I_0} U_i = \cup_{i\in I} U_i$

Proof. Let $\mathcal{B}$ be a countable open base for X. Set,

$$\mathcal{B}_0 = \left\{B \in \mathcal{B} \mid B \subset U_i \text{ for some } i \in I\right\}$$

For each $B \in \mathcal{B}_0$ choose an arbitrary $i_B \in I$ such that $B \subset U_{i_B}$. Set $I_0 = \left\{i_B \mid B \in \mathcal{B}_0\right\}$. Since $\mathcal{B}$ is countable, I_0 is also countable. It remains to show that $\cup_{i\in I_0} U_i = \cup_{i\in I} U_i$. Let $x \in \cup_{i\in I} U_i$ then exists some j such that $x \in U_j$ since $\mathcal{B}$ is an open base exists $B \subset U_j$ such that $x \in B$. By definition we see that $B \in \mathcal{B}_0$, thus $i_B \in I_0$ and then:

$$x \in B \subset U_{i_B} \subset \cup_{i\in I_0} U_i$$

The other side of the inclusion is obvious which concludes the proof. □

Corollary 2.13. *Suppose that X is second countable and that $\mathcal{B}$ is an open base for X. Then exists a countable $\mathcal{B}_0 \subset \mathcal{B}$ which is also an open base for X.*

Proof. Let $\mathcal{E}$ be a countable open base for X. Since $\mathcal{B}$ is an open base, for each $E \in \mathcal{E}$ exists $\mathcal{B}_E \subset \mathcal{B}$ such that $E = \cup_{B\in\mathcal{B}_E} B$. From Lindelöf's theorem we get that exists a countable $\mathcal{B}_E^0 \subset \mathcal{B}_E$ such that $U_{B\in\mathcal{B}_E^0} = \cup_{i\in\mathcal{B}_E} U_i$. Now set $\mathcal{B}_0 = \cup_{E\in\mathcal{E}} \mathcal{B}_E^0$. It is countable as a countable union of countable sets. Moreover, since $\mathcal{E}$ is an open base, and since each $E \in \mathcal{E}$ is a union of elements from $\mathcal{B}_0$, it is clear that $\mathcal{B}_0$ is also an open base for X which completes the proof. □

Definition 2.19 (Open base at a point)**.** Let $x \in X$. A class of neighbourhoods B_x of x is called an open base at x if for every neighbourhood U of x exists $B \in B_x$ such that $B_x \subset U$.

Definition 2.20 (First countability)**.** We say that X is first countable, or that it satisfies the first axiom of countability, if for each $x \in X$ there exists a countable open base at x.

Remark 2.5. It is clear that if X is second countable, it is also first countable.

Example 2.10. Let (X, d) be a metric space. For each $x \in X$ the collection

$$\left\{B\left(x, 1/n\right) \mid n \geq 1\right\}$$

is a countable open base at x. Thus every metric space is first countable.

2.9 Open subbases

Definition 2.21 (Open subbase)**.** Let X be a topological space. A family $\mathcal{S}$ of open subsets of X is said to be an open subbase for X if the collection of all finite intersections of elements of $\mathcal{S}$ forms an open base for X.

Proposition 2.14. *Let X and Y be topological spaces, let $\mathcal{S}$ be an open subbase for Y. Then if $f^{-1}(S)$ is open for each $S \in \mathcal{S}$ then f is continuous.*

Proof. For $S_1, \ldots, S_n$ we have that $f^{-1}(\cap_{i=1}^n S_i) = \cap_{i=1}^n f^{-1}(S_i)$. Therefore, for any finite intersection of elements of $\mathcal{S}$, in other words, for any element U of some open case $\mathcal{B}$ we have $f^{-1}(U)$ is open. The result now follows directly by Proposition 2.10. □

The above proposition shows how convenient working with subbases can be. The following will show how to easily generate topologies using the notion.

Proposition 2.15. *Let X be an aribitrary nonempty set, and let $\mathcal{S} \subset P(X)$. Set,*

$$\mathcal{B} := \left\{\cap_{i=1}^n S_i \mid n \geq 0 \text{ and } S_1, \ldots, S_n \in \mathcal{S}\right\}$$

And,

$$\tau := \left\{U \subset X \mid \forall x \in U \quad \exists B \in \mathcal{B} \ s.t.\ x \in B \subset U\right\}$$

Then τ is a topology on X, $\mathcal{B}$ is an open base for τ and $\mathcal{S}$ is an open subbase for it.

Sometimes we need to compare topologies. Let $\mathcal{T}(X)$ be the set of all topologies on a set X.

Definition 2.22 (Comparison of topologies)**.** Let $\tau_1, \tau_2 \in \mathcal{T}(X)$. We say that τ_1 is weaker than τ_2, or that τ_2 is stronger than τ_1 if $\tau_1 \subset \tau_2$.

For a simple reality check, notice that every topology is weaker than the discrete topology and stronger than the discrete topology. Also, the pair $(\mathcal{T}(X), \subset)$ form a partially ordered set.

Proposition 2.16. *Let $\mathcal{T}_0 \subset \mathcal{T}(X)$ be nonempty. Then $\mathcal{T}_0$ has a supremum and an infimum in $\mathcal{T}(X)$*

Proof. This is more of a sketch proof, but it can be verified that taking the intersection of all $\tau \in \mathcal{T}_0$ gives the infimum of $\mathcal{T}_0$, and that taking the intersection of all $\tau \in \mathcal{T}(X)$ that are stronger than every $\tau \in \mathcal{T}_0$ gives the supremum of $\mathcal{T}_0$. □

Remark 2.6. It can also be seen that the supremum of $\mathcal{T}_0$ is exactly the topology generated by $\cup_{\tau \in \mathcal{T}_0} \tau$.

2.10 Topology generated by functions

Definition 2.23 (Topology generated by functions)**.** Let $\{Y_i\}_{i \in I}$ be a family of topological spaces. For each $i \in I$ let $f_i \colon X \to Y_i$. Write $\mathcal{T}_0 \subset \mathcal{T}(X)$ for the set of all topologies with respect to which all $\{f_i\}_{i \in I}$ are continuous. The greatest lower bound of $\mathcal{T}_0$ is called the weak topology generated by $\{f_i\}_{i \in I}$.

Remark 2.7. It is easy to verify that $\tau = \cap_{\tau_0 \in \mathcal{T}_0} \tau_0$ and also that τ is generated by the set

$$S = \left\{ f_i^{-1}(U) \mid i \in I \text{ and } U \text{ is open in } Y_i \right\}$$

Example 2.11. Let Y be a topological space, let Z be a nonempty subset of Y, and let $f \colon Z \to Y$ be the inclusion map, that is $f(z) = z$ for $z \in Z$. It is easy to show that the weak topology generated by f is equal to the subspace topology induced by Y on Z.

2.11 The product topology

Definition 2.24 (Product topology)**.** The product topology on a cartesian product of topological spaces $\prod_{i \in I} X_i$ is defined to be the weak topology generated by the projections $\{\pi_i\}_{i \in I}$. Equipped with the product topology, the space X is called the product space of the spaces $\{X_i\}_{i \in I}$.

This definition is a bit abstract, but we can give a more concrete definition by setting,

$$\mathcal{S} = \left\{ \prod_{i \in I} U_i \mid \exists j \in I \text{ s.t. } \; U_i = X_i \text{ for } i \in I \setminus \{j\} \text{ and } U_j \text{ is open in } X_j \right\}.$$

Now the product topology on X is equal to the topology on X generated by $\mathcal{S}$ as a subbase. From this we can also deduce that

$$\mathcal{B} = \left\{ \prod_{i \in I} U_i \mid \text{Exists a finite } I_0 \subset I \; \text{ s.t. } U_i = X_i \text{ for } i \in I \setminus I_0 \text{ and } U_i \text{ is open in } X_i \text{ for } i \in I_0 \right\}$$

is an open base for the product topology.

Definition 2.25 (Euclidean topology)**.** The Euclidean topology is the natural topology induced on n-dimensional Euclidean space $\mathbb{R}^n$ by the Euclidean metric. The open balls, and the open boxes both form open bases for this topology.

Example 2.12. Considering the space $\mathbb{R}^n$ for a finite natural n, the Euclidean topology on it is equal to the product topology of the product of $\prod_{i=1}^{n} \mathbb{R}$ where each copy of $\mathbb{R}$ has been endowed with the standard topology. This is not true in the case of $\mathbb{R}^J$ where J is an infinite set.

Proposition 2.17. (Characteristic property of the product topology). *The product topology is the unique topology on X which satisfies the following property. Let Y be a topological space and let $f \colon Y \to X$. Then f is continuous if and only if $\pi_i \circ f$ is continuous for each $i \in I$.*

Definition 2.26 (The function algebras)**.** Let X be a topological space. We write $C(X)$ for the collection of all continuous functions from X to $\mathbb{R}$. We denote by $C_b(X)$ the set of all bounded elements of $C(X)$. It has the a natural norm defined on it, the supremum norm.

More about algebras in Section 5.

3 Complete Metric Spaces

Let (X, d) be a fixed metric space.

3.1 Definitions

Definition 3.1 (Cauchy sequence)**.** We say that a sequence $\{a_n\}_{n\geq 1} \subset X$ is a Cauchy sequence if for all $\epsilon > 0$ exists $N \geq 1$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$.

It is easy to verify that all Cauhcy sequences converge, but the converse is not always true. This leads us to formulate the following notion.

Definition 3.2 (Complete space)**.** We say that the metric space (X, d) is complete if every Cauchy sequence $\{a_n\}_{n\geq 1} \subset X$ converges to some $x \in X$.

Remark 3.1. Consider the set $(-1, 1)$ with the topology induced by $\mathbb{R}$. It is clear that the sequence $1 - \frac{1}{n}$ is a cauchy sequence, but it's limit is $1 \notin (-1, 1)$ and thus the space is not complete. However, there is a homeomorphism $x \mapsto \tan(\pi x/2)$ between $(-1, 1)$ and $\mathbb{R}$, and $\mathbb{R}$ is complete which shows that completeness is not a topological property.

3.2 Examples of complete metric spaces

3.2.1 Banach spaces

Definition 3.3 (Banach space)**.** A complete normed space is said to be a Banach space.

Example 3.1. Let X be a topological space. We will show that $C_b(X)$ is a Banach space with respect to the metric induced by the supremum norm $\|\cdot\|_\infty$. Let $\{f_n\}_{n\geq 1}$ be a Cauchy sequence in $C_b(X)$. Then, by the definition of the supremum norm we have that for any $x \in X$ that the sequence $\{f_n(x)\}_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, and since $\mathbb{R}$ is complete it also has a limit. Thus, exists $f\colon Y \to \mathbb{R}$ such that $\{f_n(x)\}_{n\geq 1} \xrightarrow{n\to\infty} f$ pointwise. Now, we see that

$$\begin{aligned}\left|f(x) - f_n(x)\right| &\leq \limsup_{m\to\infty}\left(\left|f(x) - f_m(x)\right| + \left|f_m(x) - f_n(x)\right|\right) \\ &\leq \limsup_{m\to\infty}\left\|f(x) - f_n(x)\right\|_\infty .\end{aligned}$$

Thus, $f_n \xrightarrow{n\to\infty} f$ uniformly, which implies that $f \in C_b(X)$. We have also shown that $f_n \xrightarrow{n\to\infty} f$ uniformly with respect to $\|\cdot\|$ which completes the theorem.

3.2.2 Subspace of a complete metric space

Proposition 3.1. *Suppose that X is complete and let Y be a nonempty subset of X. Then Y is complete (with respect to the metric induced by X) if and only if Y is closed in X.*

Proof. Suppose first that Y is closed. Let $\{y_n\}_{n\geq 1}$ be a Cauchy sequence in Y. Then since X is complete there exists a limit $y \in X$ to $\{y_n\}_{n\geq 1}$. Since Y is closed we have that $y \in Y$ which shows that Y is complete.

Suppose next that Y is complete. Let $\{y_n\}_{n\geq 1}$ be a sequence in Y that converges to some $x \in X$. Since it converges in X, it must be a Cauchy sequence. By the definition of the subspace metric we have that the sequence is also Chauchy in Y. Since Y is complete $\{y_n\}_{n\geq 1}$ must also have a limit in Y. From the uniqueness of the limit we have that $x \in Y$. It follows that Y is closed. $\square$

3.3 Cantor's intersection lemma

Definition 3.4 (Diameter)**.** Given a nonempty subset A of X we write

$$\operatorname{diam}(A) := \left\{ d(x_1, x_2) \mid x_1, x_2 \in X \right\}.$$

We call the number $\operatorname{diam}(A)$ the diameter of A.

The following lemma demonstrates the usefulness of the completeness property.

Lemma 3.2. (Cantor's intersection lemma for complete metric spaces). *Let $F_1, F_2, \ldots$ be nonempty closed subsets of X. Suppose that*

- *X is complete;*
- *$F_{n+1} \subset F_n$ for all $n \geq 1$;*
- *$\operatorname{diam}(F_n) \to 0$ as $n \to \infty$.*

Then $\cap_{n\geq 1} F_n = \{x\}$ for some $x \in X$.

Proof. For each $n \geq 1$ choose $x_n \in F_n$. For each $n \geq m \geq 1$ we have that $x_n, x_m \in F_n$ and thus $d(x_n, x_m) \leq \operatorname{diam}(F_n)$. Since $\operatorname{diam}(F_n) \to 0$ we have that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in X. Since X is complete exists $x \in X$ such that $x_n \to x$ as $n \to \infty$. For each $n \geq 1$ we have that $\{x_m\}_{m\geq n} \subset F_n$ and since each F_n is closed, we have that $x \in F_n$. Thus $x \in \cap_{n\geq 1} F_n$ and in particular $\cap_{n\geq 1} F_n \neq \emptyset$. Let $x, y \in \cap_{n\geq 1} F_n$. We have that $d(x, y) \leq \operatorname{diam}(F_n)$ for each $n \geq 1$. Since $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$ we have that $d(x, y) = 0$ and thus $x = y$. This implies that $\cap_{n\geq 1} F_n = \{x\}$ which completes the proof. □

3.4 The completion theorem

Definition 3.5 (Isometry)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f \colon X \to Y$ is said to be an isometry if $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. We say that X and Y are isometric if there exists an isometry from X onto Y.

Remark 3.2. Every isomorphism is continuous and injective. A surjective isomorphism is thus a homeomorphism.

Theorem 3.3. (The completion theorem for metric spaces). *Let X be a metric space. Then there exists a complete metric space $\overline{X}$ and an isometry $f \colon X \to \overline{X}$ such that $f(X)$ is dense in $\overline{X}$. Moreover, if Y is another complete metric space such that exists an isometry $g \colon X \to Y$ such that $g(X)$ is dense in Y then there exists a surjective isometry $h \colon \overline{X} \to Y$ so that $g = h \circ f$.*

Proof. To be added. □

Remark 3.3. The space $\overline{X}$ is called the completion of X. As the theorem states it is unique up isometry.

Definition 3.6 (Uniform continuity)**.** A mapping $f \colon X \to Y$ is said to be uniformly continuous if for every $\epsilon > 0$ there exists $\delta > 0$ so that $d_Y(f(x_1), f(x_2)) < \epsilon$ for all $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$.

Proposition 3.4. *Let $A \subset X$, let $f \colon X \to Y$ be uniformly continuous. Then there exists a unique $\overline{f} \colon \overline{X} \to Y$ which extends Y such that $\overline{f}$ is also continuous.*

3.5 Baire's theorem

Definition 3.7 (Nowhere dense subset). A subset A of X is said to be nowhere dense if $\operatorname{Int}\left(\overline{A}\right) = \emptyset$.

Remark 3.4. Note that a closed subset $A \subset X$ is nowhere dense if and only if $\operatorname{Int}(A) = \emptyset$.

Example 3.2. Let W be a linear subspace of $\mathbb{R}^d$ with $\dim W < d$. We will show that W is nowhere dense. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^d$. We notice that for every $v \in \mathbb{R}^d$ the map $x \mapsto \langle x, v \rangle$ is continuous. Thus the set $\left\{x \in \mathbb{R}^d \mid f(x) = 0\right\}$ is closed in $\mathbb{R}^d$ as the preimage of the closed set $\{0\}$. From this and from the fact that:

$$W = (W^{\perp})^{\perp} = \cap_{u \in W^{\perp}} \left\{x \in \mathbb{R}^d \mid \langle x, u \rangle = 0\right\},$$

it follows that W is closed in $\mathbb{R}^d$. Moreover, for every $w \in W$, $0 \neq u \in W^{\perp}$ and $\epsilon > 0$ we have that $w + u\epsilon \notin W$. Since $W^{\perp} \neq \emptyset$ that means that $\operatorname{Int}(W) = \emptyset$, which shows that W is nowhere dense.

Definition 3.8 (First category). A subset E of X is said to be of the first category if there exist nowhere dense subsets $A_1, A_2, \dots \subset X$ so that $E = \cup_{n \geq 1} A_n$. A subset of X which is not of the first category is said to be of the second category.

Theorem 3.5. (The Baire category theorem). *Suppose that X is complete, and let $E \subset X$ be of the first category. Then $\operatorname{Int}(E) = \emptyset$.*

Proof. It suffices to prove that exists $x_0 \in X$ and $\epsilon_0 > 0$ such that $B(x_0, \epsilon) \setminus E \neq \emptyset$. Since E is of the first category, there exist closed subsets $F_1, F_2, \dots$ such that $E \subset \cup_{n \geq 1} F_n$ and $\operatorname{Int}(F_n) = \emptyset$ for each $n \geq 1$. We are going to construct sequences $\{e_n\}_{n \geq 1} \subset (0, \infty)$ such that $\epsilon_n < \frac{1}{n}$ and $\overline{B}(x_n, \epsilon_n) \subset B(x_{n-1}, \epsilon_{n-1}) \setminus \cup_{k=1}^{n} F_k$.

Let $n \geq 1$ and assume that we already constructed the sequences for $1 \leq k \leq n-1$. From $B(x_{n-1}, \epsilon_{n-1}) \cap (\cup_{k=1}^{n-1} F_n) = \emptyset$ it follows that $V := B(x_{n-1}, \epsilon_{n-1}) \setminus \cup_{k=1}^{n-1} F_n \neq \emptyset$. From this, and since V is open and $\operatorname{Int}(F_n) = \emptyset$ we get that $V \setminus F_n \neq \emptyset$. Since $V \setminus F_n$ is nonempty and open we get that there exists $x_n \in X$ and $0 < \epsilon_n < \frac{1}{n}$ such that $\overline{B}(x_n, \epsilon_n) \subset V \setminus F_n = B(x_{n-1}, \epsilon_{n-1}) \setminus \cup_{k=1}^{n} F_k$. This completes the inductive construction.

From Cantor's intersection lemma it now follows that $\cap_{n \geq 1} \overline{B}(x_n, \epsilon_n) = \{x\}$ for some $x \in X$. For every $n \geq 1$ we have $x \in \overline{B}(x_n, \epsilon_n) \subset B(x_{n-1}, \epsilon_{n-1}) \setminus \cup_{k=1}^{n} F_k$. This shows that:

$$x \in B(x_0, \epsilon_0) \setminus \cup_{k=1}^{\infty} F_k \subset B(x_0, \epsilon_0) \setminus E,$$

which completes the proof of the theorem. □

The following is an immediate corollary from Baire's theorem.

Corollary 3.6. *Suppose that X is complete. Then X is of the second category as a subset of itself. Consequently, if $F_1, F_2, \dots$ are closed subsets of X with $X = \cup_{n \geq 1} F_n$, then $\operatorname{Int}(F_n) \neq \emptyset$ for some $n \geq 1$.*

Here's another useful corollary of Baire's theorem.

Corollary 3.7. *Suppose that X is complete. Let $U_1, U_2, \dots$ be open subsets of X. Suppose that U_n is dense in X for all $n \geq 1$. Then $\cap_{n \geq 1} U_n$ is also dense in X.*

Proof. For $n \geq 1$ set $F_n := X \setminus U_n$. Since U_n is dense in X for all $n \geq 1$ it follows that F_n is nowhere dense in X for all $n \geq 1$. Let $V \subset X$ be open. Then by Baire's category theorem we have that the interior of $\cup_{n=1}^{\infty} F_n$ is empty and thus

$$\emptyset \neq V \setminus \cup_{n=1}^{\infty} F_n = V \cap \cap_{n \geq 1} U_n$$

which completes the proof. □

Definition 3.9 (The sets G_δ and F_σ)**.** Let Y be a topological space. A countable intersection of open subsets of Y is called a G_δ set. A countable union of closed subsets of Y is called an F_σ set.

Definition 3.10 (Liouville number)**.** An irrational real number x is said to be a Liouville number if for every integer $n \geq 1$ there exist integers p and $q \geq 2$ so that $\left|x - \frac{p}{q}\right| < \frac{1}{q^n}$.

Example 3.3. The number $\sum_{k\geq 1} \frac{1}{10^{k!}}$ is called Liouville's constant. It is not difficult to show that it is a Liouville number.

Proposition 3.8. *Write L for the set of Liouville numbers. Then L is a dense G_δ subset of $\mathbb{R}$.*

Proof. For every $n \geq 1$ set

$$V_n := \bigcup_{q=2}^{\infty} \bigcup_{p\in\mathbb{Z}} \left(\frac{p}{q} - \frac{1}{q^n}, \frac{p}{q} + \frac{1}{q^n}\right).$$

Note that $\mathbb{Q} \subset V_n$ which means V_n is dense in $\mathbb{R}$. For each $r \in \mathbb{Q}$ denote $U_r = \mathbb{R} \setminus \{r\}$. It follows directly from the definition of Liouville numbers that:

$$L = \left(\bigcap_{n=1}^{\infty} V_n\right) \cap \left(\bigcap_{r\in\mathbb{Q}} U_r\right)$$

Now since that sets $\{V_n\}_{n=1}^{\infty}$ and $\{U_r\}_{r\in\mathbb{Q}}$ are all open and dense, and since $\mathbb{Q}$ is countable, it follows from the previous corollary that L is a dense G_δ subset of $\mathbb{R}$. This completes the proof. □

3.6 The Banach fixed-point theorem

Definition 3.11 (Contraction)**.** A mapping $f\colon X \to X$ is called a contraction of X if there exists $c \in [0,1)$ so that $d(f(x), f(y)) \leq cd(x,y)$ for all $x, y \in X$.

Theorem 3.9. (The Banach fixed-point theorem). *Suppose that X is complete and let $f\colon X \to X$ be a contraction. Then f has a unique fixed point. That is, there exists a unique $x \in X$ so that $f(x) = x$.*

Proof. First we show that f has a fixed point. Choose an arbirary $x_0 \in X$ and define a sequence $\{x_n\}_{n\geq 0}$ by setting $x_n := f(x_{n-1})$ for $n \geq 1$. It is easy to show by induction that:

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$$

Now we will show that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. Let $\epsilon > 0$. Choose $N \geq 1$ such that $c^N d(x_0, x_1)(1-c)^{-1} < \epsilon$. For $n \geq m \geq N$,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k=m}^{n-1} c^k d(x_0, x_1) \\ &\leq c^m d(x_0, x_1) \sum_{k=1}^{\infty} c^k = \frac{c^m d(x_0, x_1)}{1-c} < \epsilon \end{aligned}$$

which shows that $\{x_n\}_{n\geq 1}$ is Cauchy. Since X is complete exists $x \in X$ such that $\{x_n\}_{n\geq 1} \to x$ as $n \to \infty$. Since f is a contraction it is continuous. We get:

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(x),$$

which shows that f has a fixed point.

Next we show uniqueness. Suppose there were $y \in X$ another fixed point of f. Then

$$d(x, y) = d(f(x), f(y)) \leq cd(x, y)$$

Thus we have $(1-c)d(x, y) \leq 0$. This is only possible if $d(x, y) = 0$ thus $x = y$ which completes the proof. □

Notice that the proof of the theorem also gives an explicit way to approximate the fixed point of f.

3.7 A glimpse of Picard's theorem

The following is a simplified version of the Picard–Lindelöf theorem regarding the existence and uniqueness of solutions for ordinary differential equations, which is also sometimes called the existence and uniqueness theorem.

For $\epsilon > 0$ we set $I_\epsilon := [-\epsilon, \epsilon]$.

Theorem 3.10. (Picard's theorem). *Let $F\colon I_1 \times I_1 \to \mathbb{R}$ be continuous. Suppose that there exists $K > 0$ so that $\left|F(t, x) - F(t, y)\right| \leq K|x - y|$ for all $t, x, y \in I_1$. Then there exists $\epsilon > 0$ for which there exists a unique $f\colon I_\epsilon \to I_1$ so that,*

- *f is differentiable on I_ϵ;*
- *$f(0) = 0$*
- *$f'(t) = F(t, f(t))$ for $t \in I_\epsilon$*

Example 3.4. Suppose that $F(t, x) = 1 + x^2$. Since $\tan(0) = 0$, and on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we have

$$[\tan(x)]' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x) = F(x, \tan(x)).$$

It is clear that in this case, the map $x \mapsto \tan(x)$ is the unique solution.

4 Compactness

4.1 Definitions

Let X be a fixed topological space.

Definition 4.1 (Open cover)**.** A class $\mathcal{U} := \{U_i\}_{i\in I}$ of open subsets of a X is said to be an open cover of X if $X = \bigcup_{i\in I} U_i$. A subclass of $\mathcal{U}$ is said to be a subcover of $\mathcal{U}$ if it is in itself an open cover of X.

Definition 4.2 (Compact)**.** The space X is said to be compact if every open cover of X has a finite subcover.

Definition 4.3 (Compact subspace)**.** A subset Y of X is said to be compact if for every family of open sets $\{U_i\}_{i\in I}$ such that $Y \subset \bigcup_{i\in I} U_i$ exists a finite index set $I_0 \subset I$ such that $Y \subset \bigcup_{i\in I_0} U_i$.

Remark 4.1. Notice that from the definition of the subspace topology, a nonempty subset Y of X is compact if and only if Y is a compact space when equipped with the subspace topology.

Proposition 4.1. *Suppose that X is compact and let $F \subset X$ be closed. Then F is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of F. Since F is closed we know that $X \setminus F \cup \{U_i\}_{i\in I}$ is an open cover of X. Since X is compact exists a finite index set $I_0 \subset I$ such that $X \setminus F \cup \{U_i\}_{i\in I_0}$ is a finite open cover of X. It is clear that $F \subset \{U_i\}_{i\in I_0}$ which completes the proof. □

Proposition 4.2. *Suppose X is compact, let Y be a topological space, and let $f\colon X \to Y$ be continuous. Then $f(X)$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $f(X)$. Since f is continuous $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is compact exists a finite index set $I_0 \subset I$ such that $\{f^{-1}(U_i)\}_{i\in I_0}$ is an open cover of X. We now have:

$$f(X) = f\left(\bigcup_{i\in I_0} f^{-1}(U_i)\right) = \bigcup_{i\in I_0} f(f^{-1}(U_i)) = \bigcup_{i\in I_0} U_i$$

Which completes the proof. □

Here are some more equivalent forms of compactness that are ofter easier to apply.

Proposition 4.3. *The space X is compact if and only if for every class $\{F_i\}_{i\in I}$ of closed subsets of X with $\cap_{i\in I} F_i = \emptyset$ there exists a finite $I_0 \subset I$ with $\cap_{i\in I_0} F_i = \emptyset$.*

Proof. Assume X is compact. Let $\{F_i\}_{i\in I}$ be a family of closed subsets of X with $\cap_{i\in I} F_i = \emptyset$ then we have $\cap_{i\in I} X \setminus F_i = X$ which is a cover of X thus exists a finite $I_0 \subset I$ with $\cap_{i\in I_0} X \setminus F_i = X$ being a finite subcover of X. This implies that $\cap_{i\in I_0} F_i = \emptyset$ which completes the proof. The proof of the other direction is similar and thus omitted. □

Definition 4.4 (Finite intersection property)**.** Let S be a nonempty set. A class of subsets $\{E_i\}_{i\in I}$ of S is said to have the finite intersection property if $\cap_{i\in I_0} E_i \neq \emptyset$ for every finite $I_0 \subset I$.

Proposition 4.4. *The space X is compact if and only if every class of closed subsets of X with the finite intersection property has nonempty intersection.*

Proof. Suppose that X is compact. Let $\{F_i\}_{i\in I}$ be a class of closed subsets of X with the finite intersection property. If $\cap_{i\in I} F_i = \emptyset$, then by the previous proposition there exists a finite $I_0 \subset I$ with $\cap_{i\in I_0} F_i = \emptyset$. This contradicts $\{F_i\}_{i\in I}$ having the finite intersection property, and so we must have $\cap_{i\in I} F_i \neq \emptyset$.

Suppose next that X is not compact. By the previous proposition there exists a class $\{F_i\}_{i\in I}$ of closed subsets of X with $\cap_{i\in I} F_i = \emptyset$, so that $\cap_{i\in I_0} F_i \neq \emptyset$ for all finite $I_0 \subset I$. That is, $\{F_i\}_{i\in I}$ has the finite intersection property but $\cap_{i\in I} F_i = \emptyset$, which completes the proof of the proposition. □

Proposition 4.5. *Let $\mathcal{B}$ be an open base for X. Suppose that every open cover $\{b_i\}_{i\in i} \subset \mathcal{B}$ of X has a finite subcover. Then X is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be an aribitrary open cover of X. Since $\mathcal{B}$ is an open base for X, for every $i \in I$ exists I_i, an index set such that $\{B_j\}_{j\ I_i}$ we have $U_i = \cup_{j\in I_i} B_i$. This implies that the set

$$\mathcal{B}_0 = \left\{B_j \mid \text{for all } j \in I_i \text{ for all } i\right\}$$

is also an open cover of X. Since $\mathcal{B}_0 \subset \mathcal{B}$ there exists a finite $B_f \subset \mathcal{B}_0$ such that $\cup_{B\in\mathcal{B}_f} B = X$. By construction, for every $B \in \mathcal{B}_f \subset \mathcal{B}_0$, exists $i_B \in I$ such that $B \subset U_{i_B}$. It is clear that the index set

$$I_f = \left\{i_B \mid B \in \mathcal{B}_f\right\}$$

is finite, and by construction we have $\cup_{i\in I_f} U_i = X$. □

4.2 Closed bases

Definition 4.5 (Closed base)**.** A family $\mathcal{B}$ of closed subsets of X is called a closed base for X if the collection

$$\left\{X \setminus B \mid B \in \mathcal{B}\right\}$$

is an open base for X. Similarly, a family $\mathcal{S}$ of closed subsets of X is called a closed subbase for X if the collection $\left\{X \setminus S \mid S \in \mathcal{S}\right\}$ is an open subbase for X.

Remark 4.2. Note that if $\mathcal{S}$ is a closed subbase for X then the set $\mathcal{B}$ of all finite unions of elements of $\mathcal{S}$ forms a closed base for X. This is because by definition, the set of all finite intersections of an open subbase forms an open base. We call $\mathcal{B}$ the closed base generated by $\mathcal{S}$.

4.3 The Alexander subbase theorem

Proposition 4.6. *Let $\mathcal{B}$ be a closed base for X. Suppose that for every $\{B_i\}_{i\in I} \subset \mathcal{B}$ with the finite intersection property we have $\cap_{i\in I} B_i = \emptyset$. Then X is compact.*

In the following two theorems are let X be a fixed topological space.

Theorem 4.7. (The Alexander subbase theorem, first form). *Let $\mathcal{S}$ be an open subbase for X. Suppose that every open cover $\{S_i\}_{i\in I} \subset \mathcal{S}$ of X has a finite subcover. Then X is compact.*

Theorem 4.8. (The Alexander subbase theorem, second form). *Let $\mathcal{S}$ be a closed subbase for X. Suppose that $\cap_{i\in I} S_i \neq \emptyset$ for every $\{S_i\} i \in I \subset \mathcal{S}$ with the finite intersection property. Then X is compact.*

First, we will prove theorem Theorem 4.7 assuming Theorem 4.8, then we will prove Theorem 4.8 for completeness.

Proof. Let $\mathcal{S}$ be an open subbase for X. Now set

$$\mathcal{E} = \left\{X \setminus S \mid S \in \mathcal{S}\right\}.$$

It is clear that $\mathcal{E}$ is a closed subbase of X. Let $\{E_i\}_{i\in I}$ be a subset of $\mathcal{E}$ with the finite intersection property. By Theorem 4.8 it suffices to show that $\cap_{i\in I} E_i \neq \emptyset$ to prove that X

is compact. Assume, for the sake of contradiction, that $\cap_{i\in I} E_i = \emptyset$. Thus, it is clear that $\{X \setminus E_i\}_{i\in I}$ is not only an open cover of X, but also a subset of $\mathcal{S}$. By our assumption it has a finite subcover $\{X \setminus E_i\}_{i\in I_0}$ where $I_0 \subset I$. However, this gives that $\cap_{i\in I_0} E_i = \emptyset$ in contradiction to $\{E_i\}_{i\in I}$ having the finite intersection property. Thus we have $\cap_{i\in I} E_i \neq \emptyset$ as wanted, which completes the proof. □

Now we will prove Theorem 4.8 using Zorn's lemma.

Proof. Let $\mathcal{B}$ be the open subbase generated by $\mathcal{S}$. Now let $\mathcal{B}_0 \subset \mathcal{B}$ be with the finite intersection property. Then, by Proposition 4.6 we it suffices to show that $\cap_{B\in\mathcal{B}_0} \mathcal{B} = \emptyset$. First, let $\mathcal{P}$ be the set of all $\mathcal{B}_1$ such that $\mathcal{B}_0 \subset \mathcal{B}_1$ and $\mathcal{B}_1$ has the finite intersection property. We have that $\mathcal{P} \neq \emptyset$ because $B_0 \in \mathcal{P}$. Now for every $\mathcal{B}_x, \mathcal{B}_y \in \mathcal{P}$, write

$$\mathcal{B}_x \leq \mathcal{B}_y \iff \mathcal{B}_x \subset \mathcal{B}_y.$$

It is clear that $(\mathcal{P}, \leq)$ is a partially order set. Let $\mathcal{C}$ be a chain in $\mathcal{P}$. Denote $\mathfrak{B} := \cup_{\mathcal{B}_x\in\mathcal{C}} B_x$. It is clear that $\mathfrak{B}$ is a maximal element in $\mathcal{C}$. It is also clear that it satisfies the finite intersection property, because for every finite $\mathcal{B}_2 \subset \mathfrak{B}$ there must exist $\mathcal{B}_x \in \mathcal{C}$ such that $\mathcal{B}_2 \subset \mathcal{B}_x$, but because $\mathcal{B}_x$ has the finite intersection property, it is clear that the intersection of the elements in $\mathcal{B}_2$ is nonempty. Therefore, we have that $\mathfrak{B} \in \mathcal{P}$, which completes the conditions for Zorn's lemma. Let $\mathcal{B}_{\max}$ be the maximal element in $\mathcal{P}$ with respect to $\leq$.

Next we will show that for every $B \in \mathcal{B}_{\max}$ there exists S_B such that

$$(*) \quad S_B \in \mathcal{S} \cap \mathcal{B} \quad \text{and} \quad S_B \subset B.$$

Assume by contradiction that exists $B \in \mathcal{B}_{\max}$ for which there does not exist S_B that satisfies $(*)$. Since B is an element of the closed base $\mathcal{B}$ generated by $\mathcal{S}$, there exist $S_1, \ldots, S_n \in \mathcal{S}$ such that $B = \cup_{i=1}^n S_i$. Let $i \in \{1, 2, \ldots, n\}$, then since there does not exist S_B that satisfies $(*)$, we get that $S_i \notin \mathcal{B}_{\max}$. Now, since $\mathcal{B}_{\max}$ is maximal in $\mathcal{P}$, that must mean that $\mathcal{B}_{\max} \cup \{S_i\}$ does not satisfy the finite intersection property. Thus, exist $B_{i,1}, \ldots, B_{i,m_i}$ such that $S_i \cap \left(\cap_{j=1}^{m_i} b_{i,j}\right) = \emptyset$. Therefore, we have that

$$\emptyset \neq B \cap \left(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m_i} B_{i,j}\right) = \left(\bigcup_{k=1}^{n} S_k\right) \cap \left(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m_i} B_{i,j}\right) = \bigcup_{k=1}^{n}\left(S_k \cap \left(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m_i} B_{i,j}\right)\right) = \emptyset.$$

This contradiction shows that our assumption was false, and thus exists S_B that satisfies $(*)$ for every $B \in \mathcal{B}_{\max}$.

Finally, set

$$\mathcal{S}_0 := \left\{S_B \mid B \in \mathcal{B}_{\max}\right\}.$$

It follows that since $\mathcal{S}_0$ is a subset of $\mathcal{B}_{\max}$, that it must also satisfiy the finite intersection property. As $\mathcal{S}_0$ is also a subset of $\mathcal{S}$, we can apply the assumption of the theorem and get that $\cap_{S\in\mathcal{S}_0} S \neq \emptyset$. Thus, there exists some $x \in X$, such that $x \in S_B \subset B$ for all $B \in \mathcal{B}_{\max}$, which shows that $\cap_{B\in\mathcal{B}_{\max}} B \neq \emptyset$. Now, we just need to recall that since $\mathcal{B}_{\max} \in \mathcal{P}$, then by construction of $\mathcal{P}$ we have that $\mathcal{B}_0 \subset \mathcal{B}_{\max}$, and thus we clearly have that $\cap_{B\in\mathcal{B}_0} B \neq \emptyset$, which as we stated before, completes the proof. □

Definition 4.6 (Bounded space)**.** Let X be a metric space. We say that $A \subset X$ is bounded if exists $r > 0$ and $x \in X$ such that $A \subset B(x, r)$.

Note that it is easy to see that $A \subset X$ is bounded if and only if it has a finite diameter.

Lemma 4.9. *Let $\mathcal{S}$ be an open subbase for a topological space X. If $Y \subset X$ is a subset of X equiped with the subspace topology induced by X then $\{S \cap Y \mid S \in \mathcal{S}\}$ is an open subbase for Y.*

Proof. Let U be a nonempty subset of Y and let $y \in U$. There exists W an open set in X such that $W \cap Y = U$. Because $\mathcal{S}$ is a subbase for X exists $S_1, \dots, S_n \in \mathcal{S}$ such that $y \in \cap_{i=1}^n S_i \subset W$ and thus because $y \in Y$:

$$y \in \cap_{i=1}^n S_i \cap Y \subset W \cap Y = U$$

Because $S_i \cap Y$ are all open in Y we have that indeed $\{S \cap Y \mid S \in \mathcal{S}\}$ is an open subbase as wanted. □

4.4 The Heine–Borel theorem

4.4.1 The Heine–Borel theorem in $\mathbb{R}$

Theorem 4.10. (Heine–Borel theorem in $\mathbb{R}$). *Every closed and bounded set in $\mathbb{R}$ is compact.*

Proof. Let A be a closed and bounded set in $\mathbb{R}$. Because A is bounded we know that exist real numbers $a, b \in \mathbb{R}$ such that $a < b$ and also $A \subset [a, b]$. If we equip $[a, b]$ with the subspace topology induced on it by $\mathbb{R}$ it is not hard to see that A is closed in $[a, b]$ and thus it suffices to verify that $[a, b]$ is compact in $\mathbb{R}$. It's easy to check that the set:

$$\{(-\infty, c) \mid c \in \mathbb{R}\} \cup \{(d, \infty) \mid d \in \mathbb{R}\}$$

Is an open subbase to $\mathbb{R}$. From the lemma we have that the set:

$$S = \{[a, c) \mid a < c \leq b\} \cup \{(d, b] \mid a < d \leq b\}$$

Is an open subbase for $[a, b]$. Let $\mathcal{U} \subset S$ be an open cover of $[a, b]$, by Alexander's subbase theorem it suffices to show that $\mathcal{S}$ has a finite subcover. Since $\mathcal{U} \subset \mathcal{S}$ there exist index sets I, J such that:

$$\mathcal{U} = \{[a, c_i) \mid i \in I\} \cup \{(d_j, b] \mid j \in J\}$$

We have that $a \in [a, b]$ and $\mathcal{U}$ a cover of $[a, b]$ which means that $I \neq \emptyset$. Denote $s = \sup\{c_i\}_{i \in I}$, if we have $s \leq d_j$ for all $j \in J$ we have $s \notin \cup\mathcal{U}$ which is a contradiction. Otherwise exists $j_0 \in J$ such that $d_{j_0} < s$ and then by definition exists $i_0 \in I$ such that $d_{j_0} < c_{i_0} < s$ and then we have that $\{[a, c_{i_0}), (d_{j_0}, b]\}$ is a finite subcover of $[a, b]$ which completes the proof. □

4.4.2 Tychonoff's theorem

Theorem 4.11. (Tychonoff's theorem). *Let $\{X_i\}_{i \in I}$ be a nonempty family of compact topological spaces. Equip $\prod_{i \in I} X_i$ with the product topology. Then $\prod_{i \in I} X_i$ is compact.*

Proof. Set:

$$\mathcal{S} = \left\{ \prod_{i \in I} F_i \mid \exists i_0 \in I \text{ s.t. } (\forall i \in I \setminus \{i_0\})(F_i = X_i) \text{ and } F_{i_0} \text{ is closed in } X_{i_0} \right\}$$

This is the standard closed subbase for $\prod_{i \in I} X_i$. Let $\{S_j\}_{j \in J} \subset \mathcal{S}$ be with the finite intersection property. By Alexander's subbase theorem, second form, it suffices to prove that $\cap_{j \in J} S_j \neq \emptyset$. For every $j \in J$ exists a family $\{F_{j,i}\}_{i \in I}$ so that $F_{j,i}$ is a closed of X_i for each $i \in I$, and $S_j = \prod_{i \in I} F_{j,i}$. Thus, for every $J_0 \subset J$

$$(*) \qquad \bigcap_{j \in J_0} S_j = \left\{ \prod_{x \in I} x_i \in \prod_{i \in I} X_i \mid x_i \in F_{j,i} \text{ for all } i \in I \text{ and } j \in J_0 \right\}$$

From this, and since $\{S_j\}_{j \in J}$ has the finite intersection property, it follows that $\{F_{j,i}\}_{j \in J}$ has the finite intersection property for each $i \in I$. From this, and from proposition 4.4, and since the spaces X_i are compact, we obtain that for each $i \in I$, there exists $\tilde{x}_i \in \cap_{j \in J} F_{j,i}$. From $(*)$ it now follows that $\{\tilde{x}_i\}_{i \in I} \in \cap_{j \in J} S_j$, which completes the proof of the theorem. □

4.4.3 The general case

We can now prove the following classic result.

Theorem 4.12. (Heine–Borel theorem). *Let $d \geq 1$ be an integer, and equip $\mathbb{R}^d$ with its standard Euclidean metric. Then every closed and bounded subset of $\mathbb{R}^d$ is compact.*

First we need to prove a couple of lemmas.

Lemma 4.13. *Let $\{X_i\}_{i\in I}$ be a nonempty family of topological spaces, and equip $\prod_{i\in I} X_i$ with the product topology. Let Y be a nonempty subset of $\prod_{i\in I} X_i$. For each $i \in I$ let π_i be the coordinate projection from $\prod_{i\in I} X_i$ onto X_i, and denote by $\pi_i|_Y$ the restriction of $\prod_i$ to Y. Then the subspace topology induced by $\prod_{i\in I} X_i$ on Y is equal to the weak topology generated by $\{\pi_i|_Y\}_{i\in I}$.*

Proof. By definition of the product topology, the collection

$$\left\{\pi_i^{-1}(U) \mid i \in I \text{ and } U \text{ is open in } X_i\right\}$$

is an open subbase for the product space. By a previous lemma we have that

$$\left\{\pi_i^{-1}(U) \cap Y \mid i \in I \text{ and } U \text{ is open in } X_i\right\}$$

is an open subbase for Y with respect to the subspace topology. From this, and since $\pi_i^{-1}(E) \cap Y = \pi_i^{-1}|_Y(E)$ for each $i \in I$ and $E \subset X_i$, and now by Remark 2.4 we see that indeed the subspace topology induced by $\prod_{i\in I} X_i$ on Y is equal to the weak topology generated by $\{\pi_i|_Y\}_{i\in I}$. □

Lemma 4.14. *Let $\{X_i\}_{i\in I}$ be a nonempty family of topological spaces, and equip $\prod_{i\in I} X_i$ with the product topology. For each $i \in I$ let Y_i be a nonempty subset of X_i, and set $Y := \prod_{i\in I} Y_i$. Let τ_1 be subspace topology induced by $\prod_{i\in I} X_i$ on Y. Let τ_2 be the product topology on Y, where each Y_i is equipped with the subspace topology induced by X_i. Then $\tau_1 = \tau_2$.*

Now we can go back to prove Theorem 4.12

Proof. For each $i \in I$ let π_i be the coordinate projection from $\prod_{i\in I} X_i$ onto X_i, and denote by $\pi_i|_Y$ the restriction of $\prod_i$ to Y. From the previous lemma we have that τ_1 is equal to the weak topology generated by $\{\pi_i|_Y\}_{i\in I}$.

For each $i \in I$ let $\tilde{\pi}_i$ be the coordinate projection from Y onto Y_i. By the definition of the product topology, the collection

$$\mathcal{S} := \left\{\tilde{\pi}_i^{-1}(U) \mid i \in I \text{ and } U \text{ is open in } Y_i\right\}$$

is an open subbase for Y in respect to τ_2. We see that:

$$\mathcal{S} := \left\{(\pi|_{Y_i})^{-1}(V \cap Y_i) \mid i \in I \text{ and } V \text{ is open in } X_i\right\}$$

Now since $(\pi|_{Y_i})^{-1}(Y_i) = Y$ for all $i \in I$

$$\mathcal{S} := \left\{(\pi|_{Y_i})^{-1}(V) \mid i \in I \text{ and } V \text{ is open in } X_i\right\}$$

From this, and since $\mathcal{S}$ is an open subbase for Y with respect to τ_2, it follows that τ_2 is also equal to the weak topology generated by $\{\pi_i|_Y\}_{i\in I}$. This completes the proof of the lemma. □

4.5 Lebesgue's covering lemma

Definition 4.7 (Local compactness)**.** A topological space X is called locally compact if for any $x \in X$ exists a neighbourhood $U \subset X$ of x so that $\overline{U}$ is compact.

Example 4.1. The space $\mathbb{R}^d$ is locally compact for every $d \in \mathbb{N}$. Let $x \in \mathbb{R}^d$, since $\overline{B}(x, 1)$ is closed and bounded, it follows from the Heine–Borel theorem that it is compact.

Definition 4.8 (Sequential compactness)**.** The metric space X is said to be sequentially compact if every sequence in X has a convergent subsequence.

Definition 4.9 (Bolzano–Weierstrass property)**.** The metric space X is said to have the Bolzano–Weierstrass property if every infinite subset of X has a limit point in X.

It is important to note that in metric spaces, sequential compactness and the Bolzano Weierstrass property are both equivalent to compactness.

Definition 4.10 (Lebesgue number)**.** Let $\{U_i\}_{i\in I}$ be an open cover of X. A real number $\delta > 0$ is said to be a Lebesgue number for $\{U_i\}_{i\in I}$ if for all nonempty $A \subset X$ with $\operatorname{diam}(A) < \delta$ there exists $i \in I$ so that $A \subset U_i$.

Lemma 4.15 (Lebesgue's covering lemma)**.** *Suppose that X is a sequentially compact metric space. Let $\{U_i\}_{i\in I}$ be an open cover of X. Then $\{U_i\}_{i\in I}$ has a Lebesgue number.*

Proof. Assume by contradiction that the lemma is false, then exists $\emptyset \neq A_n$ such that:

$$\operatorname{diam}(A_n) < \frac{1}{n} \quad \text{and} \quad A_n \not\subset U_i \quad \text{for all } i \in I, \text{ for all } n \in \mathbb{N}.$$

Choose an arbitrary $x_n \in A_n$ to construct the sequence $\{x_n\}_{n\geq 1}$. Since X is sequencially compact, there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ such that $x_{n_k} \xrightarrow{k\to\infty} x$. Since $\{U_i\}_{i\in I}$ is an open cover for X, there exists some $i_0 \in I$ such that $x \in U_{i_0}$, and since U_{i_0} is open, there must exist some $\epsilon > 0$ such that $B(x, \epsilon) \subset U_{i_0}$. However, it is clear that for sufficiently large values of n we have that $x \in A_n \subset B(x, \epsilon) \subset U_{i_0}$, which shows that our assumption must be false, which completes the proof. $\square$

4.6 Total boundedness

Definition 4.11 (ϵ-net)**.** Let $\epsilon > 0$ be given. A nonempty subset A of X is said to be an ϵ-net if A is finite and $X = \cup_{a\in A} B(a, \epsilon)$.

Definition 4.12 (Total boundedness)**.** We say that X is totally bounded if it has an ϵ-net for all $\epsilon > 0$.

It is clear that a totally bounded space is also bounded.

Proposition 4.16. *Suppose that X is sequentially compact. Then X is totally bounded.*

Proof. Assume by contradiction that X is not totally bounded, then exists $\epsilon > 0$ such that $X \neq \cup_{a\in A} B(a, \epsilon)$ for any finite $A \subset X$. From this we can construct a sequence such that $x_{n+1} \notin \cup_{i=1}^{n} B(x_i, \epsilon)$, and therefore have that $d(x_n, x_m) \geq \epsilon$ for any $m > n \geq 1$. It is clear that $\{x_n\}_{n\geq 1}$ does not have a converging subsequence, in contradiction to X being sequencially compact. This proves that X is totally bounded which completes the proof. $\square$

Using Lebesgue's lemma we can also prove the following proposition:

Proposition 4.17. *Suppose that X is a compact metric space. Let (Y, d_Y) be a metric space, and let $f\colon X \to Y$ be continuous. Then f is uniformly continuous.*

Proof. Let $\epsilon > 0$. Since f is continuous the set $f^{-1}(B(f(x), \epsilon/2))$ is open for any $x \in X$ and thus the set:

$$\mathcal{U} := \{f^{-1}(B(f(x), \epsilon/2))\}_{x \in X}$$

is an open cover for X. Because X is a compact metric space it is also sequencially compact, and thus from Lebesgue's lemma we have that exists a Lebesgue number $\rho > 0$ for $\mathcal{U}$. Now let $x_1, x_2 \in X$ such that $d(x_1, x_2) < \rho$, by definition exists $x \in X$ such that $x_1, x_2 \in f^{-1}(B(f(x), \epsilon/2))$, thus:

$$d_Y(f(x_1), f(x_2)) \leq d_Y(f(x_1), f(x)) + d_Y(f(x), f(x_2)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

There is also a connection between compactness and total boundness as we see in the following proposition.

Proposition 4.18. *The metric space X is compact if and only if it is complete and totally bounded.*

Proof. To be added. □

Corollary 4.19. *Suppose that X is complete and let A be a nonempty closed subset of X. Then A is compact if and only if it is totally bounded.*

5 The Arzelà–Ascoli theorem

5.1 Algebras

Definition 5.1 (Algebra)**.** Let K be a field and A a vector space. Let $|\cdot| : A \times A \to A$ be a binary operation. Then A is called an algebra if for each $x, y, z \in V$ the following identites hold:

- Left distributiviy: $(x + y) \cdot z = x \cdot z + y \cdot z$.
- Right distributiviy: $z \cdot (x + y) = z \cdot x + z \cdot y$.
- Compatibility with scalars: $(ax) \cdot (by) = (ab)(x \cdot y)$.

Remark 5.1. These identites actually just imply that the operation is bilinear. An algebra over K is sometimes called a K-algebra and K is called the base field of A. Notice that we didn't require the operation to be associative or commutative, although some authors use the term "algebra" to refer to an associative algebra.

Definition 5.2 (K-algebra homomorphisms)**.** Given K-algebras A, B then a homomorphism of K-algebras is a K-linear map $f \colon A \to B$ such that $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in A$.

Remark 5.2. If A and B are unital then the morphism $f(1_A) = 1_B$ is called the unital homomorphism. The space of all K-algebra homomorphisms between A and B is usually written as $\mathrm{Hom}_{K\text{-alg}}(A, B)$. A K-algebra isomorphism is a bijective K-algebra homomorphism.

A subalgebra of a K-algebra A is a linear subspace of A such that all products and sums of the subspace are themselves elements of the subspace. For examples $\mathbb{R}$ with complex addition and multiplication as a subspace of the $\mathbb{R}$-algebra $\mathbb{C}$ is an example of a subalgebra.

Similarly to rings, algebras also have a concept of ideals. A left ideal L of a K-algebra A, is a linear subspace of A such that for any $x, y \in L$, $c \in K$, $z \in A$ the following three identities are satisfied:

- L is closed under addition: $x + y \in L$
- L is closed under scalar multiplication: $cx \in L$
- L is closed under vector multiplication from the left by arbitrary elements: $z \cdot x \in L$

We can similarly define a right ideal. An ideal that is both a left and a right ideal is called a two-sided ideal or simply an ideal. Notice that every ideal is a subalgebra and that in a commutative algebra any ideal is a two-sided ideal. Also notice that in contrast to an ideal of rings, here we also have a the requirement for closure under scalar multiplication and not just being a subgroup of addition. If the algebra is unital then the third requirement implies the second one.

You can also talk about extension of scalars but I don't know what that is yet.

Let (X, d) be a fixed compact metric space. Denote $C(X)$ the algebra of all continuous functions $f \colon X \to \mathbb{R}$ and $C_b(X)$ the subalgebra of all the boundded functions in $C(X)$. Because X is compact we know that the image $f(X)$ of any $f \in C(X)$ is compact and in particular bounded and thus $C_b(X) = C(X)$. This means we can set the norm $|\cdot|_\infty$ on $C(X)$. We can thus consider $C(X)$ as a metric space with the metric induced on it by $|\cdot|_\infty$. We will soon establish a useful characterisation of the compact sets in $C(X)$.

5.2 The Arzelà–Ascoli theorem

Definition 5.3 (Equicontinuity)**.** A subset $F \subset C(X)$ is called equicontinuous if for any $\varepsilon > 0$ exists a $\delta > 0$ such that $\left|f(x) - f(y)\right| < \varepsilon$ for any $f \in F$ and $x, y \in X$ with $d(x, y) < \delta$.

Theorem 5.1. (Arzelà–Ascoli theorem). *Let F be a nonempty closed subset of $C(X)$. Then F is compact if and only if it is bounded and equicontinuous.*

Remark 5.3. It is easy to see that F is bounded if and only if there exists $M > 1$ so that $|f(x)| \leq M$ for all $f \in F$ and $x \in X$.

Example 5.1. Let $K > 0$. Set $\mathbf{L}$ as the set of all $f \in C([0,1])$ such that $|f|_\infty \leq K$ that are K-Lipschitz. We will prove that $\mathbf{L}$ is compact. Given

$$f \in C([0,1]) \quad \text{and} \quad f_1, f_2, \dots \in \mathbf{L} \text{ s.t. } |f - f_n|_\infty \xrightarrow{n\to\infty} 0,$$

it is clear that

$$|f|_\infty \leq K \quad \text{and} \quad \forall x, y \in [0,1], \quad \left|f(x) - f(y)\right| = \lim_{n\to\infty} \left|f_n(x) - f_n(y)\right| \leq K|x - y|$$

which shows that $f \in \mathbf{L}$ and that means that $\mathbf{L}$ is closed in $C([0,1])$. Since $\mathbf{L}$ is a closed subset of $C([0,1])$, which is clearly also bounded and equicontinuous, from Theorem 5.1 we have that $\mathbf{L}$ is compact.

To prove Theorem 5.1 we need the following lemmas.

Lemma 5.2. *Let X be a metric space and let $A \subset X$ be totally bounded. Then every nonempty subset of A is totally bounded.*

Proof. Let $B \subset A$ be a nonempty subset of A. Let $\varepsilon > 0$, because A is totally bounded there exist $x_1, \dots, x_n \in A$ such that:

$$A \subset \bigcup_{i=1}^{n} B\left(x_i, \frac{\varepsilon}{4}\right).$$

Let $y_1, \dots, y_m$ be all the points x_i such that $B(x_i, \frac{\varepsilon}{4}) \cap B \neq \emptyset$. Now we have:

$$B \subset \bigcup_{i=1}^{m} B\left(y_i, \frac{\varepsilon}{4}\right).$$

Choose m arbirary points z_m that satisfy:

$$z_i \in B\left(y_i, \frac{\varepsilon}{4}\right) \cap B \neq \emptyset.$$

Now by the construction we have $z_1, \dots, z_m \in B$ and also:

$$B \subset \bigcup_{i=1}^{m} B\left(y_i, \frac{\varepsilon}{4}\right) \subset \bigcup_{i=1}^{m} B\left(z_i, \varepsilon\right),$$

where the last inclusion is following from the fact that:

$$\operatorname{diam} B\left(y_i, \frac{\varepsilon}{4}\right) = \frac{\varepsilon}{2} \quad \text{and} \quad z_i \in B\left(y_i, \frac{\varepsilon}{4}\right)$$

for every $1 \leq i \leq m$. ◻

Lemma 5.3. *Let $n \geq 1$ and equip $\mathbb{R}^n$ with its standard Euclidean metric. Then any nonempty bounded subset of $\mathbb{R}^n$ is totally bounded.*

Proof. Let A be a nonempty bounded subset of $\mathbb{R}^n$. Since A is bounded there exists $c \in \mathbb{R}^d$ and $r > 0$ such that $A \subset B(c, r)$. Without loss of generality we can assume that c is the origin, and then we have:

$$A \subset B(0, r) \subset \underbrace{[-r, r] \times \cdots \times [-r, r]}_{d \text{ times}} := R$$

From the previous lemma because we have $A \subset R$ and A is nonempty, it suffices to prove that R is totally bounded. Let $\epsilon > 0$, it is clear that the set:

$$N := \left\{ \left(-r + \frac{n_i \epsilon}{2} \right)_{i=1}^{d} \mid (n_i)_{i=1}^{d} \in \left[\left\lfloor \frac{2r}{\epsilon} \right\rfloor \right]^{d} \right\}$$

is an ϵ-net of R and thus R is totally bounded which completes the proof. □

We can now proceed to the proof of Theorem 5.1.

Proof. To be added. □

6 Seperation

Let X be a fixed topological space.

6.1 Definitions

6.1.1 T_1-spaces

Definition 6.1 (T_1-spaces)**.** We say that X is a T_1-space if and only if for every $x_1, x_2 \in X$ exist neighbourhoods U_1 of x_1 and U_2 of x_2 such that $x_1 \notin U_2$ and $x_2 \notin U_1$.

We can also verify that if X is a T_1-space then every topological subspace of X is also a T_1-space.

Proposition 6.1. *The space X is a T_1**-space** if and only if $\{x\}$ is closed in X for every $x \in X$.*

Proof. Suppose that X is a T_1-space. Let $x \in X$. For every $y \in X \setminus \{x\}$ exists a neighbourhood $U_y \subset X \setminus \{x\}$ the union of which gives $X \setminus \{x\}$ and then $\{x\}$ is closed as wanted. Now assume that $\{x\}$ is closed for every $x \in X$. For two points $x_1, x_2 \in X$ the sets $\{x_1\}, \{x_2\}$ are closed and thus we have $U_1 := X \setminus \{x_1\}$ neighbourhood of x_1 and $U_2 := X \setminus \{x_2\}$ neighbourhood of x_2 such that $x_1 \notin U_2$ and $x_2 \notin U_1$. □

6.1.2 Hausdorff spaces

Definition 6.2 (Hausdorff space)**.** We say that X is a Hausdorff space if for all distinct $x_1, x_2 \in X$ there exist open sets $U1, U2 \subset X$ with $x_1 \in U_1, x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

We can verify that every Hausdorff space is a T_1-space and that if X if a Hausdorff is a topological space then every subspace of X is also a Hausdorff space.

Proposition 6.2. *Let $\{X_i\}_{i\in I}$ be a nonempty family of Hausdorff spaces. Then the product space $\prod_{i\in I} X_i$ is also a Hausdorff space.*

Proof. Let $\{x_i\}_{i\in I}, \{y_i\}_{i\in I}$ be distinct points in $\prod_{i\in I} X_i$. Therefore exists $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$. Because X_{i_0} is a Hausdorff space there exist open sets $U_x, U_y \subset X_{i_0}$ with $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$. We know that the projection $\pi_{i_0} \colon \prod_{i\in I} X_i \to X_{i_0}$ is continuous and thus $\pi_{i_0}^{-1}(U_x)$ and $\pi_{i_0}^{-1}(U_y)$ are two open and disjoint sets of $\prod_{i\in I} X_i$ such that $\{x_i\}_{i\in I} \in \pi_{i_0}^{-1}(U_x)$ and $\{y_i\}_{i\in I} \in \pi_{i_0}^{-1}(U_y)$ as wanted. This shows that $\prod_{i\in I} X_i$ is a Hausdorff space which completes the proof. □

The following proposition is one of the most important properties of Hausdorff spaces.

Proposition 6.3. *Suppose that X is a Hausdorff space. Let K be a compact subset of X with $K \neq X$, and let $x \in X \setminus K$. Then there exist open sets $U, V \subset X$ so that $x \in U, K \subset V$ and $U \cap V = \emptyset$.*

Proof. First we may suppose that $K \neq \emptyset$ otherwise we could choose $U = X$ and $V = \emptyset$. Since X is Hausdorff for every $y \in K$ exist $U_y, V_y \subset X$ disjoint open sets such that $x \in U_y$ and $y \in V_y$. We have $K \subset \cup_{y\in Y} V_y$ but since K is compact exist $y_1, \ldots, y_n$ such that $K \subset \cup_{i=1}^n V_{y_i}$. We now define:

$$V := \cup_{i=1}^n V_{y_i}$$
$$U := \cap_{i=1}^n U_{y_i}$$

It is clear that both sets are open, and that $x \in U$ and $K \subset V$ and for every $i \in [n]$ we also see that:

$$Y_{y_i} \cap U \subset V_{y_i} \cap U_{y_i} = \emptyset$$

Which means that $U \cap V = \emptyset$ as wanted which completes the proof. □

Corollary 6.4. *Suppose that X is a Hausdorff space. Then every compact subset of X is closed.*

Proof. Let $K \subset X$ be compact. We may clearly assume that $K \neq X$. Given $x \in X \subset K$, it follows from the previous proposition that there exists a neighbourhood U of x which is contained in $X \setminus K$. This shows that $X \setminus K$ is a union of open sets, and so it is itself open. Thus K is closed, which completes the proof. □

One particularly useful result of this corollary is the following proposition:

Proposition 6.5. *Suppose that X is a Hausdorff space, let Y be a compact topological space, and let $f \colon Y \to X$ be a continuous bijection. Then f is a homeomorphism.*

Proof. All that's left to show is that f is an open map. Let $U \subset Y$ be open. It follows that $Y \setminus U$ is closed in a compact space and thus compact. Since f is continuous $f(Y \setminus U)$ is compact. From the previous corollary $f(Y \setminus U)$ is closed. Since f is a bijection we also have $f(Y \setminus U) = X \setminus f(U)$. This implies that U is open, so f is an open map and the proof is complete. □

7 Completely regular spaces and normal spaces

7.1 Completely regular spaces

Definition 7.1 (Seperating set)**.** We say that $C_b(X)$ separates points if for every distinct $x, y \in X$ there exists $f \in C_b(X)$ with $f(x) \neq f(y)$. In general, a set $\mathcal{S}$ of functions with domain D, is called a seperating set for D if for any two distinct elements x and y of D, there exists a function $f \in \mathcal{S}$ such that $f(x) \neq f(y)$.

Remark 7.1. It is clear that if $C_b(X)$ seperates points then X is a Hausdorff space.

The following definition strengthens this seperation property, which turns out to be quite convenient.

Definition 7.2 (Completely regular space)**.** We say that X is a completely regular space if,

(1) X is a T_1-space.

(2) for every closed subset F of X and $x \in X \setminus F$ there exists a function $f \in C_b(X)$ such that $f(X) \subset [0, 1]$ and $f(x) = 0$ and $f(y) = 1$ for all $y \in F$.

Notice that from condition (1) we have that $\{x\}$ is closed for every $x \in X$. From this follows that $C_b(X)$ seperates the points of X, and thus in particular X is a Hausdorff space.

Proposition 7.1. *Suppose that X is completely regular. Then every topological subspace of X is also completely regular.*

Proof. Let Y be a topological subspace of X. It is clear that Y is also a T_1-space. Let F be a closed subset of Y and let $x \in Y \setminus F$. Since F is closed in Y there must exist Q closed in X such that $F = Y \cap Q$. Since $x \notin Q$ and X is completely regular, there must exist $f \in C_b(X)$ such that $f(X) \subset [0, 1]$ and $f(y) = 0$ and $f(x) = 1$ for all $x \in Q$.

Now if we consider $g := f|_Y$, it is clear that $g(X) \subset [0, 1]$ and $g(y) = 0$ and $g(x) = 1$ for all $x \in F$, which shows that Y is completely regular and completes the proof. □

This next seperation property is very similar to the definition of a Hausdorff space, except that it applies to disjoint closed sets instead of points.

7.2 Normal spaces

Definition 7.3 (Normal space)**.** We say that X is a normal space if,

(1) X is a T_1-space.

(2) for every pair of disjoint closed subset F_1 and F_2 of X there exists disjoint open subsets U_1 and U_2 of X so that $F_1 \subset U_1$ and $F_2 \subset U_2$.

Remark 7.2. A topological subspace of a normal space is not necessarily normal.

Example 7.1. To be added.

Proposition 7.2. *Any metric space is normal.*

Proof. Let X be a metric space. Suppose that F_1 and F_2 are disjoint closed sets in X. Since F_1 and F_2 are disjoint closed sets we have that for any $y_1 \in F_1$ and $y_2 \in F_2$ that

$$d(y_1, F_2) > 0 \quad \text{and} \quad d(y_2, F_1) > 0.$$

Set,

$$U_1 := \bigcup_{y \in F_1} B(y, d(y, F_2)/2) \quad \text{and} \quad U_2 := \bigcup_{y \in F_2} B(y, d(y, F_1)/2).$$

It is clear that U_1 and U_2 are open sets in X and that $F_1 \subset U_1$ and $F_2 \subset U_2$.

To show that U_1 and U_2 are disjoint, we can assume by contradiction that exists $x \in U_1 \cap U_2$. Then exist $y_1 \in F_1$ and $y_2 \in F_2$ such that

$$x \in B(y_1, d(y_1, F_2)/2) \cap B(y_2, d(y_2, F_1)/2),$$

but this is clearly impossible, which completes the proof. □

Proposition 7.3. *Suppose that X is a compact Hausdorff space. Then X is normal.*

7.2.1 Urysohn's lemma

Lemma 7.4 (Urysohn's lemma)**.** *Let X be a normal topological space, and let A and B be disjoint closed subsets of X. Then there exists a continuous $f\colon X \to [0,1]$ so that $f(a) = 0$ for $a \in A$ and $f(b) = 1$ for $b \in B$.*

In order to prove Urysohn's lemma, we need another characterization of for normal spaces.

Lemma 7.5. *Let X be a T_1-space. Then X is a normal space if and only if for every closed $F \subset X$ and open $U \subset X$ such that $F \subset U$ there exists an open $V \subset X$ such that $F \subset V \subset \overline{V} \subset F$.*

We can now proceed to prove Urysohn's lemma.

Proof. To be added. □

Corollary 7.6. *Let X be a normal topological space, let A and B be disjoint closed subsets of X, and let $\alpha, \beta \in \mathbb{R}$ be with $\alpha < \beta$. Then there exists a continuous $f\colon X \to [\alpha, \beta]$ so that $f(a) = \alpha$ for $a \in A$ and $f(b) = \beta$ for $b \in B$.*

Proof. From Urysohn's lemma we know that exists a continuous function $f\colon X \to [0,1]$ such that $f(a) = 0$ for $a \in A$ and $f(b) = 1$ for $b \in B$. We can define the function

$$\begin{aligned} &g\colon [0,1] \to \mathbb{R} \\ &g(x) = (\beta - \alpha)x + \alpha. \end{aligned}$$

It is clear that the function $\varphi := f \circ g$ satisfies the conditions of the corollary and completes the proof. □

7.2.2 The Tietze extension theorem

Theorem 7.7. (The Tietze extension theorem). *Let X be a normal space and let F be a closed nonempty subset of X. Equip F with the subspace topology and let $f\colon F \to [\alpha, \beta]$ be continuous, where $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$. Then there exists a continuous $\tilde{f}\colon X \to [\alpha, \beta]$ so that $\tilde{f}(x) = f(x)$ for $x \in F$.*

The following example shows that the condition of F begin closed is necessary.

Example 7.2. Let $X = [0,1]$ and $F = (0,1]$. Set $f(x) := \sin\left(\frac{1}{x}\right)$ for $x \in F$. Then X is normal and f is continuous and bounded, but it is clear that f cannot be extended continuously to a function on X.

To prove Theorem 7.7 we need to prove the generalized form of the Weierstrass M-test we saw in analysis.

Lemma 7.8 (Weierstrass M-test)**.** *Let X be a topological space. Let $\{f_k\}_{k\geq 1} \subset C_b(X)$, and let $\{M_k\}_{k\geq 1} \subset (0,\infty)$ such that $\sum_{k\geq 1}^{\infty} M_k < \infty$ and $\left|f_k(x)\right| \leq M_k$ for all $k \geq 1$ and all $x \in X$. Then there exists a function $f \in C_b(x)$ such that $\sum_{i=1}^{k} f_i \xrightarrow{k\to\infty} f$ uniformly, and $\left|f(x)\right| \leq \sum_{k\geq 1}^{\infty} M_k$ for all $x \in X$.*

Proof. Given $\epsilon > 0$ there exists some $N \geq 1$ such that $\sum_{k\geq N} M_k < \epsilon$. Therefore, for all $n > m > N$ we have that

$$\left\|\sum_{i=1}^{n} f_i - \sum_{i=1}^{m} f_i\right\|_{\infty} \leq \left\|\sum_{i=m+1}^{n} f_i\right\|_{\infty} \leq \sum_{k\geq N} M_k < \epsilon,$$

which shows that $\{f_k\}_{k\geq 1}$ is a Cauchy space in $C_b(X)$. Since $C_b(X)$ is complete, this means that $\{f_k\}_{k\geq 1}$ converges uniformly to some $f \in C_b(X)$ which completes the proof. □

We can now go back to the proof of *Theorem* 7.7.

Proof. To be added. □

7.3 Embeddings

Definition 7.4 (Embedding)**.** Let X and Y be topological spaces. A mapping $f\colon X \to Y$ is said to be an embedding if it is a homeomorphism onto $f(X)$, where $f(X)$ is equipped with the subspace topology.

The following lemma is very useful when trying to prove a topological space is metrizable.

Proposition 7.9. *Let (X,τ) be a topological space, let (Y,d) be a metric space, and suppose that there exists an embedding $f\colon X \to Y$. Then X is metrizble.*

Proof. Denote $Z := f(X)$, and let d_Z be the metric on $Z \subset Y$ induced by d_Y. Also let $g\colon Z \to X$ be the function such that $f \circ g = \mathrm{id}_X$. Define the function$d_X$ such that for all $x_1, x_2 \in X$ we have

$$d_X(x_1,x_2) = d_Z(f(x_1), f(x_2)).$$

It is clear that d_X is a metric on X, and that f is an isometry from (X, d_X) onto (Z, d_Z) which also makes it a homeomorphism. Since f is a homeomorphism from (X,τ) to (Z, d_Z), and the composition of homoemorphisms is a homeomorphism, we have that $\mathrm{id}_X = g \circ f$ is a homeomorphism from (X,τ) to (X, d_X). This implies that τ is equal to the topology induced by d_X which shows that X is metrizable and completes the proof. □

7.3.1 ℓ^p spaces

Definition 7.5 (ℓ^1 space)**.** We define the normed space ℓ^1 with the norm N as such:

$$\ell^1 := \left\{x \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=1}^{\infty}|x_i| < \infty\right\} \quad \text{with} \quad N(x) := \sum_{i=1}^{\infty}|x_i|\,.$$

More generally, we define the normed space ℓ^p for $0 < p < \infty$ as

$$\ell^p := \left\{x \in \mathbb{R}^{\mathbb{N}} \mid \sum_{i=1}^{\infty}|x_i|^p < \infty\right\} \quad \text{with} \quad N(x) := \left(\sum_{i=1}^{\infty}|x_i|^p\right)^{\frac{1}{p}}.$$

And we can even define spaces like ℓ^∞, and see that ℓ^2 is a Hilbert space, but that is done in a functional analysis and will not be discussed in these notes.

7.3.2 The Uryshon embedding theorem

Theorem 7.10. (the Urysohn embedding theorem). *Let X be a second countable normal topological space. Then there exists an embedding $f\colon X \to \ell^1$.*

Proof. To be added. □

A very useful corollary of the Urysohn embedding theorem, coupled with the previous proposition, is the following corollary.

Corollary 7.11. *Let X be a second countable normal topological space. Then X is metrizable.*

7.4 Compactifications

7.4.1 The Stone–Čech compactification

Theorem 7.12 (Stone—Čech compactification)**.** *Let X be a completely regular space. Then there exists a compact Hausdorff space $\beta(X)$ and an embedding $\varphi\colon X \to \beta(X)$, such that*

(1) $\varphi(X)$ is dense in $\beta(X)$.

(2) for each $f \in C_b(X)$ there exists a unique $\tilde{f} \in C_b(\beta(X))$ such that $f = \tilde{f} \circ \varphi$.

Remark 7.3. The space $\beta(X)$ is called the Stone–Čech compactification of X. It is also customary to identify the space X with the space $\varphi(X)$. With this identification property (2) gives that every element of $C_b(X)$ extends uniqly to an element of $C_b(\beta(X))$.

To prove the theorem we will need the following lemma.

Lemma 7.13. *Let X be a completely regular space, and write τ for the given topology of X. Then τ is equal to the weak topology generated by $C_b(X)$.*

Proof. To be added. □

7.4.2 One-point compactification

Let X be a topological space and let ∞ be an object outside of X. Write $X_\infty := X \cup \{\infty\}$. Then the collection

$$\tau := \big\{U \mid U \subset X \text{ is open}\big\} \cup \big\{X_\infty \setminus K \mid K \subset X \text{ is compact}\big\}$$

is a topology on X_∞.

Remark 7.4. The space X_∞ is called the one-point compactification of X. The point ∞ is called the point at infinity.

Notice that ∞ is an isolated point of X_∞ if and only if X is compact.

Proposition 7.14. *For any topological space X, the space X_∞ is a compact Hausdorff space.*

Proof. To be added. □

Corollary 7.15. *Any locally compact Hausdorff space is completely regular.*

As an application of the one-point compactification, we also have the following stronger version of the previous corollary. It is an important tool in the theory of measure and integration on locally compact Hausdorff spaces.

Proposition 7.16. *Let K be a compact subset of X, and let U be an open subset of X with $K \subset U$. Then there exists a continuous $f\colon X \to [0,1]$ so that $f(x) = 0$ for $x \in K$ and $f(x) = 1$ for $x \in X \setminus U$.*

Proof. Since K and $X_\infty \setminus U$ are disjoint closed sets in the normal space X_∞, it follows from Urysohn's lemma that exists a continuous function $g\colon X_\infty \to [0,1]$ such that $g(x) = 0$ for $x \in K$ and $f(x) = 1$ for $x \in X_\infty \setminus U$. The proposition now follows from taking f to be the restriction of g on X. $\square$

Write $\mathbb{R}^2_\infty := \mathbb{R}^2 \cup \{\infty\}$. We will show that the unit sphere S^2 equipped with the topology induced on it by $\mathbb{R}^3$ is homeomorphic to $\mathbb{R}^2_\infty$. First, note that by the Heine–Borel theorem S^2 is compact. Now, set $N := (0,0,1)$ and let $\sigma\colon S^2 \to \mathbb{R}^2_\infty$ be with $\sigma(N) = \infty$ and

$$\sigma(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right) \quad \text{for } (x_1, x_2, x_3) \in S^2 \setminus \{N\}.$$

The restriction of σ to $S^2 \setminus \{N\}$ is called the stereographic projection. It is easy to see that σ is a continuous bijection. From this, since S^2 is compact, since $\mathbb{R}^2_\infty$ is Hausdorff, and by a previous proposition we have that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, so it follows that σ is a homeomorphism.

8 Connectednes

8.1 Connectedness

Definition 8.1 (Connectedness)**.** The space X is said to be connected if there does not exist disjoint nonempty open subsets U and V of X so that $X = U \cup V$. If X is not connected then we say that it is disconnected. A nonempty subset of X is said to be connected (respectively disconnected) if it is connected (respectively disconnected) as a topological subspace of X (i.e. when it is equipped with the subspace topology).

Notice that by the above definition, we can also characterize a connected space by having two disjoint nonempty closed sets U and V such that $X = U \cup V$, or be having a subset $U \subset X$ such that $U \neq X$ and $U \neq \emptyset$ and X is both open and closed.

Remark 8.1. It should be clear that for every $x \in X$ the subspace $\{x\}$ is connected.

Proposition 8.1. *Let Y be a nonempty subset of $\mathbb{R}$. Then Y is connected if and only if Y is an interval.*

Proof. First suppose that Y is not an interval. Then exist $a, b, t \in \mathbb{R}$ such that $a, b \in Y$ and $t \notin Y$ and $a < t < b$. Now set

$$U := (-\infty, t) \cap Y \quad \text{and} \quad V := (t, \infty) \cap Y.$$

It is clear that U and Y are two disjoint nonempty open subsets of Y such that $U \cap V = Y$. This shows that Y is disconnected.

Next suppose that Y is an interval. Assume by contradiction that Y is disconnected, then exist nonempty disjoint closed subsets A and B of Y such that $Y = A \cup B$. Choose arbitrary $a \in A$ and $b \in B$. Since $A \cap B = \emptyset$ we can assume without loss of generality that $a < b$. Denote $s := \sup([a, b] \cap A)$. Since A is closed in Y, it is sequencially compact and thus we have $s \in A$. Since $b \in B$ and $A \cap B = \emptyset$ we have that $s < b$. From this, and since Y is an interval we have $(s, b] \subset B$. However, since B is also sequencially compact in Y this means that $s \in B$, and thus $s \in A \cap B = \emptyset$. This contradiction shows that Y must be connected which completes the proof. ☐

Proposition 8.2. *Suppose that X is connected, let Y be a topological space, and let $f : X \to Y$ be continuous. Then $f(X)$ is connected.*

Proof. Assume by contradiction that $f(X)$ is connected. This means that exist U and V open subsets of Y such that

$$\underbrace{f(X) \subset U \cup V}_{(1)} \quad \text{and} \quad \underbrace{f(X) \cap U \neq \emptyset}_{(2)} \quad \text{and} \quad \underbrace{f(X) \cap V \neq \emptyset}_{(3)} \quad \text{and} \quad \underbrace{f(X) \cap U \cap V \neq \emptyset}_{(4)}.$$

Since f is continuous we have that $U^{-1} := f^{-1}(U)$ and $V^{-1} := f^{-1}(V)$ are open. From (1) we get that $X = U^{-1} \cup V^{-1}$. From (2) and (3) we get that U^{-1} and V^{-1} are nonempty. From (4) we have

$$\begin{aligned} U^{-1} \cap V^{-1} &= f^{-1}(U) \cap f^{-1}(V) \cap f^{-1}(f(X)) \\ &= f^{-1}(U \cap V \cap f(X)) = f^{-1}(\emptyset) = \emptyset \end{aligned}$$

which implies that X is disconnected. This contradicts our initial assumption, which shows that $f(X)$ must be connected and completes the proof. ☐

The following corollary is very useful, because it is a generalization of the mean value theorem from analysis.

Corollary 8.3. *Suppose that X is connected and let $f \in C(X)$ be given. Then $f(X)$ is an interval. In particular, for every $x, y \in X$ and $t \in \mathbb{R}$ with $f(x) < t < f(y)$ there exists $z \in X$ so that $f(z) = t$.*

In the following proposition we consider the space $\{0,1\}$ as a topological subspace of $\mathbb{R}$. It is clear that it is disconnected.

Proposition 8.4. *The space X is disconnected if and only if there exists a continuous mapping from X onto $\{0,1\}$.*

Proof. Suppose first that X is disconnected. Then exist disjoint closed nonempty set U and V such that $X = U \cup V$. Define the function $f\colon X \to \{0,1\}$ as such

$$f(x) = \begin{cases} 0, & x \in U \\ 1, & x \in V \end{cases}.$$

It is clear that f is continuous which completes the first side of this proposition.

Suppose next that there exists a continuous mapping from X onto $\{0,1\}$. If X were connected, it would follow that $\{0,1\}$ is also connected by Proposition 8.2. Since we know that $\{0,1\}$ is disconnected, it follows that X must be disconnected which completes the proof of this proposition. □

Proposition 8.5. *Let $\{Y_i\}_{i\in I}$ be a nonempty family of connected topological spaces. Then the product space $Y := \prod_{i\in I} Y_i$ is also connected.*

Proof. To be added. □

Corollary 8.6. *Let $n \geq 1$ and equip $\mathbb{R}^n$ with its standard Euclidean topology. Then $\mathbb{R}^n$ is connected.*

Proof. Since n is a finite number, we know that the standard topology on $\mathbb{R}^n$ is equal to its product topology. As we saw in Proposition 8.1 we have that $\mathbb{R}$ is connected. The corollary now immediately follows from Proposition Proposition 8.5. □

Proposition 8.7. *Let A be a connected subset of X, and let $B \subset X$ be with $A \subset B \subset \overline{A}$. Then B is connected. In particular $\overline{A}$ is connected.*

Proof. Assume by contradiction that B is disconnected. The exist nonempty disjoint open subsets U and V such that

$$\underbrace{B \subset U \cup V}_{(1)} \quad \text{and} \quad \underbrace{B \cap U \neq \emptyset}_{(2)} \quad \text{and} \quad \underbrace{B \cap V \neq \emptyset}_{(3)} \quad \text{and} \quad \underbrace{B \cap U \cap V \neq \emptyset}_{(4)}.$$

From (1) and (4) and $A \subset B$ we get that $A \subset U \cup V$ and $A \cap U \cap V \neq \emptyset$. Since U and V are open, and since we have (2) and (3) and $B \subset \overline{A}$ we get that $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$. It follows that A is disconnected which contradicts our initial assumption and completes the proof of the proposition. □

8.2 Path connectedness

Definition 8.2 (Path)**.** Let X be a fixed topological space. Let $x, y \in X$ be given. A path in X from x to y is a continuous map $\gamma\colon I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Definition 8.3. We say that X is path connected if for every $x, y \in X$ there exists a path in X from x to y. A nonempty subset of X is said to be path connected if it is path connected as a topological subspace of X (i.e. when it is equipped with the subspace topology).

Proposition 8.8. *Assume that X is path connected. Then it is also connected.*

Proof. Assume by contradiction that X is disconnected. Then there exist nonempty disjoint open subsets U and V of X such that $X = U \cup V$. Choose arbitrary $u \in U$ and $v \in V$. Since X is path connected there exists a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = u$ and $\gamma(1) = v$. However, since $[0,1]$ is an interval it follows from Proposition 8.1 that $\gamma([0,1])$ is connected. On the other hand, since $\gamma([0,1]) \subset X = U \cup V$ and $u \in \gamma([0,1]) \cap U$ and $v \in \gamma([0,1]) \cap V$ and $U \cap V = \emptyset$ it follows that $\gamma([0,1])$ is connected. This contradiction shows that X must be connected, which completes the proof. $\square$

One good pratice exercise is to prove that given a normed space $(Y,\|\cdot\|)$ and a nonempty convex $C \subset Y$, then C is path connected, and in particular, from the above proposition, connected.

8.2.1 the Topologist's sine curve

Definition 8.4 (Topologist's sine curve)**.** The topologist sine curve is defined as the set

$$T := \left\{ \left(x, \sin\tfrac{1}{x}\right) \mid x \in (0,1] \right\} \cup \{(0,0)\}.$$

The closed topologist's curve is defined as the closure of T and we get

$$\overline{T} = T \cup \big\{(0,y) \mid y \in [-1,1]\big\}\,.$$

Proposition 8.9. *The topologist's sine curve is connected but not path connected.*

Proof. To be added. $\square$

8.3 Connected components

Definition 8.5 (Connected component)**.** Let C be a connected subset of X. We say that C is a connected component of X if it is not properly contained in any connected subset of X.

Example 8.1. We have seen that $\mathbb{R}$ is connected, thus it has only one component that is itself.

Example 8.2. Equip $\mathbb{Q}$ with the subspace topology induced by $\mathbb{R}$. Let $E \subset \mathbb{Q}$ such that $|E| \geq 2$. Choose aribirary $a, b \in E$ such that $a < b$. Let $x \in (a,b) \setminus \mathbb{Q}$. Set

$$U := \big\{q \in \mathbb{Q} \mid q < x\big\} \quad \text{and} \quad V := \big\{q \in \mathbb{Q} \mid q > x\big\}\,.$$

We have that $E \subset U \cup V$ and $a \in E \cap U$ and $b \in E \cap V$ and $U \cap V = \emptyset$ which implies that the set E is disconnected. We thus have that $\big\{\{q\} \mid q \in \mathbb{Q}\big\}$ are the only connected components in $\mathbb{Q}$.

Proposition 8.10. *Let $\mathcal{C}$ be the collection of all connected components of X. For each $x \in X$ denote by C_x the union of all connected subsets of X containing x. We have the following properties,*

(1) $x \in C_x$ and $C_x \in \mathcal{C}$;

(2) $X = \cup_{C \in \mathcal{C}} C$ and $C_1 \cap C_2 = \emptyset$ for all $C_1, C_2 \in \mathcal{C}$;

(3) $A \in \mathcal{C}$ for every $A \subset X$ which is connected, open, and closed;

(4) C is closed for all $C \in \mathcal{C}$.

Definition 8.6 (Totally disconnected space)**.** We say that X is totally disconnected if its connected components are its points. That is, X is totally disconnected if each connected component C of X is of the form $C = \{x\}$ for some $x \in X$.

Definition 8.7 (Totally seperated space)**.** We say that X is totally separated if for all distinct $x, y \in X$ there exist disjoint open subsets U and V of X so that $x \in U$, $y \in V$ and $X = U \cup V$.

Remark 8.2. It is clear that every totally seperated space is a Hausdorff space.

Proposition 8.11. *Suppose that X is totally separated, then it is totally disconnected.*

Proof. Suppose that exists $Y \subset X$ such that $|Y| > 2$ and Y is a connected component of X. Since X is totally seperated, for any distinct $x, y \in Y$ there exist disjoint open subsets U and V of X so that $x \in U$, $y \in V$ and $X = U \cup V$. This clearly shows that Y is disconnected, which implies that X is totally disconnected. □

Example 8.3. As we saw in the previous example, we can show that $\mathbb{Q}$ equipped with the topology induced by $\mathbb{R}$ is totally seperated.

Example 8.4. Any discrete space is totally separated.

The following is an example of a totally disconnected space that is not totally seperated.

Example 8.5. Let ∞ and $-\infty$ be two objects outside of $\mathbb{N}$ and set $X = \mathbb{N} \cup \{\pm\infty\}$. Let τ be the topology on $\mathbb{N}$ generated by

$$\big\{\{n\} \mid n \in \mathbb{N}\big\} \cup \big\{\{\eta\} \cup (\mathbb{N} \setminus F) \mid n \in \{\pm\infty\} \text{ and } F \subset \mathbb{N} \text{ is finite}\big\}.$$

It is easy to show that (X, τ) is totally disconnected but not totally seperated.

Proposition 8.12. *Suppose that X is a T_1-space. Assume moreover that X has an open base $\mathcal{B}$ whose elements are also closed. Then X is totally separated.*

Proof. To be added. □

Proposition 8.13. *Suppose that X is compact and totally separated. Then X has an open base whose elements are also closed.*

Proof. To be added. □

8.4 Local connectedness

Definition 8.8 (Locally connected space)**.** We say that a topological space X is locally connected if for every $x \in X$ and every neighbourhood U of x there exists a connected neighbourhood V of x with $V \subset U$.

Example 8.6. Let X be a normed space, and let Y be a nonempty open subset of X. Then equip Y with the metric induced by the norm of Y. It is clear that any open ball in Y is convex, and as we saw previously, any convex subset of Y is connected. From this it follows directly that V is locally connected.

The following is an example of a locally connected space that is not connected.

Example 8.7. Set $V = (0, 1) \cup (2, 3)$ and equip V with the subspace topology induced by $\mathbb{R}$. It is easy to verify that V is locally connected but not connected.

The following is an example of a connected space that is not locally connected.

Example 8.8. Let T be the closed topologist's sine curve equipped with the subspace topology as a subspace of $\mathbb{R}$. It is connected, but it can be shown that it is not locally connected.

Proposition 8.14. *Let X be a locally connected topological space. Then the connected components of X are open sets.*

Proof. Let C be a connected component of X, and let $x \in C$ be given. Because X is locally connected there exists a connected neighbourhood U of x. Because C is a connected component, it is the union of all connected subspaces $E \subset X$ such that $x \in E$. In particular $U \subset C$. We proved that for any $x \in C$ there exists a neighbourhood U of x such that $x \subset U \subset C$. This means that C is open which completes the proof. $\square$