1 Measures and outer measures

1.1 Complete measure spaces

Exercise 1.1. Show that $(X, \mathcal{A}, \mu)$ is a complete measure space if and only if for every pair of measurable sets $A \subset B$ such that $\mu(B \setminus A) = 0$, we have that for all $C \subset X$ such that $A \subset C \subset B$ is measurable.

Solution. Let $(X, \mathcal{A}, \mu)$ be a complete measure space, and let $A \subset B$ such that $\mu(B \setminus A) = 0$. Let $C \subset X$ such that $A \subset C \subset B$. We have that $C \setminus A \subset B \setminus A$. Since $\mu(B \setminus A) = 0$ and the space is complete then $C \setminus A$ is measurable and A is measurable and $\mathcal{A}$ is and algebra, we have $C \setminus A \cup A = C$ is measurable.

Next let $A = \emptyset$. By the property of $(X, \mathcal{A}, \mu)$ we have that for any measurable set $\emptyset = A \subset B$ such $\mu(B \setminus \emptyset) = \mu(B) = 0$, we have that for any $\emptyset \subset C \subset B$ (any C) we have that C is measurable, and from $B \subset C$ and the monotonocity of μ we also have $\mu(C) \leq \mu(B) = 0$ so $\mu(C) = 0$ and that means exactly that $(X, \mathcal{A}, \mu)$ is complete.

Exercise 1.2. Let $(X, \mathcal{A}, \mu)$ be a measure space. Show that $(X, \mathcal{A}, \mu)$ has a unique minimal completion given by

$$\tilde{A} = \left\{ C \subset X \mid \exists A, B \in \mathcal{A} \text{ s.t. } A \subset C \subset B \text{ and } \mu(B \setminus A) = 0 \right\}$$

and a measure given by

$$\tilde{\mu}(C) = \mu(B)$$

for A, B, C from the definition of $\tilde{\mathcal{A}}$.

Solution. It is clear that $\emptyset \subset \tilde{\mathcal{A}}$ and since $\tilde{\mu}$ is an extension of μ we have that $\tilde{\mu}(\emptyset) = 0$.

To prove σ-additivity, let $C_1, C_2, \ldots$ be sets in $\tilde{\mathcal{A}}$. We know that exist a sequence of $\{A_n\}_{n=1}^{\infty}$ such that $A_n \subset C_n$ and $\{D_n\}_{n=1}^{\infty}$ all of measure 0 such that $C_n = A_n \cup D_n$. Therefore,

$$\biguplus C_n = \underbrace{\left(\biguplus D_n \right)}_{=0} \cup \left(\biguplus A_n \right)$$

and so

$$\mu\left(\biguplus_{i=1}^{\infty} C_n \right) = \mu\left(\biguplus_{i=1}^{\infty} A_n \right) = \sum_{i=1}^{\infty} \mu(A_n) \overset{\text{def}}{=} \sum_{i=1}^{\infty} \tilde{\mu}(C_n)$$

Exercise 1.3.

(1) Let $(X, \mathcal{A}, \mu)$ be a measure space. Assume that $\{A_n\}_{n=1}^{\infty}$ be sets with a μ-negligible intersection, that is

$$\mu(A_i \cap A_j) = 0, \quad \forall i \neq j.$$

Prove that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n).$$

(2) Assume that $\mathcal{A} = \sigma(\mathcal{E})$. Prove that

$$\mathcal{A} = \bigcup_{\substack{F \subset \mathcal{E} \\ |F| \leq \aleph_0}} \sigma(F).$$

Solution.

(1) We have that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\biguplus_{i=1}^{\infty}\left(A_i \setminus \bigcup_{j<i} A_i \cap A_j\right)\right) = \sum_{i=1}^{\infty} \mu\left(A_i \setminus \bigcup_{j<i} A_i \cap A_j\right) =$$

And we also have

$$\sum_{i=1}^{\infty} \mu\left(A_i \setminus \bigcup_{j<i} A_i \cap A_j\right) = \sum_{i=1}^{\infty} \mu\left(A_i \setminus \bigcup_{j<i} A_i \cap A_j\right) + \underbrace{\mu\left(\bigcup_{j<i} A_i \cap A_j\right)}_{=0}$$

and since these sets are disjoint it is equivalent the the sum of the unions, which are exactly A_n so,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

which completes the proof.

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