Riemann Surfaces

Based on lectures by
Notes taken by yehelip

Winter 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction **3**

1 Introduction

Definition 1.1 (Riemann surface)**.** A Riemann surface is a 1-dimensional complex manifold.

Definition 1.2 (Riemann surface)**.** A Riemann surface is a topological space X together with open subsets $\{U_k\}_{k\in I}$ of X with $\cup_{k\in I}U_k = X$ together with maps $f_i\colon U_i \to \mathbb{C}$ such that

(1) Each f_i is a homeomorphism onto its image.

(2) If $U_i \cap U_j \neq \emptyset$ then $f_i \circ f_j^{-1}\colon f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$ are *biholomorphic.*

Remark 1.1. A function $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic at p if $f'(p) = \lim_{z\to p} \frac{f(z)-f(p)}{z-p}$ exists.

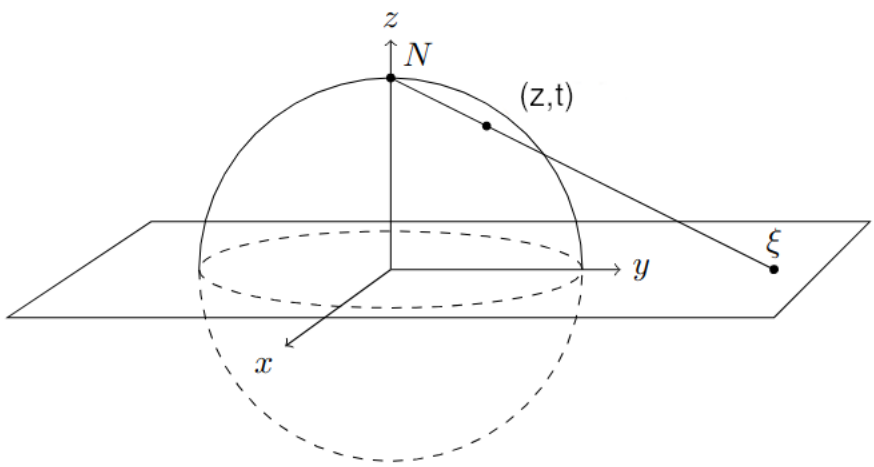
Definition 1.3 (Biholomorphism)**.** A function $f\colon \mathbb{C} \to \mathbb{C}$ is called biholomorphic if it has an inverse and both f and f' are holomorphic.

Definition 1.4 (Atlas)**.** The $\big\{(U_i, f_i)\big\}_{i\in I}$ are called an atlas of the Riemann surface.

Definition 1.5 (Chart)**.** Each individual (U_i, f_i) is called a chart of the Riemann surface.

Example 1.1. Let $U \subset \mathbb{C}$. Then U can take an atlas with one chart which is the identity map.

Example 1.2 (Riemann sphere)**.** Let $X = \big\{(z,t) \in \mathbb{C} \times \mathbb{R}\colon |z|^2 + t^2 = \mathbb{R}\big\}$. We identify $\mathbb{C}$ with the xy plane. Denote N and S the north and south poles of the sphere accordingly. We define $\pi_N\colon \mathbb{C} \to S$ such that π_N sends each point (z,t) on the sphere to its stereographic projection from N onto the plane (point ξ) as can be seen in the figure below:



We can similarly define π_S and verify that the images of the projections are $X\setminus\{N\}$ and $X\setminus\{S\}$ accordingly.

Now X is a Riemann surface with an atlas consisting of $\pi_S\colon X\setminus\{S\} \to \mathbb{C}$ and $\pi_N\colon X\setminus\{N\} \to \mathbb{C}$. We denote the Riemann sphere as $\hat{\mathbb{C}}$.

Definition 1.6 (Biholomorphism of Riemann surfaces)**.** Let $(X, (U_i, f_i))$, $(Y, (W_i, g_i))$ be two Riemann surfaces. A biholomorphism between them is a homeomorphism $X \xrightarrow{\phi} Y$ such that $g_i \circ \phi \circ f_i^{-1}$ are biholomorphisms on their domains of definition.

A main problem in Riemann surfaces was classifying certain types of Riemann surfaces up to biholomorphisms.

Theorem 1.1. (Riemann mapping theorem). *Any two proper open simply connected subsets of $\mathbb{C}$ are biholomorphic.*

A generalization of the Riemann mapping theorem is the uniformization theorem proved by Kobe in 1907.

Theorem 1.2. (Uniformization theorem). *Any simply connected Riemann surface is biholomorphic to one of the following:*

(1) $\mathbb{C}$

(2) $\hat{\mathbb{C}}$

(3) $\mathbb{H} = \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$

We will give a proof for this theorem in the end of the class.
A natural question that arises is what about non simply connected Riemann surfaces?

Theorem 1.3. (Uniformization theorem, part II). *Any connected Riemann surface is biholomorphic either to $\hat{\mathbb{C}}$ or to a quotient of $\mathbb{C}$ or $\mathbb{H}$ by a properly discontinuous torsion-free subgroup of biholomorphisms.*

Remark 1.2. Biholomorphisms of $U = \mathbb{C}$ or $\mathbb{H}$ (or any subset of $\mathbb{C}$) forms a group under composition. We denote that group by $\operatorname{Bih}(U)$.

Definition 1.7 (Properly discontinuous group)**.** A countable subgroup of $\operatorname{Bih}(U)$ is said to be properly discontinuous if for all compact $K \subseteq U$, the set $\{g \in G \colon gK \cap K \neq \emptyset\}$ is finite.

Definition 1.8 (Torsion-free group)**.** $G \subseteq \operatorname{Bih}(U)$ is torsion-free if $gp = p$ for some $p \in U$ implies g is the identity.

Remark 1.3. Notice that multiplication in gp is the group action of g on the set U. That us $gp = g(p)$.

We can know define the quotient space ${}^{U}/_{G}$ where $p \sim q$ if there exists $g \in G$ such that $gp = q$.
Introduce a topology on ${}^{U}/_{G}$ which is the coarsest topology such that the canonical projections $U \to {}^{U}/_{G}$ are continuous.
Under the assumptions that G is properly discontinuous and torsion-free, ${}^{U}/_{G}$ is a Riemann surface with the following charts. By assumptions on G, we can find for any $p \in U$ a neighbourhood W of $p \in U$ such that $\pi \colon U \to {}^{U}/_{G}$ is a homeomorphism onto its image when restricted to W.
So, restrictions of π to these neighbourhoods W give an atlas.