Practice

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1 Rings

Let R be an integral domain with a multiplicative identity. Find all the units of the polynomials ring: $R[x]$
We are looking for:
$$\{a \in R[x] | a \text{ is a unit in } R[x]\}$$
We can notice that for two polynomials $p(x)$, $q(x)$ that:
$$\deg(pq) = \deg p + \deg q$$
Let q be a unit in $R[x]$ and p be it's inverse then:
$$0 = \deg(1) = \deg(pq) = \deg p + \deg q$$
This means that:
$$\deg p + \deg q = 0$$
And since a degree of a polynomial must be a natural number the result is that all units are exactly the units in R so the set is:
$$\{a \in R[x] | a \text{ is a unit in } R\}$$

Find an invertible polynomial P in $Z_4[x]$ with $\deg P = 1$

We want to find a unit P of $Z_4[x]$, with $\deg P = 1$. Let $P = p_0 + p_1 x$ and let its inverse be $Q = q_0 + q_1 x$ since we don't need to raise the degree of P over 3 we may assume $\deg Q = 2$ then:

$$(p_0 + p_1 x)(q_0 + q_1 x) = p_0 q_0 + (p_0 q_1 + p_1 q_0)x + p_1 q_1 x^2$$

We want $p_1 q_1 = 0$ but since $p_1 \neq 0$ let us choose $p_1 = q_1 = 2$. Now we get:

$$(p_0 + 2x)(q_0 + 2x) = p_0 q_0 + 2(p_0 + q_0)x$$

So we need $p_0 + q_0 = 0$ or $p_0 + q_0 = 2$ and $p_0 q_0 = 1$. That means we can choose $p_0 = q_0 = 1$, and indeed:

$$PQ = (1 + 2x)(1 + 2x) = 1 + 4x + 4x^2 = 1$$

Since $P = Q$ it's clear that $PQ = QP = 1$ which completes the proof that $2x + 1$ is a unit in $\mathbb{Z}_4[x]$, with $\deg(1 + 2x) = 1$

Let R be a ring with a unit, and let $a \in R$ be right invertible - exists $b \in R$ such that $ab = 1$. Show that the following are equivalent:
(i) a **isn't invertible**
(ii) **exists** $c \neq b$ **such that** $ac = 1$
(iii) a **is a left zero divisor, i.e. exists** $0 \neq x \in R$ **s.t.** $ax = 0$ These can be proven using simple algebraic manipulation alone.

Let R be a finite ring with a unit. $|R| = p^2$ where p is prime. Show that R is commutative. Since R is of order p^2 we know that the additive group is either $\mathbb{Z}/p^2\mathbb{Z}$ and then we are done, or it is $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ with a center of order p, but then the quotient group $R/\mathbb{Z}(R)$ is cyclic and we can represent r, $s \in R$ as elements of those cosets. We shall write:

$$r = \alpha n + z \quad \text{and} \quad s = \alpha m + z'$$

For some $n, m \in \mathbb{Z}$ and $z, z' \in Z(R)$. Now:

$$\begin{aligned}
rs &= (\alpha n + z)(\alpha m + z') \\
&= \alpha n \alpha m + z \alpha m + \alpha n z' + z z' \\
&= \alpha m \alpha n + \alpha m z + z' \alpha n + z' z \\
&= (\alpha m + z')(\alpha n + z) \\
&= sr
\end{aligned}$$

And R is commutative.

Let R be a commutative ring with a unit and $I \triangleleft R$, Denote:

$$\sqrt{I} = \{x \in R | \exists n \in \mathbb{Z}_+ \textbf{ such that } x^n \in I\}$$

Show that $\sqrt{I} \triangleleft R$ and that $\sqrt{\sqrt{I}} = \sqrt{I}$
Let $r \in \sqrt{I}$ and $s \in R$ we know that for some n that

$$r^n = r' \in I$$

Since R is commutative we get:

$$(sr)^n = (rs)^n = r^n s^n = r' s^n \in I$$

Since I is an ideal. This means that $sr, rs \in \sqrt{I}$ so $\sqrt{I}$ is an ideal in R. For the second part of the proof:

$\underline{\sqrt{\sqrt{I}} \subseteq \sqrt{I}}$
Let $x \in \sqrt{\sqrt{I}}$, if exists an m such that:

$$x^m = x' \in \sqrt{I}$$

Then we know exists n such that we get:

$$(x)^{mn} = (x^m)^n = (x')^n \in I$$

Which means that $x \in \sqrt{I}$

$\underline{\sqrt{I} \subseteq \sqrt{\sqrt{I}}}$
Let $x \in \sqrt{I}$ then we get that:

$$x^1 = x \in \sqrt{I}$$

Which means that $x \in \sqrt{\sqrt{I}}$ and completes the proof.

If $I \neq R$ then $\sqrt{I} \neq R$

If $R \neq I$ then $\exists r \in R, r \notin I \Rightarrow 1 \notin I$ otherwise $r * 1 = r \in I$. If $\sqrt{I} = R$ then $1 \in \sqrt{I}$ but then exists $n \in \mathbb{Z}_+$ such that $1^n \in I$ which means $1 \in I$ and that is a contradiction to $1 \notin I$ so $\sqrt{I} \neq R$.

For $I = \{P(x)|P(0) = P'(0) = 0\} \triangleleft R[x]$ **find** $\sqrt{I}$

$$\begin{aligned}
&\{P(x) \in R[x] | \exists n \in \mathbb{Z}_+ \text{ s.t. } x^n \in I\} \\
=&\{P(x) \in R[x] | \exists n \in \mathbb{Z}_+ \text{ s.t. } P^n(0) = [P^n]'(0) = 0\}
\end{aligned}$$

From basic combinatorial calculations

$$\begin{aligned}
&\{P(x) \in R[x] | \exists n \in \mathbb{Z}_+ \text{ s.t. } P^n(0) = [P^n]'(0) = 0\} \\
=&\{P(x) \in R[x] | \exists n \in \mathbb{Z}_+ \text{ s.t. } a_0 = n a_0^{n-1} a_1 = 0\} \\
=&\{P(x) \in R[x] | a_0 = 0\} \\
=&\{P(x) \in R[x] | P(0) = 0\}
\end{aligned}$$

Let $R = \mathbb{Z}[\sqrt{7}]$ be the subring of $\mathbb{C}$ constructed by $\mathbb{Z}$ adjoined with $\sqrt{7}$:

$$R = \{a + b\sqrt{7} | a, b \in \mathbb{Z}\}$$

Let $I = 7R$, how many elements are in the quotient ring ${}^{R}/_{I}$?
There are exactly 49 elements in this quotient ring, and they are:

$${}^{R}/_{I} = \{a + b | a, b \in \{0, \dots, 6\}\}$$

Suppose two elements are in the same equivalence class we would get:

$$a + b\sqrt{7} - c + d\sqrt{7} \in I \Rightarrow a - c = 0 \land b - d = 0 \Rightarrow a = c \land b = d$$

So all the 49 elements are in unique equivalence classes. Now suppose we had another elements in ${}^{R}/_{I}$ that is $a + b\sqrt{7}$. We know that for $a^* = a \pmod 7$ and $b^* = b \pmod 7$ that $a^* + b^*\sqrt{7} \in {}^{R}/_{I}$. But then:

$$a + b\sqrt{7} - a^* + b^*\sqrt{7} = (a - a^*) + (b - b^*)\sqrt{7} = 7(\tilde{a} + \tilde{b}\sqrt{7}) \in I$$

For $\tilde{a}, \tilde{b} \in \mathbb{Z}$. So $a + b\sqrt{7}$ is in the equivalence class of $a^* + b^*\sqrt{7}$. That means there are no more than 49 elements in ${}^{R}/_{I}$ and thus we are done.

Find a subring S of $M_2(\mathbb{F}_7)$ that is isomorphic to R/I
The subring is:

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| a, b \in \mathbb{F}_7 \right\}$$

It is clear this set is closed under subtraction and multiplication thus it is a subring. We can also see that:

$$\varphi(a + b\sqrt{7}) = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

is a homomorphism since:

$$\varphi((a + b\sqrt{7}) + (c + d\sqrt{7})) = \begin{pmatrix} a+c & b+d \\ 0 & a+c \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \varphi(a + b\sqrt{7}) + \varphi(c + d\sqrt{7})$$

And:

$$\varphi((a + b\sqrt{7}) * (c + d\sqrt{7})) = \varphi(ac + 7bd + (ad + bc)\sqrt{7}) = \begin{pmatrix} ac + 7bd & ad + bc \\ 0 & ac + 7bd \end{pmatrix} = \begin{pmatrix} ac & ad + bc \\ 0 & ac \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} * \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \varphi(a + b\sqrt{7}) * \varphi(c + d\sqrt{7})$$

Since this is a homomorphism by the first theorem for isomorphisms we get:

$$R/_{\ker\varphi} \cong \mathrm{im}\varphi$$

It is clear that $\ker\varphi = I$ and $\mathrm{im}\varphi = S$ thus:

$$R/_I \cong S$$

Let $R = C([0,1])$ be the ring of continuous functions on $[0,1]$ with standard addition and multiplication of functions. For $a \in \mathbb{R}$ denote:

$$M_a = \{f \in R | f(a) = 0\}$$

Show that $M_a \triangleleft R$ is a maximal ideal.
First we will show it is an ideal.
Absorbs multiplication - Let $r \in R$ and $m \in M_a$ we see that:

$$(mr)(a) = m(a)r(a) = 0r(a) = r(a)m(a) = (rm)(a)$$

So $rm, mr \in M_a$
Additive subgroup - For $n, m \in M_a$ we see that $-m(a) = 0$ so $-m$ is in M_a then:

$$(n-m)(a) = n(a) - m(a) = 0$$

So $n - m \in M_a$ Which means $(M_a, +)$ is indeed an additive subgroup.
Suppose existed an ideal J that properly contains M_a. We will show it is equal to R. Let $g \in J$ such that $g \notin M_a$. Suppose $g(a) = c \neq 0$ that means that $c^{-1}g(a) = 1$. Consider the constant function $y_1(x) = 1$. since g, y_1 are continuous $y_1 - g$ is also continuous. We also notice that $(y_1 - g)(a) = y_1(a) - g(a) = 0$ so $(y_1 - g) \in M_a$. Since J is an ideal it is an additive subgroup of the ring so $(y_1 - g) + g \in J$. In other words $y_1 \in J$. We know that for any function f that $f * y_1 = f$ but since J is an ideal $f * y_1 = f \in J$ so $J = R$. This shows that M_a is a maximal ideal.

Let $\mathbb{F}$ be a field and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ different from one another. Show that for any $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ exists $f(x) \in \mathbb{F}[x]$ such that $f(\alpha_i) = \lambda_i$ for each i.
Instead of building the suggested isomorphism we instead will try to construct the polynomial directly. First we will notice that since the a_i's are all different for all $i \neq j$ we get that $a_i - a_j$ is invertibe. Now we need can in fact see that it is just going to be:

$$f(x) = \sum_{i=1}^{n} \lambda_i \prod_{i \neq j=1}^{n} (x - a_j)(a_i - a_j)^{-1}$$

We can see that it is a polynomial since it is merely a product and sum of polynomials and scalars of the field. We see that the product in the function will give us exactly 1 if $x = a_i$ since for all i we will simply get the product of scalars and their inverses. Then we multiply it by λ_i. So we get the sum of polynomials that give λ_i when $x = a_i$. Moreover whenever $x = a_j$ such that $j \neq i$ the product will be 0 since we will eventually multiply by $(a_j - a_j)(a_i - a_j)^{-1} = 0$. That means that for any a_i we get:

$$f(a_i) = \sum_{j=i}^{n} \lambda_j = \lambda_i$$

Let R be a commutative ring with unit. Show that the following are equivalent

1. R has a unique maximal ideal
2. The set of nonunits is an ideal
3. for all $x \in R$ either x is a unit or $1-x$ is a unit

$(1 \to 2)$ By Zorn's lemma according to what we have seen in class we get that each nonunit element is contained in a maximal ideal, but since it is unique, they all belong to the same ideal I in R. Suppose there was a unit in I which we shall denote x. Since I absorbs multiplication we would get $x^{-1}x \in I$ which would mean that $1 \in I$ thus $I = R$ but that can't be since I is a maximal ideal. Thus the set of all nonunit elements in R forms a ring.
$(2 \to 3)$ Let $I \triangleleft R$, be the ideal of all the nonunits in R. Let $x \in R$. Suppose both $x, 1-x$ are nonunits we would get that $x, 1-x \in I$ thus $(1-x)+x \in I$. So $1 \in I$ but that can't be since 1 is always a unit! So the assumption was false and either x or $1-x$ are units.
$(3 \to 1)$ Assume that exist two different ideals I_1, I_2 that are maximal in R. Since they are maxiaml we know that they can't include a unit element. Let $x \in I_1$, by what we stated it must be a nonunit. Now since we know that the sum of ideals is an ideal, and since I_1, I_2 are different maximal ideals, we get that $I_1 + I_2 = R$. That means that $1, 1-x \in I_1, I_2$, and since $1-(1-x) \in I_1$ and 1 must decompose into a nonzero elements in I_1, I_2 that $1-x \in I_2$ but since we said it must only contain nonunits then both $x, 1-x$ are nonunits in contradiction to (3) so (3) must imply that R has a unique maximal ideal.

Let

$$R = \left\{ \frac{a}{b} \middle| a, b \in \mathbb{Z} \wedge 3 \nmid b \right\}$$

Show that R has a unique maximal ideal M and find ${}^{R}/_{M}$
We know that the condition that for any $x \in R$ either x or $1 - x$ is a unit is equivalent to R having a unique maximal ideal so it suffices to show that instead. Let $\frac{a}{b} \in R$ such that $3 \nmid b$. If $3 \mid a$ we get that $\frac{b}{a} \in R$ and otherwise we calculate

$$1 - \frac{a}{b} = \frac{b - a}{b}$$

Now since $3 \nmid b$ and $3 \mid a$ we know that $3 \nmid b - a$ so that means that $\frac{b}{b-a} \in R$. We got what we wanted. Now we shall find ${}^{R}/_{M}$. We know that it must be the set of all non-unit elements in R, thus:

$${}^{R}/_{M} = \left\{ \frac{a}{b} \middle| 3 \mid a \wedge 3 \nmid b \right\}$$

So in fact ${}^{R}/_{M} = 3R$

The one with the homomorphism Let $\varphi : R \to S$ be a homomorphism of rings. And let $P \triangleleft S$ be a prime ideal. Show that $\varphi^{-1}(P)$ is a prime ideal of R And infer that if $R \subseteq S$ then $R \cup P$ is a prime ideal in R
Let $ab \in \varphi^{-1}(P)$. That means that $\varphi(ab) \in P$ so $\varphi(a)\varphi(b) \in P$ and since P is a prime ideal we get that either $\varphi(a) \in P$ or $\varphi(b) \in P$. Applying φ^{-1} we get that either $a \in \varphi^{-1}(P)$ or $b \in \varphi^{-1}(P)$, so $\varphi^{-1}(P)$ is a prime ideal. Now suppose $R \subseteq S$ We can define $\varphi : R \to S$:

$$\varphi(r) = r$$

And then $R \cup P = \varphi^{-1}(P)$ is a prime ideal in R.

Does the previous exercise holds if we switch between the words "prime" and "maximal"? It does not hold. For example consider $\varphi\mathbb{Z} \to \mathbb{Q}$ such that $\varphi(a) = a$. We know that $\{0\}$ is a maximal ideal in Q but $\varphi^{-1}(P) = \{0\}$ is not a maximal ideal in $\mathbb{Z}$.

Let R be a commutative integral domain with unit. Show that the following are equivalent:

1. $\langle a \rangle = \langle b \rangle$
2. $a \mid b \wedge b \mid b$
3. a and b are associates

(1 → 2)

Suppose that $\langle a \rangle = \langle b \rangle$ that means that:

$$\begin{aligned} b \in \langle a \rangle \Rightarrow b = ai, \quad i \in \mathbb{Z} \\ a \in \langle b \rangle \Rightarrow a = bj, \quad j \in \mathbb{Z} \end{aligned}$$

And since we know that $a \mid ai$ and $b \mid bj$ we get that:

$$a \mid b \wedge b \mid a$$

(2 → 3)

Suppose that $a \mid b \wedge b \mid b$, this means that exist $c, d \in R$ such that:

$$\begin{aligned} a &= cb \\ b &= da \\ \Rightarrow b &= dcb \Rightarrow dc = 1 \end{aligned}$$

Since R is commutative this implies that $dc = cd = 1$ which means that c, d are both units and the inverses of one another, so by definition a and b are associates.

(3 → 1)

Suppose a and b are associates, this means that exist $u, v \in R$ units such that:

$$\begin{aligned} a = ub \Rightarrow a \in \langle b \rangle \Rightarrow \langle a \rangle \subseteq \langle b \rangle \\ b = va \Rightarrow b \in \langle a \rangle \Rightarrow \langle b \rangle \subseteq \langle a \rangle \end{aligned}$$

Which implies that $\langle a \rangle = \langle b \rangle$ as wanted.

Let $f(x) = 2x^3 + x^2 + 2x + 2 \in \mathbb{Z}_5[x]$**. Find** $(2x+4+\langle f(x)\rangle)^{-1}$ **in the ring** $R = \mathbb{Z}_5[x] \big/ \langle f(x)\rangle$

Since we are working over $\mathbb{Z}_5[x] \big/ \langle f(x)\rangle$ we can denote $\langle f(x)\rangle$ as I and notice that in R we get:

$$2x + 4 + \langle f(x)\rangle = 2x + 4 + 0 = 2x + 4$$

Now suppose $g(x)$ is the inverse of $2x+4$, that would imply:

$$g(x)(2x+4) = 1 + I$$

Which in turn implies:

$$I = -1 + g(x)(2x+4)$$

We can find $g(x)$ by first dividing any $f(x) \in I$ by $2x+4$ over $\mathbb{Z}_5$. Let $f(x) = 2x^3 + x^2 + 2x + 2$ and then:

$$\begin{aligned} \frac{I}{2x+4} &= \frac{2x^3 + x^2 + 2x + 2}{2x+4} \\ &= x^2 + \frac{2x^2 + 2x + 2}{2x+4} \\ &= x^2 + x + \frac{-2x+2}{2x+4} \\ &= x^2 + x - 1 + \frac{1}{2x+4} \end{aligned}$$

Thus:

$$\begin{aligned} &I = (2x+4)(x^2+x-1)1 \\ &\Rightarrow -1 + I = (2x+4)(x^2+x-1) \\ &\Rightarrow 1 + I = (2x+4)(-1)^{-1}(x^2+x-1) = (2x+4)(4x^2+4x-4) \end{aligned}$$

We got that:

$$(2x+4)(4x^2+4x-4) = I + 1 = 1$$

Since we are working with a commutative ring this implies that:

$$(2x+4+\langle f(x)\rangle)^{-1} = (2x+4)^{-1} = 4x^2 + 4x - 4 + I$$

Show that $R = \mathbb{Z}[\sqrt{-2}]$ is a euclidean domain in respect to the norm:

$$N(a + b\sqrt{(-2)}) = a^2 + 2b^2$$

Let $a, b \in R$. We need to find $q, r \in R$ such that:

1. $a = bq + r$
2. $N(r) < N(b)$

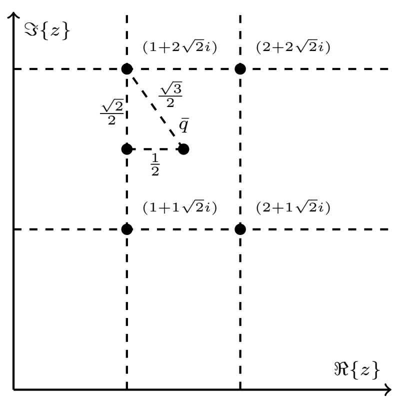
First notice that for $z = a + b\sqrt{-2} \in R$ we get:

$$N(z) = N(a + b\sqrt{-2}) = a^2 + 2b^2 = \left|a + b\sqrt{-2}\right|^2 = |z|^2$$

Where $||$ is the standard norm for complex numbers. Since the ring we are working with is a subring of $\mathbb{C}$ we can extend properties of the standard norm on complex numbers to N for example:

$$N(ab) = N(a)N(b)$$

Let $\bar{q} = \frac{a}{b} \in \mathbb{C}$, and let q be its closest element in R. By considering the geometry of the complex plane:



We can see that the maximal possible distance between q and $\bar{q}$ is $\frac{\sqrt{3}}{2}$, which means that:

$$|q - \bar{q}|^2 = N(q - \bar{q}) < \frac{3}{4}$$

Now if we define $r = b\bar{q} - bq$ we see that:

$$a = b\bar{q} \Rightarrow bq + r$$

And using the facts from earlier we get:

$$N(r) = N(b\bar{q} - bq) = N(b(\bar{q} - q)) = N(b)N(\bar{q} - q) < \frac{3}{4}N(b) < N(b)$$

As we wanted, which shows that R coupled with N is a euclidean domain.

Let R be a PID, F the field of fractions of R, and A a ring such that $R \subseteq A \subseteq F$. We shall think of the elements of F as fractions of the form $a, b \in R$, $[a, b] = \frac{a}{b}$. Show that if $[a, b] \in A$ for $a, b \in R$ that are coprime in R, then $[1, b] \in A$.

Let a, b be coprime in R. That means that $\langle a \rangle + \langle b \rangle = R$ thus exist $r_1, r_2 \in R$ such that:

$$1 = r_1 a + r_2 b$$

And now since $r_2, r_1 \in R$ they are in A as well and now:

$$\frac{a}{b} r_1 + r_2 = \frac{1}{b}(r_1 a + r_2 b) = \frac{a}{b}$$

Since A is closed under multiplication we know that $\frac{1}{b} \in A$

1.1 Show that A is a PID

Let $I \triangleleft A$, we know that $R \cap I$ is an ideal in R and since R is a PID we can look at its generator such that $\langle r \rangle = R \cup I$. We will now show that the ideal $\langle r \rangle_A$ which is the ideal generated by r in A is equal to I. It's clear that:

$$\langle r \rangle_A \subseteq I$$

Let $[a, b] \in I$, since $b \in A$ that means that

$$b * \frac{a}{b} = a \in I$$

but also we know that $a \in R$, so since $\langle r \rangle$ is a prime ideal in R it follows that:

$$\exists r' \in R \colon a = a'r \Rightarrow [a, b] = [a', b]r$$

Since $a', [1, b] \in A$ we see that $[a, b] = [a', b]r \in \langle r \rangle_A$ which shows:

$$I \subseteq \langle r \rangle_A$$

Finally:

$$I = \langle r \rangle_A$$

Which means that I is a prime ideal in A so A is a PID.

Let F be a field of character 0. A polynomial $f \in F[x]$ is called squareless if there does not exist $g(x) \in F(x)$ of positive degree such that $g^2 \mid f$. Show that f is squareless iff f, f' are coprime in $F[x]$

$(\Rightarrow)$

Let $f \in F[x]$ be squareless. Assume that f, f' are not coprime. That means that exists $0 \neq p \in F[x]$ irreducible and $q_1, q_2 \in F[x]$ such that:

$$f(x) = p(x)q_1(x)$$
$$f'(x) = p(x)q_2(x)$$

From this follows that:

$$f'(x) = [p(x)q_1(x)]' = p'(x)q_1(x) + p(x)q_1'(x) = p(x)q_2(x)$$

We notice that:

$$p'(x)q_1(x) = p(x)\left(q_2(x) - q_1'(x)\right)$$

So $p \mid p'q_1$, it can't divide p' since its different than 0 and F is a field of characteristic 0 so it must divide q_1. From this follows:

$$p \mid q_1 \Rightarrow p^2 \mid pq_1 = f$$

In contradiction to the assumption that f is squareless.

$(\Leftarrow)$

Let $f, f' \in F[x]$ be comprime, assume that exists $p \in F[x]$ such that:

$$p^2 \mid f$$

We know that exists $q \in F[x]$ such that:

$$f(x) = p^2(x)q(x)$$

But differentiating we get:

$$f'(x) = p^2(x)q'(x) + 2p(x)p'(x)q(x)$$

So we got that $p \mid f$ and $p \mid f'$ in contradiction to the assumption that they are not prime.

Let R be a PID and $\{0\} \neq I \triangleleft R$. Show that ${}^{R}/_{I}$ has a finite amount of ideals.
Since R is a PID exists $r \in R$ such that $I = \langle r \rangle$, it follows that:

$$ {}^{R}/_{I} = {}^{R}\Big/_{\langle r \rangle} = \{ar | a \in R\} $$

From the correspondence theorem for ideals we know that each ideal $A \triangleleft {}^{R}/_{I}$ exists a corresponding ideal $I \subseteq B \triangleleft R$. Since R is a PID we get that $B = \langle b \rangle$ and now since $I \subseteq B$ it follows that $b \mid r$. Since R is a PID it is a UFD, which implies that exist $p_1, \ldots, p_n$ irreducable and unique of to permutation and multiplication in units such that:

$$ r = \prod_i p_i $$

Which means that r has a finite number of divisors up to associativity of the elements, and in fact it has exactly 2^n such divisors. From the correspondence theorem that means ${}^{R}/_{I}$ has 2^n ideals if we show that ideals generated by associative elements are equal. But that's exactly what we have shown on the first question, so ${}^{R}/_{I}$ has exactly 2^n ideals, which is a finite amount.

2 Fields

Let ω be an algebraic number of an odd degree over the field F and show that $F[\omega] = F[\omega^2]$

$F[\omega^2] \subseteq F[\omega]$

We know that $\omega \in F[\omega]$ so since it is closed under multiplication we get $\omega^2 \in F[\omega]$. So now we can generate $F[\omega^2]$ in $F[\omega]$ and as we wanted we get that $F[\omega^2] \subseteq F[\omega]$

$F[\omega] \subseteq F[\omega^2]$

Because we have shown that $F[\omega^2] \subseteq F[\omega]$ we get that:

$$[F[\omega^2] \colon F] \mid [F[\omega] \colon F]$$

Assume that:

$$[F[\omega^2] \colon F] < [F[\omega] \colon F]$$

Since we know ω is of an odd degree so $F[\omega] \subseteq F[\omega]$ is odd we also get that:

$$[F[\omega^2] \colon F] < \frac{[F[\omega] \colon F]}{2}$$

Denote $[F[\omega^2] \colon F]$ as n and the minimal polynomial of ω^2 as $p(x) = a_n x^n + \cdots a_0$ as see that:

$$p(\omega^2) = a_n \omega^{2n} + \cdots + a_0 = 0$$

This is a polynomial of degree $2n$ with root ω so we get that:

$$[F[\omega] \colon F] \leq 2n$$

Which is a contradiction to:

$$n = [F[\omega^2] \colon F] < \frac{[F[\omega] \colon F]}{2}$$

So we got that:

$$[F[\omega^2] \colon F] = [F[\omega] \colon F]$$

Since we know that $F[\omega^2] \subseteq F[\omega]$ we get that:

$$\boxed{F[\omega^2] = F[\omega]}$$

Let θ be a complex root of $p(x) = x^3 - 9x + 6$, express $\frac{1}{\theta+1}$ as a rational polynomial in θ
We want to get:

$$\frac{1}{1+\theta} = \sum_{i=0}^{n} a_n x^n$$

If we multiply both sides by $1+\theta$ we get that:

$$1 = \sum_{i=0}^{n} a_n x^n + \theta \sum_{i=0}^{n} a_n x^n$$

So:

$$a_n \theta^{n+1} + \sum_{i=1}^{n} (a_n + a_{n-1}) x^n + a_0 - 1 = 0$$

We know that θ is a root of $p(x)$ so for $n = 2$ and some c we get that:

$$a_2\theta^3 + (a_2 + a_1)\theta^2 + (a_1 + a_0)\theta + a_0 - 1 = c(\theta^3 - 9\theta + 6)$$

This implies that $a_2 = c$ and we also see that:

$$a_2 + a_1 = 0 \Rightarrow a_1 = -c$$

And:

$$a_1 + a_0 = -9c \Rightarrow a_0 = -8c$$

And finally:

$$a_0 - 1 = 6c \Rightarrow c = -\frac{1}{14}$$

We now found all the constants we needed so we get:

$$\boxed{\frac{1}{1+\theta} = -\frac{1}{14}\theta^3 + \frac{9}{14}\theta - \frac{3}{7}}$$

Calculate the degree and the minimal polynomial of $\alpha = \sqrt{3+\sqrt{2}}$ over $\mathbb{Q}$
Calculate to get:

$$\begin{aligned}
&\alpha = \sqrt{3+\sqrt{2}} \\
&\Rightarrow \alpha^2 - 2 = \sqrt{2} \\
&\Rightarrow \alpha^4 - 6\alpha^2 + 7 = 0
\end{aligned}$$

Solving for the roots of the equation:

$$m(x) = x^4 - 6x^2 + 7 = 0$$

Gives:

$$x = \pm\sqrt{3 \pm \sqrt{2}}$$

Since x isn't rational it doesn't split to polynomials of degrees 1 and 3, but it might split into 2 polynomials of degree 2. We will show that is not the case. Suppose it could split into two polynomials with degree 2 then each could be written as:

$$q(x) = (x - x_i)(x - x_j) = x^2 - (x_i + x_j)x + x_i x_j$$

For $x_i \neq x_j$ roots we calculated earlier. If the polynomial is rational we get that $x_i x_j \in \mathbb{Q}$ but we can see it's not the case since for any of the roots there are we get:

$$x_i x_j = \left(\pm\sqrt{3 \pm \sqrt{2}}\right)\left(\pm\sqrt{3 \pm \sqrt{2}}\right) = \left\{3 \pm \sqrt{2}, \sqrt{7}\right\}$$

These are all not rational and therefore $q(x) \notin \mathbb{Q}[x]$ so we got that:

$$\boxed{m(x) = x^4 - 6x^2 + 7}$$

Does not split any more, which implies it is the minimal polynomial of α. We may also note that its degree is 4.

Let p be prime. Find the degree of the splitting field of $p(x) = x^p - 2$ over $\mathbb{C}$

First we will find the roots over $\mathbb{C}$ like this:

$$x^p = 2$$

So:

$$r^p e^{pi\theta} = 2$$

This means that:

$$r = \sqrt[p]{2} \quad \text{and} \quad \theta \in \left\{ \frac{2\pi}{p} k \right\}$$

For $0 \leq k \leq p - 1$.

3 Practice

Let R, S be rings and $f\colon R \to S$ be a homorphism. Let $I \triangleleft R$. Show that if f is surjective then $f(I) \triangleleft S$

We can see it is a subgroup with regards to addition because if we let $f(a), f(b) \in f(I)$ then since I is an ideal $(a-b) \in I$ so $f(a-b) = f(a) - f(b) \in I$. Now let $f(r) \in f(I)$ and $s \in S$ since f is surjective we know exists $s' \in I$ such that $f(s') = s$ and then $f(r)s = f(r)f(s') = f(rs') \in f(I)$ so we got an ideal.

Give a counter example if we don't require f to be surjective
We can choose:

$$f\colon \mathbb{Z} \to \mathbb{Q}$$
$$f(z) = z$$

And then we can send any ideal for example $\mathbb{Z}$ itself. But $\mathbb{Z}$ is not an ideal in $\mathbb{Q}$.

If $I\{0\}$ is a prime ideal in R then R is integral domain

Suppose $ab = 0 \in I$, since I we get $a \in I$ or $b \in I$. But since $I = \{0\}$ we get $a = 0$ or $b = 0$ which completes the proof.

Suppose every ideal in R is prime. Prove that if $a = bc$ for $a \neq 0$ then either b or c are units
Since $\{0\}$ is always an ideal, and in our case it is given to be prime we can assume R is an integral domain. Let $a = bc$ such that $a \neq 0$. Since we have $bc \in \langle \text{a} \rangle$ we get that either b or c are in $\langle \text{a} \rangle$, WLOG we can assume $b = ad$ and then if we substitute this in the original equation we get $a = adc$ so we get $a(1 - dc) = 0$ but since we know R is an integral domain and $a \neq 0$ we have $cd = dc = 1$ which means c is a unit.

We need to show that R is a field

All that's really left to show is that all the elements are units. We can consider the following fact for $a \neq 0$ we have $a^2 = aa$, since we are in an integral domain $a^2 \neq 0$ and then from the previous exercise we have a is a unit.

In a Euclidean domain R for every ideal $\{0\} \neq I \triangleleft R$ there is a finite amount of maximal ideals in ${}^{R}/_{I}$

We know that R is a Euclidean domain so it is a PID which means that for every ideal $I \triangleleft R$ it is generated by an element a which means we can write $I = \langle \mathrm{a} \rangle$ and then from the correspondence theorem we have that the ideals in ${}^{R}/_{I}$ are corresponding to the ideals that contain I but these correspond to the divisors of a, and since R as a Euclidean domain is also a UFD we know that that number of divisors is finite under associatives and since associate elements generate the same ideal, we get that that number is finite as needed.

Let $n \geq 2$ show that $\mathbb{Z} \big/ \langle \mathbf{n} \rangle$ has nilpotent elements different than 0 if and only if n has a prime square in it's factorization

Suppose n had a prime sqaure in its factorization we would have:

$$n = p^2 r$$

We can consider the element $pr + \mathbb{Z}$ and see that:

$$(pr + n\mathbb{Z})(pr + n\mathbb{Z}) = (p^2 r)r + n\mathbb{Z} = 0 + n\mathbb{Z}$$

From the other direction we get that if $n \big/ \langle \text{a} \rangle$ has a nilpotent element $b + \mathbb{Z} \neq 0 + \mathbb{Z}$ we would have:

$$b^2 = cn$$

For some $c \in \mathbb{Z}$ so we get:

$$b = \sqrt{c}\sqrt{n}$$

If n doesn't have prime squares then $c = nr^2$ which leads to $c \geq n$ but that can't be the case since $b < n$.

Let Ω be a field extension of F and let $a \in \Omega$ Show that a is algebraic if and only if $F(a)$ is a finite extension of F.
If $F(a)$ is a finite extension of F of degree n then we get that:

$$A = \{1, a, a^2, \ldots, a^n\}$$

Is linearly dependent so exists a polynomial of degree n with a as its root so a is algebraic. If a is algebraic then we can denote its minimal polynomial m and now we can show that ?????

Given a prime number p prove that exists an indecomposable $f(x) \in \mathbb{Q}[x]$ of degree p
We can consider the polynomial:

$$f(x) = x^p - p$$

We can use Eisenstein's criterion to show this satisfies the conditions. We see:

$$p \mid a_0 = p$$
$$p \nmid a_n = 1$$
$$p^2 \nmid a_n = 1$$

So it is indecomposable.

Define ${}^{L}/_{K}$ an algebraic extension of K.
L is an algebraic extension of K if for any $a \in L$ exists $f_a(x) \in K[x]$ such that:

$$f_a(a) = 0$$

Let Ω be a field extension of $\mathbb{Q}$ that contains a root for every polynomial in $\mathbb{Q}[x]$. Show that Ω contains an infinite algebraic extension of $\mathbb{Q}$