# Complex Analysis

Based on lectures by
Notes taken by yehelip

Winter 2025

These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

# Contents

# 1   Introduction

## 1.1   Complex numbers and the complex plane

### 1.1.1   Preliminaries

**Definition 1.1** (Complex number). A complex number is an expression of the form $x + yi$ such that $x, y \in \mathbb{R}$ and $i$ is a 'imaginary number' not in $\mathbb{R}$. We denote

$$\Re(z) := x \quad \text{and} \quad \Im(z) := y.$$

If $\Im(z) = 0$ then $z$ is said to be a real number, and if $\Re(z) = 0$ then it is said to be purely imaginary.

    The set of all complex numbers is denoted as $\mathbb{C}$ and it can be made into a field with the following operations.
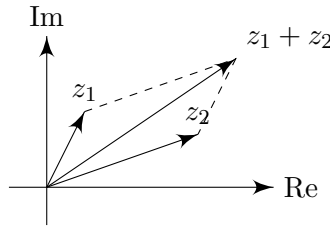
$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i \quad \text{and} \quad z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + y_1 x_2)i.$$

The field $\mathbb{C}$ is called the complex plane.

    Note that $i^2 = -1$. Also note that $T(x + yi) = (x, y)$ is a bijection between $\mathbb{C}$ and $\mathbb{R}$ and moreover, we have that $T$ is additive. That is

$$T(z_1 + z_2) = T(z_1) + T(z_2)$$

which gives complex addition a geometric meaning.



    The absolute value of a complex number $x + yi = z \in \mathbb{C}$ is defined by

$$|z| = \sqrt{x^2 + y^2}.$$

Note that $|z| = \big\|(x, y)\big\| = \big\|T(z)\big\|$ where $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^2$.

    This implies that $|z - w|$ should be considered the distance between natural numbers $z$, $w$. Because we have that $|z| = \big\|T(z)\big\|$ we also have that the triangle inequality holds:

$$|z + w| \leq |z| + |w| \text{ for all } z, w \in \mathbb{C}.$$

**Definition 1.2** (Complex conjugate). The complex conjugate of $x + yi = z \in C$ is the complex number $x - yi$. The complex conjugate of $z$ is denoted $\bar{z}$.
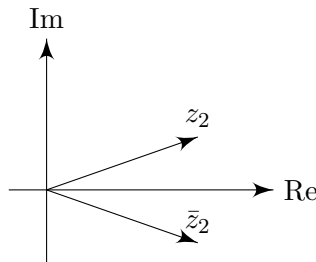
It is easy to verify that

$$\Re(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \Re(z) = \frac{z - \bar{z}}{2i} \quad \text{and} \quad |z|^2 = z\bar{z}.$$

Given $\theta$ we can denote $e^{i\theta} = \cos\theta + i\sin\theta$, and then describe any complex number $z \in \mathbb{C}$ as $re^{i\theta}$ for some $\theta \in [0, 2\pi)$ and $r > 0$. We get that $|z| = |re^{i\theta}| = r$. We also have that $\theta$ describes the angle of $z$ with the $x$-axis and it is usually denoted $\theta = \arg(z)$.

### 1.1.2 Convergence

**Definition 1.3** (Convergence)**.** We say that the sequence $\{z_n\}_{n\geq 1} \subset \mathbb{C}$ converges to some $z_0 \in \mathbb{C}$ if $|z - z_0| \xrightarrow{n\to\infty} 0$. In this case, we call $z_0$ the limit of the sequence of $\{z_n\}_{n\geq 1}$.

**Remark 1.1.** It is easy to verify that the limit is unique, and that $z_n \xrightarrow{n\to\infty} z$ if and only if $T(z_n) \xrightarrow{n\to\infty} T(z)$ in the Euclidean metric.

**Definition 1.4** (Cauchy sequence)**.** A sequence $\{z_n\}_{n\geq 1}$ is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists $N > 1$ such that for all $n, m > N$ we have that $|z_n - z_m| < \epsilon$.

**Proposition 1.1.** *The complex plane $\mathbb{C}$ is complete. That is, every Cauchy sequence converges in $\mathbb{C}$.*

*Proof.* The proof follows immediately from the known fact that $\mathbb{R}$ is complete and the previous remark. $\qquad\square$

### 1.1.3 Sets in the complex plane

**Definition 1.5** (Open disc)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$D_r(z_0) := \left\{ z \in \mathbb{C} \colon |z - z_0| < r \right\}.$$

We call $D_r(z_0)$ the open disc at center $z_0$ with radius $r$.

**Definition 1.6** (Closed disc)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$\overline{D}_r(z_0) := \left\{ z \in \mathbb{C} \colon |z - z_0| \leq r \right\}.$$

We call $\overline{D}_r(z_0)$ the closed disc at center $z_0$ with radius $r$.

**Definition 1.7** (Circle)**.** For $z_0 \in \mathbb{C}$ and $r > 0$ we set

$$C_r(z_0) := \left\{ z \in \mathbb{C} \colon |z - z_0| = r \right\}.$$

We call $C_r(z_0)$ the circle at center $z_0$ with radius $r$.

**Definition 1.8** (Interior point)**.** Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of $\Omega$ if exists $r > 0$ such that $D_r(z) \subset \Omega$.

**Definition 1.9** (Interior of a set)**.** Given $\Omega \subset \mathbb{C}$, we say that the interior of $\Omega$ is the collection of all interior points of $\Omega$. We denote the interior as $\text{Int}(\Omega)$.

**Definition 1.10** (Open set)**.** Given $\Omega \subset \mathbb{C}$, we say that $\Omega$ is an open set if $\text{Int}(\Omega) = \Omega$.

**Definition 1.11** (Closed set)**.** Given $\Omega \subset \mathbb{C}$, we say that $\Omega$ a closed set if $\Omega^c := \mathbb{C} \setminus \Omega$ is open.

**Definition 1.12** (Limit point)**.** Given $\Omega \subset \mathbb{C}$, we say that $z \in \Omega$ is an interior point of $\Omega$ if there exists a sequence $z_n$ such that $z_n \neq z$ for all $n > 1$ and $z_n \xrightarrow{n\to\infty} z$.

**Proposition 1.2.** *Let $\Omega \subset \mathbb{C}$ be given. Then $\Omega$ is closed if and only if it contains all of its limit points.*

*Proof.* Clear. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Definition 1.13** (Closure)**.** Let $\Omega \subset \mathbb{C}$ be given. The closure of $\Omega$, denoted $\overline{\Omega}$, is defined as

$$\overline{\Omega} = \Omega \cup \left\{ z \in \mathbb{C} \mid x \text{ is a limit point of } \Omega \right\}.$$

**Remark 1.2.** Note that $\Omega$ is closed if and only if $\overline{\Omega} = \Omega$.

**Definition 1.14** (Boundary)**.** The boundary of $\Omega \subset \mathbb{C}$ is denoted by $\partial\Omega$ and defined by $\partial\Omega := \Omega \setminus \mathrm{Int}(\Omega)$.

**Definition 1.15** (Diameter)**.** Given $\Omega \subset \mathbb{C}$, we define the diameter of $\Omega$ as

$$\mathrm{diam}(\Omega) := \sup \left\{ |z - w| \colon z, w \in \Omega \right\}.$$

**Definition 1.16** (Bounded set)**.** Given $\Omega \subset \mathbb{C}$, we say that $\Omega$ is bounded if $\mathrm{diam}(\Omega) < \infty$.

**Remark 1.3.** It is clear that a set $\Omega \subset \mathbb{C}$ is bounded if and only if there exists $z_0 \in \mathbb{C}$ and $r > 0$ such that $\Omega \subset D_r(z_0)$.

**Definition 1.17** (Compact set)**.** A subset $\Omega$ of $\mathbb{C}$ is said to be compact if it is closed and bounded.

**Theorem 1.3. (Bolzano–Weierstrass theorem).** *A subset $\Omega$ in $\mathbb{C}$ is compact if and only if every seqeuence $\{z_n\}_{n\geq 1}$ has a subsequence $\{z_{n_k}\}$ such that $z_{n_k} \xrightarrow{k\to\infty} z$ for some $z \in \mathbb{C}$.*

**Theorem 1.4. (Cantor's intersection lemma).** *Let $\Omega_1, \Omega_2, \ldots$ be nonempty compact subsets of $\mathbb{C}$. Suppose that $\Omega_{n+1} \subset \Omega_n$ for all $n \geq 1$, and that $\mathrm{diam}(\Omega_n) \xrightarrow{n\to\infty} 0$. Then $\cap_{n\geq 1}\Omega_n = \{z\}$ for some $z \in \mathbb{C}$.*

*Proof.* Choose $z_n \in \Omega_n$ for all $n \geq 1$. Because $\mathrm{diam}\,\Omega \xrightarrow{n\to\infty} = 0$ we have that $\{z_n\}_{n\geq 1}$ is a Cauchy sequence and therefore it converges to some $z \in \mathbb{C}$. Because $\Omega_n$ is compact for every $n \geq 1$ we get that $z \in \cap_{n\geq 1}\Omega_n$. This means that $\cap_{n\geq 1}\Omega_n \neq \emptyset$.

Let $z, w \in \Omega$. Because $\mathrm{diam}\,\Omega \xrightarrow{n\to\infty} 0$ we have that $|z - w| \leq 0$ and thus $z = w$ which implies that $\cap_{n\geq 1}\Omega_n = \{z\}$ which completes the proof. $\qquad\qquad\square$

**Definition 1.18** (Connected open set)**.** A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty open subsets $\Omega_1, \Omega_2$ such that $\Omega = \Omega_1 \cup \Omega_2$. A connected open set in $\mathbb{C}$ will be called a region.

**Definition 1.19** (Connected closed set)**.** A nonempty open set $\Omega \subset \mathbb{C}$ is said to be connected if it does not contain disjoint nonempty closed subsets $\Omega_1, \Omega_2$ such that $\Omega = \Omega_1 \cup \Omega_2$.

**Remark 1.4.** It can be shown that $\Omega$ is connected if and only if for any $z, w \in \Omega$ there exists a curve $\gamma \colon [0,1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1)$. This implies that open and closed discs, as well as circles, are connected.

### 1.1.4 Continuous functions

**Definition 1.20** (Continuous function)**.** Let $\Omega$ be a nonempty subset of $\mathbb{C}$ and let $f \colon \Omega \to \mathbb{C}$ be given. We say that $f$ is continuous at a point $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists $\delta > 0$ so that $|f(z) - f(z_0)| < \epsilon$ for all $z \in \Omega$ with $|z - z_0| < \delta$. We say that $f$ is continuous on $\Omega$ if it is continuous at every $z_0 \in \Omega$.

**Remark 1.5.** It is easy to verify that the functions $\Im$, $\Re$, $|\cdot|$, and $\theta \mapsto e^{i\theta}$ are all continuous.

**Proposition 1.5.** *The composition of continuous functions is continuous.*

**Definition 1.21** (Bounded function)**.** Let $\Omega$ be a nonempty subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that $f$ is bounded if there exists $M > 0$ so that $|f(z)| < M$ for all $z \in \Omega$. We say that $f$ attains a maximum if there exists $z_M \in \Omega$ such that $f(z) \leq f(z_M)$ for all $z \in \Omega$. We define when $f$ attains a minimum similarly.

**Proposition 1.6.** *Let $\Omega$ be a nonempty compact subset of $\mathbb{C}$, and let $f\colon \Omega \to \mathbb{C}$ be continuous. Then $f$ is bounded, and it attains its maximum and minimum on $\Omega$.*

## 1.2   Holomorphic functions

**Definition 1.22** (Holomorphic function)**.** Let $\Omega$ be a nonempty open subset of $\mathbb{C}$ and let $f\colon \Omega \to \mathbb{C}$ be given. We say that $f$ is holomorphic at a point $z \in \Omega$ if the following limit exists

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

The number $f'(z)$ is called the derivative of $f$ at $z$. It is said that $f$ is holomorphic if it is holomorphic at every $z \in \Omega$. Given a closed subset $C \subset \Omega$, we say that $f$ is holomorphic on $C$ if there exists $C \subset \Omega' \subset \Omega$ so that $\Omega'$ is open and $f$ is holomorphic on $\Omega'$.

**Definition 1.23** (Entire function)**.** We say that $f\colon \mathbb{C} \to \mathbb{C}$ is entire if it is holomorphic on $\mathbb{C}$.

**Remark 1.6.** Note that $h$ is a complex number and can approach $0$ from any direction.

**Remark 1.7.** It is also useful to notice that $f\colon \Omega \to \mathbb{C}$ is holomorphic at $z \in \Omega$ if and only if there exist $a \in \mathbb{C}$, $r > 0$ with $D_r(z) \subset \Omega$, and a function $\psi\colon D_r(0) \to \mathbb{C}$ with $\lim_{h \to 0} \psi(h) = 0$, so that

$$f(z+h) = f(z) + ah + h\psi(h) \text{ for all } h \in D_r(0).$$

From this formulation is it clear that $f$ is continuous at $z$ whenever $f$ is holomorphic at $z$.

**Example 1.1.** It follows directly from the definition that the function $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$ with $f'(z) = -1/z^2$. For all $0 \neq z \in \mathbb{C}$ we have

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{z+h} - \frac{1}{z}\right) = \lim_{h \to 0} \frac{-1}{z(z+h)} = \frac{-1}{z^2}.$$

**Example 1.2.** The function $f(z) = \bar{z}$ is not holomorphic. For any $z \in \mathbb{C}$ and $r \in \mathbb{R}$ we have that

$$\frac{f(z+t) - f(z)}{t} = 1 \quad \text{and} \quad \frac{f(z+ti) - f(z)}{ti} = -1$$

**Proposition 1.7.** *Let $\Omega \subset \mathbb{C}$ be open and let $f, g\colon \Omega \to \mathbb{C}$ be holomorphic at $z \in \Omega$. Then*

*(1) $f + g$ is holomorphic at $z$ with $(f+g)'(z) = f'(z) + g'(z)$.*

*(2) $fg$ is holomorphic at $z$ with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.*

*Proof.* We will only prove (2) because the proof of (1) is much simpler. Because $f$ and $g$ are holomorphic at $z$, they are also continuous there. Thus,

$$\lim_{h \to 0} \frac{(fg)(z+h) - (fg)(z)}{h} = \lim_{h \to 0} \left(\frac{f(z+h) - f(z)}{h}g(z+h) + f(z)\frac{g(z+h) - g(z)}{h}\right)$$
$$= f'(z)g(z) + f(z)g'(z),$$

which completes the proof.                                                                                   $\square$

**Corollary 1.8.** *It's quite easy to prove that constant function of the form $f(z) = c$ for some $c \in \mathbb{C}$ and $f(z) = z$ are holomorphic. It follows immediately from Proposition 1.7 that all polynomials, functions of the form $p(z) = \sum_{k=0}^{n} a_k z^k$ are entire, with $p'(z) = \sum_{k=1}^{n} k a_k z^{k-1}$ for all $z \in \mathbb{C}$.*

**Proposition 1.9.** *A composition of holomorphic functions at $z$ is holomorphic at $z$, with $(g \circ f)'(z) = g'(f(z))f'(z)$.*

**Corollary 1.10.** *Let $\Omega \subset \mathbb{C}$ be open and let $f, g \colon \Omega \to \mathbb{C}$ be holomorphic at $z \in \Omega$. Suppose also that $g(z) \neq 0$. Then $f/g$ is holomorphic at $z$ with*

$$(f/g)'(z) = \frac{f'(z)g(z) + f(z)g'(z)}{g(z)^2}.$$

*Proof.* Let $h \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be with $h(z) = 1/z$. We now have that

$$(f/g)'(z) = (f \cdot (h \circ g))'(z) = f'(z)(h \circ g)(z) + f(z)(h \circ g)'(z)$$
$$= f'(z)/g(z) + f(z)h'(g(z))g'(z) = f'(z)/g(z) - f(z)g(z)^{-2}g'(z).$$

$\square$

Recall that $T \colon \mathbb{C} \to \mathbb{R}^2$ is the operator $T(x + yi) = (x, y)$.

**Proposition 1.11.** *Let $\Omega \subset \mathbb{C}$ be open, let $f \colon \Omega \to \mathbb{C}$, let $u, v \colon T(\Omega) \to \mathbb{R}$ be with $f(x + yi) = u(x, y) + iv(x, y)$ for $x + iy \in \Omega$, and let $F \colon T(\Omega) \to \mathbb{R}^2$ be with $F(x, y) = (u(x, y), v(x, y))$ for $(x, y) \in T(\Omega)$. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p = (x_0, y_0)$, and suppose that $f$ is holomorphic at $z_0$. Then,*

*(1) the partial derivatives of $u$ and $v$ exist at $p$, and*

$$f'(z_0) = \frac{\partial u}{\partial x}(p) + i\frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i\frac{\partial u}{\partial y}(p);$$

*(2) The Cauchy–Riemann equations are satisfied:*

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p) \ and \ \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p).$$

*(3) $F$ is differentiable at $p$ with,*

$$dF_p = \begin{pmatrix} \frac{\partial u}{\partial x}(p) & -\frac{\partial v}{\partial x}(p) \\ \frac{\partial v}{\partial x}(p) & \frac{\partial u}{\partial x}(p) \end{pmatrix} = \begin{pmatrix} \frac{\partial v}{\partial y}(p) & \frac{\partial u}{\partial y}(p) \\ -\frac{\partial u}{\partial y}(p) & \frac{\partial v}{\partial y}(p) \end{pmatrix}.$$

**Remark 1.8.** Note that $u = \Re \circ f \circ T^{-1}$, $v = \Im \circ f \circ T^{-1}$ and $F = T \circ f \circ T^{-1}$. Thus, $F$ is the map corresponding to $f$ under the identification of $\mathbb{C}$ with $\mathbb{R}^2$ via $T$.

**Remark 1.9.** Note that from (3) we have

$$\det(dF_p) = \left( \frac{\partial u}{\partial x}(p) \right)^2 + \left( \frac{\partial v}{\partial x}(p) \right)^2.$$

From this and from (1), it follows that $\det(dF_p) > 0$ whenever $f'(z_0) \neq 0$. Moreover, we have that $\sqrt{\det(dF_p)} \cdot dF_p$ is an orthogonal matrix.

We now prove Proposition 1.11.

*Proof.*

(1) We can first let $t \to 0$ in $\mathbb{R}$ and see that

$$
\begin{aligned}
f'(z_0) &= \lim_{t \to 0} \frac{f(z+t) - f(z)}{h} \\
&= \lim_{t \to 0} \frac{u(x_0 + t, y_0) - u(x_0, y_0) + iv(x_0 + t, y_0) - iv(x_0, y_0)}{t} \\
&= \frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
f'(z_0) &= \lim_{t \to 0} \frac{f(z+it) - f(z)}{h} \\
&= \lim_{t \to 0} \frac{u(x_0, y_0 + t) - u(x_0, y_0) + iv(x_0, y_0 + t) - iv(x_0, y_0)}{t} \\
&= \frac{\partial v}{\partial y}(p) - i \frac{\partial u}{\partial y}(p)
\end{aligned}
$$

which completes the proof of (1). From the equation

$$
\frac{\partial u}{\partial x}(p) + i \frac{\partial v}{\partial x}(p) = \frac{\partial v}{\partial y}(p) - i \frac{\partial u}{\partial y}(p)
$$

(2) This is an immediate result of (1).

(3)

$\square$

The following proposition is a kind of converse to the previous proposition.

**Proposition 1.12.** *Let $\Omega \subset \mathbb{C}$ be open, let $f \colon \Omega \to \mathbb{C}$, and let $u$ and $v$ be as in Proposition 1.11. Fix $x_0 + iy_0 = z_0 \in \Omega$, write $p := (x_0, y_0)$, ans suppose that $u$ and $v$ are differentiable at $p$, that is $\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p)$ and $\frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$. Then $f$ is holomorphic at $z_0$.*

*Proof.* To be added. $\square$

## 1.3 Power series

**Definition 1.24** (Power series)**.** A power series centered at $z_0 \in \mathbb{C}$ is an expression of the form $\sum_0^\infty a_n(z - z_0)^n$, where $\{a_n\}_{n \geq 0} \subset \mathbb{C}$. Given $z \in \mathbb{C}$, we say that the power series converges at $z$ if the limit $\lim_{N \to \infty} \sum_{n=0}^N a_n(z - z_0)^n$ exists in $\mathbb{C}$. If this limit does not exist, we say that the series diverges at $z$.

**Definition 1.25** (Absolute convergence)**.** Given a power series $\sum_{n=0}^\infty a_n(z - z_0)$, we say that it converges absolutely at $z \in \mathbb{C}$ if $\sum_{n=0}^\infty |a_n| \cdot |(z - z_0)| < \infty$.

**Proposition 1.13.** *If a power series converges absolutely at $z$ then it also converges at $z$. This follows from the completeness of $\mathbb{C}$.*

In the following proposition we use the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

**Proposition 1.14** (Hadamard's theorem)**.** *Let $\sum_{n=0}^\infty a_n(z - z_0)$ be a power series, and let $0 \leq \mathbb{R} \leq \infty$ be given by*

$$
\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}.
$$

*Then for $z \in \mathbb{C}$ the series converges absolutely if $|z - z_0| < R$, and the series diverges if $|z - z_0| > R$.*

**Remark 1.10.** The number $R$ is called the radius of convergence of the power series, and the region $\{z \in \mathbb{C} \colon |z - z_0| < R\}$ is called the disc of convergence.

We now proceed to prove Proposition 1.14

*Proof.* Set $L := 1/R$. Suppose first that $0 < R \leq \infty$, so that $0 \leq L < \infty$. Let $z \in \mathbb{C}$ be such that $|z - z_0| < R$, then there exists $L < M < \infty$ so that $M|z - z_0| < 1$. By the definition of $L$ (the limsup) there exists $N \geq 1$ so that $|a_n|^{\frac{1}{n}} < M$ forall $n > N$. Thus

$$\sum_{n=0}^{\infty} |a_n| \cdot |z - z_0|^n = \sum_{n=0}^{N-1} |a_n| \cdot |z - z_0|^n + \sum_{n=N}^{\infty} \left( |a_n|^{\frac{1}{n}} \cdot |z - z_0| \right)^n$$
$$\leq \sum_{n=0}^{N-1} |a_n| \cdot |z - z_0|^n + \sum_{n=N}^{\infty} \left( M|z - z_0| \right)^n < \infty.$$

Suppose next that $0 \leq R < \infty$, so that $0 < L\infty$. Let $z \in \mathbb{C}$ be such that $|z - z_0| > R$, then similarly there exists $0 < M < L$ so that $M|z - z_0| > 1$. Then, for every $N \geq 1$ there exists $n \geq N$ so that $|a_n|^{\frac{1}{n}} > M$. For such $n$ we have

$$\left| \sum_{k=0}^{n} a_k(z - z_0)^k - \sum_{k=0}^{n-1} a_k(z - z_0)^k \right| = |a_n| \cdot |z - z_0|^n$$
$$= \left( |a_n|^{\frac{1}{n}} \cdot |z - z_0| \right)^n > \left( M|z - z_0| \right)^n > 1,$$

which shows that the partial sums do not form a Cauchy sequence. Thus the series diverges at $z$, which completes the proof. $\qquad\square$

**Example 1.3.** Considert the power series $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Because we have

$$\sqrt[n]{(2n)!} \geq \sqrt[n]{n^n} = n$$

we also have for every $n \geq 1$,

$$\left( \frac{1}{(2n)!} \right)^{\frac{1}{2n}} \leq \frac{1}{n^{\frac{1}{2}}} \quad \text{and} \quad \left( \frac{1}{(2n+1)!} \right)^{\frac{1}{2n+1}} \leq \frac{1}{n^{\frac{1}{2}}}.$$

Since $n^{-\frac{1}{2}} \xrightarrow{n \to \infty} = 0$ we get that the radius of convergence is $\infty$ for the series. The map $z \mapsto e^z$ is called the exponantial function. We also have that

$$e^z e^w = \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) + \left( \sum_{n=0}^{\infty} \frac{w^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = e^{z+w}.$$

**Example 1.4.** Consider the power series $f(z) := \sum_{n=0}^{\infty} z^n$. Since $1^{\frac{1}{n}} = 1$ we get that the radius of convergence in this case is 1. Thus $f$ defined a function from $D_1(0)$ to $\mathbb{C}$. Moreover, since we have

$$(1 - z) \sum_{n=0}^{N} z^n = 1 - z^{N+1},$$

we get for $z \in D_1(0)$ that

$$f(z) = \lim_{N \to \infty} \sum_{n=0}^{N} z^n = \lim_{N \to \infty} \frac{1 - z^{N+1}}{1 - z} = \frac{1}{1 - z}.$$

**Proposition 1.15.** *Let* $f(z) := \sum_{n=0}^{\infty}$