Topology

Based on lectures by

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These notes are not endorsed by the lecturers. I have revised them outside lectures to incorporate supplementary explanations, clarifications, and material for fun. While I have strived for accuracy, any errors or misinterpretations are most likely mine.

Contents

1 Introduction

Before getting into the main part of the course, we can first look at topology from the viewpoint it was considered hundreds of years ago.

Definition 1.1 (Geodesic triangle)**.** A geodesic triangle is the area enscribed inside 3 points on a sphere.

Theorem 1.1 (Girard's theorem)**.** *Let T be a geodesic triangle. Denote its angles θ_1, θ_2, θ_3. Then we have $\theta_1 + \theta_2 + \theta_3 > 180°$ and*

$$\text{Area}(T) = \theta_1 + \theta_2 + \theta_3 - \pi.$$

Proof. Geometric. □

Theorem 1.2 (Euler's theorem)**.** *Let P be a convex polyhydron. Denote E the number of edges in P, V the number of vertices in P, and F the number of faces of P. Then*

$$V - E + F = 2.$$

Proof. Begin by ensphering the polyhydron in the unit sphere. Put a flashlight inside the polyhydron such that the faces of P cast shadows of geodesic polygons on the sphere.

Now we trianglize all the geodesic polygons. We have

$$\boxed{2E = 3F}$$

Denote the trianlges $\sigma_1, \ldots, \sigma_F$. From Theorem 1.1 we have that

$$\sum_{i=1}^{F} \frac{\theta_1^i + \theta_2^i + \theta_3^i}{V} - \pi = 4\pi.$$

From this we get

$$2\pi V - \pi F = 4\pi \implies \boxed{2V - F = 4}$$

From these equations we can deduce the desired relation

$$\boxed{V - E + F = 2}$$

which completes the proof. □

Example 1.1. box with a hold in the middle.

2 Quotient spaces and complexes

2.1 Quotient spaces

Let (X, τ) be a topological space, and let $\sim$ be an equivalence relation on X. Denote:

$$[x] := \left\{ y \in X \mid y \sim x \right\}.$$

Also define the function

$$\begin{array}{cccc} \pi & : & X & \longrightarrow & X/\sim \\ & & x & \longmapsto & [x] \end{array}.$$

Definition 2.1 (Quotient topology)**.** We define on $X/\sim$ the quotient topology $\tau_\sim$ by $U \in \tau_\sim$ if and only if $\pi^{-1}(U) \in \tau$.

In other words, the quotient topolgoy is the topology generated by π.

Proposition 2.1 (Universal property of quotient spaces)**.** *Let X, Y be topological spaces, let $\sim$ be an equivalence relation on X, and $f\colon X \to Y$ be a function such that for every $x \sim x' \in X$ we have $f(x) = f(x')$. Then there exists $f'\colon X/\sim \to Y$ such that $f' \circ \pi = f$ where π is the quotient projection. Moreover, f' is continuous if and only if f is continuous. In a diagram it looks like this:*

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X/\sim \arrow[r, dashed, "f'"'] & Y \end{tikzcd}$$

Proof. The first part of the proof is clear, so we will focus on the equivalence of continuity between f and f'. It is clear that π is continuous so when f' is continuous we also have that $f = f' \circ \pi$ is continuous.

Next suppose f is continuous. To be added □

Example 2.1. Let $X = [0, 1]$ with the topology induced by the standard topology on $\mathbb{R}$. Let $\sim$ be the equivalence relation generated by $0 \sim 1$. Thus,

$$X/\sim \cong S^1 = \left\{ z \in \mathbb{C}^2 \colon |z| = 1 \right\}.$$

We define $f\colon X \to S^1$ by $f(t) = e^{2\pi i t}$. Since f is continuous and $f(0) = f(1)$, from the universal property of quotient spaces there exists a continuous function $f'\colon X/\sim \to S^1$ defined by $f'([x]) = f(x)$. It is clear that f is one to one and onto. To show that it is a homeomorphism we can use the following lemma

Lemma 2.2. *Let X, Y be topological spaces. Suppose that X is compact, Y is Hausdorff. Then every continuous function from X to Y is a closed transformation. In particular, if f is one to one and onto, it is a homeomorphism.*

Proof. Let C be a closed set in X. Since X is compact C is compact. Since f is continuous $f(C)$ is compact in the Hausdorff space Y and thus closed which completes the proof. □

Remark 2.1. Recall that the last part of the lemma is true because an open bijection is a homeomorphism.

Example 2.2. We will show that

$$X = \mathbb{C} \setminus \{0\} \big/ x \sim \lambda x \quad \forall 0 \neq \lambda \in \mathbb{R}$$

is homeomorphic to the unit sphere S^1. First define the function

$$\begin{array}{rcccl} f & : & \mathbb{C} \setminus \{0\} & \longrightarrow & S^1 \\ & & z & \longmapsto & \left(\frac{z}{|z|}\right)^2 \end{array}$$

The function f is continuous and satisfies $f(x) = f(\lambda x)$ for all $0 \neq \lambda \in \mathbb{R}$. Thus, from the universal property of quotient spaces there exists a continuous function $f' \colon X \to S^1$. It is clear that f is one to one and onto. We have that $X = \pi(S^1)$ because every element in $C \setminus \{0\}$ is equivalent to some element in S^1, and since π is continuous and S^1 is compact, it follows that X is also compact. It is clear that S^1 is Hausdorff, and thus from the previous lemma we have that f' is a homeomorphism between X and S^1.

From now on we denote

$$\begin{aligned} D^n &:= \left\{x \in \mathbb{R}^n \colon \|x\| \leq 1\right\}; \\ S^n &:= \partial D^{n+1} = \left\{x \in \mathbb{R}^{n+1} \colon \|x\| = 1\right\}; \\ T^n &:= (S^1)^n. \end{aligned}$$

We can see that

$$D^0 = \{0\} \text{ and } \partial D^0 = \emptyset \text{ and } S^0 = \{-1, 1\}.$$

Example 2.3. We have that $D^n \big/ S^{n-1} \cong S^n$. The equivalence relation here is $x \sim y$ if and only if $x, y \in S^{n-1}$.

Example 2.4. We have that $\mathbb{R} \big/ \mathbb{Z} \cong S^1$ where $x \sim x + k$ for all $x \in \mathbb{R}$, $k \in \mathbb{Z}$.

Example 2.5. For all $x \in \mathbb{R}^n \setminus \{0\}$ and $\lambda \in (0, \infty)$ we have:

$$S^n \cong \mathbb{R}^n \setminus \{0\} \big/ x \sim \lambda x.$$

2.2 Pastings

The following lemma will allow us to discuss more pastings of spaces.

Lemma 2.3. *Let X be a topological space, $\sim$ an equivalence relation on X, let π the quotient projection, and f a continuous function such that*

(1) f is constant on the fibers of π.

(2) f' is one to one and onto.

(3) f is closed, open, or ($X \big/ \sim$ is compact and Y is Hausdorff).

Then f' is a homeomorphism.

Example 2.6. Let $X = [0, 1] \cup [2, 3]$ and $\sim := 1 \sim 2$. Then $X \big/ \sim = [0, 2]$. We can see this by defining the function $f \colon X \to [0, 2]$ as such

$$x \mapsto \begin{cases} x, & 0 \leq x \leq 1 \\ x - 1, & 2 \leq x \leq 3 \end{cases}.$$

Example 2.7. Define

$$\mathbb{R}_n^+ = \{(x_1, \dots, x_n) \mid x_i \geq 0\}$$
$$\mathbb{R}_n^- = \{(x_1, \dots, x_n) \mid x_i \leq 0\}$$

Define $X = \mathbb{R}_n^+ \cup \mathbb{R}_n^+$ (note that we treat this as a dijoint union). We define the equivalence relation $(x_1^+, \dots, x_{n-1}^+, 0) \sim (x_1^-, \dots, x_{n-1}^-, 0)$ and then we have $\mathbb{R}^n \cong X /_\sim$.

Example 2.8. Let $X = I \times I$ where $I = [0, 1]$. We define the equivalence relation as such $(s, 0) \sim (s, 1)$ for all $0 \leq s \leq 1$. Then $X /_\sim$ is homeomorphic to a cylinder. We can see this by defining the function

$$\begin{array}{rccl} f & : & I \times I & \longrightarrow & S \times I \\ & & (s, t) & \longmapsto & (\cos 2\pi t, \sin 2\pi t, s) \end{array}$$

Example 2.9. Let $X = I \times I$. Define the equivalence relation $(s, 0) \sim (1 - s, 1)$. The space $X /_\sim$ is homeomorphic to a Mobius strip.

Example 2.10. Let $X = I \times I$. Define the equivalence relation $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, t)$. The space $X /_\sim$ is homeomorphic to a torus (specifically T^2).

2.3 CW complexes

2.3.1 Disjoint union

Definition 2.2 (Disjoint union topology)**.** Let $\{X_\alpha\}_\alpha$ be a collection of topological spaces. We define the dijoint union topology in the following manner. We say that $U \subset \coprod_\alpha X_\alpha$ is open if and only if $U \cap X_\alpha$ is open in X_α.

Example 2.11. A disjoint union of points is an open set if every X_α is endowed with the discrete topology.

Example 2.12. We have that $D^1 \coprod D^1 \cong [0, 1] \cup [2, 3]$ with the standard topology.

Proposition 2.4 (Universal property of disjoint union)**.** *Let $\{X_\alpha\}_\alpha, Y$ be topological spaces, let $f_\alpha \colon X_\alpha \to Y$ be a collection of continuous functions. Then exists a unique continuous function $f \colon \coprod_\alpha X_\alpha \to Y$ such that $f_\alpha = f \circ i_\alpha$ where $i_\alpha \colon X_\alpha \to \coprod_\alpha X_\alpha$ is the injection map. In a diagram, it looks like this:*

$$\begin{tikzcd} \coprod_\alpha X_\alpha \arrow[dr, dashed, "\exists! f"] & \\ X_\alpha \arrow[u, "i_\alpha"] \arrow[r, "f_i"'] & Y \end{tikzcd}$$

2.3.2 CW complexes

Definition 2.3 (CW complex)**.** A CW complex is a topological space X with subspaces

$$\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n = X = \bigcup_n X^n$$

such that the spaces X^n are constructed inductively in the following way. Define

$$X^{-1} := \emptyset.$$

For $n \geq 0$ assume that X^{n-1} is defined. Let $\{D_\alpha^n\}_\alpha$ be a collection of n-dimensional discs, let $\{f_\alpha \colon \partial D_\alpha^n \to X^{n-1}\}_\alpha$ be a collection of continuous functions. Define:

$$X^n := \left(X^{n-1} \amalg \coprod_\alpha D_\alpha^n\right) /_\sim$$

where $\sim$ is generated by $x \sim f_\alpha(x)$ for all α and for all $x \in \partial D^n$. We endow the space $X = \cup_n X^n$ with the topology such that $U \subset X$ open if and only if $U \cap X^n$ is open for every n. The space X is called a CW complex. The X^n subspace is called the n-skeleton of the complex. The pairs (D^n_α, f_α) are called the n-cells of the complex.

Remark 2.2. In most examples, the construction process of the complex is finite. In this case there exists n such that $X^n = X$. We call n the dimension of the complex. In particular, the n-skeleton X^n is a CW complex of dimension n (at most).

Example 2.13. A 0-dimensional complex X is a discrete set. The 0-cells are called the vertices of X.

Example 2.14. A 1-dimensional complex X is a topological graph. The 1-cells are called the edges of X. Each edge connects to vertices in its extremes.

Remark 2.3. Notice that a topological graph is not a simple graph — each edge can connect a vertice to itself, and more than a single edge can connect the same vertices.

Example 2.15. We describe a CW complex for the n-dimensional sphere S^n. Let $X^0 = \{*\}$ be with a single vertice, and a single n-cell D^n_α. Since there are no cells of dimensions $0 < i < n$ we get $X^{n-1} = \{*\}$. The pasting map of the cell D^n_α is the constant map $F_\alpha \colon \partial D^n_\alpha \to X^{n-1} = \{*\}$, and indeed we have $S^n = X^n = \{*\} \coprod D^n_\alpha /_\sim$ where $x \sim *$ for all $x \in \partial D^n_\alpha$.

Example 2.16. Using the construction from the previous example, we can construct a CW complex for the torus $T^2 = S^1 \times S^1$ in the following way. Let v be a vertice, a, b be edges, and s be 2-cell. We connect the 2-cell by TO BE CONTINUED

In general, we only require the pasting maps of CW complexes to be continuous, but in practice we will only consider very nice pasting maps. For example pasting faces of polygons.

Definition 2.4 (Polygonal complex)**.** A polygonal complex is a CW complex of dimension 2 such that the pasting maps of the 2-cells identify the 2-cell as a (regular) polygon, and paste each of its edges to an edge in X^1.

3 Topological manifolds and classification of surfaces

3.1 Closed topological spaces

Definition 3.1 (Topological manifold)**.** A topological manifold of dimension n is a Hausdorff topological space M that is locally homeomorphic to $\mathbb{R}^n$.

Definition 3.2 (Closed topological manifold)**.** A closed topological manifold of dimension n is a compact, Hausdorff topological space M that is locally homeomorphic to $\mathbb{R}^n$.

Remark 3.1. A topological space X is locally homeomorphic to a topological space Y if for all $x \in X$ there exists an neighbourhood $x \in U \subset X$ that is homeomorphic to Y.

Example 3.1. A topological manifold of dimension 0 is a finite set of points endowed with the discrete toplogy.

Example 3.2. The space S^1 is a closed topological manifold of dimension 1.

Remark 3.2. In fact, S^1 is the only connected closed topological manifold of dimension 1.

Remark 3.3. Manifolds of dimension 2 are called surfaces.

Some examples of compact surfaces are:

- The sphere S^2.
- The torus $T^2 = S^1 \times S^1$.
- The projective plane

$$\begin{aligned} P^2 &= \mathbb{R}^3 \setminus \{0\} \big/ x \sim \lambda x\,,\ \forall \lambda \in \mathbb{R} \setminus \{0\} \\ &= S^2 \Big/ (x,y) \sim (-x,-y) \\ &= D^2 \Big/ (x,y) \sim (-x,-y)\,,\ \forall (x,y) \in \partial D^2 \end{aligned}$$

  we can see it's a surface very easily from the second equivalence.
- The Klein bottle K^2.
- The orientable surface of genus g.

Remark 3.4. The map $(x, y) \mapsto (-x, -y)$ is called the antipodal map.

Remark 3.5. In general, any product of closed manifolds of dimensions u, v is a closed manifold of dimension $u + v$.

Some examples of n-dimensional topological manifolds

- The spehre S^n.
- The torus $T^n = (S^1)^n$.
- The projective n-dimensional space

$$\begin{aligned} P^n &= \mathbb{R}^{n+1} \setminus \{0\} \big/ x \sim \lambda x\,,\ \forall \lambda \in \mathbb{R} \setminus \{0\} \\ &= S^n \big/ x \sim -x \\ &= D^n \big/ x \sim -x\,,\ \forall x \in \partial D^n \end{aligned}$$

  we can see it's a surface very easily from the second equivalence.

We will see some more examples of topological manifolds of dimension 3, and see that classifying manifolds of dimension 4 is impossible in practice.

3.2 Manifolds with boundary

Definition 3.3 (Compact manifold with boundary)**.** A compact manifold with boundary is a compact, Haudorff topological space M that is locally homeomorphic to an open subset of $\overline{\mathbb{H}^n} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. The set of the points that correspond to the points in $\partial\mathbb{H} = \mathbb{R}^{n-1} \times \{0\}$ is called the boundary of the manifold, and we denote it ∂M.

Remark 3.6. The boundary ∂M is well defined. That is, it is not dependent on the choice of the local homeomorphisms. We won't prove this fact in this course.

Proposition 3.1. *Let M be a compact n-dimensional manifold with boundary. Then ∂M is a closed $n - 1$-dimensional manifold.*

Some examples of manifolds with boundary

- The closed interval $[0, 1]$ is a compact 1-dimensional manifold with boundary.
- A surface from which we remove an open disc is a compact surface with boundary, and the boundary is homeomeomorphic to S^1.
- An orientable handlebody of genus g is the . . . TO BE CONTINUTED

3.3 Triangulation

Definition 3.4 (Triangulation of a surface)**.** A triangulation of a surface is a polygonal complex (made of trianlges) that is homeomorphic to the surface.

Remark 3.7. That is like saying there exists a finite collection of triangles, and pastings of their edges (each edge is pasted to at most one other edge). If the surface is closed, each edge is pasted to exactly one other edge.

Theorem 3.2 (Radó)**.** *Every surface is can be triangulated.*

The proof of this theorem will be omitted in this course.

Remark 3.8. We have not defined what is triangularization means in 3 dimensions, but according to one definition, every manifold of dimension 3 can be triangulated.

Remark 3.9. There is no similar theorem for dimensions $n > 3$.

Remark 3.10. The question whether for every 4-dimensional there exists a CW complex homeomorphic to it is open.

3.4 Connected sum of surfaces

Exercise 3.1. Let M be a compact manifold with boundary. Suppose there exist two homeomorphic connected components $N_1, N_2 \subseteq \partial M$. Then for any homeomorphism $f\colon N_1 \to N_2$ the space

$$M \Big/ x \sim f(x), \quad \forall x \in N_1$$

is a compact manifold with boundary.

Remark 3.11. Every closed 3-dimensional manifold is the connected sum of two handlebodies on their boundary.

Definition 3.5 (Knot)**.**

Definition 3.6 (Link)**.**

Remark 3.12. Every closed 3-dimensional manifold is CONTINUE LATER

Definition 3.7 (Connected sum)**.** Let S_1, S_2 be two connected compact manifolds. The connected sum $S_1 \# S_2$ is the space we get by choosing two discs $D_1 \subset S_1$ and $D_2 \subset S_2$, and a homeomorphism $f\colon \partial D_1 \to \partial D_2$ and setting

$$S_1 \# S_2 = (S_1 \setminus D_1^\circ) \amalg (S_2 \setminus D_2^\circ)/x \sim f(x), \quad \forall x \in \partial D_1$$

Remark 3.13. The connected sum is not dependent on the choice of discs, nor the homeomorphism.

Remark 3.14. We can also define the connected of manifolds in n dimensions. In the general case, the result can be dependent on the choice of the homeomorphism - there are two options to the choice of the homeomorphism ASK YALI

Example 3.3. The manifold S^2 is the identity element of connected addition. For any manifold Σ we have $\Sigma \# S^2 = \Sigma$.

Example 3.4. The sum $T^2 \# T^2$ is the orientable closed surface of genus 2.

Example 3.5. The sum $T^2 \# T^2$ is the closed orientable surface of genus 2. We can get this by pasting edges of a haptagon as can be seen in ADD IMAGE

Example 3.6. We have $P^2 \# P^2 = K^2$ as can be seen in ADD IMAGE.

Example 3.7.

Remark 3.15. If S_1, S_2 are given by a pasting of a boundary of an n_1-gon and n_2-gon, then $S_1 \# S_2$ can be given as a pasting of the boundary of an $n_1 + n_2$-gon, with the pasting of its edges being the concatenation of the pastings of S_1, S_2.

Proposition 3.3. $P^2 \# P^2 \# + P^2 \cong K^2 \# P^2 \cong T^2 \# P^2$

Proof. TO BE ADDED. $\square$

4 Classification of surfaces

4.1 Classification of compact surfaces

Theorem 4.1 (The classification theorem of compact surfaces)**.** *Every connected surface is homeomorphic to one of the following:*

(1) S^2*;*

(2) $T^2\#\cdots\#T^2$*;*

(3) $P^2\#\cdots\#P^2$*.*

Definition 4.1 (Genus)**.** The number of summands in cases (2) and (3) is called the genus of the surface.

Theorem 4.2. *Every compact surface with triangulation is homeomorphic to*

$$\underbrace{T^2\#\cdots\#T^2}_{n\ times}\#\underbrace{P^2\#\cdots\#P^2}_{m\ times}$$

The idea of the proof of Theorem 4.1 is to show we can assume S is not homeomorphic to S^2 and then using Proposition 3.3 and Theorem 4.2 we would get the desired result.

4.2 Euler's characteristic

Definition 4.2 (Euler characteristic)**.** Let X be a CW complex with a finite amount of cells. The Euler characteristic of X is defined as the following alternating sum:

$$\chi(X) = \#\{\text{0-Cells}\} - \#\{\text{1-Cells}\} + \#\{\text{2-Cells}\} - \#\{\text{3-Cells}\} + \cdots$$

Example 4.1. A CW complex of the sphere S^2 with one 0-cell and one 2-cell gives

$$\chi(S^2) = 1 - 0 + 1 = 2$$

Example 4.2. A CW complex of the sphere S^2 with one 0-cell and one 2-cell gives

$$\chi(S^2) = 1 - 0 + 1 = 2$$