Set Theory 1

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1 Permutations

Permutation σ is a bijection from a set S onto iteself.
Every permutation can be decomposed into one or more disjoint cycles(or orbits), thus, they can also be defined by them like this:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 3 & 1 \end{pmatrix} = (3\ 4)(1\ 2\ 5)$$

1. A cycle of one element is call a **fixed point**
2. A permutation without fixed points is called a **derangement**
3. A permutation that's an orbit of 2 elements is called a **transposition**

1.1 The symmetric group

A symmetric group defined over a set is the group whose elements are all the permutation over the set, and whose group operation is the composition of functions.

reminder: A group is an algebric structure with the following charachteristics:

–Closure
–Associativity
–An Idenentity permutation exists
–$\forall \sigma \in S_n (\exists \pi : \sigma \circ \pi = Id)$

2 Hall's theorem

Hall's theorem - *In a finite bipartite graph* $G(X, Y, E)$
$\forall W \subseteq X(|W| \leq |N_g(W)|) \iff$ *An X-perfect matching exists.*

($\Leftarrow$) Suppose we have an X perfect matching M, since all verices in M for W are distince we get $\forall W \subseteq X(|W| \leq |N_g(W)|)$.
($\Rightarrow$) We'll prove by contradiction. Assume an X-perfect matching doesn't exist, we'll denote the maximal matching M,and the sets of vertices in X,Y in M S,T. An X-perfect matching doesn't exist $\Rightarrow X \setminus S \neq \emptyset$, so we can choose a vertice $u_0 \in X \setminus S$ and consider all alternating paths of the form $P = (u_0, v_1, v_2, \ldots)$ such that odd edges are not in M and even edges are in M.
Denote $A = \{u : u \in P \wedge u \in X\}, B = \{v : v \in P \wedge v \in Y\} \Rightarrow |B| \leq |A \setminus \{u_0\}|$[1] $\Rightarrow |B| < |A|$, but $B = N_g(A) \Rightarrow |N_g(A)| < |A|$.

[1] $\forall v \in B$ There's a unique vertice in A because of the matching

3 Cantor's theorem

Cantor's theorem - $|A| < |P(A)|$

We can define $f : A \to P(A)$ as such

$$f(a) = a$$

$\Rightarrow |A| \leq |P(A)|$
Assume $|A| = |P(A)|$. That means there's a bijection $g : A \to P(A)$.
consider the following set:

$$D = \{a : a \notin g(a)\}$$

since g is a bijection $\exists b \in A : f(b) = D$
If: $b \in D \Rightarrow b \notin g(b) = D \Rightarrow$ contradiction
$b \notin D = g(b) \Rightarrow b \in D \Rightarrow$ contradiction
Therefore $|A| \neq |P(A)| \Rightarrow |A| < |P(A)|$

4 Equivalence Relations

An equivalence relation is a binary relation(a set of ordered pairs) that is
–Reflexive
–Symmetric
–Transitive

4.1 Some Terminology

Suppose we have an equivalnce relation R on a set X
–Equivalence Class: $[a]_R = \{b \in X : bRa = 1\}$
–Quotient Set: $X/R = \{[a]_R : a \in X\}$
–Projection: The projection of R is $\pi : X \to X/R$ such that $\pi(x) = [x]_R$
–A Cut: A cut of X is a set with only one element of each Equivalence class.

Equivalence relations can be defined by their Quotient set. Thus they can also be defined by a function or a partition.

The numbers of partitions are known the Bell's numbers and can be calculated recursivly as such:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

Think why.

5 Kőnig's Theorem

Kőnig's Theorem - *For an index set I, if $\forall i \in I$ and κ_i, λ_i we know $\kappa_i < \lambda_i$* then $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$
We'll show this by proving for any $f : \sum_{i\in I} B_i \to \prod_{i\in I} C_i$, that f is not surjective.
Let's define the function f_i as such:

$$f_i : B_i \to C_i$$
$$f_i(x) = f(x)_i$$

Since $\forall i \in I(|B_i| < |C_i|)$ we know that $\forall i \in I(f_i \text{ is not surjective}) \Rightarrow \exists c_i \in C_i \setminus Imf_i$.

Consider the vector

$$c = < c_i : i \in I >$$

If $c \in Imf \Rightarrow \exists i \in I, b \in B_i : f(b) = c$
$\Rightarrow f(b)_i = c_i \Rightarrow f_i(b) = c_i$ but $c_i \in C_i \setminus Imf_i$
That's a contradiction so we got $|\sum_{i\in I} B_i| < |\prod_{i\in I} C_i| \Rightarrow \sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$

6 Partial Orders

A **Weak/Non-Strict** Partial Order is a homogeneous relation $\leq$ on a set P that is:
–Reflexive
–Antisymmetric[1]
–Transitive

A **Strong/Strict Partial** Order is a homogeneous relation $<$ on a set P that is:
–Irreflexive
–Asymmetric[2]
–Transitive

note: $< \bigcup \leq_{Id} = \leq$

7 Partially Ordered Sets

A Partially Ordered Set(aka poset) is a set on which a partial order is defined $(A, \leq)$
Two elements a,b are comparable $\iff a \leq b \vee b \leq a$
If two elements are incomparable they're linearly independent
A **linear/total order** is a partial order under which every pair of elements is comparable.
All ordered subsets(chains) are linearly independent of each other.

7.1 Extrema

–**Greatest Element -** *an element that's comparable and greater than all other elements*
–**Maximal Element -** *an element that doesn't have a greater element than him*
–**Upper/Lower Bounds in sets -** *a is a bound in A of* $B \subseteq A$ *if* $a \in A \wedge \forall b \in B(b \leq a)$

7.1.1 About Lattices

Let A be a partially ordered set:
A is a lattice $\iff \forall S \subseteq A(|S| = 2 \Rightarrow SupS, InfSexist)$

[1] $a \leq b \wedge b \leq a \Rightarrow a = b$
[2] $a < b \Rightarrow \neg b < a$

8 Cardinals

Cardinal numbers are the "numbers" we use to represent the cardinality of sets. All cardinal numbers are based on the size of $\mathbb{N}$ that is $\aleph_0$. This subject is rather simple, yet hard to invent from zero. Thus I encourage you to try to/look for proofs for the following:

1. $|\mathbb{N}| < |\mathbb{R}|$
2. $\aleph_0 = \aleph_0 + n = \aleph_0 \times n = \aleph_0 \times \aleph_0 = |\mathbb{Z}| = |\mathbb{Q}|$
3. $\aleph = 2^{\aleph_0} = |(0,1)^{\aleph_0}| = \aleph \times \aleph_0 = \aleph \times \aleph = |(0,1)| = |[0,1]|$
4. A plane can't be covered by $\aleph_0$ lines.
5. Let A be an infinte set, $\exists S \subseteq A : |S| = \aleph_0$
6. $\aleph = |P(\mathbb{N})| = |P(\mathbb{Q})|$
7. let A = $\{$The set of all finite subsets of $\mathbb{N}\}$ prove $|A| = \aleph_0$
8. $\aleph_0^{\aleph} = \aleph$
9. $|\mathbb{R}^{\mathbb{R}}| = |P(\mathbb{R})| = 2^{\aleph}$
10. $|$The disjoint union of $\mathbb{N}$ sets of size $\mathbb{N}| = \aleph_0$
11. $\aleph_0^{\aleph_0} = \aleph$
12. $|$The set of all invertible functions $\mathbb{R} \to \mathbb{R}| = 2^{\aleph}$
13. $|A| = |$The set of all algebric numbers$| = \aleph_0$
14. $|B| = |\mathbb{R} \setminus A| = |$The set of all transcedental number$| = \aleph$
15. $|$All subsets of $\mathbb{R}$ with cardinality $\aleph/\aleph_0|$
16. Let $\aleph_0$ people with a natural number of hats on their head guess how many hats they have. How many options are there, given only a finite number of people guessed right/wrong?

9 Schröder–Bernstein Theorem

Schröder–Bernstein Theorem - $|A| \leq |B| \wedge |B| \leq |A| \iff |A| = |B|$ There are more proofs that rely on similiar ideas. Here's one:
We're given two injective functions

$$f : A \to B$$

$$g : B \to A$$

Without loss of generality assume A,B are disjoint(Why can we do this?)
Considering the partial functions f^{-1}, g^{-1} we can create a sequence for every element of $A \bigcup B$ in the following way:

$$\ldots f^{-1}g^{-1}(a) \to g^{-1}(a) \to a \to f(a) \to g(f(a)) \ldots$$

The sequence can keep going forever to the right, but to the left it may stop eventually since the inverse functions are partial[1]. We can see that every element in $A \bigcup B$ has a sequence and that if an element appears in two sequnces they'll be identical since they're injective and by our construction. Thus those sequences form a partition of $A \bigcup B$ so it's enough to create bijections to all partitions, and we're finished. Our bijection will be:

$$h(x) = \left\{ \begin{array}{ll} f(x), & \text{for } x \in A \text{ in an A-stop} \\ g^{-1}(x), & \text{for } x \in A \text{ in a B-stop} \\ f^{-1}(x), & \text{for } x \in B \text{ in an A-stop} \\ g(x), & \text{for } x \in B \text{ in a B-stop} \end{array} \right\}$$

And of course any element of an A-stop can go one step right with f and we can get any element in any sequence by applying f^{-1}, and by that logic we get that h is the bijection we wanted and that $|A| = |B|$

[1] We'll call those who stop from the left on an element of A A-stops and the rest B-stops - even though they may not always stop!

10 Homomorphism and Isomorphosm of Ordered Sets

10.1 Homomorphisms

Let $(X, \leq_1), (Y, \leq_2)$ be partially ordered sets.
F is a Homomorphism $\iff \forall x_1, x_2 \in X(x_1 \leq_1 x_2 \Rightarrow F(x_1) \leq_2 F(x_2)$

10.2 Isomorphisms

Let $(X, \leq_1), (Y, \leq_2)$ be partially ordered sets.
F is an Isomorphism $\iff F : X \to Y$ is a bijection $\wedge \forall x_1, x_2 \in X(x_1 \leq_1 x_2 \Rightarrow F(x_1) \leq_2 F(x_2)$
An isomorphism is reflexive, symmetric and transitive so it's an equivalnce relation.
If F is an Isomorphism and the orders are total orders, F^{-1} is also an isomorphism.

10.3 Lexicographic Order

Also known as a Dictonary Order, is an order That's defined as such:
$(x_1, y_1) \leq (x_2, y_2) \iff x_1 <_x x_2 \vee (x_1 = x_2 \wedge y_1 \leq_y y_2)$
This is a partial order on $X \times Y$

11 Zorn's Lemma

Zorn's lemma - *Let F be a non-empty poset. If for every chain[2] in F exists an upper bound in F, then F has at least one maximal element.*

Proof all vector spaces have a base:
Let V be a vector space: If $V = \{0\}$ then *emptyset* is its basis. If V is finitely generated then we can add vectors from V to $\emptyset$ until it's spanning V. Suppose V is not finitely generated, let's Define F as the set of all linearly independant sets of vectors. F is partially ordered by the order of inclusion of sets. Let $C = (A_i)i \in I$ be a chain in F, $A = \bigcup_{i \in I} A_i)$. A is clearly a maximal element of the chain. Let's prove it's in F. Assume A isn't in $F \Rightarrow$ there exists a finite series of linearly dependent vectors, each is an element of a finite series of elements of C. Since that series is finite, and linearly order as a subset of C, There exists a maximal element that must contain all the vectors in the linearly independent vector series. but that element is in F so it's both linearly dependent and independent at the same time! contradiction! We get that $A \in F$ so by Zorn's lemma F has a maximal element T. That T is our basis.

11.1 Comparing Cardinals

We'll show that for every two cardinals α, β other than 0 we get $\alpha \leq \beta \vee \beta \leq \alpha$
Let A, B be two sets of cardinality α, β Define F to be the set of all ordered pairs (X, f) such that:
$-f : X \to B$ is an injective function. $(X \subseteq A)$
Now we'll define an order in the following way:

$$(X_1, f_1) \leq (X_2, f_2) \iff X_1 \subseteq X_2 \wedge f_2|_{X_1} = f_1$$

Let $C = ((X_1, f_1), (X_2, f_2), \ldots)$ be a chain in F, $(X, g) = (\bigcup A_i, \bigcup f_i)$

$$\Rightarrow \forall i((X_i, f_i) \leq (X, g)).$$

Assume g isn't a function, we get $(x, y), (x, z) \in G$:

$$\Rightarrow \exists i, j \text{ such that: } f_i(x) = y, f_j(x) = z$$

C is a chain so we without lose of generailty we get:

$$f_i \leq f_j$$

$$\Rightarrow f_j|_{X_i} = f_i$$

$$\Rightarrow f_i(x) = f_j(x)$$

$$\Rightarrow y = z$$

[2]a totally ordered subset

That means g is a function, and since it's a union of injective function, it us also injective. That means it's in F and using Zorn's lemma we get a maxima element in $F, (D, h)$. if h is injective $A \leq B$. If it's not surjective we get a contadiction to (D, h)'s maximality so it must be surjective and thus $B \leq A$
We can also prove $\alpha + \alpha = \alpha$. We know that $\alpha + \alpha = 2\alpha$ so we'll just prove $\alpha = 2\alpha$. We'll build F using bijections this time. Denote the maximal element $M = (X, g)$ if $|X| = 2\alpha$ We've finished, else we get there's a set of size $\aleph_0$ that can be mapped "bijectively" to the set of 2α contradicting M's maximality.

11.2 Corollaries

$\alpha + \beta = max\{\alpha, \beta\}$
$|A \setminus B| = |A| \iff |B| \leq |A|$
$\alpha * \alpha = \alpha$ (not a direct corollary)
$\alpha^\alpha = 2^\alpha$

12 Axiom of Choice

First let's define what is a choice function. A Choice Function - is a function from an indexed family of sets $(S_i)i \in I$ to such that $\forall i \in I(f(S_i) \in S_i)$. Now for the axiom itself- The Axiom of Choice:

$$\forall X[\emptyset \notin X \Rightarrow \exists f : X \rightarrow \bigcup X \ \forall A \in X(f(A) \in A)]$$

12.1 Nomenclature

AC - Axiom of Choice
ZF - Zermelo-Fraenkel set theory omitting AC
ZFC - ZF extended to include AC

13 Lebesgue Measure

//to be added

14 Well Order

A partially ordered set $(X, \leq)$ is well ordered

$$\iff$$

$$\forall S \subseteq X(S \neq \emptyset \to \exists b \in S(\text{b is a minimal element is S}))$$

Think about the following theorems:

1. Every finite totally ordered set is well ordered.
2. If $\leq$ is a well order then it's a linear order as well.
3. Let $(X, \leq)$ be a linearly ordered set. It's well ordered $\iff$ it doesn't include an infinite decreasing series.

We'll proceed to define two very similar terms.
Risha - If X is well ordered $A \subseteq X$(usually we mean $A \subset X$) is a Risha if $x \in A \wedge y < x \to y \in A$
Initial segment - $I_x(a) = \{x \in X : x < a\}$ aka initial segement of a in X

note: $[0, 0.5]$ in $[0, 1] \in \mathbb{R}$ is a Risha but not an initial segment. Prove a Risha and an initial segment are the same in wosets.

14.1 Some Lemmas

1. let X be a woset, $f : X \to X$ a one-to-one homomorphism $\to \forall x \in X(x \leq f(x))$
2. let $(X, \leq_x) \cong (Y, \leq_y)$ be isomorphic wosets, there's only one unique isomorphism betweem them (proof using previous theorem)
3. in a woset X a risha can't be have an isomorphism with X
4. in wosets $I_x(a) \cong I_x(b) \Rightarrow a = b$
5. let $f : X \to Y$ be an isomorphism between wosets s.t. $y_0 = f(x_0) \Rightarrow I_x(x_0) = I_y(y_0)$

14.2 A lemma about partial orders

If $(X, \leq_x), (Y, \leq_y)$ are partial orders, and $\leq_x$ is a total order. If f is an inversible homomorphism then it's an isomorphism, and $\leq_y$ is a total order.

15 Comparison of Well Ordered Sets

If X, Y are wosets then exatcly one of the following is true

1. $(X, \leq_x) \cong (Y, \leq_y)$
2. $\exists y_0 \in Y : (X, \leq_x) \cong (I_y(y_0), \leq_y)$
3. $\exists x_0 \in X : (Y, \leq_y) \cong (I_x(x_0), \leq_x)$

If $X = \emptyset \vee Y = \emptyset$ the proof is trivial. Assuming they're not empty we'll define:

$$A = \{x \in X : \exists y \in Y(I_x(x) \cong I_y(y))\}$$
$$B = \{y \in Y : \exists x \in X(I_x(x) \cong I_y(y))\}$$
$$\phi : A \to B$$
$$\phi(x) = y : I_x(x) \cong I_y(y)$$

Consider $a_1 < a_2 \in A$ and $\phi(a_1) = b_1, \phi(a_2) = b_2$. Since $I_x(a_2) \cong I_y(b_2)$ we'll mark their isomorphism α. $a_1 < a_2 \Rightarrow a_1 \in Dom\alpha \Rightarrow \alpha(a_1) \in Im\alpha = I_y(b_2) \Rightarrow \alpha(a_1) < y_2$. by one of our previous lemmas $I_x(a_1) \cong I_y(\alpha(a_1))$ and we know $I_x(a_1) \cong I_y(b_1) \Rightarrow b_1 = \alpha(a_1)$. recall that $\alpha(a_1) < y_2 \Rightarrow y_1 < y_2$. Since ϕ is a bijection and a homomorphism it's an isomorphism $\Rightarrow A \cong B$.
Bycases we'll get:

1. If $A = X, B = Y \Rightarrow (1)$.
2. If $B = Y \wedge A \subset X \neq \emptyset$ denote its minimal element $c \Rightarrow I_x(c) = A$ [1] $\Rightarrow (3)$[2]
3. If $A = X \wedge B \subset Y \neq \emptyset$ denote its minimal element $d \Rightarrow I_x(d) = B \Rightarrow (2)$
4. If $A \subset X \wedge B \subset Y \Rightarrow c \in A$[3] $\wedge\, c \in X \setminus A \Rightarrow$ contradiction.

Now we'll show only one of the conditions is true:
(2)+(3) $\Rightarrow \exists \delta : X \to I_y(d)$ *isomorphism* $\Rightarrow I_x(c) \cong I_y(\delta(c)) \Rightarrow$ Y is isomorphic to its own initial segment!
$(1) + (3)/(1) + (2) \Rightarrow$ an initial segment of X/Y is isomorphic to X/Y

[1] Think why(two sided inclusion).
[2] since $A \cong B$
[3] $I_x(c) \cong A \cong B \cong I_y(d), \phi(c) =?$

16 Ordinals

Ordinals are the generalization of ordinal numerals aimed to extend enumeration to infinite sets. The finite ordinals will be defined as such:

$$k = ord(\{0, 1, \ldots, k-1\}) = ord(I_{\mathbb{N}}(k))$$

$$ord(\emptyset) = 0$$

$$ord(\mathbb{N}) = \omega$$

By the comparability of wosets we can define an order on the ordinals as such:

$A \cong B \iff card(A) = card(B)$
$A < B \iff card(A) < card(B)$
$A > B \iff card(A) > card(B)$

Now we'll define a new set $W(\alpha)$

$$W(\alpha) = \{\beta : \beta < \alpha\}$$

–$W(\alpha)$ is a woset and $ord(W(\alpha)) = \alpha$
proof by constructing the isomorphism: $\phi : A^1 \to W(\alpha)$

$$\phi(a) = W(ord(I_A(a)))$$

–Every set of ordinals A is a woset
proof by considering every $A' \subseteq A$ has an element a, which if not already minimal, has a $W(a) \bigcup A'$ that contains the minimal element since it's a woset as a subset of a woset.

16.1 Cesare Burali-Forti Paradox

The set of all ordinals can't be well defined. Suppose it were a set, it'll be a woset, we'll denote it $O \Rightarrow ord(O) \in O \Rightarrow W(O) \in O$ but we know $ord(WO) = O$ Thus an initial segment of the set is isomorphic to it which is a contradiction.

16.2 Russell's Paradox

Russell's Paradox - *Let R be the set than contains all the sets that don't contain themselves.*
If R contains itself, it must not contain itself.
If R doesn't contain itself, then it must contain itself.
paradox.

[1] $|A| = \alpha$ and A is a woset

16.3 Kinds of Ordinals

There are two kinds of ordinals:
Successor Ordinals - *ordinals that immediatly success another ordinal*
Limit Ordinals - *the rest.*

16.4 Ordinal Arthimetic

16.4.1 addtion

Let $(X, \leq_x), (Y, \leq_y)$ be two **disjoint** wosets such that $(ord(A), ord(B) = (\alpha, \beta))$.
We denote:$(X \bigcup Y, \leq)$:

$$a \leq b \iff \begin{Bmatrix} a, b \in X & a \leq_x b \\ a, b \in Y & a \leq_y b \\ a \in X & b \in Y \end{Bmatrix}$$

As $X \oplus Y$, And by definition $\alpha + \beta = ord(X \oplus Y)$
Oridnals are associative but no commutative with addition
–$\omega + n = \omega$
–$\alpha + 0 = \alpha$
–$\omega < \omega + 1 < \omega + 2 < \ldots < \omega + k < \ldots < 2\omega$

16.4.2 multiplication

Let $(X, \leq_x), (Y, \leq_y)$ be two **disjoint** wosets such that $(ord(A), ord(B) = (\alpha, \beta))$.
We denote:$(X \times Y, \leq_{dictionary})$ As $X \odot Y$, and by definition $\alpha * \beta = ord(X \odot Y)$
It's possible to show $\omega = k\omega$ by constructing an isomorphism

$$\phi : \mathbb{N} \to \{0, 1, \ldots, k-1\} \times \mathbb{N}$$

$$\phi(n) = (\lfloor n/k \rfloor, n \bmod k)$$

–$\omega * 0 = 0$
–$\alpha * 1 = \alpha$
–$\omega < 2\omega < 3\omega < \ldots < k\omega < \ldots < \omega^2$
Ordinals are left distribuitive but not right distributive. Why?

16.4.3 Powers

$$\alpha^\gamma = \begin{pmatrix} 1 & \gamma = 0 \\ \alpha^{\gamma-1} & \gamma \text{ is a succesor ordinal} \\ min_{\delta<\gamma}\{\mu : \alpha^\delta < \mu\} & \gamma \text{ is a limit ordinal} \end{pmatrix}$$

From that we infer the biggest ordinal so far is ω^ω
Ordinals are usually expressed as polynomials of powers of ω with natural coefficients
Of course I lied earlier...
$\omega^\omega < \omega^\omega + 1 < \omega^\omega + 2 < \ldots < \omega^{\omega^\omega} < \ldots$

16.4.4 2^ω and ω^ω

By our previous definition we can conclude that 2^ω is

$$min_{\delta<\omega}\{\mu : 2^\delta < \mu\}$$

$$= min\{2^1, 2^2, \ldots, 2^k, \ldots\}$$

Since this series doesn't have an upper bound the result is the smallest infinite ordinal or $2^\omega = \omega$
By our previous definition we can conclude that ω^ω is

$$min_{\delta<\omega}\{\mu : \omega^\delta < \mu\}$$

$$= min\{\omega^1, \omega^2, \ldots, \omega^k, \ldots\}$$

Lets consider

$$X = X_1 \oplus X_2 \oplus X_3 \ldots (\forall n \in \mathbb{N}, ord(X_n) = \omega^n)$$

We see that this is what we looked for[1] thus $ord(X) = \omega^\omega$ but also this is a sum of a countable amount of groups of a countable size so surprisingly $|X| = \aleph_0$ and this is the case for sets of ordinals ω^{ω^ω} and so on...

[1] Verify this is indeed what we looked for

17 The Well Ordering Theorem

The Well Ordering Theorem(WOT) states that any set can be well ordered and is equivalent to Zorn's lemma and AC. The proof usually involves an intuitive use of Zorn's lemma. Write it.
Use WOT to prove AC.

18 The Continuum Conjecture

Let a set X of cardinality $\aleph$, and consider the set of all Risha's with a cardinality greater then $\aleph_0$[1], since X can be a woset, that set has a minimal element m and $|m| = \aleph_1$. The conjecture is that $\aleph = \aleph_1$. This was proven to be unsolvable under ZFC. We can also define an $\aleph$ greater than all $\aleph$ of the form $\aleph_n$ where $n \in \mathbb{N}$ by looking at sets $|A_n| = \aleph_n$ and at $B = \bigcup_{1 \in \mathbb{N}} \aleph_i$. $|B| > \aleph_n (\forall n \in N) \Rightarrow \exists x \in B : |I_B(x)| > \aleph_n (\forall n \in N)$. We denote the minimal element of the set of all such x's M, and $|M| = \aleph_\omega = \sum_{i \in \mathbb{N}} \aleph_i$ and after all countable ordinals we'll reach $\aleph_\Omega$, the $\aleph_1$ ordinal, and the fisrt uncountable one. The generalized Continuum Conjecture is:

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

[1] if it's empty change any one element to be maximal