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Topics: Discrete R.V.'s, PMF's, Expectation, Variance, Joint PMF

Homework and Exam Grading "Philosophy"

Answering a question is not just about getting the right answer, but also about **communicating** how you got there. This means that you should carefully define your model and notation, and provide a step-by-step explanation on how you got to your answer. Communicating clearly also means your homework should be **neat**. Take pride in your work it will be appreciated, and you will get practice thinking clearly and communicating your approach and/or ideas. To encourage you to be neat, if your homework or exam solutions are sloppy you **may NOT get full credit** even if your answer is correct.

To get full credit you need to have turned in the homework on time, and made sure to designate the location of your solutions to the various questions on the homework when you submit to Gradescope.

Q. 1 Discrete PMF Consider a discrete random variable X with PMF

$$p_X(1) = \frac{c}{3}, \;\; p_X(2) = \frac{c}{6}, \;\; p_X(5) = \frac{c}{3}$$

and 0 otherwise, where c is a positive constant.

1. Find c.
2. Compute $P(X > 2)$.
3. Compute $E[X]$ and $\mathrm{Var}(X)$.
4. Let $Y = X^2 + 1$, find the PMF of Y.
5. Compute $E[Y]$ and $E[Y^2]$.

Q. 2 Factory operation A factory owns 10 machines. On any particular day, each machine is operational with probability 0.7, independent of the status of any other machine. Let X be the random variable corresponding to the number of working machines on a particular day.

1. Describe the sample space, the range of the random variable X, and calculate the PMF of X.
2. Determine a numerical value for $P(X \geq 9)$.

Q. 3 Spider and fly A spider sits on a wall and repeatedly attempts to catch a fly. Each time it attempts, it succeeds with probability 0.3 (i.e. it misses the fly with probability 0.7). Assume that successive attempts are independent. Let Y be the random variable corresponding to the number of attempts the spider makes to first successfully catch a fly. Describe the sample space, the PMF of the random variable Y, and determine the probability that the spider successfully catches the fly within 5 attempts.

Q. 4 Evaluations with PMF The PMF for the result of any one roll of a three sided die with faces numbered 1, 2 and 3 is

$$p_X(x) = \begin{cases} \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \\ \frac{1}{4}, & x = 3 \\ 0, & \text{otherwise} \end{cases}$$

Consider a sequence of six independent rolls of this die, and let X_i be the random variable corresponding to the i^{th} roll.

1. What is the probability that exactly three of the rolls have result equal to 3?
2. What is the probability that the first roll is 1, given that exactly two of the six rolls have result of 1?
3. We are told that exactly three of the rolls resulted in 1 and exactly three resulted in 2. Given this information, what is the probability that the sequence of rolls is 121212?
4. Conditioned on the event that at least one roll resulted in 3, find the conditional PMF of the number of 3's.

Q. 5 Joint PMF The joint PMF of two random variables R and S is given by

$$P_{RS}(r,s) = \begin{cases} \frac{4}{45} & r = 1\ , s = 1 \\ \frac{6}{45} & r = 1\ , s = 2 \\ \frac{6}{45} & r = 1\ , s = 3 \\ \frac{6}{45} & r = 2\ , s = 1 \\ \frac{9}{45} & r = 2\ , s = 2 \\ \frac{9}{45} & r = 2\ , s = 3 \\ \frac{2}{45} & r = 3\ , s = 1 \\ \frac{3}{45} & r = 3\ , s = 2 \\ 0 & r = 3\ s = 3 \end{cases}$$

Let $A = \{S \neq 3\}, X = R + S, Y = R - S$.

1. Find $p_S(s)$ and $p_{S|A}(s)$.
2. Find $p_{R,Y}(r, y)$.
3. Find $p_{X|A}(x)$.

Q. 6 Resource Capacity A Poisson RV X with parameter λ has range $S_X = \{0, 1, ...\}$ and PMF:

$$p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!},\ x = 0, 1, ...$$

We will denote $X \sim \text{Poisson}(\lambda)$. Poisson distributions are used to model events in space or time. $\text{E}[X]$ and $\text{Var}(X)$ are both equal to the positive parameter λ. In this problem, we will use a Poisson RV to model the following scenario

Now imagine a cloud service provider with customers who occasionally need more computing resources than they originally purchased. To handle this, the provider sets aside some extra shared capacity, denoted as c. The excess resource needs of customers can be modeled as a Poisson random variable (RV) with an average (mean) of 2.2. Now, the provider wants to determine the minimum amount of additional capacity c to reserve so that 80% of the time, it can handle the additional resource requirements of its customers.