Discrete Random Variables: The basics

Outline:

- Discrete Random Variables (RVs): the idea and the definition
- Probability Mass Functions (PMF)
- Key types of discrete RVs and how/when to use them
  - Bernoulli(p)
  - Uniform$\{a, \ldots, b\}$
  - Binomial(n, p)
  - Geometric(p)
  - other (not discussed in class see book/class notes): Poisson(λ), Pascal(k, p),..
- Expectation of an RV

Random Variables: The idea

Random Variables (RVs): The formalism

- An RV X associates a real number $X(\omega)$ to each outcome $\omega \in \Omega$ i.e., mathematically just a **function** $X : \Omega \to \mathbb{R}$ where $\omega \to X(\omega)$
- A discrete RV has a range, denoted S_X, which is countable.

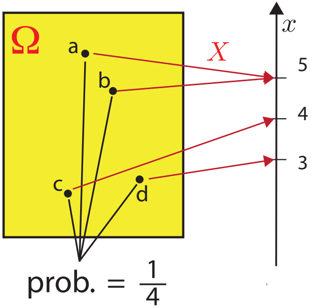
Notation: random variable X numerical value x

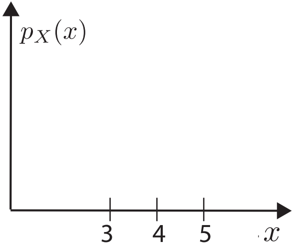
- Several RVs can be defined on the same sample space
- A function of one or more RVs on the same space is also an RV
  - meaning of $X + Y$

Probability Mass Function (PMF) of a Discrete RV X

- The "prob. law" or "distribution" of X or PMF
- If we fix some x then $X = x$ is an event

$p_X(5) =$



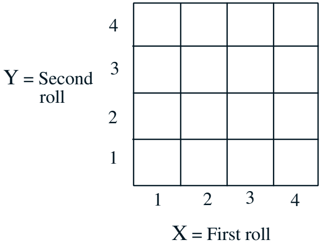


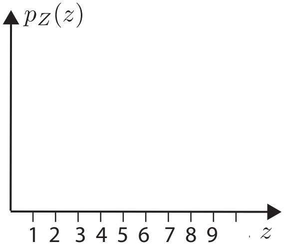
PMF: $p_X(x) = P(X = x) = P(\{\omega | X(\omega) = x\})$

Properties: $p_X(x) \geq 0, \quad \sum_x p_X(x) = 1$

PMFs and How We Compute Them

Two rolls of fair four-sided die





Let $Z = X + Y$ and find $p_Z(z)$, $\forall z$

Repeat for all z

- collect possible outcomes for which $Z = z$
- add their probabilities

$$p_Z(2) = P(Z = 2) =$$

$$p_Z(3) = P(Z = 3) =$$

The simplest random variable: Bernoulli(p) with parameter $p \in [0, 1]$

We say $X \sim$ Bernoulli(p) if $\quad S_X = \{0, 1\}$

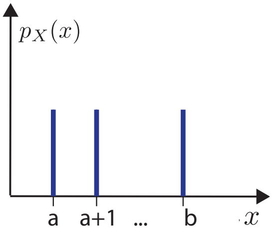
$$X = \begin{cases} 1, & \text{w.p. } p \\ 0, & \text{w.p. } 1 - p \end{cases} \qquad \begin{array}{l} p_X(1) = P(X = 1) = p \\ p_X(0) = 1 - p \end{array}$$

- $X \sim$ Bernoulli(p) means RV X has the specified PMF
  - we do not use $X =$ Bernoulli(p)
- w.p. means "with probability"
- Can be used to model an experiment with two outcomes, e.g.,
  - 0/1, success/failure, Heads/Tails.

Discrete Uniform RV : integer paramaters $a \leq b$

$X \sim \text{Uniform}\{a, a+1, \ldots, b\}$ iff $p_X(x) = \frac{1}{b-a+1}$ for $x \in S_X = \{a, a+1, \ldots, b\}$

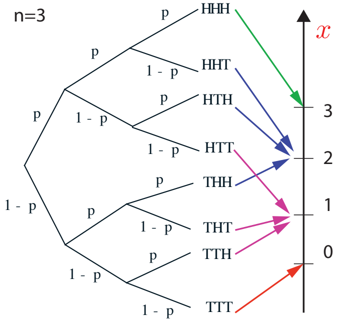
- **Experiment:** pick one of $a, a+1, \ldots, b$ at random; all equally likely
- **Sample space :** $\{a, a+1, \ldots, b\}$; $\#$ possible values $=$
- **Random variable:** $X(\omega) = \omega$
- **Models:** "complete ignorance" except for range $S_X = \{a, a+1, \ldots, b\}$



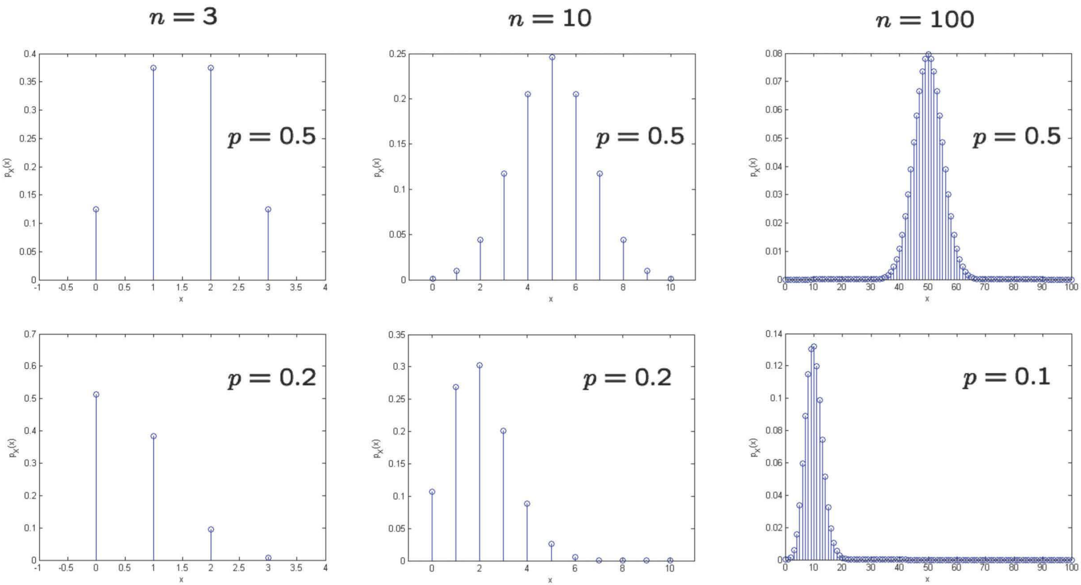
Binomial RV : paramaters: $n \in \mathbb{N}$ and $p \in [0, 1]$

$X \sim \text{Binomial}(n, p)$ iff $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$ for $x \in S_X = \{0, 1, \ldots, n\}$

- **Experiment:** n independent tosses of a coin with $P(H) = p$
- **Sample space:** sequences of H and T of length n
- **Random variable:** $X = \#$ of heads observed
- **Model:** for # of successes in a fixed number of indep trials



What does Binomial PMF look like?

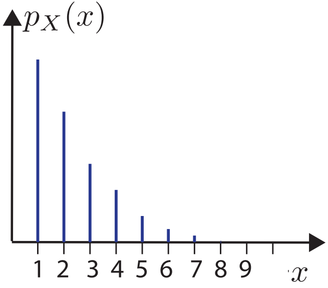


Geometric RV; parameter $p \in (0, 1]$

$X \sim \text{Geometric}(p)$ iff $p_X(x) = (1-p)^{x-1}p$ for $x \in S_X = \{1, 2, \ldots, \}$

- **Experiment:** infinite sequence of independent coin tosses $P(H) = p$
- **Sample space:** sequences of H and T
- **Random variable:** $X = \#$ of tosses until first head
- **Model:** waiting times, $\#$ of trials until a success

$p_X(x) =$



PMFs and how we use them

"Key Equation" Suppose X is a discrete RV and $A \subset \mathbb{R}$ then

$$P(X \in A) = \sum_{x \in A \cap S_X} p_X(x)$$

Example: Suppose a computer fails with prob. 1/10 on a given month independent of other months. What is prob. you see no failures in a year?

Let $T = \#$ of months until first failure, then $T \sim$

We are interested in

$$P(T \geq 13) =$$

Example: Service Facility Design - Problem

Suppose you are provisoning a service facility (e.g., cloud) to support $n = 10$ customers where each customer is independently active with probability $p = 0.2$.

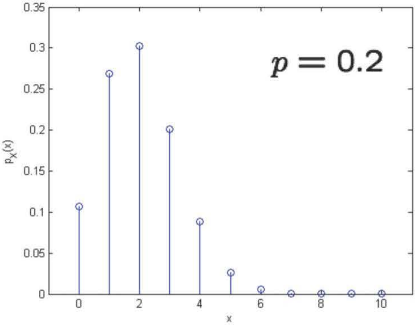
Determine the **minimum service capacity** c you should purchase to ensure your customers needs are met with probability/reliability 0.99

Example: Service Facility Design Solution

Let $N = \#$ number of active customers, then $N \sim$

We are interested in smallest c such that

$$P(N \le c) = \sum_{k=0}^{c} \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^{c} \binom{10}{k} (0.2)^k (0.8)^{10-k} \ge 0.99$$



Expectation/mean/average of an RV

Motivation: play a game 1000 times ; gain for each play modeled by RV X:

$$X = \begin{cases} 1, & \text{w.p. } \frac{2}{10} \\ 2, & \text{w.p. } \frac{5}{10} \\ 4, & \text{w.p. } \frac{3}{10} \end{cases}$$

- 'Average' gain:

Definition: $E[X] = \sum_{x \in S_X} x p_X(x)$

Interpretation: avg over large number of indep. repetitions of an experiment

Caution: For infinite sum to be well-defined require $\sum_x |x| p_X(x) < \infty$.

Expectation of a Bernoulli RV

Definition: $E[X] = \sum_{x \in S_X} x p_X(x)$

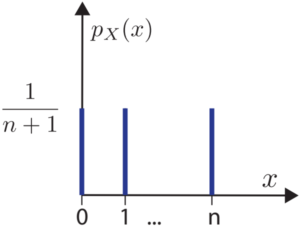
Suppose $X \sim$ Bernoulli(p), i.e., $p_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$

Then $E[X] =$

Expectation of a uniform random variable

Definition: $E[X] = \sum_{x \in S_X} x p_X(x)$

Suppose $X \sim \text{Uniform}\{0, 1, \ldots, n\}$



$$E[X] \;=\; \sum_{x \in S_X} x p_X(x) = \underbrace{0\frac{1}{n+1} + 1\frac{1}{n+1} + \cdots + n\frac{1}{n+1}}_{(0+1+\cdots+n)\frac{1}{n+1}} \;=\; \frac{(n+1)n}{2}\frac{1}{n+1} \;=\; \frac{n}{2}$$

Elementary properties of expectation

- If $X \geq 0$ then $E[X] \geq 0$

- If $a \leq X \leq b$ then $a \leq E[X] \leq b$

- If c is a constant then $E[c] = c$

Definition:
$E[X] = \sum_{x \in S_X} x p_X(x)$

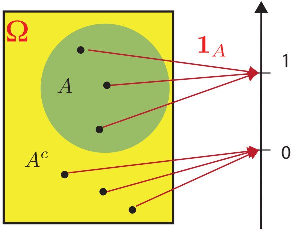
Additional slides

The simplest random variable: Bernoulli(p) with parameter $p \in [0, 1]$

We say $X \sim$ Bernoulli(p) if and only if (iff)

$$X = \begin{cases} 1, & \text{w.p. } p \\ 0, & \text{w.p. } 1-p \end{cases} \qquad \begin{array}{l} p_X(1) = P(X = 1) = p \\ p_X(0) = 1 - p \end{array}$$

- Example: Indicator RV of an event A: $\mathbf{1}_A(\omega) = 1$ iff $\omega \in A$ and 0 otherwise



Poisson RVs

A random variable $X \sim \text{Poisson}(\lambda)$ if its pmf is

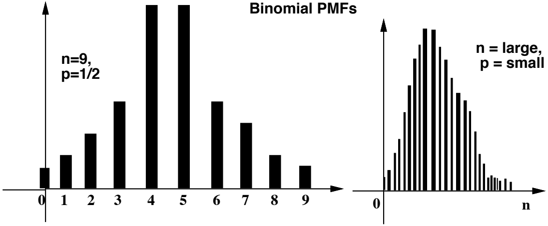
$$p_X(k) = e^{-\lambda} \frac{\lambda^x}{x!} \text{ for } x \in S_X = \{0, 1, \ldots\} \quad \text{where} \quad E[Z] = \lambda$$

Examples: Poisson distributions are used to model the number of defects on a wafer. Suppose λ is the average number of defects on a wafer with unit area?

- What is the probability of no defects?
- If the wafer area is doubled, how dramatically will the yield change?

Poisson RVs : Example

From Binomial to Poisson RV



Consider a RV $Y \sim$Binomial(n, p) where n is large, p is small, and $np = \lambda$ For example:

- Number of on-going calls in a given city
- Number of auto accidents on a given day

You can show that

$$P(Y = k) = \binom{n}{k} p^k (1-p)^{n-k} \approx e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, \ldots$$

where the right hand side is a Poisson pmf with $\lambda = np$.