HW7

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Q.. $f_z(Z) = \int_{-\infty}^{\infty} f_x(x) \cdot f(z-x)$

$f_x(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \qquad f_y(y) = \frac{1}{2}, \quad 0 \leq y \leq 2$

Zone 1: $0 \leq Z \leq 2$

$$f_z(Z) = \int_0^z \lambda e^{-\lambda x} \cdot \frac{1}{2} dx = 0.5 - 0.5 e^{-\lambda z}$$

Zone 2: $z > 2$

$$f_z(z) = \int_{z-2}^{z} \lambda e^{-\lambda x} \cdot \frac{1}{2} dx = 0.5 e^{-\lambda(z-2)} - 0.5 e^{-z\lambda}$$

$$f_z(Z) = \begin{cases} 0.5 - 0.5 e^{-\lambda z}, & 0 \leq Z \leq 2 \\ 0.5 e^{-\lambda(z-2)} - 0.5 e^{-z\lambda}, & Z > 2 \\ 0, & \text{otherwise} \end{cases}$$

Q2.

a) $f_{x|A} = \frac{1}{l}, \quad f_{x|A^c} = \frac{1}{l}$

$f_x(x) = P(A) \cdot f_{x/A}(x) + P(A^c) \cdot f_{x|A^c}(x)$

$f_x(x) = 2 \cdot \frac{1}{l} = \frac{2}{l}, \quad x \in [0, \frac{l}{2}]$

b) $f_{y|x}(y|x) = \frac{2}{x}$

c) $E[y] = \int_0^{l/2} E[y|x] \cdot f_x(x) dx$

$E[y] = \int_0^{l/2} \frac{x}{4} \cdot \frac{2}{l} dx$

$E[y] = \frac{l}{16}$

d) $Var(y) = E[Var(y|x)] + Var(E[y|x])$

$Var(y|x) = \frac{\left(\frac{x}{2}\right)^2}{12} = \frac{x^2}{48}$

$E[Var(y|x)] = \int_0^{l/2} \frac{x^2}{48} \cdot f_x(x) dx = \int_0^{l/2} \frac{x^2}{48} \cdot \frac{2}{l}$

$$Var(E[Y(X)]) = Var\left(\frac{X}{4}\right) = \left(\frac{1}{4}\right)^2 \cdot \frac{(1/2)^2}{12}$$

Q3. a) $P(T_1 < T_2) = \frac{\lambda_1}{\lambda_1+\lambda_2}$

b) $T_{min} = \min(T_1, T_2) \rightarrow \lambda_1+\lambda_2$

$$E[T_{min}] = \frac{1}{\lambda_1+\lambda_2}$$

c) $E[T_{min} \leq 3] = \int_0^3 t \cdot (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)t} \, dt$

$$E[T] = \frac{1}{\lambda} - \frac{e^{-3\lambda}}{\lambda}$$

d) $E[h(t)] = E[h(t)_{cont}] + E[h(t)_{dist}]$

$$E[h(t)_{cont}] = \int_0^3 (e^{(\lambda_1+\lambda_2)t} - 1) \cdot (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)t} \, dt$$

$$= 3\lambda_1 + 3\lambda_2 + e^{-(\lambda_1+\lambda_2)\cdot 3} - 1$$

$$E[h(t)_{dist}] = (e^{(\lambda_1+\lambda_2)\cdot 3} - 1) \cdot e^{-(\lambda_1+\lambda_2)\cdot 3}$$

$$= 1 - e^{-(\lambda_1+\lambda_2)\cdot 3}$$

$$E[h(t)] = 3\lambda_1 + 3\lambda_2$$

Q4.

$A_t = 1$ if t^{th} toss is 1, $X_1 = A_1 + A_2 \cdots + A_n$

$B_t = 1$ if t^{th} toss is 2, $X_2 = B_1 + B_2 \cdots + B_n$

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1] E[X_2]$$

$E[A_t B_t] = 0$ since $A_t + B_t$ can never both be 1.

$$E[X_1] = E[A_1 + A_2 + \cdots + A_n] = n\left(\frac{1}{k}\right)$$

$$E[X_2] = E[B_1 + B_2 + \cdots + B_n] = n\left(\frac{1}{k}\right)$$

$$E[X_1 X_2] = E\left[\sum A_i \cdot \sum B_j\right] = 0$$

$$Cov(x_1, x_2) = 0 - \left(\frac{n}{2}\right)^2 = -\left(\frac{n}{2}\right)^2$$

Q5.

a) $E[N] = np$

$Var(N) = np(1-p)$

b) $E[T] = E[x_1 + x_2 + \cdots + x_N] = E[N] \cdot E[x_i] = np \cdot \mu_x$

$Var(T) = E[Var(T|N)] + Var(E[T|N])$

$Var(T|N) = N \cdot \sigma_x^2 \rightarrow E[N \cdot \sigma_x^2] \rightarrow E[N] \cdot \sigma_x^2 \rightarrow np\, \sigma_x^2$

$E[T|N] = N \cdot \mu_x \rightarrow Var(N \cdot \mu_x) \rightarrow \mu_x^2 \cdot Var(N) \rightarrow \mu_x^2 \cdot np(1-p)$

$Var(T) = np\sigma_x^2 + \mu_x^2 \cdot np(1-p)$