HW8

Friday, March 29, 2024 5:03 PM

Q1. $X \sim N(0,1)$, $y \sim N(1,4)$, $u = x + Y$, $v = x + \alpha y$

a) $\mu_u{}^2 = \mu_x + \mu_y = 0 + 1 = 1$

$\mu_v = \mu_x + \mu_y = 0 + r = r$

$\sigma_u^2 = \sigma_x^2 + \sigma_y^2 = 1 + 4 = 5$

$\sigma^2 v = 0^2 + \alpha^2 r^2 x = 1 + \alpha^2 \cdot 4$

$Cov(u,v) = Cov(x + Y, x + \alpha y)$

$= Cov(x,x) + \alpha Cov(y,y)$

$= Var(x) + \alpha Var(y) = 1 + 4\alpha$

$$f_{u,v}(u,v) = \frac{1}{2\pi \sigma_u \sigma_v \sqrt{1-\rho^2}} \cdot \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(u-\mu_u)^2}{\sigma^2 u} - 2\rho\frac{(u-\mu_u)(v-\mu_v)}{\sigma_u \sigma_v} + \frac{(v-\mu_v)^2}{\sigma^2 v}\right]\right)$$

$$\rho = \frac{Cov(u,v)}{\sigma_u \sigma_v} = \frac{1+4\alpha}{\sqrt{5} \cdot \sqrt{1+4\alpha^2}}$$

$$f_{u,v}(u,v) = \frac{1}{2\pi\sqrt{5}\sqrt{1+4\alpha^2}\sqrt{1-\left(\frac{1+4\alpha}{\sqrt{5}\sqrt{1+4\alpha}}\right)^2}} \exp\left(-\frac{1}{2\left(1-\left(\frac{1+4\alpha}{\sqrt{5}\sqrt{1+4\alpha}}\right)^2\right)}\left[\frac{(u-1)^2}{5} - 2\left(\frac{1+4\alpha}{\sqrt{5}\sqrt{1+4\alpha}}\right)\frac{(u-1)(v-\alpha)}{\sqrt{5}\sqrt{1+4\alpha^2}} + \frac{(v-\alpha)^2}{1+4\alpha^2}\right]\right)$$

b) $Cov(u,v) = 0 \rightarrow 1 + 4\alpha = 0 \rightarrow \alpha = -1/4$

c) $E[u|V=v] = \mu_u + \frac{Cov(u,v)}{\sigma^2 v}(v - \mu_v)$

$Var(u|V=v) = \sigma^2 u - \frac{Cov(u,v)^2}{\sigma^2 v}$

$E[u|V=5] = 1 + \frac{(1+4\alpha)}{1+4\alpha^2}(5-\alpha)$

$Var(u|V=5) = 5 - \frac{(1+4\alpha)^2}{1+4\alpha^2}$

Q2. $V \sim N(7,4)$, $W \sim N(3,4)$

a) $D = V + W$

$\mu_D = \mu_V + \mu_W = 7 + 3 = 10$

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$\mu_D = \mu_V + \mu_W = 7 + 3 = 10$

$\sigma_D^2 = \sigma_V^2 + \sigma_W^2 = 4 + 4 = 8 \rightarrow \sigma_D = \sqrt{8}$

$Z = \dfrac{D - \mu_D}{\sigma_D} \rightarrow Z = 0.71$ where $D = 12$

$P(D > 12) = 1 - P(Z \leq 0.71) = 0.24$

b) $\sigma_D^2 = \sigma_V^2 + \sigma_W^2 + 2\rho_{V,W}\,\sigma_V\sigma_W$

$= 6$

$C = 10 + 1.645 \cdot \sqrt{6} = 14.04$ kbps.

Q3. $P(X \geq a) = \dfrac{E[X]}{a}$

$P(T > 60) = e^{-\lambda t} \rightarrow e^{-\frac{60}{50}}$

$E[X] = 10000 \cdot e^{-\frac{60}{50}}$

$P(X > 8000) = \dfrac{10000 \cdot e^{-\frac{60}{50}}}{8000} = 0.376$

Q4. $P(|X-\mu|^2 \sigma k) = \frac{1}{k^2} \rightarrow P(|X-\mu| \geq \sigma k) \leq 1 - 0.95 \leq 0.05$

$\frac{1}{k^2} = 0.05 \rightarrow k = \sqrt{20}$, $\sigma = \sqrt{\frac{0.25}{n}}$

$E = \sigma k$, $0.04 = \sqrt{\frac{0.25}{n}} \cdot \sqrt{20}$, $n = 3125$

$3125 \times 0.5 = \$\,1562.50$

Q5. a) $T_1 \sim N(0.5, 0.5)$; $T_2 \sim N(1, 0.5)$;

$T_3 \sim N(0.25, 0.25)$; $T_4 \sim N(0.25, 0.95)$

$T_{12} \sim N(1.5, 1)$

$$\mu_{12} = 0.5 + 1 = 1.5$$

$$\sigma^2_{12} = 0.5 + 0.5 = 1$$

$T_{34} \sim N(0.5, 1)$

$$\mu_{34} = 0.25 + 0.25 = 0.5$$

$$\sigma^2_{34} = 0.25 + 0.75 = 1$$

$P(\max(T_{12} + T_{34}) \leq 2$

$\rightarrow P(T_{12} < 2 \text{ and } T_{34} < 2) = 0.69 \cdot 0.933 = 0.65$

b) $T_{12} \sim N(1.5, 1)$; $T_{34} \sim N(0.5, 1)$

$D = D \sim N(1, 2)$

$$\mu_D = 1.5 - 0.5 = 1$$

$$\sigma^2_D = 1 + 1 = 2$$

$$Z = \frac{0 - 1}{\sqrt{2}} = -0.707$$

$P(Z < -0.707) \rightarrow 1 - P(Z < 0.707) = 0.24$

$P(T_{12} < T_{34}) = 0.24$

c) $P(\text{Total} \leq 2) = \frac{1}{4} P(T_{12} \leq 2) + \frac{3}{4} P(T_{34} \leq 2)$

$P(T_{12} \leq 2) = 0.69$, $P(T_{34} \leq 2) = 0.93$

$P(\text{Total} \leq 2) = \frac{1}{4} \cdot 0.69 + \frac{3}{4} \cdot 0.93 = 0.87$

d) $P(\text{Upper} \mid \leq 2) = \dfrac{P(\leq 2 \mid \text{Upper}) \cdot P(\text{Upper})}{P(\leq 2)}$

$$= \frac{0.69 \cdot 0.25}{0.87} = 0.20$$