פתרון שאלה 3.24 בספר

נראה שאופרטור הלפלאסיאן אינו משתנה כתוצאה מסיבוב הקואורדינאטות.

$$\nabla^2 f = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

בקואורדינאטות החדשות:

$$\nabla^2 f = \frac{\partial^2}{\partial x_R^2} + \frac{\partial^2}{\partial y_R^2}$$

נניח סיבוב בזוית θ

$$x = x_R \cdot \cos(\theta) - y_R \cdot \sin(\theta)$$
$$y = x_R \cdot \sin(\theta) + y_R \cdot \cos(\theta)$$

נחשב את הנגזרות החלקיות

$$\frac{\partial f}{\partial x_R} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x_R} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x_R} = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)$$

$$\frac{\partial f}{\partial y_R} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial y_R} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y_R} = -\frac{\partial f}{\partial x}\sin(\theta) + \frac{\partial f}{\partial y}\cos(\theta)$$

ואת הנגזרות החלקיות השניות

$$\frac{\partial^2 f}{\partial x_R^2} = \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial x_R} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial y}{\partial x_R}\right)\cos(\theta) + \left(\frac{\partial^2 f}{\partial y \partial x}\frac{\partial x}{\partial x_R} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial x_R}\right)\sin(\theta)$$
$$\frac{\partial^2 f}{\partial x_R^2} = \left(\frac{\partial^2 f}{\partial x^2}\cos(\theta) + \frac{\partial^2 f}{\partial x \partial y}\sin(\theta)\right)\cos(\theta) + \left(\frac{\partial^2 f}{\partial y \partial x}\cos(\theta) + \frac{\partial^2 f}{\partial y^2}\sin(\theta)\right)\sin(\theta)$$
$$\frac{\partial^2 f}{\partial x_R^2} = \left(\frac{\partial^2 f}{\partial x^2}\cos^2(\theta) + \frac{\partial^2 f}{\partial x \partial y}\sin(\theta)\cos(\theta)\right) + \left(\frac{\partial^2 f}{\partial y \partial x}\cos(\theta)\sin(\theta) + \frac{\partial^2 f}{\partial y^2}\sin^2(\theta)\right)$$

$$\frac{\partial^2 f}{\partial y_R^2} = -\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial y_R} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial y}{\partial y_R}\right)\sin(\theta) + \left(\frac{\partial^2 f}{\partial y \partial x}\frac{\partial x}{\partial y_R} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial y_R}\right)\cos(\theta)$$
$$\frac{\partial^2 f}{\partial y_R^2} = -\left(-\frac{\partial^2 f}{\partial x^2}\sin(\theta) + \frac{\partial^2 f}{\partial x \partial y}\cos(\theta)\right)\sin(\theta) + \left(-\frac{\partial^2 f}{\partial y \partial x}\sin(\theta) + \frac{\partial^2 f}{\partial y^2}\cos(\theta)\right)\cos(\theta)$$

$$\frac{\partial^2 f}{\partial y_R^2} = -\left(-\frac{\partial^2 f}{\partial x^2}\sin^2(\theta) + \frac{\partial^2 f}{\partial x \partial y}\cos(\theta)\sin(\theta)\right) + \left(-\frac{\partial^2 f}{\partial y \partial x}\sin(\theta)\cos(\theta) + \frac{\partial^2 f}{\partial y^2}\cos^2(\theta)\right)$$

ונסכם:

$$\frac{\partial^2 f}{\partial x_R^2}+\frac{\partial^2 f}{\partial y_R^2}=\left(\frac{\partial^2 f}{\partial x^2}\cos^2(\theta)+\frac{\partial^2 f}{\partial x\partial y}\sin(\theta)\cos(\theta)\right)+\left(\frac{\partial^2 f}{\partial y\partial x}\cos(\theta)\sin(\theta)+\frac{\partial^2 f}{\partial y^2}\sin^2(\theta)\right)+$$

$$-\left(-\frac{\partial^2 f}{\partial x^2}\sin^2(\theta)+\frac{\partial^2 f}{\partial x\partial y}\cos(\theta)\sin(\theta)\right)+\left(-\frac{\partial^2 f}{\partial y\partial x}\sin(\theta)\cos(\theta)+\frac{\partial^2 f}{\partial y^2}\cos^2(\theta)\right)=$$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$

קבלנו :

$$\frac{\partial^2 f}{\partial x_R^2}+\frac{\partial^2 f}{\partial y_R^2}=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$