Assignment 13

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1 Question 1

The complete code for the assignment is attached in a separate file named ‘Q1‘. Please refer to the attached code file for the detailed implementation.

Below is the implementation of the Local Histogram Equalization algorithm:

```python
# visualization.py
import matplotlib.pyplot as plt

def plot_images_and_histograms(original_image, enhanced_image):
    plt.figure(figsize=(12, 6))

    plt.subplot(221)
    plt.imshow(original_image, cmap='gray')
    plt.title('Original Image')
    plt.axis('off')

    plt.subplot(222)
    plt.hist(original_image.flatten(), 256, [0, 256])
    plt.title('Original Histogram')

    plt.subplot(223)
    plt.imshow(enhanced_image, cmap='gray')
    plt.title('Enhanced Image')
    plt.axis('off')

    plt.subplot(224)
    plt.hist(enhanced_image.flatten(), 256, [0, 256])
    plt.title('Enhanced Histogram')

    plt.tight_layout()
    plt.show()


# image_processing.py
import cv2
import numpy as np

def load_image(file_path):
    image = cv2.imread(file_path, cv2.IMREAD_GRAYSCALE)
    if image is None:
        raise ValueError(f"Unable to load the image. Make sure '{file_path}' is in the current directory.")
    return image

```

```python
def local_histogram_equalization(image, kernel_size=3):
    pad = kernel_size // 2
    padded_image = cv2.copyMakeBorder(image, pad, pad, pad, pad, cv2.BORDER_REFLECT)

    output = np.zeros_like(image)

    for i in range(image.shape[0]):
        for j in range(image.shape[1]):
            local_region = padded_image[i:i + kernel_size, j:j + kernel_size]

            local_hist, _ = np.histogram(local_region.flatten(), 256, [0, 256])
            local_cdf = local_hist.cumsum()
            local_cdf_normalized = (local_cdf - local_cdf.min()) * 255 / (local_cdf.max() - local_cdf.min())

            output[i, j] = local_cdf_normalized[image[i, j]]

    return output.astype(np.uint8)


# main.py
from image_processing import load_image, local_histogram_equalization
from visualization import plot_images_and_histograms

def main():
    # Load the embedded_squares image
    image_path = 'input/embedded_squares.JPG'
    try:
        image = load_image(image_path)
    except ValueError as e:
        print(f"Error: {e}")
        return

    # Apply local histogram equalization
    enhanced_image = local_histogram_equalization(image, kernel_size=15)

    # Display original and enhanced images
    plot_images_and_histograms(image, enhanced_image)


if __name__ == "__main__":
    main()
```

2 Question 2

Proof. Given the KLT transformation:

$$\mathbf{y} = \mathbf{A}(\mathbf{x} - \mathbf{m}_x)$$

where $\mathbf{y}$ is the transformed vector of $\mathbf{x}$ through the matrix $\mathbf{A}$, and $\mathbf{m}_x$ is the mean vector of $\mathbf{x}$.

It is known that the covariance matrix of $\mathbf{y}$, denoted as $\mathbf{C}_y$, is computed by:

$$\mathbf{C}_y = \mathbf{A}\mathbf{C}_x\mathbf{A}^T$$

where $\mathbf{C}_x$ is the covariance matrix of $\mathbf{x}$.

We need to prove that the eigenvalues of the covariance matrix $\mathbf{C}_y$ are the same as the eigenvalues of the covariance matrix $\mathbf{C}_x$.

Let λ be an eigenvalue of $\mathbf{C}_x$ with corresponding eigenvector $\mathbf{v}$. This means:

$$\mathbf{C}_x\mathbf{v} = \lambda\mathbf{v}$$

Consider $\mathbf{A}\mathbf{C}_x\mathbf{A}^T(\mathbf{A}\mathbf{v})$:

$$\begin{aligned}
\mathbf{A}\mathbf{C}_x\mathbf{A}^T(\mathbf{A}\mathbf{v}) &= \mathbf{A}\mathbf{C}_x(\mathbf{A}^T\mathbf{A})\mathbf{v} \\
&= \mathbf{A}(\mathbf{C}_x\mathbf{v}) \quad (\text{since } \mathbf{A}^T\mathbf{A} = \mathbf{I}, \text{ as } \mathbf{A} \text{ is orthogonal in KLT}) \\
&= \mathbf{A}(\lambda\mathbf{v}) \\
&= \lambda(\mathbf{A}\mathbf{v})
\end{aligned}$$

This shows that $\mathbf{A}\mathbf{v}$ is an eigenvector of $\mathbf{A}\mathbf{C}_x\mathbf{A}^T$ with the same eigenvalue λ.

Since $\mathbf{C}_y = \mathbf{A}\mathbf{C}_x\mathbf{A}^T$, we conclude that λ is also an eigenvalue of $\mathbf{C}_y$, with $\mathbf{A}\mathbf{v}$ as the corresponding eigenvector.

This process works in both directions: every eigenvalue of $\mathbf{C}_y$ is also an eigenvalue of $\mathbf{C}_x$.

Thus, we have proven that the eigenvalues of the covariance matrix $\mathbf{C}_y$ are the same as the eigenvalues of the covariance matrix $\mathbf{C}_x$.

Note: This proof relies on the properties of the KLT transformation, specifically that $\mathbf{A}$ is an orthogonal matrix ($\mathbf{A}^T\mathbf{A} = \mathbf{I}$). This property ensures that the eigenvalues are preserved in the transformation. $\square$

3 Question 3

We aim to prove that the erosion operation satisfies the following equation:

$$A \ominus B = \bigcap_{b \in B} (A)_{-b}$$

where $A \ominus B$ denotes the erosion of set A by set B, and $(A)_{-b}$ represents the translation of set A by $-b$.

Definition: The *erosion* of a set A by a set B is defined as:

$$A \ominus B = \{x \mid B_x \subseteq A\}$$

where $B_x = \{x + b \mid b \in B\}$, the translation of set B by x.

The *translation* of a set A by an element $-b$ is:

$$(A)_{-b} = \{x \mid x + b \in A\}$$

This is the set of all points x such that $x + b \in A$.

We now prove that $A \ominus B = \bigcap_{b \in B} (A)_{-b}$.

Proof. By the definition of erosion:

$$A \ominus B = \{x \mid B_x \subseteq A\}$$

This implies that $x \in A \ominus B$ if and only if for every $b \in B$, we have $x + b \in A$.

On the other hand, consider the intersection of translations:

$$\bigcap_{b \in B} (A)_{-b}$$

This is the set of all x that belong to all translated sets $(A)_{-b}$ for each $b \in B$. That is, $x \in \bigcap_{b \in B} (A)_{-b}$ if and only if for every $b \in B$, $x + b \in A$.

Since both sides describe the same set, we conclude that:

$$A \ominus B = \bigcap_{b \in B} (A)_{-b}$$

□

4 Question 4

(a) Validity of 2-D Discrete Fourier Transform Pair: $\delta(x, y) \Leftrightarrow 1$
To show the validity of the 2-D discrete Fourier transform pair $\delta(x, y) \Leftrightarrow 1$, we will use direct substitution into the inverse discrete Fourier transform equation:

$$\delta(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

Where $F(u, v)$ is the Fourier transform of $\delta(x, y)$, which in this case is 1. Substituting $F(u, v) = 1$ into the equation:

$$\begin{aligned}
\delta(x, y) &= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} 1 \cdot e^{j2\pi(ux/M + vy/N)} \\
&= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} e^{j2\pi(ux/M + vy/N)} \\
&= e^{j2\pi(0x/M + 0y/N)} \\
&= 1
\end{aligned}$$

The third step follows from the sifting property of the 2-D impulse, which states that the sum of exponentials over a complete period is zero except when the exponent is zero. Because we used direct substitution into the inverse discrete Fourier transform equation, and this equation and the forward transform are a Fourier transform pair, it must follow that the left side of the double arrow is the IDFT of the right: $\mathcal{F}^{-1}1 = \delta(x, y)$. This demonstrates the validity of the given Fourier transform pair $\delta(x, y) \Leftrightarrow 1$ for the 2-D discrete case.

(b) Validity of 2-D Discrete Fourier Transform Pair: $1 \Leftrightarrow MN\delta(u, v)$
To demonstrate the validity of the 2-D discrete Fourier transform pair $1 \Leftrightarrow MN\delta(u, v)$, we will start with the inverse DFT equation:

$$\mathcal{F}^{-1}\{MN\delta(u, v)\} = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} MN\delta(u, v) e^{j2\pi(ux/M + vy/N)}$$

Simplifying:

$$\begin{aligned}
\mathcal{F}^{-1}\{MN\delta(u, v)\} &= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} MN\delta(u, v) e^{j2\pi(ux/M + vy/N)} \\
&= e^{j2\pi(0x/M + 0y/N)} \\
&= 1
\end{aligned}$$

The second step follows from the sifting property of the 2-D impulse, Eq. (4-58). Because we used direct substitution into Eq. (4-68), and this equation and Eq. (4-67) are a Fourier transform pair, it must follow that the right side of the double arrow is the DFT of the left: $\mathcal{F}\{1\} = MN\delta(u, v)$.

This demonstrates the validity of the given Fourier transform pair $1 \Leftrightarrow MN\delta(u,v)$ for the 2-D discrete case.

(c) Validity of 2-D Discrete Fourier Transform Pair: $\delta(x-x_0, y-y_0) \Leftrightarrow e^{-j2\pi(\frac{ux_0}{M}+\frac{vy_0}{N})}$

To prove the validity of this Fourier transform pair, we'll demonstrate both the forward and inverse transforms.

Forward Transform Let's start with the forward discrete Fourier transform:

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

Substituting $f(x,y) = \delta(x-x_0, y-y_0)$:

$$\begin{aligned} F(u,v) &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} \delta(x-x_0, y-y_0)e^{-j2\pi(\frac{ux}{M}+\frac{vy}{N})} \\ &= e^{-j2\pi(\frac{ux_0}{M}+\frac{vy_0}{N})} \end{aligned}$$

The second step follows from the sifting property of the 2-D impulse function.

Inverse Transform Now, let's verify the inverse transform:

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})}$$

Substituting $F(u,v) = e^{-j2\pi(\frac{ux_0}{M}+\frac{vy_0}{N})}$:

$$\begin{aligned} f(x,y) &= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} e^{-j2\pi(\frac{ux_0}{M}+\frac{vy_0}{N})}e^{j2\pi(\frac{ux}{M}+\frac{vy}{N})} \\ &= \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} e^{j2\pi[\frac{u(x-x_0)}{M}+\frac{v(y-y_0)}{N}]} \\ &= \delta(x-x_0, y-y_0) \end{aligned}$$

The last step follows from the fact that the sum of complex exponentials over a complete period is zero except when the exponent is zero, which occurs when $x = x_0$ and $y = y_0$.

Thus, we have shown that:

$$\mathcal{F}\delta(x-x_0, y-y_0) = e^{-j2\pi(\frac{ux_0}{M}+\frac{vy_0}{N})} \tag{1}$$

and

$$\mathcal{F}^{-1}e^{-j2\pi(\frac{ux_0}{M}+\frac{vy_0}{N})} = \delta(x-x_0, y-y_0) \tag{2}$$

This demonstrates the validity of the given Fourier transform pair for the 2-D discrete case.

(d) Validity of the 2-D Discrete Fourier Transform Pair: $e^{j2\pi(u_0x/M+v_0y/N)} \leftrightarrow MN\delta(u-u_0, v-v_0)$

To validate the 2-D discrete Fourier transform pair provided in Table 4.4:

$$e^{j2\pi(u_0x/M+v_0y/N)} \leftrightarrow MN\delta(u-u_0, v-v_0),$$

we use direct substitution into the inverse DFT formula (4-68):

$$\begin{aligned}
\mathcal{F}^{-1}MN\delta(u-u_0, v-v_0) &= \frac{MN}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\delta(u-u_0, v-v_0)e^{j2\pi(ux/M+vy/N)} \\
&= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\delta(u-u_0, v-v_0)e^{j2\pi(ux/M+vy/N)} \\
&= e^{j2\pi(u_0x/M+v_0y/N)}.
\end{aligned}$$

The final step leverages the sifting property of the 2-D discrete impulse (Eq. (4-58)). Since direct substitution into Eq. (4-68) aligns with the definition of the Fourier transform pair (Eq. (4-67)), it confirms that:

$$\mathcal{F}e^{j2\pi(u_0x/M+v_0y/N)} = MN\delta(u-u_0, v-v_0).$$

(e) Validity of the 2-D Discrete Fourier Transform Pair:
$\cos(2\pi\mu_0x/M + 2\pi\nu_0y/N) \Leftrightarrow (MN/2)[\delta(u+\mu_0, v+\nu_0) + \delta(u-\mu_0, v-\nu_0)]$

We will show the validity of the following 2-D discrete Fourier transform pair from Table 4.4:

$$\cos(2\pi\mu_0x/M + 2\pi\nu_0y/N) \Leftrightarrow (MN/2)[\delta(u+\mu_0, v+\nu_0) + \delta(u-\mu_0, v-\nu_0)]$$

Forward Transform Let's start with the forward transform. We need to show that:

$$\mathcal{F}\{\cos(2\pi\mu_0x/M + 2\pi\nu_0y/N)\} = (MN/2)[\delta(u+\mu_0, v+\nu_0) + \delta(u-\mu_0, v-\nu_0)]$$

Using Euler's formula, we can express the cosine as:

$$\cos(2\pi\mu_0x/M + 2\pi\nu_0y/N) = \frac{1}{2}[e^{j(2\pi\mu_0x/M+2\pi\nu_0y/N)} + e^{-j(2\pi\mu_0x/M+2\pi\nu_0y/N)}]$$

The 2-D discrete Fourier transform is defined as:

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

Applying this to our function:

$$\begin{aligned}
F(u,v) &= \frac{1}{2}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}[e^{j(2\pi\mu_0x/M+2\pi\nu_0y/N)} + e^{-j(2\pi\mu_0x/M+2\pi\nu_0y/N)}]e^{-j2\pi(ux/M+vy/N)} \\
&= \frac{1}{2}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}[e^{j2\pi((\mu_0-u)x/M+(\nu_0-v)y/N)} + e^{-j2\pi((\mu_0+u)x/M+(\nu_0+v)y/N)}]
\end{aligned}$$

For the first term, when $u = \mu_0$ and $v = \nu_0$, the sum equals MN. For the second term, when $u = -\mu_0$ and $v = -\nu_0$, the sum equals MN. In all other cases, the sums are zero due to the orthogonality of complex exponentials.

Therefore:

$$F(u, v) = (MN/2)[\delta(u - \mu_0, v - \nu_0) + \delta(u + \mu_0, v + \nu_0)]$$

This proves the forward transform.

Inverse Transform For the inverse transform, we need to show:

$$\mathcal{F}^{-1}\{(MN/2)[\delta(u + \mu_0, v + \nu_0) + \delta(u - \mu_0, v - \nu_0)]\} = \cos(2\pi\mu_0 x/M + 2\pi\nu_0 y/N)$$

The 2-D inverse discrete Fourier transform is defined as:

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

Applying this to our function:

$$\begin{aligned}
f(x, y) &= \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} (MN/2)[\delta(u + \mu_0, v + \nu_0) + \delta(u - \mu_0, v - \nu_0)] e^{j2\pi(ux/M + vy/N)} \\
&= \frac{1}{2}[e^{-j2\pi(\mu_0 x/M + \nu_0 y/N)} + e^{j2\pi(\mu_0 x/M + \nu_0 y/N)}] \\
&= \cos(2\pi\mu_0 x/M + 2\pi\nu_0 y/N)
\end{aligned}$$

This proves the inverse transform.

Therefore, we have shown the validity of the given 2-D discrete Fourier transform pair.

(f) Validity of the 2-D Discrete Fourier Transform Pair:
$\sin(2\pi\mu_0 x/M + 2\pi\nu_0 y/N) \Leftrightarrow \frac{jMN}{2}[\delta(u + \mu_0, v + \nu_0) - \delta(u - \mu_0, v - \nu_0)]$

To demonstrate the validity of the 2-D discrete Fourier transform pair:

$$\sin(2\pi\mu_0 x/M + 2\pi\nu_0 y/N) \Leftrightarrow \frac{jMN}{2}[\delta(u + \mu_0, v + \nu_0) - \delta(u - \mu_0, v - \nu_0)],$$

we perform direct substitution into the forward DFT formula (Eq. (4-67)) and apply Euler's formula to rewrite the sine function in terms of exponentials:

$$
\begin{aligned}
\mathcal{F}\sin(2\pi\mu_0 x/M + 2\pi\nu_0 y/N) &= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\left[\frac{e^{j2\pi(\mu_0 x/M+\nu_0 y/N)} - e^{-j2\pi(\mu_0 x/M+\nu_0 y/N)}}{2j}\right] e^{-j2\pi(ux/M+vy/N)} \\
&= \frac{1}{2j}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{j2\pi(\mu_0 x/M+\nu_0 y/N)} e^{-j2\pi(ux/M+vy/N)} \\
&\quad - \frac{1}{2j}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} e^{-j2\pi(\mu_0 x/M+\nu_0 y/N)} e^{-j2\pi(ux/M+vy/N)} \\
&= \frac{MN}{2j}[\delta(u-\mu_0, v-\nu_0) - \delta(u+\mu_0, v+\nu_0)] \\
&= \frac{jMN}{2}[\delta(u+\mu_0, v+\nu_0) - \delta(u-\mu_0, v-\nu_0)].
\end{aligned}
$$

In the second step, the terms are expressed in terms of the Fourier transforms of exponentials. The results, as shown in the third step, correspond to the Fourier transform pair stated above. This confirms the validity of the given 2-D discrete Fourier transform pair.

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Thank you for reviewing this assignment.