

The covariance matrix is always semi positive and symmetric.

Proof:

Denote by  $X$  the matrix of observations, it consists of  $n$  (number random variables) columns, each one of length  $N$  (the number of observations). Assume that each random variable has mean zero (or otherwise that we already deducted the mean from the observations).

Then the covariance matrix  $C$  is a simple matrix product of  $X$  transposed and  $X$  (and is obviously symmetric of size  $n$  by  $n$ ).  $C = \frac{1}{N} X'X$ .

It is positively definite as any product  $X'X$ , since for any vector  $a$  of length  $n$  we have  $a'X'Xa \geq 0$ .

To show that in a specific case the covariance is semi positive it is enough to find a non-zero vector  $a$ , such that  $a'X'Xa = 0$ .

Assume that  $N < n$ , then one can always find an  $n$  dimensional vector  $a$  that will be orthogonal to all  $N$  observations (for example, having 2 observations and 3 random variables one can always find a 3 dimensional vector that is orthogonal to any two given vectors - observations)

Let's choose  $a$  orthogonal to each observation, then  $a'X' = 0$  and this completes the proof that in this case  $C$  is only semi positive definite.