A Cooperative Pursuit-Evasion Game of a High Speed Evader

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Abstract— This paper studies a pursuit-evasion game of multiple pursuers and a high speed evader for holonomic systems. A group of pursuers uses the idea of perfectly encircled formation to capture an evader. The perfectly encircled formation is a formation in which an evader does not have an instantaneous escape path and it is defined using the concept of Apollonius circle. The paper initially presents the conditions required to create a perfectly encircled formation. It is then argued that if a perfectly encircled formation shrinks over time by maintaining the connectivity between the Apollonius circles, then capture is guaranteed. This paper shows that it is not possible to maintain the connectivity while shrinking with minimum number of pursuers. Hence in this case, the capture can not be guaranteed. An escape strategy is suggested that enables an evader to escape from the perfectly encircled formation. The proposed escape strategy is evaluated through numerical simulations and future directions are discussed.

I. INTRODUCTION

Pursuit-evasion (PE) games are of popular interest and are used in many civilian and military applications. A lot of research has been carried out for the cases of pursuer and evader having equal speed and a pursuer having higher speed than that of an evader. Most of this research suggested strategies for successful capture of an evader. Contrary to those, the case of an evader having higher speed than that of a pursuer failed in observing a capturability set that ensures capture under a given set of initial conditions. This is because of the inability of a pursuer to match the abilities of a high speed evader. It has recently gained momentum in research arena and is a topic of interest for the researchers as it is more challenging and depicts the real world more accurately.

A PE game can be solved by formulating it as a differential game and solving the associated Hamilton-Jacobi-Issacs (HJI) partial differential equation [1]–[4]. Though the results are highly accurate, this approach suffers from the *curse of dimensionality* and is computationally intensive for games involving more than two players. Using missile guidance laws, PE games are modeled with simple motions under classical guidance schemes, line-of-sight (LOS), collision course (CC), proportional navigation (PN), and the condition for capturability [5] is analyzed. However, this approach only deals with motions along a straight path and does not consider complex motions that a general Unmanned Aerial Vehicle (UAV) can perform. The safe-reachable area cooperative pursuit helps in solving the PE game of pursuers and evader having equal speed in a holonomic case and the approach is found to be computationally efficient [6],

[7]. In this approach, a group of pursuers drive the safe-reachable area of an evader to zero as time progress using a decentralized strategy. The work was extended to non-holonomic systems which is more realistic and was solved using the idea of safe-reachable area and Dubins distance [8].

The inspiration for the methodology followed in this paper comes from the book by Isaacs [1], where he introduced Apollonius circle in studying PE games. The idea of using multiple pursuers is the most obvious approach for this case. Using multiple pursuers, a formation geometry was suggested that ensures successful capture of a high speed evader [9]. A more recent work used a similar approach in solving the pursuit problem [10], [11]. The work suggested the conditions on minimum number of pursuers, initial formation geometry, and strategy to be followed that ensures successful capture of an evader. This paper precisely contradicts the research done till now in the context of capturing a high speed evader in [9]–[11] and its failure is analyzed. They have analyzed the formation geometry and strategical condition required for capture but the mathematical proof is missing.

As discussed above, the safe-reachable area approach is an effective method for the case of pursuers and evader having equal speed. However, it is difficult to obtain a closed form solution for safe-reachable area in the case of high speed evader. In this paper, we propose to use the Apollonius circle with a perfectly encircled formation, carrying forward the philosophy of safe-reachable area approach. The exact solution of safe-reachable area for two players with different speeds using Hamilton-Jacobi-Bellman (HJB) equation [12] also includes the region of Apollonius circle. In this paper, the approach of Apollonius circle is found to keep the problem tractable when dealing with perfectly encircled formation. This is because it only involves the convex side of the actual safe-reachable area for an evader which directly overlaps on the Apollonius circle. We present an escape strategy that guarantees evasion in a perfectly encircled formation and its mathematical proof is derived that contradicts the necessary conditions for a successful capture mentioned in [10] and [11].

The rest of the paper is organized as follows. Section II formally describes the problem statement. Section III presents the preliminaries required to develop an escape strategy. It includes the properties of Apollonius circle and the conditions required to create a perfectly encircled formation. Section IV presents a mathematical proof of existence of an escape strategy for the case of five and more pursuers. Simulation results for the proposed escape strategy

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is presented in Section V. The concluding remarks and the scope for future work are discussed in Section VI.

II. PROBLEM STATEMENT

Consider a game of N pursuers and one evader in an open domain with the pursuers in a perfectly encircled formation. The aim of the pursuers is to capture the evader by having one of the pursuers within the radius of capture of the evader, r_c, whereas the evader tries to escape. The pursers' initial distances from the evader are greater than its radius of capture. The equations of motion of players involved in a game are given as follows,

$$
\begin{aligned}
\dot{x}_i &= V_i \cos \psi_i, \\
\dot{y}_i &= V_i \sin \psi_i, \\
\dot{\psi}_i &= \frac{u_i}{V_i},
\end{aligned} \tag{1}
$$

where $i \in \{\mathbf{P}, e\}, \mathbf{P} = \{p_1, ..., p_N\}$ is the set of pursuers, and e denotes an evader. $[x_i, y_i] \in \mathbf{R}^2$ is the position of ith player. $V_i = V_p, \forall i \in \mathbf{P}$, and V_p, V_e are the speeds of pursuer and evader respectively i.e, all pursuers have equal speed. In the case of high speed evader, $V_p < V_e$. ψ_i is the heading angle, and u_i represents lateral acceleration which acts as a control input for the ith player. There is no constraint acting on the control input which means that any vehicle can instantaneously change its orientation.

In a perfectly encircled formation, a group of pursuers try to shrink the safe-reachable area of an evader while keeping the formation intact to capture it. Whereas the evader tries to escape by creating a *gap* between Apollonius circles of any two neighboring pursuers. A perfectly encircled formation ensures *instantaneous nullification of escape path* for an evader, i.e., at that instant a straight path in any direction will intersect an Apollonius circle leading to capture. Creation of any gap will break the condition of instantaneous nullification of escape path and will aid in a successful evasion. If the evader creates a gap which makes its safe-reachable area unbounded and comes out of the perfectly encircled formation, then it is called a successful evasion.

Problem. Given the initial configuration of a perfectly encircled formation of N pursuers, $N \geq 5$, find an escape strategy for the evader that will ensure successful escape from the perfectly encircled formation.

Before going to develop an escape strategy, the required preliminaries are discussed in the following section.

III. PRELIMINARIES

A. Apollonius Circle

The idea of Apollonius circle in PE games was first discussed by Isaacs [1]. Consider a pursuer, $P[x_p, y_p]$, and an evader, $E[x_e, y_e]$, following the kinematics given in (1).

The locus of points X which take equal time for an evader and a pursuer to reach can be obtained using the relation, $\frac{PX}{V_p} = \frac{EX}{V_e}$. The locus is a circle with center, $O(\frac{x_p - \lambda^2 x_e}{1 - \lambda^2}, \frac{y_p - \lambda^2 y_e}{1 - \lambda^2})$, and radius, $r = \frac{\lambda \sqrt{(x_p - x_e)^2 + (y_p - y_e)^2}}{1 - \lambda^2}$, where $\lambda = \frac{V_p}{V_e}$, [11].

The circle is called an Apollonius circle and a typical sample is given in Fig. 1 along with the tangents from an evader. From here onwards, a tangent point will refer to the point on an Apollonius circle when joined to an evader forms a tangent line to the circle. For the case of a high speed evader the pursuer lies inside the circle whereas the evader lies outside. Any straight path of the evader that cuts through the Apollonius circle will lead to the capture. The two properties of Apollonius circle which we are going to use extensively in this paper are presented in the following lemmas.

Lemma 3.1. The angle subtended at the evader by the line joining the center of Apollonius circle and any two of the tangent points is independent of the positions of a pursuer and an evader.

Proof. From Fig. 1, let the angle subtended be α. Then,
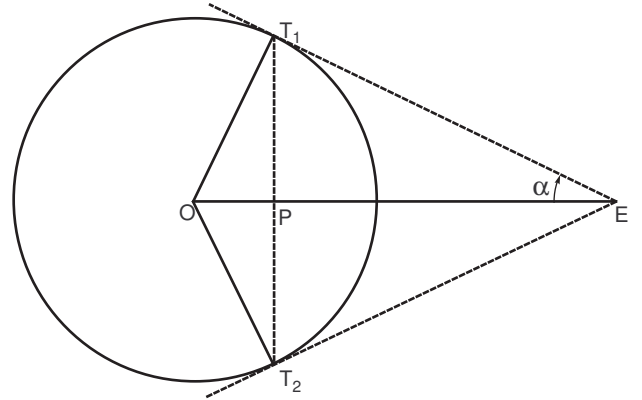
$$
\sin \alpha = \frac{OT_1}{OE}
$$

Fig. 1: Apollonius circle

Here, OT_1 is the radius of Apollonius circle and OE is the distance between the center of the circle and the evader.

$$
\begin{aligned}
\Rightarrow OE &= \sqrt{\left(\frac{x_p - \lambda^2 x_e}{1 - \lambda^2} - x_e\right)^2 + \left(\frac{y_p - \lambda^2 y_e}{1 - \lambda^2} - y_e\right)^2}, \\
&= \sqrt{\left(\frac{x_p - x_e}{1 - \lambda^2}\right)^2 + \left(\frac{y_p - y_e}{1 - \lambda^2}\right)^2}, \\
&= \frac{r}{\lambda}, \\
&= \frac{OT_1}{\lambda}
\end{aligned}
$$

$$
\Rightarrow \frac{OT_1}{OE} = \lambda \Rightarrow \sin \alpha = \lambda
$$

Hence the angle depends only on the ratio of speeds of pursuer and evader and not on their positions.

Lemma 3.2. The line joining the tangent points formed on the Apollonius circle passes through the pursuer and is perpendicular to line joining the evader and the pursuer.

Proof. From Lemma 3.1 we know $\sin \alpha = \lambda$. Moreover the points on the Apollonius circle follow the relation, $\frac{PX}{V_p} =$

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$\frac{EX}{V_e} \Rightarrow \frac{PX}{EX} = \frac{V_p}{V_e} = \lambda$. Since the tangent points lie on the circle, for the tangent point T_1

$$\frac{PT_1}{ET_1} = \lambda = \sin \alpha$$
$$\Rightarrow \angle EPT_1 = \frac{\pi}{2}$$

Moreover $\triangle PET_1$ and $\triangle PET_2$ are congruent triangles, since $ET_1 = ET_2$, $\angle PET_1 = \angle PET_2 = \alpha$, and PE forms the common side. Therefore T_1T_2 is a straight line passing through P and is perpendicular to PE.

Using the above lemmas, we will derive the necessary conditions required to form a perfectly encircled formation.

B. Perfectly Encircled Formation

A perfectly encircled formation is the one in which a set of pursuers encircle an evader in such a way that the Apollonius circle of each pursuer will have common tangent points with its neighboring pursuers and the tangent lines from the common tangent points passes through the evader. A typical example is given in Fig. 2. If an evader is trapped in a perfectly encircled formation, then it does not have an instantaneous escape path to get out of the formation. This is because any straight path of the evader to Apollonius circle will lead to capture. The conditions on the pursuers to create such a formation are discussed in Lemmas 3.3 and 3.4.
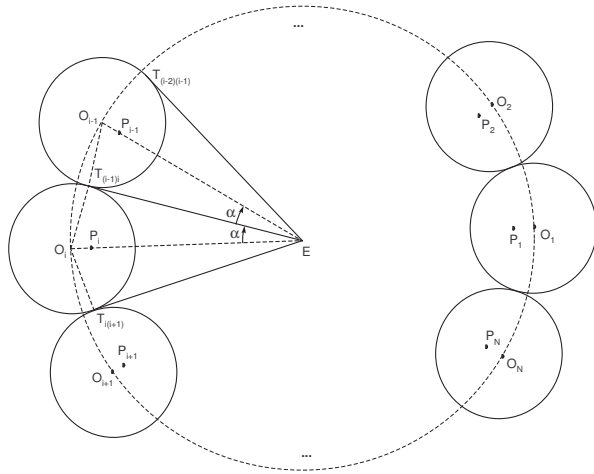

Fig. 2: Perfectly encircled formation of the pursuers around the evader

Lemma 3.3. The speed of each pursuer required to create a perfectly encircled formation using N identical pursuers is $\sin(\pi/N)$ times the speed of evader.
Proof. From Lemma 3.1, the angle subtended at the evader by the line joining the center and a tangent point of the Apollonius circle is α. In Fig. 1, $\triangle OET_1$ and $\triangle OET_2$ are congruent triangles and hence, $\angle OET_1 = \angle OET_2 = \alpha$. Therefore, $\angle T_1ET_2 = 2\alpha$, i.e. the angle subtended at the evader by the line joining the two tangent points of an Apollonius circle is 2α and is independent of the positions

of pursuer and evader.
From Fig. 2, to encircle the evader with N pursuers, we have

$$2N\alpha = 2\pi$$
$$\Rightarrow \alpha = \pi/N$$

Since $\alpha = \arcsin(\lambda)$,

$$\lambda = \sin(\pi/N)$$
$$\Rightarrow V_p = V_e \sin(\pi/N)$$

Lemma 3.4. The N pursuers required to form a perfectly encircled formation should lie equidistant from the evader.
Proof. From Fig. 2, consider $\triangle O_{i-1}T_{(i-1)i}E$ and $\triangle O_iT_{(i-1)i}E$ of the pursuers *(i-1)* and *i*.
As $\angle O_{i-1}T_{(i-1)i}E = \angle O_iT_{(i-1)i}E = \pi/2$, $\angle O_{i-1}ET_{(i-1)i} = \angle O_iET_{(i-1)i}$ and $T_{(i-1)i}E$ is the common side, both the triangles are congruent.

$$\Rightarrow O_{i-1}E = O_iE$$
$$\Rightarrow P_{i-1}E = P_iE$$

Therefore all the pursuers are equidistant from the evader.

Remark. The minimum number of pursuers required to arrive at a perfectly encircled formation to create a bounded safe-reachable area is three.

IV. PURSUIT-EVASION GAMES

The perfectly encircled formation ensures only an instantaneous nullification of escape paths for the evader. But it does not ensure successful capture. In order to capture, a group of pursuers has to make sure that the formation is maintained and the area bounded by it reduces to zero in finite time. While at the same time, the evader will try to employ a strategy that forces a *gap* between any two neighboring pursuers that will ensure a successful evasion. This is a min-max problem in terms of classical game theory.
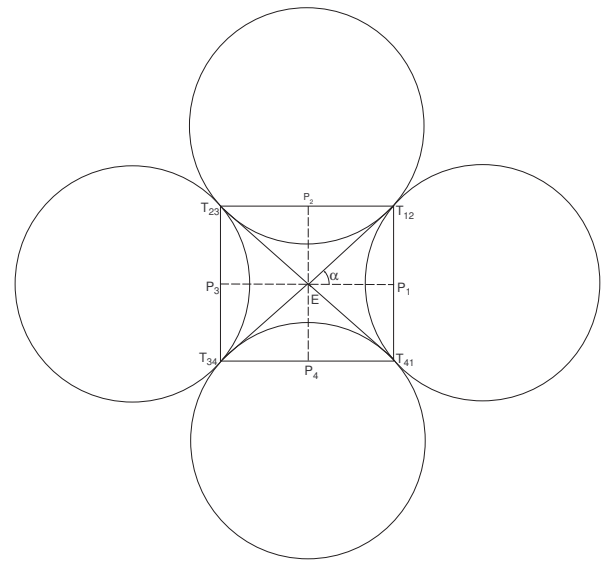

Fig. 3: Regular polygon for the case of four pursuers with Apollonius circles

A perfectly encircled formation in Fig. 2 can be transformed into a regular polygon by using the positions of the pursuer, the evader and, the tangent points involved. Since all the pursuers are equidistant and so are their tangent points, by joining the successive tangent points around the evader, a regular polygon can be created with each pursuer lying at midpoints of sides of the polygon. The safe reachable area of evader will be less than the area of the polygon because the latter includes fraction of Apollonius circle of each pursuer. This can be observed in Fig. 3 for a case of four pursuers. For a general case of N pursuers, the perfectly encircled formation can be seen as an N sided regular polygon with the vertices representing the common tangent points of the Apollonius circles and the pursuers themselves residing at midpoint of each side with evader at the center of the polygon. The same is depicted in Fig. 4 for a case of four pursuers.
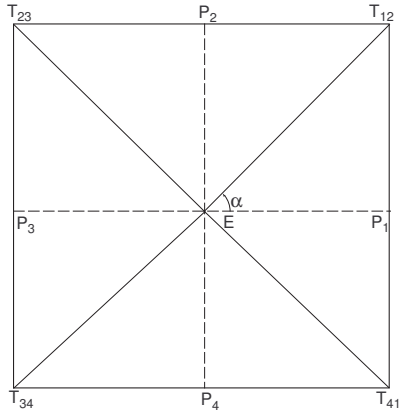

Fig. 4: Proposed regular polygon for the case of four pursuers

In terms of the proposed regular polygon of N sides, each pursuer should maintain common tangent points with its neighbors which ensures instantaneous nullification of escape path for an evader at any given time instant. The following lemma will aid in developing an escape strategy.

Lemma 4.1. In a perfectly encircled formation, if an evader moves toward a common tangent point, then the corresponding pursuers should move toward the tangent point to overcome the possibility of a gap emerging between them.

Proof. From Fig. 5, consider a situation of evader moving toward a vertex of the regular polygon formed due to N pursuers, N ≥ 3.

The evader moves to a new position E' along the line ET_{12} in a time t. Now, at least one of the pursuers of the corresponding vertex T_{12} should move toward the tangent point to capture the evader. Say, P_1 is moving toward the vertex. The triangle $\triangle EP_1T_{12}$ is same as a collision triangle that is used in missile guidance [13]. In the time t, P_1 will reach the new position P_1'. Now the line $E'P_1'$ will lie parallel to the line EP_1. From Lemmas 3.1 and 3.2, the tangent point T_{12} remains as it is. The second tangent point, T_y of the pursuer P_1 will move to T_y', $\angle P_1'E'T_y' = \alpha = \pi/N$.

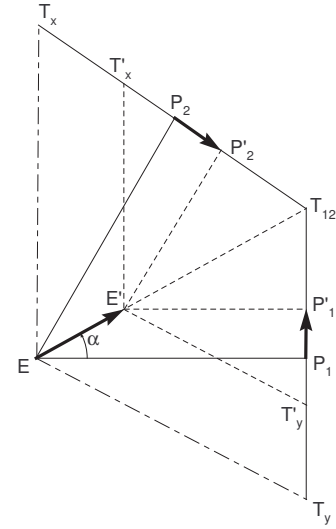Now consider the neighboring pursuer P_2. In the time

Fig. 5: Pursuit-Evasion scenario in the case of evader moving toward a tangent point

t, the pursuer P_2 has to move in such a way that its new tangent point corresponding to the vertex T_{12} has to either remain at T_{12} or on the line joining the points E' and T_{12}. This will ensure the nullification of escape path for the evader in between P_1 and P_2. Here again, the triangle $\triangle EPT_1$ is a collision triangle. Any other path will result in its corresponding tangent point at T_{12} falling away, creating a gap between P_1 and P_2, an escape path for the evader. Therefore the pursuer P_2 has to move along the line P_2T_{12}. Next, it is shown that with a perfectly encircled formation, it is not possible to capture a high speed evader.

Theorem 4.1. In a perfectly encircled formation of five or more pursuers, the movement of evader toward any of the vertex of the resulting regular polygon will eventually create an instantaneous escape path that will ensure successful evasion under any strategy of the pursuers.

Proof. Consider a situation as shown in Fig. 6 in which an evader moving toward a vertex of the regular polygon formed due to N pursuers, N ≥ 3.

The evader moved to E' along the line ET_{12} in a time t. From Lemma 4.1, since the evader is moving toward the common tangent point, the pursuers P_1 and P_2 must move toward T_{12}. Therefore, the new position of pursuer P_1 after time t is P_1' which lies on the line P_1T_{12} and the new tangent points are T_{12} and T_{31}'.

The objective of pursuer P_3 is to move in such a way that its tangent point corresponding to the vertex T_{31} will move either to the point T_{31}' or at least to any point on the line joining the points E' and T_{31}'. This will ensure nullification of an escape path for the evader in between P_1 and P_3. This means if the pursuer P_3 moves to a new position in time t, then the corresponding new tangent point, T_{31}'', must lie on the line $E'T_{31}'$. For this to happen, the new position of P_3 must lie on the line L_1 which makes an angle α with $E'T_{31}'$ as shown in Fig. 6. The possible positions that P_3 can take
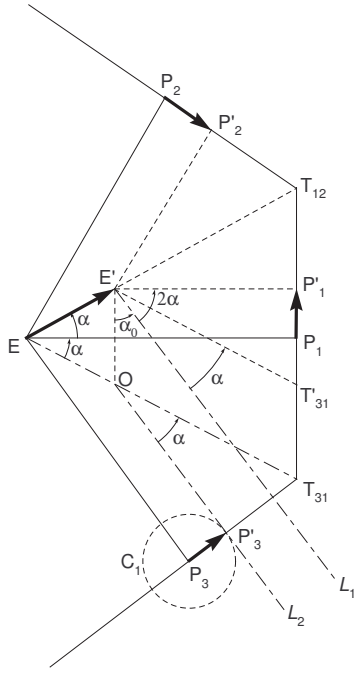
Fig. 6: Pursuit-Evasion scenario in the case of evader moving toward a tangent point of Apollonius circle

in time t can be seen as a circle C_1 around P_3 in Fig. 6. The circle intersects the line P_3T_{31} at P'_3.

The line perpendicular to EP_1 and passing through E' meets the line ET_{31} at O. The point O can be seen as the position, the evader would have taken if it had traveled along ET_{31} in the same time t. The line joining O and P'_3, L_2, lies perpendicular to P_3T_{31} and is tangent to the circle C_1 of pursuer P_3. The lines L_1 and L_2 are parallel lines, as they are at equal angle, α, with two parallel lines $E'T'_{31}$ and ET_{31}.

Since the points E, E' and T_{12} lie on a straight line, $\angle EE'T_{12} = \pi$. Let the angle created at E' by line L_1 and line $E'O$ be α_0. Now, from Fig. 6,

$$\alpha_0 + \angle EE'O + \angle L_1 E'T_{12} = \pi$$
$$\Rightarrow \alpha_0 + \pi/2 + 2\alpha = \pi$$
$$\Rightarrow \alpha_0 + 2\alpha = \pi/2$$
$$\Rightarrow \alpha_0 = \pi/2 - 2\pi/N$$

For $N \geq 5$, α_0 is a positive value and hence the line L_1 cannot intersect the circle C_1. This results in the formation of a gap between pursuers P_3 and P_1 and thereby ensures a successful evasion.

For the case of four pursuers, α_0 is zero. Because both the pursuers are not equidistant, so are their corresponding tangent points. But the corresponding tangent points and the evader will lie on the same line. This will result in the formation of an indirect gap between the two pursuers. The case of three pursuers is similar to the case of four pursuers and the mathematical proof of existence of an escape strategy for both the cases is eliminated to adhere

to the size limitation of the paper. However, the evader has to change its course in both the cases to create a direct gap for a successful escape that is again guided by simple laws in geometry.

A. Escape strategy

The escape strategy of an evader can now be summarized. In case of an evader in a perfectly encircled formation of N pursuers, for $N >= 5$, the evader will initially move toward any one of the tangent points. A gap will then be created between the corresponding pursuers that the evader uses to escape by changing its course toward the gap created. However, the evader has to create a gap big enough depending on its radius of capture to escape successfully.

V. SIMULATION RESULTS

In this section, we demonstrate the performance of the proposed strategy with an example. The simulation is conducted with a set of six pursuers around an evader starting from a perfectly encircled formation. The radius of capture for the evader is considered to be 5 m. The evader speed is fixed to 13 m/s. The initial positions of a set of pursuers in a perfectly encircled formation are at a distance of 75 m from the evader.

(a) t = 0 sec
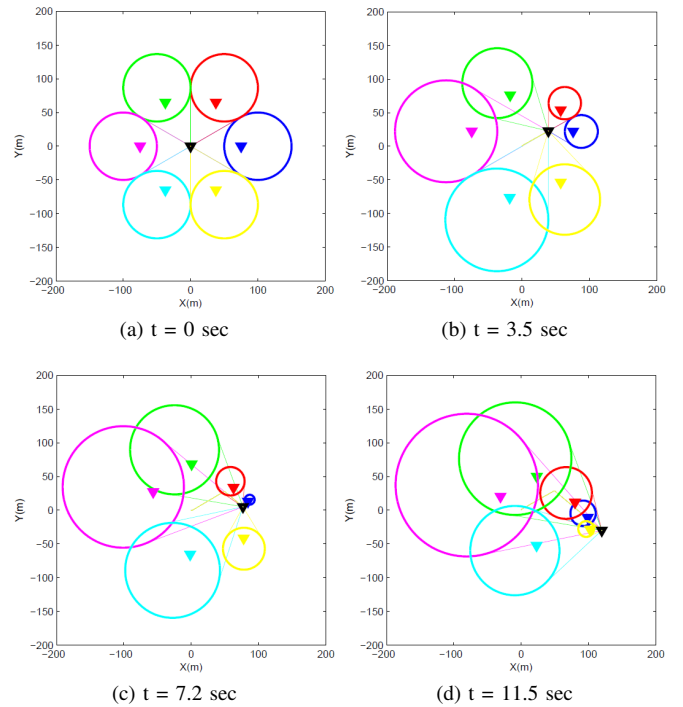
(b) t = 3.5 sec

(c) t = 7.2 sec

(d) t = 11.5 sec

Fig. 7: PE scenarios with six pursuers under the proposed strategy at different time instants

The scenarios of the game at different time instants are shown in Fig. 7. A perfectly encircled formation of six pursuers is first presented. As $N = 6$, the speed of pursuers is, $V_p = V_e \sin(\pi/6)$. The evader employs the proposed strategy and starts moving toward one of the tangent points. Due to this, two direct gaps are created between the pursuers

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that can be seen in Fig. 7 at a time instant of $t = 3.5\ sec$. The evader takes advantage of this opportunity and deviates from its initial path. It takes an escape path by moving toward the gap. This results in a successful evasion which is also shown in Fig. 7. The paths traced by the players are shown in Fig. 8
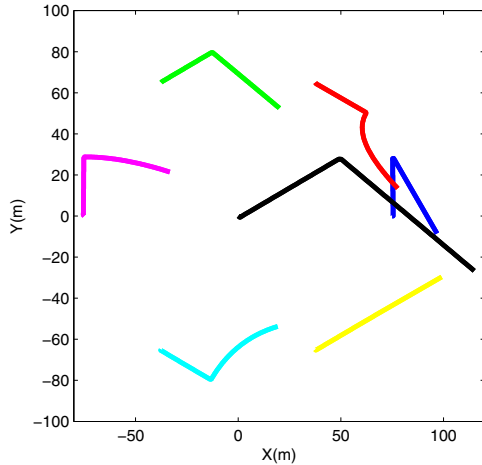

Fig. 8: Paths traced by the players

It can be observed from the simulation that the proposed strategy guides an evader successfully to escape from a perfectly encircled formation which is line with our mathematical development.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we have studied pursuit-evasion games of multiple pursuers and a single high speed evader under a perfectly encircled formation. The idea of Apollonius circle is used to describe the perfectly encircled formation. It is shown that a perfectly encircled formation only ensures instantaneous nullification of an escape path. For successful capture, the formation should remain connected until the area bounded by it shrinks to zero. For such games, it is mathematically proved that the capture is not possible with the minimum number of pursuers required to form a perfectly encircled formation. In this process, we have identified an escape strategy for an evader using the properties of Apollonius circle and the formation geometry. The idea of perfectly encircled formation and the proposed escape strategy unlock an idea of analyzing the formation which includes overlapping of Apollonius circles. This reduces the possibility of direct and indirect gaps being created during the pursuit which will be considered in the future. Also, the dependence of an escape on the evaders radius of capture is another area of interest that is being explored by us.

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