49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

Optimal Pursuit of Moving Targets using Dynamic Voronoi Diagrams

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***Abstract*— We consider Voronoi-like partitions for a team of moving targets distributed in the plane, such that each set in this partition is uniquely associated with a particular moving target in the following sense: a pursuer residing inside a given set of the partition can intercept this moving target faster than any other pursuer outside this set. It is assumed that each moving target employs its own "evading" strategy in response to the pursuer actions. In contrast to standard formulations of problems of this kind in the literature, the evading strategy does necessarily restrict the evader to be slower than its pursuer. In the special case when all moving targets employ a uniform evading strategy, the previous problem reduces to the characterization of the Zermelo-Voronoi diagram.**

I. INTRODUCTION

Voronoi-like partition problems for a set of moving generators in the plane, known in the literature as dynamic partition problems [1], constitute a class of challenging problems in dynamic computational geometry [2], [3], [4], [5], [6]. They have received a considerable amount of attention recently owing to their applicability in mobile network and multi-agent problems. One of the fundamental questions in this framework, deals with the characterization of the proximity relations between the moving generators (i.e., agents) and the points in the plane as time evolves. In contrast to the standard Voronoi partitioning problem, where all generators are stationary, the solution of the dynamic partition problem consists of a *sequence* of time-evolving Voronoi diagrams. A diagram of this time-evolving data structure at a particular instant of time is a standard Voronoi diagram with respect to the positions of the moving Voronoi generators at that time.

The work of Devillers et al. [3], [4] highlights an interesting aspect of dynamic partition problems. In particular, [3], [4] deals with the following problem: Given a set of n postmen (moving targets) that move along prescribed rays with constant speed, a set of n dogs (pursuers) going after the postmen, find the rule that assigns each dog to each postman, under the assumption that every dog is faster than every postmen. The main challenge of this problem comes from the fact that any question regarding the proximity relations between pursuers and moving targets has to be addressed with respect to a generalized distance function, in this case the minimum intercept time, rather than the usual Euclidean distance, as with the standard Voronoi partition.

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In this work, we consider a more general dynamic partitioning problem under some more realistic assumptions compared to the problem formulation in [4]. In particular, given a team of n moving targets, we consider the problem of partitioning the plane into n "capture zones" such that each element in the partition is associated with a particular moving target in the following sense: a pursuer that resides inside the i^{th} "capture zone" at a given instant in time, can intercept the i^{th} moving target faster than any other pursuer residing outside this zone. In our problem formulation, we assume that each moving target can employ its own "evading" strategy in response to the actions of its pursuer. This strategy does not necessarily constraint the target to move slower than its pursuer. If the target intercept problem is feasible, the optimal pursuit strategy is the solution of a special case of Zermelo's navigation problem [7]. After investigating the feasibility and the existence of optimal solutions of the dynamic partition problem, we propose an efficient numerical solution, which is based on the propagation of the level sets of the minimum intercept time by taking advantage of some of the properties of the optimal solutions and the reachable sets of ZNP.

In the special case when all the moving targets employ a uniform evading strategy, the solution of the proposed dynamic partition problem is reduced to the Zermelo-Voronoi diagram [8], [9]. Two interesting attempts that deal with this problem in the special case of a spatially-varying (albeit stationary) wind field have appeared in [10], [11], where purely computational/numerical solutions are presented. We propose an alternative scheme for determining the Zermelo-Voronoi diagram, which exploits the structure of the solution of the ZNP and deals with both temporally and spatially varying wind/current fields.

The rest of the paper is organized as follows. In Section II we formulate the dynamic partitioning problem based on the minimum capture time of the moving generators, and in Section III we present an efficient scheme for characterizing its solution. Section IV presents non-trivial simulation results. Finally, Section V concludes the paper with a summary of remarks.

II. PROBLEM FORMULATION

Consider a set of moving targets $\mathcal{T}(t) \triangleq \{\mathrm{x}_{\mathcal{T}}^{i}(t),\ t \geq 0, i \in \mathcal{I}\}$, where $\mathcal{I} \triangleq \{1,\ldots,n\}$, and where $\mathrm{x}_{\mathcal{T}}^{i}(t) \triangleq [x_{\mathcal{T}}^{i}(t), y_{\mathcal{T}}^{i}(t)]^{\mathsf{T}} \in \mathbb{R}^2$, is the position vector at time t of the i^{th} moving target. Each point in $\mathcal{T}(t)$ evolves with time

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according to the following equation

$$\dot{\mathsf{x}}_{\mathcal{T}}^i = u_{\mathcal{T}}^i + v_{\text{ref}}^i(t), \quad \mathsf{x}_{\mathcal{T}}^i(0) = \bar{\mathsf{x}}_{\mathcal{T}}^i, \quad i \in \mathcal{I}, \tag{1}$$

where $v_{\text{ref}}^i(t)$ is a reference velocity signal provided to the target, and $u_{\mathcal{T}}^i$ is the "evading control strategy" of the i^{th} target. Throughout this work, it is assumed that $\mathcal{T}(0)$ always consists of n distinct points. Furthermore, the equations of motion of a pursuer, whose objective is to intercept the i^{th} moving target are given by

$$\dot{\mathsf{x}}_{\mathcal{P}} = u_{\mathcal{P}}, \quad \mathsf{x}_{\mathcal{P}}(0) = \bar{\mathsf{x}}_{\mathcal{P}}, \tag{2}$$

where $\mathsf{x}_{\mathcal{P}}(t) \triangleq [x_{\mathcal{P}}(t), y_{\mathcal{P}}(t)]^{\mathsf{T}} \in \mathbb{R}^2$, and $u_{\mathcal{P}} \in \mathcal{U}_{\mathcal{P}}$ denotes the control input of the pursuer. It is assumed that the set of admissible control inputs $\mathcal{U}_{\mathcal{P}}$ consists of all measurable functions that take values in the closed unit ball. Furthermore, it is assumed that the evading strategy $u_{\mathcal{T}}^i$ employed by the i^{th} target in response to the actions of its pursuer, is a time-varying feedback control law that depends on the relative position of the i^{th} moving target from its pursuer. In particular,

$$u_{\mathcal{T}}^i = A_{\mathcal{T}}^i(t)(\mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}^i), \tag{3}$$

where $A_{\mathcal{T}}^i(t)$ is a 2×2 real, time-varying matrix.

The objective of each pursuer is to determine an admissible pursuit strategy $u_{\mathcal{P}} \in \mathcal{U}_{\mathcal{P}}$ that minimizes the time T_{f}^i such that $|\mathsf{x}_{\mathcal{P}}(t) - \mathsf{x}_{\mathcal{T}}^i(t)| > \rho$, for a given sufficiently small $\rho \geq 0$ (*radius of capturability*) and for all $t < T_{\mathsf{f}}^i$, where $|\cdot|$ denotes the Euclidean norm.

Let us consider the following state transformation $\mathsf{y}^i \triangleq \mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}^i$, where $\mathsf{y}^i \triangleq [\mathsf{y}_1^i, \mathsf{y}_2^i]^{\mathsf{T}} \in \mathbb{R}^2$. Equation (1) can then be written in the following compact form

$$\dot{\mathsf{y}}^i = \mathsf{u}^i + w^i(\mathsf{y}^i, t), \quad \mathsf{y}^i(0) = \bar{\mathsf{y}}^i, \tag{4}$$

where $\bar{\mathsf{y}}^i \triangleq \bar{\mathsf{x}}_{\mathcal{P}} - \bar{\mathsf{x}}_{\mathcal{T}}^i$, $\mathsf{u}^i \triangleq u_{\mathcal{P}}$ and $w^i \triangleq -u_{\mathcal{T}}^i - v_{\text{ref}}^i(t)$, which implies, in light of (3), that

$$w^i(\mathsf{y}^i, t) \triangleq -A_{\mathcal{T}}^i(t)\mathsf{y}^i - v_{\text{ref}}^i(t). \tag{5}$$

Thus, the pursuit strategy of the i^{th} pursuer is the solution of the following minimum-time problem.

Problem 1 (i^{th} MTP): Given the system described by equation (4), determine the control input $\mathsf{u}_*^i \in \mathcal{U}_{\mathcal{P}}$, such that

i) The trajectory $\mathsf{y}_*^i : [0, T_{\mathsf{f}}^i] \mapsto \mathbb{R}^2$ generated by the control u_*^i satisfies the boundary conditions

$$\mathsf{y}_*^i(0) = \bar{\mathsf{y}}^i, \quad |\mathsf{y}_*^i(T_{\mathsf{f}}^i)| \leq \rho. \tag{6}$$

ii) The control u_*^i minimizes along y_i^* the cost functional $J^i(\mathsf{u}^i) \triangleq T_{\mathsf{f}}^i$, where T_{f}^i is the free final time, henceforth denoted as $T_{\mathsf{f}}^i(\bar{\mathsf{y}}^i; w^i)$.

Problem 1 can be interpreted as the problem of steering an integrator from $\bar{\mathsf{y}}^i$ to the origin in minimum-time, in the presence of a temporally and spatially-varying drift term $w^i(\mathsf{y}^i, t) \triangleq -u_{\mathcal{T}}^i(\mathsf{y}^i, t) - v_{\text{ref}}^i(t)$. This problem is a special case of the Zermelo Navigation Problem (ZNP) [7].

Next, we investigate the feasibility and existence of optimal solutions of Problem 1. To this end, let $\mathfrak{R}(\bar{\mathsf{y}}^i)$ denote the reachable set from $\bar{\mathsf{y}}^i$ of the system described by (4). The following proposition provides a necessary and sufficient condition for the feasibility of the i^{th} MTP.

Proposition 1: The i^{th} MTP has a feasible solution if and only if

$$\{\mathsf{y} \in \mathbb{R}^2 : \ |\mathsf{y}| \leq \rho\} \cap \mathfrak{R}(\bar{\mathsf{y}}^i) \neq \varnothing. \tag{7}$$

Proposition 1 implies that the feasibility of the i^{th} MTP is equivalent to the system (4) being controllable to the origin. The controllability question for system (4) requires a rather detailed and careful treatment due to the existence of the drift term $w^i(\mathsf{y}^i, t)$, and the fact that the control u^i is bounded [12]. The following proposition addresses the question of complete controllability to the origin in the special case when the evading strategy $u_{\mathcal{T}}^i$ in (3) does not depend explicitly on time.

Proposition 2: Let $w(\mathsf{y}^i, t) = A_{\mathcal{T}}^i \mathsf{y}^i + v_{\text{ref}}^i(t)$, and $|v_{\text{ref}}^i(t)| < 1 - \epsilon$, where $\epsilon \in (0, 1)$, for all $t \geq 0$. Assume that

i) $\operatorname{rank} A_{\mathcal{T}}^i = 2$,
ii) no left real eigenvector e of $A_{\mathcal{T}}^i$ satisfies $\langle \mathsf{e}, v \rangle \leq 0$, for all v.
iii) there is no eigenvalue $\lambda \in \operatorname{spec}(A_{\mathcal{T}}^i)$ with $\operatorname{Re}(\lambda) > 0$.

Then i^{th} MTP has a feasible solution for every $\bar{\mathsf{y}}^i \in \mathbb{R}^2$. If, in addition, $v_{\text{ref}}^i(t) \equiv 0$, then conditions i) - iii) are also necessary for the feasibility of the i^{th} MTP.

Proof: Let $\tilde{\mathsf{u}}^i \triangleq \mathsf{u}^i - v_{\text{ref}}^i(t)$, and let $\widetilde{U}^i$ denote the set of all admissible values of $\tilde{\mathsf{u}}^i$ for a given $v_{\text{ref}}^i(t)$. It follows that $|v_{\text{ref}}^i(t)| < 1 - \epsilon$, implies that $\{u \in \mathbb{R}^2 : \ |u| < \epsilon\} \subset \widetilde{U}^i$, and therefore all conditions of Theorem 7 of [13] are satisfied. ■

The following proposition provides a sufficient condition for the existence of a solution of Problem 1.

Proposition 3: If there exists α^i, $\beta^i > 0$ such that $|A_{\mathcal{T}}^i(t)| < \alpha^i$ and $|v_{\text{ref}}^i(t)| < \beta^i$ for all $t \geq 0$ and condition (7) holds, then the i^{th} MTP has an optimal solution.

Proof: From Filippov's Theorem on the existence of solutions of minimum-time problems [14, pp. 310-317], it suffices to prove that there exists $k > 0$ such that

$$\langle \dot{\mathsf{y}}^i, \mathsf{y}^i \rangle \leq k(1 + |\mathsf{y}^i|^2). \tag{8}$$

Indeed, by virtue of the triangle and Cauchy-Schwartz inequalities, and the fact that $|\mathsf{u}^i| \leq 1$ it follows

$$\begin{aligned} \langle \dot{\mathsf{y}}^i, \mathsf{y}^i \rangle &\leq (|A_{\mathcal{T}}^i(t)||\mathsf{y}^i| + |v_{\text{ref}}^i| + |\mathsf{u}^i|)|\mathsf{y}^i| \\ &\leq \alpha^i |\mathsf{y}^i|^2 + (\beta^i + 1)|\mathsf{y}^i|. \end{aligned} \tag{9}$$

The result follows readily from the inequality $2|\mathsf{y}^i| \leq 1 + |\mathsf{y}^i|^2$. ■

Let $\mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^i)$ be the set of all initial conditions $\bar{\mathsf{x}}_{\mathcal{P}}$ of the pursuer from which the i^{th} MTP has a solution, that is,

$$\mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^i) \triangleq \bigcup_{t \geq 0, u_{\mathcal{P}} \in \mathcal{U}_{\mathcal{P}}} \{\bar{\mathsf{x}}_{\mathcal{P}} \in \mathbb{R}^2 : |\mathsf{x}_{\mathcal{P}}(t) - \mathsf{x}_{\mathcal{T}}^i(t)| \leq \rho\}. \tag{10}$$

Next, we formulate a dynamic Voronoi partitioning problem based on the minimum pursuit time of the i^{th} MTP. The space to be partitioned, denoted henceforth as $\mathfrak{A}$, is the union of $\mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^i)$ for all $i \in \mathcal{I}$.

Problem 2: Given a collection of moving targets $\mathcal{T}(t) \triangleq \{\mathsf{x}_{\mathcal{T}}^i(t) \in \mathbb{R}^2 : \ i \in \mathcal{I}\}$ which evolve according to the equations (1) and (3), and a transition cost

$$c^i(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^i) \triangleq T_{\mathsf{f}}^i(\mathsf{x} - \bar{\mathsf{x}}_{\mathcal{T}}^i; w^i), \tag{11}$$

determine a partition $\mathfrak{V} = \{\mathfrak{V}_i : \ i \in \mathcal{I}\}$ of $\mathfrak{A}$ such that

i) $\mathfrak{A} = \bigcup_{i \in \mathcal{I}} \mathfrak{V}_i$.
ii) $\overline{\mathfrak{V}_i} = \mathfrak{V}_i$, for each $i \in \mathcal{I}$.
iii) for each $\mathsf{x} \in \operatorname{int} \mathfrak{V}_i$, $c(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^i) < c(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^j)$ for $j \neq i$.

Henceforth, we shall refer to the solution of Problem 2 as the Optimal Pursuit Dynamic Voronoi Diagram (OP-DVD). The sets $\mathcal{T}(t)$ and $\mathfrak{V}^i$ constitute the set of the (moving) Voronoi generators and the Voronoi cells or Dirichlet domains of the OP-DVD respectively. Each Voronoi cell $\mathfrak{V}_i$ of OP-DVD can be interpreted as a "capture zone" from which a pursuer can intercept the i^{th} moving target faster than any other pursuer that lies within another Voronoi cell.

To this end, let us consider a pursuer traveling in the presence of spatially and temporally varying drift. It is assumed that the equations of motion of the pursuer are given by

$$\dot{\mathsf{x}}_{\mathcal{P}} = u_{\mathcal{P}} + M(t)\mathsf{x}_{\mathcal{P}} + \nu(t), \quad \mathsf{x}_{\mathcal{P}}(0) = \bar{\mathsf{x}}_{\mathcal{P}}, \tag{12}$$

where $M(t)$ is a 2×2 real, time-varying matrix and $\nu(t)$ a two-dimensional time-varying vector. The objective of the pursuer is to reach a static target $\bar{\mathsf{x}}_{\mathcal{T}}^i$ from the set $\mathcal{T}(0)$ in minimum time. Let $\mathsf{y}^i(t) \triangleq \mathsf{x}_{\mathcal{P}}(t) - \bar{\mathsf{x}}_{\mathcal{T}}^i$. Then (12) can be written as follows

$$\dot{\mathsf{y}}^i = \mathsf{u}^i + w(\mathsf{y}^i, t), \quad \mathsf{y}^i(0) = \bar{\mathsf{y}}^i, \tag{13}$$

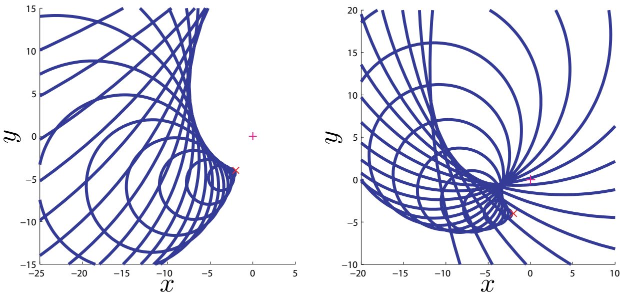
where $\mathsf{u}^i \triangleq u_{\mathcal{P}}$, and $w(\mathsf{y}^i, t) \triangleq M(t)\mathsf{y}^i + (\nu(t) + M(t)\bar{\mathsf{x}}_{\mathcal{T}}^i)$. Thus the problem of steering the pursuer from $\bar{\mathsf{x}}_{\mathcal{P}}$ to $\bar{\mathsf{x}}_{\mathcal{T}}^i$ in minimum time, is equivalent to the problem of driving the system (13) from $\bar{\mathsf{y}}^i$ to the origin, which is exactly the i^{th} MTP in the special case $w^i = w$, for all $i \in \mathcal{I}$, that is,

$$A_{\mathcal{T}}^i(t) = -M(t), \quad v_{\text{ref}}^i(t) = -(\nu(t) + M(t)\bar{\mathsf{x}}_{\mathcal{T}}^i), \tag{14}$$

for all $t \geq 0$ and $i \in \mathcal{I}$. Therefore, in the special case when $w^i = w$, for all $i \in \mathcal{I}$, then Problem 2 reduces to the Zermelo-Voronoi diagram problem [8], [9]. The Zermelo-Voronoi diagram Problem involves the construction of a Voronoi-like partition of the plane for a given set of (stationary) Voronoi generators (which in our case is the point-set $\mathcal{T}(0) = \{\bar{\mathsf{x}}_{\mathcal{T}}^i, \ i \in \mathcal{I}\}$), and a generalized distance function. In our case the generalized distance function is the minimum time from an arbitrary point x in the plane to $\mathcal{T}(0)$ in the presence of both temporally and spatially varying drift w.

Problem 3: Given a collection of fixed goal destinations $\mathcal{T} \triangleq \{\bar{\mathsf{x}}_{\mathcal{T}}^i \in \mathbb{R}^2 : \ i \in \mathcal{I}\}$ and a transition cost

$$c^i(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^i) \triangleq T_{\mathsf{f}}^i(\mathsf{x} - \bar{\mathsf{x}}_{\mathcal{T}}^i; w), \tag{15}$$



(a) Unfeasible case: $(0,0)^{\mathsf{T}} \in \mathfrak{R}^c(\bar{\mathsf{y}}^i)$ (b) Solvable case: $(0,0)^{\mathsf{T}} \in \mathfrak{R}(\bar{\mathsf{y}}^i)$

Fig. 1. The i^{th} MPT is solvable if and only if the origin (magenta cross), belongs to the reachable set of $\bar{\mathsf{y}}^i$ (red x-cross).

where T_{f}^i is the minimum time of Problem 1 with $w^i = w$, for all $i \in \mathcal{I}$, determine a partition $\mathfrak{V}^{\text{ZN}} = \{\mathfrak{V}_i^{\text{ZN}} : \ i \in \mathcal{I}\}$ of $\mathfrak{A}$ such that

i) $\mathfrak{A} = \bigcup_{i \in \mathcal{I}} \mathfrak{V}_i$.
ii) $\overline{\mathfrak{V}_i^{\text{ZN}}} = \mathfrak{V}_i^{\text{ZN}}$, for each $i \in \mathcal{I}$.
iii) for each $\mathsf{x} \in \operatorname{int} \mathfrak{V}_i$, $c^i(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^i) < c^j(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^j)$ for $j \neq i$.

Henceforth, we shall refer to the solution of Problem 3 as the Zermelo-Voronoi Diagram (ZVD).

III. Construction of the Voronoi Partitions

At this point it is not clear whether the minimum time-to-go to the origin for the i^{th} MTP enjoys the necessary properties (such as isotropy and convexity) that would allow us to associate both Problems 2 and 3 with generalized Voronoi-like partitioning problems, for which efficient computational methods exist in the literature. Therefore, we need to adopt an alternative approach. In particular, let us consider the minimum cost function $c_* : \mathfrak{A} \mapsto [0, \infty)$ with $c_*(\mathsf{x}) = \min_{i \in \mathcal{I}} c^i(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^i)$, that is, the minimum time required for a pursuer at $\bar{\mathsf{x}}_{\mathcal{P}}$ at time $t = 0$ to intercept a moving target residing at $\mathsf{x}_{\mathcal{T}}^i$ at $t = 0$. The OP-DVD can be characterized by projecting onto $\mathfrak{A}$ the intersections of the surface c_* with each cost surface c^i, $i \in \mathcal{I}$. This method can be efficiently implemented by propagating the level sets of each cost surface c^i, that is, the projection of the iso-costs of c^i onto $\mathfrak{A}$, emanating from $\bar{\mathsf{x}}_{\mathcal{T}}^i$, for all $i \in \mathcal{I}$. This can be done by employing either a fast marching or a marker-particle algorithm, as in [10], [11]. The approaches in [10], [11], however, neither take advantage of the structure of the solution of the ZNP, nor they guarantee that numerical pathogenies will not arise. In particular, as it is stressed in [15], the system (4) is not necessarily small-time locally controllable [16], and consequently the minimum time-to-go of the i^{th} MTP may undergo discontinuous jumps. Such numerical pathogenies cannot be handled effectively unless more sophisticated numerical techniques, than those presented in [10], [11], are employed (for example, approaches that find weak solutions of the HJB equation of the ZNP [15]). Instead of resorting to exhaustive computational methods, next we investigate whether there exists a more direct and efficient method to solve Problem 2, by taking advantage of the structure of the solution of the ZNP to significantly simplify the process of expanding the level sets of c^i for each $i \in \mathcal{I}$.

A. *Structure of Optimal Solutions and Reachable Sets of the i^{th} Moving Target Problem*

We first present some key results from the solution of the i^{th} MTP, (equivalently, the ZNP to the origin), that are necessary for the subsequent discussion. The reader interested in a more detailed treatment of the ZNP may refer to [16, pp. 239-247, pp. 370-373]. In particular, if the i^{th} problem is feasible, then the control u_*^i that solves the i^{th} MTP has necessarily the following structure: $\mathsf{u}_*^i = [\cos\theta_*^i, \ \sin\theta_*^i]^{\mathsf{T}}$, where θ_*^i satisfies the following differential equation [16]

$$\begin{aligned}\dot{\theta}_*^i = (a_{[1,1]}^i(t) - a_{[2,2]}^i(t))\cos\theta_*^i \sin\theta_*^i + a_{[2,1]}^i(t)\sin^2\theta_*^i \\ - a_{[1,2]}^i(t)\cos^2\theta_*^i, \quad \theta_*^i(0) = \bar{\theta}^i. \end{aligned} \tag{16}$$

where $a_{[k,\ell]}^i(t)$, $k, \ell \in \{1,2\}$, are the elements of the matrix $A_{\mathcal{T}}^i(t)$. It follows that the (candidate) optimal control u_*^i is determined up to one parameter; we thus write $\mathsf{u}_*^i(\cdot;\bar{\theta}^i)$.

To this end, let φ^i denote the solution of the differential equation (4) for $\mathsf{u}^i = \mathsf{u}_*^i(\cdot;\bar{\theta}^i)$, which we henceforth refer to as the extremal curve of the i^{th} MTP, and let $\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ denote the set of all points that can be reached by (4) from $\bar{\mathsf{y}}^i$ in time less than or equal to t. Furthermore, we define the t-level set of c^i, henceforth denoted as $\ell_t(\bar{\mathsf{y}}^i)$, to be the set of all points that can be reached from $\bar{\mathsf{y}}_i$ in minimum time t. The following proposition highlights the connection between the sets $\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ and $\ell_t(\bar{\mathsf{y}}^i)$ and the extremal curves φ^i.

Proposition 4: The sets $\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ and $\ell_t(\bar{\mathsf{y}}^i)$ are given by

$$\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i) = \bigcup_{\tau\in[0,t]} \bigcup_{\bar{\theta}^i\in\mathbb{S}^1} \varphi(\tau;\bar{\mathsf{y}}^i,\bar{\theta}^i), \tag{17}$$

$$\ell_t(\bar{\mathsf{y}}^i) \subseteq \bigcup_{\bar{\theta}^i\in\mathbb{S}^1} \varphi^i(t;\bar{\mathsf{y}}^i,\bar{\theta}^i). \tag{18}$$

Proof: It suffices to note, in light of Proposition 3, that if $\mathsf{y} \in \mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$, then there exists $\bar{\theta}^i \in \mathbb{S}^1$ such that $\mathsf{y} = \varphi^i(\tau;\bar{\mathsf{y}}^i,\bar{\theta}^i)$ for some $\tau \in [0,t]$. ∎

From Proposition 4, $\mathbb{S}^1$ admits the following decomposition

$$\mathbb{S}^1 = \Theta_*^i(t) \cup (\Theta_*^i(t))^c, \ \ \Theta_*^i(t) \cap (\Theta_*^i(t))^c = \varnothing, \ t \geq 0, \tag{19}$$

where $\Theta_*^i(t)$ denotes the set of $\bar{\theta}^i$ such that $\varphi(t;\bar{\mathsf{y}}^i,\bar{\theta}^i) \in \ell_t(\bar{\mathsf{y}}^i)$. Note that a point in $\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i) \cap \ell_t^c(\bar{\mathsf{y}}^i)$ can be reached after t units of time by means of an extremal curve that is either: 1) maximizing (locally or globally), 2) locally minimizing or 3) abnormal (that is, an extremal curve that does not satisfy the strengthened Legendre condition [12]). The following proposition provides us with a sufficient condition for determining whether an extremal curve of Problem 1 is maximizing, minimizing or abnormal [7], [16].

Proposition 5: Let $\mathsf{y}_*^i(\tau) = \varphi^i(\tau;\bar{\mathsf{y}}^i,\bar{\theta}^i)$ be the extremal curve generated by $\mathsf{u}_*^i(\tau;\bar{\theta}^i)$, for $\tau \in [0,t]$. If the functional

$$\mathsf{I}[\mathsf{y}_*^i, \mathsf{u}_*^i, \tau] \overset{\triangle}{=} 1 + \langle w^i(\mathsf{y}_*^i,\tau), \mathsf{u}_*^i \rangle, \tag{20}$$

satisfies $\mathsf{I}[\mathsf{y}_*^i, \mathsf{u}_*^i, \tau] > 0$ (< 0) for all $\tau \in [0,t]$, then y_*^i is a strong locally or globally minimizing (maximizing) curve. Furthermore, if $\mathsf{I}[\mathsf{y}_*^i, \mathsf{u}_*^i, \tau] = 0$ for all $\tau \in [0,t]$, then y_*^i is an abnormal extremal curve of the i^{th} MTP.

Using Proposition 5, one can readily determine all $\bar{\theta}^i \in (\Theta_*^i(t))^c$ that correspond to either locally or globally maximizing curves of the ZNP. Note that from the principle of optimality, if $\bar{\theta}_i \in (\Theta_*^i(t))^c$ for some $t \geq 0$, then $\bar{\theta}^i \in (\Theta_*^i(\tau))^c$ for all $\tau \geq t$. In order to further refine our knowledge of $\Theta_*^i(t)$, we need to explore the connection between the sets $\ell_t(\bar{\mathsf{y}}^i)$ and $\partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$. The following proposition, which is a direct consequence of Pontryagin's Maximum Principle, will prove useful for the subsequent discussion [12, pp. 336-337].

Proposition 6: $\ell_t(\bar{\mathsf{y}}^i) \subseteq \partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$

Proposition 6 implies that all points in the interior of $\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ correspond necessarily to $\bar{\theta}^i \in (\Theta_*^i(t))^c$. The following theorem provides us with a necessary and sufficient condition to determine whether a control law u_*^i will drive the system (4) to $\partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ after t units of time.

Theorem 1: A control law u_*^i, will steer system (4) from $\bar{\mathsf{y}}^i$ at $\tau = 0$ to $\partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ at $\tau = t$ if and only if the following equation

$$\frac{\mathrm{d}\mathsf{p}^i}{\mathrm{d}\tau} = -(A_{\mathcal{T}}^i)^{\mathsf{T}}(\tau)\mathsf{p}^i, \tag{21}$$

admits a non-trivial solution p_*^i, such that

$$\langle \mathsf{p}_*^i, \mathsf{u}_*^i \rangle \geq \langle \mathsf{p}_*^i, v \rangle, \ \text{for all } |v| \leq 1, \tag{22}$$

for almost all $\tau \in [0,t]$.

Proof: The proof follows readily from Theorem 2 in [17, p. 73-75]. ∎

Remark 2: Note that the proof of Theorem 1 is contingent upon the assumption that w^i admits the structure given in (5). In particular, as it was highlighted in [17], Theorem 1 can be interpreted as follows: Let w^i be given by (5); then Pontryagin's Maximum Principle is both a necessary and sufficient condition for the control u_*^i to drive the system (4) from $\bar{\mathsf{y}}^i$ at $\tau = 0$ to $\partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ at $\tau = t$.

Corollary 1: Let y_*^i be a strong minimizing extremal of the i^{th} MTP generated by the control law u_*^i for $\tau \in [0,t]$. Then

$$\mathsf{y}_*^i(t) \in \partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i). \tag{23}$$

Proof: It can be shown [16, p. 370-373] that a solution p_*^i of equation (21), satisfies, for almost all $\tau \in [0,t]$,

$$\mathsf{p}_*^i(\tau) = \omega(\tau)[\cos\theta_*^i(\tau), \sin\theta_*^i(\tau)]^{\mathsf{T}}, \tag{24}$$

where $\omega(\tau)$ has the same sign as $\mathsf{I}[\mathsf{y}_*^i(\tau), \mathsf{u}_*^i(\tau), \tau]$, provided that $\mathsf{I}[\mathsf{y}_*^i(\tau), \mathsf{u}_*^i(\tau), \tau] \neq 0$. Because, by hypothesis, y_*^i is a strong minimizing curve, it follows that $\mathsf{I}[\mathsf{y}_*^i(\tau), \mathsf{u}_*^i(\tau), \tau] > 0$ for all $\tau \in [0,t]$, which implies that $\omega(\tau) > 0$ for all $\tau \in [0,t]$. Furthermore, for $\mathsf{u}^i = [\cos\theta_*^i, \sin\theta_*^i]^{\mathsf{T}}$ and in light of (24), condition (22) gives

$$\omega \geq \omega\langle [\cos\theta_*^i, \sin\theta_*^i]^{\mathsf{T}}, v \rangle, \ \text{for all } |v| \leq 1, \tag{25}$$

for almost all $\tau \in [0,t]$. Since $\omega(\tau) > 0$ for all $\tau \in [0,t]$, condition (25) always holds true by virtue of the Cauchy-Schwartz inequality. ■

Corollary 2: If u_*^i corresponds to an abnormal extremal of the i^{th} MTP, then the trajectory y_*^i generated by u_*^i for $\tau \in [0,t]$, satisfies $\mathsf{y}_*^i(t) \in \partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$, if and only if there exists a non-trivial solution p_*^i of (21), such that

$$-\frac{\langle \mathsf{p}_*^i, w^i(\mathsf{y}_*^i,\tau)\rangle}{|w^i(\mathsf{y}_*^i,\tau)|^2} \geq \langle \mathsf{p}_*^i, v\rangle, \text{ for all } |v| \leq 1, \tag{26}$$

for almost all $\tau \in [0,t]$.

Proof: It suffices to note that if u_*^i corresponds to an abnormal extremal for $\tau \in [0,t]$, then $\mathsf{I}[\mathsf{y}_*^i, \mathsf{u}_*^i, \tau] = 0$, which furthermore implies that

$$\mathsf{u}_*^i(t) = -\frac{w^i(\mathsf{y}_*^i,t)}{|w^i(\mathsf{y}_*^i,t)|^2}. \tag{27}$$

By plugging equation (27) in (22) then condition (26) follows readily. ■

The only case that needs to be further investigated is when there exists a point $\mathsf{y} = \varphi(t;\bar{\mathsf{y}}^i,\bar{\theta}^i) \in \partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ but $\mathsf{y} \notin \ell_t(\bar{\mathsf{y}}^i)$. This situation appears when the boundary of $\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ consists of points that can be reached in time t as well as in time shorter than t. The following proposition provides a necessary condition that will allow us to complete the characterization of $\ell_t(\bar{\mathsf{y}}^i)$.

Proposition 7: $\mathsf{y} \in \ell_t(\bar{\mathsf{y}}^i)$ only if $\mathsf{y} \notin \ell_\tau(\bar{\mathsf{y}}^i)$ for all $\tau \in [0,t)$.

The complete characterization of $\ell_t(\bar{\mathsf{y}}^i)$ can therefore be achieved by characterizing only the level sets $\ell_\tau(\bar{\mathsf{y}}^i)$ for $\tau \in [0,t]$, which requires, in turn, the computation of $\partial\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ for $\tau \in [0,t]$. This can be carried out by making use of Corollaries 1-2.

B. Topological Properties of the OP-DVD

In this section we examine a fundamental topological property of the OP-DVD, namely the relative position of $\bar{\mathsf{x}}_{\mathcal{T}}^i$ with respect to its associated Voronoi cell $\mathfrak{V}_i$. The following two propositions present sufficient conditions for $\bar{\mathsf{x}}_{\mathcal{T}}^i$ to be, respectively, an interior and a boundary point of its associated Voronoi cell $\mathfrak{V}_i$.

Proposition 8: Let the set $\mathcal{T}(0)$ consisting of distinct points in $\mathbb{R}^2$, and let the system (4) be small-time locally controllable at the origin. Then $\bar{\mathsf{x}}_{\mathcal{T}}^i \in \mathrm{int}(\mathfrak{V}_i)$.

Proof: Let $\mathfrak{A}_{\leq t}(\bar{\mathsf{x}}_{\mathcal{T}}^i) \triangleq \bigcup_{\tau\in[0,t],u_{\mathcal{P}}\in\mathcal{U}_{\mathcal{P}}}\{\bar{\mathsf{x}}_{\mathcal{P}} \in \mathbb{R}^2 : |\mathsf{x}_{\mathcal{P}}(\tau) - \mathsf{x}_{\mathcal{T}}^i(\tau)| \leq \rho\}$. From small-time local controllability, $\bar{\mathsf{y}}^i \in \mathrm{int}\,\mathfrak{R}_{\leq t}(\bar{\mathsf{y}}^i)$ for all $t \geq 0$ [18, p. 34], which implies that $\bar{\mathsf{x}}_{\mathcal{T}}^i \in \mathrm{int}\,\mathfrak{A}_{\leq t}(\bar{\mathsf{x}}_{\mathcal{T}}^i)$ for all $t \geq 0$. Therefore, there exists $\varepsilon > 0$, such that $B_\varepsilon(\mathsf{x}_{\mathcal{T}}^i) \triangleq \{\mathsf{x} : |\mathsf{x} - \mathsf{x}_{\mathcal{T}}^i| < \varepsilon\} \subset \mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^i)$. Assume that $\bar{\mathsf{x}}_{\mathcal{T}}^i \in \mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^j)$, for $j \in \mathcal{J} \subset \mathcal{I}$ and $i \neq j$. Then $B_\varepsilon(\bar{\mathsf{x}}_{\mathcal{T}}^i) \cap \mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^j) \neq \varnothing$, for $j \in \mathcal{J}$. By hypothesis, $\mathcal{T}(0)$ consists of distinct points, which implies that for sufficient small ε $\mathsf{x}_{\mathcal{T}}^j \notin B_\varepsilon(\mathsf{x}_{\mathcal{T}}^i)$, for all $j \in \mathcal{J}$. We wish to show that there exists $\varepsilon > 0$, such that for all $\mathsf{x} \in B_\varepsilon(\mathsf{x}_{\mathcal{T}}^i) \cap \mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^j)$, the minimum time-to-go from x to any point $\mathsf{x}_{\mathcal{T}}^j$, where $j \in \mathcal{J}$, is strictly greater than the minimum time-to-go to $\mathsf{x}_{\mathcal{T}}^i$. Let $\bar{\tau} \triangleq \min_{j\in\mathcal{J}} \inf\{T_{\mathsf{f}}^j(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^i) : \mathsf{x} \in B_\varepsilon(\bar{\mathsf{x}}_{\mathcal{T}}^i) \cap \mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^j)\}$. From small-time local controllability it follows that for every $\tau' < \bar{\tau}$, there exists sufficient small $\varepsilon = \varepsilon(\tau') > 0$ such that $B_\varepsilon(\bar{\mathsf{x}}_{\mathcal{T}}^i) \subset \mathfrak{A}_{\leq\tau'}(\bar{\mathsf{x}}_{\mathcal{T}}^i)$. Therefore, by construction, $T_{\mathsf{f}}^i(\mathsf{x}, \bar{\mathsf{x}}_{\mathcal{T}}^i) < \bar{\tau}$ for all $\mathsf{x} \in B_\varepsilon(\bar{\mathsf{x}}_{\mathcal{T}}^i)$, thus completing the proof. ■

Proposition 9: Let $\mathsf{x}_{\mathcal{P}}(T_{\mathsf{f}}^i) = \mathsf{x}_{\mathcal{T}}^i(T_{\mathsf{f}}^i)$, for all $i \in \mathcal{I}$, in (6) (equivalently, let $\rho = 0$ in (6)), and let $\bar{\mathsf{x}}_{\mathcal{T}}^i \in \partial\mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^i)$, then $\bar{\mathsf{x}}_{\mathcal{T}}^i \in \partial\mathfrak{V}_i$.

Proof: The proof follows readily from the fact that $\partial\mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^i) \cap \mathfrak{V}_i \ni \bar{\mathsf{x}}_{\mathcal{T}}^i$ is non-empty and $\mathfrak{V}_i \subseteq \mathfrak{A}(\bar{\mathsf{x}}_{\mathcal{T}}^i)$. ■

Another fundamental topological property of a standard Voronoi diagram is the connectedness of its Voronoi cells; something which does not hold true, in general, for the Voronoi cells of the OP-DVD as it will be demonstrated in Section IV. The connectedness of the ZVD in the special case when the winds/currents do not depend explicitly on time is conjectured in [11].

IV. Simulation Results

In this section we present simulation results to illustrate the previous developments. First, we consider the OP-DVD problem for a team of n moving targets where each target employs a feedback "evading" strategy of the form $u_{\mathcal{T}}^i = -A_{\mathcal{T}}^i(\mathsf{x}_{\mathcal{P}} - \mathsf{x}_{\mathcal{T}}^i)$, where

$$A_{\mathcal{T}}^i = \begin{pmatrix} \kappa^i & \lambda^i \\ -\lambda^i & \kappa^i \end{pmatrix}, \quad \kappa^i \neq 0, \lambda^i \in \mathbb{R},$$

and furthermore let $v_{\mathrm{ref}}^i(t) \equiv 0$. In this case equation (4) admits a closed form solution. In particular, $\theta_*^i(t) = \bar{\theta}^i + \lambda^i t$, which implies [19]

$$z^i(T_{\mathsf{f}}^i) = \mathrm{e}^{(\kappa^i - j\lambda^i T_{\mathsf{f}}^i)}\left(\bar{z}^i + \mathrm{e}^{(j\bar{\theta}^i)}(1 - \mathrm{e}^{(-\kappa^i t)})/\kappa^i\right), \tag{28}$$

where $z(t) \triangleq \mathsf{y}_1^i(t) + j\mathsf{y}_2^i(t)$, $\bar{z} \triangleq \bar{\mathsf{y}}_1^i + j\bar{\mathsf{y}}_2^i$, and where $j \triangleq \sqrt{-1}$. For $\mathsf{y}^i(T_{\mathsf{f}}^i) = 0$, equation (28) implies that

$$\bar{z}^i + \mathrm{e}^{(j\bar{\theta}^i)}(1 - \mathrm{e}^{(-\kappa^i t)}/\kappa^i) = 0. \tag{29}$$

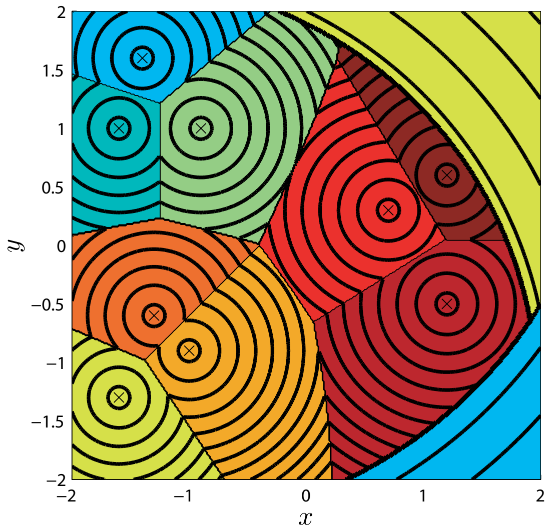
Equation (29) defines a system of two equations for $\bar{\theta}^i$ and T_{f}^i. Note that neither $\bar{\theta}^i$ nor T_{f}^i depend on λ^i (this would not be the case if $\mathsf{y}^i(T_{\mathsf{f}}^i) \neq 0$). After some algebraic manipulations, it follows from (29) that

$$\bar{\theta}^i = \arctan(\bar{\mathsf{y}}_2^i/\bar{\mathsf{y}}_1^i), \tag{30}$$

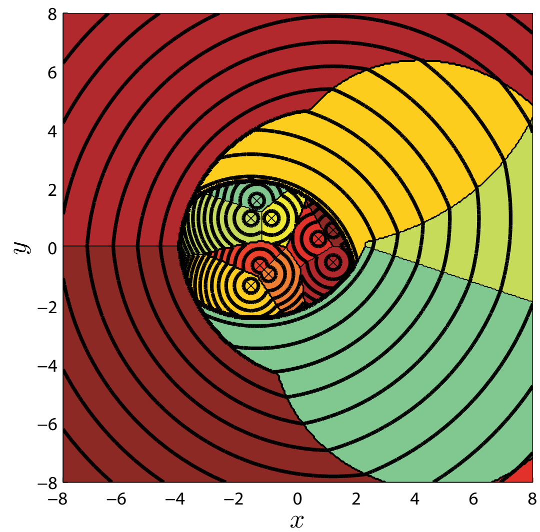
$$T_{\mathsf{f}}^i = -1/\kappa^i \ln(1 \pm |\kappa^i||\bar{\mathsf{y}}^i|). \tag{31}$$

Note that both equations (30)-(31) admit two solutions at most. Therefore, T_{f}^i is the least non-negative solution of (31) and $\bar{\theta}^i$ is the corresponding solution of (30). It follows from (31) that the i^{th} MTP is always feasible when $\kappa^i < 0$, and it is feasible if and only if $|\bar{\mathsf{y}}^i| < 1/|\kappa^i|$ when $\kappa^i > 0$.

The OP-DVD diagram for a particular scenario is illustrated in Fig. 2. Here, we consider a team of ten moving



(a) The OP-DVD close to $\mathcal{T}(0)$.



(b) The OP-DVD away from $\mathcal{T}(0)$.

Fig. 2. OP-DVD diagram of $\mathcal{T}(t)$, for a set of ten moving targets with different feedback "evading" strategies.

targets that are divided into three subgroups such that all the targets that belong to the same subgroup employ the same evading strategy. Figure 2(a) illustrates the OP-DVD in the vicinity of the initial positions of the targets, whereas Fig. 2(b) illustrates OP-DVD over a larger area. It is interesting to note that the Voronoi cells of this OP-DVD are disconnected sets.

V. Conclusion

In this article we have formulated a new dynamic partitioning problem for a finite set of moving targets, with respect to the minimum time required for a pursuer to intercept each of the moving targets. It is assumed that each moving target employs a time-varying feedback "evading" control strategy in response to its pursuer's actions. In the special case when all moving targets adopt the same "evading" strategy, the problem reduces to the Zermelo-Voronoi diagram problem. We have presented an efficient scheme for the construction of the solution of this partition problem, by exploiting the structure of the solution of a special case of Zermelo's navigation problem.

Acknowledgement: This work has been supported in part by NASA (award no. NNX08AB94A). The first author also acknowledges support from the A. Onassis Public Benefit Foundation.

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