Fuzzy Control Systems

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Contents

Summary:
This chapter presents a perspective of fuzzy control systems. Fuzzy control is a form of intelligent control characterized by the use of expert knowledge on the control strategy and/or the behavior of the controlled plant. This expert knowledge is represented by means of IF-THEN rules and linguistic variables. Attributes or values of these linguistic variables are linguistic terms associated with fuzzy sets, a generalization of ordinary ("crisp") sets. Fuzzy set theory is the theoretical basis underlying information processing in fuzzy control systems. From the systems theory's view, a fuzzy controller is a static nonlinear transfer element incorporated into a control loop. This gives rise to methods for analysis and systematic design. Fuzzy systems may perform different tasks within an automatic control system leading to different structural schemes. Nowadays, fuzzy control systems are successfully applied in many technical and non-technical fields. The application of fuzzy control systems is supported by numerous hardware and software solutions.

1 Introduction

Fuzzy control has been a new paradigm of automatic control since the introduction of fuzzy sets by L. A. Zadeh in 1965. Its rationale can be summarized by the statement of Zadeh "As complexity rises, precise statements lose meaning and meaningful statements lose precision." Thus, fuzzy control is an attempt to meet the challenges of increasing complexity of the processes to be controlled and of the tasks to be solved by automatic control systems.

To be more concrete, fuzzy control may be an advantageous alternative to conventional control techniques if

- the process to be controlled exhibits a pronounced nonlinear behavior,
- no mathematical model of the process is available because the modeling effort is unacceptably high or the process is not well understood,
- expert knowledge plays a key role in controlling the process and should be acquired and used for automatic control, or
- a multidimensional nonlinear relationship (e. g. a control law) should be represented such that it can be understood and modified easily.

Fuzzy control systems may be considered under various aspects: A fuzzy controller may be seen as a nonlinear controller described by linguistic rules rather than differential equations. Or a fuzzy control system may be seen as the implementation of the control strategy of a human expert. Understanding the functioning of fuzzy control systems, i. e. the information processing taking place within the fuzzy control system and its interaction with the plant and other components of the automatic control system requires knowledge of fuzzy logic and control theory.

The aim of this chapter, therefore, is

- to introduce the basic ideas of fuzzy control by means of a simple example (Section 2),
- to provide the essential theoretical bases of fuzzy systems (Section 3), and
- to discuss the control issues of fuzzy control (Section 4).

2 Fuzzy Control - A Simple Example

2.1 Example

In the following section, a simple and illustrative example will be used to explain information processing in fuzzy systems:

Example 1 (Control of room temperature) The temperature of a room equipped with a hot water heating should be controlled by adjusting the position of the valve at the radiator (see Fig. 1). A human being would use meta-rules, such as

If things are not OK but change in the right direction then maintain present settings

or more specifically

If the temperature is too warm but decreases, then leave valve position unchanged or

If the temperature is too cold and decreases, then increase the valve opening significantly.

Starting from these meta-rules, an experienced user would develop a set of control rules which are more specific regarding the linguistic description of the values of temperature, temperature change, and change of valve position.

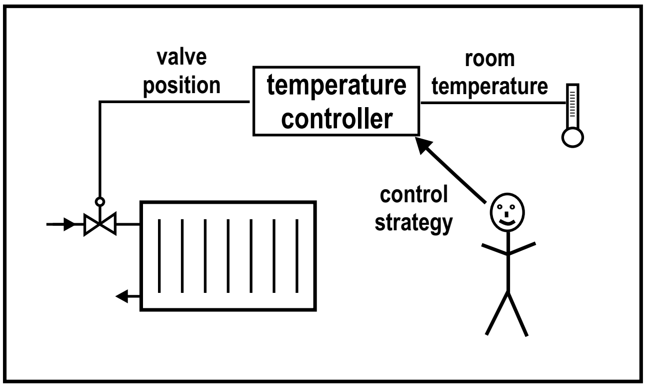


Figure 1: Schematic representation of the control for Example 1

Fuzzy systems provide a means to represent and process expert knowledge as stated in the example above. By treating them as knowledge-based systems, the separation of knowledge representation and information processing is realized. Knowledge is represented in form of rules and the meaning of the expressions or symbols appearing in them. In Example 1, the knowledge consists of the control rules and the meaning of the linguistic labels describing the values of temperature, temperature change, and change of valve position (valve change for short). The formal concepts for knowledge representation in fuzzy systems are *linguistic variables* and *fuzzy IF-THEN rules* presented in the next subsection.

The *inference engine* forms the core of the information processing components. As part of a control system, a fuzzy system/controller usually processes numerical inputs to numerical outputs. Therefore, *fuzzification* and *defuzzification* supplement inference:

- Fuzzification: Transformation of numerical values, e. g. measurements, into a fuzzy representation of the input situation,
- Inference: Transformation of the fuzzy input representation into a fuzzy decision, and
- Defuzzification: Transformation of the fuzzy decision into a real decision, e. g. a real value of a manipulated variable.

Fig. 2 shows a schematic representation of a fuzzy system whose components will be described in the following using Example 1 as a basis.

2.2 Fuzzy Sets, Linguistic Variables and Fuzzy IF-THEN Rules

By means of Example 1, it will be shown first how the formal concepts of a linguistic variable with their linguistic terms and membership functions and of a fuzzy rule are used to represent the available knowledge. The notion of a *linguistic variable* formalizes the practices of many domains to describe the values of certain variables in terms of natural language. For Example 1, a linguistic variable is the *temperature deviation* expressing the discrepancy between the desired and the actual room temperature, denoted as $T_{Dev} = T_{desired} - T_{room}$, with the linguistic terms *TOO WARM, OK* and *TOO COLD*. A second one is the *temperature change* ΔT with the terms $\{INCR, EQUAL, DECR\}$ where *INCR* stands for INCREASING and *DECR* for DECREASING. *Temperature change* is the difference

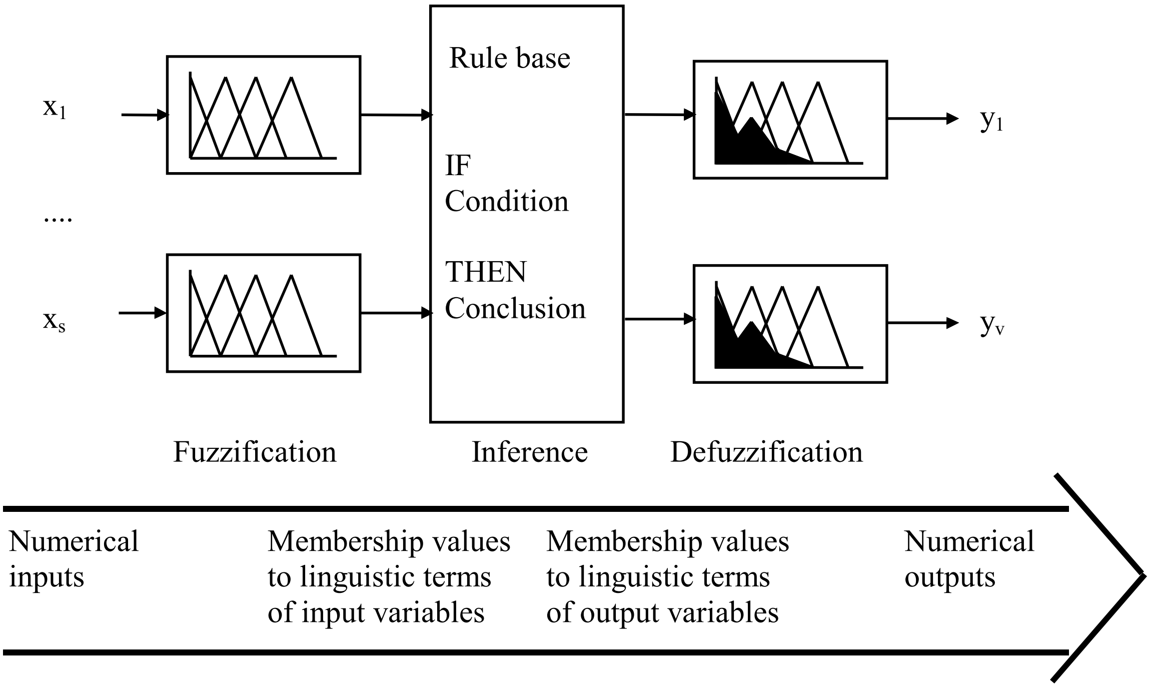


Figure 2: Structure of a fuzzy system with numerical inputs $x_1, \dots, x_s$ and numerical outputs $y_1, \dots, y_v$

between the current room temperature and the room temperature at the last time instant. Assuming a constant sampling time the temperature difference is proportional to the temperature trend. It should be noted that the expression of temperature in the example has two different 'meanings'. Specification is done by using different linguistic terms: *TOO WARM* specifies a difference between the desired and the room temperature and *INCR* a temperature change. In contrast to this, the term *WARM* could characterize the room temperature itself.

A third linguistic variable is the *valve change y* of the radiator, the output of the fuzzy controller. Valve change is an incremental variable, such that the actual valve opening results from the previous opening plus valve change both expressed in percent. The term set {*NB*, *NS*, *ZE*, *PS*, *PB*} uses standardized names with the typical abbreviations *B* (big), *M* (medium), *N* (negative), *P* (positive), *S* (small), *ZE* (zero). These abbreviations will be combined in names as *NB* (negative big) and so on. In practical applications, a linguistic variable usually has between two and seven linguistic terms. This corresponds to the result of psychological investigations stating that human beings differentiate a maximum of five to seven objects at the same time.

The next problem is to define the 'meaning' of each linguistic term. In many real-world problems, the decision, whether a given x (e. g. a temperature deviation) satisfies a certain property A (e. g. *TOO COLD*) or not is impossible or not reasonable. In the example above, a human being would not consider a small temperature deviation (e. g. $T_{Dev} = 0.01$ K) as *TOO COLD*, whereas deviations of 2 K or 5 K probably would be felt as *TOO COLD*, to a certain extent at least. In other words, the membership of x in the subset A should be a matter of degree as x satisfies the property up to a certain degree.

According to set theory, each ordinary subset A of the universe of discourse X is determined by its characteristic function $\mu_A : X \to \{0, 1\}$. This means that $\mu_A(x) = 1$ when x is an element of A and zero when it is not. The value of $\mu_A(x)$ can be interpreted as the truth value of a proposition 'x is an element of A' relating set theory with logic.

Zadeh proposed to introduce a *fuzzy set* as a generalization of ordinary ("crisp") sets. This means that the proposition of 'x is an element of A' is no longer true or false, but may be true with a certain degree (*fuzzy truth value*). The characteristic function $\mu_A(x)$ or *membership function* of the fuzzy (sub-) set A is allowed to assume real values between 0 and 1:

$$\mu_A : X \rightarrow [0,1]. \tag{1}$$

Such fuzzy sets associated with the linguistic terms in Example 1 are depicted in Fig. 3 and Fig. 4, respectively.

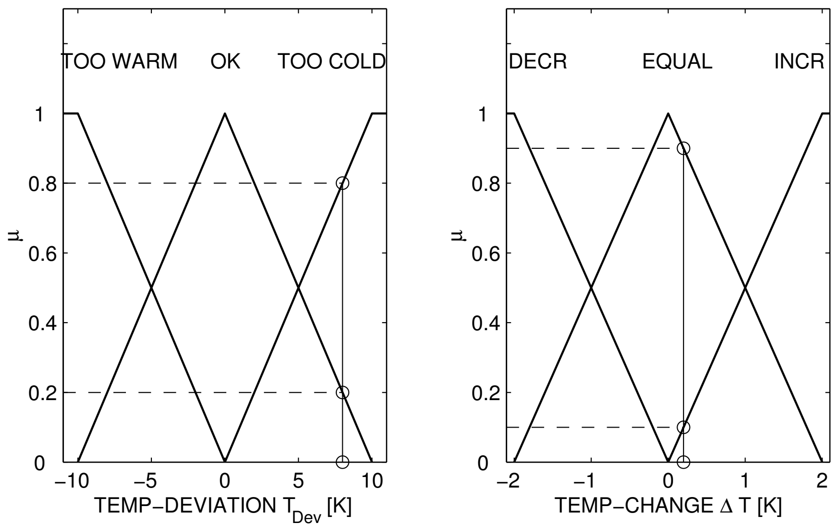


Figure 3: Membership functions and fuzzification for *temperature deviation* (left) and *temperature change* (right)

Statements as, for example, 'T_{Dev} is $TOO\ COLD$' are called *fuzzy propositions* as the truth value of such a statement is a matter of degree. It is determined by the membership degree of T_{Dev} in the fuzzy set labeled *TOO COLD*. Using connectives such as AND and OR, compound fuzzy propositions can be formed.

A *fuzzy IF-THEN rule* or fuzzy conditional statement is expressed as

IF <fuzzy proposition> THEN <fuzzy proposition>

where <fuzzy proposition> is a simple or compound fuzzy proposition. For Example 1 a fuzzy rule of a controller might be 'IF *temperature deviation* is *TOO COLD* AND *temperature change* is *INCR* THEN *valve change* is *ZE*'. The IF part is called premise, condition or antecedent, the THEN part conclusion or consequence.

In Example 1, the rule premises contain the two linguistic input variables *temperature deviation* T_{Dev} and *temperature change* ΔT, while the conclusions contain the linguistic output variable *valve change* y. The complete rule base is shown in Table 1.

2.3 Fuzzification - From Measurements to a Fuzzy Representation of the Input Situation

The inputs of a fuzzy system, especially a fuzzy controller, are (crisp) values of some variables, e. g. measurement signals. The vector of these values characterizes an input situation which may be the system state, for example. Likewise, rule premises specify such input situations, but this specification uses linguistic terms for the values of the input variables. In order to determine the

Table 1: Rule base of Example 1

	y	T_{Dev} *TOO WARM*	*OK*	*TOO COLD*
ΔT	*INCR*	R_1: *NB*	R_2: *NS*	R_3: *ZE*
	EQUAL	R_4: *NS*	R_5: *ZE*	R_6: *PS*
	DECR	R_7: *ZE*	R_8: *PS*	R_9: *PB*

degree of fulfillment of the (compound) fuzzy proposition in the premise the truth values of the simple propositions have to be known. It is the task of fuzzification to provide these values.

If the input variables assume the values of $T_{Dev} = 8$ K and $\Delta T = 0.2$ K, fuzzification in Example 1results in

$$\begin{array}{lll} \textit{temperature deviation } T_{Dev}: & \mu_{TOO\ WARM}(8) = 0.0, \quad \mu_{OK}(8) = 0.2, \quad \mu_{TOO\ COLD}(8) = 0.8, \\ \textit{temperature change } \Delta T: & \mu_{INCR}(0.2) = 0.1, \quad \mu_{EQUAL}(0.2) = 0.9, \quad \mu_{DECR}(0.2) = 0.0. \end{array}$$

A graphic explanation is depicted in Fig. 3.

This transformation is unique, but not one to one in general. In the example, only $\Delta T \geq 2$ can be deduced for a given $\mu_{INCR}(\Delta T) = 1$, $\mu_{EQUAL}(\Delta T) = 0$.

2.4 Inference - From a Fuzzy Input Representation to a Fuzzy Decision

Inference essentially consists of three steps, namely,

- aggregation,
- activation, and
- accumulation.

Aggregation A rule premise in general is a compound fuzzy proposition (e. g. an AND connection of two propositions with T_{Dev} and ΔT). Its degree of fulfillment μ_{P_k} results from the aggregation of the truth values of the simple propositions given by fuzzification. The operations have to be chosen in accordance with the connectives (AND, OR) between simple propositions. The connective AND is related to the *intersection*, OR to the *union* of two (fuzzy) sets.

Table 2: Intersection (left) and union (right) of crisp sets defined by their characteristic functions

	$\mu_B(x) = 0$	$\mu_B(x) = 1$
$\mu_A(x) = 0$	$\mu_{A\cap B}(x) = 0$	$\mu_{A\cap B}(x) = 0$
$\mu_A(x) = 1$	$\mu_{A\cap B}(x) = 0$	$\mu_{A\cap B}(x) = 1$

	$\mu_B(x) = 0$	$\mu_B(x) = 1$
$\mu_A(x) = 0$	$\mu_{A\cup B}(x) = 0$	$\mu_{A\cup B}(x) = 1$
$\mu_A(x) = 1$	$\mu_{A\cup B}(x) = 1$	$\mu_{A\cup B}(x) = 1$

The definitions in Table 2, valid for crisp sets, have to be generalized for fuzzy sets. According to the original proposal of Zadeh, the *intersection* $A \cap B$ and the *union* $A \cup B$ can be defined *point-wise* using the respective membership degrees

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \tag{2}$$

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}. \tag{3}$$

For Example 1, the results of aggregation using a minimum for the AND connective are given in Table 3.

Table 3: Aggregation for Example 1 with the minimum operation ($T_{Dev} = 8$ K, $\Delta T = 0.2$ K)

		T_{Dev}		
		$\mu_{TOO\ WARM}(T_{Dev}) = 0.0$	$\mu_{OK}(T_{Dev}) = 0.2$	$\mu_{TOO\ COLD}(T_{Dev}) = 0.8$
ΔT	$\mu_{INCR}(\Delta T) = 0.1$	$\mu_{P_1} = 0.0$	$\mu_{P_2} = 0.1$	$\mu_{P_3} = 0.1$
	$\mu_{EQUAL}(\Delta T) = 0.9$	$\mu_{P_4} = 0.0$	$\mu_{P_5} = 0.2$	$\mu_{P_6} = 0.8$
	$\mu_{DECR}(\Delta T) = 0.0$	$\mu_{P_7} = 0.0$	$\mu_{P_8} = 0.0$	$\mu_{P_9} = 0.0$

Activation A fuzzy IF-THEN rule is a connection of two (compound) fuzzy propositions. Hence, this connective has to be interpreted within the framework of set theoretic or logical operators. The simplest interpretation is that of the conjunction of premise and conclusion, such that the appropriate operation is the minimum. Thus, the result of activation μ_{C_k} of a rule k is the minimum of the degree of fulfillment μ_{P_k} and the fuzzy set in the conclusion. In other words, the fuzzy set in the conclusion is clipped to μ_{P_k}.

In Example 1, activation leads to nonempty fuzzy sets as depicted in Fig. 4a for rules 2,3,5, and 6, only.

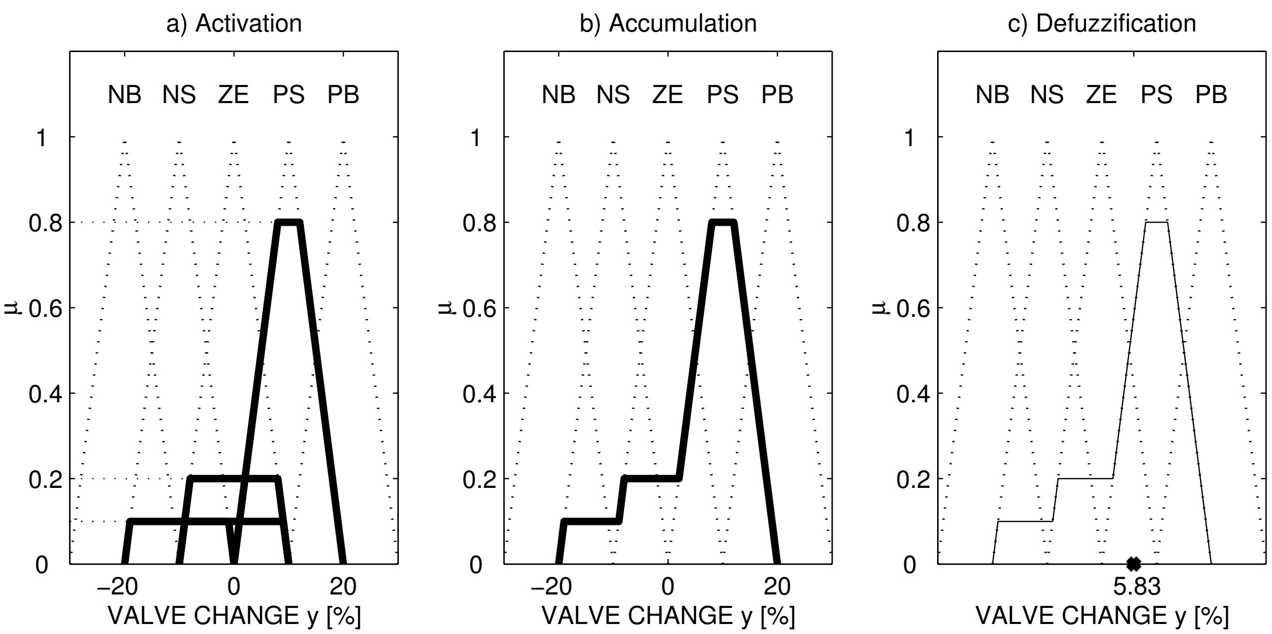


Figure 4: Results of a) activation (only positive for rules R_2, R_3, R_5, R_6), b) accumulation, and c) defuzzification for Example 1

Accumulation Usually, a rule base is interpreted as a disjunction of rules i. e. rules are seen as independent "experts". Accumulation has the task to combine the individual "expert statements", which actually are fuzzy sets of recommended output values. Consequently, an appropriate accumulation operation is the maximum. The maximum of the activated rule conclusions $\max_Y\{\mu_{C_k}(y)\}$ is a fuzzy set $\mu(y)$ over the domain Y of the output variable. The membership degree $\mu(y)$ can be interpreted as the degree to which the value y is suggested by the "expert committee" to be the real output value.

In Example 1, accumulation results in a fuzzy set $\mu(y)$ as shown in Fig. 4b.

To summarize, inference yields the output fuzzy set as the maximum of all clipped fuzzy sets of the linguistic terms of the output variable. The clipping results from applying the minimum operation between each output fuzzy set referred to in the rule conclusion and the degree of fulfillment of the rule premise. This degree of fulfillment is calculated from the results of fuzzification by applying the minimum for the AND and maximum for the OR connective. This inference scheme, using minimum and maximum operations is called max-min inference.

2.5 Defuzzification - From a Fuzzy Decision to a Real Decision

As inference results in a fuzzy set, the task of defuzzification is to find the numerical value which "best" comprehends the information contained in this fuzzy set. A frequently used method is the so-called Center-of-Gravity defuzzification (CoG; also called Center-of-Area defuzzification COA):

$$y = \frac{\int_Y \mu(y) y \, dy}{\int_Y \mu(y) dy}, \tag{4}$$

which chooses the y-coordinate of the center of gravity of the area below the graph $\mu(y)$. This defuzzification can be interpreted as a weighted mean, i. e. each value y is weighted with $\mu(y)$ and the integral in the denominator serves for normalization. In Example 1, defuzzification of $\mu(y)$ using Center-of-Gravity defuzzification yields a value of 5.83 %, as shown in Fig. 4c.

The characteristic surface of the fuzzy controller or control surface, that is the graphic representation of the function $y(T_{Dev}, \Delta T)$, is depicted in Fig. 5. Here, the tasks of fuzzification, inference, and defuzzification have been performed for all possible combinations of $T_{Dev}, \Delta T$ in the universe of discourse (with some reasonable discretization).

3 Fuzzy Logic-related Issues in Fuzzy Control

In the previous section, only the simplest possible type of a fuzzy system has been discussed. In this section the theoretical fundamentals of fuzzy systems are introduced. This introduction goes as far as necessary for the understanding of key concepts of knowledge representation and information processing in fuzzy systems. Furthermore, it is intended to reveal alternatives to choices made in the previous section and again uses Example 1 for illustration.

3.1 Fuzzy Sets and Operations

A fuzzy set A with membership function $\mu_A(x)$ is completely characterized by the set of pairs $A = \{(x, \mu_A(x)) \,|\, x \in X\}$. For finite universes of discourse, this notation of fuzzy sets may be appropriate. In fuzzy control, where the universes of discourse are continuous or contain a large number of elements

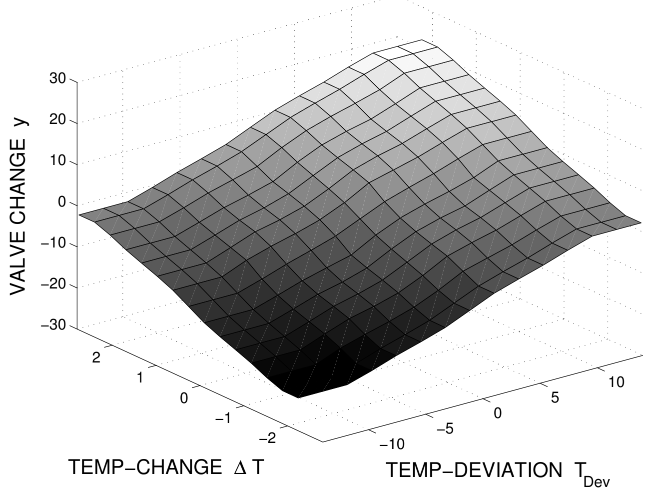


Figure 5: Control surface of the fuzzy controller in Example 1

(due to discretization), fuzzy sets are more reasonably expressed in the parametric form. Common types of membership functions include triangular, trapezoidal, or Gaussian functions. In addition, a parametric form is a prerequisite for parameter optimization in fuzzy systems (see *Optimization of Fuzzy Systems*).

Parametric membership functions Trapezoidal membership functions (see Fig. 6c) are given by

$$\mu(x) = \begin{cases} 0 & \text{if } x \leq m_1 - a \\ 1 + \dfrac{1}{a}(x - m_1) & \text{if } m_1 - a < x \leq m_1 \\ 1 & \text{if } m_1 < x \leq m_2 \\ 1 - \dfrac{1}{b}(x - m_2) & \text{if } m_2 < x \leq m_2 + b \\ 0 & \text{if } x > m_2 + b. \end{cases} \tag{5}$$

Special cases of trapezoidal membership functions are rectangular ($a = b = 0$, Fig. 6a), triangular ($m = m_1 = m_2$, Fig. 6b), and singleton ($m = m_1 = m_2, a = b = 0$, Fig. 6d) membership functions. Examples of smooth membership functions are Gaussian

$$\mu(x) = e^{-\dfrac{(x-m)^2}{2a^2}} \tag{6}$$

(Fig. 6e) and $\cos^2$ membership functions (Fig. 6f):

$$\mu(x) = \begin{cases} 0 & \text{if } x < m - a \\ \cos^2(\dfrac{\pi}{2a}(u - m)) & \text{if } m - a \leq x \leq m + a \\ 0 & \text{if } x > m + a. \end{cases} \tag{7}$$

Properties of fuzzy sets In analogy with ordinary set theory, fuzzy set theory defines certain properties of fuzzy sets. A fuzzy set is called *convex*, if

$$\forall x_1, x_2 \in X \, \forall \lambda \in [0,1] : \mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}. \tag{8}$$

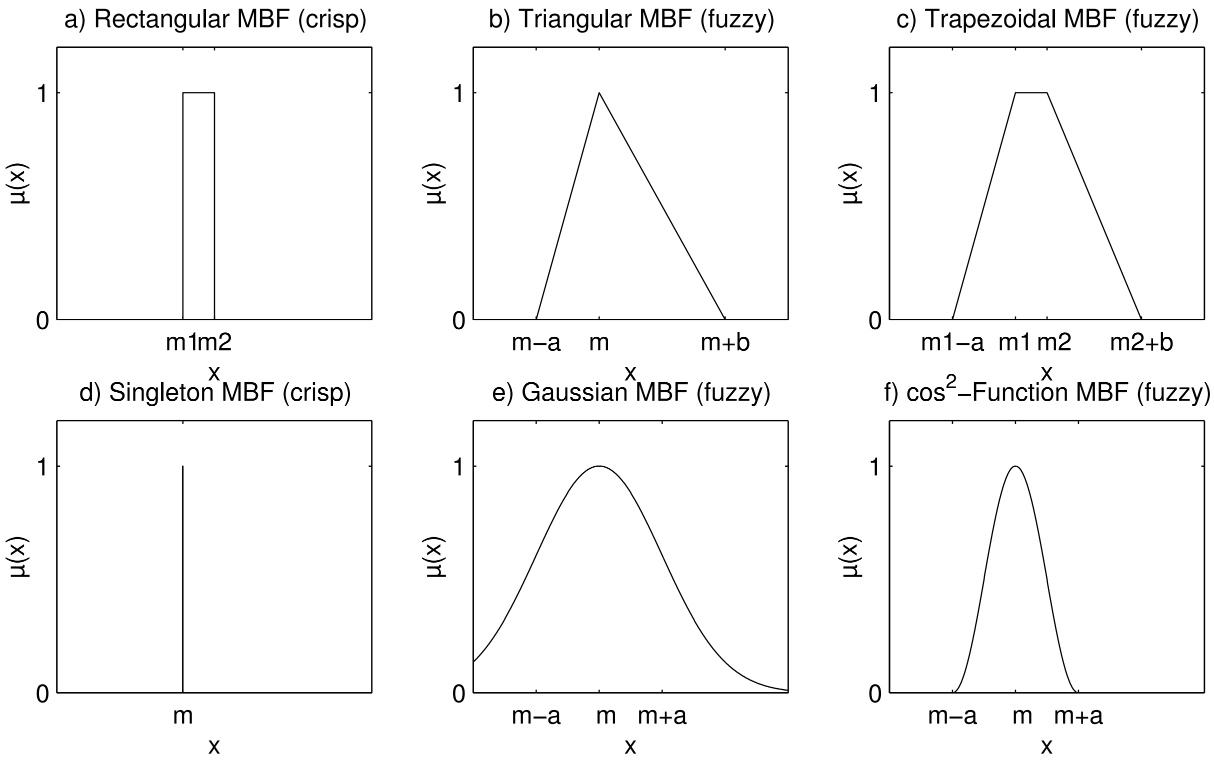


Figure 6: Examples of parametric membership functions

The *support* of a fuzzy set A is the ordinary set that contains all elements of the domain of A with a non-zero membership degree in A and will be denoted as $S(A)$, i. e. $S(A) = \{x \in X \,|\, \mu_A(x) > 0\}$. For convex fuzzy sets, the support is a convex set, e. g. an interval for scalar, real x. The ordinary set consisting of all elements of the domain of A with a membership degree in A of at least α is called α-*cut* and denoted as A^α, i. e. $A^\alpha = \{x \in X \,|\, \mu_A(x) \geq \alpha\}$. The least upper bound of the membership function is called the *height* of the fuzzy set, i. e. $\mathrm{hgt}(A) = \sup_{x \in X} \mu_A(x)$. A fuzzy set with $\mathrm{hgt(A)} = 1$ is said to be *normal*. In the case of a discrete universe, the *cardinality* of a fuzzy set A is $|A| = \sum_{i=1}^{n} \mu_A(x_i)$ (n number of elements of the universe). In the case of a continuous universe, it is $|A| = \int_X \mu_A(x)dx$.

Operations on fuzzy sets Notions like equality and inclusion of two fuzzy sets are directly derived from ordinary set theory. In contrast to this, different definitions are possible for connectives like intersection, union, or complement, which only coincide for the special case of ordinary sets. This may be an advantage, as it allows for a flexible and context-dependent choice of them, but it is also a challenge for the user to choose the appropriate ones for an application at hand.

A fuzzy set A is a *subset* of a fuzzy set B, i. e. $A \subseteq B$, if for all x of the universe $\mu_A(x) \leq \mu_B(x)$. Two fuzzy sets A and B are *equal*, if $A \subseteq B$ and $B \subseteq A$ hold for all x in X.

To define fuzzy set connectives, the definitions of connectives for ordinary set are restated first using the concept of a characteristic function. The intersection of two ordinary sets A, B is defined as the set of all elements contained in both sets

$$\begin{aligned} A \cap B &= \{x \,|\, x \in X \text{ and } \mu_{A \cap B}(x) = 1\} = \{x \,|\, x \in X \text{ and } x \in A \text{ and } x \in B\} \\ &= \{x \,|\, x \in X \wedge \mu_A(x) = 1 \wedge \mu_B(x) = 1\}. \end{aligned} \tag{9}$$

The union of A, B is defined as the set of those elements which are contained in one of both sets at least

$$
\begin{aligned}
A \cup B &= \{x \mid x \in X \text{ and } \mu_{A\cup B}(x) = 1\} = \{x \mid x \in X \text{ and } (x \in A \text{ or } x \in B)\} \\
&= \{x \mid x \in X \wedge (\mu_A(x) = 1 \vee \mu_B(x) = 1)\}.
\end{aligned} \tag{10}
$$

The complement of ordinary set A is defined as the set of all elements not contained in A

$$
A^c = \{x \mid x \in X \text{ and } \mu_{A^c}(x) = 1\} = \{x \mid x \in X \text{ and } x \notin A\} = \{x \mid x \in X \text{ and } \mu_A(x) = 0\}. \tag{11}
$$

The definitions of intersection and union of fuzzy sets (2), (3) as given in Section 2 obviously generalize the operations defined for ordinary sets as well as the definition of the complement of a fuzzy set A^c according to

$$
\mu_{A^c}(x) = 1 - \mu_A(x). \tag{12}
$$

As in ordinary set theory, where the pair of a set and its complement forms a partition of the universe, the pair of fuzzy sets (A, A^c) is called a fuzzy partition of X, since $\mu_A(x) + \mu_{A^c}(x) = 1$. More generally, a m-tuple $(A_1, \ldots, A_m)$ of non-zero fuzzy sets with

$$
\forall x \in X : \sum_{i=1}^{m} \mu_{A_i}(x) = 1 \tag{13}
$$

is called a *fuzzy partition* of X. In fuzzy control or modeling it is often advisable to define the fuzzy sets of the linguistic terms of a linguistic variable such that they form a fuzzy partition.

In ordinary set theory $(\mathcal{P}(X), \cap, \cup, ()^c)$, where $\mathcal{P}(X)$ denotes the power set of X (i. e. the set of all subsets of X), represents a Boolean algebra (i. e. a complemented distributive lattice). The set of all fuzzy sets on X, $\mathcal{F}(X)$, together with the operations (2), (3), and (12) shares most of the properties of a Boolean algebra, i. e.

Commutativity:	$A \cup B = B \cup A$, $A \cap B = B \cap A$
Associativity:	$A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
Idempotence:	$A \cup A = A$, $A \cap A = A$
Distributivity:	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
De-Morgans Laws:	$(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

The only Boolean law of ordinary set theory, which is no longer true is the excluded-middle law, is

$$
A \cap A^c \neq \emptyset, \; A \cup A^c \neq X.
$$

Hence, $(\mathcal{F}(X), \min, \max, 1 - ())$ has the properties of a distributive lattice. Therefore, the operations in (2), (3), and (12) play a special role in fuzzy set theory. From a practical point of view, they lead to very simple algorithms. However, they are often too simplistic models of fuzzy set operations (see Example 2) and are, as mentioned above, not the only possible extensions of ordinary set operations. Below, a second example will be considered to illustrate differences between definitions of the above operations on fuzzy sets.

Example 2 (Selection of an optimal route) To get from point A to B by car different routes can be chosen. A fuzzy system containing rules like

IF the *distance* between A and B is *SHORT* AND the *driving speed* is *FAST*
THEN the route is *GOOD*

should make a decision for finding the best of several routes. From an optimization point of view, the fuzzy system selects a route which is optimal for both sub-conditions *distance* and *driving speed* (optimal solution of a multi-criteria optimization problem) or an appropriate compromise route (optimal solution of a single-criterion optimization problem).

Assume, there are two candidate routes with their respective membership degrees in the fuzzy sets *SHORT* and *FAST*: $\mu_{SHORT}(\text{route } 1) = 0.7$, $\mu_{SHORT}(\text{route } 2) = 0.5$, $\mu_{FAST}(\text{route } 1) = 0.5$, and $\mu_{FAST}(\text{route } 2) = 0.5$. Which is the best route? Intuitively, route 1 would be preferred. But with minimum for the intersection there is no preference for a route because both routes belong to the fuzzy set of *SHORT* AND *FAST* routes with degree 0.5.

Possible definitions for intersection operations can be derived within the framework of triangular norms. A function $\top : [0,1] \times [0,1] \to [0,1]$ is called *t-norm*, if for all $u,v,w \in [0,1]$ the following conditions hold:

$$\begin{array}{rll} i. & \top(u,1) = u,\ \top(u,0) = 0 & \text{(boundary conditions)}, \\ ii. & u \le v \Rightarrow \top(u,w) \le \top(v,w) & \text{(monotonicity)}, \\ iii. & \top(u,v) = \top(v,u) & \text{(commutativity)}, \\ iv. & \top\big(u,\top(v,w)\big) = \top\big(\top(u,v),w\big) & \text{(associativity)}. \end{array}$$

Applying De Morgan's Law and the negation (12), a t-conorm $\perp$ with $\perp(u,v) = 1 - \top(u,v)$ can be defined as respective union operation for any t-norm. The neutral element of a t-conorm is zero. Also, the conditions *ii.–iv.* hold.

Generally, t-norms and t-conorms observe certain of the Boolean laws only, but never all. This may restrict the possibility of transforming compound expressions into simpler ones. In fuzzy control, however, this plays a minor role.

Examples of particular t-norm/t-conorm pairs are the minimum/maximum $\top_m/\perp_m$ as in (2), (3), the algebraic product/algebraic sum $\top_a/\perp_a$, the bounded difference/bounded sum $\top_b/\perp_b$ and the drastic product/drastic sum $\top_d/\perp_d$ given by:

$$\top_m(u,v) \stackrel{\text{def}}{=} \min\{u,v\} \qquad \perp_m \stackrel{\text{def}}{=} \max\{u,v\} \tag{14a}$$

$$\top_a(u,v) \stackrel{\text{def}}{=} uv \qquad \perp_a(u,v) \stackrel{\text{def}}{=} u+v-uv \tag{14b}$$

$$\top_b(u,v) \stackrel{\text{def}}{=} \max\{0,u+v-1\} \qquad \perp_b(u,v) \stackrel{\text{def}}{=} \min\{1,u+v\} \tag{14c}$$

$$\top_d(u,v) \stackrel{\text{def}}{=} \begin{cases} u, & v=1 \\ v, & u=1 \\ 0, & \text{otherwise} \end{cases} \qquad \perp_d(u,v) \stackrel{\text{def}}{=} \begin{cases} u, & v=0 \\ v, & u=0 \\ 1, & \text{otherwise} \end{cases} \tag{14d}$$

Considering two fuzzy sets A,B, a t-norm $\top$ induces a connective for the intersection $A \cap_\top B$ by the point-wise definition of

$$\mu_{A\cap_\top B}(x) = \top\left(\mu_A(x), \mu_B(x)\right). \tag{15}$$

Analogously, the connective for the union $A \cup_\perp B$ is induced by a t-conorm $\perp$:

$$\mu_{A\cup_\perp B}(x) = \perp\left(\mu_A(x), \mu_B(x)\right). \tag{16}$$

In Fig. 7 the intersection and union of crisp and fuzzy sets are compared with the operations for fuzzy sets exemplarily being the minimum and algebraic product for intersection and maximum and algebraic sum for union.

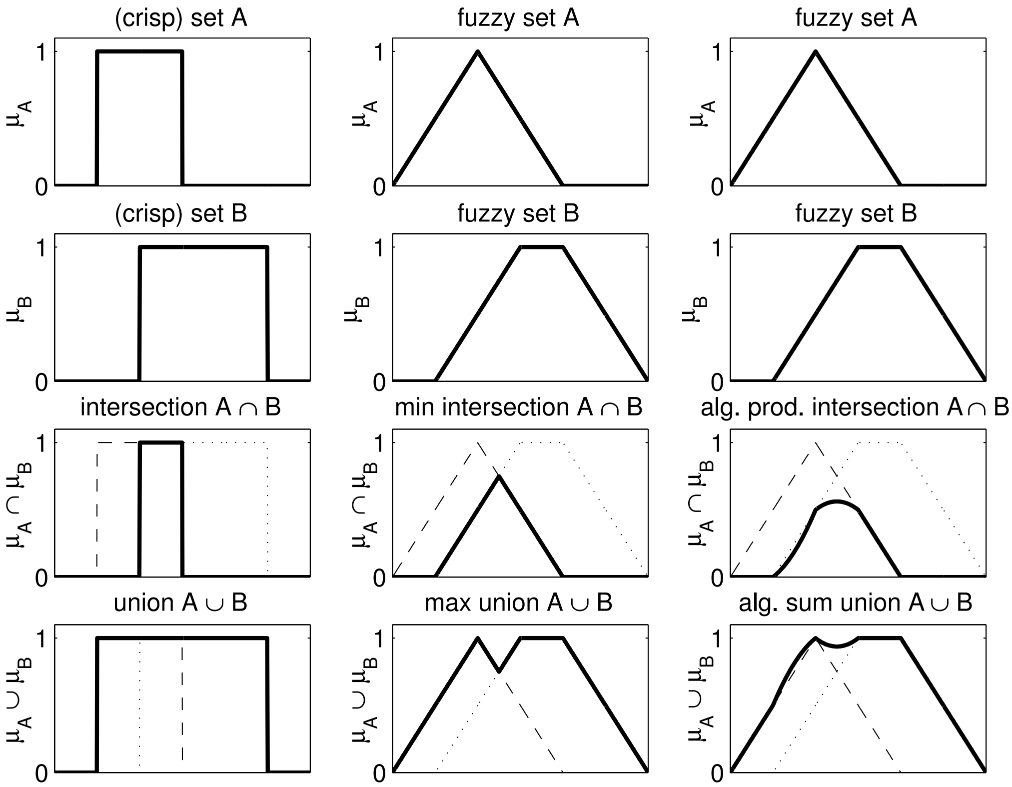


Figure 7: Intersection and union of ordinary and fuzzy sets

Generally, the following relations hold for every t-norm $\top$ and t-conorm $\bot$, respectively:

$$\top_d(u,v) \leq \top(u,v) \leq \top_m(u,v) \leq \bot_m(u,v) \leq \bot(u,v) \leq \bot_d(u,v). \tag{17}$$

Hence, the drastic product gives the lower bound and the minimum the upper bound for every t-norm, while the maximum represents the lower and the drastic sum the upper bound for every t-conorm.

The following example is a continuation of Example 2.

Example 3 Using the algebraic product for intersection, route 1 and route 2 belong to the fuzzy set of *SHORT* AND *FAST* routes with a degree of 0.35 and 0.25, respectively. Obviously, the algebraic product operator leads to the preference suggested by intuition. Assume, there is a third route with $\mu_{SHORT}(\text{route } 3) = 1$ and $\mu_{FAST}(\text{route } 3) = 0.35$, then this route also belongs to the fuzzy set of *SHORT* AND *FAST* routes with a degree of 0.35. Routes 1 and 3 are not distinguishable, which again may contradict intuition as route 3 is much shorter, but only a little slower.

In practical problems, it may be important to have other than the given t-norms and t-conorms, which are more appropriate to the situation under consideration. Here, parametric families of t-norms allow us to adapt the intersection and union operations. An example is the Yager family with a parameter p:

$$\top_p(u,v) \stackrel{\text{def}}{=} 1 - \min\left\{[(1-u)^p + (1-v)^p]^{1/p}, 1\right\}, \quad \bot_p(u,v) \stackrel{\text{def}}{=} \min\left\{[u^p + v^p]^{1/p}, 1\right\}. \tag{18}$$

$\top_p$ increases from $\top_0 = \top_d$ to $\top_\infty = \top_m$, including $\top_1 = \top_b$. Analogously, the dual $\bot_p$ decreases from $\bot_0 = \bot_d$ to $\bot_\infty = \bot_m$. Hence, the Yager family covers the whole range of t-norms and t-conorms, respectively, in the sense of the inequalities (17).

In ordinary set theory complementation together with intersection or union suffice to define all further connectives for two sets (as two operations suffice to define all operations of Boolean logic). In contrast to this, there are no unique definitions for these operations in fuzzy set theory, and, in addition, further possibilities of aggregating fuzzy sets exists. Of these general aggregation operations, the so-called averaging operators play an important role in decision making or fuzzy control.

An *averaging operator N* is a function $N:[0,1]\times[0,1]\to[0,1]$ satisfying for all $u,v,w\in[0,1]$

$$
\begin{array}{rll}
i. & N(0,0)=0,\ N(1,1)=1 & \text{(boundary conditions)},\\
ii. & u\le v\Rightarrow N(u,w)\le N(v,w) & \text{(monotonicity)},\\
iii. & N(u,v)=N(v,u) & \text{(commutativity)},\\
iv. & N(u,u)=u & \text{(idempotency)},\\
v. & N \text{ is continuous.} &
\end{array}
$$

Note, that averaging operators are not associative. For every averaging operator N the relation $\top_m(u,v)\le N(u,v)\le\bot_m(u,v)$ holds. Examples of averaging operators are the arithmetic mean, the harmonic mean, the generalized p-mean, or, more generally, the *ordered weighted averaging* (OWA) operators introduced by Yager. An OWA operator of dimension n is a mapping $f:R^n\to R$ that has an associated vector $\mathbf{w}=(w_1,\dots,w_n)^T$ such that $w_i\in[0,1]$ and $w_1+\cdots+w_n=1$. Furthermore, $f(u_1,\dots,u_n)=\sum_j w_j b_j$ where b_j is the j-th largest of the u_i. Special cases are the maximum, the arithmetic mean, and the minimum:

$$
\begin{aligned}
f^*:\mathbf{w}=\mathbf{w}^*=(1,0,\dots,0)^T \quad &\text{and}\quad f^*(u_1,\dots,u_n)=\max\{u_1,\dots,u_n\},\\
f_{\text{avg}}:\mathbf{w}=\mathbf{w}_{\text{avg}}=(1/n,\dots,1/n)^T \quad &\text{and}\quad f_{\text{avg}}(u_1,\dots,u_n)=1/n\sum_i u_i,\\
f_*:\mathbf{w}=\mathbf{w}_*=(0,0,\dots,1)^T \quad &\text{and}\quad f_*(u_1,\dots,u_n)=\min\{u_1,\dots,u_n\}.
\end{aligned}
$$

This is to continue Example 3.

Example 4 If the fuzzy set of *SHORT* AND *FAST* routes is created using the arithmetic mean f_{avg}, then route 1, route 2, and route 3 belong to it with a degree of 0.6, 0.675, and 0.5, respectively. These degrees reflect the preference according to intuition as sketched above.

3.2 Types of Rule-based Fuzzy Systems

Rule-based fuzzy systems can be differentiated according to the types of fuzzy IF-THEN rules. There are three types of practical relevance:

1. rules of the type 'IF $x_1=A_{1,i_1}$ AND ... AND $x_s=A_{s,i_s}$ THEN $y=B_i$' leading to *linguistic* or *Mamdani-type fuzzy systems* as discussed in Section 2,

2. rules of the type 'IF $x_1=A_{1,i_1}$ AND ... AND $x_s=A_{s,i_s}$ THEN $y=f_i(x_1,\dots,x_s)$' used in the so-called *Takagi-Sugeno-type fuzzy systems* (note, that the arguments of f_i which in most cases is a linear function are not necessarily the same as in the premise.), and

3. rules of the type 'IF $x_1=A_{1,i_1}$ AND ... AND $x_s=A_{s,i_S}$ THEN $y=c_i$', where c_i is a constant, being a special case of the other two rule types. The resulting fuzzy systems are called *singleton fuzzy systems*.

Here, the A_{j,i_j} and B_i denote linguistic terms of the linguistic variables x_j and y, respectively. In addition, a fuzzy rule may be assigned an individual weight or confidence value (from the interval [0,1]) according to the degree of certainty or confidence.

For Example 1, as alternative to the Mamdani-type fuzzy controller presented in Section 2 (**Example 1a**)

- **Example 1b**: a Mamdani-type fuzzy controller with the rule base in Table 1, the algebraic product as intersection, the bounded sum as union, and the output membership functions as shown in Fig. 4,
- **Example 1c**: a singleton fuzzy controller where the constants in the conclusion are labeled according to Fig. 9b ($c_1 = -20, c_2 = c_4 = -10, c_3 = c_5 = c_7 = 0, c_6 = c_8 = 10, c_9 = 20$), with the rule base in Table 1, and the algebraic product as intersection,
- **Example 1d**: a Takagi-Sugeno-type fuzzy controller with linear functions $f_i(x_1, x_2) = a_{i,0} + a_{i,1}x_1 + a_{i,2}x_2$ in the conclusions, the rule base in Table 4, and the algebraic product as intersection,

will be considered in the following subsection. All controllers posses input membership functions as shown in Fig. 3. These versions cover the most important types of fuzzy controllers in practical applications.

Table 4: Rule base for a Takagi-Sugeno-type fuzzy controller (Example 1d)

$\mathbf{a}_i = (a_{i,0}, a_{i,1}, a_{i,2})^T$		T_{Dev}		
		TOO WARM	OK	TOO COLD
ΔT	INCR	$\mathbf{a}_1 = (-10, 0.5, 2.5)^T$	$\mathbf{a}_2 = (-5, 1.0, 2.5)^T$	$\mathbf{a}_3 = (0, 0.5, 2.5)^T$
	EQUAL	$\mathbf{a}_4 = (-5, 0.5, 5)^T$	$\mathbf{a}_5 = (0, 1.0, 5)^T$	$\mathbf{a}_6 = (5, 0.5, 5)^T$
	DECR	$\mathbf{a}_7 = (0, 0.5, 2.5)^T$	$\mathbf{a}_8 = (5, 1.0, 2.5)^T$	$\mathbf{a}_9 = (10, 0.5, 2.5)^T$

3.3 Information Processing in Fuzzy Systems

In Section 2, information processing in fuzzy systems is explained by means of Example 1. In order to make the descriptions simple, no alternatives to the applied operations have been presented. But this description and Section 3.1 imply that there are many alternatives. These alternatives concern mainly the interpretation of connectives in fuzzy propositions and of fuzzy rules leading to different inference schemes and strategies for defuzzification. Practically important alternatives will be briefly described below.

Several inference mechanisms have been developed within the framework of approximate reasoning. Mainly, they base either on the *compositional rule of inference* (using fuzzy set theory considerations) or on the *generalized modus ponens* (analogous to the classical *modus ponens*). These inference mechanisms differ in the interpretation of rules (logical implication vs. conjunction), the operations used for logical or set connectives, the order of inference steps (composition based vs. individual-rule based) etc. However, most fuzzy control algorithms used in practice employ an inference mechanism based on individual-rule inference where rules are interpreted as conjunctions.

Aggregation In aggregation, different t-norms or t-conorms can be applied to model the connectives between fuzzy propositions. However, in fuzzy control, where the premises usually contain AND connectives only, the algebraic product is the most popular choice. For Example 1 aggregation with the algebraic product yields the results shown in Table 5. They differ significantly from the minimum aggregation in Table 3.

Table 5: Aggregation for Example 1b-d with the algebraic product operation

		T_{Dev}		
		$\mu_{TOO\ WARM}(T_{Dev}) = 0.00$	$\mu_{OK}(T_{Dev}) = 0.20$	$\mu_{TOO\ COLD}(T_{Dev}) = 0.80$
ΔT	$\mu_{INCR}(\Delta T) = 0.1$	$\mu_{P_1} = 0.00$	$\mu_{P_2} = 0.02$	$\mu_{P_3} = 0.08$
	$\mu_{EQUAL}(\Delta T) = 0.9$	$\mu_{P_4} = 0.00$	$\mu_{P_5} = 0.18$	$\mu_{P_6} = 0.72$
	$\mu_{DECR}(\Delta T) = 0.0$	$\mu_{P_7} = 0.00$	$\mu_{P_8} = 0.00$	$\mu_{P_9} = 0.00$

Activation As fuzzy IF-THEN rules in fuzzy control are usually interpreted as a conjunction of the premise and the conclusion, a t-norm is the appropriate operator. Mainly, the algebraic product is used resulting in a scaling of the fuzzy set in the conclusion by the degree of premise fulfillment. Results for the Example 1b are depicted in Fig. 8a. They also differ from Example 1a (Fig. 4a).

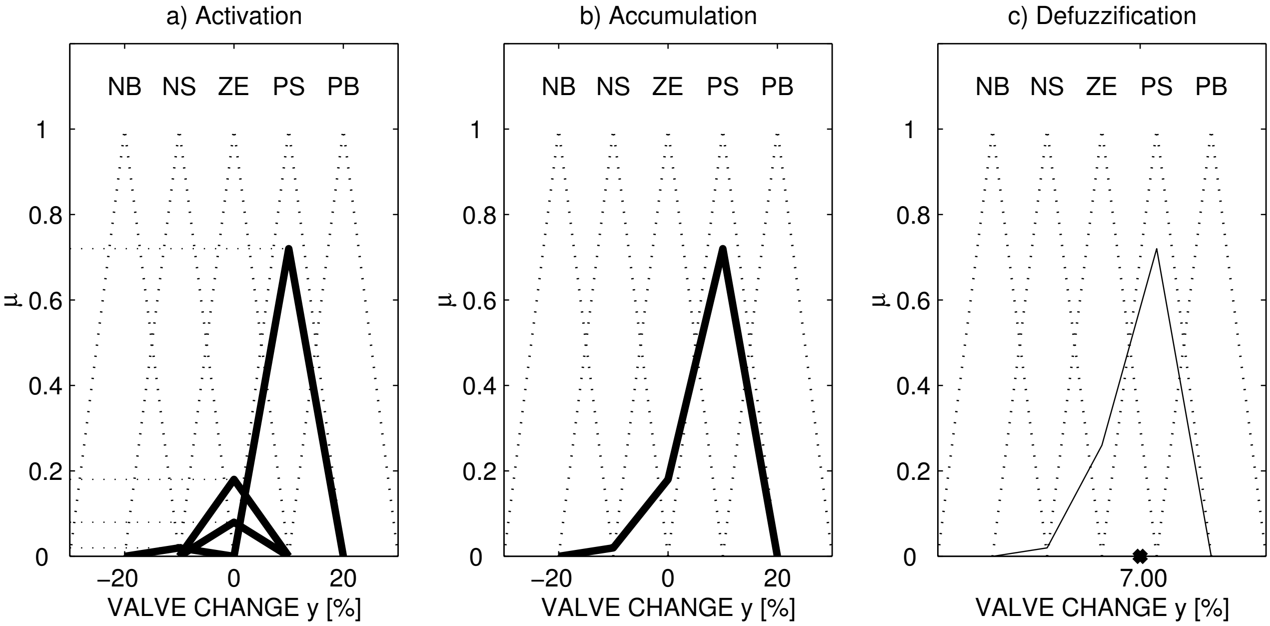


Figure 8: Results of inference and defuzzification using the product operator and bounded sum operator: a) aggregation, b) accumulation, c) defuzzification (Example 1b)

In the case of rules with additional confidence values these values are connected with the degree of premise fulfillment by a t-norm, usually the algebraic product.

Accumulation If a rule base is interpreted as a disjunction of rules, the appropriate operator is a t-conorm. In this case popular alternatives are the maximum and the bounded sum operators. An alternative, motivated by the "expert" metaphor, is to combine the "expert opinions" by an averaging operator, e. g. an ordered weighted averaging (OWA) operator. A possible choice is a weighted mean of the activated fuzzy sets in the rule conclusions where the weights may be equal (no preference of experts) or the confidence values.

The application of the bounded sum in Example 1b gives a fuzzy set with $\mu(y)$ as shown in Fig. 8b.

To summarize, inference yields the following output fuzzy set in case of t-conorm accumulation

$$\mu(y) = \underbrace{\overset{r}{\underset{k=1}{\perp}} \mu_{C_k}(y)}_{\text{accumulation}} = \overset{r}{\underset{k=1}{\perp}} \underbrace{\top\left(\mu_{P_k}, \mu_{B_{i(k)}}(y)\right)}_{\text{activation}} = \overset{r}{\underset{k=1}{\perp}} \top\left(\underbrace{\overset{s}{\underset{j=1}{\top}} \mu_{A_{j,i_j(k)}}(x_j)}_{\text{aggregation}}, \mu_{B_{i(k)}}(y)\right), \tag{19}$$

where $\overset{r}{\underset{k=1}{\perp}}$ denotes the r-ary extension of a t-conorm, $\overset{s}{\underset{j=1}{\top}}$ the s-ary extension of a t-norm (r number of rules, s number of inputs). In case of an OWA accumulation, the output fuzzy set is given by

$$\mu(y) = f\left(\top\left(\overset{s}{\underset{j=1}{\top}} \mu_{A_{j,i_j(k)}}(x_j), \mu_{B_{i(k)}}(y)\right), \ldots, \top\left(\overset{s}{\underset{j=1}{\top}} \mu_{A_{j,i_j(k)}}(x_j), \mu_{B_{i(k)}}(y)\right)\right), \tag{20}$$

with $f(\cdot)$ being an OWA operator.

Several inference schemes, named after the used operations, are especially popular in control applications, namely sum-prod inference, max-prod inference, and max-min inference. Aggregation is mostly performed with the same operator as activation. For differences between these inference schemes and the practical consequences see *Analysis and Stability of Fuzzy Systems*.

Defuzzification The understanding of what value "best" comprehends the meaning of the output fuzzy set depends on the application. Therefore, several methods are employed. For applications whit a continuously changing control signal being needed, the Center-of-Gravity (CoG) or Bisector-of-Area (BOA) method are used. For non-convex fuzzy sets, these defuzzification methods may lead to output values which have a very small or even a zero degree of membership in the inferred fuzzy set, i. e. which are only a little or not supported by any of the rules. Maximum methods which select only output values with a maximum membership degree prevent this effect. As several values may have a maximum membership degree, a preference for example leads to the Leftmost-Maximum or Rightmost-Maximum methods. The so-called Mean-of-Maximum method, however, which takes the mean of all output values with maximum membership degree, may also lead to little supported values in case of a non-convex fuzzy set. In comparison to CoG or BoA, maximum methods are computationally less complex. Their disadvantage is the generally discontinuous control signal produced. In Table 6 the formulae of the mentioned defuzzification methods are summarized.

Defuzzification results for different methods applied to the inference results of Example 1a (Fig. 4b, min-max inference) are shown in Fig. 9a. In general, the different defuzzification methods yield different output values that are considered to best represent the inferred fuzzy set with $\mu(y)$.

A concept which generalizes these methods is the *inference filter* introduced by H. Kiendl. It allows a continuous transition between the defuzzification methods mentioned above. The idea is to first filter the membership function $\mu(y)$ to obtain an attractiveness function $\hat{\mu}(y)$, and to secondly choose the maximum value $y = \arg\max \hat{\mu}(y)$, that is the most "attractive" value, as output value. The impulse response $h(y)$ of the inference filter is a parameterized function symmetric to zero. This function changes its shape from a δ-function via a triangular function to a rectangular function of a specified width by adjusting two parameters. The resulting output value changes from the value obtained with the Mean-of-Maximum method via the value obtained by the Center-of-Gravity method to the mean of the support of $\mu(y)$ by changing the shape parameter of $h(y)$.

For Example 1b, the control surface of the Mamdani-type fuzzy controller with sum-prod inference and Center-of-Gravity defuzzification is shown in Fig. 10a. Here, CoG defuzzification is a reasonable choice. For bounded sum-prod inference, it leads to a value of 7 % as shown in Fig. 8c, which differs significantly from 5.83% in Section 2 (Fig. 4c).

Table 6: Defuzzification methods (s. t.: such that, sup: supremum, inf: infimum)

Method	Formula
Center-of-Gravity (CoG)	$y = \dfrac{\int_Y \mu(y) y dy}{\int_Y \mu(y) dy}$
Bisector-of-Area (BoA)	$y \text{ s.t. } \int_{y_{\min}}^{y} \mu(y) dy = \int_{y}^{y_{\max}} \mu(y) dy$
Mean-of-Maxima (MoM)	$y = \dfrac{\int_{y':\mu(y')=\sup_Y \mu(y)} y dy}{\int_{y':\mu(y')=\sup_Y \mu(y)} dy}$
Leftmost-Maximum (LM)	$y = \inf\{y' \in Y \mid \mu(y') = \sup \mu(y)\}$
Rightmost-Maximum (RM)	$y = \sup\{y' \in Y \mid \mu(y') = \sup \mu(y)\}$

For singleton and Takagi-Sugeno-type fuzzy controllers, defuzzification is integrated in activation and accumulation. The output is calculated as a weighted mean of the singleton or function values, respectively, where the weights are the degrees of rule fulfillment, i. e.

$$y = \frac{\sum_{k=1}^{r} \mu_{P_k} c_k}{\sum_{k=1}^{r} \mu_{P_k}} \qquad \text{(singleton fuzzy system)} \tag{21}$$

$$y = \frac{\sum_{k=1}^{r} \mu_{P_k} f_k(\mathbf{x})}{\sum_{k=1}^{r} \mu_{P_k}} \qquad \text{(Takagi-Sugeno-type fuzzy system).} \tag{22}$$

The control surface of the singleton fuzzy controller in Example 1c is given by

$$y(T_{Dev}, \Delta T) = \frac{\sum_{k=1}^{9} \mu_{P_k}(T_{Dev}, \Delta T) c_k}{\sum_{k=1}^{9} \mu_{P_k}}. \tag{23}$$

The result is $y(8, 0.2) = 7$ with algebraic product aggregation (Table 5) (terms with non-zero elements of $\mu_{P_k}(8, 0.2)$ are $c_2\mu_{P_2} = -0.2, c_3\mu_{P_3} = 0, c_5\mu_{P_5} = 0, c_6\mu_{P_6} = 7.2$; $\sum_{k=1}^{9} \mu_{P_k} = 1$) and defuzzification is shown in Fig 9b.

For the Takagi-Sugeno-type fuzzy controller in Example 1d the control surface is given by

$$\begin{aligned} y(T_{Dev}, \Delta T) &= \frac{\sum_{k=1}^{9} \mu_{P_k}(a_{k,0} + a_{k,1} T_{Dev} + a_{k,2} \Delta T)}{\sum_{k=1}^{9} \mu_{P_k}} \\ &= \tilde{a}_0(T_{Dev}, \Delta T) + \tilde{a}_1(T_{Dev}, \Delta T) T_{Dev} + \tilde{a}_2(T_{Dev}, \Delta T) \Delta T \end{aligned} \tag{24}$$

(Fig. 10b). This control surface is similar to the one of the Mamdani-type fuzzy controller (Example 1b) in Fig 10a. The main difference lies in the extrapolation behavior outside the region $[-10\text{ K}, 10\text{ K}] \times [-2\text{ K}, 2\text{ K}]$. Whereas the Mamdani-type controller extrapolates with constant values, the Takagi-Sugeno type controller extrapolates linearly. For the values $T_{Dev} = 8$ and $\Delta T = 0.2$, $\tilde{a}_0(8, 0.2) = 3.5$, $\tilde{a}_1(8, 0.2) = 0.6$, $\tilde{a}_2(8, 0.2) = 4.75$ and $y(8, 0.2) = 9.25$ are obtained.

Under the assumptions of

- sum-prod inference,

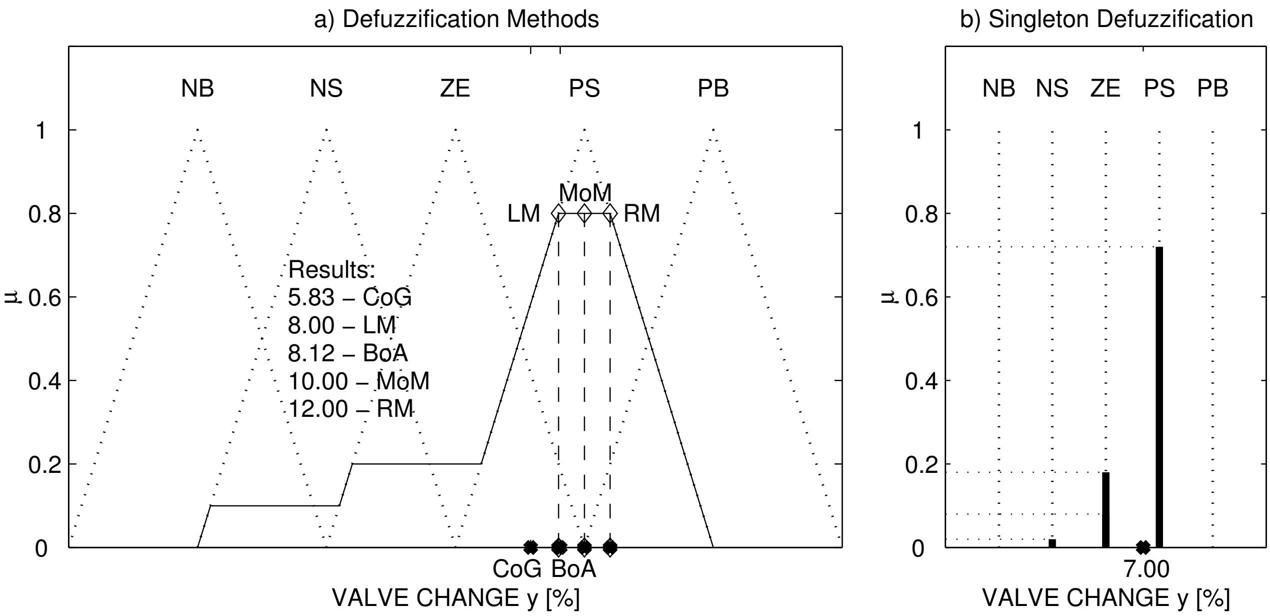


Figure 9: a) Comparison of different defuzzification methods from Table 6 for Example 1a; b) Results of algebraic product aggregation and defuzzification for the singleton system (Example 1c)

- output fuzzy sets with equal cardinality, and
- CoG defuzzification,

a Mamdani-type fuzzy system is equivalent to a singleton fuzzy system, where the singletons result from an individual CoG defuzzification of the output fuzzy sets. It explains the equivalence between the control surfaces of Examples 1b and c (10a). These assumptions are very reasonable in many practical applications resulting in a very simple implementation of the fuzzy controller. Moreover, the resulting control surface is piece-wisely multi-linear, a convenient form for the analysis of the control loop (see *Analysis and Stability of Fuzzy Systems*).

Two-way fuzzy controller and hyperinference The so-called *two-way fuzzy controller* also introduced by Kiendl is another type of fuzzy system. The rule base of this fuzzy system contains positive rules as stated above and negative rules expressing explicit warnings or vetoes.

For a two-way fuzzy controller the inference scheme contains an additional processing step – the so-called *hyperinference*. Inference is performed separately for positive and negative rules, resulting in two fuzzy sets $\mu^+(y)$ for positive and $\mu^-(y)$ for negative rules, respectively. Hyperinference combines both fuzzy sets by applying a certain strategy, namely, the strong, the weak, or the fuzzy veto.

4 Control Issues in Fuzzy Control

4.1 Structures in Fuzzy Control

In Example 1, the fuzzy controller is directly incorporated in the basic control loop. That is, its inputs are the control error, called *temperature deviation*, and the discrete derivative of the controlled variable, called *temperature change*, and its output is the change of the manipulated variable, called *valve change*. Hence, the resulting control law is of PI type. The fuzzy controller needs external

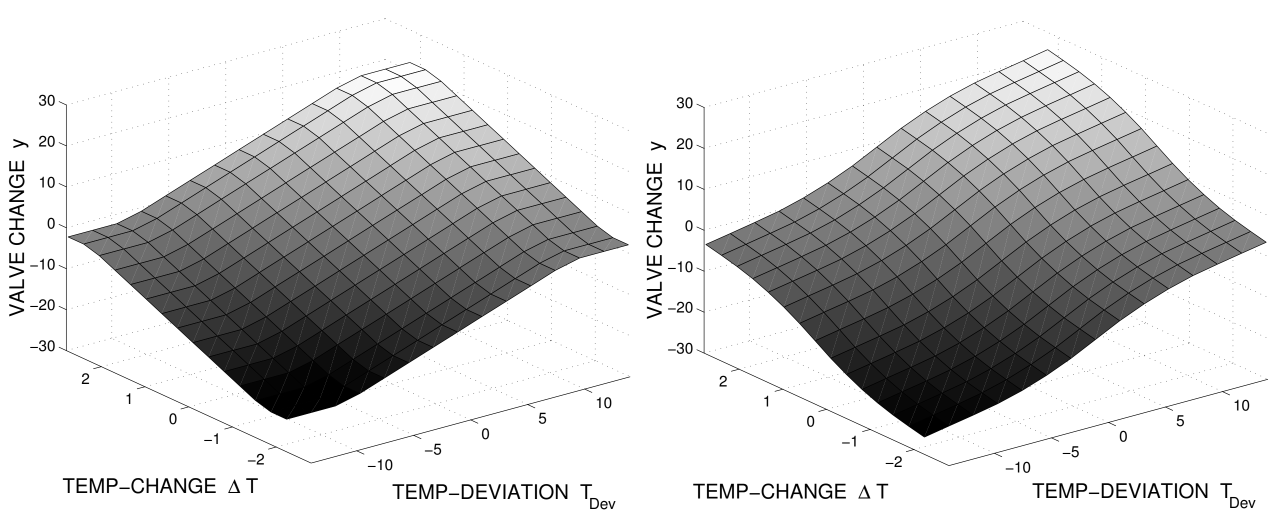


Figure 10: Control surface of the fuzzy controller a) Mamdani-type with sum-prod inference and CoG defuzzification (Example 1b), b) Takagi-Sugeno-type fuzzy controller (Example 1d)

elements to form the derivative (or difference), to integrate (or to sum) its output, and adapt inputs and output to signal ranges (scaling, normalization, and denormalization). Additional components for an implementation include an analog-digital and a digital-analog converter, possibly a filter to suppress measurement noise, elements for an anti-windup scheme, etc. The components necessary for pre- and post-processing depends on the application and function of the fuzzy controller.

Apart from direct fuzzy control as in Example 1, fuzzy controllers can perform certain additional functions in control systems. An overview of typical structural schemes of control systems incorporating fuzzy controllers is given in Table 7.

In the classification of control structures with a fuzzy controller as given in Table 7, there is no clear cut between the classes. For instance, a Mamdani-type fuzzy controller which adapts the parameters of a linear controller as a gain scheduler according to a desired reference trajectory can be realized as a Takagi-Sugeno-type fuzzy system which is directly incorporated in the control loop.

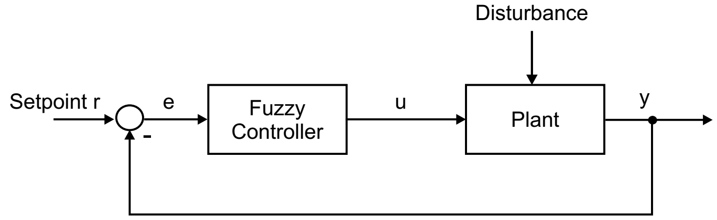
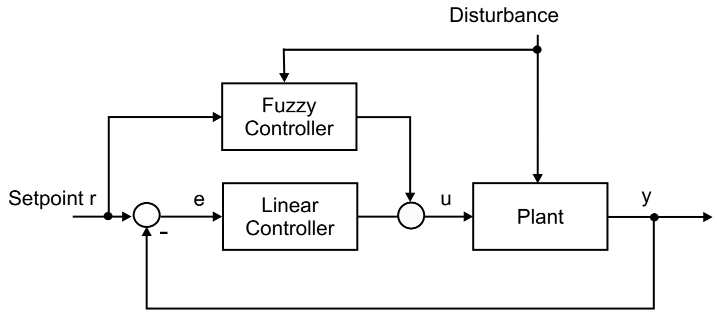
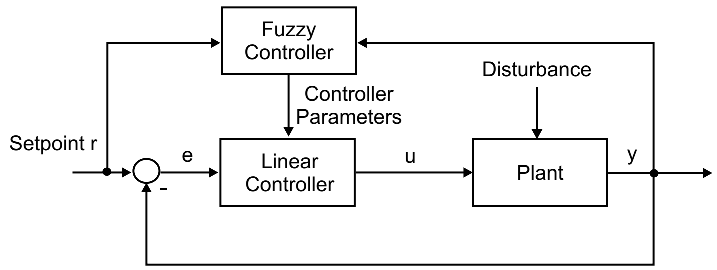
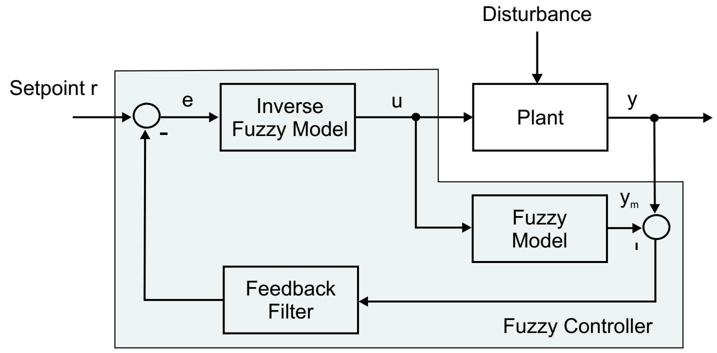
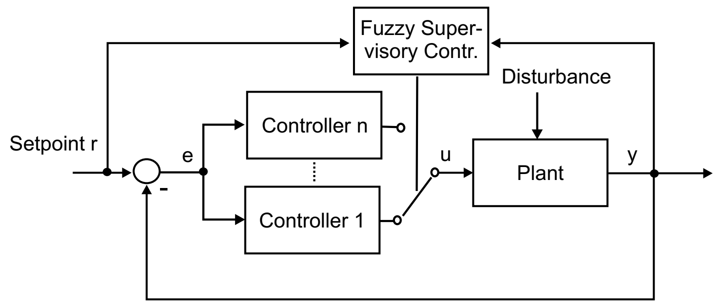
4.2 Design of Fuzzy Control Systems

The design of fuzzy control systems covers

- the determination of inputs and outputs,
- the definition of the respective linguistic variables including the linguistic terms and their fuzzy sets,
- the compilation of the rule base, and
- the choice of the inference strategy and defuzzification method.

Whereas the latter task requires knowledge of fuzzy theory and can therefore hardly be realized automatically, the first three tasks can be handled by different approaches including heuristic, systematic and partially automatic methods. The differences between theses approaches will be explained for fuzzy control systems (Table 8 and Fig. 11), but similar approaches also exist for non-control applications.

Table 7: Structures of control systems with a fuzzy controller

<table>
<tr><th>Description</th><th>Structural scheme</th></tr>
<tr><td>Direct fuzzy control:
Fuzzy controller replaces conventional controller, e. g. a PID controller, or manual control by operator.</td><td>
</td></tr>
<tr><td>Feedforward fuzzy control:
Fuzzy controller modifies the manipulated variable to compensate a measurable disturbance or a change of the setpoint.</td><td>
</td></tr>
<tr><td>Fuzzy parameter-adaptive control:
Fuzzy controller adapts parameters of linear controller to changing operating conditions (gain scheduling), or changes controller structure (switching D part). The basic control loop with a linear controller, e. g. PID controller, is left unchanged.</td><td>
</td></tr>
<tr><td>Fuzzy model-based control:
The controller incorporates a fuzzy plant model. Two different schemes exist: Internal model control (IMC) as shown, and model-predictive control (MPC) containing additionally an optimizer.</td><td>
</td></tr>
<tr><td>Fuzzy supervisory control:
Supervisory control works on the level of human operation. Its general task is to improve process performance (e. g. stable operation, maximize product yield, minimize energy consumption) accomplished by e. g. the appropriate setting of set-points for basic control loops and switching or reparameterization of basic controllers.</td><td>
</td></tr>
</table>

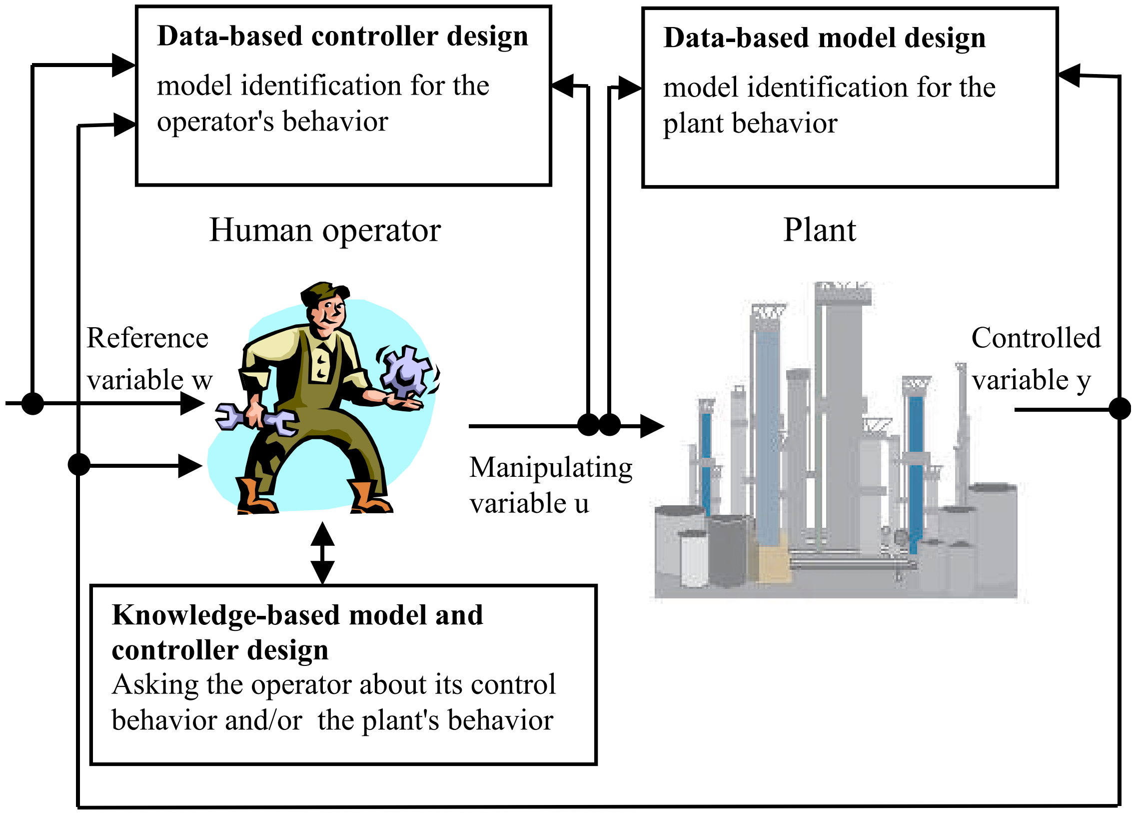


Figure 11: Different data-based and knowledge-based strategies for model and controller design

In the early years of fuzzy control, the primary approach was to interview human process operators and to formalize the knowledge gained in the form of rules and membership functions of a fuzzy system (knowledge-based controller design). Here, knowledge subsumes all types of qualitative (rules, structure, properties) and quantitative information. This was not always successful for certain reasons. To begin with, human knowledge contains unconscious parts, which can hardly be formalized by rules. Moreover, many rules will be context sensitive, which results in incomplete rule premises with missing context information. Furthermore, the complexity of human strategies may be very high, quantification in the form of membership functions is difficult, and so on. Alternatively, the control engineer who develops the fuzzy controller knows or learns how to control the process and captures his knowledge. This may be advantageous as it prevents the communication losses between operator and control engineer. Moreover, the control engineer may be more conscious of the control strategy, which makes it easy to determine important variables, formulate rules, etc.

An alternative to this knowledge-based approach is to record data of process measurements, the set-points, and the actions of the human process operator (manipulated variable). In the so-called data-based controller design, a model of the human control behavior is established by identification of a fuzzy system with a similar input-output behavior. Existing a-priori knowledge will be often used to obtain appropriate structures for fuzzy systems (inputs, outputs, possible rules). The structures may result in Mamdani-type fuzzy controllers (especially if the conclusions are directly referred to manipulating variables) or Takagi-Sugeno-type fuzzy controllers (especially if the conclusions are switching operations between different control strategies). An comprehensive overview of data-based techniques for model or controller design is given in *Data-Based Fuzzy Modeling*.

Both approaches discussed do not require any plant model. This seems to be advantageous for a simpler design, but it restricts the possibilities to simulate, to optimize, and to supervise fuzzy systems similar to human strategies. As an example, a human process operator certainly has some perceptions

Table 8: Different data-based and knowledge-based strategies for model and controller (contr.) design ($\surd$ - information given for the task, ? - aim of the task)

	Data		Knowledge		Model		Behavior
	Plant	Contr.	Plant	Contr.	Plant	Contr.	Closed loop
Knowledge-based controller design				$\surd$		?	
Data-based controller design		$\surd$		($\surd$)		?	
Model-based controller design					$\surd$	?	$\surd$
Knowledge-based model design			$\surd$		?		
Data-based model design	$\surd$		($\surd$)		?		
Analysis of closed-loop systems					$\surd$	$\surd$	?

about the expected plant behavior and is able to detect deviations from it.

In contrast to these model-free approaches, a model-based controller design does not formalize human strategies of problem solution but is based on a given problem description and a description of goals. In case of fuzzy control, a fuzzy controller will be developed by means of a given model of the plant and a definition of the desired closed-loop behavior (e. g. stability, performance, robustness). Different approaches to designing for a model-based controller are used in practical applications:

As an example, the design can be accomplished by offline parameter optimization as presented in detail in *Optimization of Fuzzy Systems*. The desired closed-loop behavior may be a minimal difference (described by an integral criterion) between a given setpoint and the controlled variable. In optimization, simulation-based techniques, which simulate the closed control loop with a given plant model and a fuzzy controller, are very popular. The closed-loop behavior desired is only used for evaluation and the choice of the best controller.

An other model-based approach is model-predictive fuzzy control (see *Model-based Predictive Control*). Here, one or more future values of the manipulated variable are found by an online optimization using the fuzzy plant model and the desired closed-loop behavior. In this case, an explicit controller structure is not necessary.

If the plant is modeled by a Takagi-Sugeno-type fuzzy system with linear functions as rule conclusions, a Takagi-Sugeno-type controller may be designed using techniques from linear control design (see *Control of Linear Multivariable Systems*) as e. g. pole placement. Each rule describes a fuzzy region where a linear model, for which a controller is designed, is valid. This approach is called Parallel Distributed Compensation. As the stability of the local controller designs does not imply global stability more advanced techniques start with a formalization of the desired behavior of the closed-loop system (e. g. a Lyapunov function to ensure the stability of the system, see *Lyapunov Stability*). Here, appropriate controller parameters are found by optimization (e. g. by solving Linear Matrix Inequalities, see *Controller Design by LMI Approach*).

The plant model required for a model-based controller design can be obtained by knowledge–based or data-based modeling. Here, human knowledge or data of the plant behavior are obtained instead of the knowledge or data of the control behavior. From a mathematical point of view, the techniques for modeling the plant are the same as for modeling the control behavior. Modern data-based approaches are able to find structures (inputs, outputs, rules) and parameters. Further elements of the fuzzy system (fuzzy operators, types of membership functions, defuzzification method) are often fixed. Data-based approaches preferably use Takagi-Sugeno-type fuzzy systems with linear functions as rule conclusions. This makes well-known parameter estimation methods applicable. The remaining task is to find an appropriate fuzzy representation (rules and membership functions) for the rule premises (see *System Identification using Fuzzy Models*).

4.3 Analysis of Fuzzy Control Systems

Most fuzzy systems are nonlinear systems of a high complexity making the analysis of behavior and properties an important and nontrivial task. There are several methods to analyze fuzzy systems and their interactions with other systems which are described in *Analysis and Stability of Fuzzy Systems*. Here, only the basic ideas shall be presented.

The overall objective of the analysis is to prove whether the performance specifications are met. Issues in the analysis of fuzzy control systems can be divided into fuzzy logic and control and system theory aspects.

The fuzzy logic aspect takes into account properties of the fuzzy system itself, for example the consistency and completeness of the rule base, correctness of inference. For this purpose, methods have been developed on the basis e. g. the theories of approximate reasoning and fuzzy relational equations.

From the control and system theory points of view, properties of the overall system consisting of the plant and the control system, including the fuzzy system, are of interest, e. g. stability, robustness, control performance. Many analysis techniques transform a fuzzy system into an equivalent nonlinear function to describe the input-output behavior (see *Closed-loop behavior*). After this transformation, methods from nonlinear system theory can be readily applied (see *Basic Nonlinear Control Systems*, *Control of Nonlinear Systems*). Here, the typical structures of fuzzy controllers result in representations common to control theory, for example in linear parameter-varying (LPV) systems or in systems consisting of parallel linear and nonlinear subsystems.

Another class of methods is specific for fuzzy systems. These methods typically represent the overall control system and the plant as a fuzzy system. Analysis is performed on the level of linguistic terms or fuzzy sets and, thus, results have the form of fuzzy statements. For this reason, such methods are not well accepted.

Provided that a sufficiently accurate plant model exists, extensive simulation studies, apart from theoretical analysis, are a means for assessing the performance of the control system. Thus, properties which cannot be proved theoretically are still checked.

Analysis should accompany system design and for this purpose a plant model is needed. However, a rational for using a fuzzy control system is the lack of such a model. In this situation, an online supervision and analysis is applicable only. The supervisor might contain formal mathematical knowledge and rule-based expert knowledge to evaluate the control system.

4.4 Applications

Since the 1980s, many applications of fuzzy systems in automatic control have been reported. This section highlights some principal aspects of fuzzy control system applications. An overview on fields of applications and descriptions of examples are contained in *Fuzzy System Applications*.

Applications of fuzzy systems can be found in various technical and non-technical domains. The number of reported applications is higher for technical domains where the processes are difficult to model due to complexity, nonlinearity, time-variance or uncertainty, and where experienced operators are an essential part of the process control. Examples include the production of raw materials (e. g. iron and steel production, cement production), environmental engineering (e. g. waste incineration, water treatment plants), biotechnological processes, power systems, autonomous robots, automotive systems. When the objective of automatic control is to enhance functionality and user friendliness without high performance requirements, e. g. consumer products and household appliances, fuzzy systems have been applied as well.

In the non-technical domains the dominating fields of applications are medicine and biomedical engineering, image processing, and pattern recognition. These domains also are characterized by a high complexity, uncertainty regarding underlying dependencies and laws as well as measurements. In addition, expert knowledge often plays an important role in the successful solution of the problems.

From the application point of view, two questions are of major importance:

1. Which type of tasks in an automatic control system can be reasonably solved by a fuzzy system/ controller?

On the basic level, the use of fuzzy systems for tasks such as direct control or signal processing may be attractive, if only imprecise or qualitative measurements are available, the process is nonlinear, human control has to be imitated, or the controller functionality should be enhanced.

If the process is equipped with a basic automation system, but it is time-variant, or operated in different regimes, then fuzzy systems may be appropriate for tasks, such as parameter adaptation, quality control, and supervisory control.

On higher levels, where tasks of scheduling, supervision or multi-objective control have to be performed by operators processing quantitative and qualitative information, fuzzy systems may be an essential part of an automation solution.

According to the main task to be accomplished by the fuzzy system, the following main classes of applications can be distinguished:

- Control,
- diagnosis, classification, and pattern recognition,
- modeling and forecasting,
- decision support.

2. What are the specific prerequisites and advantages of the application of fuzzy systems/ controller?

The prerequisite for the successful implementation of any automatic control concept is knowledge about the process and in control engineering. Furthermore, appropriate control hardware (sensors, actuators, programmable logical controllers - PLC, process control systems - PCS, etc.) and software are necessary. This statement holds for fuzzy systems as well. Regarding the knowledge, the specificity lies in the form of knowledge representation (linguistic variables and rules). The present status regarding hardware and software, as described in Section 4.5, allows an easy implementation and integration of fuzzy control systems. Nevertheless, basic knowledge of fuzzy system theory and the specific design and analysis methods is a necessary prerequisite.

The advantages of fuzzy system applications are manifold. First, fuzzy systems help to automate processes which could not be automated before or to improve the performance of existing automation systems. This often leads to improved quality, reduction of energy or material consumption, inventory or maintenance costs. Moreover, this may improve the operability of processes for human operators. Secondly, by using dedicated software tools and/or function blocks of control hardware, fuzzy systems can be implemented faster than comparable control algorithms implemented in C or other computer languages. Thirdly, fuzzy systems are a transparent means for documenting control strategies, provided that single rule bases remain manageable. This makes them easy to modify and debug at a later time. Furthermore, fuzzy systems may provoke users to iteratively optimize the automation system by deriving process knowledge and operation experience.

To summarize, fuzzy systems have become a tool of the control engineer, which is readily available through theoretical and technological developments. Fuzzy systems are not a panacea for all control problems, but successful applications demonstrate the usefulness of fuzzy technology.

4.5 Hardware and Software for Fuzzy Control Systems

Different CAD (Computer Aided Design) tools for fuzzy systems are on the market, which support the design of fuzzy systems. They are available for almost all types of hardware and software platforms. In the following, only some typical features shall be mentioned.

Usually, there is a graphic user interface for the design of membership functions, the editing of rules (as a list of rules or as a table), the choice of fuzzy operations, and defuzzification methods. The analysis of fuzzy systems often is restricted to the plot of control surfaces or to protocol the reactions of the fuzzy system to given inputs. Some CAD tools contain more or less basic simulation facilities or an interface to other simulation tools. A few tools also support the design of fuzzy systems by searching for rules or by means of a parameter optimization for membership functions.

Most fuzzy CAD tools support the export of the fuzzy systems designed to other hardware and software platforms. The latter step is of great practical relevance to the acceptance of fuzzy systems. Different strategies are possible:

- The fuzzy system will be automatically translated into the source code of programming languages (e. g. languages of the international standard IEC 61131-3 or specialized Fuzzy Control Language of the IEC 61131-7 for process control systems, or ANSI C for personal computers or workstations). Here, the procedures of fuzzification, inference, and defuzzification are stepwisely implemented. This strategy allows for an easy integration into existing systems and it does not require any additional hardware effort. The running time in routine use is not very fast. This strategy is most important for practical solutions.

- The fuzzy system is transformed into a look-up table with characteristic points and (optionally) interpolation between them. Here, an equivalent functional description of the control surface is implemented instead of the procedures of fuzzification, inference, and defuzzification. As discussed before, the integration in existing systems is possible without any additional hardware effort. The run time is very fast. The solution might require a lot of memory to store characteristic points for multiple-input fuzzy systems. It is mainly used for smaller fuzzy systems and for low-cost hardware solutions, e. g. microcontrollers.

- The fuzzy system will be exported as assembler code for a special structured Fuzzy ASIC (Application Specific Integrated Circuit). Such ASICs are extremely fast, but restrictions regarding the structure of the fuzzy systems may occur. This solution requires additional hardware effort and is not commonly used.

4.6 Further Concepts Based on Fuzzy Sets Applied to Control Systems

The idea of fuzzy sets has promoted many fields of scientific research. New concepts and methods have been developed, often extending existing ones, by allowing for gradual truth values or gradual set membership, by employing fuzzy numbers or fuzzy relations, or by describing relationships by IF-THEN rules.

As the kind of task performed by an automatic control system is manifold, more fuzzy set-based concepts are being applied nowadays than fuzzy controls. As mentioned in the previous section, fuzzy systems are used for classification and pattern recognition, modeling, and decision support. In the following sections, two examples shall be presented, namely, fuzzy Petri nets, a concept for modeling manufacturing and control systems, and fuzzy decision making, a concept useful in higher-level control. The application of fuzzy systems to system identification is the subject of *System Identification using Fuzzy Models*.

Fuzzy Petri nets result from combining Petri nets with fuzzy sets. Petri nets are used to model and analyze systems which are characterized by parallelism, concurrency and sequencing of processes, e. g. complex manufacturing systems, communication systems or control systems of such systems (see *Discrete Event Systems*). Petri nets consist of places representing conditions or situations, transitions representing events or activities and arcs connecting places and transitions. Places can contain tokens. Their flow through the net is controlled by the switching of the transitions. A marking, the distribution of tokens, represents the system state. Models of such complex systems are uncertain to some degree due to simplifications or abstractions. In particular, the difficulty of making precise distinctions leads to vagueness, which can be dealt with by fuzzy sets. Furthermore, impreciseness occurs in situations, where the choice between alternatives is left unspecified. This impreciseness is described within the framework of possibility distributions, and thus, is strongly related to fuzzy sets, too. Hence, it seems natural to introduce fuzziness in the Petri net model.

Several concepts of fuzzy Petri nets have been proposed. An example is the fuzzy Petri net model on the basis of Place/Transition (P/T) nets, which can be considered a combination of several P/T nets valid for specific process situations. This type of fuzzy Petri net can be used to model the control of complex manufacturing systems. Another example is the fuzzy Petri net model in which places are interpreted as fuzzy propositions, transitions as fuzzy rules. This type can be applied as a model of a fuzzy reasoning process, e. g. of fuzzy expert systems.

Fuzzy decision making refers to (multiple criteria) decision-making, where some of the constituents of the decision problem are not known or not described precisely. A decision problem is characterized by a set of alternatives, a set of constraints on the choice between alternatives, and a performance or objective function (or a set thereof) associating the gain (or loss) to each alternative. The alternatives can be described by discrete or continuous variables. In many real world problems the value of the objective function for a certain alternative is not known precisely, e. g. this function is given by a set of expert rules. Moreover, the constraints are not necessarily "hard" (ordinary sets), but more reasonably modeled with fuzzy sets.

One way to solve such fuzzy decision problems is to exploit their symmetry with regard to objectives and constraints. The basic idea can be described as follows. Both, objectives and constraints, can be represented as fuzzy sets over the set of alternatives and, thus, be treated identically when finding a decision. The fuzzy set to represent an objective function measures the degree with which the alternatives fulfill this objective, i. e. it is the fuzzy set of "good solutions" for this objective. Assuming multiple objectives, a "good compromise solution" is the alternative which is "as good as possible" for all objectives. Thus, an intersection of the fuzzy sets of the objectives gives the appropriate fuzzy set measuring the joint fulfillment of all objectives for each alternative. The same reasoning applies to the constraints resulting in a fuzzy set measuring the degree to which the constraints are jointly met by each of the alternatives. Combining both fuzzy sets by intersection and searching for the maximum gives the optimal decision.

Fuzzy decision making has been successfully applied to various control problems, e. g. scheduling in manufacturing systems, or selection of control strategies in traffic control systems, power plants, etc.

The same principle of extending existing methods by fuzzification is used in many fields, e. g. fuzzy classification, fuzzy clustering, fuzzy optimization, or fuzzy expert systems.

5 Conclusions

Fuzzy control has emerged as a new paradigm in automatic control. It aims at controlling of systems of high complexity, for which only vague or fuzzy knowledge on the structure and the parameters is available. A sound theoretical background has been developed since the first paper was written by L. A. Zadeh in 1965. The theoretical achievements cover fuzzy logic-related issues, namely, different methods and algorithms for fuzzification, inference, and defuzzification, as well as control issues, in particular design and analysis methods. In addition, several tools have been developed to support the design and implementation of fuzzy control systems. Meanwhile, many successful applications of fuzzy systems as part of the automatic control system of technical and non-technical systems have been reported.

Nevertheless, there are some open problems, e. g.

- the evaluation of the efficiency of fuzzy control systems by appropriate criterions,
- the proof of global stability and expected performance of these systems,
- the acceptance of fuzzy control theory by control engineers in practice,
- the automatic extraction of expert knowledge for different processes.

These problems will be the subject of research activities in the next years.

Glossary

Accumulation: The third step of inference, where the results of rule activation, i. e. the resulting fuzzy sets, are combined in the output fuzzy set.

Activation: The second step of inference, where the conclusions of the rule are activated individually according to the degree of premise fulfillment resulting in a fuzzy set per rule.

Aggregation: The first step of inference, where the degree of premise fulfillment is calculated.

Control error: The difference between the values of the reference signal and the controlled variable (symbol e, $e(t) = r(t) - y(t)$).

Control surface: Graphic representation of the multivariate nonlinear relationship described by a fuzzy controller.

Controlled variable: The output signal of a controlled plant, which is to assume certain desired values (symbol y).

Defuzzification: The conversion of a fuzzy set into a numerical quantity.

Fuzzification: The conversion of a numerical quantity into degrees of membership of linguistic terms of a linguistic variable.

Fuzzy associative map: The (multidimensional) representation of a rule base showing the (weighted) associations of a combination of linguistic terms of the inputs with a linguistic term of the output (fuzzy rules).

Fuzzy control: A control concept on the basis of fuzzy systems theory.

Fuzzy IF-THEN rule: A rule with a premise (condition) and a conclusion (consequence), where at least the premise must be a fuzzy proposition.

Fuzzy logic: In a narrow sense, a generalization of the Boolean logic, i. e. of the two-valued logic to a multi-valued logic. In a broader sense, the theory of fuzzy sets in all its facets (logical, set-theoretical, relational and epistemic).

Fuzzy model: A model on the basis of fuzzy systems theory.

Fuzzy Petri net: A modification of classical Petri nets to describe discrete-event processes, where marks are described in terms of memberships ranging from zero to unity, and/or transitions are described in terms of fuzzy rules.

Fuzzy proposition: A sentence 'v is A' where v is a variable taking values from some universal set V, and A is a linguistic term labeling a fuzzy set on V.

Fuzzy relation: A fuzzy set on a Cartesian product space, which generalizes the notion of the classical relation.

Fuzzy set: A set of ordered pairs $(x, \mu(x))$, where x is an element of the universal set X and $\mu(x)$ is a membership function, whose values lie in the interval [0,1].

Fuzzy system: A system using linguistic rules or fuzzy relations to map numerical input quantities to numerical output quantities, the steps of this process being fuzzification, inference, and defuzzification.

Inference: The procedure which determines the membership degrees of the linguistic terms of the output variables from the membership degrees of the linguistic terms of the input variables using the given set of rules.

Mamdani-type fuzzy system: A fuzzy system with IF-THEN rules, in which conditions and conclusions contain fuzzy propositions.

Manipulated variable (control signal): The output signal of a controller, which is applied to a controlled plant in order to make it respond in a certain desired way (symbol u).

Linguistic term: A label denoting vague or imprecise concepts, such as 'small', 'large', etc. The meaning of a linguistic term is defined by a fuzzy set.

Linguistic variable: A variable whose values are described by linguistic terms.

Membership function: The characteristic function of a fuzzy set A, $\mu_A(x) : X \to [0,1]$, indicating the degree of membership of a (numerical) element x in the fuzzy set A.

Plant: The object (system) that is to be controlled.

Reference signal (Setpoint): The signal, expressed in the same unit as the controlled variable, that sets the desired values of the controlled variable (symbol r).

Rule base: A set of rules describing the knowledge available for reaching specific objectives.

Singleton fuzzy system: A fuzzy system with IF-THEN rules, in which only the conditions contain fuzzy propositions, whereas the conclusions contain constants.

Takagi-Sugeno-type fuzzy system: A fuzzy system with IF-THEN rules, in which only the conditions contain fuzzy propositions, whereas the conclusions contain expressions which can be a function (e. g. of the variables of the premise) or a conventional control law.

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