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Pursuit, Evasion and Defense in the Plane

Selina Pan†, Haomiao Huang†, Jerry Ding†, Wei Zhang†, Dušan M. Stipanović, and Claire J. Tomlin

***Abstract*— Multi-player games are important for analyzing complex real-world applications that involve both cooperative and adversarial agents, but computational complexity complicates solving such games. We study a modified pursuit-evasion game with multiple pursuers and a single evader, played in a convex domain with an exit through which the evader may escape. We present a strategy whereby one pursuer acts as a defender, utilizing a multi-mode switching strategy to prevent the evader from escaping while the other pursuers subsequently capture the evader. The strategy requires each pursuer to have knowledge only of its Voronoi neighbors and the evader, and runs in real time. The existence and uniqueness of the players' trajectories are proved using non-smooth analysis, and it is also shown that the evader can never reach the exit regardless of its control inputs, resulting in eventual capture. Simulation results are presented demonstrating the algorithm.**

I. Introduction

Complex adversarial games involving multiple agents have many applications in robotics and automation. In addition to security and military applications, game solutions are useful in robust control systems with external disturbances; for example, pursuit-evasion strategies for both aircraft collision avoidance [1], [2] and autopilot safety analysis [3].

An important variant of pursuit-evasion games are situations in which the evader may win by exiting the game region, analogous to such games in reachability analysis, where the objective is not merely to avoid some portion of the state-space, but also arrive safely in a goal set [4], [5]. Some complete solutions may be computed using Hamilton-Jacobi equations [1], [5], [6]; however, such methods are computationally very expensive. Model predictive control (MPC) [7], [8] and optimization [9] have also been employed, assuming certain opponent behavior, minimizing a cost with respect to the player states over a time horizon, and obtaining solutions more quickly than Hamilton-Jacobi-based methods. However, there are no optimality or correctness guarantees due to prediction uncertainty.

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S. Pan is with the Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA slpan@berkeley.edu

H. Huang is with the Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, USA haomiao@stanford.edu

W. Zhang is with the Department of Electrical and Computer Engineering, The Ohio State University, Columbus, OH 43210, USA zhang@ece.osu.edu

J. Ding and C. J. Tomlin are with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720, USA {jding, tomlin}@EECS.Berkeley.edu

D. M. Stipanović is with the Control and Decision Group, Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801, USA dusan@illinois.edu

†These authors contributed equally to this work.

One recent approach to obtaining fast, feasible solutions in pursuit-evasion is to use the evader's Voronoi cell area as a value function for the pursuers to minimize [10]. The Voronoi cell of a player corresponds to the set of all points that player can reach before any other player, assuming equal speeds [11]. Early Voronoi pursuit formulations required knowledge of the evader's control law [12]; however, [10] guaranteed capture for multiple pursuers without this knowledge. There, the pursuers jointly minimized the area of the evader's Voronoi cell, guaranteeing capture by reducing the area to zero.

In this work, we present a Voronoi-based pursuit-evasion-defense game with many pursuers and a single evader. The game occurs in a planar convex polytope with an exit on the boundary through which the evader may escape to win the game. The pursuit strategy we propose assigns one pursuer the objective of *defense*, and the rest of the pursuers the objective of *capture*. Defense is performed by switching between the Voronoi strategy described in [10] and an exit guarding strategy, while capture is performed solely using the Voronoi strategy. The combined strategy generates pursuer control laws in real-time, and requires each pursuer to have knowledge of the states of only its Voronoi neighbors and the evader. We show that, if the game begins in a configuration such that the evader's Voronoi region does not intersect the exit, an evader victory will always be prevented. A single pursuer guarantees that the evader cannot escape, and more than one pursuer guarantees capture. Empirical results also suggest that a single pursuer is sufficient for capture, although a formal proof cannot be currently provided.

It is worth noting that there is strong interest in the control community to use biological systems to inspire automation design. In particular, one may draw inspiration from the animal kingdom in designing multi-player pursuit-evasion strategies. Our work has been inspired by research on predatory behavior of African lions [13]: this paper, with its theme of regions with exits, finds a parallel in the territorial defense employed by African lions [14].

This paper is organized as follows: First, we provide a concise problem formulation in Section II. We go on to describe the pursuit strategy in Section III. Sections IV and V show the existence of strategies and proofs of guaranteed escape prevention and capture, respectively. Simulation results are presented in Section VI, with conclusions and future work presented in Section VII.

II. Problem Formulation

We consider a multiple-player pursuit-evasion game with N_p pursuers and an evader that occurs in the interior of a convex polytope D in $\mathbb{R}^2$, with an exit region E. The goal of the pursuers is to capture the evader while preventing the evader from escaping the game domain through the exit. Capture is defined as having at least one pursuer come within some distance r_c of the evader, and escape is defined as the evader entering region E. Let $\mathbf{e} \in \mathbb{R}^2$ be the position of the evader and $\mathbf{p}_i \in \mathbb{R}^2$ be the position of pursuer i. The positions of the players are constrained to lie within the region D with equations of motion

$$\begin{aligned} \dot{\mathbf{e}} &= \mathbf{u}_e,\ \mathbf{e}(0) = \mathbf{e}_0, \\ \dot{\mathbf{p}}_i &= \mathbf{u}_{p,i},\ \mathbf{p}_i(0) = \mathbf{p}_{i,0}, \end{aligned} \tag{1}$$

where $\mathbf{u}_e$ and $\mathbf{u}_{p,i}$ are the respective velocity control inputs of the evader and pursuer i, and $\mathbf{e}_0$, $\mathbf{p}_{i,0} \in D$ are the initial evader and pursuer positions. The velocity inputs are assumed to be subject to identical speed constraints, namely

$$||\mathbf{u}_e(t)||_2 \leq v_{max},\ ||\mathbf{u}_{p,i}(t)||_2 \leq v_{max},\ \forall t \geq 0,$$

for some maximum speed v_{max}.

The minimum separation distance between the evader and pursuers at any given time t is defined as

$$d_{min}(t) \equiv \min_i ||\mathbf{p}_i(t) - \mathbf{e}(t)||_2.$$

In the rest of the paper, we will write, as shorthand, the Euclidean norm $||\cdot||_2$ simply as $||\cdot||$. Assuming a finite radius of capture $r_c > 0$, the desired capture condition for the pursuers is given by

$$d_{min}(T) \leq r_c, \tag{2}$$

for some $T < \infty$. In order to achieve this capture condition, each pursuer is allowed to select a pursuit strategy $\mu_{p,i}(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_{N_p})$, based upon observations of the evader and pursuer positions at each time instant, resulting in the closed-loop system dynamics:

$$\begin{aligned} &\dot{\mathbf{e}} = \mathbf{u}_e,\ \mathbf{e}(0) = \mathbf{e}_0, \\ &\dot{\mathbf{p}}_i = \mu_{p,i}(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_{N_p}),\ \mathbf{p}_i(0) = \mathbf{p}_{i,0},\ i = 1, ..., N_p. \end{aligned} \tag{3}$$

To ensure the existence and uniqueness of trajectories under (3), we will impose certain technical conditions on the choice of evader input $\mathbf{u}_e$ and the pursuit strategies $\mu_{p,i}$. In particular, we say that an evader input $\mathbf{u}_e$ is admissible if it is piecewise continuous in time, satisfies $||\mathbf{u}_e|| \leq v_{max}$, and $\mathbf{e}(t) \in D$, $\forall t \geq 0$ under the evader dynamics $\dot{\mathbf{e}} = \mathbf{u}_e$, $\mathbf{e}(0) = \mathbf{e}_0 \in D$. We say that a choice of pursuit strategies $\mu_{p,i}$, $i = 1, ..., N_p$ is admissible if:

1) For each i, $\mu_{p,i}$ is a piecewise continuous function of $(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_{N_p})$ and satisfies $||\mu_{p,i}|| \leq v_{max}$;
2) For every admissible choice of evader input $\mathbf{u}_e$, a solution to (3) exists and is unique, in the sense of Filippov [15], and satisfies $\mathbf{p}_i(t) \in D$, for every initial condition $\mathbf{p}_{i,0} \in D$ and $t \geq 0$ such that $d_{min}(t) > r_c$.

We define the exit E as a connected subset of a facet of the polytope D, with the endpoints E_1 and E_2, as shown in Figure 1. In order to prevent the evader from exiting the domain through the exit E, it is necessary that the pursuers keep the joint configuration $(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_{N_p})$ within the following set before capture occurs:

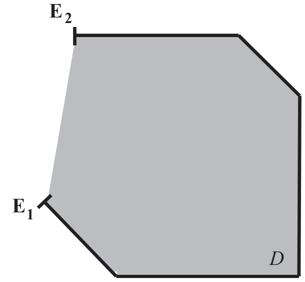

Fig. 1. **Illustration of the game region and the exit, with the endpoints E_1 and E_2 of the exit highlighted.**

$$\begin{aligned} D_p^* := \big\{ &(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_{N_p}) \in D^{N_p+1} : (\exists i, ||\mathbf{p}_i - \mathbf{x}|| \leq ||\mathbf{e} - \mathbf{x}||, \\ &\forall \mathbf{x} \in E) \wedge ||\mathbf{p}_i - \mathbf{e}|| > r_c, \forall i \big\}. \end{aligned} \tag{4}$$

The problem statement is as follows.

Problem 1: For any initial configuration $(\mathbf{e}, \mathbf{p}_{1,0}, ..., \mathbf{p}_{N_p,0}) \in D_p^*$, find an admissible choice of pursuit strategy $\mu_{p,i}$ for each pursuer i such that, regardless of any admissible choice of evader input $\mathbf{u}_e$, the capture condition (2) is satisfied for some $T < \infty$ and the trajectories of (3) satisfy $(\mathbf{e}, \mathbf{p}_1, ..., \mathbf{p}_{N_p})(t) \in D_p^*, \forall t \in (0, T)$.

Without loss of generality, we can assume that $v_{max} = 1$ for the rest of this paper by appropriate re-scaling of the dynamics (3), the polytope D, and the capture radius r_c.

III. Pursuer Strategy

The pursuer strategy we propose to capture the evader while preventing its escape is based on assigning one pursuer, called the defender, to defend the exit while the other pursuers attempt to capture the evader. The pursuers utilize a pursuit strategy that attempts to minimize the area of the Voronoi cell of the evader, while the defender utilizes a switching strategy that alternates between pursuit and a specialized defense strategy that prevents the evader's Voronoi cell from intersecting the exit.

First, we will describe some mathematical properties of this dual pursuit strategy necessary for the proof of finite time capture and escape prevention. Let $\mathscr{V}(D) = \{V_e, V_{p,i}\}$ be the Voronoi partition of D generated by the points $\{\mathbf{e}, \mathbf{p}_1, ...\mathbf{p}_{N_p}\}$. We denote the evader's Voronoi partition as V_e and the partition for each pursuer i as $V_{p,i}$, defined as

$$\begin{aligned} V_e &= \{\mathbf{x} \in D : \|\mathbf{x} - \mathbf{e}\| < \|\mathbf{x} - \mathbf{p}_i\|, \forall i \leq N\}, \\ V_{p,i} &= \big\{\mathbf{x} \in D : \|\mathbf{x} - \mathbf{p}_i\| \leq \\ &\qquad \min\{\|\mathbf{x} - \mathbf{e}\|, \|\mathbf{x} - \mathbf{p}_j\|\}, \forall j \neq i\big\}, i \leq N. \end{aligned}$$

The edge shared by V_e and $V_{p,i}$ is called a *line of control*, denoted by B_i, where L_i is the length of B_i. Note that B_i is not contained in V_e, since if both a pursuer and evader arrive at the same point at the same time the evader is captured. We denote by A the area of the Voronoi cell V_e containing the evader. It can be easily verified that the area A depends only on the locations of the players and that this dependence is smooth whenever the locations are in D.

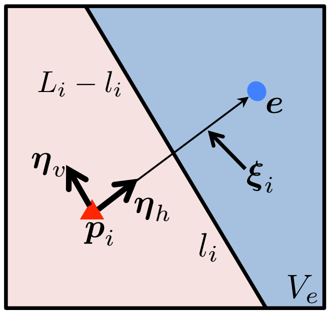

Fig. 2. **Figure showing the evader's Voronoi partition and the necessary vectors and definitions for the pursuit strategy.**

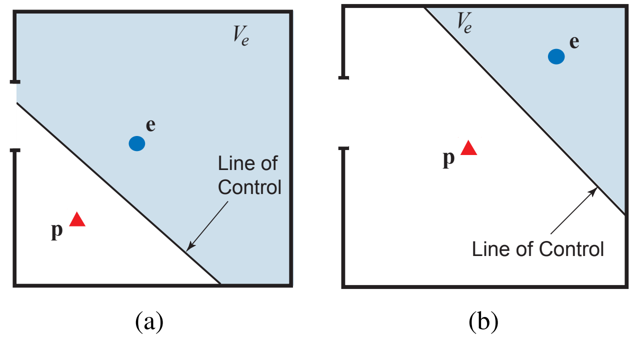

Fig. 3. **Illustrations showing the evader's Voronoi cell V_e (a) with exit intersection and consequent escape and (b) in the desired case of the pursuer preventing evader Voronoi cell exit intersection.**

A. Pursuit Strategy

We first consider the pursuit case, based on a strategy that attempts to minimize A at every instant in time, as described in [10]. The time derivative of A is given by

$$\frac{dA}{dt} = \frac{\partial A}{\partial \mathbf{e}}\dot{\mathbf{e}} + \sum_{i=1}^{N} \frac{\partial A}{\partial \mathbf{p}_i}\dot{\mathbf{p}}_i \,. \tag{5}$$

This joint objective can be decoupled into the individual objectives of minimizing $\frac{\partial A}{\partial \mathbf{p}_i}\dot{\mathbf{p}}_\mathbf{i}$ for each pursuer i. Let $\mathscr{N}_e$ be the set of pursuers that share a Voronoi boundary with the evader. Since A depends only on the Voronoi neighbors of the evader, we have $\frac{\partial A}{\partial \mathbf{p}_i} = 0$ for all $i \notin \mathscr{N}_e$. Thus, for any pursuer i which is not a Voronoi neighbor of the evader, we may simply set $\mathbf{u}_i = \frac{\mathbf{e}-\mathbf{p}_i}{||\mathbf{e}-\mathbf{p}_i||}$. On the other hand, for each pursuer $i \in \mathscr{N}_e$, the choice of pursuit strategy which minimizes (5) is given by:

$$\mathbf{u}_i^* = \mu_{p,i}^0(\mathbf{e}, \mathbf{p}_1, \ldots, \mathbf{p}_{N_p}) \triangleq \frac{-\frac{\partial A}{\partial \mathbf{p}_i}}{||\frac{\partial A}{\partial \mathbf{p}_i}||},$$

where we make use of the assumption that $v_{max} = 1$. Furthermore, we introduce a local coordinate system in which $\xi_i = \mathbf{e} - \mathbf{p}_i$ represents the vector pointing from the location of pursuer i towards the location of the evader. Define $\eta_h^i = \frac{\xi_i}{||\xi||}$ and let $\eta_v^i \in \mathbb{R}^2$ be the unit vector orthogonal to η_h^i. Thus, the change in area of the evader's cell is

$$\frac{\partial A}{\partial \mathbf{p}_i} = D_h^i A \cdot \eta_h^i + D_v^i A \cdot \eta_v^i \,, \tag{6}$$

where $D_h^i A$ and $D_v^i A$ can be shown to be

$$D_h^i A = -\frac{L_i}{2}, \; D_v^i A = \frac{l_i^2 - (L_i - l_i)^2}{2||\xi_i||}.$$

Recall that L_i is the length of the line of control B_i; we also denote l_i as the length of the segment of B_i opposite η_v, as shown in Figure 2. A more detailed derivation of the analytical expression of $\mu_{p,i}^0$ can be found in [10].

B. Defense Strategy

Now we examine the defense case based on a blocking scheme in which the defender takes advantage of the properties of the Voronoi decomposition. As the players have equal speeds, the evader's Voronoi partition V_e represents the set of points that the evader can reach before any pursuer. Therefore, if the intersection of the evader's Voronoi cell V_e and the exit E is non-empty, the evader may be able to escape if r_c is sufficiently small, as shown in Figure 3.

For the purposes of defense, we will take a conservative approach that prevents any intersection of V_e with E, through the control actions of a single defender. First, consider a modification of the definition of D_p^* for the case of a single evader and a single defender:

$$\begin{aligned} D_p := \{ & (\mathbf{e}, \mathbf{p}) \in D^2 : ||\mathbf{p} - \mathbf{x}|| \le ||\mathbf{e} - \mathbf{x}||, \forall \mathbf{x} \in E \\ & \wedge ||\mathbf{p} - \mathbf{e}|| > r_c \} \,. \end{aligned}$$

Note that under the assumptions on the initial configuration $(\mathbf{e}, \mathbf{p}_{1,0}, \ldots, \mathbf{p}_{N_p,0})$, there exists at least one player i such that $(\mathbf{e}, \mathbf{p}_{i,0}) \in D_p$. By assigning this player the role of the defender throughout the pursuit-evasion game, we can reduce the exit defense problem to constructing an admissible control policy $\mu_{p,i}^D$ such that the configuration $(\mathbf{e}, \mathbf{p}_i)$ remains in D_p at all times, regardless of any admissible choice of evader input. For the rest of this section, we will omit the player index i for the defender, so as to simplify notation.

The proposed defense strategy is a switching control law which activates when the defender and the evader are equidistant from one of the endpoints of the exit E. In particular, consider the switching boundaries $S_j \subset D_p$ for

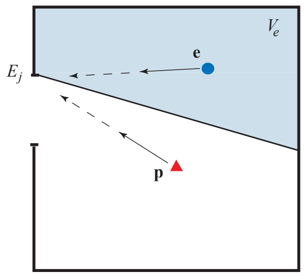

Fig. 4. **Illustrations showing E_j and the corresponding defense strategy.**

$j = 1,2$ defined as

$$S_j := \{(\mathbf{e},\mathbf{p}) \in D_p : ||\mathbf{E}_j - \mathbf{p}|| = ||\mathbf{E}_j - \mathbf{e}||\}.$$

Together with the boundary of the capture set, $S_1 \cup S_2$ forms the boundary which D_p shares with its complement in $D^2 \subset \mathbb{R}^4$. It can be checked that $S_1 \cap S_2 = \emptyset$.

For configurations of the evader and defender lying on the boundary S_j, we propose the following defense strategy.

$$\mu_p^j(\mathbf{e},\mathbf{p}) = \frac{\mathbf{E}_j - \mathbf{p}}{||\mathbf{E}_j - \mathbf{p}||},\ (\mathbf{e},\mathbf{p}) \in S_j,\ j = 1,2,$$

Intuitively, this corresponds to the defender moving towards the exit point $\mathbf{E}_j$ at maximum velocity in order to cut off the evader from the exit (see Figure 4). The overall strategy of the defender is defined as

$$\mu_p^D(\mathbf{e},\mathbf{p}) = \begin{cases} \mu_p^0(\mathbf{e},\mathbf{p}), & (\mathbf{e},\mathbf{p}) \in D_p \setminus (S_1 \cup S_2) \\ \mu_p^j(\mathbf{e},\mathbf{p}), & (\mathbf{e},\mathbf{p}) \in S_j. \end{cases} \quad (7)$$

Notice that the vector field defined by μ_p^D on the configuration space of evader and pursuer positions is only piecewise continuous. As such, we will discuss trajectories under (7) in terms of the Filippov solution concept.

IV. Existence and Uniqueness of Trajectories

We begin with some mathematical preliminaries. First, observe that the surface S_j is characterized by the zero level set of a function $s_j : D^2 \to \mathbb{R}$ defined as

$$s_j(\mathbf{e},\mathbf{p}) = ||\mathbf{E}_j - \mathbf{p}|| - ||\mathbf{E}_j - \mathbf{e}||,\ j = 1,2.$$

Since s_j is smooth on D^2, S_j is a smooth submanifold of D^2. Clearly, in the interior of the domain D_p, we have $s_j < 0$. In order to characterize the behavior of the solution trajectories along the switching boundaries S_j, it is necessary to extend the exit guarding strategy μ_p^j into the complement of D_p, so that it is defined on a set of configurations $(\mathbf{e},\mathbf{p}) \in D^2$ with non-zero measure. In particular, consider a set S_j^+ in the complement of D_p as defined by

$$S_j^+ := \{(\mathbf{e},\mathbf{p}) \in D^2 \setminus D_p : ||\mathbf{E}_j - \mathbf{p}|| < ||\mathbf{E}_j - \mathbf{e}|| + \varepsilon, \\ \wedge ||\mathbf{p} - \mathbf{e}|| > r_c\},$$

where $\varepsilon > 0$ characterizes the width of the region S_j^+. From the definitions of the domains D and D_p, it follows that S_j^+ is an open subset of D^2, and shares the boundary S_j with D_p. Furthermore, one can show by geometric arguments that, in a bounded configuration space D^2, there exists a choice of ε sufficiently small such that S_1^+ and S_2^+ are disjoint. Finally, it can be observed that $s_j > 0$ on S_j^+.

In the interior of D_p, where $s_j < 0$, we define a vector field $f^- : D_p^\circ \times [0,\infty) \to \mathbb{R}^4$ as

$$f^-(\mathbf{e},\mathbf{p},t) = \begin{bmatrix} \mathbf{u}_e(t) \\ \mu_p^0(\mathbf{e},\mathbf{p}) \end{bmatrix}.$$

On the set $S_j \cup S_j^+$, where $s_j \geq 0$, we define a vector field $f_j^+ : S_j \cup S_j^+ \times [0,\infty) \to \mathbb{R}^4$ as

$$f_j^+(\mathbf{e},\mathbf{p},t) = \begin{bmatrix} \mathbf{u}_e(t) \\ \mu_p^j(\mathbf{e},\mathbf{p}) \end{bmatrix}.$$

By the smoothness of the maps μ_p^0 and μ_p^j, both f^- and f_j^+ are continuously differentiable with respect to $(\mathbf{e},\mathbf{p})$ on their respective domains. Now consider a piecewise continuous vector field f defined on the open domain $D_p \cup S_1^+ \cup S_2^+$ (relative to the configuration space D^2):

$$f(\mathbf{e},\mathbf{p},t) = \begin{cases} f^-(\mathbf{e},\mathbf{p},t), & (\mathbf{e},\mathbf{p}) \in D_p \setminus (S_1 \cup S_2) \\ f_j^+(\mathbf{e},\mathbf{p},t), & (\mathbf{e},\mathbf{p}) \in S_j \cup S_j^+,\ j = 1,2. \end{cases} \quad (8)$$

We associate with each point $(\mathbf{e},\mathbf{p})$ in the domain of f the following set-valued map [15]:

$$F(\mathbf{e},\mathbf{p},t) = \bigcap_{\delta > 0} \bigcap_{\nu(N)=0} \overline{\text{co}}\{f(B_\delta(\mathbf{e},\mathbf{p}) \setminus N, t)\}, \quad (9)$$

where $\overline{\text{co}}$ is the convex closure of a set, $B_\delta(\mathbf{e},\mathbf{p}) \subset \mathbb{R}^4$ is a ball of radius δ centered on $(\mathbf{e},\mathbf{p})$, ν is the Lebesgue measure on $\mathbb{R}^4$, and N represents any set of measure zero. For the piecewise continuous vector field in (8), it can be shown that for every point in the interior of D_p and the sets S_j^+, $j = 1,2$, this map is simply given by the singleton $f(\mathbf{e},\mathbf{p},t)$, whereas on the switching boundary S_j, the set $F(\mathbf{e},\mathbf{p},t)$ is the convex hull of the vector fields f^- and f_j^+:

$$F(\mathbf{e},\mathbf{p},t) = \text{co}\left\{f_*^-(\mathbf{e},\mathbf{p},t), f_j^+(\mathbf{e},\mathbf{p},t)\right\},\ (\mathbf{e},\mathbf{p}) \in S_j,$$

where f_*^- denotes the continuous extension of the vector field f^- onto S_j (this is possible by the smoothness of μ_p^0).

The Filippov solution under the vector field f is then defined as follows [15].

Definition 1: An absolutely continuous function $(\mathbf{e}(\cdot), \mathbf{p}(\cdot)) : [0,T] \to \mathbb{R}^4$ is said to be a *Filippov solution* to

$$(\dot{\mathbf{e}}, \dot{\mathbf{p}}) = f(\mathbf{e}, \mathbf{p}, t), \ (\mathbf{e}(0), \mathbf{p}(0)) \in D_p \cup S_1^+ \cup S_2^+ \tag{10}$$

if the following differential inclusion is satisfied

$$(\dot{\mathbf{e}}(t), \dot{\mathbf{p}}(t)) \in F(\mathbf{e}(t), \mathbf{p}(t), t) \tag{11}$$

for almost every $t \in [0,T]$.

Since the vector field f is measurable and bounded, we have by Theorem 8 in Chapter 2, Section 7 of [15] that there exists a Filippov solution to (10) from any initial condition $(\mathbf{e}(0), \mathbf{p}(0)) \in D_p \cup S_1^+ \cup S_2^+$. It turns out that the solution is also unique. In order to show this, consider a function $g_j^- : D_p^\circ \times [0,\infty) \to \mathbb{R}$, which characterizes the time derivative of s_j along trajectories under the vector field f^-:

$$g_j^-(\mathbf{e}, \mathbf{p}, t) = \frac{ds_j}{dt}|_{f^-} = \\ -\frac{(\mathbf{E}_j - \mathbf{p})^T}{||\mathbf{E}_j - \mathbf{p}||}\mu_p^0(\mathbf{e}, \mathbf{p}) + \frac{(\mathbf{E}_j - \mathbf{e})^T}{||\mathbf{E}_j - \mathbf{e}||}\mathbf{u}_e(t).$$

We note that if $g_j^-(\mathbf{e}, \mathbf{p}, t) > 0$, then the vector field f^- at $(\mathbf{e}, \mathbf{p}, t)$ is pointing towards the boundary S_j, whereas $g_j^- < 0$ implies that f^- is pointing towards the interior of the set D_p. By the smoothness of μ_p^0, the function g_j^- can be extended continuously up to the boundary S_j. Similarly, define a function $g_j^+ : S_j \cup S_j^+ \times [0,\infty) \to \mathbb{R}$ as

$$g_j^+(\mathbf{e}, \mathbf{p}, t) = \frac{ds_j}{dt}|_{f_j^+} = -1 + \frac{(\mathbf{E}_j - \mathbf{e})^T}{||\mathbf{E}_j - \mathbf{e}||}\mathbf{u}_e(t).$$

Now observe that given $||\mathbf{u}_e|| \le 1$, we have $g_j^+ \le 0$, where equality is achieved if and only if $\mathbf{u}_e(t) = \frac{(\mathbf{E}_j - \mathbf{e})}{||\mathbf{E}_j - \mathbf{e}||}$. Furthermore, it can be checked that $g_j^- > 0$ whenever $g_j^+ = 0$ on S_j. This implies that everywhere on the switching boundary S_j, we have either a vector pointing from S^- to S_j^+ or vice versa. Then by Theorem 3 in Chapter 2, Section 10 of [15], there exists a unique Filippov solution to (10) for every initial condition on $D_p \cup S_1^+ \cup S_2^+$.

V. Proof of Defense Invariance and Capture

By the discussions of the preceding section, the switching policy μ_p^D for the defender, combined with the Voronoi pursuit strategy $\mu_{p,i}^0$ for the other pursuers, is an admissible choice of control strategy. We will now proceed to show that this control strategy in fact maintains D_p as an invariant set for the defender, while ensuring finite-time capture of the evader through the Voronoi pursuit strategy.

A. Invariance of Defender Set

First, we will show that the defense strategy μ_p^D prevents the trajectories of the evader and pursuer from exiting the domain D_p through the switching boundaries S_j.

Proposition 1: For every initial condition $(\mathbf{e}(t_0), \mathbf{p}(t_0)) \in S_j$, $t_0 \ge 0$, and admissible evader input $\mathbf{u}_e$, the Filippov solution to (10) on a time interval $[t_0, t_1]$ satisfies either $(\mathbf{e}(t), \mathbf{p}(t)) \in D_p, \forall t \in [t_0, t_1]$, or $||\mathbf{e}(T) - \mathbf{p}(T)|| \le r_c$ for some $T \in [t_0, t_1]$ and $(\mathbf{e}(t), \mathbf{p}(t)) \in D_p, \forall t \in [t_0, T)$.

Proof: Let $(\mathbf{e}(t_0), \mathbf{p}(t_0)) \in S_j$, $t_0 \ge 0$, and $\mathbf{u}_e$ be an admissible evader input. We consider several cases.

Case 1: $g_j^-(\mathbf{e}(t_0), \mathbf{p}(t_0), t_0) < 0$. By the fact that μ_p^0 is continuously differentiable on D_p and the assumption that $\mathbf{u}_e$ is piecewise continuous, g_j^- is continuous in $(\mathbf{e}, \mathbf{p})$ and piecewise continuous in t. By separately considering each time interval on which $\mathbf{u}_e$ is continuous, we can assume that g_j^- is also continuous in t. This implies that there exists $r > 0$ and $\Delta t > 0$ such that for every $(\mathbf{e}, \mathbf{p}) \in B_r(\mathbf{e}(t_0), \mathbf{p}(t_0)) \cap D_p^\circ$ and $|t - t_0| < \Delta t$, we have $g_j^-(\mathbf{e}, \mathbf{p}, t) < 0$. By the observation that $g_j^+ \le g_j^-$ on S_j, this also implies $g_j^+(\mathbf{e}, \mathbf{p}, t) < 0$ on the neighborhood of interest. Thus, both f^- and f_j^+ in a neighborhood of $(\mathbf{e}(t_0), \mathbf{p}(t_0), t_0)$ is directed away from the boundary S_j and into the interior of D_p. Hence, for a time interval $[t_0, t_0 + \delta]$, $\delta > 0$ sufficiently small, the Filippov solution starting from $(\mathbf{e}(t_0), \mathbf{p}(t_0))$ lies in D_p.

Case 2: $g_j^-(\mathbf{e}(t_0), \mathbf{p}(t_0), t_0) > 0$. As in the previous case, the continuity of g_j^- implies that there exists $r > 0$ and $\Delta t > 0$ such that for every $(\mathbf{e}, \mathbf{p}) \in B_r(\mathbf{e}(t_0), \mathbf{p}(t_0)) \cap D_p^\circ$ and $|t - t_0| < \Delta t$, we have $g_j^-(\mathbf{e}, \mathbf{p}, t) > 0$. Given that $g_j^+ \le 0$ on S_j, the vector fields f^- and f_j^+ are both directed towards the boundary in a neighborhood of $(\mathbf{e}(t_0), \mathbf{p}(t_0), t_0)$. Hence, for a time interval $[t_0, t_0 + \delta]$ sufficiently small, the Filippov solution is described by a vector field f^* in the convex hull of f^- and f_j^+ such that $(\mathbf{e}(t), \mathbf{p}(t)) \in S_j, \forall t \in [t_0, t_0 + \delta]$ [15]. In particular, one can show that f^* is given by

$$f^*(\mathbf{e}, \mathbf{p}, t) = \lambda(\mathbf{e}, \mathbf{p}, t) f_j^+(\mathbf{e}, \mathbf{p}, t) + (1 - \lambda(\mathbf{e}, \mathbf{p}, t)) f^-(\mathbf{e}, \mathbf{p}, t),$$

where $\lambda(\mathbf{e}, \mathbf{p}, t) = \frac{g_j^-(\mathbf{e}, \mathbf{p}, t)}{g_j^-(\mathbf{e}, \mathbf{p}, t) - g_j^+(\mathbf{e}, \mathbf{p}, t)}$.

Case 3: $g_j^-(\mathbf{e}(t_0), \mathbf{p}(t_0), t_0) = 0$. As observed during the discussion on the uniqueness of solutions, we have in this case that $g_j^+(\mathbf{e}(t_0), \mathbf{p}(t_0), t_0) < 0$. The analysis for this scenario is somewhat more involved due to the fact that g_j^- may be positive, negative, or zero in any neighborhood of $(\mathbf{e}(t_0), \mathbf{p}(t_0), t_0)$. However, it can be shown, through the use of the implicit function theorem, that the Filippov solution to (10) in a neighborhood of $(\mathbf{e}(t_0), \mathbf{p}(t_0), t_0)$ is confined to $S_j \cup D_p^\circ$ (see Theorem 3 in Chapter 2, Section 10 of [15]).

Combining these results, we have that the Filippov solution on any time interval $[t_0, t_1]$ does not exit the domain D_p through the boundaries S_1 and S_2. Together with the fact that the pursuit strategy μ_p^0 prevents trajectories from exiting D_p through the boundaries of D^2 (see Lemma 2 of [10]), the desired conclusion follows. ■

Using this proposition, it is then straightforward to prove the following invariance result.

Theorem 1: For every initial condition $(\mathbf{e}(0), \mathbf{p}(0)) \in D_p$, and admissible evader input $\mathbf{u}_e$, the Filippov solution to (10) satisfies either $(\mathbf{e}(t), \mathbf{p}(t)) \in D_p$, $\forall t \in [0, \infty)$, or $||\mathbf{e}(T) - \mathbf{p}(T)|| \leq r_c$ for some $T < \infty$ and $(\mathbf{e}(t), \mathbf{p}(t)) \in D_p$, $\forall t \in [0, T)$.

Proof: Assume without loss of generality that $(\mathbf{e}(0), \mathbf{p}(0)) \in D_p \setminus (S_1 \cup S_2)$. We can solve for the trajectory $(\mathbf{e}(\cdot), \mathbf{p}(\cdot))$ using the vector field f^- until the first time instant $t_1^* \geq 0$ such that $(\mathbf{e}(t_1^*), \mathbf{p}(t_1^*)) \in S_j$ for some $j = 1, 2$ or $||\mathbf{e}(t_1^*) - \mathbf{p}(t_1^*)|| = r_c$ (note again that μ_p^0 prevents trajectories from exiting the configuration space D^2). If the latter condition is not satisfied, namely the evader is not captured, then we can solve for the trajectory $(\mathbf{e}(\cdot), \mathbf{p}(\cdot))$ starting from $(\mathbf{e}(t_1^*), \mathbf{p}(t_1^*))$ until a time $t_2^* > t_1^*$ at which $(\mathbf{e}(t_2^*), \mathbf{p}(t_2^*)) \notin S_j$. By Proposition 1, we have either $(\mathbf{e}(t), \mathbf{p}(t)) \in D_p$, $\forall t \in [t_1^*, t_2^*]$ or $||\mathbf{e}(T) - \mathbf{p}(T)|| \leq r_c$ for some $T \in [t_1^*, t_2^*]$. Thus, if the evader is not captured during that time, then $(\mathbf{e}(t_2^*), \mathbf{p}(t_2^*)) \in D_p \setminus (S_1 \cup S_2)$ and the same process repeats. The desired conclusion then follows by induction. ■

B. Guaranteed Capture

With $N > 1$ pursuers, there are sufficient pursuers to have a single defender protecting the exit while the other pursuers purely utilize the pursuit strategy. In this case, capture is guaranteed via a straightforward extension of Theorem 1 from [10], which states that a bound can be found on the area $A(t) = A(\mathbf{e}(t), \mathbf{p}(t))$ of V_e at time t of the form

$$A(t) \leq A(0) + \alpha_1 - \alpha_2 t, \tag{12}$$

where α_1 and α_2 are positive constants defined by the geometry of the game region and $A(0)$ is the initial area of the evader's Voronoi region. As time increases, this upper bound approaches 0, guaranteeing capture of the evader.

The defender in our modified game provides one of the boundaries of the evader's Voronoi partition. Note that this boundary may change in such a way as to temporarily increase the area A, resulting in the following modification to Equation (12):

$$A(t) \leq A(0) + \hat{A}(t) + \alpha_1 - \alpha_2 t,$$

where $\hat{A}(t)$ is the additional area introduced through the defender's actions. However, as the defender strategy forces the evader to remain within the game region we have that $\hat{A}(t)$ must be bounded and finite, with $\hat{A}(t) \leq \hat{A}_{max}$ for some constant $\hat{A}_{max}$, thus maintaining the original capture result.

Remark 1: Although the proof here is presented with the role of defender and pursuers permanently assigned, we conjecture that the overall result of escape prevention and capture still hold when all pursuers are allowed to freely switch between defense and pursuit. This conjecture agrees well with our simulation results. However, its formal proof has yet to be found.

VI. Results

Simulation results are presented here demonstrating the pursuit and defense strategies. The simulations are conducted in a 10 x 10 square, with a maximum speed of 1 for all players, and time steps of 0.01. For the purposes of these simulations the trajectory of the evader is controlled by human input.

Figures 5 shows an example game played with three pursuers and one evader. The far left pursuer acts as the defender, and as the evader's Voronoi region (highlighted in blue) approaches the exit it switches to the appropriate defense strategy. Note the sharp kink in the defender's trajectory, visible in Figure 5(d). The other pursuers utilize the Voronoi pursuit strategy. The natural cooperation created through the Voronoi intersections causes the pursuers to surround the evader, eventually collapsing its Voronoi area. With nowhere to go, the evader is captured.

VII. Conclusions and Future Work

In this paper we have presented a pursuit-evasion game with exit defense for one evader and multiple pursuers in a bounded, convex polytope. We have proposed and demonstrated a switching pursuer strategy that switches between pursuit via minimizing the evader's Voronoi cell, and defending the exit. This strategy guarantees the evader cannot escape, and also guarantees capture with more than one pursuer. We plan to extend this work in the future in various ways, including adding more exits and operating in non-convex game domains with obstacles. We also hope to connect the results with biological data about predator-prey relations, particularly those of African lions (Panthera leo) in hunting and territorial defense [14].

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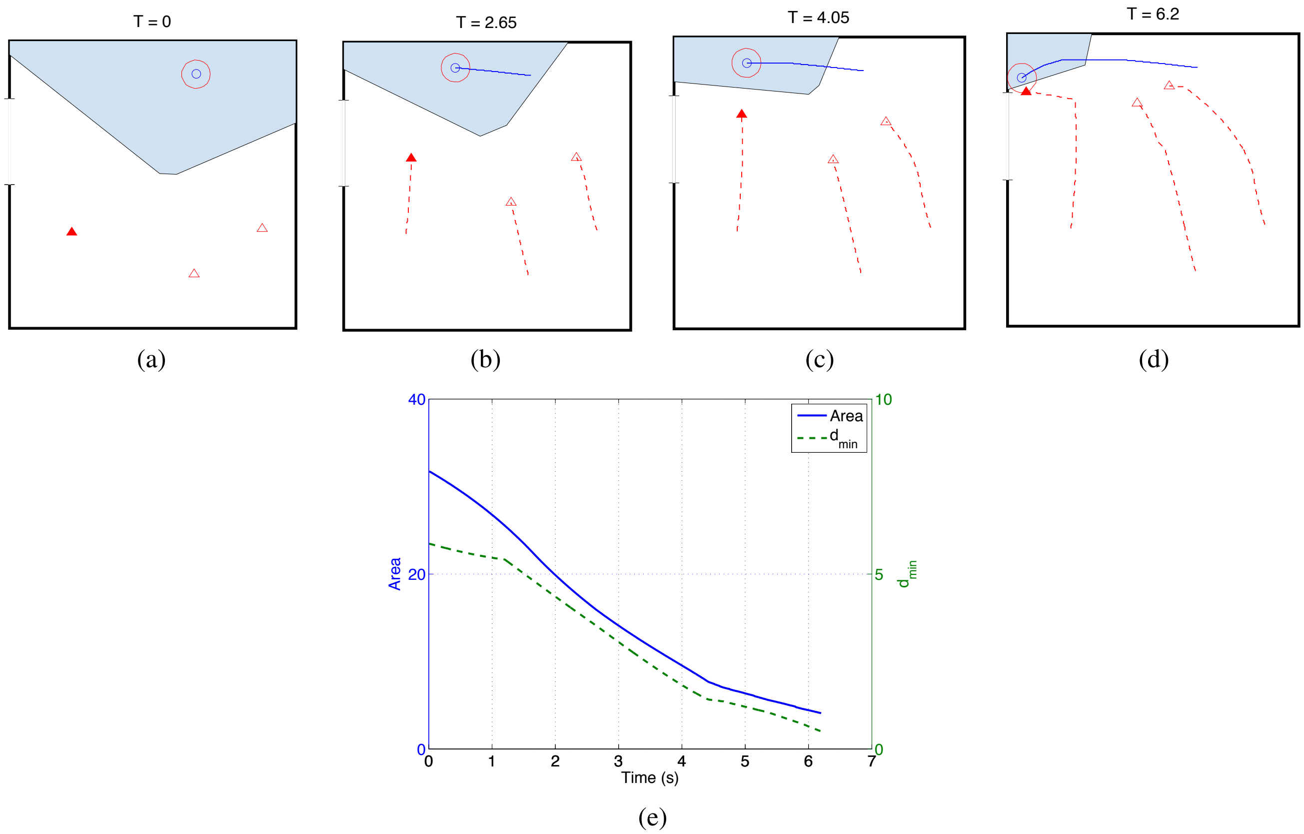

Fig. 5. **Simulation results for one evader (blue circle) versus three pursuers (red triangles, where solid triangle is the defender), showing the (a) the initial configuration, (b-d) the trajectories of the players through the game, and (e) the area of V_e and d_{min} over time.**

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