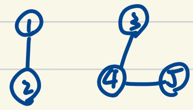
4. Consider a graph that has k connected components $S_1, S_2, \dots, S_k$, and let the j-th element of the vector $u^{(i)}$: $(u^{(i)})_j = \begin{cases} 1, & \text{if } v_j \in S_i \\ 0, & \text{otherwise.} \end{cases}$

Note that:

① The vectors $u^{(i)}$ and $u^{(j)}$ are orthogonal if $i \neq j$ (e.g.,



$u^{(1)} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $u^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$) $(u^{(1)})^T u^{(2)} = 0$

② And $Lu^{(i)} = 0$ for $i \in \{1, 2, \dots, k\}$ (from problem 2.)

(0 is the eigenvalue of the vector of 1_S)

From ②, we can see that there are at least k zero eigenvalues of L.

To prove there are at most k zero eigenvalues of L:

If there exist an vector u that is an eigenvector of L and the corresponding eigenvalue is 0, then $u^T L u = u^T \cdot 0 \cdot u = 0$.

But observe that:

$u^T L u = \underbrace{\frac{1}{2} \sum_{i,j : (v_i, v_j) \in E} (u_i - u_j)^2}_{\text{from lecture slides}} = 0$ only when $u_i = u_j$, $\forall (v_i, v_j) \in E$

$\Rightarrow$ All vertices in a connected component should have equal value.

For such a vector u, it can always be expressed by a linear combination of $u^{(1)}, u^{(2)}, \dots, u^{(k)}$:

$u = \sum_{i=1}^{k} a_i u^{(i)}$,

so u is not orthogonal to $u^{(1)}, u^{(2)}, \dots$, and $u^{(k)}$.

$\Rightarrow$ There are at most k eigenvectors of L that have an eigenvalue of 0.

($\because$ L is symmetric, and eigenvectors of a symmetric matrix are orthogonal.)

$\Rightarrow$ The number of connected components is the number of zero eigenvalues that L has. (the nullity of L)