1. Consider a graph with m edges and n vertexs, and let the incidence matrix $F_{m\times n} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & & \vdots \\ \vdots & & \ddots & \\ f_{m1} & \cdots & & f_{mn} \end{bmatrix}$

$(F^T F)_{kj}$ will be $\sum_{i=1}^{n} f_{ik} \cdot f_{ij}$

$\downarrow$

the k-th row, j-th column element

① If $k=j$. $\sum_{i=1}^{n} f_{ik} \cdot f_{ij} = \sum_{i=1}^{n} f_{ik}^2 = \deg(v_k)$ $\because$ f_{ik} is either $1, -1$, or 0

$f_{ik}^2 = \begin{cases} 1 & \text{if } (v_i, v_k) \in E \\ 0 & \text{otherwise} \end{cases}$

The matrix that collects all $k=j$ entries will thus be D.

② For $k \neq j$, $\sum_{i=1}^{n} f_{ik} \cdot f_{ij} = \begin{cases} -1 & \text{if } e_i \text{ is the edge connecting } v_k \text{ and } v_j \\ 0 & \text{otherwise} \end{cases} \Rightarrow -A$

$F^T F$ = matrix of ① + matrix of ② $= D - A$

2. ① $L = F^T F$

$x^T F^T F x = (Fx)^T (Fx) = \|Fx\|^2 \geq 0$

$\therefore$ L is p.s.d. $\quad \downarrow$ vector

② $L = D - A$

sum of row i of D: $\sum_j D_{ij} = \deg(v_i)$

sum of row i of A: $\sum_j A_{ij} = |\{v_j \in V : (v_i, v_j) \in E\}|$

$= \deg(v_i)$

sum of row i of L: $\sum_j D_{ij} - A_{ij} = 0$

The above is equivalent to $L \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \underset{\uparrow \atop \text{eigenvalue}}{0} \cdot \underset{\nearrow \atop \text{eigenvector}}{\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}$

(Multiplication by the vector of 1s sums over each row)

3. (a) # non-zero entries: A, F, L $\quad n = |V|, m = |E|$

$A_{n\times n}$: $2m$

$F_{m\times n}$: $2m$

$L_{n\times n}$: $n + 2m$

(b) $\sum_{v_i \in V} \deg(v_i) = \sum_{v_i \in V} |N_{v_i}| = \sum_{v_i \in V} |\{v_j \in V : (v_i, v_j) \in E\}|$

$= |\{v_i \in V, v_j \in V : (v_i, v_j) \in E\}|$

$= 2|E|$

[$\because$ (v_i, v_j) and (v_j, v_i) refer to the same edge

$\therefore$ Need a factor 2