1. (a) $P: \mathbb{R}^n \to W \subseteq \mathbb{R}^n$

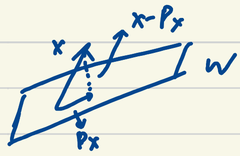
Let $u = \{u_1, u_2, \cdots, u_n\}$ be a basis of W and write matrix $A = [u_1\ u_2 \cdots u_n]$.

Then P can be written as $A(A^TA)^{-1}A^T$

Let x be any vector in $\mathbb{R}^n$

$\|P(x)\|_2^2 = \|Px\|_2^2$ $\qquad$ $\langle Px, x-Px \rangle = 0$

$= \langle Px, Px \rangle$ $\qquad$ $\Rightarrow \langle Px, x \rangle = \langle Px, Px \rangle$



$= \langle Px, x \rangle$

$\leq \|Px\|_2^2 \cdot \|x\|_2^2$

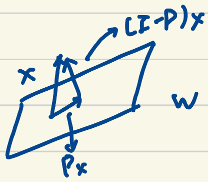
$\|Px\|_2 \cdot \|Px\|_2 \leq \|Px\|_2 \cdot \|x\|_2$

$\|Px\|_2 \leq \|x\|_2$

$\|Px\|_2 \leq \|P\|_2 \|x\|_2 \leq \|x\|_2$

$\Rightarrow \|P\|_2 \leq 1$

(b) $P^\perp = I_n - P$



$x = Px + (I_n - P)x$

$\langle Px, (I_n - P)x \rangle$

$= \langle Px, I_n x \rangle - \langle Px, Px \rangle$

$= \langle Px, I_n x \rangle - \langle Px, x \rangle$

$= \langle Px, I_n x \rangle - \langle Px, I_n x \rangle$

$= 0$

① To show that $I_n - P$ is a projection, we demonstrate that $(I_n - P) \circ (I_n - P) = (I_n - P)$

$(I_n - P) \cdot (I_n x - Px)$

$= (I_n - 2P + P^2) \cdot x$

$= (I_n - 2P + P) \cdot x$

$= (I_n - P) \cdot x$

$\Rightarrow (I_n - P) \circ (I_n - P) = (I_n - P)$

② To show that $I_n - P$ is orthogonal, we demonstrate that $I_n - P = (I_n - P)^T$

$(I_n - P)x$

$= I_n x - Px$

$= I_n x - P^T x$ (given that P is an orthogonal projection)

$= (I_n - P^T)x$

$= (I_n - P)^T x$

$\Rightarrow (I_n - P) = (I_n - P)^T$

$P^T P x = (I_n - P)Px = Px - P^2 x = Px - Px = 0$ $\qquad$ $\therefore P^T P = T_0$

$P P^T x = Px - P^2 x = 0$ $\qquad$ $\therefore P P^T = T_0$ where T_0 is the zero transformation.

2. $C_n = I_n - \frac{1}{n} 1_n 1_n^T$

(a) show that C_n is symmetric and positive semidefinite

① symmetric : $C_n = C_n^T$

$$(I_n - \frac{1}{n} 1_n 1_n^T)^T = I_n^T - \frac{1}{n}(1_n 1_n^T)^T$$
$$= I_n - \frac{1}{n} 1_n 1_n^T$$

② positive semidefinite : $x^T C_n x \geq 0$, $\forall x \in \mathbb{R}^n$

$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_j$

$$x^T (I_n - \frac{1}{n} 1_n 1_n^T) x$$
$$= x^T \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} \frac{1}{n}\sum_{j=1}^{n} x_j \\ \frac{1}{n}\sum_{j=1}^{n} x_j \\ \vdots \\ \frac{1}{n}\sum_{j=1}^{n} x_j \end{bmatrix} \right) = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{j=1}^{n} x_j = \sum_{i=1}^{n} x_i^2 - x_i \cdot \bar{x}$$
$$= \sum_{i=1}^{n} x_i^2 - 2x_i \cdot \bar{x} + \bar{x}^2 + x_i \cdot \bar{x} - \bar{x}^2$$
$$= \sum_{i=1}^{n} (x_i - x)^2 + x_i \bar{x} - \bar{x}^2$$
$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 + n\bar{x}^2 - n\bar{x}^2$$
$$= \sum_{i=1}^{n} (x_i - \bar{x})^2 \geq 0$$

If 0 and 1 are the eigenvalues of C_n,
then $\det(C_n) = 0$ and $\det(C_n - I_n) = 0$.

$\because$ $(C_n - I_n\lambda)x = 0$ implies $\det(C_n - \lambda I_n) = 0$ (non-trivial solutions)

Obviously, when $\lambda = 1$, $|C_n - I_n| = \begin{vmatrix} -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & & \ddots & \vdots \\ -\frac{1}{n} & \cdots & & -\frac{1}{n} \end{vmatrix} = 0$

$$\lambda = 0, \quad |C_n| = \begin{array}{l} \begin{vmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & & & \vdots \\ \vdots & & \ddots & \vdots \\ -\frac{1}{n} & \cdots & & 1-\frac{1}{n} \end{vmatrix} \end{array} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \begin{array}{l} n-1 \\ \text{rows} \end{array}$$

add the first $n-1$ rows
to the last row

$$= \begin{vmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & & \vdots \\ \vdots & & \ddots & \vdots \\ \underbrace{1-\frac{n-1}{n}-\frac{1}{n} \quad \cdots \quad 1-\frac{n-1}{n}-\frac{1}{n}}_{=0 \cdots 0} \end{vmatrix} = 0$$

(b) The hyperplane $\langle \mathbb{1}_n, x \rangle = x_1 + x_2 + \cdots + x_n = 0$

The normal of this hyperplane is $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$.

Let u be any vector $\in \mathbb{R}^n$.

To prove that C_n is an projection onto the hyperplane, we need to show that

① $(\mathbb{1}_n)^T \cdot C_n u = 0$ ② $C_n^2 u = C_n u$

①

$$(\mathbb{1}_n)^T \cdot C_n u = \mathbb{1}_n^T \cdot (I_n - \frac{1}{n}\mathbb{1}_n \mathbb{1}_n^T) u$$
$$= \mathbb{1}_n^T \cdot (u - \frac{1}{n}\mathbb{1}_n \mathbb{1}_n^T u)$$
$$= \mathbb{1}_n^T u - \frac{1}{n}\underbrace{\mathbb{1}_n^T \mathbb{1}_n}_{=n} \mathbb{1}_n^T u$$
$$= \mathbb{1}_n^T u - \mathbb{1}_n^T u$$
$$= 0$$

②

$$C_n^2 u$$
$$= (I_n - \frac{1}{n}\mathbb{1}_n \mathbb{1}_n^T)(I_n - \frac{1}{n}\mathbb{1}_n \mathbb{1}_n^T) u$$
$$= (I_n - \frac{2}{n}\mathbb{1}_n \mathbb{1}_n^T + \frac{1}{n^2}\mathbb{1}_n \underbrace{\mathbb{1}_n^T \mathbb{1}_n}_{n} \mathbb{1}_n^T) u$$
$$= (I_n - \frac{1}{n}\mathbb{1}_n \mathbb{1}_n^T) u$$
$$= C_n u$$

In addition, we have shown that $C_n = C_n^T$ in (a), and therefore C_n is an orthogonal projection.

(c)

$$C_m X = \begin{bmatrix} x_{11} - \frac{1}{m}\sum_{i=1}^m x_{i1} & x_{12} - \frac{1}{m}\sum_{i=1}^m x_{i2} & \cdots & x_{1n} - \frac{1}{m}\sum_{i=1}^m x_{in} \\ x_{21} - \frac{1}{m}\sum_{i=1}^m x_{i1} & x_{22} - \frac{1}{m}\sum_{i=1}^m x_{i2} & & \vdots \\ \vdots & & \ddots & \\ x_{m1} - \frac{1}{m}\sum_{i=1}^m x_{i1} & x_{m2} - \frac{1}{m}\sum_{i=1}^m x_{i2} & \cdots & x_{mn} - \frac{1}{m}\sum_{i=1}^m x_{in} \end{bmatrix}$$

Sum of column $j = \sum_{i=1}^m x_{ij} - \sum_{i=1}^m x_{ij} = 0$

$$X C_n = \begin{bmatrix} x_{11} - \frac{1}{n}\sum_{i=1}^n x_{1i} & x_{12} - \frac{1}{n}\sum_{i=1}^n x_{1i} & \cdots & x_{1n} - \frac{1}{n}\sum_{i=1}^n x_{1i} \\ x_{21} - \frac{1}{n}\sum_{i=1}^n x_{2i} & x_{22} - \frac{1}{n}\sum_{i=1}^n x_{2i} & & \vdots \\ \vdots & & \ddots & \\ x_{m1} - \frac{1}{n}\sum_{i=1}^n x_{mi} & x_{m2} - \frac{1}{n}\sum_{i=1}^n x_{mi} & \cdots & x_{mn} - \frac{1}{n}\sum_{i=1}^n x_{mi} \end{bmatrix}$$

Sum of row $j = \sum_{i=1}^n x_{ji} - \sum_{i=1}^n x_{ji} = 0$

3. (a) $\pi_w \circ \pi_w = \pi_w$

$$\pi_w(\pi_w(x)) = c + P_v([c + P_v(x-c)] - c)$$
$$= c + P_v(P_v(x-c))$$
$$= c + P_v^2(x-c)$$
$$= c + P_v(x-c) \quad (\because P \text{ is an orthogonal projection})$$
$$= \pi_w(x)$$

(b) $\|\pi_w x - \pi_w y\|_2$
$$= \|c + P_v(x-c) - (c + P_v(y-c))\|_2$$
$$= \|P_v(x-c) - P_v(y-c)\|_2$$
$$= \|P_v(x) + P_v^{\perp}(c) - c - P_v(y) - P_v^{\perp}(c) + c\|_2$$
$$= \|P_v(x) - P_v(y)\|_2$$

(c) $\pi_w^{\perp}(\pi_w^{\perp} x)$
$$= \pi_w^{\perp}((I_n - \pi_w) x)$$
$$= \pi_w^{\perp}(x - c - P_v(x-c))$$
$$= x - c - P_v(x-c) - \pi_w(x - c - P_v(x-c))$$
$$= x - c - P_v(x-c) - c - P_v(x - c - P_v(x-c) - c)$$
$$= x - c - P_v(x - \cancel{c}) - c - P_v(\cancel{x} - c) + P_v^2(\cancel{x} - c) + \cancel{P_v c}$$
$$= x + P_v x$$
$$\neq I_n - \pi_w(x) = I_n - c - P_v x + P_v c$$

(d) $\pi_w^{\perp}(\pi_w^{\perp}(x))$
$$= x + P_v(x)$$

$$-\pi_w^{\perp}(-\pi_w^{\perp}(x))$$
$$= -\pi_w^{\perp}(-x + \pi_w(x))$$
$$= -\pi_w^{\perp}(-x + c + P_v(x-c))$$
$$= -(-x + c + P_v(x-c)) + \pi_w(-x + c + P_v(x-c))$$
$$= x - \cancel{c} - P_v(x-c) + \cancel{c} + P_v(-x + \cancel{c} + P_v(x-c) - \cancel{c})$$
$$= x - P_v(\cancel{x} - c) - P_v x + P_v(\cancel{x} - c)$$
$$= x - P_v x$$

(e) $(-\pi_w^{\perp})^3 x$
$$= -\pi_w^{\perp}(x - P_v x)$$
$$= -(x - P_v x) + \pi_w(x - P_v x)$$
$$= -x + P_v x + c + P_v(x - P_v x - c)$$
$$= -x + P_v x + c - P_v c$$

$(-\pi_w^{\perp})^4 x$
$$= -\pi_w^{\perp}(-x + P_v x + c - P_v c)$$
$$= x - P_v x - c + P_v c + \pi_w(-x + P_v x + c - P_v c)$$
$$= x - P_v x - \cancel{c} + P_v c + \cancel{c} + P_v(-x + P_v x + \cancel{c} - P_v c - \cancel{c})$$
$$= x - P_v x + \cancel{P_v c} - \cancel{P_v x} + \cancel{P_v x} - \cancel{P_v c}$$
$$= (-\pi_w^{\perp 2})^2 x$$

$\therefore \forall k = 2n,\ n \in \mathbb{N}^+ \quad (-\pi_w^{\perp k})^2 = -\pi_w^{\perp k}$ holds true.

4. (a) In SVD, the subspace is built from the space spanned by a set of vectors derived from linear combinations of all data points. However, in this subspace fitting algorithm, the subspace is built directly from the selected data points — one in this case ($n=1$) — but not from all data points.