Homework 1

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1 Sheaves

Exercise 1.1: Let A be an abelian group, and define the constant presheaf associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

Proof. Let $\mathscr{F}$ denote the constant presheaf. By Proposition 1.1, it suffices to prove that for any $x \in X$, $\mathscr{F}_x^+ \cong \mathscr{A}_x$. Since $\mathscr{F}_x \cong \mathscr{F}_x^+$, we only need to prove that $\mathscr{F}_x \cong \mathscr{A}_x$. Indeed, $\mathscr{F}_x = \varinjlim \mathscr{F}(U) = \varinjlim A \cong A$. For each open set U, let V_U be the connected component containing x and $V_U \subseteq U$, then $\mathscr{A}(V_U) \cong A$. So $\mathscr{A}_x = \varinjlim_U \mathscr{A}(U) = \varinjlim_U \mathscr{A}(V_U) \cong \varinjlim A \cong A$. Thus, $\mathscr{F}_x^+ \cong \mathscr{F}_x \cong \mathscr{A}_x$ for all $x \in X$. It follows that $\mathscr{A} \cong \mathscr{F}^+$ is the sheaf associated to this presheaf. □

Exercise 1.2: (a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker\varphi)_P = \ker(\varphi_P)$ and $(\mathrm{im}\ \varphi)_P = \mathrm{im}\ (\varphi_P)$.

(b) Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.

(c) Show that a sequence $\cdots \to \mathscr{F}_{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}_i \xrightarrow{\varphi^i} \mathscr{F}_{i+1} \to \cdots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

Proof. (a) We first claim that $\varinjlim \ker\varphi(U) = \ker \varinjlim \varphi(U)$ and $\varinjlim \mathrm{im}\ \varphi(U) = \mathrm{im}\ \varinjlim \varphi(U)$, where U runs through all open neighborhoods of P. Indeed, let $\mathscr{A}(U) = \ker\varphi(U)$, $\mathscr{B}(U) = \mathrm{im}\ \varphi(U)$ and consider the short exact sequence

$$0 \to \mathscr{A}(U) \to \mathscr{F}(U) \xrightarrow{\varphi(U)} \mathscr{B}(U) \to 0.$$

Apply $\varinjlim$, we then obtain an exact sequence

$$0 \to \varinjlim \mathscr{A}(U) \to \varinjlim \mathscr{F}(U) \xrightarrow{\varinjlim \varphi(U)} \varinjlim \mathscr{B}(U) \to 0,$$

as the set of neighborhoods of P is directed. So $\varinjlim \ker\varphi(U) = \varinjlim \mathscr{A}(U) = \ker \varinjlim \varphi(U)$. Similarly, $\varinjlim \mathrm{im}\ \varphi(U) = \varinjlim \mathscr{B}(U) = \mathrm{im}\ \varinjlim \varphi(U)$.

Then $(\ker\varphi)_P = \varinjlim \ker\varphi(U) = \ker \varinjlim \varphi(U) = \ker\varphi_P$, where U runs through all open neighborhoods of P.

Let $\mathscr{F} : U \mapsto \mathrm{im}\ \varphi(U)$, then $\mathrm{im}\ \varphi = \mathscr{F}^+$ by definition, so $(\mathrm{im}\ \varphi)_P = \mathscr{F}_P^+ = \mathscr{F}_P = \varinjlim \mathrm{im}\ \varphi(U) = \mathrm{im}\ \varinjlim \varphi(U) = \mathrm{im}\ \varphi_P$, where U runs through all open neighborhoods of P.

(b) This simply follows from (a) and Proposition 1.1.

(c) $\Rightarrow$: Suppose the sequence $\cdots \to \mathscr{F}_{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}_i \xrightarrow{\varphi^i} \mathscr{F}_{i+1} \to \cdots$ is exact. Then $\operatorname{im}\varphi^{i-1} = \ker\varphi^i$ for each i and $(\operatorname{im}\varphi^{i-1})_P = (\ker\varphi^i)_P$ for each i and each P. By (a), we have $\operatorname{im}\varphi_P^{i-1} = \ker\varphi_P^i$ for each i and each P. So for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

$\Leftarrow$: Suppose for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups. Then $\operatorname{im}\varphi_P^{i-1} = \ker\varphi_P^i$ for each i. By (a), we have that $(\operatorname{im}\varphi^{i-1})_P = (\ker\varphi^i)_P$ for each i and each P. It follows that $\operatorname{im}\varphi^{i-1} = \ker\varphi^i$ for each i by Proposition 1.1. Thus, the sequence $\cdots \to \mathscr{F}_{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}_i \xrightarrow{\varphi^i} \mathscr{F}_{i+1} \to \cdots$ is exact. $\square$

Exercise 1.3: (a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ and an open set U such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Proof. (a) $\Rightarrow$: Suppose φ is surjective, then φ_P is surjective for all P by Exercise 1.2(b). For each P, let s_P be the image of s in $\mathscr{G}_P$. Since φ_P is surjective, there exists $t_P \in \mathscr{F}_P$ s.t. $\varphi_P(t_P) = s_P$. By the property of colimit, we know that there exists a neighborhood of P, say U_i, and $t_i \in \mathscr{F}(U_i)$ s.t. t_P is the image of t_i in $\mathscr{F}_P$, i.e. $\rho(t_i) = t_P$.

Consider the following commutative diagram

$$\begin{tikzcd} \mathscr{F}(U_i) \arrow[r, "\varphi(U_i)"] \arrow[d, "\rho"'] & \mathscr{G}(U_i) \arrow[d, "\rho'"] \\ \mathscr{F}_P \arrow[r, "\varphi_P"] & \mathscr{G}_P \end{tikzcd}$$

we must have $\rho'(s|_{U_i}) = s_P$. Also $\langle U_i, \varphi(U_i)(t_i)\rangle$ and $\langle U_i, s|_{U_i}\rangle$ have the same image in $\mathscr{G}_P$. So there exists a neighborhood V_i of P contained in U_i such that $\varphi(V_i)(t_i|_{V_i}) = \varphi(U_i)(t_i)|_{V_i} = (s|_{U_i})|_{V_i} = s|_{V_i}$. Hence, by replace U_i with a small enough neighborhood of P, we may assume that $\varphi(U_i)(t_i) = s|_{U_i}$. Since U is covered by neighborhoods of all its points P, we obtain the desired result.

$\Leftarrow$: To show φ is surjective, it is sufficient to show that φ_P is surjective for all points P by Exercise 1.2(b). Let $s_P \in \mathscr{G}_P$, then there exists a neighborhood U of P and $s \in \mathscr{G}(U)$ such that s_P is the image of s in $\mathscr{G}_P$. Then by the hypothesis, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$, for all i. Choose a U_i containing P and let t_P be the image of t_i in $\mathscr{F}_P$. Then $\varphi(t_P) = s_P$. So φ_P is surjective for each P and it follows that φ is surjective.

(b) Let $X = \mathbb{C}$. Let $\mathscr{O}_X$ be the sheaf of holomorphic functions, i.e. for each open subset $U \subseteq \mathbb{C}$,

$$\mathscr{O}_X(U) = \{f | f \text{ is holomorphic on } U\},$$

and $\mathscr{O}_X^*$ be the sheaf of nonzero holomorphic functions, i.e. for each open subset $U \subseteq \mathbb{C}$,

$$\mathscr{O}_X^*(U) = \{f | f \text{ is holomorphic on } U \text{ and } f(z) \neq 0,\ \forall z \in U\}.$$

The restriction maps are both defined to be the usual restriction.

Define $\varphi(U) : \mathscr{O}_X(U) \to \mathscr{O}_X^*(U)$ by $f \mapsto e^f$. We then obtain a morphism $\varphi : \mathscr{O}_X \to \mathscr{O}_X^*$. For each $P \in \mathbb{C}$, consider $\varphi_P : \mathscr{O}_{X,P} \to \mathscr{O}_{X,P}^*$. This map must be surjective. Indeed, for each

$f(z) \in \mathscr{O}^*_{X,P}$, there exists a neighborhood Ω of P such that $f(z)$ is nowhere vanishing on Ω. Moreover, we may require Ω to be simply connected. Then there exists a holomorphic function $g(z)$ on Ω such that $f(z) = e^{g(z)}$ [See Elias M. Stein, Complex Analysis, Chapter 3, Theorem 6.2]. Thus, by Exercise 1.2(b), we conclude that the morphism φ is surjective.

However, let $U = \mathbb{C} - \{0\}$. We know that $\varphi(U)$ is not surjective as $\log z$ can not be defined over U. $\square$

Exercise 1.4: (1) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Then the induced map $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.

(2) If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then there is a natural map $\mathrm{im}\ \varphi \to \mathscr{G}$, which is injective and thus $\mathrm{im}\ \varphi$ can be naturally identified with a subsheaf of $\mathscr{G}$.

Proof. (1) Since $\varphi(U)$ is injective, we know that $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is injective by the exactness of direct limit. Since $\mathscr{F}^+_P \cong \mathscr{F}_P$ and $\mathscr{G}^+_P \cong \mathscr{G}_P$, then $\varphi^+_P : \mathscr{F}^+_P \to \mathscr{G}^+_P$ is also injective. We conclude that $\varphi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ is injective by Exercise 1.2(b).

(2) Let $\mathscr{H} : U \mapsto \mathrm{im}\ \varphi(U)$ be the presheaf image of φ and $\psi(U) : \mathscr{H}(U) \to \mathscr{G}(U)$ be the natural imbedding. Then by (1), we have $\psi^+ : \mathscr{H}^+ \to \mathscr{G}^+$ is injective. However, $\mathscr{H}^+ = \mathrm{im}\ \varphi$ by definition and $\mathscr{G}^+ = \mathscr{G}$ since $\mathscr{G}$ is a sheaf. Hence, we conclude that $\mathrm{im}\ \varphi \to \mathscr{G}$ is injective and $\mathrm{im}\ \varphi$ can be identified with a subsheaf of $\mathscr{G}$. $\square$

Exercise 1.5: Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

Proof. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. By Proposition 1.1, φ is an isomorphism if and only if $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ is an isomorphism for each P if and only if φ_P is injective and surjective for each P if and only if φ is injective and surjective by Exercise 1.2(b). $\square$

Exercise 1.6: (a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0.$$

(b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

Proof. (a) Let $\mathscr{H} : U \mapsto \mathscr{F}(U)/\mathscr{F}'(U)$ be a presheaf, then the quotient sheaf $\mathscr{F}/\mathscr{F}' = \mathscr{H}^+$ by definition. Let $\psi : \mathscr{F} \to \mathscr{H}$ be the natural morphism defined by the quotient map $\psi(U) : \mathscr{F}(U) \to \mathscr{F}(U)/\mathscr{F}'(U) = \mathscr{H}(U)$, which is surjective. By Proposition-Definition 1.2, there exists a unique $\theta : \mathscr{H} \to \mathscr{H}^+ = \mathscr{F}/\mathscr{F}'$ up to isomorphism. We consider the following commutative diagram,

$$\begin{tikzcd} & \mathscr{H}^+ = \mathscr{F}/\mathscr{F}' & \\ \mathscr{F} \arrow[ur, "\varphi"] \arrow[rr, "\psi"] & & \mathscr{H} \arrow[ul, "\theta"'] \end{tikzcd}$$

where $\varphi = \theta \circ \psi$.

Then for any point P and open neighborhoods U of P, we must have $\varinjlim_U \psi(U) : \varinjlim_U \mathscr{F}(U) \to \varinjlim_U \mathscr{H}(U)$ is surjective by the exactness of direct limit, i.e. $\psi_P : \mathscr{F}_P \to \mathscr{H}_P$ is surjective. Since $\theta_P : \mathscr{H}_P \to \mathscr{H}^+_P$ is an isomorphism, we conclude that $\varphi_P = \theta_P \circ \psi_P : \mathscr{F}_P \to \mathscr{H}^+_P = \mathscr{F}_P/\mathscr{F}'_P$

is surjective and $\ker(\varphi_P) = \mathscr{F}'_P$. By Exercise 1.2$(b)$, $\varphi : \mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective. By Exercise 1.2(a), $(\ker\varphi)_P = \ker(\varphi_P) = \mathscr{F}'_P$ for all points P. Again, by Proposition 1.1, we conclude that $\ker\varphi \cong \mathscr{F}'$ and this gives a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \xrightarrow{\varphi} \mathscr{F}/\mathscr{F}' \to 0.$$

(b) For any points P, we have a short exact sequence

$$0 \to \mathscr{F}'_P \to \mathscr{F}_P \to \mathscr{F}''_P \to 0,$$

by Exercise 1.2(c). So $\mathscr{F}''_P \cong \mathscr{F}_P/\mathscr{F}'_P = (\mathscr{F}/\mathscr{F}')_P$. So, $\mathscr{F}'' \cong \mathscr{F}/\mathscr{F}'$ by Proposition 1.1. Consider the morphism $\varphi : \mathscr{F} \to \mathscr{F}/\mathscr{F}'$ in (a), we know that $\mathscr{F}' \cong \ker\varphi$ is a subsheaf of $\mathscr{F}$ by (a). □

Exercise 1.7: *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

(a) *Show that* $\mathrm{im}\ \varphi \cong \mathscr{F}/\ker\varphi$.

(b) *Show that* $\mathrm{coker}\ \varphi \cong \mathscr{G}/\mathrm{im}\ \varphi$.

Proof. (a) Consider the induced homomorphism $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ on stalk at point P, we have a short exact sequence of abelian groups

$$0 \to \ker(\varphi_P) \to \mathscr{F}_P \to \mathrm{im}\ (\varphi_P) \to 0.$$

By Exercise 1.2(a), $\ker(\varphi_P) = (\ker\varphi)_P$ and $\mathrm{im}\ (\varphi_P) \to (\mathrm{im}\ \varphi)_P$, this exact sequence is equivalent to

$$0 \to (\ker\varphi)_P \to \mathscr{F}_P \to (\mathrm{im}\ \varphi)_P \to 0.$$

By Exercise 1.2(c), we know that the short sequence

$$0 \to \ker\varphi \to \mathscr{F} \to \mathrm{im}\ \varphi \to 0$$

is exact. So by Exercise 1.6(b), we must have $\mathrm{im}\ \varphi \cong \mathscr{F}/\ker\varphi$.

(b) We first claim that $\mathrm{coker}\ (\varphi_P) = (\mathrm{coker}\ \varphi)_P$. Indeed, let $\mathscr{A}(U) = \mathrm{im}\ \varphi(U)$, $\mathscr{B}(U) = \mathrm{coker}\ \varphi(U)$ and consider the short exact sequence with $\ker\psi(U) = \mathrm{im}\ \varphi(U)$

$$0 \to \mathscr{A}(U) \to \mathscr{G}(U) \xrightarrow{\psi(U)} \mathscr{B}(U) \to 0.$$

Apply $\varinjlim$, we then obtain an exact sequence

$$0 \to \varinjlim \mathscr{A}(U) \to \varinjlim \mathscr{G}(U) \xrightarrow{\varinjlim \psi(U)} \varinjlim \mathscr{B}(U) \to 0,$$

as the set of neighborhoods of P is directed. So

$$\begin{aligned}\varinjlim \mathrm{coker}\ \varphi(U) &= \varinjlim(\mathscr{G}(U)/\mathrm{im}\ \varphi(U)) = \varinjlim(\mathscr{G}(U)/\ker\psi(U)) = \varinjlim \mathscr{B}(U) = \varinjlim \mathscr{G}(U)/\varinjlim \mathscr{A}(U)\\ &= \varinjlim \mathscr{G}(U)/\varinjlim \mathrm{im}\ \varphi(U) = \varinjlim \mathscr{B}(U)/\mathrm{im}\ \varinjlim \varphi(U) = \mathrm{coker}\ \varinjlim \varphi(U).\end{aligned}$$

Let $\mathscr{H}$ be the presheaf cokernel of φ, then $\mathscr{H}^+ = \mathrm{coker}\ \varphi$. So $(\mathrm{coker}\ \varphi)_P = \mathscr{H}^+_P = \mathscr{H}_P = \varinjlim \mathscr{H}(U) = \varinjlim \mathrm{coker}\ \varphi(U) = \mathrm{coker}\ \varinjlim \varphi(U) = \mathrm{coker}\ (\varphi_P)$, where U runs through all open neighborhoods of P.

Consider the induced homomorphism $\varphi_P : \mathscr{F}_P \to \mathscr{G}_P$ on stalk at point P, we have coker $(\varphi_P) = \mathscr{G}_P/\text{im }(\varphi_P)$ by definition. This gives a short exact sequence

$$0 \to \text{im }(\varphi_P) \to \mathscr{G}_P \to \text{coker }(\varphi_P) \to 0.$$

Since im $(\varphi_P) = (\text{im }\varphi)_P$ and coker $(\varphi_P) = (\text{coker }\varphi)_P$, we have another short exact sequence

$$0 \to (\text{im }\varphi)_P \to \mathscr{G}_P \to (\text{coker }\varphi)_P \to 0.$$

So, by Exercise 1.2(c), we have a short exact sequence

$$0 \to \text{im }\varphi \to \mathscr{G} \to \text{coker }\varphi \to 0,$$

which implies that coker $\varphi \cong \mathscr{G}/\text{im }\varphi$ by Exercise 1.6(b). ☐

Exercise 1.10: *Direct Limit.* Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limit of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathscr{G}$ and a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$, compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathscr{F}_i \to \mathscr{G}$ such that for each i, the original map $\mathscr{F}_i \to \mathscr{G}$ is obtained by composing the maps $\mathscr{F}_i \to \varinjlim \mathscr{F}_i \to \mathscr{G}$.

Proof. Let $\mathscr{H}$ be the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$, then we have $\varinjlim \mathscr{F}_i = \mathscr{H}^+$ by definition. Given a sheaf $\mathscr{G}$ and a collection of morphisms $\varphi_i : \mathscr{F}_i \to \mathscr{G}$, for each open subset U, we have a collection of homomorphisms of abelian groups $\varphi_i(U) : \mathscr{F}_i(U) \to \mathscr{G}(U)$. Then we have a commutative diagram of abelian groups

$$\begin{tikzcd}
\mathscr{H}(U) = \varinjlim \mathscr{F}_i(U) \arrow[rr, dotted, "\exists\,!\quad \psi(U)"] & & \mathscr{G}(U) \\
& \mathscr{F}_i(U) \arrow[ul, "\alpha_i(U)"'] \arrow[ur, "\varphi_i(U)"] \arrow[d, "\rho_i^j(U)"'] & \\
& \mathscr{F}_j(U) \arrow[uul, bend left, "\alpha_j(U)"] \arrow[uur, bend right, "\varphi_j(U)"'] &
\end{tikzcd}$$

It is routine to check $\psi : \mathscr{H} \to \mathscr{G}$ is a morphism of presheaves from the following diagram

$$\begin{tikzcd}
& \mathscr{G}(U) \arrow[rr, "s_{UV}"] & & \mathscr{G}(V) \\
\mathscr{H}(U) \arrow[ur, "\psi(U)"] \arrow[rr, "r_{UV}"'] & & \mathscr{H}(V) \arrow[ur, "\psi(V)"] & \\
& \mathscr{F}_i(U) \arrow[ul, "\alpha_i(U)"] \arrow[u, dotted, "\varphi_i(U)"] \arrow[rr, "\rho_{UV}"] & & \mathscr{F}_i(V) \arrow[ul, "\alpha_i(V)"'] \arrow[uu, "\varphi_i(V)"']
\end{tikzcd}$$

where all faces are commutative except the top one. Apply Proposition-Definition 1.2 to the morphism ψ, we obtain a commutative diagram

$$\begin{tikzcd}
& \mathscr{H}^+ \arrow[dr, dotted, "\exists\, !\, \theta"] & \\
\mathscr{H} \arrow[ur, "\beta"] \arrow[rr, "\psi"] & & \mathscr{G}
\end{tikzcd}$$

So for each open set U, we have a commutative diagram of abelian groups

$$\begin{tikzcd}
& \mathscr{H}^+(U) \arrow[dr, dotted, "\exists\, !\, \theta(U)"] & \\
\mathscr{H}(U) \arrow[ur, "\beta(U)"] \arrow[rr, "\psi(U)"] & & \mathscr{G}(U)
\end{tikzcd}$$

Then, by composing $\alpha_i(U)$ and $\beta(U)$, we obtain a commutative diagram of abelian groups

$$\begin{tikzcd}
\mathscr{H}^+(U) \arrow[rr, dotted, "\exists\, !\quad \theta(U)"] & & \mathscr{G}(U) \\
& \mathscr{F}_i(U) \arrow[ul, "\alpha_i'(U)"'] \arrow[ur, "\varphi_i(U)"] \arrow[d, "\rho_i^j(U)"'] & \\
& \mathscr{F}_j(U) \arrow[uul, bend left, "\alpha_j'(U)"] \arrow[uur, bend right, "\varphi_j(U)"'] &
\end{tikzcd}$$

This gives a commutative diagram of sheaves in the same manner

$$\begin{tikzcd}
\varinjlim \mathscr{F}_i \arrow[rr, dotted, "\exists\, !\quad \theta"] & & \mathscr{G} \\
& \mathscr{F}_i \arrow[ul, "\alpha_i'"'] \arrow[ur, "\varphi_i"] \arrow[d, "\rho_i^j"'] & \\
& \mathscr{F}_j \arrow[uul, bend left, "\alpha_j'"] \arrow[uur, bend right, "\varphi_j"'] &
\end{tikzcd}$$

Hence, $\varinjlim \mathscr{F}_i$ is a direct limit in the category of sheaves on X $\square$

Exercise 1.11: Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.

Proof. We use the notation $[\![1, n]\!]$ to denote $\{1, 2, \cdots, n\}$. We prove the case $\{\mathscr{F}_i\}_{i\in I}$ is a direct system over a directed index set, i.e. I is directed.

Let $\mathscr{H} : U \mapsto \varinjlim_i \mathscr{F}_i(U)$ be a presheaf, then $\varinjlim_i \mathscr{F}_i = \mathscr{H}^+$ is a sheaf. Let $\{U_j\}_{j\in J}$ be an open cover of U. Since X is noetherian, then U is quasi-compact. Then there exists a finite subcover $\{U_j\}_{j\in[\![1,n]\!]}$ of U, i.e. $U \subseteq \bigcup_{j=1}^n U_j$. Let $s \in \mathscr{H}(U)$, such that for each $j \in J$, $s|_{U_j} = 0 \in \mathscr{H}(U_j)$. By the property of colimits, we know that for each $j \in [\![1, n]\!]$, there exists $i(j) \in I$ and an element $s_{i(j),j} \in \mathscr{F}_{i(j)}(U_j)$ such that $s|_{U_j}$ is the image of $s_{i(j),j}$ and $s_{i(j),j} = 0$ in $\mathscr{F}_{i(j)}(U_j)$. Since I is directed, there exists a $k \in I$, such that $i(j) \leqslant k$ for all j. Let $s_{k,j}$ be the image of $s_{i(j),j}$ in $\mathscr{F}_k(U_j)$ for each j. By the sheaf property, $\exists s_k \in \mathscr{F}_k(U)$, such that $s_k|_{U_j} = s_{k,j} = 0$. Thus $s_k = 0 \in \mathscr{F}_k(U)$. By replacing k with a large enough index, we may assume that s is the image of s_k in $\mathscr{H}(U)$. So $s = 0$. Thus $\mathscr{H}$ satisfies the sheaf property (3).

For each j, let $s_j \in \mathscr{H}(U_j)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in [\![1, n]\!]$. Then there exist $l(j) \in I$ and an element $s_{l(j),j} \in \mathscr{F}_{l(j)}(U_j)$ such that s_j is the image of $s_{l(j),j}$ in $\mathscr{H}(U_j)$. Since I is directed, there exists $k \in I$ such that $l(j) \leqslant k$ for all j. Let $s_{k,j}$ be the image of $s_{l(j),j}$ in $\mathscr{F}_k(U_j)$. Then s_j is the image of $s_{k,j}$ in $\mathscr{H}(U_j)$ for each j. Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{H}(U_j) \arrow[r] & \mathscr{H}(U_i \cap U_j) & \mathscr{H}(U_i) \arrow[l] \\
\mathscr{F}_k(U_j) \arrow[r] \arrow[u] & \mathscr{F}_k(U_i \cap U_j) \arrow[u] & \mathscr{F}_k(U_i) \arrow[l] \arrow[u]
\end{tikzcd}$$

and the fact that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, we may assume $s_{k,j}|_{U_i \cap U_j} = s_{k,i}|_{U_i \cap U_j}$ for each i, j by replacing k with a large enough index. By the sheaf property (4), there exists $s_k \in \mathscr{F}_k(U)$ such that $s_k|_{U_j} = s_{k,j}$ for all $j \in [\![1, n]\!]$. Let s be the image of s_k in $\mathscr{H}(U)$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{H}(U) \arrow[r] & \mathscr{H}(U_j) \\
\mathscr{F}_k(U) \arrow[r] \arrow[u] & \mathscr{F}_k(U_j) \arrow[u]
\end{tikzcd}$$

we must have that $s|_{U_j} = s_j$ for all $j \in [\![1, n]\!]$. By the sheaf property (3), we know that $s|_{U_j} = s_j$ for all $j \in J$. This proves the sheaf property (4). Thus $\mathscr{H} : U \mapsto \varinjlim \mathscr{F}_i$ is already a sheaf. It follows that $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$. $\square$

Exercise 1.12: *lnrerse Limit.* Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \to \varprojlim \mathscr{F}_i(U)$ is a sheaf. It is called the inverse limit of the system $\{\mathscr{F}_i\}$, and is denoted by $\varprojlim \mathscr{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

Proof. Let $(\psi_i^j : \mathscr{F}_j \to \mathscr{F}_i)_{j \geqslant i}$ be the indexed family of morphisms of sheaves associated to $\{\mathscr{F}_i\}_{i \in I}$. We denote the presheaf $U \to \varprojlim \mathscr{F}_i(U)$ by $\mathscr{H}$.

Let $\{U_j\}_{j \in J}$ be an open cover of U. Let $s = (s_k)_{k \in I} \in \mathscr{H}(U)$ such that $s|_{U_j} = 0$ for all $j \in J$, where $s_k \in \mathscr{F}_k(U)$ satisfying $\psi_l^k(U)(s_k) = s_l$ for all $k \geqslant l$. Then consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{H}(U) \arrow[r] \arrow[d] & \mathscr{H}(U_j) \arrow[d] \\
\mathscr{F}_k(U) \arrow[r] & \mathscr{F}_k(U_j)
\end{tikzcd}$$

we must have $s_k|_{U_j} = 0$ for all $k \in I$ and $j \in J$. So $s_k \in \mathscr{F}_k(U)$ must be 0 for all k since $\mathscr{F}_k$ is a sheaf. So $s = 0$ and $\mathscr{H}$ thus satisfies the sheaf property (3).

Now let $s_j = (s_{j,k})_{k \in I} \in \mathscr{H}(U_j)$ for all $j \in J$ such that $s_j|_{U_i \cap U_j} = s_i|_{U_i \cap U_j}$, where $s_{j,k} \in \mathscr{F}_k(U_j)$ satisfying $\psi_l^k(U_j)(s_{j,k}) = s_{j,l}$ for all $k \geqslant l$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{H}(U_j) \arrow[r] \arrow[d] & \mathscr{H}(U_i \cap U_j) \arrow[d] & \mathscr{H}(U_i) \arrow[l] \arrow[d] \\
\mathscr{F}_k(U_j) \arrow[r] & \mathscr{F}_k(U_i \cap U_j) & \mathscr{F}_k(U_i) \arrow[l]
\end{tikzcd}$$

we must have $s_{j,k}|_{U_i \cap U_j} = s_{i,k}|_{U_i \cap U_j}$ for all $i, j \in J$ and $k \in I$ since $s_j|_{U_i \cap U_j} = s_i|_{U_i \cap U_j}$ for all $i, j \in J$. Thus there exists $s_k \in \mathscr{F}_k(U)$ such that $s_k|_{U_j} = s_{j,k}$ since $\mathscr{F}_k$ is a sheaf. Further $\psi_l^k(U)(s_k)|_{U_j} = \psi_l^k(U_j)(s_k|_{U_j}) = \psi_l^k(U_j)(s_{j,k}) = s_{j,l} = s_l|_{U_j}$ for all $j \in J$ and $k \geqslant l$. This implies that $\psi_l^k(U)(s_k) = s_l$ for all $k \geqslant l$. So take $s = (s_k)_{k \in I}$, we must have $s \in \mathscr{H}(U)$. Then consider the following diagram

$$\begin{tikzcd}
\mathscr{H}(U) \arrow[r] \arrow[d] & \mathscr{H}(U_j) \arrow[d] \\
\mathscr{F}_k(U) \arrow[r] & \mathscr{F}_k(U_j)
\end{tikzcd}$$

we must have that the image of $s|_{U_j} - s_j$ in $\mathscr{F}_k(U_j)$ is 0 for all k. Thus $s|_{U_j} = s_j$ by the property of inverse limit. This proves that $\mathscr{H}$ satisfies the sheaf property (4). Hence, $\mathscr{H} : U \mapsto \varprojlim \mathscr{F}_i(U)$ is a sheaf.

Argue in the same manner as in Exercise 1.10, we know that it has the universal property of an inverse limit in the category of sheaves. ☐

Exercise 1.14: *Support.* Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The *support* of s, denoted $\mathrm{Supp}(s)$, is defined to be $\{P \in U | s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that $\mathrm{Supp}(s)$ is a closed subset of U. We define the *support* of $\mathscr{F}$, $\mathrm{Supp}(\mathscr{F})$, to be $\{P \in X | \mathscr{F}_P \neq 0\}$. It need not be a closed subset.

Proof. Let $P \in U - \mathrm{Supp}(s)$, then $s_P = 0$. So there exists an open neighborhood V of P contained in U such that $s|_V = 0$. So for any $Q \in V$, we have $s_Q = (s|_V)_Q = 0$, which means $Q \notin \mathrm{Supp}(s)$. So $V \subseteq X - \mathrm{Supp}(s)$. Hence, $U - \mathrm{Supp}(s)$ is an open set. It follows that $\mathrm{Supp}(s) = U^c \cap (U - \mathrm{Supp}(s))^c$ is closed. ☐

Exercise 1.17: *Skyscraper Sheaves.* Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \overline{\{P\}}$, and 0 elsewhere, where $\overline{\{P\}}$ denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf." Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\overline{\{P\}}$, and $i : \overline{\{P\}} \to X$ is the inclusion.

Proof. First, we prove that for any nonempty open subset U, if $U \cap \overline{\{P\}} \neq \varnothing$, we have $P \in U$. Indeed, if $P \notin U$, then $\overline{\{P\}} \setminus U$ is a closed set containing P in the subpace $\overline{\{P\}}$, so $\overline{\{P\}} \setminus U \supseteq \overline{\{P\}}$, a contradiction. Thus, for any $Q \in \overline{\{P\}}$, $(i_P(A))_Q = \varinjlim_{U \ni Q} i_P(A)(U) = \varinjlim_{\overline{\{P\}} \supseteq U \ni Q} i_P(A)(U) = \varinjlim_{U \ni P} i_P(A)(U) = \varinjlim_{U \ni P} A = A$. If $Q \notin \overline{\{P\}}$, then there exists an open neighborhood of Q such that $U \subset X - \overline{\{P\}}$ as $X - \overline{\{P\}}$ is open. So $(i_P(A))_Q = \varinjlim_{X - \overline{\{P\}} \supseteq U \ni Q} i_P(A)(U) = 0$ as $P \notin U$.

By definition, $(i_*A)(U) = A(i^{-1}(U)) = A(U \cap \overline{\{P\}})$. Now we work in the closed subspace $\overline{\{P\}}$. If $P \in U$, then $\{P\} \subseteq U \cap \overline{\{P\}} \subseteq \overline{\{P\}}$. Thus, $U \cap \overline{\{P\}}$ is connected as $\{P\}$ is connected. So, $A(U \cap \overline{\{P\}}) = A$. If $P \notin U$, then $U \cap \overline{\{P\}} = \varnothing$ as we discussed above. So $A(U \cap \overline{\{P\}}) = 0$. Thus, we conclude that $i_*(A) = i_P(A)$. ☐

Exercise 1.18: *Adjoint Property of f^{-1}.* Let $f : X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1} f_* \mathscr{F} \to \mathscr{F}$, and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_* f^{-1} \mathscr{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y,

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

Hence we say that f^{-1} is a left adjoint of f_* and that f_* is a right adjoint of f^{-1}.

Proof. Let $\mathscr{A}$ be the presheaf $U \mapsto \varinjlim_{V \supseteq f(U)} (f_*\mathscr{F})(V)$. Then $\mathscr{A}(U) = \varinjlim_{V \supseteq f(U)} (f_*\mathscr{F})(V) = \varinjlim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V)) = \varinjlim_{f^{-1}(V) \supseteq U} \mathscr{F}(f^{-1}(V)) = \varinjlim_{W \supseteq U} \mathscr{F}(W)$, where W run through all preimage of open subsets containing $f(U)$. The restriction map associated to $\mathscr{A}$ is defiend to be $\rho_{UV} : \mathscr{A}(U) \to \mathscr{A}(V)$ by $[\langle s, W\rangle] \mapsto [\langle s|_U, U\rangle]$, where $V \subseteq U$ and W is the preimage of an open subset containing $f(U)$. Define $\varphi(U) : \mathscr{A}(U) \to \mathscr{F}(U)$ by $[\langle s, W\rangle] \mapsto [\langle s|_U, U\rangle]$, then $\varphi : \mathscr{A} \to \mathscr{F}$ is a morphism of presheaves as one can verify. By Proposition-Definition 1.2, we have a unique morphism $\mathscr{A}^+ \to \mathscr{F}$ of sheaves up to isomorphism, i.e. a morphism $\alpha : f^{-1}f_*\mathscr{F} \to \mathscr{F}$.

Let $\mathscr{B}$ be the presheaf $V \mapsto \varinjlim_{U \supseteq f(f^{-1}(V))} G(U)$. Then $\mathscr{B}^+(V) = (f^{-1}\mathscr{G})(f^{-1}(V)) = f_*(f^{-1}\mathscr{G})(V)$, so $\mathscr{B}^+ = f_*f^{-1}\mathscr{G}$. We then have a natural morphism $\mathscr{G} \to \mathscr{B}$ defined by $\psi(V) : \mathscr{G}(V) \to \varinjlim_{U \supseteq f(f^{-1}(V))} \mathscr{G}(U)$, where $\psi(V) : \langle s, V\rangle \mapsto [\langle s, V\rangle]$. So the composition $\mathscr{G} \to \mathscr{B} \to \mathscr{B}^+$ gives a natural map $\beta : \mathscr{G} \to f_*f^{-1}\mathscr{G}$ as desired.

Define

$$\Phi : \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) \to \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F})$$

by $\varphi \mapsto f_*\varphi \circ \beta$ and

$$\Gamma : \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}) \to \mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F})$$

by $\psi \mapsto \alpha \circ f^{-1}\psi$. We then have $\Gamma \circ \Phi = \mathrm{id}$ and $\Phi \circ \Gamma = \mathrm{id}$. It follows that

$$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$

□

Exercise 1.19: *Extending a Sheaf by Zero.* Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.

(a) Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathscr{F}$ instead of $i_*\mathscr{F}$, and say "consider $\mathscr{F}$ as a sheaf on X", when we mean "consider $i_*\mathscr{F}$."

(b) Now let $\mathscr{F}$ be a sheaf on U. Let $j_!(\mathscr{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!(\mathscr{F})$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!(\mathscr{F})$ the sheaf obtained by extending $\mathscr{F}$ by zero outside U.

(c) Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0.$$

Proof. (a) If $P \in Z$, then $(i_*\mathscr{F})_P = \varinjlim_{V \ni P} (i_*\mathscr{F})(V) = \varinjlim_{V \ni P} \mathscr{F}(i^{-1}(V)) = \varinjlim_{V \ni P} \mathscr{F}(V \cap Z) = \mathscr{F}_P$. If $P \notin Z$, i.e. $P \in U$, then there exists open neighborhoods V of P contained in U. So $(i_*\mathscr{F})_P = \varinjlim_{U \supseteq V \ni P} (i_*\mathscr{F})(V) = \varinjlim_{U \supseteq V \ni P} \mathscr{F}(i^{-1}(V)) = \varinjlim_{U \supseteq V \ni P} \mathscr{F}(V \cap Z) = \varinjlim_{U \supseteq V \ni P} \mathscr{F}(\varnothing) = 0$.

(b) Let $\mathscr{H}$ be the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. If $P \in U$, then $(j_!(\mathscr{F}))_P = \mathscr{H}^+_P = \mathscr{H}_P = \varinjlim_{U \supseteq V \ni P} \mathscr{H}(V) = \varinjlim_{U \supseteq V \ni P} \mathscr{F}(V) = \mathscr{F}_P$. If $P \notin U$, then $(j_!(\mathscr{F}))_P = \mathscr{H}^+_P = \mathscr{H}_P = \varinjlim_{V \ni P} \mathscr{H}(V) = \varinjlim_{V \ni P} 0 = 0$ since for any open neighborhoods V of P, $V \nsubseteq U$. By Proposition 1.1 we know that $j_!(\mathscr{F})$ is the unique sheaf on X which has this property up to isomorphism.

Let $\mathscr{G}$ be the presheaf $V \mapsto \varinjlim_{W \supseteq i(V)} (j_! \mathscr{F})(W)$ on U. For any subset $V \subseteq U$, we have $\mathscr{G}(V) = \varinjlim_{W \supseteq V} (j_! \mathscr{F})(W) = (j_! \mathscr{F})(V) = \mathscr{F}(V)$, so $\mathscr{G} = \mathscr{F}$ is a sheaf. Then $(j_! \mathscr{F})|_U = \mathscr{G}^+ = \mathscr{G} = \mathscr{F}$.

(c) By (a) and (b), we know that

$$i_*(\mathscr{F}|Z)_P = \begin{cases} (\mathscr{F}|_Z)_P, & \text{if } P \in Z \\ 0, & \text{if } P \in U \end{cases} = \begin{cases} \mathscr{F}_P, & \text{if } P \in Z \\ 0, & \text{if } P \in U \end{cases}$$

and

$$j_!(\mathscr{F}|U)_P = \begin{cases} (\mathscr{F}|_U)_P, & \text{if } P \in U \\ 0, & \text{if } P \in Z \end{cases} = \begin{cases} \mathscr{F}_P, & \text{if } P \in U \\ 0, & \text{if } P \in Z \end{cases}$$

For each $P \in X$, this gives a short exact sequence of stalks at P:

$$0 \to j_!(\mathscr{F}|_U)_P \to \mathscr{F}_P \to i_*(\mathscr{F}|_Z)_P \to 0.$$

So, by Exercise 1.2(c), there is an exact sequence of sheaves on X,

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0.$$

□

Exercise 1.21: *Some Examples of Sheaves on Varieties.* Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathscr{O}_x$ be the sheaf of regular functions on X (1.0.1).

(a) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathscr{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathscr{I}_Y(U)$ is a sheaf. It is called the **sheaf of ideals** $\mathscr{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathscr{O}_X$.

(b) If Y is a subvariety, then the quotient sheaf $\mathscr{O}_X/\mathscr{I}_Y$ is isomorphic to $i_*(\mathscr{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathscr{O}_Y$ is the sheaf of regular functions on Y.

Proof. (a) Let $\{U_i\}_{i \in I}$ be an open covering of U, i.e. $U \subseteq \bigcup_{i \in I} U_i$. Let $f \in \mathscr{I}_Y(U)$ such that $f|_{U_i} = 0$ in $\mathscr{I}_Y(U_i)$. Then $f = 0$ since f is a regular function. So $\mathscr{I}_Y$ satisfies the sheaf property (3). Let $f_i \in \mathscr{I}_Y(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i.j \in I$. We can define $f(x)$ to be $f_i(x)$ if $x \in U_i$. Then for all $x \in Y \cap U$, we have $x \in Y \cap U_i$ and $f_i(x) = 0$ for some i. So $f(x) = 0$ and it follows that $f \in \mathscr{I}_Y(U)$. This proves the sheaf property (4). Hence, $\mathscr{I}_Y$ is indeed a sheaf.

(b) For each U, consider $\varphi(U) : \mathscr{O}_X(U) \to i_*(\mathscr{O}_Y)(U) = \mathscr{O}_Y(U \cap Y)$ defined by restriction. Its kernel is simply $\mathscr{I}_Y(U)$. So we have a short exact sequence of rings

$$0 \to \mathscr{I}_Y(U) \to \mathscr{O}_X(U) \to i_*(\mathscr{O}_Y)(U) \to 0,$$

which defines a sequence of sheaves

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to i_*(\mathscr{O}_Y) \to 0.$$

We need to prove that it is exact.

If $P \notin Y$, since Y is closed, there are some open neighborhoods U of P contained in Y^c, then $\mathscr{I}_{Y,P} = \varinjlim_{U \ni P} \mathscr{I}_Y(U) = \varinjlim_{U \ni P} \mathscr{O}_X(U) = \mathscr{O}_{X,P}$, where U runs through all open neighborhood of P contained in Y^c such that $U \cap Y = \varnothing$. By Exercise 1.19, $i_*(\mathscr{O}_Y)_P = 0$. So we obtain a short exact sequence

$$0 \to \mathscr{I}_{Y,P} \to \mathscr{O}_{X,P} \to i_*(\mathscr{O}_Y)_P \to 0.$$

If $P \in Y$, then By Exercise 1.19, $i_*(\mathscr{O}_Y)_P = \mathscr{O}_{Y,P}$. So we have another short exact sequence

$$0 \to \mathscr{I}_{Y,P} \to \mathscr{O}_{X,P} \to \mathscr{O}_{Y,P} \to 0.$$

Thus, for each points $P \in X$, we have a short exact sequence

$$0 \to \mathscr{I}_{Y,P} \to \mathscr{O}_{X,P} \to i_*(\mathscr{O}_Y)_P \to 0.$$

By Exercise 1.2(c), the sequence

$$0 \to \mathscr{I}_Y \to \mathscr{O}_X \to i_*(\mathscr{O}_Y) \to 0$$

is exact and we conclude that $\mathscr{O}_X/\mathscr{I}_Y \cong i_*(\mathscr{O}_Y)$ by Exercise 1.6(b). □

Exercise 1.22: *Glueing Sheaves.* Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i,j an isomorphism $\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathscr{F}_j|_{U_i \cap U_j}$ such that

(1) for each i, $\varphi_{ii} = \mathrm{id}$, and

(2) for each i,j,k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$.

Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$ such that for each i,j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by glueing the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.

Proof. For each nonempty open set U, we define

$$\mathscr{F}(U) = \left\{ (f_i) \in \prod_i \mathscr{F}_i(U_i \cap U) : \varphi_{ij}(U \cap U_i \cap U_j)(f_i|_{U \cap U_i \cap U_j}) = f_j|_{U \cap U_i \cap U_j} \text{ for all } i,j \right\},$$

and the restriction map $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ by $(f_i) \mapsto (f_i|_{U_i \cap V})$, where $V \subseteq U$. Then we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$ since $\mathscr{F}_i$ are sheaves. Thus, $\mathscr{F}$ is a presheaf.

Let $\{V_j\}$ be an open covering of U.

Uniqueness: Let $f = (f_i) \in \mathscr{F}(U)$ such that $f|_{V_j} = (f_i|_{U_i \cap V_j}) = 0$ for all j. Thus $f_i|_{U_i \cap V_j} = 0$ for all i,j. Since $\{U_i \cap V_j\}$ is an open cover of $U_i \cap U$, so $f_i \in \mathscr{F}_i(U_i \cap U)$ is 0 since $\mathscr{F}_i$ is a sheaf. So f is 0.

Gluabillity: Let $f_j = (f_{j,i}) \in \mathscr{F}(V_j)$ such that $f_j|_{V_j \cap V_k} = f_k|_{V_j \cap V_k}$ for all j,k. By the definition of the restriction map, we have $(f_{j,i}|_{U_i \cap V_j \cap V_k}) = (f_{k,i}|_{U_i \cap V_j \cap V_k})$ for all j,k, i.e. $f_{j,i}|_{U_i \cap V_j \cap V_k} = f_{k,i}|_{U_i \cap V_j \cap V_k}$ for all i,j,k. Since $\mathscr{F}_i$ is a sheaf, there exists some $g_i \in \mathscr{F}_i(U_i \cap U)$ such that $g_i|_{U_i \cap V_j} = f_{j,i}$ for each i. Let $f = (g_i) \in \prod_i \mathscr{F}_i(U_i \cap U)$. Since $f_j = (f_{j,i}) \in \mathscr{F}(V_j)$, we have $\varphi_{il}(V_j \cap U_i \cap U_l)(f_{j,i}|_{V_j \cap U_i \cap U_l}) = f_{j,l}|_{V_j \cap U_i \cap U_l}$ for all i,l. Then, $\varphi_{il}(U \cap U_i \cap U_l)(g_i|_{U \cap U_i \cap U_l})|_{V_j} = \varphi_{il}(V_j \cap U_i \cap U_l)(g_i|_{V_j \cap U_i \cap U_l}) = \varphi_{il}(V_j \cap U_i \cap U_l)(f_{j,i}|_{V_j \cap U_i \cap U_l}) = f_{j,l}|_{V_j \cap U_i \cap U_l} = g_l|_{V_j \cap U_i \cap U_l} = (g_l|_{U \cap U_i \cap U_l})|_{V_j}$. Thus, $\varphi_{il}(U \cap U_i \cap U_l)(g_i|_{U \cap U_i \cap U_l}) = (g_l|_{U \cap U_i \cap U_l})$ and it follows that $f = (g_i) \in \mathscr{F}(U)$. We have $f|_{V_j} = (g_i|_{U_i \cap V_j}) = (f_{j,i}) = f_j$ for all j.

Thus, $\mathscr{F}$ is a sheaf.

For each fixed i and open subset $V \subseteq U_i$,

$$\mathscr{F}|_{U_i}(V) = \mathscr{F}(V) = \left\{ (f_k) \in \prod_k \mathscr{F}_k(U_k \cap V) : \varphi_{kj}(V \cap U_k \cap U_j)(f_k|_{V \cap U_k \cap U_j}) = f_j|_{V \cap U_k \cap U_j} \text{ for all } k, j \right\}.$$

We define $\psi_i(V) : \mathscr{F}|_{U_i}(V) \to \mathscr{F}_i(V)$ by $(f_i) \mapsto f_i$. We now show that $\psi_i(V)$ is surjective. Since φ_{ji} is an isomorphism of sheaves, we have a isomorphism of sections $\varphi_{ji}(U_j \cap V) : \mathscr{F}_j(U_j \cap V) \to \mathscr{F}_i(U_j \cap V)$. Thus, for a given f_i, there exists a unique $f_j \in \mathscr{F}_j(U_j \cap V)$ for each j such that $\varphi_{ji}(U_j \cap V)(f_j) = f_i|_{U_j \cap V}$. Indeed, $f_j = \varphi_{ij}(U_j \cap V)(f_i|_{U_j \cap V})$. Thus, for any k, j, we have

$$\begin{aligned}
\varphi_{jk}(V \cap U_k \cap U_j)(f_j|_{V \cap U_k \cap U_j}) &= \varphi_{ik}(V \cap U_k \cap U_j) \circ \varphi_{ji}(V \cap U_k \cap U_j)(f_j|_{V \cap U_k \cap U_j}) \\
&= \varphi_{ik}(V \cap U_k \cap U_j)(\varphi_{ji}(V \cap U_j)(f_j)|_{V \cap U_k \cap U_j}) \\
&= \varphi_{ik}(V \cap U_k \cap U_j)(f_i|_{V \cap U_j \cap U_k}) \\
&= \varphi_{ik}(V \cap U_k)(f_i|_{V \cap U_k})|_{V \cap U_k \cap U_j} = f_k|_{V \cap U_k \cap U_j}
\end{aligned}$$

Let $f = (f_i)$, then $f \in \mathscr{F}|_{U_i}(V)$. Hence, we have $\psi_i(V)(f) = f_i$ and it follows that $\psi_i(V)$ is surjective. By the construction of f, we know that such a f is unique. So $\psi_i(V)$ is injective. We now obtain an isomorphism of sheaves $\psi_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i$. By our definition of $\psi_i(V)$, we have that for each i, j and open subset $V \subseteq U_i \cap U_j$, $\psi_j(V) = \varphi_{ij} \circ \psi_i(V)$ on $U_i \cap U_j$. So, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$ for each i, j.

The uniqueness of $\mathscr{F}$ is clear once we reduce to stalks. $\square$