Algebraic Geometry: Midterm

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Question 1 (40 points): *Explain the following concepts:*

(1) *Sheaves and a stalk of a sheaf,*
(2) *Spectrum of a ring,*
(3) *Morphisms between locally ringed spaces,*
(4) *Reduced schemes, irreducible schemes and integral schemes,*
(5) *Finite type morphisms and finite morphisms,*
(6) *Open immersions and closed immersions,*
(7) *Dimension of a scheme,*
(8) *Fiber products of schemes.*

Proof. (1) Let X be a topological space. Let $\mathfrak{Top}(X)$ be the category such that the objects of $\mathfrak{Top}(X)$ are open subsets of X and the only morphisms are inclusions, i.e.

$$\mathrm{Hom}(V,U)=\begin{cases}\varnothing, \text{ if } V \nsubseteq U \\ \{i_{VU}\}, \text{ if } V \subseteq U\end{cases},$$

where $i_{VU} : V \hookrightarrow U$ is the inclusion map.

A sheaf is a contravariant functor $\mathscr{F}$ from $\mathfrak{Top}(X)$ to a fixed category $\mathfrak{C}$ satisfying the following two conditions

Uniqueness: If U is an open subset and $\{V_i\}$ is an open covering of U and if $s, t \in \mathscr{F}(U)$ such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$.

Gluability: If U is an open subset and $\{V_i\}$ is an open covering of U and if for each i, $s_i \in \mathscr{F}(V_i)$ are elements such that $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there exists $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$.

Let p be a point of X. The stalk of a sheaf $\mathscr{F}$ at p, denoted by $\mathscr{F}_p$, is defined to be $\mathscr{F}_p = \varinjlim_U \mathscr{F}(U)$, where U runs through all open neighborhoods of p.

(2) Let A be a ring. Spec $A = \{$ prime ideals of $A\}$ is a topological space endowed with Zariski topology. For an open set $U \subseteq \mathrm{Spec}\ A$, there is a sheaf on Spec A defined by

$$\mathcal{O}_{\mathrm{Spec}\ A}(U)=\left\{ s: U \to \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}} \;\middle|\; \begin{array}{l}\text{for each } \mathfrak{p}\in U, s(\mathfrak{p})\in A_{\mathfrak{p}} \text{ and} \\ \exists \text{ a neighborhood } V \text{ of } \mathfrak{p} \text{ contained in } U \text{ and } a, f \in A \\ \text{such that } \forall \mathfrak{q}\in V, f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}\end{array}\right\}.$$

The spectrum of A is $(\mathrm{Spec}\ A, \mathcal{O}_{\mathrm{Spec}\ A})$.

(3) A morphism of locally ringed spaces is a morphism $(f, f^{\#})$ of ringed spaces, i.e. a continuous map $f : X \to Y$ and a morphism $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y, such that for each

point $P \in X$, the induced map of local rings $f_P^{\#} : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ is a local homomorphism of local rings.

(4) A reduced scheme X is a scheme such that $\mathcal{O}_X(U)$ has no nilpotent elements for every open subsets $U \subseteq X$.

An irreducible scheme X is a scheme such that $\mathrm{sp}(X)$ is irreducible.

An integral scheme X is a scheme such that $\mathcal{O}_X(U)$ is an integral domain for every open subsets $U \subseteq X$.

(5) A morphism $f : X \to Y$ of schemes is of finite type if there exists an open affine covering of Y, $V_i = \mathrm{Spec}\ B_i$, such that for each i, $f^{-1}(V_i)$ can be covered by a finite number of open affine subsets $U_{ij} = \mathrm{Spec}\ A_{ij}$, where A_{ij} is a finitely generated B_i-algebra.

A morphism $f : X \to Y$ of schemes is finite if there exists an open affine covering of Y, $V_i = \mathrm{Spec}\ B_i$, such that for each i, $f^{-1}(V_i) = \mathrm{Spec}\ A_i$ for some ring A_i, where A_i is a B_i-algebra which is a finitely generated B_i-module.

(6) An open immersion $f : X \to Y$ is a morphism of schemes which induces an isomorphism between X and an open subscheme of Y.

A closed immersion $f : X \to Y$ is a morphism of schemes such that it induces a homeomorphism between $\mathrm{sp}(X)$ and a closed subset of $\mathrm{sp}(Y)$ and the induced map $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves is surjective.

(7) The dimension of a scheme X is its dimension as a topological space, i.e. the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X.

(8) Let S be a scheme and X, Y are schemes over S. The fibred product of X and Y over S is the scheme $X \times_S Y$ together with morphisms $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ satisfying the following universal property:

For any given morphisms $X \to S$, $Y \to S$, any scheme Z over S, the given morphisms $f : Z \to X$ and $g : Z \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd},$$

there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$, i.e. we have a commutative diagram

$$\begin{tikzcd}
& Z \arrow[d, dotted, "\theta"'] \arrow[ddl, "f"'] \arrow[ddr, "g"] & \\
& X \times_S Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

$\square$

Question 2 (10 points): *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism between sheaves. Show that ϕ is surjective if and only if the induced morphism on stalks $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$ is surjective for all p.*

Proof. We first claim that for each p, $(\text{im } \phi)_p = \text{im } \phi_p$. Let $\mathscr{H} : U \mapsto \text{im } \phi(U)$ be a presheaf. Then, we have $(\text{im } \phi)_p = \mathscr{H}_p^+ = \mathscr{H}_p$. It suffices to prove that $\mathscr{H}_p = \text{im } \phi_p$. Let $x \in \text{im } \phi_p$, then there exists $y \in \mathscr{F}_p$ such that $\phi_p(y) = x$. By the property of direct limit, there exists open neighborhoods U, V of p and $s \in \mathscr{F}(U)$, $t \in \mathscr{G}(V)$ such that $s_p = y$ and $t_p = x$, where s_p and t_p are the image of s and t in stalks respectively. Then there exists $W \subseteq U \cap V$ containing p. By shirinking W, we may assume that $\phi(W)(s|_W) = t|_W$. Thus, $x = (t|_W)_p \in \mathscr{H}_p$. Thus, $\text{im } \phi_p \subseteq \mathscr{H}_p$. Conversely, take $x \in \mathscr{H}_p$. Then there exists an open neighborhood of p and $t \in \mathscr{H}(U) = \text{im } \phi(U)$ such that $t_p = x$. Then, there exists $s \in \mathscr{F}(U)$ such that $\phi(U)(s) = t$. Passing to the stalks, we obtain that $\phi_p(s_p) = t_p = x$. Thus, $x \in \text{im } \phi_p$. Thus, $\mathscr{H}_p \subseteq \text{im } \phi_p$. Thus, $\mathscr{H}_p = \text{im } \phi_p$ and it follows that $(\text{im } \phi)_p = \text{im } \phi_p$

Now, by Proposition 1.1, ϕ is surjective if and only if $\text{im } \phi = \mathscr{G}$ if and only if $(\text{im } \phi)_p = \text{im } \phi_p = \mathscr{G}_p$ for all p. This means that ϕ is surjective if and only if the induced morphism on stalks $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$ is surjective for all p. $\square$

Question 3 (10 points): *Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subset X$, show that the set $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf*

$$\mathscr{H}om : U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$$

is a sheaf.

Proof. For any $V \subseteq U$, $\mathscr{F}|_U(V) = \mathscr{F}(V)$ and $\phi \in \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a family of compatible homomorphisms $\phi(V) : \mathscr{F}(V) \to \mathscr{G}(V)$ of abelian groups. Suppose $\phi, \psi \in \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, then we can define a binary operation $+$ on $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ by $(\phi + \psi)(V) = \phi(V) + \psi(V)$. We now illustrate that it is well-defined. Since ϕ, ψ are morphisms of sheaves, then for any $W \subseteq V \subseteq U$ and any $s \in \mathscr{F}(V)$, we have $\phi(V)(s)|_W = \phi(W)(s|_W)$ and $\psi(V)(s)|_W = \psi(W)(s|_W)$. Thus, $(\phi + \psi)(V)(s)|_W = \phi(V)(s)|_W + \psi(V)(s)|_W = \phi(W)(s|_W) + \psi(W)(s|_W) = (\phi + \psi)(W)(s|_W)$. This means that $\phi + \psi \in \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$. Clearly, $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a group with identity $0 = \{0 : \mathscr{F}(V) \to \mathscr{G}(V)\}_V$ and the inverse of ϕ is given by $-\phi = \{-\phi(V) : \mathscr{F}(V) \to \mathscr{G}(V)\}_V$. Since $\phi(V)$ and $\psi(V)$ are homomorphisms of abelian groups, we see that $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is abelian.

We point out that the restriction map $\rho_{UV} : \mathscr{H}om(U) \to \mathscr{H}om(V)$ is defined by $\phi \mapsto \phi|_V$, where $\phi|_V(W) = \phi(W)$ for all $W \subseteq V \subseteq U$. Let U be an open subset of X and $\{V_i\}$ an open covering of U. Observe that

(1) Suppose $\phi \in \mathscr{H}om(U)$, i.e. $\phi : \mathscr{F}|_U \to \mathscr{G}|_U$ is a morphism of sheaves. If $\phi|_{V_i} = 0$, then $\phi = 0$. Indeed, take $W \subseteq U$, then $W = \bigcup_i (W \cap V_i)$. Consider the following commutative diagram

$$\begin{tikzcd} \mathscr{F}(W) \arrow[r, "\phi(W)"] \arrow[d] & \mathscr{G}(W) \arrow[d] \\ \mathscr{F}(W \cap V_i) \arrow[r, "\phi(W \cap V_i)"] & \mathscr{G}(W \cap V_i) \end{tikzcd} .$$

For any $s \in \mathscr{F}(W)$, $\phi(W)(s)|_{W\cap V_i} = \phi(W \cap V_i)(s|_{W\cap V_i}) = \phi|_{V_i}(W \cap V_i)(s|_{W\cap V_i}) = 0$ as $\phi|_{V_i} = 0$ for all i. So, $\phi(W)(s) = 0$ since $\mathscr{G}$ is a sheaf. This implies that $\phi(W) = 0$. As $W \subseteq U$ is arbitrary, we see that $\phi = 0$.

(2) Suppose $\phi_i \in \mathscr{H}om(V_i)$, i.e. $\phi_i : \mathscr{F}|_{V_i} \to \mathscr{G}|_{V_i}$ is a morphism of sheaves, such that $\phi_i|_{V_i\cap V_j} = \phi_j|_{V_i\cap V_j}$. Then there exists $\phi \in \mathscr{H}om(U)$ such that $\phi|_{V_i} = \phi_i$ for all i. Indeed, we first define $\phi(W) = \phi_i(W)$ if $W \subseteq V_i$. It is well-defined since if $W \subseteq V_i \cap V_j$, then $\phi_i(W) = \phi_i|_{V_i\cap V_j}(W) = \phi_j|_{V_i\cap V_j}(W) = \phi_j(W)$.

For any $W \subseteq U$, we can define $\phi(W)$ which makes the following diagram commute

$$\begin{tikzcd}
\mathscr{F}(W) \arrow[r, "\phi(W)"] \arrow[d] & \mathscr{G}(W) \arrow[d] \\
\mathscr{F}(W \cap V_i) \arrow[r, "\phi(W\cap V_i)"] & \mathscr{G}(W \cap V_i)
\end{tikzcd}.$$

Indeed, for any $s \in \mathscr{F}(W)$, set $t_i = \phi(W \cap V_i)(s|_{W\cap V_i}) = \phi_i(W \cap V_i)(s|_{W\cap V_i}) \in \mathscr{G}(W \cap V_i)$ for each i. Note that $W = \bigcup_i (W \cap V_i)$ and $\mathscr{G}$ is a sheaf. There exists $t \in \mathscr{G}(W)$ such that $t|_{W\cap V_i} = t_i$. Now, we define $\phi(W)$ by $s \mapsto t$. Then $\phi(W) : \mathscr{F}(W) \to \mathscr{G}(W)$ is a group homomorphism making the diagram commute. We now check that $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism. It suffices to prove that, for any $V \subseteq W \subseteq U$ and any $s \in \mathscr{F}(W)$, $\phi(W)(s)|_V = \phi(V)(s|_V)$. Considering the following commutative diagram

$$\begin{tikzcd}
\mathscr{G}(W) \arrow[r] \arrow[d] \arrow[dr] & \mathscr{G}(V) \arrow[d] \\
\mathscr{G}(W \cap V_i) \arrow[r] & \mathscr{G}(V \cap V_i)
\end{tikzcd},$$

we have $[\phi(W)(s)|_V]|_{V\cap V_i} = (\phi(W)(s)|_{W\cap V_i})|_{V\cap V_i} = \phi(W\cap V_i)(s|_{W\cap V_i})|_{V\cap V_i} = \phi_i(W\cap V_i)(s|_{W\cap V_i})|_{V\cap V_i} = \phi_i(V \cap V_i)(s|_{V\cap V_i}) = \phi(V \cap V_i)(s|_{V\cap V_i}) = [\phi(V)(s|_V)]|_{V\cap V_i}$. Since $\mathscr{G}$ is a sheaf, we see that $\phi(W)(s)|_V = \phi(V)(s|_V)$. Thus, ϕ is indeed a morphism of sheaves and $\phi \in \mathscr{H}om(U)$. By our construction, we have $\phi|_{V_i} = \phi_i$.

Thus, $\mathscr{H}om$ is indeed a sheaf. □

Question 4 (10 points): *Let $\phi : A \to B$ be a homomorphism of rings, and let $f : Y = \mathrm{Spec}B \to X = \mathrm{Spec}A$ be the induced morphism of affine schemes. Show that ϕ is injective if and only if the map of sheaves $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is dominant, i.e. $f(Y)$ is dense in X.*

Proof. Suppose that ϕ is injective. We first show that for any $g \in A$, $f^\#(D(g))$ is injective. Let $\overline{g} \in B$ be the image of g via ϕ. Consider the map $\overline{\phi} : A_g \to B_{\overline{g}}$ by $a/g^n \mapsto \phi(a)/\phi(g)^n$. We have that $\overline{\phi}$ is injective. Indeed, if $\phi(a)/\phi(g)^n = 0$, there exists $k \in \mathbb{N}$ such that $\phi(g)^k\phi(a) = 0$, then $g^k a = 0$ since φ is injective. Thus, $a/g^n = 0$ and it follows that $\overline{\phi}$ is injective. Correspondingly, $f^\#(D(g)) : \mathcal{O}_X(D(g)) \to \mathcal{O}_Y(D(\overline{g})) = \mathcal{O}_Y(f^{-1}(D(g))) = (f_*\mathcal{O}_Y)(D(g))$ is injective. Note that $D(g)$ is a base for the topology on X. For any $\mathfrak{p} \in X$, we have that $f^\#_{\mathfrak{p}} = \varinjlim_{D(g)\ni\mathfrak{p}} f^\#(D(g))$, where $D(g)$ runs through all principal open subsets containing $\mathfrak{p}$, is also injective since direct limit functor is exact. So, we conclude that $f^\#$ is injective.

Conversely, if $f^\#$ is injective. By taking global section, we know that $\phi : A \to B$ is injective.

In this case, we may identity A with a subring of B. We now show that $f(Y)$ is dense in X. We claim that for any $U \subseteq X$, we have $\overline{U} = V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p})$. Indeed, we have $U \subseteq V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p})$, and so $\overline{U} \subseteq V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p})$. Conversely, let $\overline{U} = V(\mathfrak{a})$, then for any $\mathfrak{q} \in V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p})$, we have $\mathfrak{q} \supset \bigcap_{\mathfrak{p}\in U} \mathfrak{p} \supset \mathfrak{a}$ since $\mathfrak{p} \in U \Rightarrow \mathfrak{p} \supset \mathfrak{a}$. Thus, $V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p}) \subseteq V(\mathfrak{a}) = \overline{U}$. So, we finish the proof of the claim. It follows that $\overline{f(Y)} = V(\bigcap_{\mathfrak{p}\in f(Y)} \mathfrak{p}) = V(\bigcap_{\mathfrak{q}\in Y} f(\mathfrak{q})) = V(\bigcap_{\mathfrak{q}\in Y}(\mathfrak{q} \cap A)) = V(\mathrm{nil}(B) \cap A) = V(\mathrm{nil}(A)) = V(0) = X$. Thus, f is dominant. $\square$

Question 5 (10 points): *Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Give a morphism $f : X \to \mathrm{Spec} A$, we have an associated map on sheaves $f^\# : \mathcal{O}_{\mathrm{Spec} A} \to f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map*

$$\alpha : \mathrm{Hom}_{\text{schemes}}(X, \mathrm{Spec} A) \to \mathrm{Hom}_{\text{rings}}(A, \Gamma(X, \mathcal{O}_X)).$$

Show that α is bijective.

Proof. By the Proposition 2.3, we know that there exists a bijection

$$\beta : \mathrm{Hom}_{\text{schemes}}(\mathrm{Spec}\ B, \mathrm{Spec}\ A) \to \mathrm{Hom}_{\text{rings}}(A, B),$$

by $(f, f^\#) \mapsto f^\#(\mathrm{Spec}\ A)$. Thus, if X is an affine scheme, then α is bijective.

Note that for each $x \in X$, there exists an open neighborhood U_x such that $(U_x, \mathcal{O}_X|_{U_x})$ is an affine scheme. We may write $X = \bigcup_i U_i$, where each U_i is an affine scheme. Let $U = U_i$. Define $\rho_i : \mathrm{Hom}_{\text{schemes}}(X, \mathrm{Spec}\ A) \to \mathrm{Hom}_{\text{schemes}}(U, \mathrm{Spec}\ A)$ by $(f, f^\#) \to (f|_U, f^\#|_U)$, where, for each $V \subseteq \mathrm{Spec}\ A$, $f^\#|_U(V)$ is defined by the following commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\mathrm{Spec}\ A}(V) \arrow[r, "f^\#(V)"] \arrow[d, equal] & f_*\mathcal{O}_X(V) \arrow[r, equal] \arrow[d, "\mathrm{Res}"] & \mathcal{O}_X(f^{-1}(V)) \arrow[d, "\mathrm{Res}"] \\
\mathcal{O}_{\mathrm{Spec}\ A}(V) \arrow[r, "f^\#|_U(V)"] & (f|_U)_*(\mathcal{O}_X|_U)(V) \arrow[r, equal] & \mathcal{O}_X(f^{-1}(V) \cap U)
\end{tikzcd}$$

This implies that the map

$$\rho : \mathrm{Hom}_{\text{schemes}}(X, \mathrm{Spec}\ A) \to \prod_i \mathrm{Hom}_{\text{schemes}}(U_i, \mathrm{Spec}\ A)$$

defined by $(f, f^\#) \mapsto ((f|_{U_i}, f^\#|_{U_i}))_i$ is injective. Now, consider the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\text{schemes}}(X, \mathrm{Spec}\ A) \arrow[r, "\alpha"] \arrow[d, "\rho"] & \mathrm{Hom}_{\text{rings}}(A, \Gamma(X, \mathcal{O}_X)) \arrow[d, "\gamma"] \\
\prod_i \mathrm{Hom}_{\text{schemes}}(U_i, \mathrm{Spec}\ A) \arrow[r, "\prod \beta_i"] & \prod_i \mathrm{Hom}_{\text{rings}}(A, \Gamma(U_i, \mathcal{O}_X|_{U_i}))
\end{tikzcd}$$

we conclude that α is injective since β_i are isomorphisms.

Here, γ is defined by $\gamma_i : \mathrm{Hom}_{\mathrm{rings}}(A, \Gamma(X, \mathcal{O}_X)) \to \mathrm{Hom}_{\mathrm{rings}}(A, \Gamma(U_i, \mathcal{O}_X|_{U_i}))$ by $\varphi \mapsto \mathrm{Res}_{X,U_i} \circ \varphi$, i.e. the composition $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(U_i)$.

Now, we are going to prove that α is surjective. Let $\varphi \in \mathrm{Hom}_{\mathrm{rings}}(A, \Gamma(X, \mathcal{O}_X))$, then there exists a unique $f_i = (f_i, f_i^{\#}) \in \mathrm{Hom}_{\mathrm{schemes}}(U_i, \mathrm{Spec}\ A)$ for each i such that $\beta_i(f_i) = \gamma_i(\varphi)$. We now want to glue these f_i to a morphism $f \in \mathrm{Hom}_{\mathrm{schemes}}(X, \mathrm{Spec}\ A)$.

First, for any affine open subset $W \subseteq U_i \cap U_j$, the image of $f_i|_W$ in $\mathrm{Hom}_{\mathrm{rings}}(A, \Gamma(W, \mathcal{O}_X|_W))$ is $\mathrm{Res}_{U_i,W} \circ \beta(f_i) = \mathrm{Res}_{U_i,W} \circ \gamma_i(\varphi) = \mathrm{Res}_{U_i,W} \circ \mathrm{Res}_{X,U_i} \circ \varphi = \mathrm{Res}_{X,W} \circ \varphi$. Similarly, the image of $f_j|_W$ in $\mathrm{Hom}_{\mathrm{rings}}(A, \Gamma(W, \mathcal{O}_X|_W))$ is $\mathrm{Res}_{X,W} \circ \varphi$. Thus, $f_i|_W$ and $f_j|_W$ have the image in $\mathrm{Hom}_{\mathrm{rings}}(A, \Gamma(W, \mathcal{O}_X|_W))$. Thus $f_i|_W = f_j|_W$ for any affine subset $W \subseteq U_i \cap U_j$. We conclude that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for any i, j. This is because ρ is injective, where we replace X by $U_i \cap U_j$ and each U_i by W.

Set $Y = \mathrm{Spec}\ A$. We may define $f(x) = f_i(x)$ if $x \in U_i$ for some i, so $f : X \to Y$ is a continuous map. Since $(f_i)_*(\mathcal{O}_X|_{U_i})(V) = (\mathcal{O}_X|_{U_i})(f_i^{-1}(V)) = \mathcal{O}_X(f_i^{-1}(V))$ for each $V \subseteq Y$, then $f_i^{\#} : \mathcal{O}_Y \to (f_i)_*(\mathcal{O}_X|_{U_i})$ induces a map of rings $f_i^{\#}(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f_i^{-1}(V))$ for each $V \subseteq Y$. Note that $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$. For each $t \in \mathcal{O}_Y(V)$, let $s_i = f_i^{\#}(V)(t)$, then there exists a unique $s \in \mathcal{O}_X(f^{-1}(V))$ such that $s|_{f_i^{-1}(V)} = s_i$. Define $f^{\#}(V)(t) = s$, we then obtain a ring homomorphism $f^{\#}(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$, which gives us a morphism of sheaves $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ with $f^{\#}|_{U_i} = f_i^{\#}$. Thus, we have a morphism of schemes $f = (f, f^{\#})$ such that $f|_{U_i} = f_i$. Thus, $\gamma_i(\alpha(f) - \varphi) = \gamma_i(\alpha(f)) - \gamma_i(\varphi) = \beta_i(\rho(f)) - \gamma_i(\varphi) = \beta_i(f_i) - \gamma_i(\varphi) = 0$ for each i. Since $\mathcal{O}_X$ is a sheaf, we know that $\alpha(f) - \varphi = 0$. Thus, α is surjective. $\square$

Question 6 (10 points): *Show that a morphism $f : X \to Y$ between schemes is locally of finite type if and only if for every open affine subset $V = \mathrm{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \mathrm{Spec} A_j$, where each A_j is a finitely generated B-algebra.*

Proof. Suppose f is locally of finite type. We first observe that

(1) **If $U = \mathrm{Spec}\ A \subseteq Y$ is an affine open subset satisfying that $f^{-1}(U)$ can be covered by open affine subsets $\mathrm{Spec}\ A_i$ with each A_i a finitely generated A-algebra, then for any $g \in A$, there exist some rings B_i such that $f^{-1}(\mathrm{Spec}\ A_g)$ can be covered by open affine subsets $\mathrm{Spec}\ B_i$ with each B_i a finitely generated A_g-algebra.** Indeed, $\mathrm{Spec}\ A_g = D(g)$ is a principal open subset in $\mathrm{Spec}\ A$. Then $f^{-1}(D(g)) = f^{-1}(U \cap D(g)) = \bigcup_i (f^{-1}(D(g)) \cap \mathrm{Spec}\ A_i)$, where each A_i is a finitely generated A-algebra. Let $f_i = f|_{\mathrm{Spec}\ A_i}$ and $\varphi_i : A \to A_i$ be the ring homomorphism induced by $f_i : \mathrm{Spec}\ A_i \to \mathrm{Spec}\ A$. Then, for any $g \in A$, we have $f^{-1}(D(g)) \cap \mathrm{Spec}\ A_i = D(\varphi_i(g))$, where $D(\varphi_i(g))$ is a principal open subset in $\mathrm{Spec}\ A_i$. Indeed, $f^{-1}(D(g)) \cap \mathrm{Spec}\ A_i = f_i^{-1}(D(g)) = f_i^{-1}(U - V(g)) = f_i^{-1}(U) - f_i^{-1}(V(g)) = \mathrm{Spec}\ A_i - V(\varphi_i(g)) = D(\varphi_i(g))$. So, $f^{-1}(D(g)) \cap \mathrm{Spec}\ A_i = D(\varphi_i(g)) \cong \mathrm{Spec}\ (A_i)_{\varphi_i(g)}$ is a princial subset in $\mathrm{Spec}\ A_i$ and $(A_i)_{\varphi_i(g)}$ is a finitely generated A_g-algebra.

(2) **If $(f_1, \cdots, f_n) = A$ and each $\mathrm{Spec}\ A_{f_i}$ is an affine scheme satisfying that $f^{-1}(\mathrm{Spec}\ A_{f_i})$ can be covered by $U_{ij} = \mathrm{Spec}\ A_{ij}$ with A_{ij} a finitely generated A_{f_i}- algebra, then $f^{-1}(\mathrm{Spec}\ A)$ can be covered by $U_{ij} = \mathrm{Spec}\ A_{ij}$ with A_{ij} a finitely generated A-algebra.** Indeed, since $(f_1, \cdots, f_n) = A$, we see that $\bigcap V(f_i) = V(f_1, \cdots, f_n) = V(1) = \varnothing$. Thus, $\bigcup D(f_i) = \mathrm{Spec}\ A$. Let $U = \mathrm{Spec}\ A$, then $U \subseteq \bigcup_{i=1}^n \mathrm{Spec}\ A_{f_i}$ and $f^{-1}(U) \subseteq \bigcup_{i=1}^n f^{-1}(\mathrm{Spec}\ A_{f_i}) = \bigcup_{i=1}^n \bigcup_j \mathrm{Spec}\ A_{ij}$. Since A_{f_i} is a finitely generated A-algebra, we see that A_{ij} is a finitely generated A-algebra.

Since f is locally of finite type, there exists an open affine covering $V_i = \mathrm{Spec}\ B_i$ of Y such that each $f^{-1}(V_i)$ can be covered by open subsets $U_{ij} = \mathrm{Spec}\ A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. Since $V = \mathrm{Spec}\ B$ is quasi-compact, we can cover $\mathrm{Spec}\ B$ with a finite number of principal open sets Spec B_{g_j}, each of which is principal in some Spec B_i. Indeed, given any point $\mathfrak{p} \in \text{Spec } B \cap \text{Spec } B_i$, we can find an open neighborhood of $\mathfrak{p}$ in Spec $B \cap$ Spec B_i that is simultaneously principal open in both Spec B and Spec A_i. Let Spec $A_f \cong D(f)$ be a principal open subset of Spec A contained in Spec $B \cap$ Spec B_i and containing $\mathfrak{p}$. Let Spec $(B_i)_g \cong D(g)$ be a principal open subset of Spec B_i contained in Spec B_f and containing $\mathfrak{p}$. Then $g \in \Gamma(\text{Spec } B_i, \mathcal{O}_X)$ restricts to an element $g' \in \Gamma(\text{Spec } B_f, \mathcal{O}_X) = B_f$. Then, by using a similar argument in (1), we know that $D(g) = D(g) \cap \text{Spec } B_f = D(g')$, where $D(g')$ is a principal open subset in Spec B_f, so Spec $(B_i)_g \cong$ Spec $(B_f)_{g'}$. If $g' = h/f^n$ with $h \in B$, then Spec $(B_f)_{g'} =$ Spec $B_{fh} \cong D(fh)$ since $(B_f)_{g'} = B_{fh}$. This implies that Spec $(B_i)_g$ is also a principal open set in B.

By (1), each Spec B_{g_j} satisfies that $f^{-1}(\text{Spec } B_{g_j})$ can be covered by open affine subsets Spec A_i with A_i a finitely generated B_{g_j}-algebra. Since Spec B is covered by all of Spec B_{g_j}, so all g_j generate B. By (2), we see that $f^{-1}(V)$ can be covered by open affine subsets $U_j = \text{Spec} A_j$, where each A_j is a finitely generated B-algebra. We are done.

The other direction is trivial. □

Question 7 (10 points): *Using valuative criterion to show the following claims:*

1. *A composition of proper morphisms is proper,*
2. *Products of proper morphisms are proper. (If $f : X \to Y$ and $f' : X' \to Y'$ are two morphisms of schemes over S, then the product morphism is $f \times f' : X \times_S X' \to Y \times_S Y'$.)*

Proof. (1) Suppose $f : X \to Y$ and $g : Y \to Z$ are two proper morphisms, which are of finite type. We claim that $g \circ f$ is also of finite type. Indeed, Let $W =$ Spec C be an affine open subset of Z. Since g is of finite type, then $g^{-1}(W)$ can be covered by a finite number of affine open subset $V_i =$ Spec B_i, where B_i is a finitely generated C-algebra. Again, f is of finite type, then each $f^{-1}(V_i)$ can be covered by a finite number of affine open subsets $U_{ij} =$ Spec A_{ij}, where each A_{ij} is a finitely generated B_i-algebra. Thus, A_{ij} is a finitely generated C-algebra. So, $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \subseteq \bigcup_i f^{-1}(V_i) \subseteq \bigcup_{i,j} U_{ij}$, where the number of U_{ij} is finite. Thus, $g \circ f$ is of finite type.

Now, for every valuation ring R and $K = \text{Frac}(R)$, consider the following commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[dd] & X \arrow[d, "f"] \\ & Y \arrow[d, "g"] \\ T \arrow[r] & Z \end{tikzcd},$$

where $T =$ Spec R and $U =$ Spec K. Let $U \to Y$ be the composed map $U \to X \xrightarrow{f} Y$. Since g is proper, by Valuative Criterion of Properness, there exists a unique morphism $\alpha : T \to Y$ making the following diagram commute

$$\begin{tikzcd} U \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ T \arrow[r] \arrow[ur, dotted, "\alpha"] & Z \end{tikzcd}$$

Again, since f is proper, there exists a unique morphism $\beta : T \to X$ such that the following diagram commute

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r, "\alpha"] \arrow[ur, dotted, "\beta"] & Y \end{tikzcd}$$

Thus, $\beta : T \to X$ is the unique morphism such that the following diagram commute

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "g\circ f"] \\ T \arrow[r] \arrow[ur, dotted, "\beta"] & Z \end{tikzcd}$$

Since $g \circ f$ is of finite type, by Valuative Criterion for Properness, we see that $g \circ f$ is proper.

(2) By definition, $f \times f'$ is the unique morphism making the following diagram commute

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times_S X' \arrow[l, "p"'] \arrow[r, "p'"] \arrow[d, "f\times f'"] & X' \arrow[d, "f'"] \\ Y & Y \times_S Y' \arrow[l, "q"'] \arrow[r, "q'"] & Y' \end{tikzcd} .$$

Now, for every valuation ring R and $K = \mathrm{Frac}(R)$, let $T = \mathrm{Spec}\ R$ and $U = \mathrm{Spec}\ K$. For any given morphisms $\phi : U \to X \times_S X'$ and $\psi : T \to Y \times_S Y'$ making the following diagram commute

$$\begin{tikzcd} U \arrow[r, "\phi"] \arrow[d, "i"'] & X \times_S X' \arrow[d, "f\times f'"] \\ T \arrow[r, "\psi"] & Y \times_S Y' \end{tikzcd} ,$$

composed with p, q and p', q' respectively, we obtain two commutative diagrams

$$\begin{tikzcd} U \arrow[r, "p\circ\phi"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r, "q\circ\psi"] \arrow[ur, dotted, "x"] & Y \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "p'\circ\phi"] \arrow[d, "i"'] & X' \arrow[d, "f'"] \\ T \arrow[r, "q'\circ\psi"] \arrow[ur, dotted, "x'"] & Y' \end{tikzcd}$$

Since f and f' are proper, there exist unique morphism $x : T \to X$ and $x' : T \to X'$ making the above two diagram commute by Valuative Criterion for Properness.

By the universal property of fibred product, there exists a unique morphism $h : T \to X \times_S X'$

making the following diagram commute

$$\begin{tikzcd}
X & X \times_S X' \arrow[l, "p"'] \arrow[r, "p'"] & X' \\
& T \arrow[ul, "x"] \arrow[u, dotted, "h"] \arrow[ur, "x'"'] &
\end{tikzcd}$$

These commutative diagrams tell us that $p \circ (h \circ i) = p \circ \phi$ and $p' \circ (h \circ i) = p' \circ \phi$, which implies that $h \circ i = \phi$ by the universal property of fibred product. Similarly, we have $f \circ x = q \circ \psi$ and $f' \circ x' = q' \circ \psi$. Now, considering the following commutative diagram

$$\begin{tikzcd}
& T \arrow[dl, "x"'] \arrow[d, "h"] \arrow[dr, "x'"] & \\
X \arrow[d, "f"'] & X \times_S X' \arrow[l, "p"'] \arrow[r, "p'"] \arrow[d, "f \times f'"] & X' \arrow[d, "f'"] \\
Y & Y \times_S Y' \arrow[l, "q"'] \arrow[r, "q'"] & Y'
\end{tikzcd},$$

we have that $q' \circ (f \times f') \circ h = q' \circ \psi$ and $q \circ (f \times f') \circ h = q \circ \psi$. Thus, $(f \times f') \circ h = \psi$ by the universal property of fibred product.

Thus, there exists a unique morphism $h : T \to X \times_S X'$ making the following diagram commute

$$\begin{tikzcd}
U \arrow[r, "\phi"] \arrow[d, "i"'] & X \times_S X' \arrow[d, "f \times f'"] \\
T \arrow[r, "\psi"] \arrow[ur, dotted, "h"] & Y \times_S Y'
\end{tikzcd}$$

It remains to prove that $f \times f'$ is of finite type. We claim that morphisms of finite type are stable under base extension, i.e. if the morphism $f : X \to S$ of schemes is of finite type, then for any $g : Z \to S$, the second projection $p_2 : X \times_S Z \to Z$ is of finite type.

Let $\{S_i\}$ be an affine open cover of S. By the construction of fibred product, we see that $X \times_S Z = \bigcup_i f^{-1}(S_i) \times_{S_i} g^{-1}(S_i)$. Note that the finite type property is affine local, it reduces to the case that S is affine, say $S = \text{Spec } R$. Take an open affine subset $V \subseteq Z$ with $V = \text{Spec } C$. By the universal property, we see that $(p_2)^{-1}(V) = X \times_S V$. Since f is of finite type, $X = f^{-1}(S)$ can be covered by a finite number of affine open subsets $U_i = \text{Spec } A_i$, where each A_i is a finitely generated B-algebra. Thus, $(p_2)^{-1}(V)$ can be covered by a finite number of $U_i \times_S V = \text{Spec } (A_i \otimes_R C)$, where $A_i \otimes_R C$ is a finitely generated $B \otimes_R C$-algebra.

Considering the morphism $Y \times_S Y' \to Y$, we know that $X \times_Y (Y \times_S Y') \to (Y \times_S Y')$ is of finite type. Thus, $f \times \text{id} : X \times_S Y' \to Y \times_S Y'$ is of finite type. Similarly, $\text{id} \times f' : X \times_S X' \to X \times_S Y'$ is also of finite type. Now, observe that $f \times f'$ is the composed map of $X \times_S X' \xrightarrow{\text{id} \times f'} X \times_S Y' \xrightarrow{f \times \text{id}} Y \times_S Y'$.

In (1), we have already proved that the composition of morphisms of finite type is also of finite type. Thus, $f \times f'$ is of finite type.

By Valuative Criterion for Properness, we conclude that $f \times f'$ is proper. $\square$