Line bundles and Divisors

Jing YE

November 21, 2020

Contents

1 Vector Bundles

1.1 Schemes as functors

Let S be a fixed scheme. The category $(\mathbf{Sch}/S)$ of schemes over S (or of S-schemes) is the category whose objects are the morphisms $X \to S$ of schemes, and whose morphisms $\mathrm{Hom}(X \to S, Y \to S)$ are the morphisms $X \to Y$ of schemes with the property that

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

The morphism $X \to S$ is called the **structural morphism** of the S-scheme X (and often is silently omitted from the notation). The scheme S is also sometimes called the **base scheme**. In case $S = \text{Spec } R$ is an affine scheme, one also speaks about R-schemes or schemes over R instead. For S-schemes X and Y we denote the set of morphisms $X \to Y$ in the category of S-schemes by $\text{Hom}_S(X, Y)$ or by $\text{Hom}_R(X, Y)$, if $S = \text{Spec } R$ is affine. By definition the S-scheme $\text{id}_S : S \to S$ is a final object in the category $(\mathbf{Sch}/S)$.

It is natural to attach to a scheme X the contravariant functor

$$h_X : (\mathbf{Sch}) \to (\mathbf{Sets}),$$

$$T \mapsto h_X(T) := \text{Hom}_{(\mathbf{Sch})}(T, X), \qquad \text{(on objects)},$$

$$(f : T' \to T) \mapsto (\text{Hom}(T, X) \to \text{Hom}(T', X),\ g \mapsto g \circ f), \quad \text{(on morphisms)}.$$

Equivalently, we have a covariant functor

$$h_X^{\text{opp}} : (\mathbf{Sch})^{\text{opp}} \to (\mathbf{Sets}),$$

$$T \mapsto h_X(T) := \text{Hom}_{(\mathbf{Sch})}(T, X), \qquad \text{(on objects)},$$

$$(f^{\text{opp}} : T \to T') \mapsto (\text{Hom}(T, X) \to \text{Hom}(T', X),\ g \mapsto g \circ f), \quad \text{(on morphisms)}.$$

The set $\text{Hom}_{(\mathbf{Sch})}(T, X)$ is called the set of **T-valued points** of X. Usually we simply write $X(T)$ instead of $h_X(T) = \text{Hom}_{(\mathbf{Sch})}(T, X)$. If $T = \text{Spec } A$ is an affine scheme, we also set $X(A) := X(\text{Spec } A)$. More generally, we might consider an arbitrary functor $F : (\mathbf{Sch})^{\text{opp}} \to (\mathbf{Sets})$ as a "geometric object" and we call $F(T)$ the set of T-valued points of F.

Now let $\mathcal{C}$ be an arbitrary category (we will mainly use the examples that $\mathcal{C}$ is the category $(\mathbf{Sch}/S)$ for some scheme S). As above we define for every object X of $\mathcal{C}$ the functor

$$h_X^{\text{opp}} : \mathcal{C}^{\text{opp}} \to (\mathbf{Sets}),$$

$$T \mapsto h_X(T) := \text{Hom}_{\mathcal{C}}(T, X),$$

$$(u^{\text{opp}} : T \to T') \mapsto (h_X(u) : h_X(T) \mapsto h_X(T'),\ x \mapsto x \circ u).$$

If $f : X \to Y$ is a morphism in $\mathcal{C}$, for every object T the composition

$$h_f(T) : h_X^{\text{opp}}(T) \to h_Y^{\text{opp}}(T),$$

$$g \mapsto f \circ g$$

defines a morphism $h_f : h_X^{\text{opp}} \to h_Y^{\text{opp}}$ of functors, i.e. a natural transformation between h_X^{opp} and h_Y^{opp}. We obtain a covariant functor $X \mapsto h_X^{\text{opp}}$ from $\mathcal{C}$ to the category $\widehat{\mathcal{C}}$ of functors $\mathcal{C}^{\text{opp}} \to \mathbf{Sets}$.

Recall the Yoneda's lemma in category theory,

Theorem 1.1 (Yoneda's lemma). *Let $X \in \text{obj}(\mathcal{C})$ and $F : \mathcal{C} \to (\mathbf{Sets})$ be a covariant functor. Then the map*

$$y : \text{Hom}_{\widehat{\mathcal{C}}}(\text{Hom}_{\mathcal{C}}(X, -), F) \to F(X),$$

$$\alpha \mapsto \alpha(X)(\text{id}_X).$$

is bijective and functorial in X.

Proof. For $\xi \in F(X)$ we define the map $\alpha_\xi(Y) : \text{Hom}_{\mathcal{C}}(X, Y) \to F(Y)$ by $f \to F(f)(\xi)$ for $f \in \text{Hom}_{\mathcal{C}}(X, Y)$. Then $x : \xi \mapsto \alpha_\xi$ is an inverse map. The functoriality is clear. $\square$

In particular, if X is an S-scheme and $F : \mathcal{C}^{\mathrm{opp}} \to (\mathbf{Sets})$ be a covariant functor. Then the map

$$y : \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X^{\mathrm{opp}}, F) \to F(X),$$

$$\alpha \mapsto \alpha(X)(\mathrm{id}_X).$$

is bijective and functorial in X.

If we apply the Yoneda's lemma to the special case $F = h_Y^{\mathrm{opp}}$ for an object Y of $\mathcal{C}$, we see that the functor $X \mapsto h_X^{\mathrm{opp}}$ induces a bijection

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X^{\mathrm{opp}}, h_Y^{\mathrm{opp}}).$$

That is, the functor $X \mapsto h_X^{\mathrm{opp}}$ is fully faithful.

Definition 1.2. *A functor $F : \mathcal{C}^{\mathrm{opp}} \to (\mathbf{Sets})$ is called **representable** if there exist an object X and an isomorphism $\xi : h_X \xrightarrow{\sim} F$.*

In the sequel we will often not distinguish between an object X and the representing functor h_X^{opp}.

Example 1.3 (Fiber products of functors). *In the category $\widehat{\mathcal{C}}$, fiber products always exist: Let F, G, and H be contravariant functors from $\mathcal{C}$ to $(\mathbf{Sets})$, and let $F \to H$ and $G \to H$ be morphisms of functors. We set for every object $T \in \mathcal{C}$,*

$$(F \times_H G)(T) := F(T) \times_{H(T)} G(T),$$

where the right hand side denotes the fiber product in the category of sets. Then $(F \times_H G)(T)$ is clearly functorial in T and we obtain a functor $F \times_H G \in \widehat{\mathcal{C}}$. The projections

$$p(T) : (F \times_H G)(T) \to F(T) \text{ and } q(T) : (F \times_H G)(T) \to G(T)$$

define morphisms of functors $p : F \times_H G \to F$ and $q : F \times_H G \to G$. Then $(F \times_H G, p, q)$ is the fiber product of F and G over H in $\mathcal{C}$.

If F, G, and H are representable, say $F \cong h_X$, $G \cong h_Y$, and $H \cong h_S$, the fiber product $F \times_H G$ in $\mathcal{C}$ is representable by an object Z if and only if $X \times_S Y$ exists in $\mathcal{C}$ and in this case $Z = X \times_S Y$. Indeed, the universal property of $(X \times_S Y, p, q)$ says that for every object T in $\mathcal{C}$ the morphisms p and q yield a bijection (automatically functorial in T).

$$h_{X \times_S Y}(T) = \mathrm{Hom}(T, X \times_S Y) \xrightarrow{\sim} \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, Y) = h_X(T) \times_{h_S(T)} h_Y(T).$$

1.2 Zariski sheaves, Zariski coverings of functors

Definition 1.4. *Let $F : (\mathbf{Sch}/S)^{\mathrm{opp}} \to (\mathbf{Sets})$ be a (covariant) functor. If $j : U \to X$ is a morphism of S-schemes and $\xi \in F(X)$. We write $j^*\xi = \xi|_U = F(j^{\mathrm{opp}})(\xi) \in F(U)$, where $F(j^{\mathrm{opp}}) : F(X) \to F(U)$ is induced by the morphism $j^{\mathrm{opp}} : X \to U$.*

*We say that F is a **sheaf for the Zariski topology** or a **Zariski sheaf** on $(\mathbf{Sch}/S)$ if the usual sheaf axioms are satisfied, i.e. for every S-scheme X and for every open covering $X = \bigcup_{i \in I} U_i$, we have*

(Sh) Given $\xi_i \in F(U_i)$ for all $i \in I$ such that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$ for all $i, j \in I$, there exists a unique element $\xi \in F(X)$ such that $\xi|_{U_i} = \xi_i$ for all $i \in I$.

We can reformulate the proposition on gluing of morphisms as

Proposition 1.5. *Every representable functor $F : (\mathbf{Sch}/S)^{\mathrm{opp}} \to (\mathbf{Sets})$ is a sheaf for the Zariski topology.*

Let's now recall the definition of subfunctors.

Definition 1.6. *Let $\mathcal{C}$ be a category, and let $F : \mathcal{C} \to \mathbf{Sets}$ be a covariant (resp. contravariant) functor. A covariant (resp. contravariant) functor $G : \mathcal{C} \to \mathbf{Sets}$ is a **subfunctor** of F if*

(1) *For all objects X of $\mathcal{C}$, $G(X) \subseteq F(X)$, and*

(2) *For all morphisms $f : X \to Y$ of $\mathcal{C}$, $G(f)$ is the restriction of $F(f)$ to $G(X)$ (resp. $G(Y)$).*

This relation is often written as $G \subseteq F$.

Definition 1.7.

(1) *Let $H \subseteq F$ be a subfunctor. We say that $H \subseteq F$ is **representable by open immersions** if for all pairs (T, ξ), where T is a scheme and $\xi \in F(T)$, there exists an open subscheme $U_\xi \subseteq T$ with the following property:*

$(*)$ *A morphism $f : T' \to T$ factors through U_ξ if and only if $f^*\xi \in H(T')$.*

(2) *Let I be a set. For each $i \in I$, let $F_i \subseteq F$ be a subfunctor. We say that the collection $(F_i)_{i \in I}$ **covers** F if for every $\xi \in F(T)$ there exists an open covering $T = \bigcup_{i \in I} U_i$ such that $\xi|_{U_i} \in F_i(U_i)$.*

(3) *We call the family $(f_i : F_i \to F)_{i \in I}$ a **Zariski open covering** of F if the collection $(F_i)_{i \in I}$ **covers** F and all $F_i \subset F$ are representable by open immersions.*

Remark. If $F_i \subset F$ is representable by open immersions for all i, then to check $(F_i)_{i \in I}$ covers F, it suffices to check $F(T) = F_i(T)$ whenever T is the spectrum of a field.

We now show that every Zariski sheaf that has a Zariski covering by representable functors is itself representable.

Theorem 1.8. *Let $F : (\mathbf{Sch}/S)^{\mathrm{opp}} \to (\mathbf{Sets})$ be a functor such that*

(1) *F is a sheaf for the Zariski topology,*

(2) *F admits a Zariski open covering $(f_i : F_i \to F)_{i \in I}$ with each F_i is representable.*

Then F is representable.

Proof. Let the F_i be represented by S-schemes X_i. We will show that we may glue the schemes X_i to a scheme X that represents F. The morphisms $F_i \to F$ are representable by open immersions, i.e. $F_i \times_F X \to X$ is an open immersion. Recall that if $U \to X$ is an open immersion, then $\mathrm{Hom}(-, U) \to \mathrm{Hom}(-, X)$ is a monomorphism. Therefore, for all S-schemes T the maps $F_i(T) \times_{F(T)} X(T) \to X(T)$ is injective. So, $F_i(T) \to F(T)$ are injective if we take $X(T) = F(T)$.

For all $i, j \in I$ and all T we can therefore identify $(F_i \times_F F_j)(T)$ with $F_i(T) \cap F_j(T) \subseteq F(T)$. In virtue of this identification, the functors $F_i \times_F F_j$ and $F_j \times_F F_i$ are equal. Let $X_{\{i,j\}}$ be a scheme that represents this functor. Likewise we identify for $i, j, k \in I$ the functors $F_i \times_F F_j \times_F F_k$, $F_j \times_F F_i \times_F F_k$, etc. and write $X_{\{i,j,k\}}$ for them.

The morphisms $X_{\{i,j\}} \to X_i$, induced by the projections $F_i \times_F F_j \to F_i$, are open immersions and we denote their images by U_{ij}. These immersions induce isomorphisms $\psi_{i,j} : X_{\{i,j\}} \xrightarrow{\sim} U_{ij}$. We set $\varphi_{ji} = \psi_{j,i} \circ \psi_{i,j}^{-1} : U_{ij} \cong U_{ji}$ and claim that the tuple i, j, $((X_i)_{i\in I}, (U_{ij}), (\varphi_{ij}))$ is a gluing datum of schemes.

We have to check that

(1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$. Indeed, $\varphi_{ij} \circ \varphi_{ji} = \psi_{i,j} \circ \psi_{j,i}^{-1} \circ \psi_{j,i} \circ \psi_{i,j}^{-1} = \mathrm{id}$.

(2) the cocycle condition, that is,

$$\varphi_{ji}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$$

and

$$(*) \quad \varphi_{kj} \circ \varphi_{ji} = \varphi_{ki} \text{ on } U_{ij} \cap U_{ik}$$

for all $i, j, k \in I$.

For the open subscheme $U_{ij} \cap U_{ik}$ of X_i and for all T we then have $(U_{ij} \cap U_{ik})(T) = U_{ij}(T) \cap U_{ik}(T) = X_{\{i,j,k\}}(T)$ and there is a commutative diagram

$$\begin{tikzcd}
U_{ij} \cap U_{ik} \arrow[r, "\varphi_{ji}|_{U_{ij}\cap U_{ik}}"] \arrow[d, "\cong"] & U_{ji} \cap U_{jk} \arrow[d, "\cong"] \\
X_{\{i,j,k\}} \arrow[r, "\mathrm{id}"] & X_{\{i,j,k\}}
\end{tikzcd}$$

Therefore it suffices to show that the equality $(*)$ holds for the corresponding morphisms between the functors $X_{\{i,j,k\}}$. But this is obvious by definition of φ_{ji}.

Let X be the S-scheme that results from gluing the X_i with respect to this gluing datum. As F is a Zariski sheaf, the open immersions $f_i : X_i \to F$ can be glued to an S-isomorphism $f : X \to F$ and thus F is represented by X. $\square$

1.3 Spectrum of quasi-coherent $\mathcal{O}_X$-algebras

Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. We can define a functor F associated to $(S, \mathcal{A})$ by

$$F : (\mathbf{Sch}/S)^{\mathrm{opp}} \to (\mathbf{Sets}),$$

$$(f^{\mathrm{opp}} : S \to T) \mapsto \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathbf{Alg})}(\mathcal{A}, f_*\mathcal{O}_T),$$

where $f : T \to S$ is an S-scheme. We may write $F(T) = \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathbf{Alg})}(\mathcal{A}, f_*\mathcal{O}_T) = \{\text{all pairs } (f, \varphi)\}$, where $\varphi : \mathcal{A} \to f_*\mathcal{O}_T$, which corresponds to $f^*\mathcal{A} \to \mathcal{O}_T$, is a morphism of $\mathcal{O}_S$-algebras.

Lemma 1.9. *Let F be the functor associated to $(S, \mathcal{A})$ above. Let $g : S' \to S$ be a morphism of schemes. Set $\mathcal{A}' = g^*\mathcal{A}$. Let F' be the functor associated to $(S', \mathcal{A}')$ above. Identify $(\mathbf{Sch}/S')^{\mathrm{opp}}$ as a subcategory of $(\mathbf{Sch}/S)^{\mathrm{opp}}$ and let $F|_{S'}$ be the restriction of F to the subcategory $(\mathbf{Sch}/S')^{\mathrm{opp}}$. Then there is a canonical isomorphism*

$$F' \cong h_{S'}^{\mathrm{opp}} \times_{h_S^{\mathrm{opp}}} F \cong F|_{S'}$$

of functors.

Proof. Let $f' : T \to S'$ and $f : T \to S$ obtained by $f = g \circ f'$. Then, we see that

$$\begin{aligned}\mathrm{Hom}_{(\mathcal{O}'_S\text{-}\mathbf{Alg})}(\mathcal{A}', f'_*\mathcal{O}_T) &\simeq \mathrm{Hom}_{(\mathcal{O}_T\text{-}\mathbf{Alg})}((f')^*\mathcal{A}', \mathcal{O}_T)\\ &\simeq \mathrm{Hom}_{(\mathcal{O}_T\text{-}\mathbf{Alg})}((f')^*g^*\mathcal{A}, \mathcal{O}_T)\\ &\simeq \mathrm{Hom}_{(\mathcal{O}_T\text{-}\mathbf{Alg})}(f^*\mathcal{A}, \mathcal{O}_T)\\ &\simeq \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathbf{Alg})}(\mathcal{A}, f_*\mathcal{O}_T)\end{aligned}$$

For any $f' : T \to S'$, and $f = g \circ f'$,

$$\begin{aligned}(h_{S'}^{\mathrm{opp}} \times_{h_S^{\mathrm{opp}}} F)(T) &= h_{S'}^{\mathrm{opp}}(T) \times_{h_S^{\mathrm{opp}}(T)} F(T)\\ &\cong \mathrm{Hom}(T, S') \times_{\mathrm{Hom}(T,S)} F(T)\\ &= \{(f', (f, \varphi)) : g^*(f') = g \circ f' = f\}\\ &= \{\text{all pairs } (f', \varphi') | \varphi' : \mathcal{A}' \to f'_*\mathcal{O}_T\} = F|_{S'}(T)\\ &= F'(T).\end{aligned}$$

Thus, we see that $F' \cong h_{S'}^{\mathrm{opp}} \times_{h_S^{\mathrm{opp}}} F \cong F|_{S'}$. ☐

Lemma 1.10. *Let F be the functor associated to $(S, \mathcal{A})$ above. If S is affine, then F is representable by the affine scheme* $\mathrm{Spec}(\Gamma(S, \mathcal{A}))$.

Proof. Let $A = \Gamma(S, \mathcal{A})$ and $S = \mathrm{Spec}\ R$. Let $f : T \to S$ be a S-scheme. Then,

$$\begin{aligned}F(T) &\simeq \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathbf{Alg})}(\mathcal{A}, f_*\mathcal{O}_T)\\ &\simeq \mathrm{Hom}_R(\mathcal{A}(S), \Gamma(S, f_*\mathcal{O}_T))\\ &= \mathrm{Hom}_R(A, \Gamma(T, \mathcal{O}_T))\\ &\simeq \mathrm{Hom}_S(T, \mathrm{Spec}(A)).\end{aligned}$$

☐

Lemma 1.11. *Let F be the functor associated to $(S, \mathcal{A})$ above. Then, F is a sheaf for the Zariski topology.*

Proof. Suppose that T is an S-scheme, that $T = \bigcup_{i \in I} U_i$ is an open covering, and that $(f_i, \varphi_i) \in F(U_i)$ such that $(f_i, \varphi_i)|_{U_i \cap U_j} = (f_j, \varphi_j)|_{U_i \cap U_j}$, i.e. $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ and $\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$. This implies that the morphisms $f_i : U_i \to S$ glue to a morphism of schemes $f : T \to S$ such that $f|_{U_i} = f_i$. Thus $f_i^*\mathcal{A} = f^*\mathcal{A}|_{U_i}$. Indeed, let $g_i : U_i \hookrightarrow T$ be an open immersion, then $f_i = f|_{U_i} = f \circ g_i$. Thus, $f_i^*\mathcal{A} = g_i^*(f^*\mathcal{A}) = f^*\mathcal{A}|_{U_i}$.

By assumption the morphisms $\varphi_i : f_i^*\mathcal{A} \to \mathcal{O}_T$, or $\varphi_i : f^*\mathcal{A}|_{U_i} \to \mathcal{O}_T$ agree on $U_i \cap U_j$. Hence, these glue to a morphism of $\mathcal{O}_T$-algebras $f^*\mathcal{A} \to \mathcal{O}_T$. This proves that F satisfies the sheaf condition with respect to the Zariski topology. ☐

Lemma 1.12. *Let F be the functor associated to $(S, \mathcal{A})$ above. Then, F admits a Zariski open covering $(F_i \to F)_{i \in I}$ with each functor F_i representable.*

Proof. Let $S = \bigcup_{i \in I} U_i$ be an affine open covering. Let $F_i \subseteq F$ be the subfunctor consisting of those pairs $(f : T \to S, \varphi)$ such that $f(T) \subseteq U_i$. More precisely, $F_i : (\mathbf{Sch}/S)^{\mathrm{opp}} \to (\mathbf{Sets})$ is defined by

$$
\begin{aligned}
F_i(T) &= \begin{cases} \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathbf{Alg})}(\mathcal{A}, f_*\mathcal{O}_T), & \text{if } f(T) \subseteq U_i, \\ \varnothing, & \text{otherwise.} \end{cases} \\
&= \begin{cases} \{\text{all pairs } (f : T \to S, \varphi : \mathcal{A} \to f_*\mathcal{O}_T)\}, & \text{if } f(T) \subseteq U_i, \\ \varnothing, & \text{otherwise.} \end{cases}
\end{aligned}
$$

We now identify $(\mathbf{Sch}/U_i)^{\mathrm{opp}}$ to be a subcategory of $(\mathbf{Sch}/S)^{\mathrm{opp}}$. We have to show each F_i is representable. Indeed, let $G_i : (\mathbf{Sch}/U_i)^{\mathrm{opp}} \to (\mathbf{Sets})$ be the functor associated to $(U_i, \mathcal{A}|_{U_i})$. Then, we see that $G_i \cong h_{U_i}^{\mathrm{opp}} \times_{h_S^{\mathrm{opp}}} F_i \cong F_i|_{U_i}$ by Lemma 1.9, where $F_i|_{U_i}$ is the restriction of F_i to the subcategory $(\mathbf{Sch}/U_i)^{\mathrm{opp}}$ of $(\mathbf{Sch}/S)^{\mathrm{opp}}$. By Lemma 1.10, $F_i|_{U_i} \cong G_i$ is representable, we see that F_i is also representable. Indeed, if $G_i \cong \mathrm{Hom}_{(\mathbf{Sch}/U_i)^{\mathrm{opp}}}(-, X)$ for some U_i-scheme X, then $\mathrm{Hom}_{(\mathbf{Sch}/S)^{\mathrm{opp}}}(T, X) = \varnothing = F_i(T)$ if $f(T) \nsubseteq U_i$.

Next we show that $F_i \subset F$ is representable by open immersions. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. It follows from the definition of F_i that given $a : T' \to T$ we have $a^*(f, \varphi) = (f \circ a, a^*\varphi) \in F_i(T')$ if and only if $f(a(T')) \subseteq U_i$ if and only if $a(T') \subset V_i$.

Finally, we show that the collection $(F_i)_{i \in I}$ covers F. Let $(f : T \to S, \varphi) \in F(T)$. Consider $V_i = f^{-1}(U_i)$. Since $S = \bigcup_{i \in I} U_i$ is an open covering of S, we see that $T = \bigcup_{i \in I} V_i$ is an open covering of T. Suppose $g_i : V_i \to T$ is the natural open immersion. Then, we see that $g_i(V_i) \subseteq U_i$ and $(f \circ g_i)(V_i) \subseteq U_i$. So, $(f, \varphi)|_{V_i} = (f \circ g_i, g_i^*\varphi) \in F_i(V_i)$.

Thus, $(F_i \to F)_{i \in I}$ is a Zariski open covering with each functor F_i representable. ☐

To summarize, we have the following proposition.

Proposition 1.13. *Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. Then there exists an S-scheme* $\mathrm{Spec}(\mathcal{A})$ *such that for all S-schemes $f : T \to S$ there are bijections, functorial in T,*

$$\mathrm{Hom}_S(T, \mathrm{Spec}(\mathcal{A})) \xrightarrow{\sim} \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathbf{Alg})}(\mathcal{A}, f_*\mathcal{O}_T). \tag{1.3.1}$$

In other words, $\mathrm{Spec}(\mathcal{A})$ *represents the functor*

$$(\mathbf{Sch}/S)^{\mathrm{opp}} \to (\mathbf{Sets}),$$

$$(f^{\mathrm{opp}} : S \to T) \to \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathbf{Alg})}(\mathcal{A}, f_*\mathcal{O}_T).$$

Definition 1.14. *The S-scheme $h : \mathrm{Spec}(\mathcal{A}) \to S$ above is called the **spectrum** of $\mathcal{A}$ and h is called its **structure morphism**.*

Corollary 1.15. *Let $h : \mathrm{Spec}(\mathcal{A}) \to S$ be the spectrum of $\mathcal{A}$ above. Then for every affine open subset U of S, there exists isomorphisms of affine U-schemes*

$$h^{-1}(U) \cong \mathrm{Spec}\ \mathcal{A} \times_S U \cong \mathrm{Spec}(\Gamma(U, \mathcal{A})).$$

It follows that $h_*\mathcal{O}_{\mathrm{Spec}\ \mathcal{A}} = \mathcal{A}$.

Proof. Suppose $i : U \hookrightarrow S$ is the natural open immersion. Let $F_U \subseteq F$ be the subfunctor of F corresponding to pairs (f, φ) over schemes T with $f(T) \subseteq U$. Then, by the proof of Lemma 1.12 and Lemma 1.10, we see that F_U is represented by $\mathrm{Spec}(\Gamma(U, \mathcal{A}))$.

Note that for every $f : T \to S$,

$$\begin{aligned}(\mathrm{Spec}\ \mathcal{A} \times_S U)(T) &= (\mathrm{Spec}\ \mathcal{A})(T) \times_{S(T)} U(T) \\ &\simeq (\mathrm{Spec}\ \mathcal{A})(T) \times_{\mathrm{Hom}(T,S)} \mathrm{Hom}(T, U) \\ &= \{\text{all pairs } ((f, \varphi), f')|f = i \circ f',\ \text{where } f' : T \to U\} \\ &\simeq \{\text{all pairs } (f, \varphi)|f(T) \subseteq U\} \\ &= F_U(T).\end{aligned}$$

Thus, $F_U \cong \mathrm{Spec}\ \mathcal{A} \times_S U \cong h^{-1}(U)$. Thus, we conclude that

$$h^{-1}(U) \cong \mathrm{Spec}\ \mathcal{A} \times_S U \cong \mathrm{Spec}(\Gamma(U, \mathcal{A})).$$

□

Let $\mathcal{B}$ be another quasi-coherent O_S-algebra. Then applying (1.3.1) to $T = \mathrm{Spec}\ \mathcal{B}$ and since $f_*\mathcal{O}_{\mathrm{Spec}(\mathcal{B})} = \mathcal{B}$, we obtain a functorial isomorphism

$$\mathrm{Hom}_{(O_S\text{-}\mathbf{Alg})}(\mathcal{A}, \mathcal{B}) \cong \mathrm{Hom}_S(\mathrm{Spec}\ \mathcal{B}, \mathrm{Spec}\ \mathcal{A}). \tag{1.3.2}$$

So $\mathcal{A} \mapsto \mathrm{Spec}\ \mathcal{A}$ is a fully faithful functor from the category of quasi-coherent O_S-algebras into the category of S-schemes.

Corollary 1.16. *If $g : X \to S$ is a morphism of schemes, there is an isomorphism of X-schemes, functorial in $\mathcal{A}$,*

$$\mathrm{Spec}\ g^*\mathcal{A} \cong \mathrm{Spec}\ \mathcal{A} \times_S X. \tag{1.3.3}$$

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
T \arrow[dr, dotted] \arrow[drr] \arrow[ddr, "f"'] & & \\
& \mathrm{Spec}\ \mathcal{A} \times_S X \arrow[r] \arrow[d] & \mathrm{Spec}\ \mathcal{A} \arrow[d] \\
& X \arrow[r, "g"] & S
\end{tikzcd}$$

we have

$$\mathrm{Hom}_X(T, \mathrm{Spec}\ \mathcal{A} \times_S X) = \mathrm{Hom}_S(T, \mathrm{Spec}\ \mathcal{A})$$

for every X-scheme $f : T \to X$. Therefore both X-schemes represent the same functor on X-schemes given by

$$(f : T \to X) \mapsto \mathrm{Hom}_{(O_X\text{-}\mathbf{Alg})}(g^*\mathcal{A}, f_*\mathcal{O}_T) = \mathrm{Hom}_{(\mathcal{O}_S\text{-}\mathbf{Alg})}(\mathcal{A}, g_*(f_*\mathcal{O}_T)).$$

□

1.4 The symmetric algebra of an $\mathcal{O}_X$-module

We first recall the tensor algebra and symmetric algebra of a module. Let A be a ring and M an A-module.

Definition 1.17. *The **tensor algebra** of M is the A-algebra*

$$T_A(M) = \bigoplus_{n \geqslant 0} T^n(M), \qquad \text{with } T^n(M) := M^{\otimes n} := M \otimes_A \cdots \otimes_A M,$$

where the product is given by

$$(m_1 \otimes \cdots \otimes m_n, m'_1 \otimes \cdots \otimes m'_{n'}) \mapsto m_1 \otimes \cdots \otimes m_n \otimes m'_1 \otimes \cdots \otimes m'_{n'}.$$

Note that $T^0(M) = A, T^1(M) = M$, and $(T^n M)(T^m M) \subseteq T^{m+n} M$. So, $T_A(M)$ is a graded A-algebra.

Definition 1.18. *Let I be the two-sided ideal of $T_A(M)$ generated by the elements $m \otimes m' - m' \otimes m$, with $m, m' \in M$. We define the **symmetric algebra** of M to be*

$$\mathrm{Sym}_A(M) := T_A(M)/I.$$

Remark. *Let $I_n \subseteq T^n(M)$ be the A-submodule generated by all elements of the form*

$$m_1 \otimes \cdots \otimes m_n - m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}$$

with $m_i \in M$ and $\sigma \in S_n$. Define

$$\mathrm{Sym}^n(M) := T^n(M)/I_n.$$

Then $I = \bigoplus_{n \geqslant 0} I_n$ and $\mathrm{Sym}_A(M) = \bigoplus_{n \geqslant 0} \mathrm{Sym}^n(M)$. This is a commutative graded A-algebra. Moreover, we have $\mathrm{Sym}^0(M) = A$ and $\mathrm{Sym}^1(M) = M$.

Proposition 1.19. *The A-algebra $\mathrm{Sym}_A(M)$ and the A-linear map $i : M = \mathrm{Sym}^1(M) \hookrightarrow \mathrm{Sym}(M)$ satisfy the following universal property:*

For every commutative A-algebra B, composition with i yields a bijection

$$\mathrm{Hom}_{A\text{-}\mathbf{Alg}}(\mathrm{Sym}_A(M), B) \xrightarrow{\sim} \mathrm{Hom}_{A\text{-}\mathbf{Mod}}(M, B), \tag{1.4.1}$$

given by

$$\varphi \mapsto \varphi \circ i.$$

If $u : M \to N$ is an A-linear map, applying (1.4.1) to $B = \mathrm{Sym}_A(N)$, we see that u induces an A-algebra (graded) homomorphism

$$\mathrm{Sym}(u) : \mathrm{Sym}_A(M) \to \mathrm{Sym}_A(N).$$

Thus, we have a functor

$$\mathrm{Sym} : A\text{-}\mathbf{Mod} \to A\text{-}\mathbf{CommGrdAlg},$$

where A-**Mod** is the category of A-modules and A-**CommGrdAlg** is the category of commutative graded A-algebras. Again, by (1.4.1), Sym is left adjoint to the forgetful functor

$$\mathrm{For} : A\text{-}\mathbf{Alg} \to A\text{-}\mathbf{Mod}.$$

Proposition 1.20.
(1) *If $\varphi : A \to B$ is a ring homomorphism, then there is an isomorphism of B-algebras*

$$\mathrm{Sym}_A(M) \otimes_A B \xrightarrow{\sim} \mathrm{Sym}_B(M \otimes_A B)$$

which is functorial in M and compatible with grading.

(2) *There is an isomorphism of graded A-algebras*

$$\mathrm{Sym}_A(M \oplus M') \xrightarrow{\sim} \mathrm{Sym}_A(M) \otimes_A \mathrm{Sym}_A(M'),$$

which is functorial in M and M'.

(3) *If M is a free A-module with basis $\{x_1, \cdots, x_r\}$, then there is a unique isomorphism of graded A-algebras*

$$A[T_1, \cdots, T_r] \xrightarrow{\sim} \mathrm{Sym}_A(M), \qquad \text{with } T_i \mapsto x_i.$$

In particular, $\mathrm{Sym}^n(M)$ *is a free A-module of rank* $\binom{r+n-1}{n}$.

We now generalize the construction to ringed spaces. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{E}$ an $\mathcal{O}_X$-module on X.

Definition 1.21. *Consider the presheaf* $\mathrm{PSym}(\mathcal{E}) : U \mapsto \mathrm{Sym}_{\mathcal{O}_X(U)}(\mathcal{E}(U))$. *Then the sheafification of* $\mathrm{PSym}(\mathcal{E})$ *is called the* ***symmetric algebra of*** $\mathcal{E}$, *denoted by* $\mathrm{Sym}(\mathcal{E})$.

Lemma 1.22. *Let $\mathcal{F}$ be a presheaf on X, then for any point $P \in X$, there exists a neighborhood V of P such that $\mathcal{F}^+(W) \cong \mathcal{F}(W)$ for any $W \subseteq V$.*

Proof. By definition, $\mathcal{F}^+(U)$ is the set of functions $s : U \to \bigcup_{P \in U} \mathcal{F}_P$ such that

(i) for each $P \in U$, $s(P) \in \mathcal{F}_P$, and

(ii) for each $P \in U$, there is a neighborhood V of P, contained in U, and an element $t \in \mathcal{F}(V)$, such that for all $Q \in V$, the image t_Q of t in $\mathcal{F}_Q$ is equal to $s(Q)$.

So, for any $W \subseteq V$, we have $\mathcal{F}^+(W) = \{s : V \to \bigcup_{P \in W} \mathcal{F}_P | s(Q) = (t|_W)_Q \text{ for all } Q \in W\} \cong \mathcal{F}(W)$. ∎

Proposition 1.23. *The sheaf* $\mathrm{Sym}(\mathcal{E})$ *on X is a commutative graded $\mathcal{O}_X$-algebra and*

$$\mathrm{Sym}(\mathcal{E}) = \bigoplus_{n \geqslant 0} \mathrm{Sym}^n(\mathcal{E}).$$

Proposition 1.24. *For every commutative $\mathcal{O}_X$-algebra $\mathcal{A}$, we have a bijection,*

$$\mathrm{Hom}_{\mathcal{O}_X\text{-}\mathbf{Alg}}(\mathrm{Sym}(\mathcal{E}), \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{A}),$$

which is functorial in $\mathcal{A}$ and in $\mathcal{E}$.

Proposition 1.25. *Let $X = \operatorname{Spec}(A)$ be an affine scheme, M an A-module, and let $\mathcal{E} = \widetilde{M}$ be the associated quasi-coherent $\mathcal{O}_X$-module. Then for elements $f \in A$, we have*

$$\operatorname{Sym}_A(M)_f = \operatorname{Sym}_{A_f}(M_f) = \Gamma(D(f), \operatorname{Sym}(\mathcal{E})).$$

Thus, $\operatorname{Sym}(\mathcal{E})$ *is the quasi-coherent $\mathcal{O}_X$-algebra associated to* $\operatorname{Sym}_A(M)$*.*

Proof. By Proposition 1.20(1), take $B = A_f$, we see that $\operatorname{Sym}_A(M)_f = \operatorname{Sym}_{A_f}(M_f)$.

For any point $P \in X$, by Lemma 1.22, there exists a neighborhood V of P such that $\operatorname{Sym}(\mathcal{E})(W) = \operatorname{PSym}(\mathcal{E})(W) = \operatorname{Sym}_{\mathcal{O}_X(W)}(\mathcal{E}(W))$ for all $W \subseteq V$. In particular, we can take $W = D(f)$ for some $f \in A$, then we see that $\operatorname{Sym}(\mathcal{E})(D(f)) = \operatorname{Sym}_{A_f}(M_f) = \operatorname{Sym}_A(M)_f$. Since a sheaf is determined by its values on a basis of open subsets, we see that $\operatorname{Sym}(\mathcal{E})$ is the quasi-coherent $\mathcal{O}_X$-algebra associated to $\operatorname{Sym}_A(M)$. Thus, $\Gamma(D(f), \operatorname{Sym}(\mathcal{E})) = \operatorname{Sym}_A(M)_f = \operatorname{Sym}_{A_f}(M_f)$ for all $f \in A$. ☐

Corollary 1.26. *Let X be a scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_X$-module. Then* $\operatorname{Sym}(\mathcal{E})$ *is a quasi-coherent $\mathcal{O}_X$-algebra.*

Proof. Take an affine open cover $\{U_i\}_i$ of X such that $U_i = \operatorname{Spec}(A_i)$ and $\mathcal{E}|_{U_i} = \widetilde{M_i}$ for some A_i-module M_i. Then $\operatorname{Sym}(\mathcal{E})|_{U_i} = \operatorname{Sym}(\mathcal{E}|_{U_i}) = \widetilde{\operatorname{Sym}_{A_i}(M_i)}$. Thus, $\operatorname{Sym}(\mathcal{E})$ is a quasi-coherent $\mathcal{O}_X$-algebra. ☐

Proposition 1.27. *Let $f : T \to X$ be a morphism of schemes. Then,*

$$f^* \operatorname{Sym}(\mathcal{E}) \cong \operatorname{Sym}(f^*\mathcal{E}).$$

1.5 Quasi-coherent bundles

Let X be a scheme.

Definition 1.28. *For every quasi-coherent O_X-module $\mathcal{E}$, we set*

$$\mathbb{V}(\mathcal{E}) := \operatorname{Spec}(\operatorname{Sym}(\mathcal{E})).$$

We obtain a contravariant functor $\mathcal{E} \mapsto \mathbb{V}(\mathcal{E})$ from the category of quasi-coherent $\mathcal{O}_X$-modules into the category of X-schemes.

Let $h : \mathbb{V}(\mathcal{E}) \to X$ be the structure morphism. Then we see that

$$h_*\mathcal{O}_{\mathbb{V}(\mathcal{E})} = \operatorname{Sym}(\mathcal{E}) = \bigoplus_{n \geqslant 0} \operatorname{Sym}^n(\mathcal{E}).$$

Let $\mathcal{E}$ and $\mathcal{F}$ be two quasi-coherent $\mathcal{O}_X$-modules. By the functoriality of $\mathbb{V}(-)$, a morphism $\mathcal{F} \to \mathcal{E}$ induces a morphism $\mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{F})$. Moreover, $\operatorname{Hom}_{O_X}(\mathcal{F}, \mathcal{E}) \hookrightarrow \operatorname{Hom}_X(\mathbb{V}(\mathcal{E}), \mathbb{V}(\mathcal{F}))$ is injective.

Definition 1.29. *We call a morphism of X-schemes $\mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{F})$ **linear**, if it is induced by an $\mathcal{O}_X$-linear morphism $\mathcal{F} \to \mathcal{E}$, i.e. for every points $x \in X$, $\mathcal{F}_x \to \mathcal{E}_x$ is an $\mathcal{O}_{X,x}$-linear homomorphism.*

Let $h : \mathbb{V}(\mathcal{E}) \to X$ and $g : \mathbb{V}(\mathcal{F}) \to X$ be the structure morphisms. Recall that $h_*\mathcal{O}_{\mathbb{V}(\mathcal{E})} = \mathrm{Sym}(\mathcal{E})$ and $g_*\mathcal{O}_{\mathbb{V}(\mathcal{F})} = \mathrm{Sym}(\mathcal{F})$. The following commutative diagram

$$\begin{tikzcd}
\mathbb{V}(\mathcal{E}) \arrow[rr, "f"] \arrow[dr, "h"'] & & \mathbb{V}(\mathcal{F}) \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

induces a commutative diagram of morphisms, the pushforwards on X,

$$\begin{tikzcd}
\mathrm{Sym}(\mathcal{E}) \arrow[d, equal] & & \mathrm{Sym}(\mathcal{F}) \arrow[d, equal] \\
h_*\mathcal{O}_{\mathbb{V}(\mathcal{E})} & & g_*\mathcal{O}_{\mathbb{V}(\mathcal{F})} \arrow[ll, "g_*(f^\#)"'] \\
& \mathcal{O}_X \arrow[ul, "h^\#"] \arrow[ur, "g^\#"'] &
\end{tikzcd}$$

Equivalently, An X-morphism $f : \mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{F})$ induces a morphism $\varphi : \mathrm{Sym}(\mathcal{F}) \to \mathrm{Sym}(\mathcal{E})$ of $\mathcal{O}_X$-algebras.

Proposition 1.30. *The morphism $f : \mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{F})$ is linear if and only if the induced morphism $\varphi : \mathrm{Sym}(\mathcal{F}) \to \mathrm{Sym}(\mathcal{E})$ preserves the grading, i.e. φ induces morphisms of $\mathcal{O}_X$-modules $\varphi_n : \mathrm{Sym}^n(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{E})$.*

Proof. Note that $\mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{F})$ is linear means it is induced by an $\mathcal{O}_X$-linear morphism $\mathcal{F} \to \mathcal{E}$, i.e. for every points $x \in X$, $\mathcal{F}_x \to \mathcal{E}_x$ is an $\mathcal{O}_{X,x}$-linear homomorphism. So, $\mathbb{V}(\mathcal{E}) \to \mathbb{V}(\mathcal{F})$ is linear if and only if $(\varphi_1)_x$ is $\mathcal{O}_{X,x}$-linear if and only if $\varphi_x : \mathrm{Sym}(\mathcal{F}_x) \to \mathrm{Sym}(\mathcal{E}_x)$ preserves the grading. We are done. $\square$

Definition 1.31. *Let X be a scheme. A* ***quasi-coherent bundle*** *over X is a pair $(h : V \to X, h_*\mathcal{O}_V = \bigoplus_{n\geqslant 0} \mathcal{E}_n)$ consisting of*

(1) *an X-scheme $h : V \to X$ such that $V \cong \mathrm{Spec}\,\mathcal{A}$ for some quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{A}$,*

(2) *and a grading of quasi-coherent $\mathcal{O}_X$-algebras $h_*\mathcal{O}_V = \bigoplus_{n\geqslant 0} \mathcal{E}_n$ such that $\mathcal{E}_0 = \mathcal{O}_X$ and the morphisms $\mathrm{Sym}^n(\mathcal{E}_1) \to \mathcal{E}_n$ are isomorphisms for all $n \geqslant 0$.*

Definition 1.32. *A* ***morphism***

$$\left(h : V \to X, h_*\mathcal{O}_V = \bigoplus_{n\geqslant 0} \mathcal{E}_n\right) \to \left(g : W \to X, g_*\mathcal{O}_W = \bigoplus_{n\geqslant 0} \mathcal{F}_n\right)$$

of quasi-coherent bundles over X is a morphism $f : V \to W$ of X-schemes, i.e.

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "h"'] & & W \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

such that the morphism $f^{\#} : \mathcal{O}_W \to f_*\mathcal{O}_V$ induces a morphism

$$\varphi = g_*(f^{\#}) : g_*\mathcal{O}_W \to g_*f_*\mathcal{O}_V = h_*\mathcal{O}_V,$$

which preserves the grading, i.e. we have induced morphisms $\varphi_n : \mathcal{F}_n \to \mathcal{E}_n$ for all n.

Clearly, we have

Proposition 1.33. *The functor $\mathcal{E} \to \mathbb{V}(\mathcal{E})$ yields an anti-equivalence between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of quasi-coherent bundles over X.*

Definition 1.34. *If $h : V \to X$ and $g : W \to X$ are quasi-coherent bundles over X, then a morphism of X-schemes $f : V \to W$ is called **linear** if it is a morphism of quasi-coherent bundles.*

Corollary 1.35. *Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module on X. If $g : X' \to X$ is a scheme morphism, then there is an isomorphism of quasi-coherent bundles over X', functorial in $\mathcal{E}$,*

$$\mathbb{V}(g^*\mathcal{E}) \xrightarrow{\sim} \mathbb{V}(\mathcal{E}) \times_X X'. \tag{1.5.1}$$

In particular, for every open $U \subseteq X$, we have $\mathbb{V}(\mathcal{E}|_U) = \mathbb{V}(\mathcal{E})|_U := h^{-1}(U)$, where $h : \mathbb{V}(\mathcal{E}) \to X$ is the structure morphism.

Proof. This follows from the existence of identities, functorial in $\mathcal{E}$,

$$\begin{aligned} \mathbb{V}(g^*\mathcal{E}) &= \operatorname{Spec}(\operatorname{Sym}(g^*\mathcal{E})) \\ &= \operatorname{Spec}(g^* \operatorname{Sym}(\mathcal{E})) \\ &= \operatorname{Spec}(\operatorname{Sym}(\mathcal{E})) \times_X X' \\ &= \mathbb{V}(\mathcal{E}) \times_X X' \end{aligned}$$

Let $h : \mathbb{V}(\mathcal{E}) \to X$ be the structure morphism. Then, for every open $U \subseteq X$, we have $\mathbb{V}(\mathcal{E}|_U) = \mathbb{V}(i^*\mathcal{U}) = \mathbb{V}(\mathcal{E}) \times_X U = h^{-1}(U) = \mathbb{V}(\mathcal{E})|_U$, where $i : U \hookrightarrow X$ is the natural open immersion. ☐

Proposition 1.36. *Let $\mathcal{E}$ be a quasi-coherent $\mathcal{O}_X$-module. Then for every X-scheme $h : T \to X$ there is a bijection, functorial in T,*

$$\operatorname{Hom}_X(T, \mathbb{V}(\mathcal{E})) \xrightarrow{\sim} \Gamma(T, (h^*\mathcal{E})^\vee). \tag{1.5.2}$$

In other words, the X-scheme $\mathbb{V}(\mathcal{E})$ represents the functor $T \mapsto \Gamma(T, (h^*\mathcal{E})^\vee)$.

Proof. This follows from the existence of identities, functorial in $h : T \to X$,

$$\begin{aligned}\mathrm{Hom}_X(T, \mathbb{V}(\mathcal{E})) &= \mathrm{Hom}_{(\mathcal{O}_X\text{-}\mathbf{Alg})}(\mathrm{Sym}(\mathcal{E}), h_*\mathcal{O}_T) \\ &= \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, h_*\mathcal{O}_T) \\ &= \mathrm{Hom}_{\mathcal{O}_T}(h^*\mathcal{E}, \mathcal{O}_T) \\ &= \mathscr{H}om_{\mathcal{O}_T}(h^*\mathcal{E}, \mathcal{O}_T)(T) \\ &= \Gamma(T, (h^*\mathcal{E})^\vee).\end{aligned}$$

□

Definition 1.37. *Let X be a scheme and $T_1, \cdots, T_n$ be n indeterminates. Set $I = \{T_1^{a_1} \cdots T_n^{a_n} | a_i \in \mathbb{N} \text{ for all } i\}$. The quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{O}_X[T_1, \cdots, T_n]$ is defined to be the direct sum of copies of $\mathcal{O}_X$ indexed by I, i.e. $\mathcal{O}_X[T_1, \cdots, T_n] = \mathcal{O}_X^{(I)} = \bigoplus_{i \in I} \mathcal{O}_X$.*

Lemma 1.38. *Let $X = \mathrm{Spec}\ A$ be an affine scheme, then for any $f \in A$, we have*

$$\mathcal{O}_X[T_1, \cdots, T_n](D(f)) = \mathcal{O}_X(D(f))[T_1, \cdots, T_n].$$

Proof. Keep I as above. Then $\mathcal{O}_X[T_1, \cdots, T_n] = \mathcal{O}_X^{(I)} = \widetilde{A}^{(I)} = \widetilde{A^{(I)}}$. So, $\mathcal{O}_X[T_1, \cdots, T_n](D(f)) = \widetilde{A^{(I)}}(D(f)) = (A^{(I)})_f = A_f^{(I)} = \widetilde{A}(D(f))^{(I)} = \mathcal{O}_X(D(f))^{(I)} = \mathcal{O}_X(D(f))[T_1, \cdots, T_n]$. □

Definition 1.39. *Let $(X, \mathcal{O}_X)$ be a scheme and $n \geqslant 0$. The scheme*

$$\mathbb{A}_X^n = \mathrm{Spec}(\mathcal{O}_X[T_1, \cdots, T_n])$$

*is called the **affine n-space** over X.*

*If $X = \mathrm{Spec}\ R$ is affine, then we also call this **affine n-space** over R and we denote it by $\mathbb{A}_R^n$. For $R = \mathbb{Z}$, we may simply write it as $\mathbb{A}^n$.*

Proposition 1.40. *Let R be a ring. Then $\mathbb{A}_R^n = \mathrm{Spec}(R[T_1, \cdots, T_n])$.*

Proof. It follows from $\mathbb{A}_R^n = \mathrm{Spec}(\mathcal{O}_{\mathrm{Spec}\ R}[T_1, \cdots, T_n]) = \mathrm{Spec}(\mathcal{O}_{\mathrm{Spec}\ R}[T_1, \cdots, T_n](\mathrm{Spec}\ R)) = \mathrm{Spec}(\mathcal{O}_{\mathrm{Spec}\ R}(\mathrm{Spec}\ R)[T_1, \cdots, T_n]) = \mathrm{Spec}(R[T_1, \cdots, T_n])$. □

Proposition 1.41. *Let X be a scheme. Then*

$$\mathbb{A}_X^n = \mathbb{A}^n \times_{\mathrm{Spec}\ \mathbb{Z}} X.$$

Proof. Consider the structure map $g : X \to \mathrm{Spec}\ \mathbb{Z}$. Let $\mathcal{A} = \mathcal{O}_{\mathrm{Spec}\ \mathbb{Z}}[T_1, \cdots, T_n]$. Then, we see that $\mathbb{A}^n = \mathrm{Spec}\ \mathcal{A}$ and $g^*\mathcal{A} = g^*\mathcal{O}_{\mathrm{Spec}\ \mathbb{Z}}[T_1, \cdots, T_n] = \mathcal{O}_X[T_1, \cdots, T_n]$. So, by Corollary 1.16, we see that $\mathbb{A}_X^n = \mathrm{Spec}(\mathcal{O}_X[T_1, \cdots, T_n]) = \mathrm{Spec}(g^*\mathcal{A}) = \mathrm{Spec}\ \mathcal{A} \times_{\mathrm{Spec}\ \mathbb{Z}} X = \mathbb{A}^n \times_{\mathrm{Spec}\ \mathbb{Z}} X$. □

Example 1.42. *Let X be a scheme. Then $\mathbb{A}_X^n = \mathbb{V}(\mathcal{O}_X^{\oplus n})$. Indeed, for any point $x \in X$, we have $\mathrm{Sym}(\mathcal{O}_X^{\oplus n})_x = \mathrm{Sym}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}^{\oplus n}) = \mathcal{O}_{X,x}[T_1, \cdots, T_n] = (\mathcal{O}_X[T_1, \cdots, T_n])_x$. Thus, $\mathrm{Sym}(\mathcal{O}_X^{\oplus n}) = \mathcal{O}_X[T_1, \cdots, T_n]$. So, $\mathbb{A}_X^n = \mathrm{Spec}(\mathcal{O}_X[T_1, \cdots, T_n]) = \mathrm{Spec}(\mathrm{Sym}(\mathcal{O}_X^{\oplus n})) = \mathbb{V}(\mathcal{O}_X^{\oplus n})$.*

If $X = \mathrm{Spec}\ A$ is affine, then $\mathbb{A}_X^n = \mathrm{Spec}(A[T_1, \cdots, T_n])$. A linear endomorphism (resp. automorphism) $\mathbb{A}_X^n \to \mathbb{A}_X^n$ is given by an $\mathcal{O}_X$-linear morphism (resp. automorphism) $\mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n}$, *which is determined by its global section, i.e. an A-linear map (resp. automorphism) $A^{\oplus n} \to A^{\oplus n}$. Let $\{x_1, \cdots, x_n\}$ be a basis for $A^{\oplus n}$, then this map can be written as*

$$x_i \mapsto \sum_j a_{ij} x_j$$

for suitable $a_{ij} \in A$.

Recall that

$$A[T_1, \cdots, T_n] \to \mathrm{Sym}_A(A^{\oplus n}), \quad T_i \mapsto x_i$$

gives an isomorphism of graded A-algebras. Moreover, an A-linear map (resp. automorphism) $A^{\oplus n} \to A^{\oplus n}$ is equivalent to an endomorphism (resp. automorphism) $\mathrm{Sym}_A(A^{\oplus n}) \to \mathrm{Sym}_A(A^{\oplus n})$ preserving grading. So, a linear endomorphism (resp. automorphism) $\mathbb{A}^n_X \to \mathbb{A}^n_X$ is equivalently given by a linear endomorphism (resp. automorphism) $\theta : A[T_1, \cdots, T_n] \to A[T_1, \cdots, T_n]$, i.e. $\theta(a) = a$ for all $a \in A$ and $\theta(T_i) = \sum_j a_{ij} T_j$ for suitable $a_{ij} \in A$.

1.6 Vector bundles and locally free sheaves

In complex geometry, we see that holomorphic vector bundles are corresponding to locally free sheaves of finite rank. We will define the notion of vector bundles over schemes first and then establish an analogy in algebraic geometry.

Let X be a scheme. For an X-scheme $f : Y \to X$ and for $U \subseteq X$ open , we write $Y|_U$ for the U-scheme $f^{-1}(U)$.

Definition 1.43. *Let $\pi : V \to X$ be a X-scheme and $n \geqslant 0$ be an integer. We call (U, c_U) a* ***local trivialization*** *or a* ***bundle chart*** *for V if $U \subseteq X$ be an open subset and $c_U : V|_U \xrightarrow{\sim} \mathbb{A}^n_U = U \times_{\mathrm{Spec}\ \mathbb{Z}} \mathbb{A}^n$ is an isomorphism of U-schemes.*

Two bundle charts (U_1, c_1) and (U_2, c_2) are said to be ***compatible*** *if the automorphisms $c_2 \circ c_1^{-1} : \mathbb{A}^n_{U_1 \cap U_2} \to \mathbb{A}^n_{U_1 \cap U_2}$ is linear.*

A ***vector bundle atlas*** *for V is a system $\mathcal{A} = \{(U_i, c_i) | i \in I\}$ of bundle charts which are compatible with each other and cover X, i.e. $c_i \circ c_j^{-1} : \mathbb{A}^n_{U_i \cap U_j} \to \mathbb{A}^n_{U_i \cap U_j}$ is linear for all i, j and $X = \bigcup_{i \in I} U_i$.*

Two vector bundle atlases $\mathcal{A}$ and $\mathcal{B}$ are called ***equivalent*** *if every chart of $\mathcal{A}$ is compatible with every chart of $\mathcal{B}$, i.e. $\mathcal{A} \cup \mathcal{B}$, also form a vector bundle atlas. We denote the equivalence class of $\mathcal{A}$ by $[\mathcal{A}]$.*

A ***vector bundle structure*** *for V is an equivalence class $[\mathcal{A}]$ of vector bundle atlas.*

Definition 1.44. *Let $n \geqslant 0$ be an integer. A (**geometric**) **vector bundle** of rank n over X is a pair $(V, [\mathcal{A}])$ consisting of*

(1) *an X-scheme $\pi : V \to X$*

(2) *and a vector bundle structure $[\mathcal{A}]$ of vector bundle atlas $\mathcal{A}$.*

Definition 1.45. *A* ***morphism*** *$(V, [\mathcal{A}]) \to (V', [\mathcal{A}'])$ of geometric vector bundles over X is an X-morphism $f : V \to V'$ such that $c' \circ f \circ c^{-1} : \mathbb{A}^n_{U \cap U'} \to \mathbb{A}^n_{U \cap U'}$ is linear for all bundle charts $(U, c) \in \mathcal{A}$ and $(U', c') \in \mathcal{A}'$.*

Proposition 1.46. *Let X be a scheme. The functor $\mathcal{E} \mapsto \mathbb{V}(\mathcal{E})$ yields an anti-equivalence of the category of locally free $\mathcal{O}_X$-modules of rank n and the category of geometric vector bundles of rank n over X.*

Proof. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank n. Then there exists an open cover $X = \bigcup_i U_i$ such that $\mathcal{E}|_{U_i} \cong (\mathcal{O}_X|_{U_i})^{\oplus n} = \mathcal{O}_{U_i}^{\oplus n}$ for all i. So, $\mathbb{V}(\mathcal{E})|_{U_i} \cong \mathbb{V}(\mathcal{E}|_{U_i}) \cong \mathbb{V}(\mathcal{O}_{U_i}^{\oplus n}) \cong \mathbb{A}_{U_i}^{\oplus n}$. So, by Proposition 1.33, there exists an open covering $X = \bigcup_i U_i$ and linear isomorphisms of U_i-schemes $c_i : \mathbb{V}(\mathcal{E})|_{U_i} \xrightarrow{\sim} \mathbb{A}_{U_i}^n$.

Conversely, suppsoe $(V, [\mathcal{A}])$ is a vector bundle of rank n with structure map $h : V \to X$ and $\mathcal{A} = \{(U_i, c_i) | i \in I\}$. Define $\mathcal{E}_V$ to be the $\mathcal{O}_X$-module as follows.

Define a presheaf $\mathscr{S}(V/X)$ on X by attaching to an open subset $U \subseteq X$ the set of sections of V over U, that is, the set of morphisms $s : U \to V|_U$ such that $h \circ s = \mathrm{id}_U$. The restriction maps of $\mathscr{S}(V/X)$ are given by the restriction of scheme morphisms. Recall that scheme morphisms can be glued, so $\mathscr{S}(V/X)$ is a sheaf on X. Choose a bundle chart (U_i, c_i), we see that $V|_{U_i} \cong \mathbb{A}_{U_i}^n = \mathbb{V}(\mathcal{O}_{U_i}^{\oplus n}) = \mathbb{V}((\mathcal{O}_X|_{U_i})^{\oplus n})$. Set $\mathcal{E} = \mathcal{O}_{U_i}^{\oplus n}$. By Proposition 1.36, we see that an element in $\Gamma(U_i, \mathcal{E}^\vee)$ corresponds to a section $z : U_i \to \mathbb{V}(\mathcal{E})$ of the structure morphism $f : \mathbb{V}(\mathcal{E}) \to U_i$, i.e., $f \circ z = \mathrm{id}_{U_i}$. So, $\mathscr{S}(V/X)|_{U_i} = \mathscr{S}(V/U_i) = \mathcal{E}^\vee \cong (\mathcal{O}_{U_i}^\vee)^{\oplus n} \cong \mathcal{O}_{U_i}^{\oplus n}$. This shows that $\mathscr{S}(V/X)$ is locally free of rank n. Set $\mathcal{E}_V := \mathscr{S}(V/X)$. We see that $\mathbb{V}(\mathcal{E}_V)|_{U_i} \cong \mathbb{V}(\mathcal{E}_V|_{U_i}) \cong \mathbb{V}(\mathscr{S}(V/X)|_{U_i}) \cong \mathbb{V}(\mathcal{O}_{U_i}^{\oplus n}) \cong V|_{U_i}$. Thus, $V = \mathbb{V}(\mathcal{E}_V)$. Thus, $V \mapsto \mathcal{E}_V$ is the inverse of $\mathcal{E} \mapsto \mathbb{V}(\mathcal{E})$. $\square$

1.7 Torsors and non-abelian cohomology

Definition 1.47. *Let X be a topological space, G a sheaf of groups on X and let T be a sheaf of sets on X. We say T is a **G-sheaf** or G **acts on** T if there is given a morphism of sheaves $G \times T \to T$ such that for every open subset $U \subseteq X$ the map $G(U) \times T(U) \to T(U)$ is a left action of group $G(U)$ on the set $T(U)$.*

Definition 1.48. *A **morphism** of G-sheaves is a morphism of sheaves $\varphi : T \to T'$ such that $\varphi_U : T(U) \to T'(U)$ is $G(U)$-equivariant for all open subsets $U \subseteq X$, i.e. $\varphi_U(g \cdot x) = g \cdot \varphi_U(x)$ for all $g \in G(U)$ and $x \in T(U)$.*

These definitions give us a category of G-sheaves. We now recall some types of group actions.

Let G be a group and S a non-empty set. The action of G on S is called

1. **transitive** if for each pair $x, y \in S$, there exists a $g \in G$ such that $g \cdot x = y$.
2. **faithful** if, given $g, h \in G$, $g \cdot x = h \cdot x$ for every $x \in S$ implies $g = h$.
3. **free** (or **semiregular**) if, given $g, h \in G$, the existence of an $x \in S$ with $g \cdot x = h \cdot x$ implies $g = h$.
4. **regular** (or **simply transitive** or **sharply transitive**) if it is both transitive and free; this is equivalent to saying that for every two $x, y \in S$, there exists a unique $g \in G$ such that $g \cdot x = y$.

Definition 1.49. *A G-sheaf T is called a **G-torsor** (for the Zariski topology) if it satisfies the following two properties.*

(a) The group $G(U)$ acts regularly on $T(U)$ for every open subset $U \subseteq X$.

(b) There exists an open covering $\mathcal{U} = (U_i)_i$ of X such that $T(U_i) \neq \varnothing$ for all i.

Example 1.50. *A sheaf of groups G acts on itself by left multiplication is a G-torsor. This torsor is called the **trivial** G**-torsor**. A G-torsor T is isomorphic to the trivial G-torsor if and only if $T(X) \neq \varnothing$ because any $t \in T(X)$ yields an isomorphism $G(U) \to T(U)$, $g \mapsto g \cdot t|_U$.*

Definition 1.51. *Let T be a G-torsor and let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X. We say that $\mathcal{U}$ **trivializes** T if $T|_{U_i}$ is isomorphic to the trivial $G|_{U_i}$-torsor for each $i \in I$. Equivalently, $T(U_i) \neq \varnothing$ for all $i \in I$.*

Definition 1.52. *A pair (H, e) consisting of a set H and an element $e \in H$ is called a **pointed set**. The element e is called a **base point** or **distinguished element**.*

*A **homomorphism** $\gamma : (H, e) \to (H', e')$ of pointed sets is a map $\gamma : H \to H'$ with $\gamma(e) = e'$. The **kernel** of γ is defined by $\ker \gamma = \{h \in H : \gamma(h) = e'\}$. Note that a homomorphism of pointed sets with trivial kernel is not necessarily injective.*

Asequence

$$(H_1, e_1) \xrightarrow{\gamma_1} (H_2, e_2) \xrightarrow{\gamma_2} (H_3, e_3)$$

*of pointed sets is called **exact** if $\mathrm{im}\,(\gamma_1) = \ker(\gamma_2)$.*

We now start to introduce non-abelian Čech cohomology. To ease the notation, for two sections $s \in \Gamma(U, G)$ and $t \in \Gamma(V, G)$ we will often write $st \in \Gamma(U \cap V, G)$ instead of $s|_{U\cap V} t|_{U\cap V}$, and $s = t$ instead of $s|_{U\cap V} = t|_{U\cap V}$.

Definition 1.53. *Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of a topological space X. A **Čech 1-cocycle** of G on $\mathcal{U}$ is a tuple $\theta = (g_{ij})_{i,j\in I}$, where $g_{ij} \in G(U_i \cap U_j)$, such that the cocycle condition*

$$g_{kj} g_{ji} = g_{ki} \tag{1.7.1}$$

holds for all i, j, k. This implies $g_{ii} = 1$ and $g_{ij} = g_{ji}^{-1}$ for all $i, j \in I$.

Definition 1.54. *Two Čech 1-cocycles $\theta = (g_{ij})_{i,j\in I}$ and $\theta' = (g'_{ij})_{i,j\in I}$ on $\mathcal{U}$ are called **cohomologous** if there exist $h_i \in G(U_i)$ for all i such that we have*

$$h_i g_{ij} = g'_{ij} h_j \tag{1.7.2}$$

for all $i, j \in I$. This is easily checked to be an equivalence relation on the set of Čech 1-cocycles of G on $\mathcal{U}$.

Definition 1.55. *The equivalence classes are called **cohomology classes**, and the set of cohomology classes of Čech 1-cocycles on $\mathcal{U}$ is called the **(first) Čech cohomology** of G on $\mathcal{U}$ and is denoted by $\check{H}^1(\mathcal{U}, G)$. This is a pointed set in which the distinguished element is the cohomology class of the cocycle (g_{ij}) with $g_{ij} = 1$ for all i, j.*

Definition 1.56. *We say that a covering $V = (V_j)_{j\in J}$ of X is a **refinement** of a covering $U = (U_i)_{i\in I}$ if there exists a map $: J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for all $j \in J$. Two coverings $\mathcal{U}$ and $\mathcal{V}$ are called **equivalent** if each one is a refinement of the other one.*

If $g = (g_{ii'})$ is a Čech 1-cocycle on U, then the tuple $\tau^*(g)_{jj'} = g_{\tau(j)\tau(j')}|_{V_j \cap V_{j'}}$ is a Čech 1-cocycle on V. So, we have a induced map

$$\tau^* : \check{H}^1(\mathcal{U}, G) \to \check{H}^1(\mathcal{V}, G).$$

Moreover, as in the case of abelian Čech cohomology, this map is independent of the choice of τ. If $\mathcal{U}$ and $\mathcal{V}$ are equivalent, we use the isomorphisms described above to identify $\check{H}^1(\mathcal{U}, G) = \check{H}^1(\mathcal{V}, G)$.

Definition 1.57. *Let G be a sheaf of groups on a topological space. The pointed set*

$$\check{H}^1(X, G) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, G),$$

where $\mathcal{U}$ runs through the set of equivalence classes of open coverings of X, is called the ***(first) Čech cohomology*** *of G on X.*

We denote by $H^1(X, G)$ the set of isomorphism classes of G-torsors. This is a pointed set, where the distinguished element is the isomorphism class of the trivial G-torsor.

Proposition 1.58. *There is an isomorphism*

$$H^1(X, G) \cong \check{H}^1(X, G) \tag{1.7.3}$$

of pointed sets.

Proof. Let T be a G-torsor and let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of X that trivializes T, i.e. $T(U_i) \neq \emptyset$ for all i. Set $U_{ij} = U_i \cap U_j$ for all $i, j \in I$. Choose elements $t_i \in T(U_i)$. As G acts regularly, there exists a unique element $g_{ij} \in G(U_{ij})$ such that $g_{ij} t_j = t_i$. We have $g_{kj} g_{ji} t_i = t_k = g_{ki} t_i$ and thus $g_{kj} g_{ji} = g_{ki}$. For a different choice of elements t_i we obtain a cohomologous 1-cocycle. Indeed, suppose t_i' is another choice, then there exists $h_i \in G(U_i)$ such that $h_i t_i = t_i'$. So, we see that $g_{ij}' h_j t_j = h_i t_i$, i.e. $h_i^{-1} g_{ij}' h_j t_j = t_i$. Since $G(U_i)$ acts regularly on $T(U_i)$

We obtain a map of pointed sets

$$c_{G,\mathcal{U}} : H^1(\mathcal{U}, G) := \left\{T \in H^1(X, G); T \text{ is trivialized by } \mathcal{U}\right\} \to \check{H}^1(\mathcal{U}, G).$$

By taking inductive limits we obtain a map of pointed sets,

$$c_G : H^1(X, G) \to \check{H}^1(X, G)$$

We now claim that The maps $c_{G,\mathcal{U}}$ are isomorphisms of pointed sets. In particular, c_G is an isomorphism.

We define an inverse of $c_{G,\mathcal{U}}$ as follows. Let (g_{ij}) be a representative of a 1-cocycle in $\check{H}^1(\mathcal{U}, G)$. For $V \subseteq X$ open, we set

$$T(V) = \left\{(t_i) \in \prod_i G\left(U_i \cap V\right) : t_i t_j^{-1} = g_{ij}\right\} \tag{1.7.4}$$

Endowed with the obvious restriction maps, we see that T is a sheaf. We define a G-action on T via $g \cdot (t_i)_i = (t_i g^{-1})_i$. For a fixed $k \in I$ and for $V \subseteq U_k$ the map $G(V) \to T(V)$, $g \mapsto (g_{ik} g^{-1})_i$, defines an isomorphism of $G|_{U_k}$-sheaves $G|_{U_k} \to T|_{U_k}$ whose inverse is given by $(t_i) \mapsto t_k^{-1}$. So, we see that $G|_{U_k}$ acts regularly on $T|_{U_k}$. Since $T(U_i)$ is non-empty for each i. Thus T is a G-torsor which is trivialized by $\mathcal{U}$. If (g_{ij}) is replaced by a cohomologous cocycle $(g_{ij}') = (h_i g_{ij} h_j^{-1})$ with associated G-torsor T', then $(t_i) \mapsto (h_i t_i)_i$ defines an isomorphism $T \xrightarrow{\sim} T'$ of G-torsors. So, this map is well-defined. Clearly, it is the inverse of $c_{G,\mathcal{U}}$.

□

We state the following theorem without proof.

Theorem 1.59. *The exact sequence of sheaves of groups*

$$1 \longrightarrow G' \xrightarrow{\varphi} G \xrightarrow{\psi} G'' \longrightarrow 1$$

induces a long exact sequence of pointed sets

$$1 \to G'(X) \to G(X) \to G''(X) \xrightarrow{\delta} \check{H}^1(X, G') \to \check{H}^1(X, G) \to \check{H}^1(X, G'')$$

Moreover, we have:

(1) *Assume that G' is a subgroup sheaf of the center of G (in particular it is a sheaf of abelian groups). Then δ is a homomorphism of groups, and $\check{H}^1(\varphi)$ induces an injection of the group* $\mathrm{Coker}(\delta)$ *into the pointed set $\check{H}^1(X, G)$.*

(2) *Assume that G', G, and G'' are abelian sheaves. Then the above sequence is an exact sequence of abelian groups.*

1.8 Vector bundles and GL_n-torsors

We may define vector bundles of rank n over a ringed space $(X, \mathcal{O}_X)$, where $n \geqslant 0$ is an integer, simply by $\mathcal{O}_X$-modules that are locally isomorphic to $\mathcal{O}_X^{\oplus n}$. Given a locally free $\mathcal{O}_X$-module $\mathcal{E}$ of rank n, we can define the sheaf of isomorphisms $\mathscr{I}\mathrm{som}(\mathcal{O}_X^{\oplus n}, \mathcal{E})$ by the sheafification of the presheaf given by

$$U \mapsto \mathrm{Isom}_{\mathcal{O}_U}(\mathcal{O}_U^{\oplus n}, \mathcal{E}|_U) = \mathrm{Isom}(\Gamma(U, \mathcal{O}_U)^{\oplus n}, \Gamma(U, \mathcal{E})).$$

In particular, we define the sheaf of automorphisms $\mathrm{GL}_n(\mathcal{O}_X) = \mathscr{I}\mathrm{som}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X^{\oplus n})$. Define an action of $\mathrm{GL}_n(\mathcal{O}_X)(U)$ on $\mathscr{I}\mathrm{som}(\mathcal{O}_X^{\oplus n}, \mathcal{E})(U)$ by $(g, u) \mapsto u \circ g^{-1}$ and this action is regular. Note that $\mathcal{E}$ is locally free of rank n, there exists an open covering $X = \bigcup_i U_i$ such that $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$ for all i. Thus, we see that $\mathscr{I}\mathrm{som}(\mathcal{O}_X^{\oplus n}, \mathcal{E})(U_i) \neq \varnothing$ for all i. So, $\mathscr{I}\mathrm{som}(\mathcal{O}_X^{\oplus n}, \mathcal{E})$ is a $GL_n(\mathcal{O}_X)$-torsor.

Proposition 1.60. *The map of pointed sets*

$$\alpha : \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{locally free } \mathcal{O}_X\text{-modules of rank } n \end{matrix} \right\} \to H^1(X, GL_n(\mathcal{O}_X))$$

given by

$$\mathcal{E} \mapsto \mathscr{I}\mathrm{som}(\mathcal{O}_X^{\oplus n}, \mathcal{E}).$$

is bijective.

Proof. By proposition 1.58, we see that $H^1(X, GL_n(\mathcal{O}_X)) \cong \check{H}^1(X, GL_n(\mathcal{O}_X))$. So, we may identify $H^1(X, GL_n(\mathcal{O}_X))$ with $\check{H}^1(X, GL_n(\mathcal{O}_X))$ via this isomorphism. We now define an inverse to α as follows. Let $\Theta \in H^1(X, \mathrm{GL}_n(\mathcal{O}_X))$ be represented by a Čech cocycle (g_{ij}) on an open covering $(U_i)_i$ of X. We set $\mathscr{S}_i := \mathcal{O}_{U_i}^n$ and glue these modules using $g_{ij} : \mathscr{S}_{j|U_i \cap U_j} \xrightarrow{\sim} \mathscr{S}_{i|U_i \cap U_j}$. The cocycle condition $g_{ik} = g_{ij} g_{jk}$ ensures that there exists an $\mathscr{O}_X$-module $\mathcal{E} = \mathcal{E}_\Theta$ and isomorphisms $t_i : \mathcal{E}_{|U_i} \xrightarrow{\sim} \mathscr{S}_i$ such that $g_{ij} = t_i \circ t_j^{-1}$ after restricting to $U_i \cap U_j$. The isomorphism class of $\mathscr{E}_\Theta$ does not depend on the choice of $(U_i)_i$ and not on the choice of the representing cocycle (g_{ij}). Using the explicit definition of a torsor attached to a 1-cocycle, it is immediate that this defines an inverse to α. $\square$

We now turn to line bundles and the Picard group. Let $(X, \mathcal{O}_X)$ be a scheme. Recall that there is a one-to-one correspondence between line bundles over X and locally free $\mathcal{O}_X$-modules of rank 1, i.e. invertible $\mathcal{O}_X$-modules. Let $\mathrm{Pic}(X)$ be the set of isomorphism classes of invertible $\mathcal{O}_X$-modules.

Proposition 1.61. *Let $\mathcal{L}, \mathcal{M}$ be two invertible $\mathcal{O}_X$-modules. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is also an invertible $\mathcal{O}_X$-module.*

Clearly if $\mathcal{L}_1, \mathcal{L}_2, \mathcal{M}_1, \mathcal{M}_2$ are invertible $\mathcal{O}_X$-modules with $\mathcal{L}_1 \cong \mathcal{L}_2$ and $\mathcal{M}_1 \cong \mathcal{M}_2$. Then $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_1 \cong \mathcal{L}_2 \otimes_{\mathcal{O}_X} \mathcal{M}_2$. So, We may define a product on $\mathrm{Pic}(X)$ by

$$[\mathcal{L}] \cdot [\mathcal{M}] = [\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}].$$

Proposition 1.62. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-modules. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$.*

We see that $[\mathcal{L}] \cdot [\mathcal{L}^\vee] = [\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee] = [\mathcal{O}_X]$. Thus, each $[\mathcal{L}]$ admits an inverse $[\mathcal{L}^\vee]$, i.e. $[\mathcal{L}]^{-1} = [\mathcal{L}^\vee]$. So far we see that $\mathrm{Pic}(X)$ is a group.

Definition 1.63. *The group $\mathrm{Pic}(X)$ with the product defined by tensor product is called the **Picard group** of X.*

Corollary 1.64. *Let X be a scheme. Then*

$$\alpha : \mathrm{Pic}(X) \to H^1(X, \mathcal{O}_X^*)$$

is an isomorphism.

Proof. It follows from proposition 1.60 immediately. ☐

In $n = 1$ case, $H^1(X, \mathcal{O}_X^*)$ as the pointed set of isomorphic classes of $\mathcal{O}_X^*$-torsors coincides with the cohomology group $H^1(X, \mathcal{O}_X^*)$ and the Čech cohomology group $\check{H}^1(X, \mathcal{O}_X^*)$.

2 Divisors on Schemes

2.1 Schemeatically dense open subschemes and associated points

Definition 2.1. *Let A be a ring. A non-zero divisor $f \in A$ is called a **regular** element.*

Definition 2.2. *Let A be a Noetherian ring. Let M be an A-module. We define the **annihilator** of $x \in M$ by*

$$\mathrm{Ann}(x) := \{a \in A : ax = 0\}.$$

Definition 2.3. *A prime ideal $\mathfrak{p}$ of A of the form $\mathrm{Ann}(x)$ for some $x \in M$ is called an **associated prime** of M. The set of associated primes is denoted by $\mathrm{Ann}_A(M)$ or simply $\mathrm{Ann}(M)$.*

Lemma 2.4. *Let M be a module over a Noetherian ring A.*

(a) If $\mathrm{Ass}_A M = \varnothing$, then $M = 0$.

(b) Let S be a multiplicative subset of A. Then the elements of $\mathrm{Ass}_{A[S^{-1}]}(M[S^{-1}])$ are exactly the primes of the form $\mathfrak{p}[S^{-1}]$ with $\mathfrak{p} \in \mathrm{Ass}_A(M)$ and $\mathfrak{p} \cap S = \varnothing$.

Corollary 2.5. *Let A be a Noetherian ring. Then the following properties are true.*

(*a*) *The set of zero divisors of A is equal to the union of the associated prime ideals of A.*

(*b*) *The minimal prime ideals of A belong to* $\operatorname{Ass}(A)$.

Lemma 2.6. *Let M be a finitely generated module over a Noetherian ring A. Then there exists a chain of submodules*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

of M such that each successive quotient M_{i+1}/M_i is isomorphic to $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of A.

Corollary 2.7. *Let A be a Noetherian ring, and let M be a finitely generated A-module. Then* $\operatorname{Ass}(M)$ *is a finite set.*

Definition 2.8. *Let X be a locally Noetherian scheme, and let*

$$\operatorname{Ass}(\mathcal{O}_X) := \{x \in X | \mathfrak{m}_x \in \operatorname{Ass}_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x})\},$$

where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. The points of $\operatorname{Ass}(\mathcal{O}_X)$ *are called the* ***associated points*** *of X.*

Remark. For any open subset $U \subseteq X$, we have $\operatorname{Ass}(\mathcal{O}_X) \cap U = \operatorname{Ass}(\mathcal{O}_U)$. *If X is affine, then* $\operatorname{Ass}(\mathcal{O}_X) = \operatorname{Ass}(\mathcal{O}_X(X))$. *The generic points of X are associated points of X. By Corollary 2.7,* $\operatorname{Ass}(\mathcal{O}_X)$ *is a locally finite set.*

Definition 2.9. *An associated point that is not a generic point of X is called an* ***embedded point*** *of X.*

Lemma 2.10. *Let X be a locally Noetherian scheme, U an open subset of X, and $i : U \hookrightarrow X$ the canonical inclusion. Then the canonical homomorphism $\mathcal{O}_X \to i_*\mathcal{O}_U$ is injective if and only if* $\operatorname{Ass}(\mathcal{O}_X) \subseteq U$.

Proof. As $\mathcal{O}_X \to i_*\mathcal{O}_U$ is injective if and only if $\mathcal{O}_X(V) \to \mathcal{O}_U(V \cap U)$ is injective, i.e. $\mathcal{O}_V(V) \to \mathcal{O}_V(V \cap U)$ is injective for all open subsets $V \subseteq X$, we may suppose $X = \operatorname{Spec} A$ and restrict ourselves to showing that $A \to \mathcal{O}_X(U)$ is injective if and only if $\operatorname{Ass}(A) \subseteq U$.

Let us first suppose that $\operatorname{Ass}(A) \subseteq U$. Let $a \in A$ be such that $a|_U = 0$. If $a \neq 0$, then there exists a $\mathfrak{p} = \operatorname{Ann}(ab) \in \operatorname{Ass}(aA) \subseteq \operatorname{Ass}(A) \subseteq U$. We see that $a = 0$ in $A_\mathfrak{p}$ as $a|_U = 0$ and $\mathfrak{p} \in U$. Thus, there exists an $s \in A \backslash \mathfrak{p}$ such that $sa = 0$. Hence $s \in \operatorname{Ann}(ab) = \mathfrak{p}$, which is absurd. Consequently, $a = 0$ in A.

Let us now suppose that $A \to \mathcal{O}_X(U)$ is injective. Suppose there exists a $\mathfrak{p} = \operatorname{Ann}(a) \in \operatorname{Ass}(A)$ with $\mathfrak{p} \notin U$. Recall that $\mathfrak{q} \notin \overline{\{\mathfrak{p}\}} \Leftrightarrow$ there exists a open neighborhood of $\mathfrak{q}$, say $U_\mathfrak{q}$, such that $\mathfrak{p} \notin U_\mathfrak{q}$. Thus, for any point $\mathfrak{q} \in U$, we see that $\mathfrak{q} \notin \overline{\{\mathfrak{p}\}}$. Equivalenlty, $\mathfrak{p} \not\subseteq \mathfrak{q}$, i.e. $\mathfrak{p} \cap (A - \mathfrak{q}) \neq \varnothing$. Take $s \in \mathfrak{p} \cap (A - \mathfrak{q})$, then $sa = 0$. Hence $a_\mathfrak{q} = 0$. Consequently, $a|_U = 0$. The injectivity of $A \to \mathcal{O}_X(U)$ shows that $a = 0$. Then $\mathfrak{p} = A$, a contradiction. ☐

Definition 2.11. *Let A be a ring. We denote $R(A)$ the multiplicative group of the regular elements of A. We let* $\operatorname{Frac}(A)$ *denote the* ***total ring of fractions*** *of A, the localization of A with respect to $R(A)$. It is a ring containing A as a subring.*

Lemma 2.12. *Let X be a scheme. For any open subset U of X, let*

$$\mathcal{R}_X(U) := \{a \in \mathcal{O}_X(U) | a_x \in R(\mathcal{O}_{X,x}), \forall x \in U\}.$$

Then $\mathcal{R}_X$ is a sheaf on X, and $\mathcal{R}_X(U) = R(\mathcal{O}_X(U))$ if U is affine. Moreover, there exists a unique presheaf of algebras $\mathcal{K}'_X$ on X containing $\mathcal{O}_X$, verifying the following properties:

(a) For any open subset U of X, we have $\mathcal{K}'_X(U) = \mathcal{O}_X(U)[\mathcal{R}_X(U)^{-1}]$. In particular, $\mathcal{K}'_X(U) = \mathrm{Frac}(\mathcal{O}_X(U))$ if U is affine.

(b) For any open subset U of X, the canonical homomorphism $\mathcal{K}'(U) \to \prod_{x \in U} \mathcal{K}'_{X,x}$ is injective.

(c) If X is locally Noetherian, then for any $x \in X$, $\mathcal{K}'_{X,x} \simeq \mathrm{Frac}(\mathcal{O}_{X,x})$.

Definition 2.13. *Let X be a scheme. We denote the sheaf of algebras associated to the presheaf $\mathcal{K}'_X$ by $\mathcal{K}_X$, and we call it the **sheaf of stalks of meromorphic functions** on X.*

*An element of $\mathcal{K}_X(X)$ is called a **meromorphic function** on X. We denote the subsheaf of invertible elements of $\mathcal{K}_X$ by $\mathcal{K}_X^*$.*

Proposition 2.14. *Let X be a locally Noetherian scheme, and let U be an open subset of X containing $\mathrm{Ass}(\mathcal{O}_X)$. Let $i : U \to X$ denote the canonical injection. Then the canonical homomorphism $\mathcal{K}_X \to i_*\mathcal{K}_U$ is an isomorphism.*

Definition 2.15. *A regular local ring A is a noetherian local ring $(A, \mathfrak{m})$ with dimension equal to $\dim_k \mathfrak{m}/\mathfrak{m}^2$, where $k = A/\mathfrak{m}$. Note that $\mathfrak{m}/\mathfrak{m}^2$ is a k-vector space.*

Lemma 2.16. *Suppose A is a regular local ring of dimension one. Let t be a regular parameter, i.e. $\mathfrak{m} = (t)$. Then A is a domain and every element $x \in A - \{0\}$ can be uniquely written as $x = t^r u$ with $r \geqslant 0$ and $u \in A^\times$.*

This makes A into a discrete valuation ring by seeting $v(x) = r$. And t is the uniformizer.

Proof. We know that regular local rings are noetherian domains from commutative algebra. By applying Krull intersection theorem to $\mathfrak{m} = (t)$, we see that $\bigcap_{k \geqslant 0} m^k = \bigcap_{k \geqslant 0}(t^k) = 0$. Let $r = \sup\{k \geqslant 0 : x \in (t^k)\}$. Then r must be finite. Write $x = t^r u$ for some $u \in R - (t)$. Since $R^\times = R - \mathfrak{m}$, we see that $u \in R^\times$. For the uniqueness, suppose $t^r u = t^s w$ for some $s \leqslant r$. Then $t^{r-s} = u^{-1} w \in R^\times$, which implies that $r - s = 0$ and $u = w$ as A is a domain. ☐

Corollary 2.17. *In dimension one, we have*

$$\text{regular local ring} \Leftrightarrow \text{discrete valuation ring.}$$

Definition 2.18. *We say a scheme X is **regular in codimension one** if every local ring $\mathcal{O}_{X,x}$ satisfying $\dim \mathcal{O}_{X,x} = 1$ is regular.*

Throughout this notes, we assume X satisfies the following condition
$(*)$ X is a noetherian integral separated scheme which is regular in codimension one.

2.2 Cartier Divisors

Definition 2.19. *Let X be a scheme. We denote the group $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ by $\mathrm{Div}(X)$. The elements of $\mathrm{Div}(X)$ are called **Cartier divisors** on X. The group law on $\mathrm{Div}(X)$ is denoted additively.*

Proposition 2.20. *Let $D \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ be a Cartier divisor, then D can be represented by a system $\{(U_i, f_i)_i\}$, where the U_i are open subsets of X forming a covering of X, $f_i \in \mathcal{K}_X^*(U_i)$ with $f_i|_{U_i \cap U_j}(f_j|_{U_i \cap U_j})^{-1} \in \mathcal{O}_X(U_i \cap U_j)^*$ for every i, j.*

Proof. By Lemma 1.22, we can find an open covering $\mathcal{U} = \{U_i\}_i$ such that $(\mathcal{K}_X^*/\mathcal{O}_X^*)(U_i) = \mathcal{K}_X^*(U_i)/\mathcal{O}_X^*(U_i)$. Then, we see that $D|_{U_i} \in (\mathcal{K}_X^*/\mathcal{O}_X^*)(U_i) = \mathcal{K}_X^*(U_i)/\mathcal{O}_X^*(U_i)$. So, we may find $f_i \in \mathcal{K}_X^*(U_i)$ such that $D|_{U_i} = f_i\mathcal{O}_X^*(U_i)$. Thus, by restricting on $U_i \cap U_j$, we see that $f_i|_{U_i \cap U_j}\mathcal{O}_X^*(U_i \cap U_j) = D|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}\mathcal{O}_X^*(U_i \cap U_j)$. We are done. □

Remark. (2) *Two systems $\{(U_i, f_i)_i\}$ and $\{(V_j, g_j)_j\}$ represent the same Cartier divisor if on $U_i \cap V_j$, f_i and g_j differ by a multiplicative factor in $\mathcal{O}_X(U_i \cap V_j)^*$. Let D_1, D_2 be two Cartier divisors, represented by $\{(U_i, f_i)_i\}$ and $\{(V_j, g_j)_j\}$, respectively.*

Definition 2.21. *Let $f \in \Gamma(X, K_X^*)$; its image in $\mathrm{Div}(X)$ is called a* ***principal Cartier divisor*** *and denoted by (f).*

Definition 2.22. *We say that two Cartier divisors D_1 and D_2 are* ***linearly equivalent*** *if $D_1 - D_2$ is principal, denoted by $D_1 \sim D_2$.*

We denote by $\mathrm{CaCl}(X)$ the group of isomorphism classes of Cartier divisors modulo the linear equivalence relation. For a Cartier divisor D, represented by $\{(U_i, f_i)_i\}$, we can associate a subsheaf $\mathcal{O}_X(D) \subseteq \mathcal{K}_X$ defined by

$$\mathcal{O}_X(D)|_{U_i} = f^{-1}\mathcal{O}_X|_{U_i}.$$

2.3 Weil divisors

Definition 2.23. *Let X satisfy $(*)$. A* ***prime divisor*** *on X is an integral closed subscheme of codimension one.*

Remark. By Exercise 3.11(*c*)*, there is a bijection between*

$$\{\text{ reduced closed subschemes of } X\} \leftrightarrow \{\text{ closed subsets of } X\}$$

given by associating a closed subset with the corresponding induced reduced scheme structure.

Thus, for any scheme X, there is a bijection between

$$\{\text{ integral closed subschemes of } X\} \leftrightarrow \{\text{ irreducible closed subsets of } X\},$$

where we associate the induced reduced scheme structure with an irreducible closed subset.

Equivalently, a prime divisor is a closed irreducible subset $Y \subseteq X$ of codimension one. Indeed, given such $Y \subseteq X$ we associate it with the corresponding induced reduced closed subscheme. So, Y has a unique generic point $\eta \in Y$ by Exercise 2.9.

Definition 2.24. $\mathrm{Div}\, X$ *is the free abelian group generated by the prime divisors on X. By convention,* $\mathrm{Div}\, X = 0$ *if no prime divisors exist.*

Definition 2.25. *A* ***Weil divisor*** *is an element of* $\mathrm{Div}\, X$*. We write a divisor as $D = \sum n_i Y_i$ where the Y_i are prime divisors, and the n_i are integers, and only finitely many $n_i \neq 0$.*

If all the $n_i \geqslant 0$, we say that D is ***effective****.*

Remark. (1) *If X is an integral scheme with generic point ξ then $\mathcal{O}_{X,\xi} = K(X)$ is the function field of X.*

(2) *For every $x \in X$ there is an injection of rings $\mathcal{O}_{X,x} \to K(X)$ such that $K(X)$ is the quotient field of all these local rings. Indeed, take an open affine neighborhood $U = \text{Spec } A$ of $x = \mathfrak{p}$, by Exercise 3.6, we see that $K(X) = \text{Frac}(A)$. The injection is simply given by $A_{\mathfrak{p}} \hookrightarrow \text{Frac}(A)$.*

(3) *If two sections $s \in \mathcal{O}_X(U), t \in \mathcal{O}_X(V)$ (with U, V nonempty) have the same stalks at ξ then $s|_{U \cap V} = t|_{U \cap V}$.*

(4) *If $f \in K(X)$, we call f a **rational function** on X. The **domain of definition** of f is the union of all open sets U occurring in the equivalence class $[\langle U, f_U \rangle] \in \mathcal{O}_{X,\xi}$. This is a nonempty open set. If the domain of definition of f is V, there is a unique section of $\mathcal{O}_X(V)$ whose stalk at ξ is f. We also denote this section by f.*

(5) *Suppose U is an open subset of X. We say $f \in K$ is **regular** on U if $f \in \mathcal{O}_X(U)$.*

Lemma 2.26. *If X is a scheme satisfying $(*)$ and $Y \subseteq X$ a closed irreducible subset with generic point η, then $\dim \mathcal{O}_{X,\eta} = \text{codim}(Y, X)$.*

Proof. By Exercise 3.20(e), it reduces to the affine case $X = \text{Spec } A$. Suppose η corresponds to a prime ideal $\mathfrak{p} \subseteq A$. Then $Y = \overline{\{\eta\}} = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. We only need to show that $\dim A_{\mathfrak{p}} = \text{codim}(V(\mathfrak{p}), \text{Spec } A)$. Indeed, since $V(\mathfrak{p}) \cong \text{Spec } A/\mathfrak{p}$, $\text{codim}(V(\mathfrak{p}), \text{Spec } A) = \dim \text{Spec } A - \dim V(\mathfrak{p}) = \dim A - \dim A/\mathfrak{p} = \text{ht}(\mathfrak{p}) = \dim A_{\mathfrak{p}}$. ☐

Remark. If Y is a prime divisor on X, let $\eta \in Y$ be its generic point. Then $\dim \mathcal{O}_{X,\eta} = \text{codim}(Y, X) = 1$ so $\mathcal{O}_{X,\eta}$ is a discrete valuation ring. We denote the discrete valuation by v_Y. Note that $K(X) = \text{Frac}(\mathcal{O}_{X,\eta})$.

Definition 2.27. *If Y is a prime divisor on X, let $\eta \in Y$ be its generic point. We call the corresponding discrete valuation v_Y on $K(X)$ the **valuation** of Y.*

*If $f \in K(X)^*$ is a nonzero rational function on X then $v_Y(f)$ is an integer. If it is positive, we say that f has a **zero** along Y of that order; if it is negative, we say f has a **pole** along Y, of order $-v_Y(f)$.*

Lemma 2.28. *If Y is a prime divisor on X with generic point η and $f \in K(X)^*$, then $v_Y(f) \geqslant 0$ if and only if η belongs to the domain of definition of f, and $v_Y(f) > 0$ if and only if $f_\eta \in \mathfrak{m}_\eta$, where $\mathfrak{m}_\eta$ is the maximal ideal of $\mathcal{O}_{X,\eta}$ and f_η is the stalk of $f \in \mathcal{O}_X(U)$ at η.*

Proof. $v_Y(f) \geqslant 0 \Leftrightarrow f \in \mathcal{O}_{X,\eta}$. This means that $f = [\langle U, f \rangle] \in \mathcal{O}_{X,\eta}$ with $\eta \in U$. The second statement follows from $v_Y(f) > 0 \Leftrightarrow f \in \mathfrak{m}_\eta$ ☐

Remark. If X satisfies $(*)$ then so does any open subset $U \subseteq X$. If Y is a prime divisor of X then $Y \cap U$ is a prime divisor of U, provided it is nonempty.

Moreover the assignment $Y \mapsto Y \cap U$ is injective since $Y = \overline{Y \cap U}$ (provided of course $Y \cap U \neq \varnothing$). If Y is a prime divisor of U then $\overline{Y}$ is a prime divisor of X and $\overline{Y} \cap U = Y$, so in fact there is a bijection between prime divisors of U and prime divisors of X meeting U.

Lemma 2.29. *Let X satisfy $(*)$, and let $f \in K(X)^*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y.*

Proof. Let $U = \operatorname{Spec} A$ be an affine open subset of X such that $f \in \mathcal{O}_X(U)$. Then, U is contained in the domain of definition of f. Then $Z = X - U$ is a proper closed subset of X. Since X is noetherian we can write $Z = Z_1 \cup \cdots \cup Z_n$ for closed irreducible Z_i. Thus Z can contain at most a finite number of prime divisors of X, since any closed irreducible subset of Z of codimension one in X must be one of the Z_i.

It suffices to show that there are only finitely many prime divisors $Y \subseteq X$ meeting U with $v_Y(f) > 0$ as we can apply the same argument to $v_Y(f^{-1})$. If $Y \cap U \neq \varnothing$ then U must contain the generic point η of Y. Hence $v_Y(f) \geqslant 0$. Recall that $X_f = \{x \in X | f_x \notin \mathfrak{m}_x\}$. So, $v_Y(f) > 0$ iff $\eta \in U - X_f = U - D(f)$, which is a proper closed subset of U. Indeed, $D(f) = \varnothing \Leftrightarrow V(f) = \operatorname{Spec} A \Leftrightarrow f \in \bigcap \mathfrak{p} \Leftrightarrow f$ is nilpotent. However, $f \neq 0$ and therefore cannot be nilpotent as $\mathcal{O}_X(U)$ is an integral domain. Thus, we see that $U - D(f)$ is a proper closed subset of the noetherian space U. Hence $v_Y(f) > 0$ iff $Y \cap U \subseteq U - D(f)$. But $U - D(f)$ is a proper closed subset of the noetherian space U, hence contains only finitely many closed irreducible subsets of codimension one of U. It follows that there are only finitely many Y meeting U such that $v_Y(f) > 0$ as $Y \mapsto Y \cap U$ is injective by the above remark. □

Definition 2.30. *Let X satisfy $(*)$ and let $f \in K(X)^*$. We define the* ***divisor*** *of f, denoted (f), by*

$$(f) = \sum v_Y(f) \cdot Y,$$

where the sum is taken over all prime divisors of X. This is a finite sum, hence it is a divisor. We use the convention that if no divisors exist the sum is zero. Any divisor which is equal to the divisor of a function is called a ***principal divisor.***

Remark. *If $f, g \in K(X)^*$, then $(f/g) = (f) - (g)$ because of the properties of valuations. Therefore, we have homomorphism*

$$\varphi : K(X)^* \to \operatorname{Div} X$$

given by

$$f \mapsto (f)$$

and the image, which consists of the principal divisors, is a subgroup of $\operatorname{Div} X$*, denoted by* $\operatorname{Princ} X$*.*

Definition 2.31. *Let X satisfy $(*)$. Two divisors D and D' are said to be* ***linearly equivalent****, written $D \sim D'$, if $D - D'$ is a principal divisor.*

The group $\operatorname{Div} X$ *of all divisors divided by the subgroup of principal divisors is called the* ***divisor class group*** *of X and is denoted* $\operatorname{Cl} X$*, i.e.* $\operatorname{Cl} X = \operatorname{Div} X / \operatorname{Princ} X$*.*

Definition 2.32. *A ring A is* ***normal*** *if it is integrally closed in* $\operatorname{Frac}(A)$*.*

Remark. (1) *Recall that in commutative algebra, we have A is normal if and only if $A_{\mathfrak{p}}$ is normal for all $\mathfrak{p} \in \operatorname{Spec} A$. Thus,* $\operatorname{Spec} A$ *is normal if and only if A is normal.*

(2) *A is a UFD if and only if every prime ideal of height* 1 *is principal.*

(3) *By Serre's criterion, a noetherian normal scheme is regular in codimension one. If A is a normal noetherian domain then $X =$* $\operatorname{Spec} A$ *has the property $(*)$ since any affine scheme is separated. A prime divisor $Y \subseteq X$ is $V(\mathfrak{p})$ for a prime ideal $\mathfrak{p}$ of height* 1*.*

(4) *If Q is the quotient field of A, there is a commutative diagram*

$$\begin{tikzcd}
\mathcal{O}_{X,\mathfrak{p}} \arrow[r] \arrow[d] & \mathcal{O}_{X,0} \arrow[d] \\
A_{\mathfrak{p}} \arrow[r] & Q
\end{tikzcd}$$

The discrete valuation v_Y on Q with valuation ring $A_{\mathfrak{p}}$ is defined on elements of $A_{\mathfrak{p}}$ by

$$v_Y(a/s) = \max\{k \geqslant 0 | a/s \in \mathfrak{p}^k A_{\mathfrak{p}}\}.$$

Proposition 2.33. *Let A be a noetherian domain. Then A is a unique factorization domain if and only if $X = \operatorname{Spec} A$ is normal and $\operatorname{Cl} X = 0$.*

Proof. Recall that a UFD is normal. Thus, $X = \operatorname{Spec} A$ is normal if A is a UFD.

It remains to show that if A is normal, then every prime ideal of height 1 is principal if and only if $\operatorname{Cl} X = 0$.

$\Rightarrow$:Suppose every prime ideal of height 1 is principal. Consider a prime divisor $Y \subseteq X = \operatorname{Spec} A$. Then $Y = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of height 1 as Y is a irreducible closed subset of codimension 1. So, $\mathfrak{p}$ is principal, say $\mathfrak{p} = (f)$ with $f \in A$. Considering f as a global section in $\mathcal{O}_{\operatorname{Spec} A}(X)$, we claim that $(f) = Y$. By the preceding notes for any prime divisor $Z = V(\mathfrak{q})$, $v_Z(f)$ is the largest $k \geqslant 0$ such that $f/1 \in \mathfrak{q}^k A_{\mathfrak{q}}$. It is easy to see that for $Z \neq Y$ we must have $v_Z(f) = 0$. Indeed, $\mathfrak{q}$ is principal for the same reason and $\sqrt{\mathfrak{p}} \neq \sqrt{\mathfrak{q}}$, so $f \notin \mathfrak{q}$. Thus, $f/1 \notin \mathfrak{q} A_{\mathfrak{q}}$. $\square$

3 Vector bundles on $\mathbb{P}^1$