## Algebraic Geometry: Midterm

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Question 1 (40 points): Explain the following concepts:

- (1) Sheaves and a stalk of a sheaf,
- (2) Spectrum of a ring,
- (3) Morphisms between locally ringed spaces,
- (4) Reduced schemes, irreducible schemes and integral schemes,
- (5) Finite type morphisms and finite morphisms,
- (6) Open immersions and closed immersions,
- (7) Dimension of a scheme,
- (8) Fiber products of schemes.

*Proof.* (1) Let X be a topological space. Let  $\mathfrak{Top}(X)$  be the category such that the objects of  $\mathfrak{Top}(X)$  are open subsets of X and the only morphisms are inclusions, i.e.

$$\operatorname{Hom}(V, U) = \begin{cases} \emptyset, & \text{if } V \nsubseteq U \\ \{i_{VU}\}, & \text{if } V \subseteq U \end{cases},$$

where  $i_{VU}:V\hookrightarrow U$  is the inclusion map.

A sheaf is a contravariant functor  $\mathscr{F}$  from  $\mathfrak{Top}(X)$  to a fixed category  $\mathfrak{C}$  satisfying the following two conditions

Uniqueness: If U is an open subset and  $\{V_i\}$  is an open covering of U and if  $s, t \in \mathscr{F}(U)$  such that  $s|_{V_i} = t|_{V_i}$  for all i, then s = t.

Gluability: If U is an open subset and  $\{V_i\}$  is an open covering of U and if for each  $i, s_i \in \mathscr{F}(V_i)$  are elements such that  $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ , then there exists  $s \in \mathscr{F}(U)$  such that  $s|_{V_i} = s_i$ .

Let p be a point of X. The stalk of a sheaf  $\mathscr{F}$  at p, denoted by  $\mathscr{F}_p$ , is defined to be  $\mathscr{F}_p = \underline{\lim}_U \mathscr{F}(U)$ , where U runs through all open neighborhoods of p.

(2) Let A be a ring. Spec  $A = \{$  prime ideals of  $A\}$  is a topological space endowed with Zariski topology. For an open set  $U \subseteq \operatorname{Spec} A$ , there is a sheaf on Spec A defined by

$$\mathcal{O}_{\mathrm{Spec}\ A}(U) = \left\{ s: U \to \coprod_{\mathfrak{p} \in U} A_{\mathfrak{p}} \middle| \begin{array}{l} \text{for each } \mathfrak{p} \in U, s(\mathfrak{p}) \in A_{\mathfrak{p}} \text{ and} \\ \exists \text{ a neighborhood } V \text{ of } \mathfrak{p} \text{ contained in } U \text{ and } a, f \in A \\ \text{such that } \forall \mathfrak{q} \in V, f \notin \mathfrak{q} \text{ and } s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}} \end{array} \right\}.$$

The spectrum of A is (Spec A,  $\mathcal{O}_{\text{Spec }A}$ ).

(3) A morphism of locally ringed spaces is a morphism  $(f, f^{\#})$  of ringed spaces, i.e. a continuous map  $f: X \to Y$  and a morphism  $f^{\#}: \mathcal{O}_Y \to f_*\mathcal{O}_X$  of sheaves of rings on Y, such that for each

point  $P \in X$ , the induced map of local rings  $f_P^\#: \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$  is a local homomorphism of local rings.

(4) A reduced scheme X is a scheme such that  $\mathcal{O}_X(U)$  has no nilpotent elements for every open subsets  $U \subseteq X$ .

An irreducible scheme X is a scheme such that sp(X) is irreducible.

An integral scheme X is a scheme such that  $\mathcal{O}_X(U)$  is an integral domain for every open subsets  $U \subseteq X$ .

(5) A morphism  $f: X \to Y$  of schemes is of finite type if there exists an open affine covering of Y,  $V_i = \text{Spec } B_i$ , such that for each i,  $f^{-1}(V_i)$  can be covered by a finite number of open affine subsets  $U_{ij} = \text{Spec } A_{ij}$ , where  $A_{ij}$  is a finitely generated  $B_i$ -algebra.

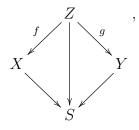
A morphism  $f: X \to Y$  of schemes is finite if there exists an open affine covering of Y,  $V_i = \text{Spec } B_i$ , such that for each i,  $f^{-1}(V_i) = \text{Spec } A_i$  for some ring  $A_i$ , where  $A_i$  is a  $B_i$ -algebra which is a finitely generated  $B_i$ -module.

(6) An open immersion  $f: X \to Y$  is a morphism of schemes which induces an isomorphism between X and an open subscheme of Y.

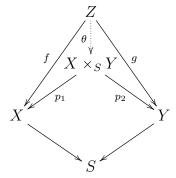
A closed immersion  $f: X \to Y$  is a morphism of schemes such that it induces a homeomorphism between  $\operatorname{sp}(X)$  and a closed subset of  $\operatorname{sp}(Y)$  and the induced map  $f^{\#}: \mathcal{O}_{Y} \to f_{*}\mathcal{O}_{X}$  of sheaves is surjective.

- (7) The dimension of a scheme X is its dimension as a topological space, i.e. the supremum of all integers n such that there exists a chain  $Z_0 \subset Z_1 \subset \cdots \subset Z_n$  of distinct irreducible closed subsets of X.
- (8) Let S be a scheme and X, Y are schemes over S. The fibred product of X and Y over S is the scheme  $X \times_S Y$  together with morphisms  $p_1 : X \times_S Y \to X$  and  $p_2 : X \times_S Y \to Y$  satisfying the following universal property:

For any given morphisms  $X \to S$ ,  $Y \to S$ , any scheme Z over S, the given morphisms  $f: Z \to X$  and  $g: Z \to Y$  such that the following diagram commutes



there exists a unique morphism  $\theta: Z \to X \times_S Y$  such that  $f = p_1 \circ \theta$  and  $g = p_2 \circ \theta$ , i.e. we have a commutative diagram



Question 2 (10 points): Let  $\phi : \mathscr{F} \to \mathscr{G}$  be a morphism between sheaves. Show that  $\phi$  is surjective if and only if the induced morphism on stalks  $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$  is surjective for all p.

Proof. We first claim that for each p,  $(\text{im }\phi)_p = \text{im }\phi_p$ . Let  $\mathscr{H}: U \mapsto \text{im }\phi(U)$  be a presheaf. Then, we have  $(\text{im }\phi)_p = \mathscr{H}_p^+ = \mathscr{H}_p$ . It suffices to prove that  $\mathscr{H}_p = \text{im }\phi_p$ . Let  $x \in \text{im }\phi_p$ , then there exists  $y \in \mathscr{F}_p$  such that  $\phi_p(y) = x$ . By the property of direct limit, there exists open neighborhoods U, V of p and  $s \in \mathscr{F}(U)$ ,  $t \in \mathscr{G}(V)$  such that  $s_p = y$  and  $t_p = x$ , where  $s_p$  and  $t_p$  are the image of s and t in stalks respectively. Then there exists  $W \subseteq U \cap V$  containing p. By shirinking W, we may assume that  $\phi(W)(s|_W) = t|_W$ . Thus,  $x = (t|_W)_p \in \mathscr{H}_p$ . Thus, im  $\phi_p \subseteq \mathscr{H}_p$ . Conversely, take  $x \in \mathscr{H}_p$ . Then there exists an open neighborhood of p and  $t \in \mathscr{H}(U) = \text{im }\phi(U)$  such that  $t_p = x$ . Then, there exists  $s \in \mathscr{F}(U)$  such that  $\phi(U)(s) = t$ . Passing to the stalks, we obtain that  $\phi_p(s_p) = t_p = x$ . Thus,  $x \in \text{im }\phi_p$ . Thus,  $\mathscr{H}_p \subseteq \text{im }\phi_p$ . Thus,  $\mathscr{H}_p = \text{im }\phi_p$  and it follows that  $(\text{im }\phi)_p = \text{im }\phi_p$ 

Now, by Proposition 1.1,  $\phi$  is surjective if and only if im  $\phi = \mathcal{G}$  if and only if  $(\text{im }\phi)_p = \text{im }\phi_p = \mathcal{G}_p$  for all p. This means that  $\phi$  is surjective if and only if the induced morphism on stalks  $\phi_p : \mathscr{F}_p \to \mathscr{G}_p$  is surjective for all p.

**Question 3 (10 points):** Let  $\mathscr{F},\mathscr{G}$  be sheaves of abelian groups on X. For any open set  $U \subset X$ , show that the set  $\operatorname{Hom}(\mathscr{F}|_U,\mathscr{G}|_U)$  of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf

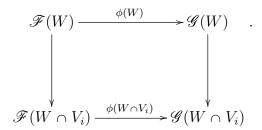
$$\mathscr{H}om: U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$$

is a sheaf.

Proof. For any  $V \subseteq U$ ,  $\mathscr{F}|_U(V) = \mathscr{F}(V)$  and  $\phi \in \operatorname{Hom}(\mathscr{F}|_U,\mathscr{G}|_U)$  is a family of compatible homomorphisms  $\phi(V): \mathscr{F}(V) \to \mathscr{G}(V)$  of abelian groups. Suppose  $\phi, \psi \in \operatorname{Hom}(\mathscr{F}|_U,\mathscr{G}|_U)$ , then we can define a binary operation + on  $\operatorname{Hom}(\mathscr{F}|_U,\mathscr{G}|_U)$  by  $(\phi + \psi)(V) = \phi(V) + \psi(V)$ . We now illustrate that it is well-defined. Since  $\phi, \psi$  are morphisms of sheaves, then for any  $W \subseteq V \subseteq U$  and any  $s \in \mathscr{F}(V)$ , we have  $\phi(V)(s)|_W = \phi(W)(s|_W)$  and  $\psi(V)(s)|_W = \psi(W)(s|_W)$ . Thus,  $(\phi + \psi)(V)(s)|_W = \phi(V)(s)|_W + \psi(V)(s)|_W = \phi(W)(s|_W) + \psi(W)(s|_W) = (\phi + \psi)(W)(s|_W)$ . This means that  $\phi + \psi \in \operatorname{Hom}(\mathscr{F}|_U,\mathscr{G}|_U)$ . Clearly,  $\operatorname{Hom}(\mathscr{F}|_U,\mathscr{G}|_U)$  is a group with identity  $0 = \{0: \mathscr{F}(V) \to \mathscr{G}(V)\}_V$  and the inverse of  $\phi$  is given by  $-\phi = \{-\phi(V): \mathscr{F}(V) \to \mathscr{G}(V)\}_V$ . Since  $\phi(V)$  and  $\psi(V)$  are homomorphisms of abelian groups, we see that  $\operatorname{Hom}(\mathscr{F}|_U,\mathscr{G}|_U)$  is abelian.

We point out that the restriction map  $\rho_{UV}: \mathscr{H}om(U) \to \mathscr{H}om(V)$  is defined by  $\phi \mapsto \phi|_V$ , where  $\phi|_V(W) = \phi(W)$  for all  $W \subseteq V \subseteq U$ . Let U be an open subset of X and  $\{V_i\}$  an open covering of U. Observe that

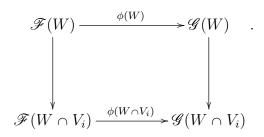
(1) Suppose  $\phi \in \mathcal{H}om(U)$ , i.e.  $\phi : \mathcal{F}|_U \to \mathcal{G}|_U$  is a morphism of sheaves. If  $\phi|_{V_i} = 0$ , then  $\phi = 0$ . Indeed, take  $W \subseteq U$ , then  $W = \bigcup_i (W \cap V_i)$ . Consider the following commutative diagram



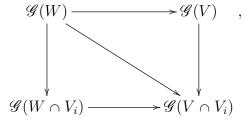
For any  $s \in \mathscr{F}(W)$ ,  $\phi(W)(s)|_{W \cap V_i} = \phi(W \cap V_i)(s|_{W \cap V_i}) = \phi|_{V_i}(W \cap V_i)(s|_{W \cap V_i}) = 0$  as  $\phi|_{V_i} = 0$  for all i. So,  $\phi(W)(s) = 0$  since  $\mathscr{G}$  is a sheaf. This implies that  $\phi(W) = 0$ . As  $W \subseteq U$  is arbitrary, we see that  $\phi = 0$ .

(2) Suppose  $\phi_i \in \mathscr{H}om(V_i)$ , i.e.  $\phi_i : \mathscr{F}|_{V_i} \to \mathscr{G}|_{V_i}$  is a morphism of sheaves, such that  $\phi_i|_{V_i \cap V_j} = \phi_j|_{V_i \cap V_j}$ . Then there exists  $\phi \in \mathscr{H}om(U)$  such that  $\phi|_{V_i} = \phi_i$  for all i. Indeed, we first define  $\phi(W) = \phi_i(W)$  if  $W \subseteq V_i$ . It is well-defined since if  $W \subseteq V_i \cap V_j$ , then  $\phi_i(W) = \phi_i|_{V_i \cap V_j}(W) = \phi_j|_{V_i \cap V_j}(W) = \phi_j(W)$ .

For any  $W \subseteq U$ , we can define  $\phi(W)$  which makes the following diagram commute



Indeed, for any  $s \in \mathscr{F}(W)$ , set  $t_i = \phi(W \cap V_i)(s|_{W \cap V_i}) = \phi_i(W \cap V_i)(s|_{W \cap V_i}) \in \mathscr{G}(W \cap V_i)$  for each i. Note that  $W = \bigcup_i (W \cap V_i)$  and  $\mathscr{G}$  is a sheaf. There exists  $t \in \mathscr{G}(W)$  such that  $t|_{W \cap V_i} = t_i$ . Now, we define  $\phi(W)$  by  $s \mapsto t$ . Then  $\phi(W) : \mathscr{F}(W) \to \mathscr{G}(W)$  is a group homomorphism making the diagram commute. We now check that  $\phi : \mathscr{F} \to \mathscr{G}$  is a morphism. It suffices to prove that, for any  $V \subseteq W \subseteq U$  and any  $s \in \mathscr{F}(W)$ ,  $\phi(W)(s)|_{V} = \phi(V)(s|_{V})$ . Considering the following commutative diagram



we have  $[\phi(W)(s)|_V]|_{V \cap V_i} = (\phi(W)(s)|_{W \cap V_i})|_{V \cap V_i} = \phi(W \cap V_i)(s|_{W \cap V_i})|_{V \cap V_i} = \phi_i(W \cap V_i)(s|_{W \cap V_i})|_{V \cap V_i} = \phi_i(V \cap V_i)(s|_{V \cap V_i})|_{V \cap V_i} = \phi(V \cap V_i)(s|_{V \cap V_i})|_{V \cap V_i$ 

Thus,  $\mathcal{H}om$  is indeed a sheaf.

Question 4 (10 points): Let  $\phi: A \to B$  be a homomorphism of rings, and let  $f: Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$  be the induced morphism of affine schemes. Show that  $\phi$  is injective if and only if the map of sheaves  $f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Y$  is injective. Show furthermore in that case f is dominant, i.e. f(Y) is dense in X.

Proof. Suppose that  $\phi$  is injective. We first show that for any  $g \in A$ ,  $f^{\#}(D(g))$  is injective. Let  $\overline{g} \in B$  be the image of g via  $\phi$ . Consider the map  $\overline{\phi}: A_g \to B_{\overline{g}}$  by  $a/g^n \mapsto \phi(a)/\phi(g)^n$ . We have that  $\overline{\phi}$  is injective. Indeed, if  $\phi(a)/\phi(g)^n = 0$ , there exists  $k \in \mathbb{N}$  such that  $\phi(g)^k \phi(a) = 0$ , then  $g^k a = 0$  since  $\varphi$  is injective. Thus,  $a/g^n = 0$  and it follows that  $\overline{\phi}$  is injective. Correspondingly,  $f^{\#}(D(g)): \mathcal{O}_X(D(g)) \to \mathcal{O}_Y(D(\overline{g})) = \mathcal{O}_Y(f^{-1}(D(g))) = (f_*\mathcal{O}_Y)(D(g))$  is injective. Note that D(g) is a base for the topology on X. For any  $\mathfrak{p} \in X$ , we have that  $f^{\#} = \varinjlim_{D(g) \ni \mathfrak{p}} f^{\#}(D(g))$ , where D(g) runs through all principal open subsets containing  $\mathfrak{p}$ , is also injective since direct limit functor is exact. So, we conclude that  $f^{\#}$  is injective.

Conversely, if  $f^{\#}$  is injective. By taking global section, we know that  $\phi: A \to B$  is injective.

In this case, we may identity A with a subring of B. We now show that f(Y) is dense in X. We claim that for any  $U \subseteq X$ , we have  $\overline{U} = V(\bigcap_{\mathfrak{p} \in U} \mathfrak{p})$ . Indeed, we have  $U \subseteq V(\bigcap_{\mathfrak{p} \in U} \mathfrak{p})$ , and so  $\overline{U} \subseteq V(\bigcap_{\mathfrak{p} \in U} \mathfrak{p})$ . Conversely, let  $\overline{U} = V(\mathfrak{a})$ , then for any  $\mathfrak{q} \in V(\bigcap_{\mathfrak{p} \in U} \mathfrak{p})$ , we have  $\mathfrak{q} \supset \bigcap_{\mathfrak{p} \in U} \mathfrak{p} \supset \mathfrak{a}$  since  $\mathfrak{p} \in U \Rightarrow \mathfrak{p} \supset \mathfrak{a}$ . Thus,  $V(\bigcap_{\mathfrak{p} \in U} \mathfrak{p}) \subseteq V(\mathfrak{a}) = \overline{U}$ . So, we finish the proof of the claim. It follows that  $\overline{f(Y)} = V(\bigcap_{\mathfrak{p} \in f(Y)} \mathfrak{p}) = V(\bigcap_{\mathfrak{q} \in Y} f(\mathfrak{q})) = V(\bigcap_{\mathfrak{q} \in Y} (\mathfrak{q} \cap A)) = V(\operatorname{nil}(B) \cap A) = V(\operatorname{nil}(A)) = V(0) = X$ . Thus, f is dominant.

Question 5 (10 points): Let A be a ring and let  $(X, \mathcal{O}_X)$  be a scheme. Give a morphism  $f: X \to \operatorname{Spec} A$ , we have an associated map on sheaves  $f^{\#}: \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$ . Taking global sections we obtain a homomorphism  $A \to \Gamma(X, \mathcal{O}_X)$ . Thus there is a natural map

$$\alpha: \operatorname{Hom}_{\operatorname{schemes}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\operatorname{rings}}(A, \Gamma(X, \mathcal{O}_X)).$$

Show that  $\alpha$  is bijective.

*Proof.* By the Proposition 2.3, we know that there exists a bijection

$$\beta: \operatorname{Hom}_{\operatorname{schemes}}(\operatorname{Spec} B, \operatorname{Spec} A) \to \operatorname{Hom}_{\operatorname{rings}}(A, B),$$

by  $(f, f^{\#}) \mapsto f^{\#}(\operatorname{Spec} A)$ . Thus, if X is an affine scheme, then  $\alpha$  is bijective.

Note that for each  $x \in X$ , there exists an open neighborhood  $U_x$  such that  $(U_x, \mathcal{O}_X|_{U_x})$  is an affine scheme. We may write  $X = \bigcup_i U_i$ , where each  $U_i$  is an affine scheme. Let  $U = U_i$ . Define  $\rho_i$ : Hom<sub>schemes</sub> $(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\operatorname{schemes}}(U, \operatorname{Spec} A)$  by  $(f, f^{\#}) \to (f|_U, f^{\#}|_U)$ , where, for each  $V \subseteq \operatorname{Spec} A$ ,  $f^{\#}|_U(V)$  is defined by the following commutative diagram

This implies that the map

$$\rho: \operatorname{Hom}_{\operatorname{schemes}}(X, \operatorname{Spec} A) \to \prod_i \operatorname{Hom}_{\operatorname{schemes}}(U_i, \operatorname{Spec} A)$$

defined by  $(f, f^{\#}) \mapsto ((f|_{U_i}, f^{\#}|_{U_i}))_i$  is injective. Now, consider the following diagram

$$\operatorname{Hom}_{\operatorname{schemes}}(X,\operatorname{Spec}\,A) \xrightarrow{\alpha} \operatorname{Hom}_{\operatorname{rings}}(A,\Gamma(X,\mathcal{O}_X))$$

$$\downarrow^{\rho} \qquad \qquad \downarrow^{\gamma} \qquad \qquad \downarrow^{\gamma}$$

$$\prod_{i} \operatorname{Hom}_{\operatorname{schemes}}(U_i,\operatorname{Spec}\,A) \xrightarrow{\prod \beta_i} \prod_{i} \operatorname{Hom}_{\operatorname{rings}}(A,\Gamma(U_i,\mathcal{O}_X|_{U_i}))$$

we conclude that  $\alpha$  is injective since  $\beta_i$  are isomorphisms.

Here,  $\gamma$  is defined by  $\gamma_i$ : Hom<sub>rings</sub> $(A, \Gamma(X, \mathcal{O}_X)) \to \text{Hom}_{\text{rings}}(A, \Gamma(U_i, \mathcal{O}_X|_{U_i}))$  by  $\varphi \mapsto \text{Res}_{X,U_i} \circ \varphi$ , i.e. the composition  $A \to \mathcal{O}_X(X) \to \mathcal{O}_X(U_i)$ .

Now, we are going to prove that  $\alpha$  is surjective. Let  $\varphi \in \operatorname{Hom}_{\operatorname{rings}}(A, \Gamma(X, \mathcal{O}_X))$ , then there exists a unique  $f_i = (f_i, f_i^{\#}) \in \operatorname{Hom}_{\operatorname{schemes}}(U_i, \operatorname{Spec} A)$  for each i such that  $\beta_i(f_i) = \gamma_i(\varphi)$ . We now want to glue these  $f_i$  to a morphism  $f \in \operatorname{Hom}_{\operatorname{schemes}}(X, \operatorname{Spec} A)$ .

First, for any affine open subset  $W \subseteq U_i \cap U_j$ , the image of  $f_i|_W$  in  $\operatorname{Hom}_{\operatorname{rings}}(A, \Gamma(W, \mathcal{O}_X|_W))$  is  $\operatorname{Res}_{U_i,W} \circ \beta(f_i) = \operatorname{Res}_{U_i,W} \circ \gamma_i(\varphi) = \operatorname{Res}_{U_i,W} \circ \operatorname{Res}_{X,U_i} \circ \varphi = \operatorname{Res}_{X,W} \circ \varphi$ . Similarly, the image of  $f_j|_W$  in  $\operatorname{Hom}_{\operatorname{rings}}(A, \Gamma(W, \mathcal{O}_X|_W))$  is  $\operatorname{Res}_{X,W} \circ \varphi$ . Thus,  $f_i|_W$  and  $f_j|_W$  have the image in  $\operatorname{Hom}_{\operatorname{rings}}(A, \Gamma(W, \mathcal{O}_X|_W))$ . Thus  $f_i|_W = f_j|_W$  for any affine subset  $W \subseteq U_i \cap U_j$ . We conclude that  $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$  for any i, j. This is because  $\rho$  is injective, where we replace X by  $U_i \cap U_j$  and each  $U_i$  by W.

Set  $Y = \operatorname{Spec} A$ . We may define  $f(x) = f_i(x)$  if  $x \in U_i$  for some i, so  $f : X \to Y$  is a continuous map. Since  $(f_i)_*(\mathcal{O}_X|_{U_i})(V) = (\mathcal{O}_X|_{U_i})(f_i^{-1}(V)) = \mathcal{O}_X(f_i^{-1}(V))$  for each  $V \subseteq Y$ , then  $f_i^\# : \mathcal{O}_Y \to (f_i)_*(\mathcal{O}_X|_{U_i})$  induces a map of rings  $f_i^\#(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f_i^{-1}(V))$  for each  $V \subseteq Y$ . Note that  $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$ . For each  $t \in \mathcal{O}_Y(V)$ , let  $s_i = f_i^\#(V)(t)$ , then there exists a unique  $s \in \mathcal{O}_X(f^{-1}(V))$  such that  $s|_{f_i^{-1}(V)} = s_i$ . Define  $f^\#(V)(t) = s$ , we then obtain a ring homomorphism  $f^\#(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ , which gives us a morphism of sheaves  $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$  with  $f^\#|_{U_i} = f_i^\#$ . Thus, we have a morphism of schemes  $f = (f, f^\#)$  such that  $f|_{U_i} = f_i$ . Thus,  $\gamma_i(\alpha(f) - \varphi) = \gamma_i(\alpha(f)) - \gamma_i(\varphi) = \beta_i(\rho(f)) - \gamma_i(\varphi) = \beta_i(f_i) - \gamma_i(\varphi) = 0$  for each i. Since  $\mathcal{O}_X$  is a sheaf, we know that  $\alpha(f) - \varphi = 0$ . Thus,  $\alpha$  is surjective.  $\square$ 

Question 6 (10 points): Show that a morphism  $f: X \to Y$  between schemes is locally of finite type if and only if for every open affine subset  $V = \operatorname{Spec} B$  of Y,  $f^{-1}(V)$  can be covered by open affine subsets  $U_i = \operatorname{Spec} A_i$ , where each  $A_i$  is a finitely generated B-algebra.

*Proof.* Suppose f is locally of finite type. We first observe that

- (1) If  $U = \operatorname{Spec} A \subseteq Y$  is an affine open subset satisfying that  $f^{-1}(U)$  can be covered by open affine subsets  $\operatorname{Spec} A_i$  with each  $A_i$  a finitely generated A-algebra, then for any  $g \in A$ , there exist some rings  $B_i$  such that  $f^{-1}(\operatorname{Spec} A_g)$  can be covered by open affine subsets  $\operatorname{Spec} B_i$  with each  $B_i$  a finitely generated  $A_g$ -algebra. Indeed,  $\operatorname{Spec} A_g = D(g)$  is a principal open subset in  $\operatorname{Spec} A$ . Then  $f^{-1}(D(g)) = f^{-1}(U \cap D(g)) = \bigcup_i (f^{-1}(D(g)) \cap \operatorname{Spec} A_i)$ , where each  $A_i$  is a finitely generated A-algebra. Let  $f_i = f|_{\operatorname{Spec} A_i}$  and  $\varphi_i : A \to A_i$  be the ring homomorphism induced by  $f_i : \operatorname{Spec} A_i \to \operatorname{Spec} A$ . Then, for any  $g \in A$ , we have  $f^{-1}(D(g)) \cap \operatorname{Spec} A_i = D(\varphi_i(g))$ , where  $D(\varphi_i(g))$  is a principal open subset in  $\operatorname{Spec} A_i$ . Indeed,  $f^{-1}(D(g)) \cap \operatorname{Spec} A_i = f_i^{-1}(D(g)) = f_i^{-1}(U V(g)) = f_i^{-1}(U) f_i^{-1}(V(g)) = \operatorname{Spec} A_i V(\varphi_i(g)) = D(\varphi_i(g))$ . So,  $f^{-1}(D(g)) \cap \operatorname{Spec} A_i = D(\varphi_i(g)) \cong \operatorname{Spec} (A_i)_{\varphi_i(g)}$  is a princial subset in  $\operatorname{Spec} A_i$  and  $(A_i)_{\varphi_i(g)}$  is a finitely generated  $A_g$ -algebra.
- (2) If  $(f_1, \dots, f_n) = A$  and each Spec  $A_{f_i}$  is an affine scheme satisfying that  $f^{-1}(\operatorname{Spec} A_{f_i})$  can be covered by  $U_{ij} = \operatorname{Spec} A_{ij}$  with  $A_{ij}$  a finitely generated  $A_{f_i}$  algebra, then  $f^{-1}(\operatorname{Spec} A)$  can be covered by  $U_{ij} = \operatorname{Spec} A_{ij}$  with  $A_{ij}$  a finitely generated A-algebra. Indeed, since  $(f_1, \dots, f_n) = A$ , we see that  $\bigcap V(f_i) = V(f_1, \dots, f_n) = V(1) = \emptyset$ . Thus,  $\bigcup D(f_i) = \operatorname{Spec} A$ . Let  $U = \operatorname{Spec} A$ , then  $U \subseteq \bigcup_{i=1}^n \operatorname{Spec} A_{f_i}$  and  $f^{-1}(U) \subseteq \bigcup_{i=1}^n f^{-1}(\operatorname{Spec} A_{f_i}) = \bigcup_{i=1}^n \bigcup_j \operatorname{Spec} A_{ij}$ . Since  $A_{f_i}$  is a finitely generated A-algebra, we see that  $A_{ij}$  is a finitely generated A-algebra.

Since f is locally of finite type, there exists an open affine covering  $V_i = \text{Spec } B_i$  of Y such that each  $f^{-1}(V_i)$  can be covered by open subsets  $U_{ij} = \text{Spec } A_{ij}$ , where each  $A_{ij}$  is a finitely generated  $B_i$ -algebra. Since V = Spec B is quasi-compact, we can cover Spec B with a finite

number of principal open sets Spec  $B_{g_j}$ , each of which is principal in some Spec  $B_i$ . Indeed, given any point  $\mathfrak{p} \in \operatorname{Spec} B \cap \operatorname{Spec} B_i$ , we can find an open neighborhood of  $\mathfrak{p}$  in Spec  $B \cap \operatorname{Spec} B_i$  that is simultaneously principal open in both Spec B and Spec  $A_i$ . Let Spec  $A_f \cong D(f)$  be a principal open subset of Spec A contained in Spec  $B \cap \operatorname{Spec} B_i$  and containing  $\mathfrak{p}$ . Let Spec  $(B_i)_g \cong D(g)$  be a principal open subset of Spec  $B_i$  contained in Spec  $B_f$  and containing  $\mathfrak{p}$ . Then  $g \in \Gamma(\operatorname{Spec} B_i, \mathcal{O}_X)$  restricts to an element  $g' \in \Gamma(\operatorname{Spec} B_f, \mathcal{O}_X) = B_f$ . Then, by using a similar argument in (1), we know that  $D(g) = D(g) \cap \operatorname{Spec} B_f = D(g')$ , where D(g') is a principal open subset in Spec  $B_f$ , so  $\operatorname{Spec} (B_i)_g \cong \operatorname{Spec} (B_f)_{g'}$ . If  $g' = h/f^n$  with  $h \in B$ , then  $\operatorname{Spec} (B_f)_{g'} = \operatorname{Spec} B_{fh} \cong D(fh)$  since  $(B_f)_{g'} = B_{fh}$ . This implies that  $\operatorname{Spec} (B_i)_g$  is also a principal open set in B.

By (1), each Spec  $B_{g_j}$  satisfies that  $f^{-1}(\operatorname{Spec} B_{g_j})$  can be covered by open affine subsets Spec  $A_i$  with  $A_i$  a finitely generated  $B_{g_j}$ -algebra. Since Spec B is covered by all of Spec  $B_{g_j}$ , so all  $g_j$  generate B. By (2), we see that  $f^{-1}(V)$  can be covered by open affine subsets  $U_j = \operatorname{Spec} A_j$ , where each  $A_j$  is a finitely generated B-algebra. We are done.

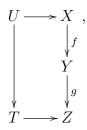
The other direction is trivial.

Question 7 (10 points): Using valuative criterion to show the following claims:

- 1. A composition of proper morphisms is proper,
- 2. Products of proper morphisms are proper. (If  $f: X \to Y$  and  $f': X' \to Y'$  are two morphisms of schemes over S, then the product morphism is  $f \times f': X \times_S X' \to Y \times_S Y'$ .)

Proof. (1) Suppose  $f: X \to Y$  and  $g: Y \to Z$  are two proper morphisms, which are of finite type. We claim that  $g \circ f$  is also of finite type. Indeed, Let  $W = \operatorname{Spec} C$  be an affine open subset of Z. Since g is of finite type, then  $g^{-1}(W)$  can be covered by a finite number of affine open subset  $V_i = \operatorname{Spec} B_i$ , where  $B_i$  is a finitely generated C-algebra. Again, f is of finite type, then each  $f^{-1}(V_i)$  can be covered by a finite number of affine open subsets  $U_{ij} = \operatorname{Spec} A_{ij}$ , where each  $A_{ij}$  is a finitely generated  $B_i$ -algebra. Thus,  $A_{ij}$  is a finitely generated C-algebra. So,  $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) \subseteq \bigcup_i f^{-1}(V_i) \subseteq \bigcup_{i,j} U_{ij}$ , where the number of  $U_{ij}$  is finite. Thus,  $g \circ f$  is of finite type.

Now, for every valuation ring R and  $K = \operatorname{Frac}(R)$ , consider the following commutative diagram



where  $T = \operatorname{Spec} R$  and  $U = \operatorname{Spec} K$ . Let  $U \to Y$  be the composed map  $U \to X \stackrel{f}{\longrightarrow} Y$ . Since g is proper, by Valuative Criterion of Properness, there exists a unique morphism  $\alpha: T \to Y$  making the following diagram commute

$$\begin{array}{ccc} U \longrightarrow Y \\ \downarrow & & \downarrow g \\ T \longrightarrow Z \end{array}$$

Again, since f is proper, there exists a unique morphism  $\beta: T \to X$  such that the following

diagram commute

$$\begin{array}{c}
U \longrightarrow X \\
\downarrow \beta & \downarrow f \\
T \xrightarrow{\alpha} Y
\end{array}$$

Thus,  $\beta: T \to X$  is the unique morphism such that the following diagram commute

$$\begin{array}{ccc} U \longrightarrow X \\ \downarrow & & \downarrow g \circ f \\ T \longrightarrow Z \end{array}$$

Since  $g \circ f$  is of finite type, by Valuative Criterion for Properness, we see that  $g \circ f$  is proper.

(2) By definition,  $f \times f'$  is the unique morphism making the following diagram commute

$$X \stackrel{p}{\longleftarrow} X \times_S X' \stackrel{p'}{\longrightarrow} X'$$

$$\downarrow f \qquad \qquad \downarrow f \times f' \qquad \qquad \downarrow f'$$

$$Y \stackrel{q}{\longleftarrow} Y \times_S Y' \stackrel{q'}{\longrightarrow} Y'$$

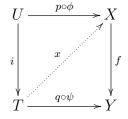
Now, for every valuation ring R and  $K = \operatorname{Frac}(R)$ , let  $T = \operatorname{Spec}(R)$  and  $U = \operatorname{Spec}(R)$ . For any given morphisms  $\phi: U \to X \times_S X'$  and  $\psi: T \to Y \times_S Y'$  making the following diagram commute

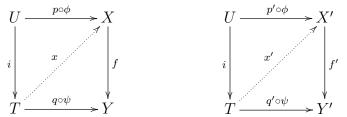
$$U \xrightarrow{\phi} X \times_S X'$$

$$\downarrow \downarrow \qquad \qquad \downarrow f \times f'$$

$$T \xrightarrow{\psi} Y \times_S Y'$$

composed with p, q and p', q' respectively, we obtain two commutative diagrams

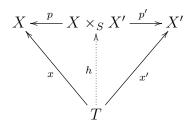




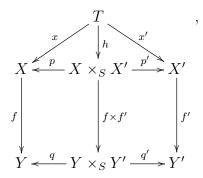
Since f and f' are proper, there exist unique morphism  $x:T\to X$  and  $x':T\to X'$  making the above two diagram commute by Valuative Criterion for Properness.

By the universal property of fibred product, there exists a unique morphism  $h: T \to X \times_S X'$ 

making the following diagram commute

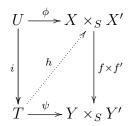


These commutative diagrams tell us that  $p \circ (h \circ i) = p \circ \phi$  and  $p' \circ (h \circ i) = p' \circ \phi$ , which implies that  $h \circ i = \phi$  by the universal property of fibred product. Similarly, we have  $f \circ x = q \circ \psi$  and  $f' \circ x' = q' \circ \psi$ . Now, considering the following commutative diagram



we have that  $q' \circ (f \times f') \circ h = q' \circ \psi$  and  $q \circ (f \times f') \circ h = q \circ \psi$ . Thus,  $(f \times f') \circ h = \psi$  by the universal property of fibred product.

Thus, there exists a unique morphism  $h: T \to X \times_S X'$  making the following diagram commute



It remains to prove that  $f \times f'$  is of finite type. We claim that morphisms of finite type are stable under base extension, i.e. if the morphism  $f: X \to S$  of schemes is of finite type, then for any  $g: Z \to S$ , the second projection  $p_2: X \times_S Z \to Z$  is of finite type.

Let  $\{S_i\}$  be an affine open cover of S. By the construction of fibred product, we see that  $X \times_S Z = \bigcup_i f^{-1}(S_i) \times_{S_i} g^{-1}(S_i)$ . Note that the finite type property is affine local, it reduces to the case that S is affine, say  $S = \operatorname{Spec} R$ . Take an open affine subset  $V \subseteq Z$  with  $V = \operatorname{Spec} C$ . By the universal property, we see that  $(p_2)^{-1}(V) = X \times_S V$ . Since f is of finite type,  $X = f^{-1}(S)$  can be covered by a finite number of affine open subsets  $U_i = \operatorname{Spec} A_i$ , where each  $A_i$  is a finitely generated B-algebra. Thus,  $(p_2)^{-1}(V)$  can be covered by a finite number of  $U_i \times_S V = \operatorname{Spec} (A_i \otimes_R C)$ , where  $A_i \otimes_R C$  is a finitely generated  $B \otimes_R C$ -algebra.

Considering the morphism  $Y \times_S Y' \to Y$ , we know that  $X \times_Y (Y \times_S Y') \to (Y \times_S Y')$  is of finite type. Thus,  $f \times \operatorname{id}: X \times_S Y' \to Y \times_S Y'$  is of finite type. Similarly,  $\operatorname{id} \times f': X \times_S X' \to X \times_S Y'$  is also of finite type. Now, observe that  $f \times f'$  is the composed map of  $X \times_S X' \xrightarrow{\operatorname{id} \times f'} X \times_S Y' \xrightarrow{f \times \operatorname{id}} Y \times_S Y'$ .

In (1), we have already proved that the composition of morphisms of finite type is also of	nnite
type. Thus, $f \times f'$ is of finite type.	
By Valuative Criterion for Properness, we conclude that $f \times f'$ is proper.	