Algebraic Geometry Homework 3

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3 First Properties of Schemes

Lemma 1: Suppose Spec A and Spec B are affine open subschemes of a scheme X. Then Spec $A \cap$ Spec B is the union of open sets that are simultaneously principal open subschemes of Spec A and Spec B.

Proof. Given any point $\mathfrak{p} \in$ Spec $A \cap$ Spec B, we produce an open neighborhood of $\mathfrak{p}$ in Spec $A \cap$ Spec B that is simultaneously principal open in both Spec A and Spec B. Let Spec $A_f \cong D(f)$ be a principal open subset of Spec A contained in Spec $A \cap$ Spec B and containing $\mathfrak{p}$. Let Spec $B_g \cong D(g)$ be a principal open subset of Spec B contained in Spec A_f and containing $\mathfrak{p}$. Then $g \in \Gamma(\text{Spec } B, \mathscr{O}_X)$ restricts to an element $g' \in \Gamma(\text{Spec } A_f, \mathscr{O}_X) = A_f$. Then Spec $B_g = D(g')$ in Spec A_f, so Spec $B_g \cong$ Spec $(A_f)_{g'}$. If $g' = g''/f^n$ with $g'' \in A$, then Spec $(A_f)_{g'} =$ Spec $A_{fg''} \cong D(fg'')$ since $(A_f)_{g'} = A_{fg''}$. This implies that Spec B_g is also a principal open set in A. ☐

Lemma 2 (Affine Communication Lemma): Let P be some property enjoyed by some affine open subsets of a scheme X, such that

(i) if an affine open subset Spec $A \hookrightarrow X$ has property P then for any $f \in A$, Spec $A_f \hookrightarrow X$ does too.

(ii) if $(f_1, \cdots, f_n) = A$, and Spec $A_{f_i} \hookrightarrow X$ has P for all i, then so does Spec $A \hookrightarrow X$.

Suppose that $X = \bigcup_{i \in I}$ Spec A_i, where Spec A_i has property P. Then every affine open subset of X has P too.

Proof. Let Spec A be an affine subscheme of X. Since Spec A is quasi-compact, we can cover Spec A with a finite number of principal open sets Spec A_{g_j}, each of which is principal in some Spec A_i. This is possible by Lemma 1. By (i), each Spec A_{g_j} has P. By (ii), Spec A has P. ☐

Lemma 3: *Let $f : X \to Y$ be a morphism of schemes and $U =$ Spec A an affine subscheme of Y with $f^{-1}(U) = \bigcup_i$ Spec A_i for some rings A_i. Let $f_i = f|_{\text{Spec } A_i}$ and $\varphi_i : A \to A_i$ be the ring homomorphism induced by $f_i :$ Spec $A_i \to$ Spec A. Then, for any $g \in A$, we have $f^{-1}(D(g)) \cap$ Spec $A_i = D(\varphi_i(g))$, where $D(\varphi_i(g))$ is a principal open subset in* Spec A_i.

Proof. $f^{-1}(D(g)) \cap \text{Spec } A_i = f_i^{-1}(D(g)) = f_i^{-1}(U - V(g)) = f_i^{-1}(U) - f_i^{-1}(V(g)) = \text{Spec } A_i - V(\varphi_i(g)) = D(\varphi_i(g))$. ☐

Exercise 3.1: Show that a morphism $f : X \to Y$ is locally of finite type if and only if for every open affine subset $V =$ Spec B of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_i =$ Spec A_i, where each A_i is a finitely generated B-algebra.

Proof. We will use the Affine Communication Lemma to prove this Exercise. We first check that condition (1), (2) in this Lemma are satisfied. Here, the property P enjoyed by $U = \text{Spec } A$ is that $f^{-1}(U)$ can be covered by open affine subsets Spec A_i with each A_i a finitely generated A-algebra.

Suppose $U = \text{Spec } A \subseteq Y$ has property P mentioned above, then for any $g \in A$, Spec $A_g = D(g)$ is a principal open subset in Spec A. Then $f^{-1}(D(g)) = f^{-1}(U \cap D(g)) = \bigcup_i (f^{-1}(D(g)) \cap \text{Spec } A_i)$, where each A_i is a finitely generated A-algebra with the associated map $\varphi_i : A \to A_i$. Note that $f^{-1}(D(g)) \cap \text{Spec } A_i = D(\varphi_i(g)) \cong \text{Spec } (A_i)_{\varphi_i(g)}$ is a princial subset in Spec A_i and $(A_i)_{\varphi_i(g)}$ is a finitely generated A_g-algebra.

Suppose $(f_1, \cdots, f_n) = A$ and Spec A_{f_i} has P, i.e. each $f^{-1}(\text{Spec } A_{f_i})$ can be covered by $U_{ij} = \text{Spec } A_{ij}$ with A_{ij} a finitely generated A_{f_i}- algebra. So, $U = \text{Spec } A \subseteq \bigcup_{i=1}^n \text{Spec } A_{f_i}$ and $f^{-1}(U) \subseteq \bigcup_{i=1}^n f^{-1}(\text{Spec } A_{f_i}) = \bigcup_{i=1}^n \bigcup_j \text{Spec } A_{ij}$. Since A_{f_i} is a finitely generated A-algebra, we see that A_{ij} is a finitely generated A-algebra.

If f is locally of finite type, then by Affine Communication Lemma, we get the desired result.

The other direction is trivial. □

Exercise 3.2: A morphism $f : X \to Y$ of schemes is quasi-compact if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for every open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

Proof. We first verify that

(1) If $U = \text{Spec } A$ is an affine open subset of Y such that $f^{-1}(U)$ is compact, then for any $g \in A$, $D(g) = \text{Spec } A_g$ satisfies $f^{-1}(D(g))$ is compact. Indeed, since $f^{-1}(U)$ is compact, we can find a finite affine open cover, say $f^{-1}(U) = \bigcup_{i=1}^n \text{Spec } A_i$. Then $f^{-1}(D(g)) = f^{-1}(U \cap D(g)) = \bigcup_{i=1}^n (f^{-1}(D(g)) \cap \text{Spec } A_i)$ with the associated map $\varphi_i : A \to A_i$. Note that $f^{-1}(D(g)) \cap \text{Spec } A_i = D(\varphi_i(g))$, we see that $f^{-1}(D(g)) = \bigcup_{i=1}^n D(\varphi_i(g))$ is quasi-compact as principal open subsets are always quasi-compact.

(2) If $A = (f_1, \cdots, f_n)$ and each $f^{-1}(\text{Spec } A_{f_i})$ is quasi-compact, then $f^{-1}(\text{Spec } A)$ is also quasi-compact. Indeed, Spec $A = \bigcup_{i=1}^n D(f_i) = \bigcup_{i=1}^n \text{Spec } A_f$, then $f^{-1}(\text{Spec } A) = f^{-1}(\bigcup_{i=1}^n D(f_i)) = \bigcup_{i=1}^n f^{-1}(\text{Spec } A_{f_i})$ is quasi-compact.

If f is quasi-compact, then by Affine Communication Lemma, we see that $f^{-1}(V)$ is quasi-compact for all open affine subset $V \subseteq Y$.

The other direction is trivial. □

Exercise 3.3: (a) Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

(b) Conclude from this that f is of finite type if and only if for every open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B-algebra.

(c) Show also if f is of finite type, then for every open affine subset $V = \text{Spec } B \subseteq Y$, and for every open affine subset $U = \text{Spec } A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

Proof. (a) If f is of finite type, then there exists a covering of Y by open affine subsets $V_i = \text{Spec } B_i$, such that for each i, $f^{-1}(V_i)$ can be covered by finite number of open affine subsets $U_{ij} = \text{Spec } A_{ij}$, where each A_{ij} is a finitely generated B_i-algebra. So $f^{-1}(V_i)$ is quasi-compact because each Spec A_{ij} is quasi-compact. Thus, f is locally of finite type and quasi-compact.

The other direction is trivial.

(b) f is of finite type, then f is locally of finite type and quasi-finite. By Exercise 3.1, for every open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B-algebra. By Exercise 3.2, $f^{-1}(V)$ is quasi-compact. Thus, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \text{Spec } A_j$.

Conversely, suppose for every open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B-algebra. Then f is locally of finite type. Moreover, $f^{-1}(V)$ is quasi-compact since each Spec A_j is quasi-compact and finite union of quasi-compact subsets is still quasi-compact. Thus, f is quasi-compact. It follows that f is of finite type.

(c) By (a), f is locally of finite type and quasi-compact. So, for every open affine subset $V = \text{Spec } B \subseteq Y$, $f^{-1}(V)$ can be covered by finitely many open affine subsets $U_i = \text{Spec } A_i$, where each A_i is a finitely generated B-algebra, i.e. $f^{-1}(V) = \bigcup_{i=1}^n U_i$.

Now, observe that

(1) Suppose $U = \text{Spec } A \subseteq f^{-1}(V)$ satisfies that A is a finitely generated B-algebra. Then for any $f \in A$, A_f is a finitely generated B-algebra since A_f is a finitely generated A-algebra.

(2) If $U = \text{Spec } A \subseteq f^{-1}(V)$ and $(f_1, \cdots, f_n) = A$ such that each A_{f_i} is a finitely generated B-algebra, we must have A is a finitely generated B-algebra. Indeed, first $f : U \to V$ induces a ring homomorphism $B \to A$, which gives A a B-algebra structure.

Since $(f_1, \cdots, f_n) = A$, we have $1 = \sum_{i=1}^n c_i f_i$ for some $c_i \in A$. We write $A_{f_i} = B[a_{i1}, \cdots, a_{in_i}]$ as a B-algebra, where $a_{ij} = r_{ij}/f_i^{k_j}$ with $r_{ij} \in A$ for $j = 1, 2, \cdots, n_i$. We let $C \subseteq A$ be the B-subalgebra generated by all of f_i and r_{ij}. Clearly $C_{f_i} \subseteq A_{f_i}$ for all f_i.

Conversely, take any $a \in A_{f_i}$, we can write $a = h(r_{i1}/f_i^{k_1}, \cdots, r_{in_i}/f_i^{k_{n_i}}) = h_i/f_i^t$, where $h_i = h_i(r_{i1}, \cdots, r_{in_i}, f_i)$ with coefficients in B. Thus, $a \in C_{f_i}$. This implies that $C_{f_i} = A_{f_i}$ for all f_i. Then $(A/C)_{f_i} \cong A_{f_i}/C_{f_i} = 0$ for all f_i. Thus, $A/C = 0$ and it follows that $A = C$ is a finitely generated B-algebra. Indeed, it suffices to show that if $R_{f_i} = 0$ for all f_i, then $R = 0$, where R is an A-algebra. We take $r \in R$, then $r/1 = 0$ in all R_{f_i}, then $f_i^{k_i} r = 0$ in A. We may take $k = \max_i k_i$, then $f_i^k r = 0$ for all i. Since $(f_1, \cdots, f_n) = A$, we see that $(f_1^k, \cdots, f_n^k) = A$ as $D(f_i) = D(f_i^k)$. So, there exists $b_i \in A$ such that $\sum_{i=1}^n b_i f_i^k = 1$, and it follows that $r = \sum_{i=1}^n b_i f_i^k r = 0$.

Now, by Affine Communication Lemma, we conclude that for every open affine subset $V = \text{Spec } B \subseteq Y$, and for every open affine subset $U = \text{Spec } A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra. $\square$

Exercise 3.4: Show that a morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ is affine, equal to Spec A, where A is a finite generated B-module.

Proof. The restriction $f|_{\text{Spec } A} : \text{Spec } A \to \text{Spec } B$ of f on Spec A induces a ring homomorphism $\varphi : B \to A$, which gives A a B-algebra structure.

We first observe that

(1) If $V = \text{Spec } B$ is an open affine subset of Y and $f^{-1}(V) = \text{Spec } A$ is affine such that A is a finite generated B-module, then for any $g \in B$, $f^{-1}(\text{Spec } B_g) = f^{-1}(D(g)) = D(\varphi(g)) = \text{Spec } A_{\varphi(g)}$ is affine. Moreover, $A_{\varphi(g)} = A[g^{-1}]$ is a finitely generated B_g-module.

(2) Suppose $V = \text{Spec } B$ is an open affine subset of Y, $(f_1, \cdots, f_n) = B$ and each $f^{-1}(\text{Spec } B_{f_i}) = \text{Spec } B_i$ is affine, where B_i is a finitely generated B_{f_i}-module. Then, $\text{Spec } B = \bigcup_{i=1}^n D(f_i) = \bigcup_{i=1}^n \text{Spec } B_{f_i}$. So, $f^{-1}(V) = \bigcup_{i=1}^n f^{-1}(\text{Spec } B_{f_i}) = \bigcup_{i=1}^n \text{Spec } B_i$.

Let $Z = f^{-1}(V)$ and $h = f|_Z : Z \to \text{Spec } B$ be the restriction of f on Z. Let $A = \mathscr{O}_X|_Z(Z) = \Gamma(Z, \mathscr{O}_X)$. Then, by Exercise 2.4, we have a ring homomorphism $\varphi : B \to A$. Set $g_i = \varphi(f_i)$. Since $(f_1, \cdots, f_n) = B$, we can write $1_B = \sum_{i=1}^n b_i f_i$. So, $1_A = \varphi(1_B) = \varphi(\sum_{i=1}^n b_i f_i) = \sum_{i=1}^n \varphi(b_i) g_i$, which implies that $(g_1, \cdots, g_n) = A$. Let $Z_{g_i} = \{x \in Z : (g_i)_x \notin \mathfrak{m}_x\}$, where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathscr{O}_{X,x}$. Then by Exercise 2.16, we see that $Z_{g_i} \cap \text{Spec } B_i = D(\overline{g_i})$, where $\overline{g_i}$ is the image of g_i in B_i. By Lemma 3, $f^{-1}(D(f_i)) \cap \text{Spec } B_i = D(\overline{g_i}) = Z_{g_i} \cap \text{Spec } B_i$. So, we see that $Z_{g_i} = \bigcup_{i=1}^n (Z_{g_i} \cap \text{Spec } B_i) = \bigcup_{i=1}^n (f^{-1}(D(f_i)) \cap \text{Spec } B_i) = \text{Spec } B_i$ for all i. Thus, Z_{g_i} is affine. By Exercise 2.17, the Criterion for Affineness, we see that $Z = f^{-1}(V)$ is affine. Thus, $Z = f^{-1}(V) = \text{Spec } A$. It follows that $Z_{g_i} = \text{Spec } B_i = D(g_i)$ and $B_i = \mathscr{O}_X(D(g_i)) \cong A_{g_i} = A_{f_i}$, which is a finitely generated B_{f_i}-module for each i. Then we must have A is a finitely generated B-module.

Indeed, we may write $A_{f_i} = \sum_{i=1}^{n_i} B_{f_i} x_{ij}$ for some $x_{ij} = a_{ij}/f_i^{k_j}$ with $a_{ij} \in A$. Let $C \subseteq A$ be the B-module generated by all of a_{ij} and f_i, which is a finitely generated B-module. Then $C_{f_i} \subseteq A_{f_i}$. Conversely, let $a \in A_{f_i}$, then $a = \sum_{j=1}^n (b_j a_{ij}/f_i^{t_j}) \in C_{f_i}$, where $b_j \in B$. Thus, $C_{f_i} = A_{f_i}$ for all i. Thus, we have $A = C$ by using a same type of argument as in Exercise 3.3. We conclude that A is a finitely generated B-module as desired.

Now, if f is finite, then for every open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V) = \text{Spec } A$ is affine, where A is a finite generated B-module by Affine Communication Lemma.

The other direction is trivial. □

Exercise 3.5: A morphism $f : X \to Y$ is quasi-finite if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

(a) Show that a finite morphism is quasi-finite.

(b) Show that a finite morphism is closed, i.e., the image of any closed subset is closed.

(c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

Proof. (a) For any $y \in Y$, there exists an open affine neighborhood of y, say $V = \text{Spec } B$. By Exercise 3.4, $f^{-1}(V) = \text{Spec } A$ for some ring A, where A is a finitely generated B-module. Consider the restriction of f on $U := \text{Spec } A$, which is still denoted by f, i.e. $f : \text{Spec } A \to \text{Spec } B$. Then it induces a ring homomorphism $\varphi : B \to A$. The fibre $f^{-1}(y) = U_y = U \times_V \text{Spec } k(y) = \text{Spec } (A \otimes_B k(y))$. Since A is a finitely generated B-module, we have that $A \otimes_B k(y)$ is a finite dimensional $k(y)$-vector space. Indeed, $A \cong B^{\oplus n}/I$ gives a short exact sequence

$$0 \to I \to B^{\oplus n} \to A \to 0.$$

Tensoring with $- \otimes_B k(y)$, we have a right exact sequence

$$I \otimes_B k(y) \to k(y)^n \xrightarrow{\psi} A \otimes_B k(y) \to 0.$$

Thus, $A \otimes_B k(y) \cong k(y)^n/\ker\psi$ is a finite dimensional $k(y)$-vector space. Now, each ideal of $A \otimes_B k(y)$ is a subspace, so it satisfies d.c.c, which means that $A \otimes_B k(y)$ is Artinian. Hence, $A \otimes_B k(y)$ is semi-local and every prime ideal of $A \otimes_B k(y)$ is a maximal ideal. Thus, Spec $(A \otimes_B k(y))$ is a finite set.

(b) We first deal with the case that X and Y are affine, say $X = \text{Spec } B$ and $Y = \text{Spec } A$. Then f induces a homomorphism of rings $\varphi : A \to B$ and the finiteness of f implies that B is a finitely generated A-module. So, B is integral over $f(A)$.

Choose a closed subset $V(\mathfrak{b})$ in B, set $\mathfrak{a} = \varphi^{-1}(\mathfrak{b})$. Since $\varphi : A \to B$ is integral, we have that $\widetilde{\varphi} : A/\mathfrak{a} \to B/\mathfrak{b}$ is also integral. Moreover, $\widetilde{\varphi}$ is injective. By Krull-Cohen-Seidenberg theorem, we know that the induced map $\widetilde{f} : \text{Spec}(B/\mathfrak{b}) \to \text{Spec}(A/\mathfrak{a})$ is surjective. Thus $f(V(\mathfrak{b})) = V(\mathfrak{a})$. Thus, $f(V(\mathfrak{b}))$ is closed and hence f is a closed map.

Now, we turn to the general case. Since f is a finite morphism, there exists a covering of Y by open affine subsets $V_i = \text{Spec } B_i$, such that for each i, $f^{-1}(V_i)$ is affine, equal to Spec A_i, where A_i is a B_i-algebra which is a finitely generated B_i-module. Let $f_i = f|_{\text{Spec } A_i}$, then we have some morphisms $f_i : \text{Spec } A_i \to \text{Spec } B_i$ of affine schemes. Let C be a closed subset in X, then $C \cap \text{Spec } A_i$ is closed in Spec A_i. Thus, $W_i := f(C) \cap V_i = f(C \cap \text{Spec } A_i) = f_i(C \cap \text{Spec } A_i)$ is closed in $V_i = \text{Spec } B_i$. Then $V_i - W_i$ is open in V_i. So $V_i - W_i$ is open in Y since V_i is open. Thus, $f(C)^c = \bigcup_i (f(C)^c \cap V_i) = \bigcup_i (V_i - W_i)$ is open in Y. This implies that $f(C)$ is closed. □

Exercise 3.6: Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. It is called the **function field** of X, and is denoted by $K(X)$. Show also that if $U = \text{Spec } A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

Proof. Since ξ is the generic point of scheme X, then any open affine subset $U \cong \text{Spec } A$ contains ξ, where A is a ring. Then ξ is a generic point of Spec A, whose corresponding prime ideal will be denoted by $\mathfrak{p}$.

We claim that $\mathfrak{p} \subseteq \mathfrak{p}' \Leftrightarrow \mathfrak{p}' \in \overline{\{\mathfrak{p}\}}$. Indeed, if $\mathfrak{p}' \in \overline{\{\mathfrak{p}\}}$, then for any ideal I, $V(I) \supset \{\mathfrak{p}\}$ implies that $\mathfrak{p}' \in V(I)$. Take $I = \mathfrak{p}$, then $\mathfrak{p}' \in V(\mathfrak{p})$. Hence, $\mathfrak{p}' \supset \mathfrak{p}$. Conversely, if $\mathfrak{p} \subset \mathfrak{p}'$, and for any ideal I with $\mathfrak{p} \in V(I)$, we have that $I \subset \mathfrak{p} \subset \mathfrak{p}'$ implies $\mathfrak{p}' \in V(I)$. Thus, $\mathfrak{p}' \in \overline{\{\mathfrak{p}\}}$.

By this claim, we see that for any $\mathfrak{p}' \in \text{Spec } A$, we have $\mathfrak{p} \subset \mathfrak{p}'$, i.e. $\mathfrak{p}$ is a minimal prime ideal. Since X is integral, we see that $A \cong \mathcal{O}_X(U)$ is an integral domain. Thus, $\mathfrak{p} = (0)$. $\mathcal{O}_\xi \cong A_\mathfrak{p} = A_{(0)} = \text{Frac}(A)$ is a field.

□

Lemma 4: If a morphism $f : Z \to Y = \text{Spec } A$ between integral schemes is dominant, then the corresponding map $\varphi = f^\#(Y) : A \to \Gamma(Z, \mathcal{O}_Z)$ is injective.

Proof. First we observe that $f : Z \to Y$ is dominant if and only if for any non-empty open subset $U \subseteq Y$, we have $f^{-1}(U)$ is open non-empty subset of Z. Indeed, $Y = \overline{f(Z)} \Leftrightarrow U \cap f(Z) \neq \varnothing \Leftrightarrow f^{-1}(U) \neq \varnothing$ for any nonempty open subset $U \subseteq Y$. So, we can conclude that the composition of dominant maps is still dominant.

We now deal with the special case $Z = \text{Spec } B$. Suppose given a dominant morphism f : Spec $B \to$ Spec A between integral schemes. Then f induced a map $\varphi = f^\#(Y) : A \to B$ with A, B integral domains. Observe that a principal open subset $D(g)$ in Spec A is empty if and only if $g \in \mathfrak{p}$ for all $\mathfrak{p} \in \text{Spec } A$ if and only if $g \in \sqrt{(0)} = (0)$ if and only if $g = 0$, as A is a domain. Since f is dominant, then $f^{-1}(D(g)) \neq \varnothing$ for $g \in A$ nonzero . But $f^{-1}(D(g)) = D(\varphi(g))$, so $\varphi(g) \neq 0$. This shows that φ is injective.

For the general case, we can consider an open affine subset Spec $B \subseteq Z$. Since Z is integral, it is irreducible by Proposition 3.1. Then any non-empty open subset of Z is dense in Z; in particular, the inclusion Spec $B \hookrightarrow Z$ is dominant. Thus, the map Spec $B \to Y$ is dominant. We know that the composite map $A \to \Gamma(Z, \mathcal{O}_Z) \to \Gamma(\text{Spec } B, \mathcal{O}_Z) \cong B$ is injective by previous argument. So $A \to \Gamma(Z, \mathcal{O}_Z)$ must be injective since its kernel is contained in the kernel of the composite map. □

Lemma 5: *Let Z be an integral scheme, then for every open subset U of Z, $\mathcal{O}_Z(U)$ is an integrally closed domain.*

Proof. Let $U = \bigcup_i \text{Spec } B_i$ with B_i is integrally closed in its field of fractions. Recall that A is integrally closed if and only if $A_\mathfrak{p}$ is integrally closed for all $\mathfrak{p} \in \text{Spec } A$. By the normality of Z, we see that $(B_i)_{\mathfrak{p}}$ is integrally closed for all $\mathfrak{p} \in \text{Spec } B_i$. Indeed, for any $\mathfrak{p} \subseteq \text{Spec } B_i \subseteq Z$, we have $\mathscr{O}_{Z,\mathfrak{p}} = \mathscr{O}_{\text{Spec } B_i,\mathfrak{p}} \cong (B_i)_{\mathfrak{p}}$ is integrally closed.

By Exercise 2.9, Z has a unique generic point since it is irreducible closed in itself. Let ξ be the generic point of Z. Then $\xi \in U$. We may regard $\mathscr{O}_Z(U)$ as a subring of each B_i. Indeed, if the image of a section $s \in \mathscr{O}_Z(U)$ in $\mathscr{O}_{Z,\xi}$ is zero, then there exists an open neighborhood $V \subseteq U$ of ξ such that $s|_V = 0$. Thus $s|_{V \cap \text{Spec } B_i} = 0$ for all i. Since ξ is the generic point of Z, we see that $V \cap \text{Spec } B_i$ is non-empty for all i as $\xi \in V \cap \text{Spec } B_i$. This implies that $s = 0$ in $\text{Frac}(B_i) = \mathscr{O}_{\text{Spec } B_i,\xi}$ by Exercise 3.6. Since B_i is a domain, $s = 0$ in $B_i = \mathscr{O}_Z(\text{Spec } B_i)$. Hence $s = 0$, i.e. $\mathscr{O}_Z(U) \to \mathscr{O}_{Z,\xi}$ is injective. Now, for any B_i, the injective map $\mathscr{O}_Z(U) \to \mathscr{O}_{Z,\xi}$ factor through $\mathscr{O}_Z(U) \to \mathscr{O}_Z(\text{Spec } B_i) \to \mathscr{O}_{Z,\xi}$ naturally. Thus, $\mathscr{O}_Z(U) \to \mathscr{O}_Z(\text{Spec } B_i) \cong B_i$ is injective. This map induces an injective map between fields of fractions $\text{Frac}(\mathscr{O}_Z(U)) \to \text{Frac}(B_i)$, so if $a \in \text{Frac}(\mathscr{O}_Z(U))$ is integral over $\mathscr{O}_Z(U)$, it is also integral over B_i and hence lies in B_i for each i as B_i is integrally closed. Thus $a \in \mathscr{O}_Z(U)$ by sheaf properties, which means that $\mathscr{O}_Z(U)$ is integrally closed. ☐

Exercise 3.8: *Normalization.* A scheme is **normal** if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \text{Spec } A$ of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \text{Spec } \widetilde{A}$. Show that one can glue the schemes $\widetilde{U}$ to obtain a normal integral scheme $\widetilde{X}$, called the **normalization** of X. Show also that there is a morphism $\widetilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$. If X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism. This generalizes (I, Ex. 3.17).

Proof. We first deal with the affine case $X = \text{Spec } A$ and $\widetilde{X} = \text{Spec } \widetilde{A}$, where $\widetilde{A}$ is the integral closure of A in its field of fractions. We now show that $\widetilde{X}$ is normal. Indeed, $\widetilde{A}$ is integrally closed, so all of $\mathscr{O}_{\tilde{X},\mathfrak{p}} \cong (\widetilde{A})_{\mathfrak{p}}$ is integrally closed. This shows that $\widetilde{X}$ is a normal scheme. The natural inclusion $A \to \widetilde{A}$ gives us a morphism $\widetilde{X} \to X$.

We show this satisfies the universal property. Let Z be a normal integral scheme and $f : Z \to X$ dominant. We then have an inclusion $\varphi = f^{\#}(X) : A \to \Gamma(Z, \mathscr{O}_Z)$ by Lemma 4. By Lemma 5, $\mathscr{O}_Z(Z)$ is integrally closed. Thus, $A \to \mathscr{O}_Z(Z)$ factors through an injective homomorphism $\widetilde{A} \to \mathscr{O}_Z(Z)$ uniquely, i.e. there exists a unique injective map $\widetilde{A} \to \mathscr{O}_Z(Z)$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[d] \arrow[dr] & \\ \widetilde{A} \arrow[r] & \mathscr{O}_Z(Z). \end{tikzcd}$$

Indeed, φ can be extended to $\text{Frac}(A)$ uniquely as an injective homomorphism $\text{Frac}\,\varphi : \text{Frac}(A) \to \text{Frac}(\mathscr{O}_Z(Z))$. Take $x \in \widetilde{A}$, then $x \in \text{Frac}(A)$ and x is integral over A. Thus, $(\text{Frac}\,\varphi)(x)$ is integral over $\mathscr{O}_Z(Z)$. Thus, $(\text{Frac}\,\varphi)(x) \in \mathscr{O}_Z(Z)$ as $\mathscr{O}_Z(Z)$ is integrally closed. So, we may define $\widetilde{A} \to \mathscr{O}_Z(Z)$ by $x \mapsto (\text{Frac}\,\varphi)(x))$. The uniqueness is clear.

So, Exercise 2.4 gives us a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d] & X \\ \widetilde{X} \arrow[ur] & \end{tikzcd},$$

proving the universal property for the affine case.

We now deal with the general case. Observe that if $\pi : \widetilde{X} \to X$ satisties the universal property, then for any open subset $U \subset X$, $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ also satisfies the universal property. Suppose $X = \bigcup_{i \in I} U_i$ with $U_i = \text{Spec } A_i$. Let $\widetilde{U_i} = \text{Spec } \widetilde{A_i}$ and $U_{ij} = U_i \cap U_j$. Then $\pi_i : \widetilde{U_i} \to U_i$ satisfies the universal property. Let $\widetilde{U_{ij}} = \pi_i^{-1}(U_{ij}) \subseteq \widetilde{U_i}$, then $\widetilde{U_{ij}} \to U_{ij}$ satisties the universal property for all $i, j \in I$. So, the universal property tells us that we have a unique isomorphism $\varphi_{ij} : \widetilde{U_{ij}} \xrightarrow{\sim} \widetilde{U_{ji}}$. Moreover, we have $\varphi_{ij}(\widetilde{U_{ij}} \cap \widetilde{U_{ik}}) = \widetilde{U_{ji}} \cap \widetilde{U_{jk}}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $\widetilde{U_{ij}} \cap \widetilde{U_{ik}}$ for each $i, j, k \in I$ for the same reason. Thus, by Glueing Lemma, we see that $\widetilde{U_i}$ can be glued to a scheme, say $\widetilde{X}$, along φ_{ij} and we have a collection of morphisms $\{\psi_i : \widetilde{U_i} \to \widetilde{X}\}_{i \in I}$ such that

(1) ψ_i is an isomorphism of $\widetilde{U_i}$ onto an open subscheme of $\widetilde{X}$,

(2) the $\psi_i(\widetilde{U_i})$ cover $\widetilde{X}$,

(3) $\psi_i(\widetilde{U_{ij}}) = \psi_i(\widetilde{U_i}) \cap \psi_j(\widetilde{U_j})$ and

(4) $\psi_i = \psi_j \circ \varphi_{ij}$ on $\widetilde{U_{ij}}$ for each $i, j \in I$.

$\widetilde{X}$ is clearly normal since each $\widetilde{U_i}$ is normal. We now can prove the universal property for $\widetilde{X} \to X$. Let $f : Z \to X$ be a dominant morphism. Let $Z_i = f^{-1}(U_i)$ and $f_i = f|_{Z_i}$. Then, for each i, we have a commutative diagram

$$\begin{tikzcd}
Z_i \arrow[r, "f_i"] \arrow[d, "h_i"'] & U_i \arrow[r, hook] & X \\
\widetilde{U_i} \arrow[ur, "\pi_i"'] \arrow[d, hook, "\psi_i"'] & & \\
\widetilde{X} \arrow[uurr, "\pi"'] & &
\end{tikzcd}$$

by universal property for affine case. Since $\widetilde{U_{ji}} \to U_{ji}$ satisfies the universal property, we have $\varphi_{ij} \circ h_i|_{Z_i \cap Z_j} = h_j|_{Z_i \cap Z_j}$. So, $\psi_i \circ h_i|_{Z_i \cap Z_j} = \psi_j \circ h_j|_{Z_i \cap Z_j}$ by (4). Thus, $\psi_i \circ h_i$ can be glued to a unique morphism $h : Z \to \widetilde{X}$ such that $f = \pi \circ h$. $\square$

Exercise 3.10: *Fibres of a Morphism.*

(*a*) If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $\text{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

Proof. (*a*) Let $p : X_y = X \times_Y \text{Spec } k(y) \to X$ be the projection map. Note that if $V \subseteq Y$ is an affine open subset, then $f^{-1}(V) = X \times_Y V$, we see that $X_y = (X \times_Y V) \times_V \text{Spec } k(y) = f^{-1}(V)_y$. So, we may assume that $Y = \text{Spec } A$ is affine. Again, note that if U is an open affine subset of X, then $p^{-1}(U) = U \times_Y \text{Spec } k(y) = U_y$. It reduces to the case that $X = \text{Spec } B$ is affine. Let $\mathfrak{p}$ be the prime ideal in A corresponding to y. First, by some basic facts of localization, we know that the homomorphism $B \to B \otimes_A A_{\mathfrak{p}}$ induces a homeomorphism

$$\text{Spec } (B \otimes_A A_{\mathfrak{p}}) \cong \{\mathfrak{q} \in \text{Spec } B : \mathfrak{q} \cap \varphi(A - \mathfrak{p}) = \varnothing\},$$

where $\varphi = f^{\#}(Y) : A \to B$. Now, consider the surjective map $A_{\mathfrak{p}} \to \kappa(\mathfrak{p})$, we have a surjective map $\phi : B \otimes_A A_{\mathfrak{p}} \to B \otimes_A \kappa(\mathfrak{p})$, which induces a homeomorphism $\text{Spec } (B \otimes_A \kappa(\mathfrak{p})) \cong V(\ker \phi)$. So, $\text{Spec } (B \otimes_A \kappa(\mathfrak{p})) \cong \{\mathfrak{q}B_{\mathfrak{p}} : \mathfrak{q}B_{\mathfrak{p}} \supseteq \ker \phi, \mathfrak{q} \cap \varphi(A - \mathfrak{p}) = \varnothing\} \cong \{\mathfrak{q} \in \text{Spec } B : \mathfrak{q} \supseteq \mathfrak{p}B, \mathfrak{q} \cap \varphi(A - \mathfrak{p}) = \varnothing\} = \{\mathfrak{q} \in \text{Spec } B : \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}\} = f^{-1}(\mathfrak{p})$. Thus, $\text{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$.

$\square$

Exercise 3.11: *Closed Subschemes.*

(a) Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

$*(b)$ If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image ofthe closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$. [Hints: First show that Y can be covered by a finite number of open affine subsets of the form $D(f_i) \cap Y$, with $f_i \in A$. By adding some more f_i with $D(f_i) \cap Y = \varnothing$, if necessary, show that we may assume that the $D(f_i)$ cover X. Next show that $f_1, \cdots, f_r$ generate the unit ideal of A. Then use (Ex. 2.17b) to show that Y is affine, and (Ex. 2.18d) to show that Y comes from an ideal $\mathfrak{a} \subseteq A$.] Note: We will give another proof of this result using sheaves of ideals later (5.10).

(c) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

(d) Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the **scheme-theoretic image** of f if Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Proof.

□

Exercise 3.12: *Closed Subschemes of* Proj S.

(a) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to Proj T, and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.

(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined as image of the closed immersion $\operatorname{Proj} S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.

We will see later (5.16) that every closed subscheme of X comes from a homogeneous ideal I of S (at least in the case where S is a polynomial ring over S_0).

Proof.

□

Exercise 3.18: *Constructible Sets.* Let X be a Zariski topological space. A **constructible subset** of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.

(a) A subset of X is **locally closed** if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.

(b) Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.

(c) A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.

(d) If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.

Proof.

(a) Let $\mathfrak{L}$ be the set of finite disjoint union of locally closed subsets.

First, by definition, every locally closed subset is constructible. By de Morgan's law, it is clearly that all finite disjoint union of locally closed subsets are constructible. So, $\mathfrak{L} \subseteq \mathfrak{F}$.

Conversely, first note that every open subset is locally closed, thus they are in $\mathfrak{L}$. Similarly, every closed subset is locally closed. Thus, if $U = \coprod_i (U_i \cap W_i^c), V = \coprod_j (V_j \cap Z_j^c)$ are two locally closed subsets, where U_i, W_i, V_j, Z_j are all open, then $U \cap V = \coprod_i (U_i \cap W_i^c) \cap \coprod_j (V_j \cap Z_j^c) = \coprod_{i,j} ((U_i \cap V_j) \cap (Z_j^c \cap W_i^c)) \in \mathfrak{L}$ since $U_i \cap V_j$ is open and $Z_j^c \cap W_i^c$ is closed. So, finite intersection of elements of $\mathfrak{L}$ is still in $\mathfrak{L}$. Since $U^c = \bigcap_i (U_i \cap W_i^c)^c = \bigcap_i (U_i^c \cup W_i)$ is a finite intersection of locally closed subsets of $\mathfrak{L}$, it is in $\mathfrak{L}$ as we argued previously. Thus, $\mathfrak{L}$ satisfies (1), (2) and (3) at the same times. Hence, $\mathfrak{F} \subseteq \mathfrak{L}$.

(b) If a constructible subset contains the generic point of X, then it is trivially dense.

Conversely, let U be a constructible subset dense in X. Then $U = \coprod_{i=1}^n (U_i \cap V_i^c)$ for some open subsets U_i and V_i. So, $\overline{U} = \bigcup_{i=1}^n \overline{U_i \cap V_i^c} = X$. Since X is irreducible, we know that $\overline{U_i \cap V_i^c} = X$ for some i. Thus, $X = \overline{U_i} = \overline{V_i^c} = V_i^c$. This implies that $V_i = \varnothing$. Thus, U contains an open subsets. However, every open subsets of X must contain the generic point, we're done.

(c)

(d) Let $U = \coprod_{i=1}^n (U_i \cap V_i^c)$ be a constructible subset with U_i and V_i open. Then $f^{-1}(U) = f^{-1}(\coprod_{i=1}^n (U_i \cap V_i^c)) = \coprod_{i=1}^n (f^{-1}(U_i) \cap f^{-1}(V_i)^c)$ is a constructible subset of X. $\square$

Exercise 3.20: *Dimension.* Let X be an integral scheme of finite type over a field k (not necessarily algebraically closed). Use appropriate results from (I, §1) to prove the following.

(a) For any closed point $P \in X$, $\dim X = \dim \mathscr{O}_p$, where for rings, we always mean the Krull dimension.

(b) Let $K(X)$ be the function field of X (Ex. 3.6). Then $\dim X = \operatorname{tr.deg} K(X)/k$.

(c) If Y is a closed subset of X, then $\operatorname{codim}(Y, X) = \inf\{\dim \mathscr{O}_{X,P} | P \in Y\}$.

(d) If Y is a closed subset of X, then $\dim Y + \operatorname{codim}(Y, X) = \dim X$.

(e) If U is a nonempty open subset of X, then $\dim U = \dim X$.

(f) If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.

Proof. (a) This is a local property, so we can find an open affine neighborhood of P, say $U = \operatorname{Spec} A$, such that $\dim X = \dim U$. Then we may reduce to the case $X = \operatorname{Spec} A$. Since X is an integral scheme, A must be an integral domain. A closed point P corresponds to a maximal ideal $\mathfrak{p}$ of A, so $\dim A/\mathfrak{p} = 0$. Thus, $\dim X = \dim \operatorname{Spec} A = \dim A = \dim A/\mathfrak{p} + \operatorname{ht}(\mathfrak{p}) = 0 + \dim A_{\mathfrak{p}} = \dim \mathscr{O}_P$.

$\dim \operatorname{Spec} A = \dim A$ because $V(\mathfrak{p})$ is irreducible if and only if $\mathfrak{p}$ is prime.

(b) Let P be a closed point and $U = \operatorname{Spec} A$ be an affine open neighborhood of P. Then $K(X) \cong \operatorname{Frac}(A)$ by Exercise 3.6. Let $\mathfrak{p}$ be the maximal ideal of A corresponding to P. Then $A_{\mathfrak{p}} \cong \mathscr{O}_P$. So, we see that $\operatorname{Frac}(\mathscr{O}_P) \cong K(X)$. Thus, $\dim X = \dim \mathscr{O}_P = \operatorname{tr.deg} \operatorname{Frac}(\mathscr{O}_P)/k = \operatorname{tr.deg} K(X)/k$. $\square$