Algebraic Geometry Homework 2

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2 Schemes

Lemma 1 (Glueing of the morphisms): *Let X, Y be two locally ringed spaces. Suppose $\{U_i\}_i$ is an open covering of X and $f_i : U_i \to Y$ are morphisms such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for every i, j. Then there exists a unique morphism $f : X \to Y$ such that $f|_{U_i} = f_i$.*

Proof. We may define $f(x) = f_i(x)$ if $x \in U_i$ for some i, so $f : X \to Y$ is a continuous map. Since $(f_i)_*(\mathscr{O}_X|_{U_i})(V) = (\mathscr{O}_X|_{U_i})(f_i^{-1}(V)) = \mathscr{O}_X(f_i^{-1}(V))$ for each $V \subseteq Y$, then $f_i^{\#} : \mathscr{O}_Y \to (f_i)_*(\mathscr{O}_X|_{U_i})$ induces a map of rings $f_i^{\#}(V) : \mathscr{O}_Y(V) \to \mathscr{O}_X(f_i^{-1}(V))$ for each $V \subseteq Y$. Note that $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$. For each $t \in \mathscr{O}_Y(V)$, let $s_i = f_i^{\#}(V)(t)$, then there exists a unique $s \in \mathscr{O}_X(f^{-1}(V))$ such that $s|_{f_i^{-1}(V)} = s_i$. Define $f^{\#}(V)(t) = s$, we then obtain a ring homomorphism $f^{\#}(V) : \mathscr{O}_Y(V) \to \mathscr{O}_X(f^{-1}(V))$, which gives us a morphism of sheaves $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ with $f^{\#}|_{U_i} = f_i^{\#}$. Thus, we have a morphism of schemes $f = (f, f^{\#})$ such that $f|_{U_i} = f_i$. $\square$

Exercise 2.1: Let A be a ring, let $X = \text{Spec } A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathscr{O}_X|_{D(f)})$ is isomorphic to Spec A_f.

Proof. Let $S = \{f^n : n \geqslant 0\}$, then $A_f = A[S^{-1}]$.

First, the natural map $\varphi : A \to A_f$ induces a bijection

$$\psi : \text{Spec}(A_f) \to \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \cap S = \varnothing\} \subseteq \text{Spec}(A)$$

by $\mathfrak{q} \mapsto \varphi^{-1}(\mathfrak{q})$. Note that $\{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \cap S = \varnothing\} = \{\mathfrak{p} \in \text{Spec}(A) : f \notin \mathfrak{p}\} = D(f)$. We have already known that this map is continuous. Let I be an ideal of A_f, then $\psi(V(I)) = V(\varphi^{-1}(I)) \cap \psi(\text{Spec}(A_f)) = V(\varphi^{-1}(I)) \cap D(f)$, which is a closed set of the subspace $D(f)$. So, we have a homeomorphism of topological spaces

$$\psi : \text{Spec}(A_f) \to D(f).$$

Since $D(g)$ form a base for the topology on X, then $D(fg) = D(g) \cap D(f)$ form a base for the topology on $D(f)$. Now, let $D(h) \subset D(f)$ and $\overline{h}$ be the image of h in A_f. Then $\mathscr{O}_X|_{D(f)}(D(h)) = \mathscr{O}_X(D(h)) \cong A_h \cong (A_f)_{\overline{h}} \cong \mathscr{O}_{\text{Spec } A_f}(D(\overline{h})) = \mathscr{O}_{\text{Spec } A_f}(\psi^{-1}(D(h))) = \psi_*\mathscr{O}_{\text{Spec } A_f}(D(h))$. Thus, we know that for each open set U in $D(f)$, we have $\mathscr{O}_X|_{D(f)}(U) \cong \psi_*\mathscr{O}_{\text{Spec } A_f}(U)$ by the sheaf properties. This gives us a isomorphism $\psi^{\#} : \mathscr{O}_X|_{D(f)} \to \psi_*\mathscr{O}_{\text{Spec } A_f}$ of sheaves on $D(f)$. For any $\mathfrak{q} \in \text{Spec } A_f$, $\psi^{\#}_{\mathfrak{q}} : A_{\psi(\mathfrak{q})} \to (A_f)_{\mathfrak{q}}$ is a isomorphism of local rings since ψ is a homeomorphism. So it is a local homomorphism.

So, $(\psi, \psi^{\#}) : (\text{Spec } A_f, \mathscr{O}_{\text{Spec } A_f}) \to (D(f), \mathscr{O}_X|_{D(f)})$ is an isomorphism of locally ringed spaces. $\square$

Exercise 2.2: Let $(X, \mathscr{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathscr{O}_X|_U)$ is a scheme. We call this the **induced scheme structure** on the open set U, and we refer to $(U, \mathscr{O}_X|_U)$ as an **open subscheme** of X.

Proof. For any point $p \in U$, there exists an open neighborhood $V \subseteq X$ such that $(V, \mathscr{O}_X|_V)$ is an affine scheme. Then there is an isomorphism $(f, f^{\#}) : (V, \mathscr{O}_X|_V) \to (\text{Spec } A, \mathscr{O}_{\text{Spec } A})$ of locally ringed spaces for some ring A. Consider $f(p) \in f(V \cap U) \subseteq \text{Spec } A$, there exists a principal open subset $D(h) \subseteq f(U \cap V)$ such that $f(p) \in D(h)$ since $f(U \cap V)$ is open. By Exercise 2.1, we know that $(D(f), \mathscr{O}_{\text{Spec } A}|_{D(f)})$ is isomorphic to Spec A_f as locally ringed spaces. Then $W = f^{-1}(D(h))$ is an open subset containing p and $(W, \mathscr{O}_X|_W) \cong (D(h), \mathscr{O}_{\text{Spec } A}|_{D(h)}) \cong (\text{Spec } A_f, \mathscr{O}_{\text{Spec } A_f})$. Thus, $(U, \mathscr{O}_X|_U)$ is a scheme. □

Exercise 2.3: *Reduced Schemes.* A scheme $(X, \mathscr{O}_X)$ is **reduced** if for every open set $U \subseteq X$, the ring $\mathscr{O}_X(U)$ has no nilpotent elements.

(*a*) Show that $(X, \mathscr{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathscr{O}_{X,P}$ has no nilpotent elements.

(*b*) Let $(X, \mathscr{O}_X)$ be a scheme. Let $(\mathscr{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathscr{O}_X(U)_{\text{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathscr{O}_X)_{\text{red}})$ is a scheme. We call it the **reduced scheme** associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $X_{\text{red}} \to X$, which is a homeomorphism on the underlying topological spaces.

(*c*) Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \to Y$.

Proof. (*a*) Suppose $(X, \mathscr{O}_X)$ is reduced. Let x_P be an element of $\mathscr{O}_{X,P}$ such that $x_P^n = 0$ for some n. Then there exists an open neighborhood U of P and a section $x \in \mathscr{O}_X(U)$ such that x_P is the image of x in $\mathscr{O}_{X,P}$ and $x^n = 0$. Since $\mathscr{O}_X(U)$ has no nilpotent element, we know that $x = 0$. Thus $x_P = 0$. So $\mathscr{O}_{X,P}$ has no nilpotent element for every $P \in X$.

Conversely, suppose $\mathscr{O}_{X,P}$ has no nilpotent element for each $P \in X$. Then, for each U, let $x \in \mathscr{O}_X(U)$ such that $x^n = 0$. Let $P \in U$, we must have $x_P = 0$ as $x_P^n = 0$. So, there exists $V_P \subseteq U$ containing P such that $x|_{V_P} = 0$ by definition. When P runs through points in U, $\{V_P\}_{P \in U}$ forms an open cover of U. By the sheaf property, we know that $x = 0$. Thus, we conclude that $(X, \mathscr{O}_X)$ is reduced.

(*b*) Let $N(A)$ be the nilradical of ring A. Let $\mathscr{N}_X$ be the presheaf $U \mapsto N(\mathscr{O}_X(U))$.

Let $x \in X$, then there exists an open neighborhood U of x such that $(U, \mathscr{O}_X|_U)$ is isomorphic to $(\text{Spec } A, \mathscr{O}_{\text{Spec } A})$ for some ring A. We want to show that $(U, (\mathscr{O}_X|_U)_{\text{red}})$ is isomorphic to $(\text{Spec } A/N, \mathscr{O}_{\text{Spec } A/N})$ as locally ringed spaces, where N is the nilradical of A. It suffices to show that $(\text{Spec } A, (\mathscr{O}_{\text{Spec } A})_{\text{red}}) \cong (\text{Spec } A/N, \mathscr{O}_{\text{Spec } A/N})$ as locally ringed spaces.

Let $\mathscr{N}(U) = N(\mathscr{O}_{\text{Spec } A}(U))$, then $\mathscr{N}$ is a subsheaf of ideals of $\mathscr{O}_{\text{Spec } A}$ and $(\mathscr{O}_{\text{Spec } A})_{\text{red}} = \mathscr{O}_{\text{Spec } A}/\mathscr{N}$. Consider the natural quotient map $\varphi : A \to A/N$. It induces a continuous bijective map of topological spaces

$$\psi : \text{Spec } A/N \to V(N),$$

by sending each prime ideal of A/N to a prime ideal of A that contains N, i.e. its preimage under φ. Since $\psi(V(J)) = v(\varphi^{-1}(J))$, which means that ψ is a closed map. Thus, ψ is a homeomorphism. Noticing that $V(N) = \text{Spec } A$ as N is the intersection of all prime ideals, we have a homeomorphism of topological space

$$\psi : \text{Spec } A/N \to \text{Spec } A.$$

For any $f \in A$, let $\overline{f}$ be the image of f in A/N. Then, consider the surjective map $A_f \to (A/N)_{\overline{f}}$ with kernel N_f. Note that $\mathscr{O}_{\text{Spec } A}(D(f)) \cong A_f$ and $(A/N)_{\overline{f}} \cong \mathscr{O}_{\text{Spec } A/N}(D(\overline{f})) = \mathscr{O}_{\text{Spec } A/N}(\psi^{-1}(D(f))) = \psi_* \mathscr{O}_{\text{Spec } A/N}(D(f))$. We have a surjective map

$$\psi^{\#}(D(f)) : \mathscr{O}_{\text{Spec } A}(D(f)) \to \psi_* \mathscr{O}_{\text{Spec } A/N}(D(f))$$

with $(\ker \psi^{\#})(D(f)) \cong N_f$. Since $(\ker \psi^{\#})(D(f)) \cong N_f = N(A_f) \cong N(\mathscr{O}_{\text{Spec } A}(D(f))) = \mathscr{N}(D(f))$, we conclude that $\ker \psi^{\#} = \mathscr{N}$ since $\{D(f)\}_{f \in A}$ form a base for the topology on Spec A. Now, consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{\text{Spec } A}(U) \arrow[r, "\psi^{\#}(U)"] \arrow[d, "\text{Res}_{U,D(f)}"'] & \psi_* \mathscr{O}_{\text{Spec } A/N}(U) \arrow[d, "\text{Res}_{U,D(f)}"] \\
\mathscr{O}_{\text{Spec } A}(D(f)) \arrow[r, "\psi^{\#}(D(f))"] & \psi_* \mathscr{O}_{\text{Spec } A/N}(D(f))
\end{tikzcd},$$

we conclude that $\psi^{\#}(U)$ is surjective for all open set $U \subseteq \text{Spec } A$ by the sheaf properties and the fact that $\{D(f)\}_{f \in A}$ form a base for the topology on Spec A.

So, $\psi^{\#} : \mathscr{O}_{\text{Spec } A} \to \psi_* \mathscr{O}_{\text{Spec } A/N}$ is surjective with kernel sheaf $\mathscr{N}$. Thus, $(\mathscr{O}_{\text{Spec } A})_{\text{red}} = \mathscr{O}_{\text{Spec } A}/\mathscr{N} \cong \psi_* \mathscr{O}_{\text{Spec } A/N}$. The stalk of this isomorphism of sheaves at each point $P \in \text{Spec } A/N$ is an isomorphism of local rings, which must be a local homomorphism, since ψ is a homeomorphism. We conclude that $(\text{Spec } A, (\mathscr{O}_{\text{Spec } A})_{\text{red}}) \cong (\text{Spec } A/N, \mathscr{O}_{\text{Spec } A/N})$ as locally ringed spaces. This proves that $(U, (\mathscr{O}_X|_U)_{\text{red}})$ is an affine scheme and it follows that $(X, (\mathscr{O}_X)_{\text{red}})$ is a scheme.

Let $i : X \to X$ be the identity map. Let $\mathscr{O}_X/\mathscr{N}_X$ be the sheaf associated to the presheaf $\mathscr{F} : U \mapsto \mathscr{O}_X(U)/\mathscr{N}_X(U)$. Note that $i_*((\mathscr{O}_X)_{\text{red}}) = i_*(\mathscr{O}_X/\mathscr{N}_X) = \mathscr{O}_X/\mathscr{N}_X$. Then, we have a natural quotient map $i^{\#} : \mathscr{O}_X \to \mathscr{O}_X/\mathscr{N}_X$ obtained by the composition $\mathscr{O}_X \to \mathscr{F} \to \mathscr{F}^+ = \mathscr{O}_X/\mathscr{N}_X$. The stalk $\mathscr{O}_{X,P} \to \mathscr{O}_{X,P}/\mathscr{N}_{X,P}$ at P is clearly a local homomorphism. Then $(i, i^{\#}) : X_{\text{red}} = (X, (\mathscr{O}_X)_{\text{red}}) \to (X, \mathscr{O}_X) = X$ is a morphism of schemes as desired.

(c) For each open set U, let $\mathscr{F} : U \mapsto \mathscr{O}_Y(U)/\mathscr{N}_Y(U)$ be a presheaf. Then $\mathscr{F}^+ = \mathscr{O}_Y/\mathscr{N}_Y = (\mathscr{O}_Y)_{\text{red}}$. Consider the map $f^{\#}(U) : \mathscr{O}_Y(U) \to f_* \mathscr{O}_X(U)$, define $\mathscr{F}(U) \to f_* \mathscr{O}_X(U)$ to be $x + \mathscr{N}_Y(U) \mapsto f^{\#}(U)(x)$, where $x \in \mathscr{O}_Y(U)$. This map is well-defined as X is reduced. This gives us a morphism $\mathscr{F} \to f_* \mathscr{O}_X$. By the universal property of sheafication, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_Y \arrow[r, "f^{\#}"] \arrow[d] & f_* \mathscr{O}_X \\
\mathscr{F} \arrow[ur] \arrow[r, "\theta"] & \mathscr{O}_Y/\mathscr{N}_Y \arrow[u, dotted, "\exists\ \psi"']
\end{tikzcd}$$

Thus, $f^\#$ can be decomposed as shown in the following diagram

$$\begin{tikzcd}
\mathscr{O}_Y \arrow[rr, "f^\#"] \arrow[dr, "i^\#"'] & & f_*\mathscr{O}_X \\
& (\mathscr{O}_Y)_{\mathrm{red}} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

So, we can take g to be $(f, \psi) : (X, \mathscr{O}_X) \to (Y, (\mathscr{O}_Y)_{\mathrm{red}})$. $\square$

Exercise 2.4: Let A be a ring and let $(X, \mathscr{O}_X)$ be a scheme. Given a morphism $f : X \to \mathrm{Spec}\ A$, we have an associated map on sheaves $f^\# : \mathscr{O}_{\mathrm{Spec}\ A} \to f_*\mathscr{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathscr{O}_X)$. Thus there is a natural map

$$\alpha : \mathrm{Hom}_{\mathfrak{Sch}}(X, \mathrm{Spec}\ A) \to \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathscr{O}_X)).$$

Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

Proof. By the Proposition 2.3, we know that there exists a bijection

$$\beta : \mathrm{Hom}_{\mathfrak{Sch}}(\mathrm{Spec}\ B, \mathrm{Spec}\ A) \to \mathrm{Hom}_{\mathfrak{Rings}}(A, B),$$

by $(f, f^\#) \mapsto f^\#(\mathrm{Spec}\ A)$. Thus, if X is an affine scheme, then α is bijective.

Note that for each $x \in X$, there exists an open neighborhood U_x such that $(U_x, \mathscr{O}_X|_{U_x})$ is an affine scheme. We may write $X = \bigcup_i U_i$, where each U_i is an affine scheme. Define ρ_i : $\mathrm{Hom}_{\mathfrak{Sch}}(X, \mathrm{Spec}\ A) \to \mathrm{Hom}_{\mathfrak{Sch}}(U, \mathrm{Spec}\ A)$ by $(f, f^\#) \to (f|_U, f^\#|_U)$, where, for each $V \subseteq \mathrm{Spec}\ A$, $f^\#|_U(V)$ is defined by the following commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{\mathrm{Spec}\ A}(V) \arrow[r, "f^\#(V)"] \arrow[d, equal] & f_*\mathscr{O}_X(V) \arrow[r, equal] \arrow[d, "\mathrm{Res}"] & \mathscr{O}_X(f^{-1}(V)) \arrow[d, "\mathrm{Res}"] \\
\mathscr{O}_{\mathrm{Spec}\ A}(V) \arrow[r, "f^\#|_U(V)"] & (f|_U)_*(\mathscr{O}_X|_U)(V) \arrow[r, equal] & \mathscr{O}_X(f^{-1}(V) \cap U)
\end{tikzcd}$$

This implies that the map

$$\rho : \mathrm{Hom}_{\mathfrak{Sch}}(X, \mathrm{Spec}\ A) \to \prod_i \mathrm{Hom}_{\mathfrak{Sch}}(U_i, \mathrm{Spec}\ A)$$

defined by $(f, f^\#) \mapsto ((f|_{U_i}, f^\#|_{U_i}))_i$ is injective. Now, consider the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{Sch}}(X, \mathrm{Spec}\ A) \arrow[r, "\alpha"] \arrow[d, "\rho"] & \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathscr{O}_X)) \arrow[d, "\gamma"] \\
\prod_i \mathrm{Hom}_{\mathfrak{Sch}}(U_i, \mathrm{Spec}\ A) \arrow[r, "\prod \beta_i"] & \prod_i \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(U_i, \mathscr{O}_X|_{U_i}))
\end{tikzcd}$$

we conclude that α is injective.

Here, γ is defined by $\gamma_i : \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathscr{O}_X)) \to \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(U_i, \mathscr{O}_X|_{U_i}))$ by $\varphi \mapsto \mathrm{Res}_{X,U_i} \circ \varphi$, i.e. the composition $A \to \mathscr{O}_X(X) \to \mathscr{O}_X(U_i)$.

Now, we are going to prove that α is surjective. Let $\varphi \in \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathscr{O}_X))$, then there exists a unique $f_i = (f_i, f_i^{\#}) \in \mathrm{Hom}_{\mathfrak{Sch}}(U_i, \mathrm{Spec}\ A)$ for each i such that $\beta_i(f_i) = \gamma_i(\varphi)$. We now want glue these f_i to a morphism $f \in \mathrm{Hom}_{\mathfrak{Sch}}(X, \mathrm{Spec}\ A)$. First, for any affine open subset $W \subseteq U_i \cap U_j$, the image of $f_i|_W$ in $\mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(W, \mathscr{O}_X|_W))$ is $\mathrm{Res}_{U_i,W} \circ \beta(f_i) = \mathrm{Res}_{U_i,W} \circ \gamma_i(\varphi) = \mathrm{Res}_{U_i,W} \circ \mathrm{Res}_{X,U_i} \circ \varphi = \mathrm{Res}_{X,W} \circ \varphi$. Similarly, the image of $f_j|_W$ in $\mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(W, \mathscr{O}_X|_W))$ is $\mathrm{Res}_{X,W} \circ \varphi$. Thus, $f_i|_W$ and $f_j|_W$ have the image in $\mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(W, \mathscr{O}_X|_W))$. Thus $f_i|_W = f_j|_W$ for any affine subset $W \subseteq U_i \cap U_j$. We conclude that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for any i, j. This is because ρ is injective, where we replace X by $U_i \cap U_j$ and each U_i by W.

Set $Y = \mathrm{Spec}\ A$. We may define $f(x) = f_i(x)$ if $x \in U_i$ for some i, so $f : X \to Y$ is a continuous map. Since $(f_i)_*(\mathscr{O}_X|_{U_i})(V) = (\mathscr{O}_X|_{U_i})(f_i^{-1}(V)) = \mathscr{O}_X(f_i^{-1}(V))$ for each $V \subseteq Y$, then $f_i^{\#} : \mathscr{O}_Y \to (f_i)_*(\mathscr{O}_X|_{U_i})$ induces a map of rings $f_i^{\#}(V) : \mathscr{O}_Y(V) \to \mathscr{O}_X(f_i^{-1}(V))$ for each $V \subseteq Y$. Note that $f^{-1}(V) = \bigcup_i f_i^{-1}(V)$. For each $t \in \mathscr{O}_Y(V)$, let $s_i = f_i^{\#}(V)(t)$, then there exists a unique $s \in \mathscr{O}_X(f^{-1}(V))$ such that $s|_{f_i^{-1}(V)} = s_i$. Define $f^{\#}(V)(t) = s$, we then obtain a ring homomorphism $f^{\#}(V) : \mathscr{O}_Y(V) \to \mathscr{O}_X(f^{-1}(V))$, which gives us a morphism of sheaves $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ with $f^{\#}|_{U_i} = f_i^{\#}$. Thus, we have a morphism of schemes $f = (f, f^{\#})$ such that $f|_{U_i} = f_i$. Thus, $\gamma_i(\alpha(f) - \varphi) = \gamma_i(\alpha(f)) - \gamma_i(\varphi) = \beta_i(\rho(f)) - \gamma_i(\varphi) = \beta_i(f_i) - \gamma_i(\varphi) = 0$ for each i. Since $\mathscr{O}_X$ is a sheaf, we know that $\alpha(f) - \varphi = 0$. Thus, α is surjective. ☐

Exercise 2.7: Let X be a scheme. For any $x \in X$, let $\mathscr{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the **residue field** of x on X to be the field $k(x) = \mathscr{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of Spec K to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof. Let $\varphi : \mathrm{Spec}\ K \to X$ be a morphism, let $x \in X$ be the point $\varphi(0)$, where 0 is the unique point of Spec K. Consider $\varphi^{\#} : \mathscr{O}_X \to \varphi_*\mathscr{O}_{\mathrm{Spec}\ K}$, the stalk $\varphi_0^{\#} : \mathscr{O}_{X,\varphi(0)} \to \mathscr{O}_{\mathrm{Spec}\ K,0}$ at 0 is simply the local homomorphism $\mathscr{O}_{X,x} \to K$, which iduces an inclusion map $k(x) \hookrightarrow K$.

Conversely, $k(x) \to K$ gives us a morphism of affine scheme $f : \mathrm{Spec}\ K \to \mathrm{Spec}\ k(x)$. The natural quotient map $\mathscr{O}_{X,x} \to k(x)$ also gives us a morphism $g : \mathrm{Spec}\ k(x) \to \mathrm{Spec}\ \mathscr{O}_{X,x}$. Now, take an open neighborhood U of x such that $U \cong \mathrm{Spec}\ A$ for some ring A as affine schemes. We know that $\Gamma(U, \mathscr{O}_X|_U) \cong A$. The map $A \to \mathscr{O}_{X,x}$ obtained by composition $A \to \Gamma(U, \mathscr{O}_X|_U) \to \mathscr{O}_{X,x}$ induces a morphism $\mathrm{Spec}\ \mathscr{O}_{X,x} \to \mathrm{Spec}\ A \cong U \hookrightarrow X$. So the compostion of the above three maps

$$\mathrm{Spec}\ K \to \mathrm{Spec}\ k(x) \to \mathrm{Spec}\ \mathscr{O}_{X,x} \to X$$

gives us the desired morphism Spec $K \to X$. ☐

Exercise 2.8: Let X be a scheme. For any point $x \in X$, we define the **Zariski tangent space** T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the ring of *dual numbers* over k. Show that to give a k-morphism of Spec $k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e., such that $k(x) = k$), and an element of T_x.

Proof. Let $\pi : X \to \mathrm{Spec}\ k$ be the scheme over k.

First, Spec $k[\varepsilon]/\varepsilon^2 = \{(\varepsilon)/\varepsilon^2\}$. Let $f : \mathrm{Spec}\ k[\varepsilon]/\varepsilon^2 \to X$ be a k-morphism and x be the image in X of the unique point in Spec $k[\varepsilon]/\varepsilon^2$. Then the morphism $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_{\mathrm{Spec}\ k[\varepsilon]/\varepsilon^2}$ of sheaves induces a local homomorphism

$$f_x^\# : \mathscr{O}_{X,x} \to \mathscr{O}_{\text{Spec } k[\varepsilon]/\varepsilon^2, (\varepsilon)/\varepsilon^2} \cong (k[\varepsilon]/\varepsilon^2)_{(\varepsilon)/\varepsilon^2} \cong k[\varepsilon]/\varepsilon^2.$$

So, we have

$$k(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x \hookrightarrow \frac{k[\varepsilon]/\varepsilon^2}{(\varepsilon)/\varepsilon^2} \cong k[\varepsilon]/(\varepsilon) \cong k.$$

Since X is over k, we know that $k(x)$ is over k. Thus $k(x) = k$.

Conversely, if there exists a point $x \in X$ such that $k(x) = k$. By Exercise 2.7, there exists a morphism $f : \text{Spec } k \to X$. We have $\pi \circ f = \text{id}_{\text{Spec k}}$ since Spec k is a singleton. Since Spec $k[\varepsilon]/\varepsilon^2$ is a scheme over k, we have a morphism $g : \text{Spec } k[\varepsilon]/\varepsilon^2 \to \text{Spec } k$. So $g \circ f : \text{Spec } k[\varepsilon]/\varepsilon^2 \to X$ is a morphism of schemes and $\pi \circ f \circ g = g$. Thus, $g \circ f : \text{Spec } k[\varepsilon]/\varepsilon^2 \to X$ is a k-morphism as desired.

□

Exercise 2.9: If X is a topological space, and Z an irreducible closed subset of X, a **generic point** for Z is a point ξ such that $Z = \overline{\{\xi\}}$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Proof. We use $\overline{\cdot}^U$ to represent the closure taken in U with respect to the subspace topology. Let $x \in Z$. Then there exists an open neighborhood U of x such that U is an affine scheme, i.e. $f : U \xrightarrow{\sim} \text{Spec } A$ for some ring A. Then $U \cap Z \neq \varnothing$. Since Z is closed, thus $U \cap Z$ is closed in U. So, we may assume that $U \cap Z = f^{-1}(V(\mathfrak{p}))$ for some prime ideal $\mathfrak{p}$ of A. Now, suppose $V(I)$ is a closed set containing $\mathfrak{p}$, then $\mathfrak{p} \supset I$. So, $V(\mathfrak{p}) \subseteq V(I)$ and $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$. Thus, if we set $\xi = f^{-1}(\mathfrak{p})$, we have $U \cap Z = f^{-1}(V(\mathfrak{p})) = \overline{\{\xi\}}^U = \overline{\{\xi\}} \cap U$. Noticing that $U \cap Z$ is an open subset of Z, we must have $U \cap Z$ is irreducible and dense in Z. Thus, $Z = \overline{U \cap Z} \subseteq \overline{\{\xi\}} \cap \overline{U} \subseteq \overline{\{\xi\}}$. Since $\xi \in U \cap Z \subseteq Z$, we conclude that $Z = \overline{\{\xi\}}$.

Suppose there exists another point, say ζ, such that $Z = \overline{\{\zeta\}}$. We must have $\zeta \in U \cap Z$ since the complement of $U \cap Z$ in Z is closed. So, let $\mathfrak{q} = f(\zeta)$, then we have $V(\mathfrak{p}) = V(\mathfrak{q})$. Thus, $\mathfrak{p} = \mathfrak{q}$ and it follows that $\zeta = \xi$. □

Exercise 2.12: *Glueing Lemma.* Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possible infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that

(1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$, and

(2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that

(1) ψ_i is an isomorphism of X_i onto an open subscheme of X,

(2) the $\psi_i(X_i)$ cover X,

(3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and

(4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

We say that X is obtained by **glueing** the schemes X_i along the isomorphisms φ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and φ_{ij} are all empty. Then the scheme X is called the **disjoint union** of the X_i, and is denoted $\coprod_i X_i$.

Proof. We first define an equivalence relation $\sim$ on the disjoint union $\coprod_i X_i$ by $x \sim y \Leftrightarrow x \in U_{ij} \subseteq X_i, y \in U_{ji} \subseteq X_j$ and $y = \varphi_{ij}(x)$. As a topological space, set $X = \coprod_i X_i / \sim$ and endow X with the quotient topology. We define the morphisms $\psi_i : X_i \to X$ by the composition $X_i \hookrightarrow \coprod_i X_i \to X$ as continuous maps from X_i to X. Then any subset $V \subseteq X$ is open if and only if $\psi_i^{-1}(V)$ is open in X_i for all i by our construction. Let $Y_i = \psi_i(X_i)$ and $\mathscr{O}_{Y_i} = (\psi_i)_*\mathscr{O}_{X_i}$. Since for all $j \neq i$, $\psi_j^{-1}(X_i) = U_{ji}$ is open in X_j and $\psi_i^{-1}(X_i) = X_i$ is open in X_i, we conclude that $(Y_i, \mathscr{O}_{Y_i})$ is an open subscheme of X once we prove that X is a scheme such that $\mathscr{O}_X|_{Y_i} = \mathscr{O}_{Y_i}$. (2) and (3) clearly hold by the construction.

We then have continuous maps $\psi_i : X_i \to X$ such that $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij} by our construction. Indeed, for any $x \in U_{ij}$, $x \sim \varphi_{ij}(x)$ in $\coprod_i X_i$, so $\psi_i(x) = \psi_j(\varphi_{ij}(x))$. Thus, (4) holds.

Let $V \subseteq Y_i \cap Y_j$, then $\psi_i^{-1}(V) \subseteq U_{ij}$ for all i, j. Thus $\mathscr{O}_{X_i}(\psi_i^{-1}(V)) \cong \mathscr{O}_{X_j}(\psi_j^{-1}(V))$ by using the isomorphism φ_{ij}. Then there exists an isomorphism $f_{ij} : \mathscr{O}_{Y_i}|_{Y_i \cap Y_j} \xrightarrow{\sim} \mathscr{O}_{Y_j}|_{Y_i \cap Y_j}$. Note that $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$ for each i, j, k tells us that for each i, j, k, we have $f_{jk} \circ f_{ij} = f_{ik}$ on $Y_i \cap Y_j \cap Y_k$. By Exercise 1.22, there exists a unique sheaf $\mathscr{O}_X$ on X such that $\mathscr{O}_X|_{Y_i} = \mathscr{O}_{Y_i}$. Then it is clear that $(X, \mathscr{O}_X)$ is a scheme and that the ψ_i induce isomorphisms $X_i \cong Y_i$. Thus, (1) holds. $\square$

Exercise 2.13: A topological space is **quasi-compact** if every open cover has a finite subcover.

(a) Show that a topological space is noetherian if and only if every open subset is quasi-compact.

(b) If X is an affine scheme, show that $\operatorname{sp}(X)$ is quasi-compact, but not in general noetherian. We say a scheme X is **quasi-compact** if $\operatorname{sp}(X)$ is.

(c) If A is a noetherian ring, show that sp(Spec A) is a noetherian topological space.

Proof. (a) If X is noetherian and V an open subset of X. Then V is noetherian as a subspace. Indeed, for any descending chain of closed subsets $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$, we have that $V_i = X_i \cap V$ for some X_i closed in X. Now, $X_1 \supset X_1 \cap X_2 \supset \cdots \supset \bigcap_{i=1}^n X_i \supset \cdots$ is stable, which implies that the descending chain of $X_i \cap V$ is stable because $X_i \cap V = \bigcap_{j=1}^i (X_i \cap V) = (\bigcap_{j=1}^i X_i) \cap V$.

Suppose $V = \bigcup_{i \in I} V_i$, where V_i is open in V. Then $V^c = \bigcap_{i \in I} V_i^c$. Let $J \cong \mathbb{N}^*$ be a countable subset of I. Consider the descending chain of closed subsets

$$V_1^c \supset V_1^c \cap V_2^c \supset \cdots \supset \bigcap_{i \in J} V_i^c \supset \bigcap_{i \in I} V_i^c.$$

So, by the descending chain condition, there exists n such that $V_1^c \supset V_1^c \cap V_2^c \supset \cdots \supset \bigcap_{i=1}^n V_i^c = \cdots = \bigcap_{i \in I} V_i^c = V^c$, i.e. $V^c = \bigcap_{i=1}^n V_i^c$ for some n. Thus, $V = \bigcup_{i=1}^n V_i$ and it follows that V is quasi-compact.

Conversely, suppose for any subset V of X, V is quasi-compact. Let $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$ be an ascending chain of open subsets in X. Let $V = \bigcup_{i=1}^\infty V_i$, then V is an open subset of X. Since V is quasi-compact by our htpothesis, there exists n such that $\bigcup_{i=1}^n V_i = \bigcup_{i=1}^\infty V_i$. So, for any $j \geqslant n+1$, $V_j \subseteq \bigcap_{i=1}^n V_i = V_n$. Thus, any ascending chain of open subsets of X is stable. Thus, X is noetherian.

(b) Suppose $X = \operatorname{Spec} A$ for some ring A. We first claim that $\sqrt{(f)} \subseteq \sqrt{\sum (f_i)} \Leftrightarrow f \in \sqrt{\sum (f_i)}$.

$\Rightarrow$: $f \in \sqrt{(f)} \subseteq \sqrt{\sum (f_i)}$

$\Leftarrow$: Set $\mathfrak{a} = \sum_{i \in I} (f_i)$. For any $\mathfrak{p} \supseteq \mathfrak{a}$, we have $\mathfrak{p} \supseteq \sqrt{\mathfrak{a}}$ as for any $g \in \sqrt{\mathfrak{a}}$, there exists $k \in \mathbb{N}$ such that $g^k \in \mathfrak{a} \subseteq \mathfrak{p}$, i.e. $g \in \mathfrak{p}$ as $\mathfrak{p}$ is prime. For any $h \in \sqrt{(f)}$ and $h^r \in (f) \subseteq \sqrt{\mathfrak{a}} \subseteq \mathfrak{p}$ for some r. So $h \in \mathfrak{p}$ for all $\mathfrak{p} \in V(\mathfrak{a})$. Then $\sqrt{(f)} \subseteq \sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$.

Then $D(f) \subseteq \bigcup D(f_i) \Leftrightarrow V(f) \supseteq \bigcap V(f_i) = V(\sum (h_i)) \Leftrightarrow \sqrt{(f)} \subseteq \sqrt{\sum (f_i)} \Leftrightarrow f \in \sqrt{\sum (f_i)} \Leftrightarrow f^r \in \sum (f_i)$ for some r. So If $D(f)$ is covered by an infinite union of principal open sets $\bigcup_{j \in I} D(f_j)$, then there exists a finite subset $J \subseteq I$ such that $D(f) \subseteq \bigcup_{j\in J} D(f_j)$. This implies that $D(f)$ is quasi-compact since $\{D(f)\}_{f\in A}$ is a base for the topology. In particular, $D(1) = \mathrm{sp}(X)$ is quasi-compact.

Suppose Spec A is noetherian, let $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_n \subset \cdots$ be an acending chian of prime ideals, then $V(\mathfrak{p}_1) \supset V(\mathfrak{p}_2) \supset \cdots \supset V(\mathfrak{p}_n) \supset \cdots$ is a decending chain of closed subsets of Spec A. Then, there exists some n such that $V(\mathfrak{p}_1) \supset V(\mathfrak{p}_2) \supset \cdots \supset V(\mathfrak{p}_n) = V(\mathfrak{p}_{n+1}) = \cdots$. Thus, $\sqrt{\mathfrak{p}_1} \subset \sqrt{\mathfrak{p}_2} \subset \cdots \subset \sqrt{\mathfrak{p}_n} = \sqrt{\mathfrak{p}_{n+1}} = \cdots$. Since, $\sqrt{\mathfrak{p}} = \mathfrak{p}$ for any prime ideals $\mathfrak{p}$, we conclude that A satisfies a.c.c for prime ideals. Thus, if we take $A = k[X_1, X_2, \cdots]$ with infinitely many variables, where k is a field, then $(X_1) \subset (X_1, X_2) \subset \cdots$ gives us an ascending chain of prime ideals which is not stable. Thus, Spec A is not noetherian in this case.

(c) Let $V(\mathfrak{a}_1) \supset V(\mathfrak{a}_2) \supset \cdots \supset V(\mathfrak{a}_n) \supset \cdots$ be a descending chain of closed subsets in sp(Spec A), then we have an ascending chain of ideals $\sqrt{\mathfrak{a}_1} \subset \sqrt{\mathfrak{a}_2} \subset \cdots \subset \sqrt{\mathfrak{a}_n} \subset \cdots$. Since A is noetherian, there exists some n such that $\sqrt{\mathfrak{a}_1} \subset \sqrt{\mathfrak{a}_2} \subset \cdots \subset \sqrt{\mathfrak{a}_n} = \sqrt{\mathfrak{a}_{n+1}} = \cdots$. So $V(\mathfrak{a}_1) \supset V(\mathfrak{a}_2) \supset \cdots \supset V(\mathfrak{a}_n) = V(\mathfrak{a}_{n+1}) = \cdots$. We conclude that sp(Spec A) is noetherian. □

Exercise 2.14: (b) Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \mathrm{Proj}\ T | \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of Proj T, and show that φ determines a natural morphism $f : U \to \mathrm{Proj}\ S$.

(c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \mathrm{Proj}\ T$ and the morphism $f : \mathrm{Proj}\ T \to \mathrm{Proj}\ S$ is an isomorphism.

Proof.

□

Exercise 2.16: Let X be a scheme, let $f \in \Gamma(X, \mathscr{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathscr{O}_x$.

(a) If $U = \mathrm{Spec}\ B$ is an open affine subscheme of X, and if $\overline{f} \in B = \Gamma(U, \mathscr{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$. Conclude that X_f is an open subset of X.

(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathscr{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$. [Hint:Use an open affine cover of X.]

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied, for example, if sp(X) is noetherian.) Let $b \in \Gamma(X_f, \mathscr{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

(d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathscr{O}_{X_f}) \cong A_f$.

Proof. (a) We may identify $\mathscr{O}_{X,\mathfrak{p}}$ with $B_\mathfrak{p}$ for any $\mathfrak{p} \in U$ by Proposition 2.2(a). Thus, $\mathfrak{m}_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$ in this setting. Moreover, the stalk $f_\mathfrak{p} = \overline{f}_\mathfrak{p}$ is simply the image $\overline{f}/1$ in $B_\mathfrak{p}$. If $\mathfrak{p} \in D(\overline{f})$, then $\overline{f} \notin \mathfrak{p}$, so $1/\overline{f} \in B_\mathfrak{p}$. Thus, $\overline{f}_\mathfrak{p}$ is invertible. We conclude that $f_\mathfrak{p} \notin \mathfrak{m}_\mathfrak{p}$. So, $D(\overline{f}) \subseteq U \cap X_f$. Conversely, let $\mathfrak{p} \in U \cap X_f$, we have that $f_\mathfrak{p} = \overline{f}_\mathfrak{p} \notin \mathfrak{m}_\mathfrak{p}$. This means that $\overline{f}/1$ is invertible. Thus, $1/\overline{f} \in B_\mathfrak{p}$ and it follows that $\overline{f} \notin \mathfrak{p}$. Thus, $\mathfrak{p} \in D(\overline{f})$. This tells us that $D(\overline{f}) = U \cap X_f$. We can find an open cover of X_f, say $\{U_i\}_i$ such that each $U_i = \mathrm{Spec}\ B_i$ for some ring B_i. Since $U_i \cap X_f$ is open (because it is open in U_i) as we argue above, we conclude that X_f is open.

(b) Since X is quasi-compact, we can find a finite open affine cover $\{U_i\}_{i=1,2,\cdots,m}$ of X, say $U_i = \mathrm{Spec}\ B_i$ with $B_i = \Gamma(U_i, \mathrm{Spec}\ _X|_{U_i})$. Let $f_i \in B_i$ be the restriction of f. Consider the map $\psi_i : \mathscr{O}_X(X) \to (B_i)_{f_i}$ obtained by the composition of restriction maps $\mathscr{O}_X(X) \to \mathscr{O}_X(X_f) \to \mathscr{O}_X(X_f \cap U_i) = \mathscr{O}_X(D(f_i)) \cong (B_i)_{f_i}$, we know that $a_i := \psi_i(a) = 0$ in $(B_i)_{f_i}$. Then $f_i^{k_i} a_i = 0$ for some k_i. Let $n = \max_i k_i$, then $f_i^n a_i = 0$ for all i. Then we can glue $f_i^n a_i$ to obtain that $f^n a = 0$ since $\mathscr{O}_X$ is a sheaf.

(c) We consider the $b|_{D(f_i)}$, the restriction of b on $X_f \cap U_i = D(f_i)$, and the following commutative diagram

$$\begin{tikzcd} \mathscr{O}_X|_{U_i}(U_i) \arrow[r, "\mathrm{Res}"] \arrow[d, "\cong"'] & \mathscr{O}_X|_{U_i}(D(f_i)) \arrow[d, "\cong"] \\ B_i \arrow[r, "x \mapsto x/1"] & (B_i)_{f_i} \end{tikzcd}$$

. We know that $b|_{D(f_i)}$ is of the form $a_i/f_i^{n_i}$. Thus, there exists $b_i \in \mathscr{O}_X|_{U_i}(U_i)$ such that $b_i|_{D(f_i)} = f_i^{n_i} b|_{D(f_i)}$ for large enough n_i. Let $n = \max_i n_i$ and replace n_i with n, i.e. $b_i|_{D(f_i)} = f_i^n b|_{D(f_i)}$.

Let $f_{ij} = f|_{U_i \cap U_j} = f_i|_{U_i\cap U_j} = f_j|_{U_i \cap U_j}$, $X_{f_{ij}} = X_f \cap U_i \cap U_j$ and $U_{ij} = U_i \cap U_j$. We see that $b_i|_{X_{f_{ij}}} - b_j|_{X_{f_{ij}}} = (f_i^n b|_{D(f_i)})|_{X_{f_{ij}}} - (f_j^n b|_{D(f_j)})|_{X_{f_{ij}}} = (f^n b)|_{X_{f_{ij}}} - (f^n b)|_{X_{f_{ij}}} = 0$. So by ($b$), there exists m such that $f_{ij}^m (b_i|_{U_{ij}} - b_j|_{U_{ij}}) = 0$, i.e. $(f_i^m b_i)|_{U_{ij}} = (f_j^m b_j)|_{U_{ij}}$. Then, we can glue $f_i^m b_i$ to an element a since $\mathscr{O}_X$ is a sheaf. Then $a|_{X_f} = f^{m+n} b$.

(d) Let $\overline{f}$ be the image of f in $\Gamma(X_f, \mathscr{O}_{X_f})$. We first show that $\overline{f}$ is invertible. For any $x \in X_f$, we know that $\overline{f}_x = f_x \notin \mathfrak{m}_x$. Thus, $\overline{f}_x$ is invertible in $\mathscr{O}_{X,x}$. Then, there exists a neighborhood U_x of x contained in X_f such that $\overline{f}|_{U_x}$ is invertible. We call its inverse g_{U_x}. Then we can glue g_x to a section $g \in \Gamma(X_f, \mathscr{O}_{X_f})$ as $\mathscr{O}_{X_f}$ is a sheaf. Since $(\overline{f} g - 1)|_{U_x} = \overline{f}|_{U_x} g_x - 1 = 0$. Thus $\overline{f} g = 1$. Thus, $\overline{f}$ is invertible. Then by the universal property of localization, there exists a map $\psi : A_f \to \Gamma(X_f, \mathscr{O}_{X_f})$ such that the following commutative diagram commutes

$$\begin{tikzcd} A = \Gamma(X, \mathscr{O}_X) \arrow[r, "x \mapsto x/1"] \arrow[dr, "\mathrm{Res}"'] & A_f \arrow[d, "\psi"] \\ & \Gamma(X_f, \mathscr{O}_{X_f}) \end{tikzcd}$$

By (b), for any $b \in \Gamma(X_f, \mathscr{O}_{X_f})$, there exists some $n > 0$ and $a \in A$ such that $\mathrm{Res} : a \mapsto \overline{f}^n b$. Thus $\psi(a/f^n) = \mathrm{Res}(a)\,\mathrm{Res}(f^n)^{-1} = \overline{f}^n b \cdot \overline{f}^{-n} = b$, i.e. ψ is surjective.

It remains to show that ψ is injective. Let $\psi(a/f^m) = 0$, then $\overline{a}\overline{f}^{-m} = 0$, so $\overline{a} = 0$, where $\overline{a} = \mathrm{Res}(a)$. By ($b$), we know that $f^n a = 0$ for some n. This means that $a/f^m = 0 \in A_f$.

We conclude that $A_f \cong \Gamma(X_f, \mathscr{O}_{X_f})$. □

Exercise 2.17: *A Criterion for Affineness.* (a) Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

(b) A scheme X is affine if and only if there is a finite set of elements $f_1, \cdots, f_r \in A = \Gamma(X, \mathscr{O}_X)$, such that the open subsets X_{f_i} , are affine, and $f_1, \cdots, f_r$ generate the unit ideal in A. [Hint: Use (Ex. 2.4) and (Ex. 2.16d) above.]

Proof. (a) We denote $f_i : f^{-1}(U_i) \to U_i$ and inverse by g_i. Then, we must have $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$ because their inverses $f_i|_{f^{-1}(U_i \cap U_j)}$ and $f_j|_{f^{-1}(U_i \cap U_j)}$ are the same. Thus, we can glue these g_i to a morphism $g : Y \to X$ by Lemma 1. Then $(f \circ g)|_{U_i} = f|_{f^{-1}(U_i)} \circ g_i = f_i \circ g_i = 1$. Thus, $f \circ g = 1$. Similaly, $g \circ f = 1$. This means that f is an isomorphism.

(b) Suppose X is affine. Then we may assume that $X = \text{Spec } A$ for some ring A. Moreover $A = \Gamma(X, \mathscr{O}_X)$. Let $r = 1$ and take $f_1 = 1$, then $X_{f_1} = D(f_1) = X$ is affine.

Conversely, suppose there exists a finite set of elements $f_1, \cdots, f_r \in A = \Gamma(X, \mathscr{O}_X)$, such that the open subsets X_{f_i} , are affine, and $f_1, \cdots, f_r$ generate the unit ideal in A. By Exercise 2.16(d), we have that $X_{f_i} \cong \text{Spec } A_{f_i}$. By Exercise 2.4, the identity $A \to \Gamma(X, \mathscr{O}_X)$ corresponds a morphism of schemes $X \to \text{Spec } A$. Now, since $\sum(f_i) = (1)$, we see that $V(1) = V(\sum(f_i)) = \bigcap V(f_i)$. Thus, $\text{Spec } A = \bigcup_{i=1}^r D(f_i)$. Note that $D(f_i) \cong \text{Spec } A_{f_i}$ by Exercise 2.1. We see that $D(f_i) \cong X_{f_i}$ as schemes. By (a), we conclude that $X \cong \text{Spec } A$. Thus, X is affine. ☐

Exercise 2.18: In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

(a) Let A be a ring, $X = \text{Spec } A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y = \text{Spec } B \to X = \text{Spec } A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is injective. Show furthermore in that case f is **dominant**, i.e., $f(Y)$ is dense in X.

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is surjective.

(d) Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is surjective, then φ is surjective. [Hint: Consider $X' = \text{Spec } (A/\ker\varphi)$ and use (b) and (c).]

Proof. (a) First, note that $D(f) = D(f^n)$ for all $n > 0$ since $f \notin \mathfrak{p} \Leftrightarrow f^n \notin \mathfrak{p}$ for any prime ideal $\mathfrak{p}$. Thus, if f is nilpotent, then $f^n = 0$ for some n. So, $D(f) = D(f^n) = D(0) = \varnothing$. Conversrly, of $D(f) = \varnothing = D(0)$, we have $V(f) = V(0)$. So, $\sqrt{(f)} = \sqrt{0} = \text{nil}\, A$ and it follows that $f \in \sqrt{(f)} = \text{nil}\, A$ is nilpotent.

(b) Suppose that φ is injective. It is sufficient to show that for each $\mathfrak{p} \in X$, $f^{\#}_{\mathfrak{p}} : \mathscr{O}_{X,\mathfrak{p}} \to (f_*\mathscr{O}_Y)_{\mathfrak{p}}$ is injective. However, $f^{\#}_{\mathfrak{p}}$ is simply the homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ of A-modules, which is injective by the exactness of localization. Here, $B_{\mathfrak{p}} = B[\varphi(A\backslash\mathfrak{p})^{-1}] \cong B \otimes_A A_{\mathfrak{p}}$ as an A-module. Indeed, $\mathscr{O}_{X,\mathfrak{p}} \cong A_{\mathfrak{p}}$ and $(f_*\mathscr{O}_Y)_{\mathfrak{p}} = \varinjlim_{D(g)\ni\mathfrak{p}} \mathscr{O}_Y(f^{-1}(D(g))) = \varinjlim_{g\notin\mathfrak{p}} \mathscr{O}_Y(D(\varphi(g))) \cong \varinjlim_{g\in A\backslash\mathfrak{p}} B_{\varphi(g)} \cong B_{\mathfrak{p}}$. So, we conclude that $f^{\#}$ is injective.

Conversely, if $f^{\#}$ is injective. By taking global section, we know that $\varphi : A \to B$ is injective.

In this case, we may identity A with a subring of B. We now show that $f(Y)$ is dense in X. We claim that for any $U \subseteq X$, we have $\overline{U} = V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p})$. Indeed, we have $U \subseteq V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p})$, and so $\overline{U} \subseteq V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p})$. Conversely, let $\overline{U} = V(\mathfrak{a})$, then for any $\mathfrak{q} \in V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p})$, we have $\mathfrak{q} \supset \bigcap_{\mathfrak{p}\in U} \mathfrak{p} \supset \mathfrak{a}$ since $\mathfrak{p} \in U \Rightarrow \mathfrak{p} \supset \mathfrak{a}$. Thus, $V(\bigcap_{\mathfrak{p}\in U} \mathfrak{p}) \subseteq V(\mathfrak{a}) = \overline{U}$. So, we finish the proof of the claim. It follows that $\overline{f(Y)} = V(\bigcap_{\mathfrak{p}\in f(Y)} \mathfrak{p}) = V(\bigcap_{\mathfrak{q}\in Y} f(\mathfrak{q})) = V(\bigcap_{\mathfrak{q}\in Y}(\mathfrak{q} \cap A)) = V(\text{nil}(B) \cap A) = V(\text{nil}(A)) = V(0) = X$.

(c) Since φ is surjective, then $A/\ker\varphi \cong B$, so we conclude that $\text{Spec } B \cong V(\ker\varphi)$ as topological spaces via f. By (b), we know that $f^{\#}_{\mathfrak{p}}$ is simply the homomorphism $A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ of A-modules, which is surjective by the exactness of localization again. Thus, $f^{\#}$ is surjective.

(d) Similar to (c), we know that $f^{\#}_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is surjective for all $\mathfrak{p} \in X$. Thus, $\varphi : A \to B$ is surjective by the exactness of localization. ☐

Exercise 2.19: *Let A be a ring. Show that the following conditions are equivalent:*

(i) Spec A *is disconnected;*

(ii) *there exist nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called* ***orthogonal idempotents****);*

(iii) *A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.*

Proof.

□