Algebraic Geometry Homework 4

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4 Separated and Proper Morphisms

Exercise 4.1: Show that a finite morphism is proper.

Proof. Suppose $f : X \to Y$ is finite, then f is of finite type by definition. By Exercise 3.3(a), f is quasi-compact. Thus, the diagonal morphism $\Delta : X \to X \times_Y X$ is also quasi-compact by the construction of fibred product. From the proof of Valuativa Criterion for Separatedness, we see that the noetherian condition can be replaced by the quasi-compactness of Δ. Thus, by Corollary 4.6(f) and the finiteness of f, we may assume that X and Y are affine. Then f is separated by Proposition 4.1. Now, by Exercise 3.5(b), we see that f is closed. Let $Y' \to Y$ be a morphism and $f' = f \times \mathrm{id} : X' = X \times_Y Y' \to Y' \cong Y \times_Y Y'$ be the corresponding morphism obtained by base extension. We claim that f' is also finite. By the construction of fibred product and finiteness is an affine local property, we may assume that Y and Y' are affine, say $Y = \mathrm{Spec}\ B$ and $Y' = \mathrm{Spec}\ C$. Then, by the finiteness of f, we see that X is also affine, say $X = \mathrm{Spec}\ A$, where A is a finitely generated B-module. So, $f'^{-1}(Y') = \mathrm{Spec}\ A \times_{\mathrm{Spec}\ B} \mathrm{Spec}\ C = \mathrm{Spec}\ (A \otimes_B C)$ is affine and $A \otimes_B C$ is a finitely generated C-module. Indeed, if $x_1, \cdots, x_n$ generate A as a B-module, then $x_1 \otimes 1, \cdots, x_n \otimes 1$ generate $A \otimes_B C$ as a C-module. This implies that f' is also finite, hence closed. Thus, f is universally closed. Therefore, f is proper.

□

Exercise 4.3: Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

Proof. By the universal property of fibered product, $U \times_S V = p^{-1}(U) \cap q^{-1}(V)$, where p, q are the first and second projection $X \times_S X \to X$ respectively. Thus, $\Delta^{-1}(U \times_S V) = \Delta^{-1}p^{-1}(U) \cap \Delta^{-1}q^{-1}(V) = U \cap V$. Since X is a separated scheme over S, we know that $\Delta : X \to X \times_S X$ is a closed immersion. $\Delta|_{U \cap V} : U \cap V \to U \times_S V$ is also a closed immersion. $U \times_S V$ is affine since U, V and S are affine. By Exercise 3.11(b), we see that $U \cap V$ is affine.

□

Exercise 4.4: Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure (Ex. 3.11d) is proper over S. We refer to this result by saying that "the image of a proper scheme is proper." [Hint: Factor f into the graph morphism $\Gamma_f : X \to X \times_S Y$ followed by the second projection p_2, and show that Γ_f is a closed immersion.]

Proof. We first show that $f(Z)$ is closed. Since Z is a closed subscheme of X, we have a closed immersion $Z \to X$ and the composed map $Z \to X \to S$ is proper. Let $Z \to Y$ be the composed map of $Z \to X \xrightarrow{f} Y$. Then $Z \to S$ factors through Y. Since $Y \to S$ is separated, by Corollary 4.8(e), we see that $Z \to Y$ is proper. Thus, $f(Z)$ is closed in Y as $Z \to Y$ is closed. This gives a closed immersion $f(Z) \to Y$ by Exercise 3.11(d).

So, we see that the map $Z \to Y$ factors as $Z \to f(Z) \to Y$. Considering the following commutative diagram

$$\begin{tikzcd}
 & f(Z) \arrow[dr] & \\
Z \arrow[ur, "f|_Z"] \arrow[r] \arrow[dr] & X \arrow[r, "f"] \arrow[d] & Y \arrow[dl] \\
 & S &
\end{tikzcd}$$

and denote the composed map $f(Z) \to S$ by g. Since $f(Z) \to Y$ is a closed immersion, we see that $f(Z) \to Y$ is separated and of finite type by Corollary 4.6(a) and Exercise 3.13(a). Since $Y \to S$ is of finite type and separated by the hypothesis, we see that g is of finite type and separated by Corollary 4.6(b) and Exercise 3.13(c). It remains to prove that g is universally closed. For any base extension $T \to S$, consider the following diagram

$$\begin{tikzcd}
T \times_S Z \arrow[r] \arrow[d, "f'_T"'] & Z \arrow[d, "f'"] \\
T \times_S f(Z) \arrow[r] \arrow[d, "g_T"'] & f(Z) \arrow[d, "g"] \\
T \arrow[r] & S
\end{tikzcd},$$

where $f' = f|_Z$, $f'_T = \mathrm{id} \times f'$ and $g_T = \mathrm{id} \times g$. It suffices to show that g_T is closed for all base extension $T \to S$. Take a closed subset $W \subseteq T \times_S f(Z)$ and $\Omega = T \times_S f(Z) - W$. Then, ${f'_T}^{-1}(W)$ is a closed subset of $T \times_S Z$. Since $Z \to S$ is proper, we see that $g_T \circ f'_T$ is proper, hence closed. Note that f'_T is surjective as f' is. We have $W = f'_T({f'_T}^{-1}(W))$. Thus, $g_T(W) = g_T \circ f'_T({f'_T}^{-1}(W))$ is closed. It follows that g is universally closed. We conclude that $f(Z)$ is proper over S.

$\square$

Exercise 4.8: Let $\mathscr{P}$ be a property of morphisms of schemes such that:

(a) a closed immersion has $\mathscr{P}$;

(b) a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;

(c) $\mathscr{P}$ is stable under base extension.

Then show that:

(d) a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;

(e) if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$;

(f) If $f : X \to Y$ has $\mathscr{P}$, then $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ has $\mathscr{P}$.

[Hint: For (e), consider the graph morphism $\Gamma_f : X \to X \times_Z Y$ and note that it is obtained by base extension from the diagonal morphism $\Delta : Y \to Y \times_Z Y$.]

Proof. (d) Let $f : X \to Y$ be a morphism of S-schemes having property $\mathscr{P}$. Then for any S-scheme Y', $Y \times_S Y'$ is a Y-scheme. Thus, by (b), $X \times_Y (Y \times_S Y') \to Y \times_S Y'$ has property $\mathscr{P}$, i.e. $f \times \mathrm{id} : X \times_S Y' \to Y \times_S Y'$ has $\mathscr{P}$. Similarly, $\mathrm{id} \times f : Y' \times_S X \to Y' \times_S Y$ has $\mathscr{P}$. Now, if $f' : X' \to Y'$ has $\mathscr{P}$, then $X \times_S X' \to Y \times_S X' \to Y \times_S Y'$ also has $\mathscr{P}$ by (b).

(e) Since g is separated, $\Delta : Y \to Y \times_Z Y$ is a closed immersion. Consider the following commutative diagram,

$$\begin{tikzcd}
& X \arrow[d, dotted, "\Gamma_f"] \arrow[ddl, "\mathrm{id}"'] \arrow[ddr, "f"] & \\
& X \times_S Y \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X \arrow[dr, "g\circ f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

and note that Γ_f is obtained by base extension from the diagonal morphism $\Delta : Y \to Y \times_Z Y$ and p_2 is obtained by base extension from $g \circ f$. We have that Γ_f and p_2 has $\mathscr{P}$. Thus, $f = p_2 \circ \Gamma_f$ has $\mathscr{P}$ by (b).

(f) Note that by the construction of X_{red}, $X_{\mathrm{red}} \to X$ is a closed immersion, which has $\mathscr{P}$. Thus, the composed $X_{\mathrm{red}} \to Y$ obtained by composing $X_{\mathrm{red}} \to X$ with $f : X \to Y$ also has $\mathscr{P}$. Note that $Y_{\mathrm{red}} \to Y$ is separated since it is a closed immersion, we conclude that $X_{\mathrm{red}} \to Y_{\mathrm{red}}$ has $\mathscr{P}$ by (e). $\square$

Exercise 4.9: Show that a composition of projective morphisms is projective. [Hint: Use the Segre embedding defined in (1, Ex. 2.14) and show that it gives a closed immersion $\mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^{rs+r+s}$.] Conclude that projective morphisms have properties (a) – (f) of (Ex. 4.8) above.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be projective morphisms. Then they factor as $X \to \mathbf{P}^r_Y \to Y$ and $Y \to \mathbf{P}^s_Z \to Z$. Consider the composed map $\mathbf{P}^r_Y \to \mathbf{P}^s_Z$ and $\mathbf{P}^r_Y = \mathbf{P}^r_{\mathbb{Z}} \times_{\mathrm{Spec}\ \mathbb{Z}} Y = \mathbf{P}^r_{\mathbb{Z}} \times_{\mathrm{Spec}\ \mathbb{Z}} Z \times_Z Y = \mathbf{P}^r_Z \times_Z Y = (\mathbf{P}^r_Z \times_Z \mathbf{P}^s_Z) \times_{\mathbf{P}^s_Z} Y$. We know that $\mathbf{P}^r_Y \to \mathbf{P}^s_Z$ factors as $\mathbf{P}^r_Y \to \mathbf{P}^r_Z \times \mathbf{P}^s_Z \to \mathbf{P}^s_Z$. Now, we claim that there exists a closed immersion $\mathbf{P}^r_Z \times_Z \mathbf{P}^s_Z \to \mathbf{P}^{rs+r+s}_Z$.

We first deal with the case $Z = \mathrm{Spec}\ \mathbb{Z}$. We know that $\mathbf{P}^r_{\mathbb{Z}} = \mathrm{Proj}\ \mathbb{Z}[x_0, \cdots, x_r]$, $\mathbf{P}^s_{\mathbb{Z}} = \mathrm{Proj}\ \mathbb{Z}[y_0, \cdots, y_s]$, $\mathbf{P}^{mn+m+n} = \mathrm{Proj}\ \mathbb{Z}[z_{00}, \cdots, z_{rs}]$. Since the map $f_{ij} : D_+(x_i) \times_{\mathrm{Spec}\ \mathbb{Z}} D_+(y_j) \to D_+(z_{ij})$ induced by the surjective homomorphism

$$\mathbb{Z}[z_{00}, \cdots, z_{rs}]_{(z_{ij})} \to \mathbb{Z}[x_0, \cdots, x_r]_{(x_i)} \otimes_{\mathbb{Z}} \mathbb{Z}[y_0, \cdots, y_r]_{(y_j)}$$

by $z_{kl}/z_{ij} \mapsto x_k/x_i \otimes y_l/y_j$ is a closed immersion. Then f_{ij} glues to a closed immersion $\mathbf{P}^r_{\mathrm{Spec}\ \mathbb{Z}} \times_{\mathrm{Spec}\ \mathbb{Z}} \mathbf{P}^s_{\mathrm{Spec}\ \mathbb{Z}} \to \mathbf{P}^{rs+r+s}_{\mathrm{Spec}\ \mathbb{Z}}$. Note that $\mathbf{P}^r_Z \times_Z \mathbf{P}^s_Z \to \mathbf{P}^{rs+r+s}_Z$ is obtained from $\mathbf{P}^r_{\mathrm{Spec}\ \mathbb{Z}} \times_{\mathrm{Spec}\ \mathbb{Z}} \mathbf{P}^s_{\mathrm{Spec}\ \mathbb{Z}} \to \mathbf{P}^{rs+r+s}_{\mathrm{Spec}\ \mathbb{Z}}$ by base extension, we see that it is a closed immersion by Exercise 3.11(a). For the same reason, $\mathbf{P}^r_Y = (\mathbf{P}^r_Z \times_Z \mathbf{P}^s_Z) \times_{\mathbf{P}^s_Z} Y \to \mathbf{P}^r_Z \times_Z \mathbf{P}^s_Z$ is a closed immersion since $Y \to \mathbf{P}^s_Z$ is a closed immersion. Note that $X \to \mathbf{P}^r_Z$ is a closed immersion and the composition of closed immersions is still a closed immersion. We conclude that $X \to \mathbf{P}^{rs+r+s}_Z$ is a closed immersion. Thus, $g \circ f$ is projective.

$\square$