Design and Analysis of Algorithms
Greedy Algorithms

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Topics

- **Coin Changing**
- **Interval Scheduling and Partitioning**

Coin Changing

Goal. Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest coins.

Ex. 34¢

Cashier's algorithm. At each iteration, and coin of the largest value that does not take us past the amount to be paid.

Ex. $2.89

Cashier's Algorithm

At each iteration, add coin of the largest value that does not take us past the amount to be paid.

Cashiers-Algorithm $(x, c_1, c_2, \ldots, c_n)$
Sort n coin denominations so that $0 < c_1 < c_2 < \cdots < c_n$
$S \leftarrow \emptyset$
while $x > 0$
$k \leftarrow$ largest coin denomination c_k such that $c_k \leq x$
if no such k, Return "no solution"
else
$x \leftarrow x - c_k$
$S \leftarrow S \cup \{k\}$
Return S

Is Cashier's Algorithm optimal?

Properties of Optimal Solution

Property. Number of pennies ≤ 4.

Pr. Replace 5 pennies with 1 nickel.

Property. Number of nickels ≤ 1.

Property. Number of quarters ≤ 3.

Property. Number of nickels + number of dimes ≤ 2.

Pr.

- Replace 3 dimes and 0 nickels with 1 quarter and 1 nickel;
- Replace 2 dims and 1 nickel with 1 quarter.
- Recall: at most 1 nickel.

Analysis of Cashier's Algorithm

Theorem. Cashier's algorithm is optimal for U.S. coins: 1, 5, 10, 25, 100.

Pf.

- Consider optimal way to change $c_k \leq x \leq c_{k+1}$: greedy takes coin k.
- We claim that any optimal solution must also take coin k.
  - if not, it needs enough coins of type $c_1, \ldots, c_{k-1}$ to add up to x.
- Problem reduces to coin-changing $x - c_k$ cents, which, by induction, is optimally solved by Cashier's algorithms.

Cashier's Algorithm for Other Denominations

Is Cashier's algorithm optimal for any set of denominations?

No. Consider U.S. postage: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

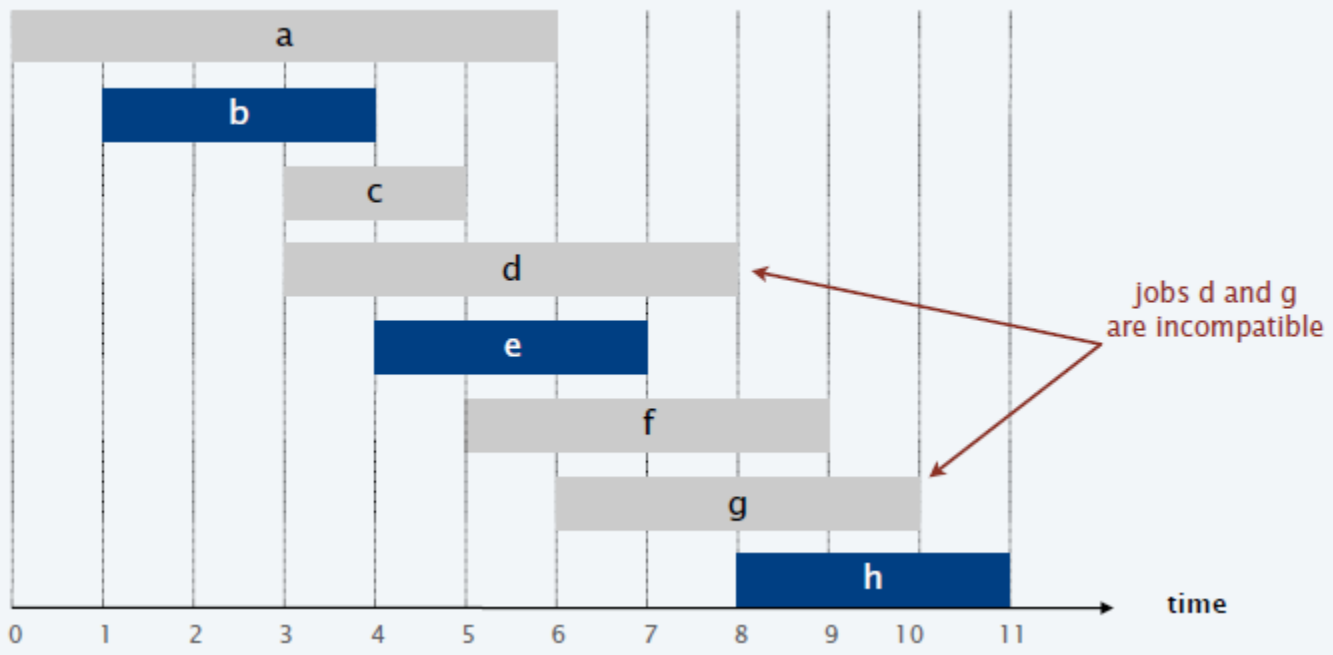
- Cashier's algorithm: 140¢=100 + 34 + 1 + 1 + 1 + 1 + 1 + 1.
- Optimal: 140¢=70 + 70.

It may not even lead to a feasible solution if $c_1 > 1$： 7, 8, 9.

- Cashier's algorithm: 15¢=9 + ?.
- Optimal: 15¢=7+8.

Interval Scheduling

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of s_j.
- [Earliest finish time] Consider jobs in ascending order of f_j.
- [Shortest interval] Consider jobs in ascending order of $f_j - s_j$.
- [Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

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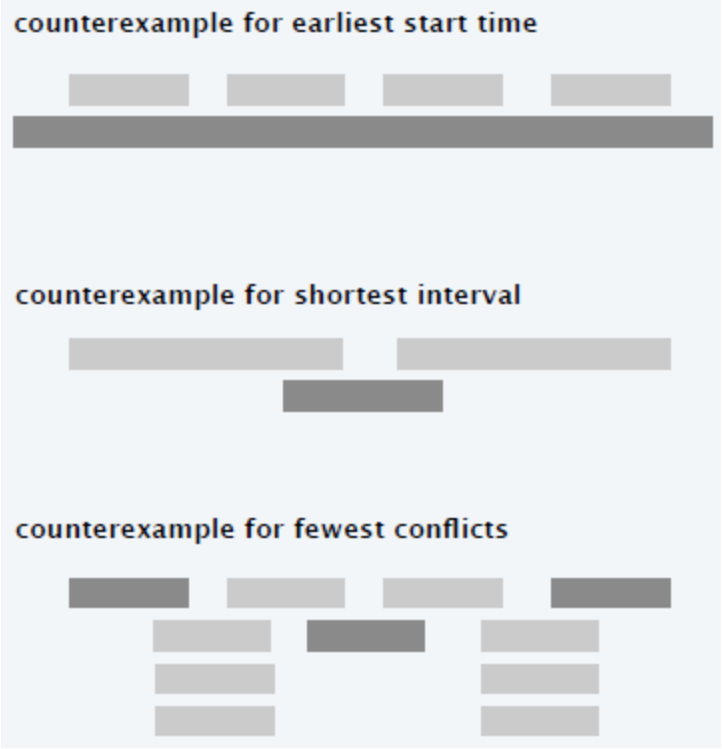


Interval Scheduling: Earliest-Finish-Time-First Algorithm

Earliest-Finish-Time-First $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

Sort jobs by finish time so that $f_1 \leq f_2 \leq \cdots \leq f_n$

$A \leftarrow \emptyset$ (set of jobs selected)

for $j = 1$ to n

If job j is compatible with A

$A \leftarrow A \cup \{j\}$

Return A

The Earliest-Finish-Time-First algorithm is optimal.

Proposition. Can implement Earliest-Finish-Time-First in $O(nlogn)$ time.

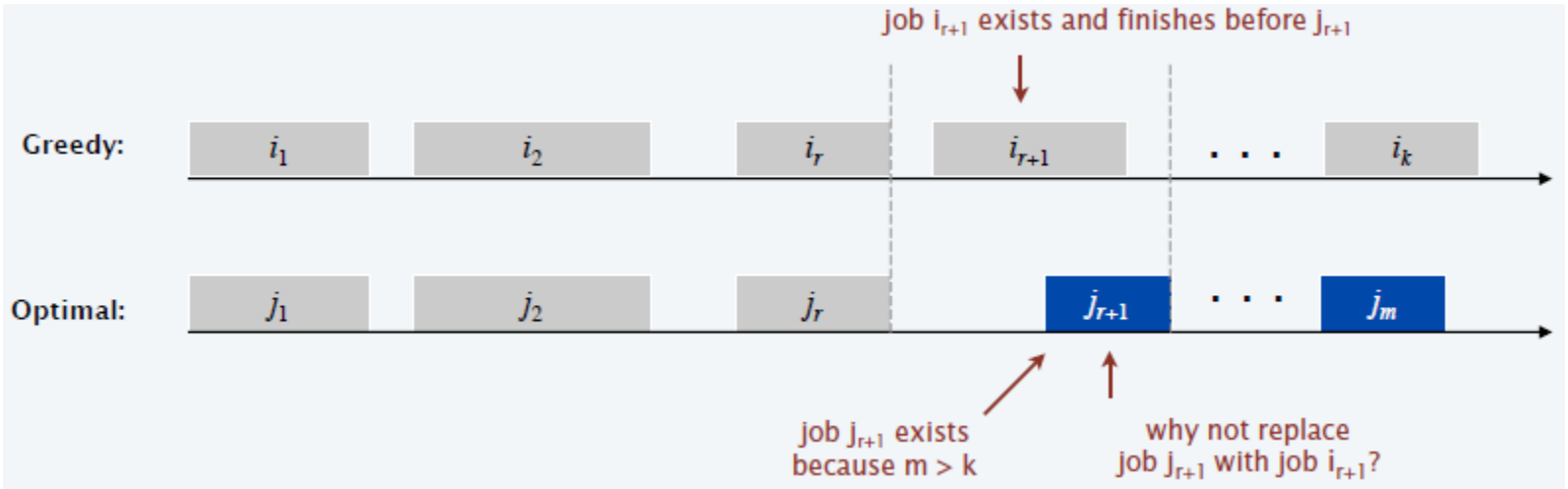
- Keep track of job j^* that was added last to A
- Job j is compatible with A iff $s_n \geq f_{j^*}$.
- Sorting by finish time takes $O(nlogn)$ time.

Interval Scheduling: Analysis of Earliest-Finish-Time-First Algorithm

Theorem. The Earliest-Finish-Time-First algorithm is optimal.

Pf.

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots, i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots, j_m$ denote set of jobs in an optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



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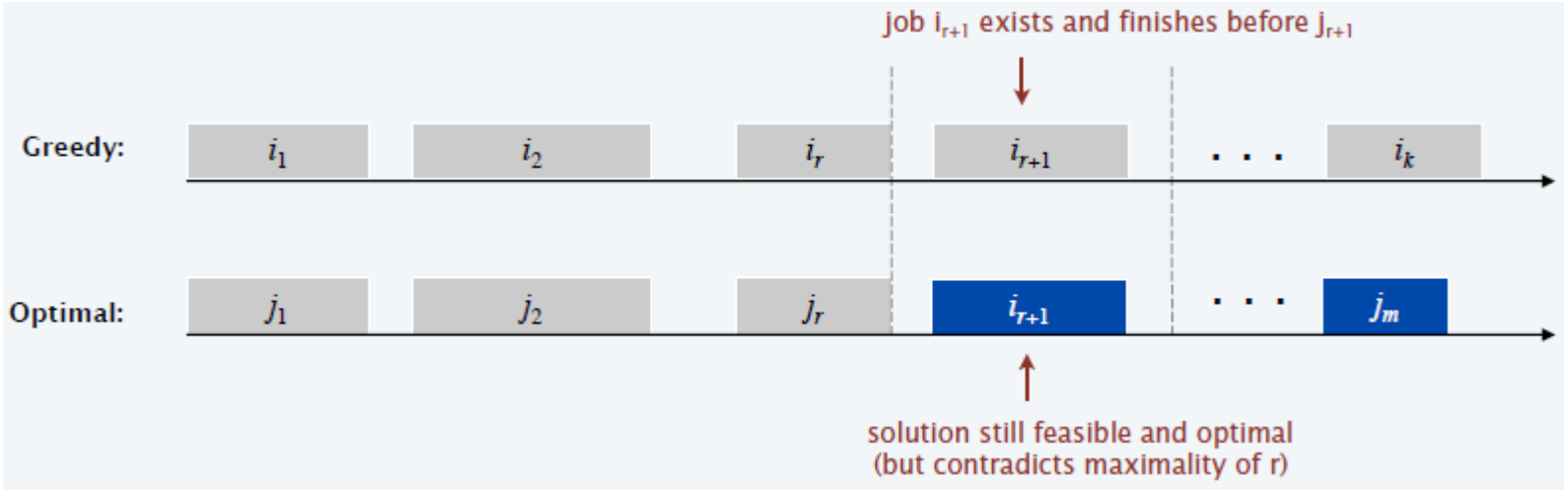
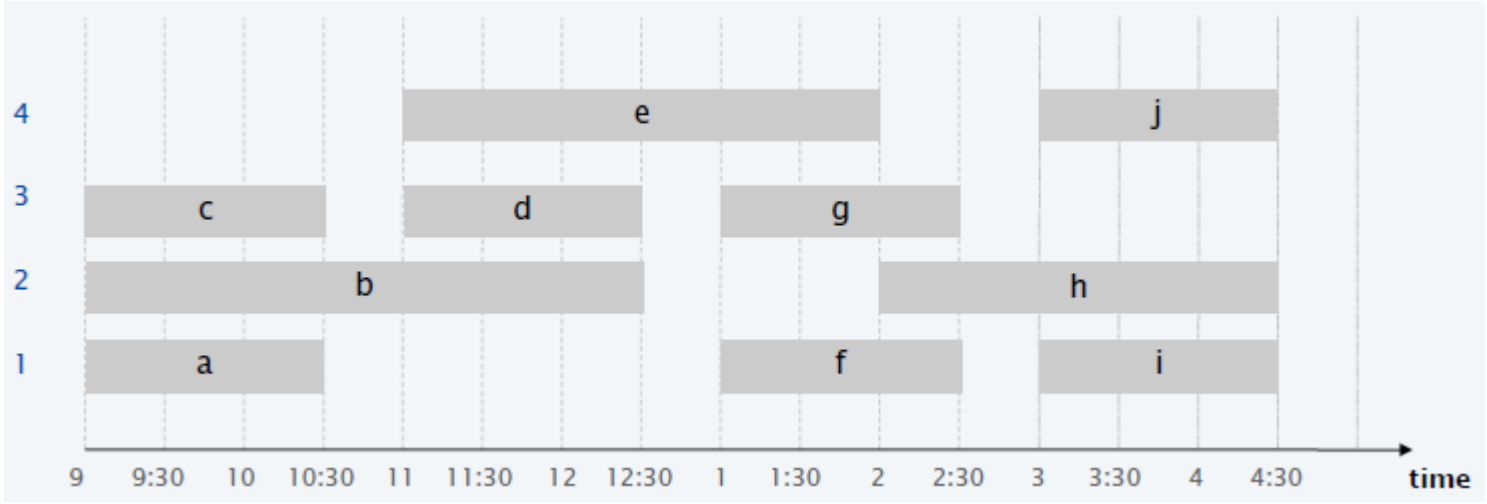


Interval Partitioning

Interval Partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two lectures occur at the same time in the same room.

Ex. This schedule uses 4 classrooms to schedule 10 lectures.



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- Lecture j starts at s_j and finishes at f_j.
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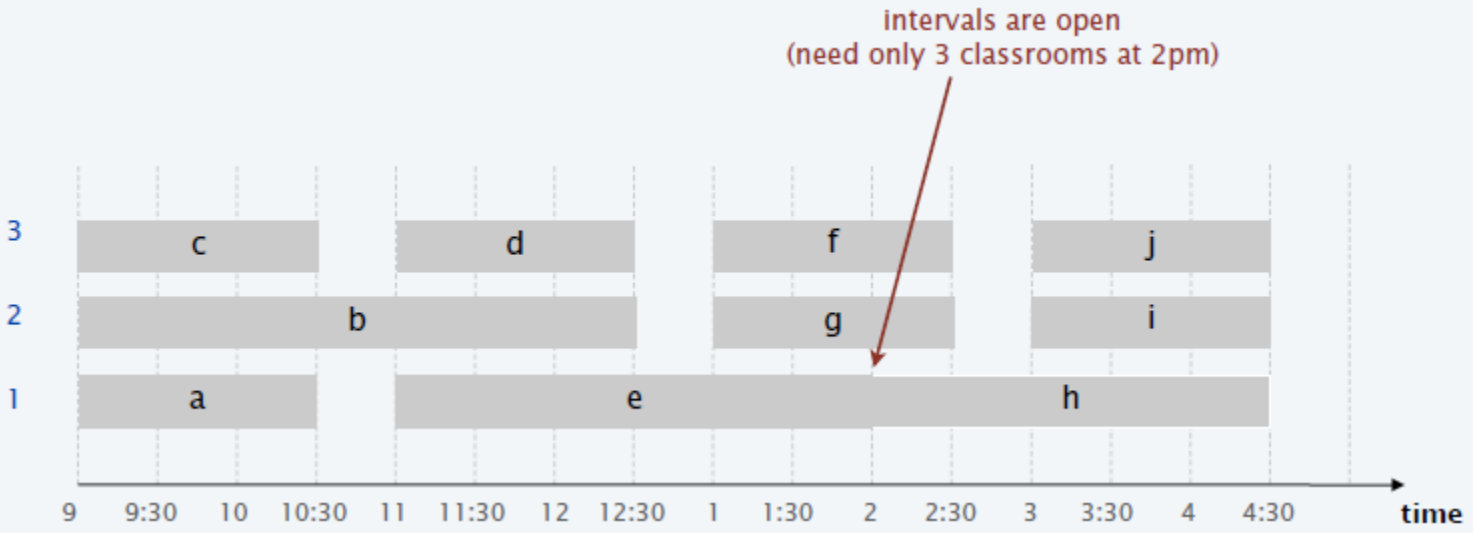

Interval Partitioning: Greedy Algorithm

Greedy template. Consider lectures in some natural order. Assign each lecture to an available classroom; allocate a new classroom if none are available.

- [Earliest start time] Consider lectures in ascending order of s_j.
- [Earliest finish time] Consider lectures in ascending order of f_j.
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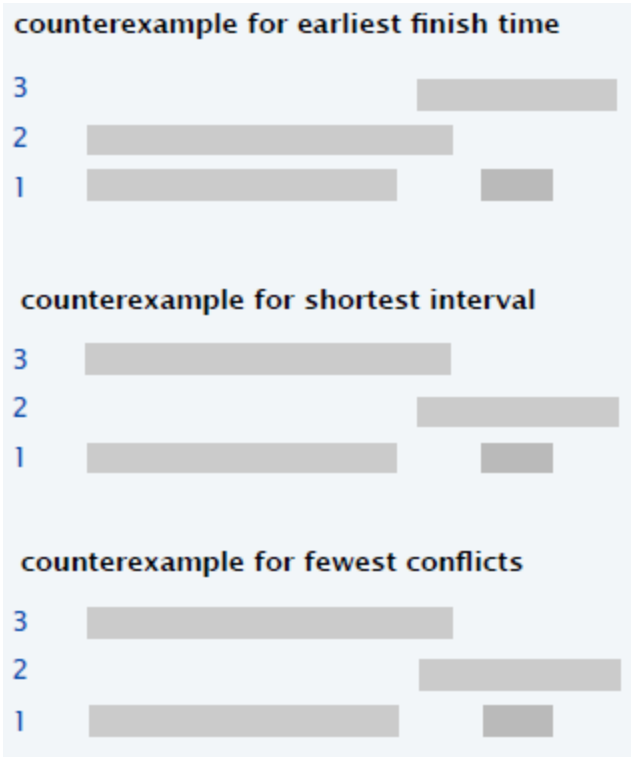


Interval Partitioning: Earliest-Start-Time-First Algorithm

Earliest-Start-Time-First $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$
Sort lectures by start time so that $s_1 \leq s_2 \leq \cdots \leq s_n$
$d \leftarrow 0$ (the number of allocated classrooms)
for $j = 1$ to n
if lecture j is compatible with some classroom
    Schedule lecture j in any such classroom k.
else
    Allocate a new classroom $d + 1$.
    Schedule lecture j in classroom $d + 1$.
    $d \leftarrow d + 1$
Return schedule.

Interval Partitioning: Earliest-Start-Time-First Algorithm

Proposition. The Earliest-Start-Time-First algorithm can be implemented in $O(nlogn)$ time.

Pf. Store classrooms in a priority queue (key=finish time of its last lecture).

- To determine whether lecture j is compatible with some classroom, compare s_j to key of min classroom k in priority queue.
- To add lecture j to classroom k, increase key of classroom k to f_j.
- Total number of priority queue operation is $O(n)$.
- Sorting by start time takes $O(nlogn)$ time.

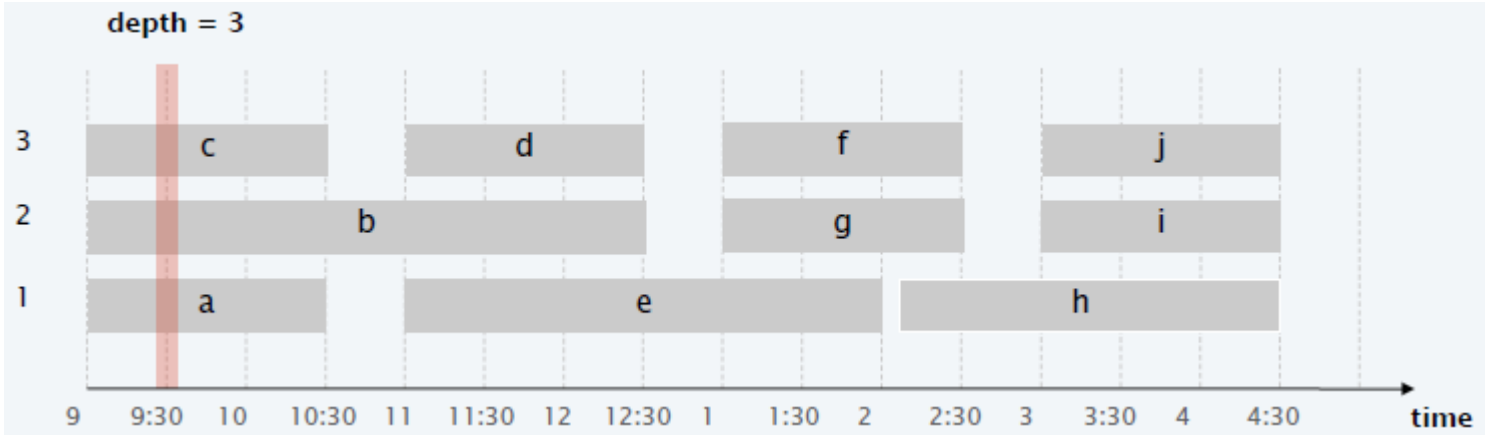
Remark. This implementation chooses a classroom k whose finish time of its last lecture is the earliest.

Interval Partitioning: Lower Bound on Optimal Solution

Def. The depth of a set of open intervals is the maximum number of intervals that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Does minimum number of classrooms needed always equal depth?
Earliest-Start-Time-First algorithm finds a schedule whose number of classrooms equals the depth.



Interval Partitioning: Analysis of Earliest-Start-Time-First Algorithm

Observation. The Earliest-Start-Time-First algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Earliest-Start-Time-First algorithm is optimal.

Pf.

- Let d = number of classrooms that the algorithm allocates.
- Classroom d is opened because we needed to schedule a lecture, say j, that is incompatible with all $d - 1$ other classrooms.
- These d lectures each end after s_j.
- Since we sorted by start time, all these incompatibilities are caused by the lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation $\Rightarrow$ all schedules use $\geq d$ classrooms.