Design and Analysis of Algorithms
Network Flow

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Topics

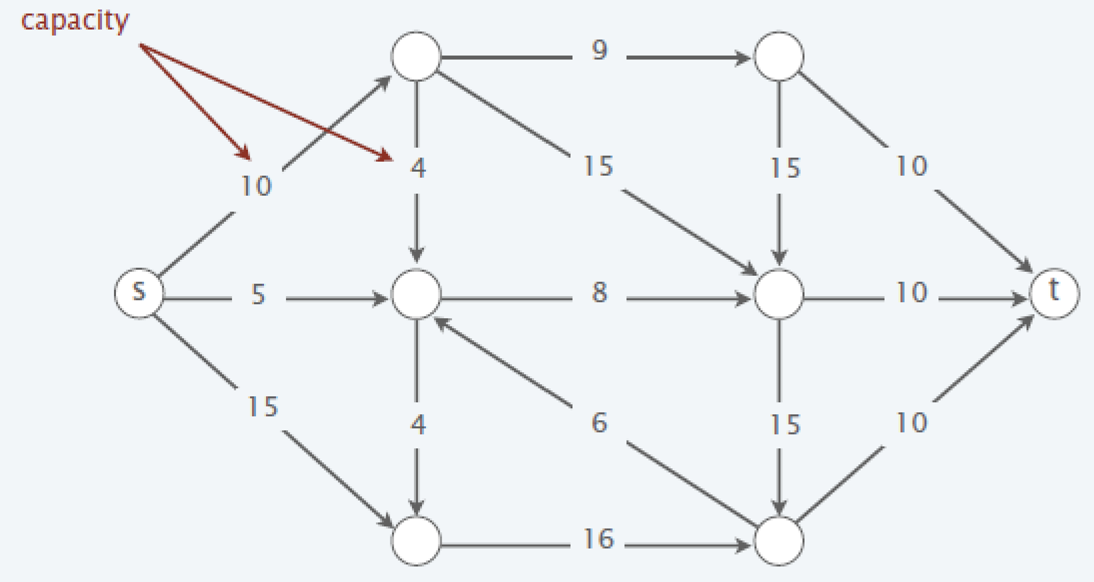
- Max-Flow and Min-Cut Problems
- Ford-Fulkerson Algorithm
- Max-Flow Min-Cut Theorem
- Capacity-Scaling Algorithm
- Shortest Augmenting Paths

Flow Network

A flow network is a tuple $G = (V, E, s, t, c)$.

- Digraph (V, E) with source $s \in V$ and sink $t \in V$.
- Non-negative capacity $c(e)$ for each $e \in E$.

Intuition. Material flowing through a transportation network; material originates at source and is sent to sink.

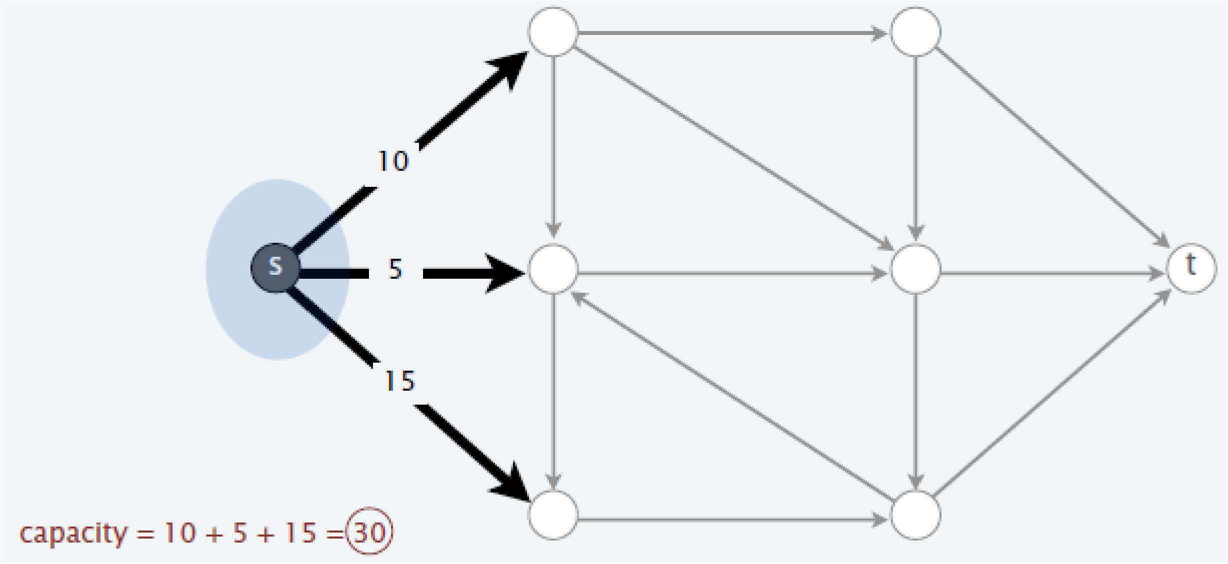


Minimum-Cut Problem

Def. An st-cut (cut) is a partition (A, B) of the vertices with $s \in A$ and $t \in B$.

Def. Its capacity is the sum of the capacities of the edges from A to B.

$$cap(A, B) = \sum_{e \text{ out of } A} c(e)$$



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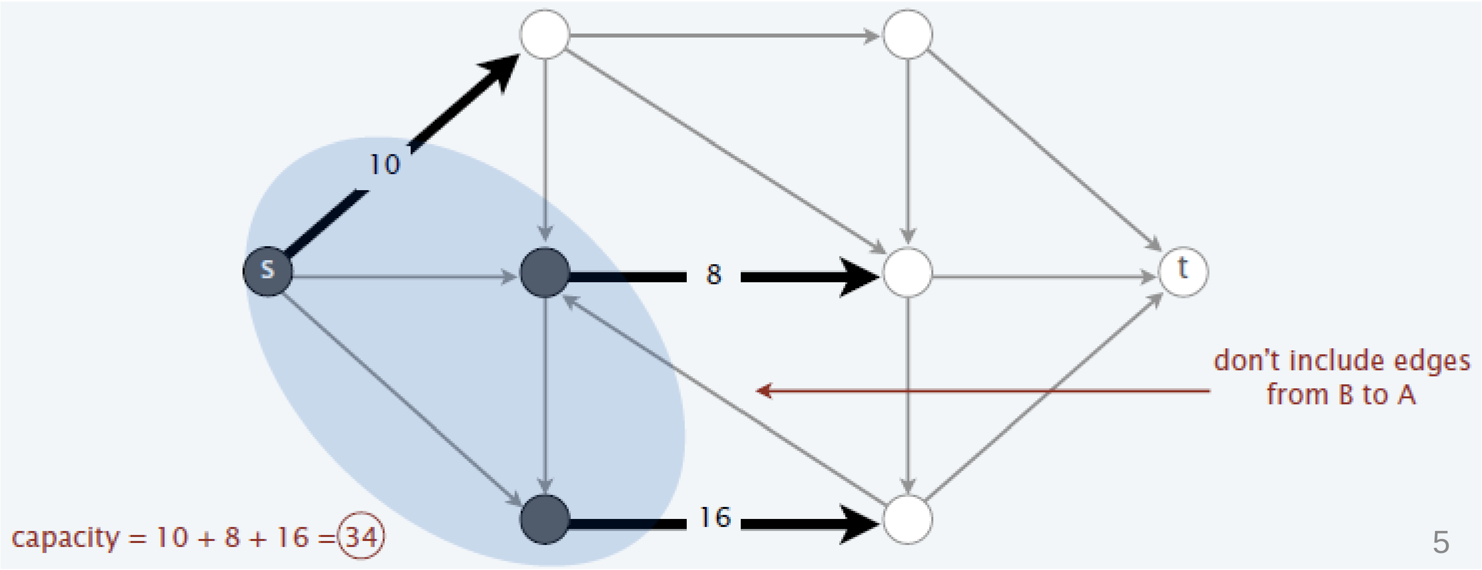

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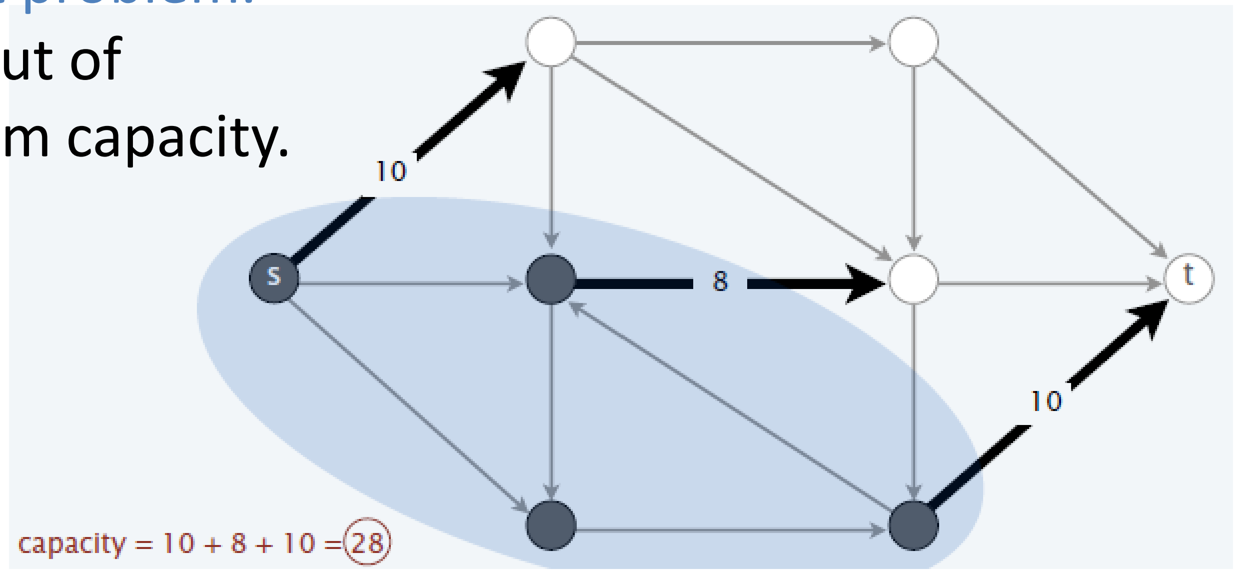
Min-cut problem.

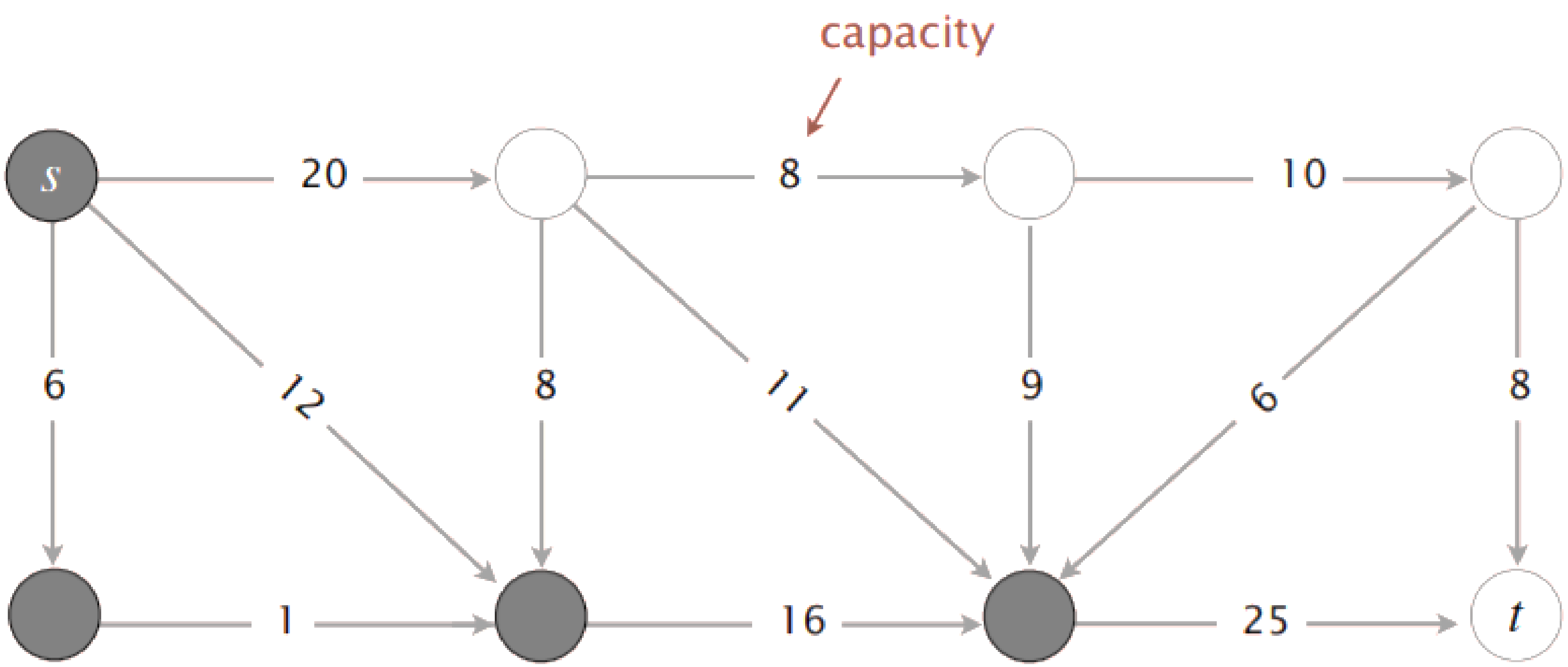
Find a cut of minimum capacity.



Minimum-Cut Problem

What is the capacity of the given *st*-cut?

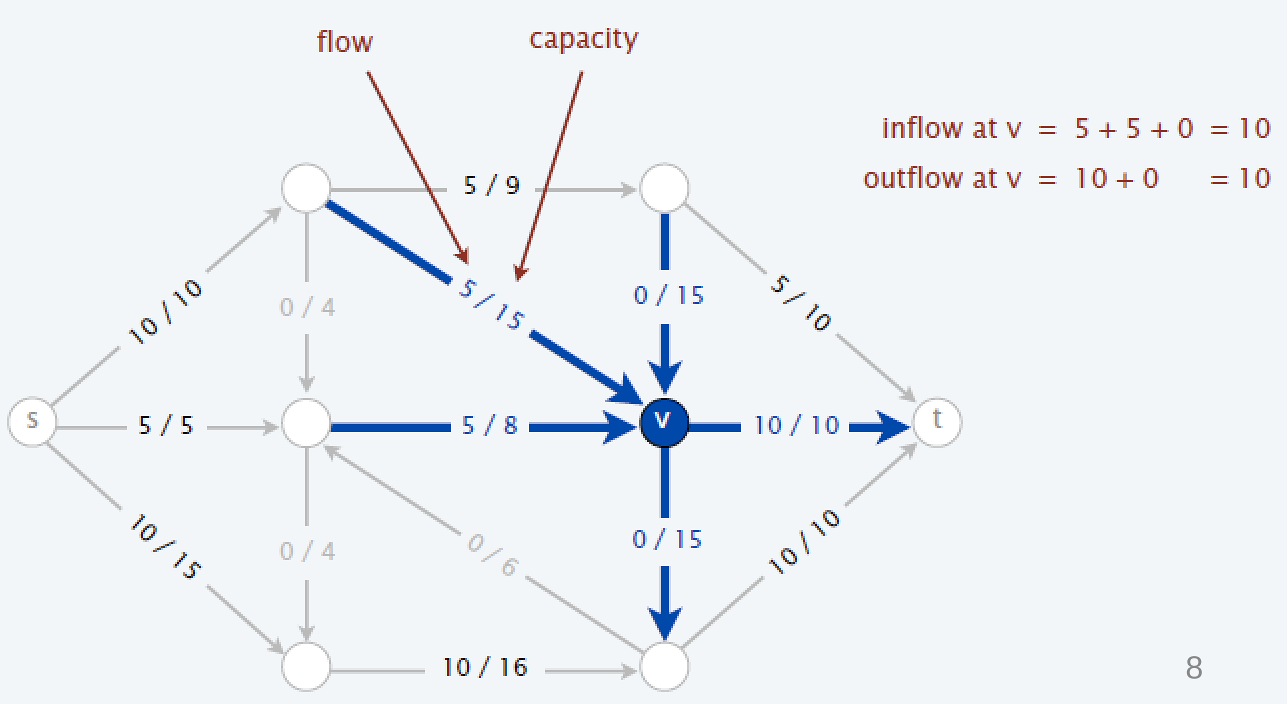
$$cap(A, B) = 45 \; (20 + 25)$$



Maximum-Flow Problem

Def. An st-flow (flow) f is a function that satisfies:

- For each $e \in E$: $0 \le f(e) \le c(e)$ [capacity]
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [flow conservation]



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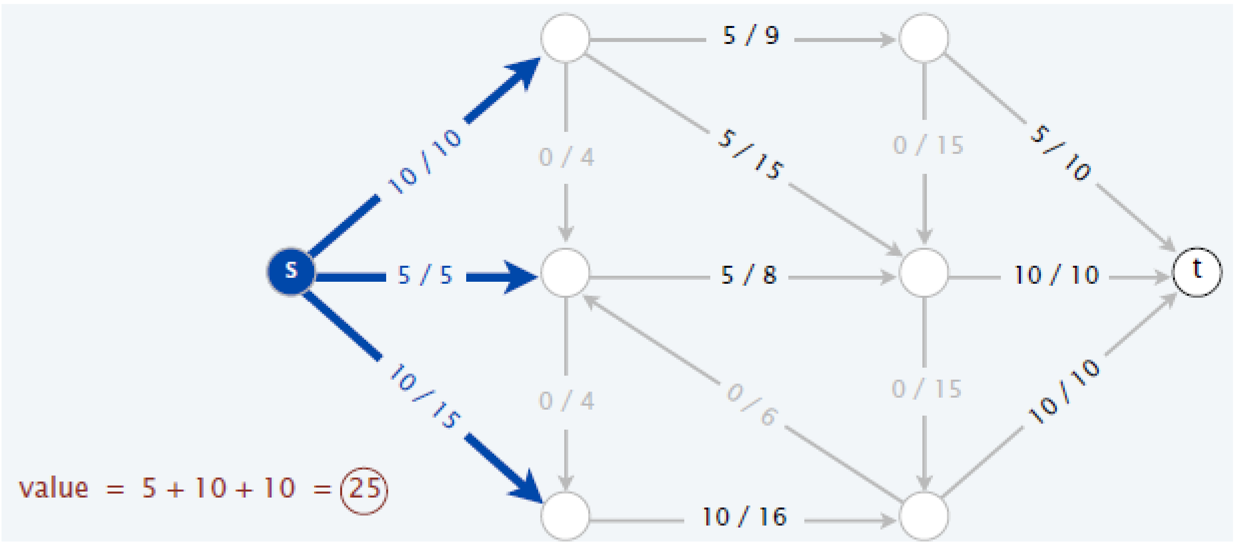

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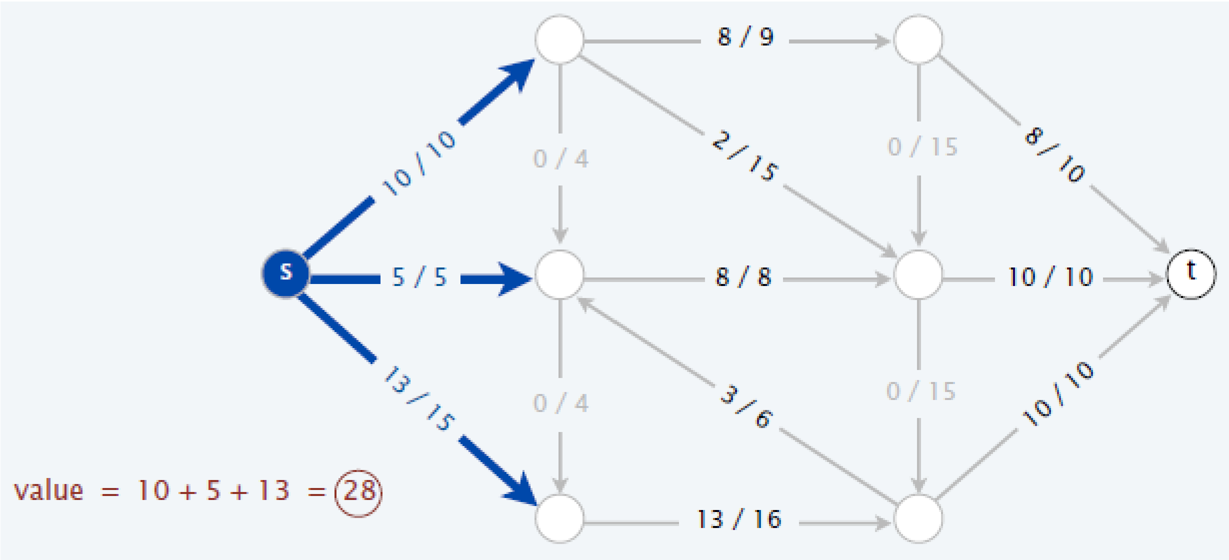
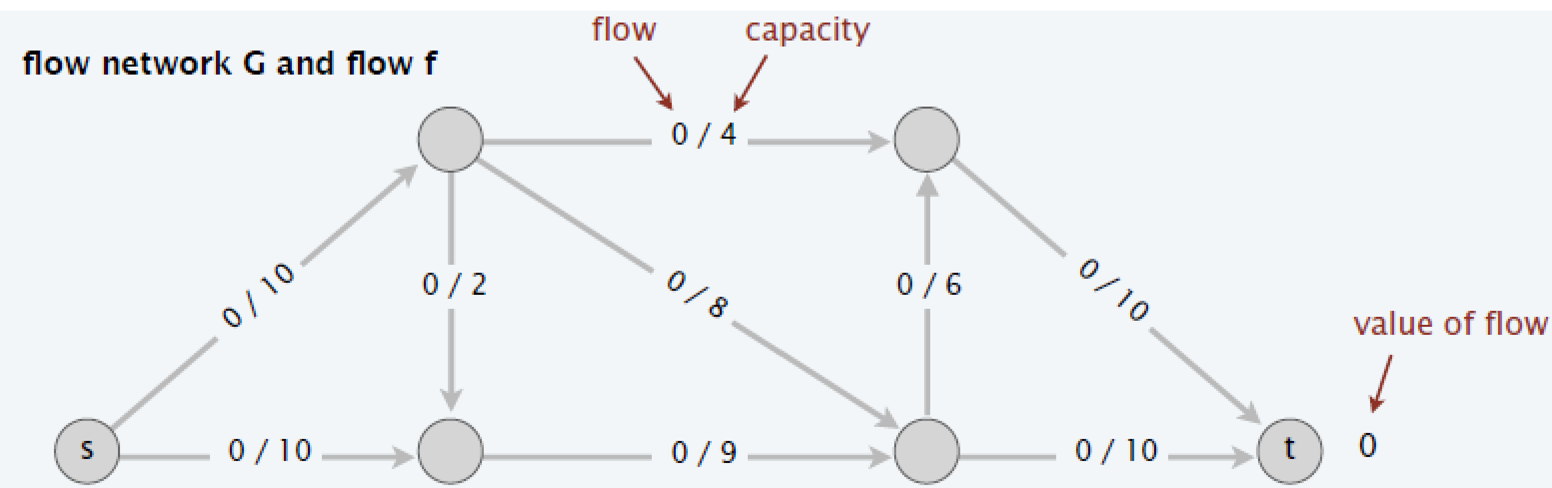
Max-flow problem. Find a flow of maximum value.



Towards a Max-Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \to t$ path P where each edge has $f(e) \leq c(e)$.
- Augment flow along path P.
- Repeat until get stuck.



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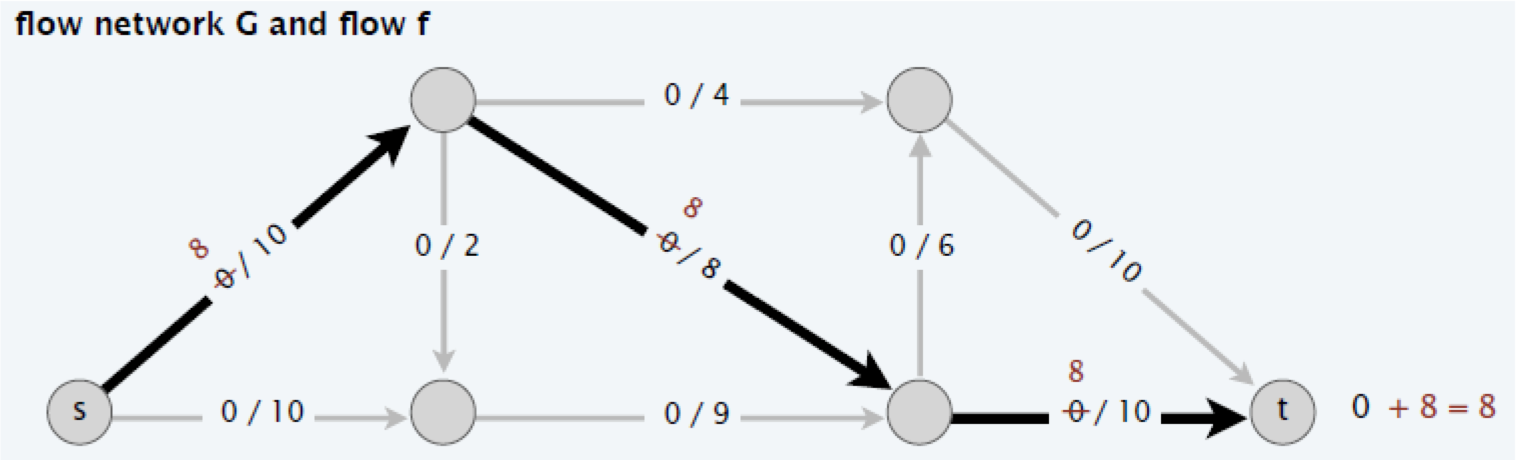

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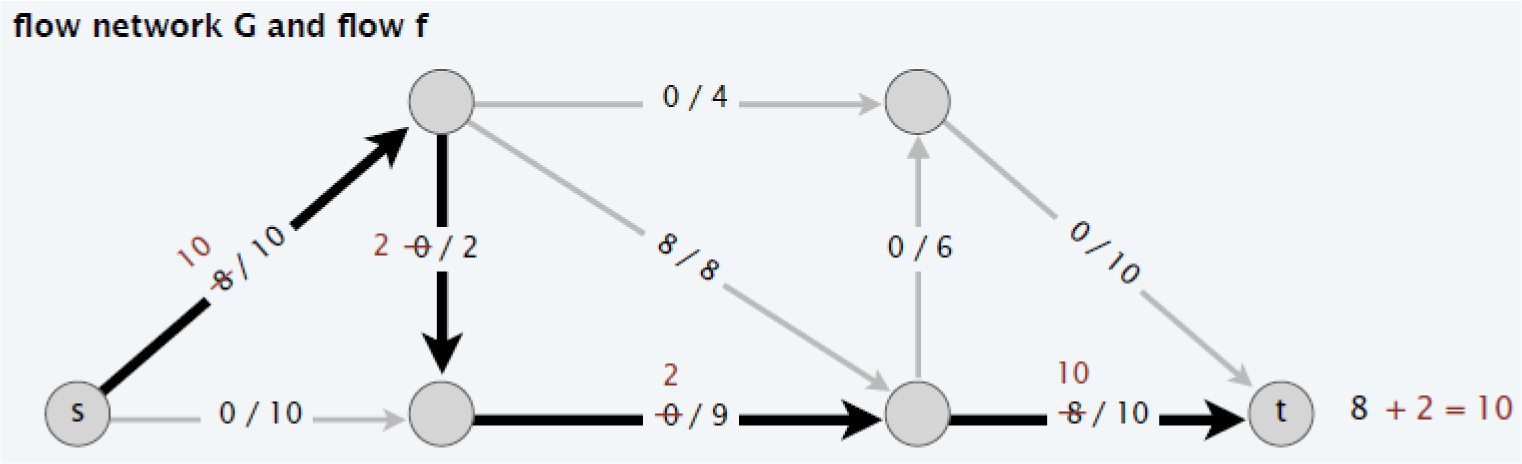

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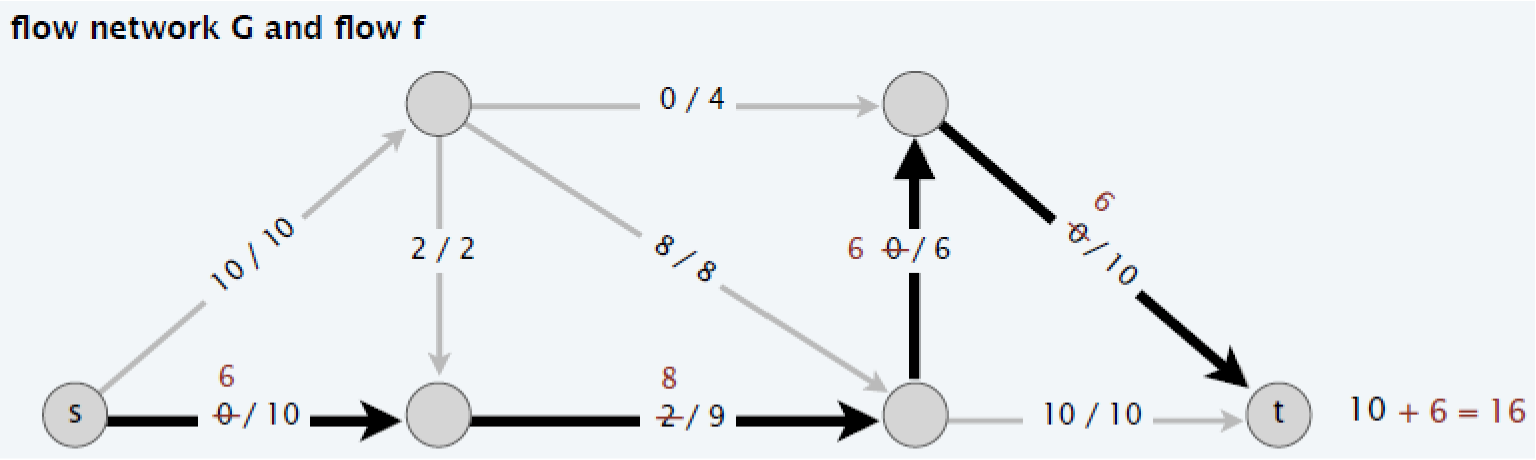

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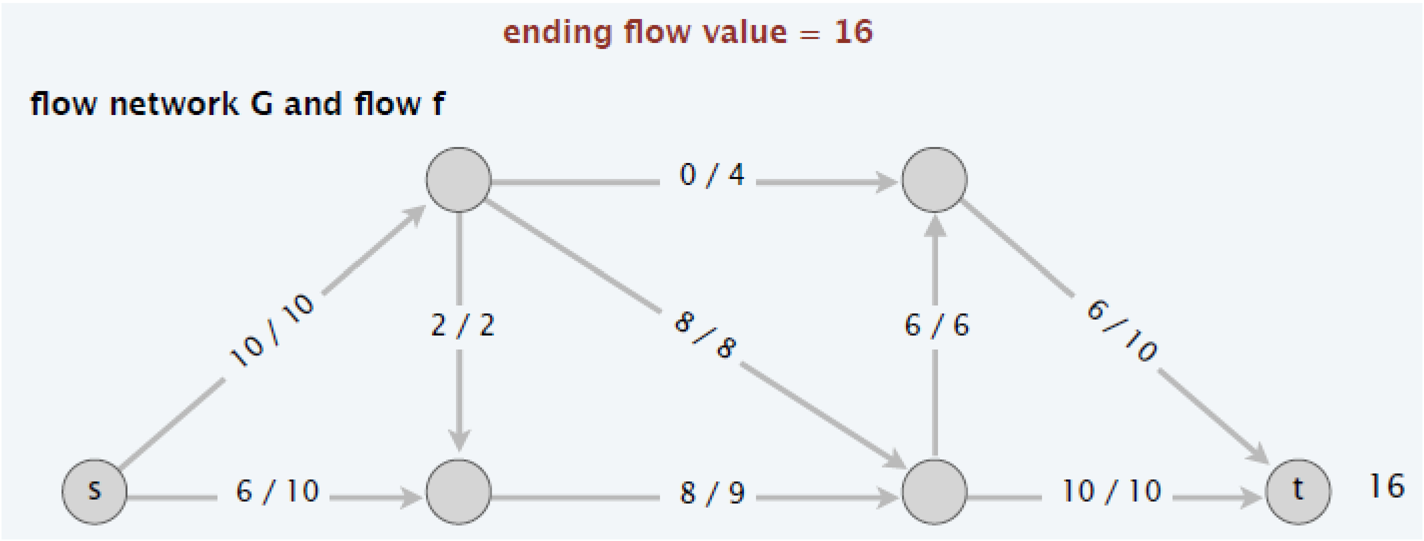

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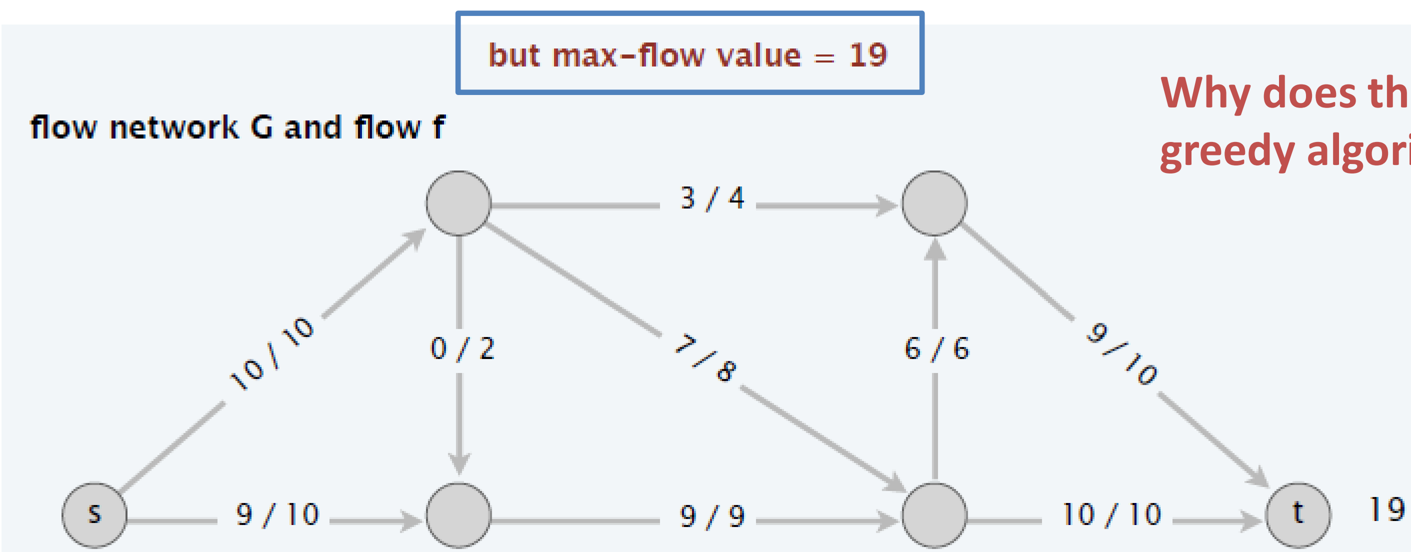


Why does the greedy algorithm fail?

Why the Greedy Algorithm Fails

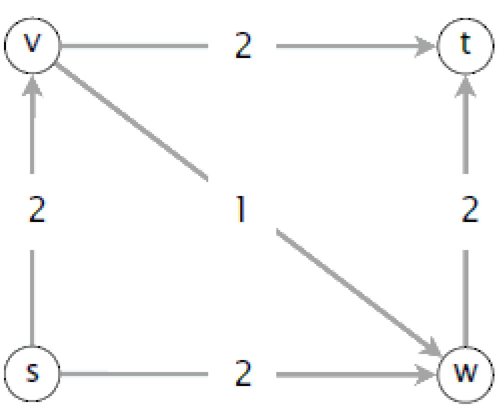
Q. Why does the greedy algorithm fail?

A. Once greedy algorithm increases flow on an edge, it never decrease it.

Ex.

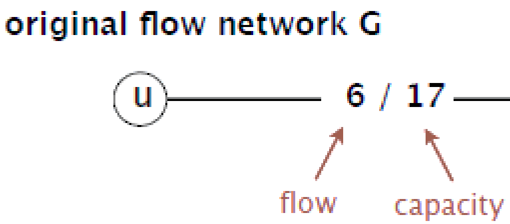
- The max flow is unique; flow on edge （v, w） is zero.
- Greedy algorithm could choose $s \to v \to w \to t$ for first augmenting path.

flow network G



Need some mechanism to “undo” bad decision.

Residual Network

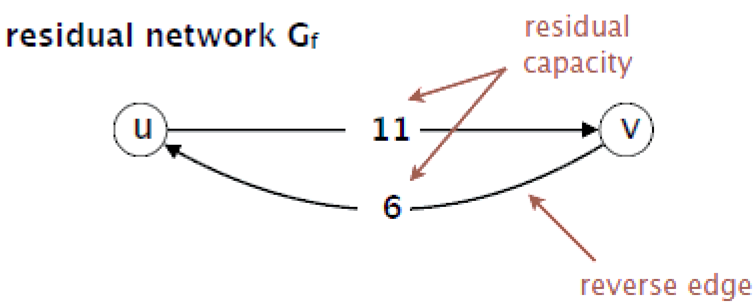


Original edge. $e = (u, v) \in E$.

- Flow $f(e)$.
- Capacity $c(e)$.

Reverse edge. $e^{reverse} = (v, u)$.

- "Undo" flow sent.



Residual capacity.

$$c_f(e) = \begin{cases} c(e) - f(e) & if\ e \in E \\ f(e) & if\ e^{reverse} \in E \end{cases}$$

Residual network. $G_f = (V, E_f, s, t, c_f)$.

- $E_f = \{e: f(e) < c(e)\} \cup \{e^{reverse}: f(e) > 0\}$.

Edges with positive residual capacity

Augmenting Path

Def. An augmenting path is a simple $s \to t$ path in the residual network G_f.

Def. The bottleneck capacity of an augmenting path P is the minimum residual capacity of any edge in P.

Key property. Let f be a flow and let P be an augmenting path in G_f. Then, after call Augment, the resulting f' is a flow and $val(f') = val(f) + bottleneck(G_f, P)$.

Augmenting Path

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Augment (f, c, P)

$b \leftarrow$ bottleneck capacity of path P.

For each edge $e \in P$

  If $(e \in E)$

    $f[e] \leftarrow f[e] + b$.

  Else

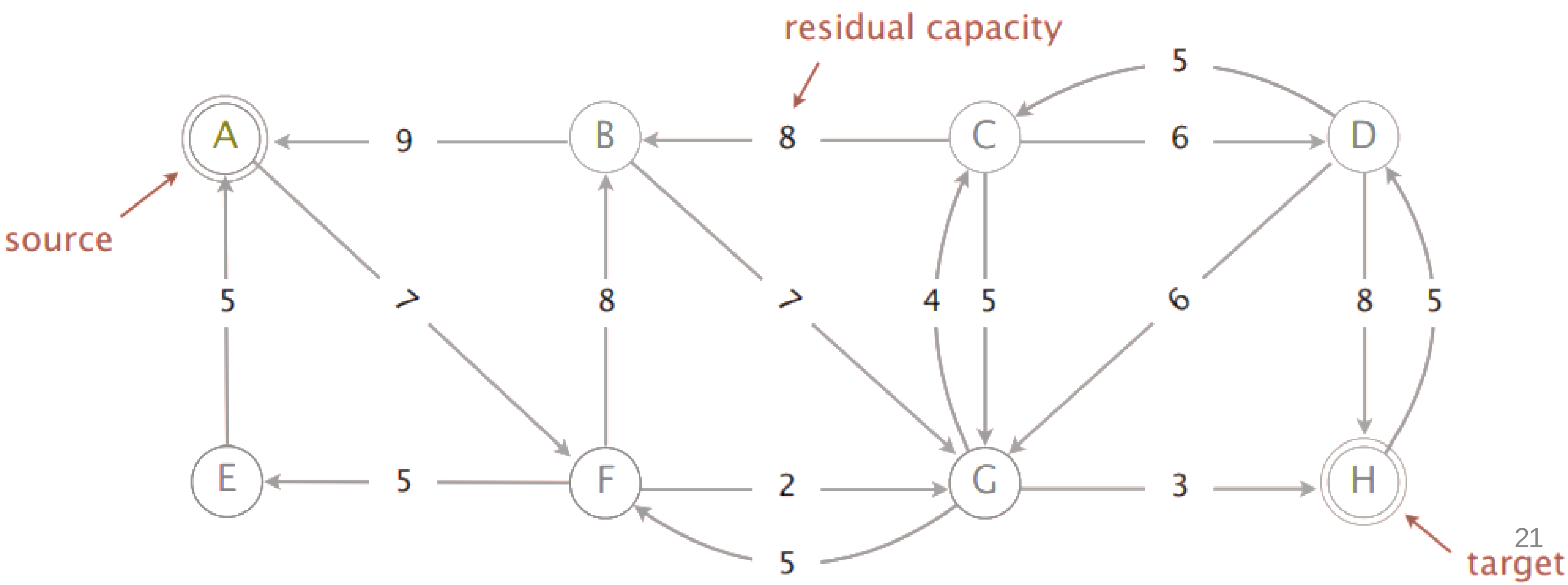
    $f[e^{reverse}] \leftarrow f[e^{reverse}] - b$.

Return f.

Augmenting Path

Which is the augmenting path of highest bottleneck capacity?

A. A → F → G → H
B. A → F → B → G → H
C. A → F → B → G → C → D → H



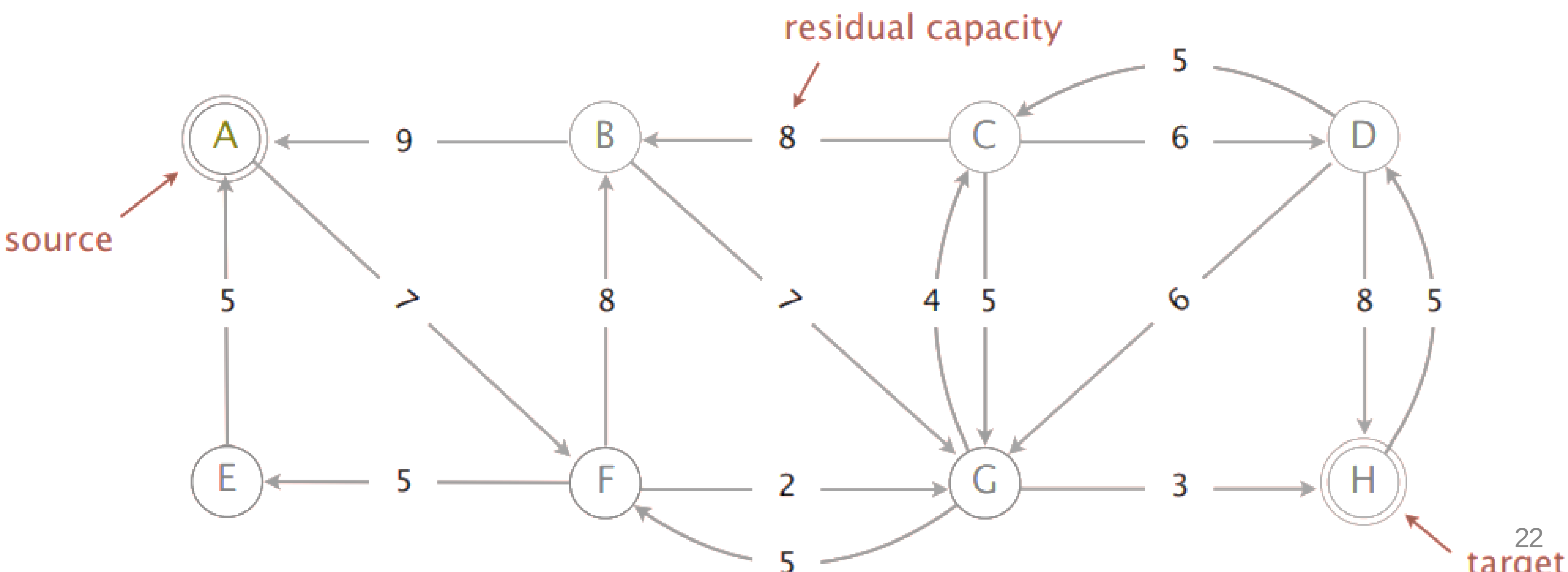
Augmenting Path

Which is the augmenting path of highest bottleneck capacity?

A. A → F → G → H (2)

B. A → F → B → G → H (3)

C. A → F → B → G → C → D → H (4)



Ford-Fulkerson Algorithm

Ford-Fulkerson augmenting path algorithm.

- Start with $f(e) = 0$ for each edge $e \in E$.
- Find an $s \to t$ path P in the residual network G_f.
- Augment flow along path P.
- Repeat until you get stuck.

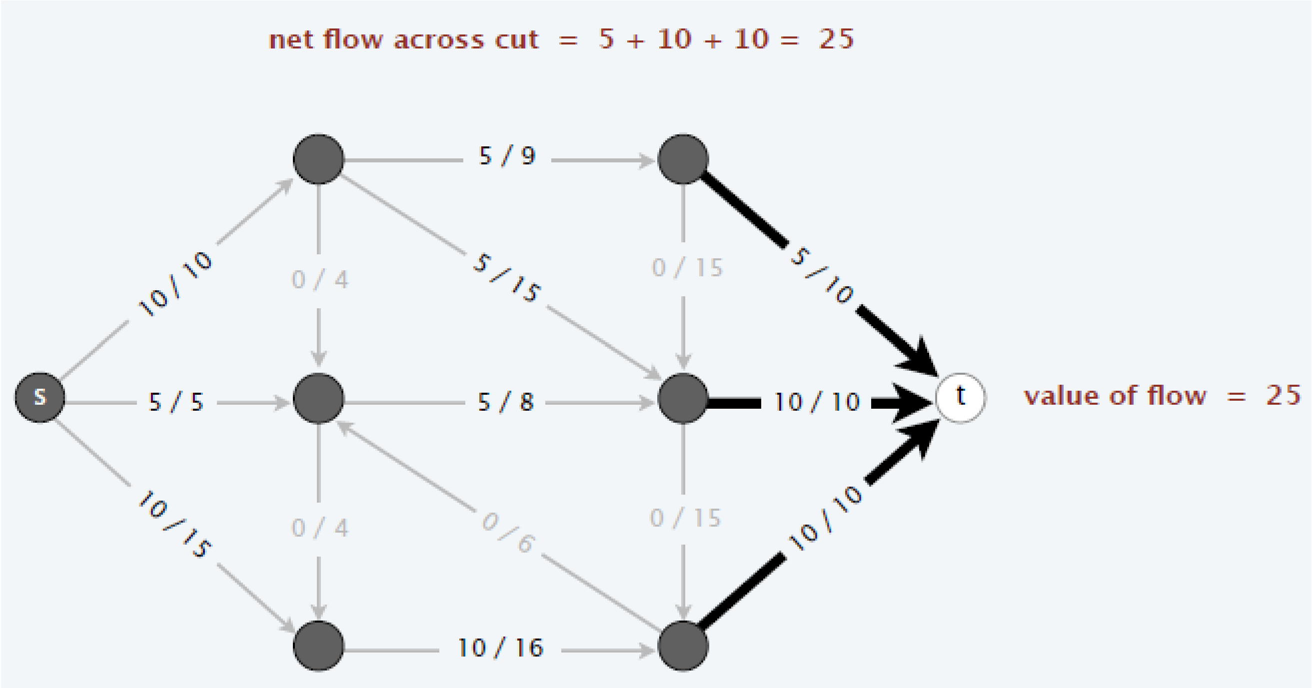
Ford-Fulkerson (G)

For each edge $e \in E$: $f[e] \leftarrow 0$.
$G_f \leftarrow$ residual network of G with respect to f.
While (there exists an $s \to t$ augmenting path P in G_f
  $f \leftarrow$ Augment(f, c, P).
  Update G_f.
Return f.

Relationship between flows and cuts

Flow value lemma. Let f be any flow and let (A, B) be any cut. Then, the value of the flow f equals the net flow across the cut (A, B).

$$val(f) = \sum_{e\ out\ of\ A} f(e) - \sum_{e\ in\ to\ A} f(e)$$



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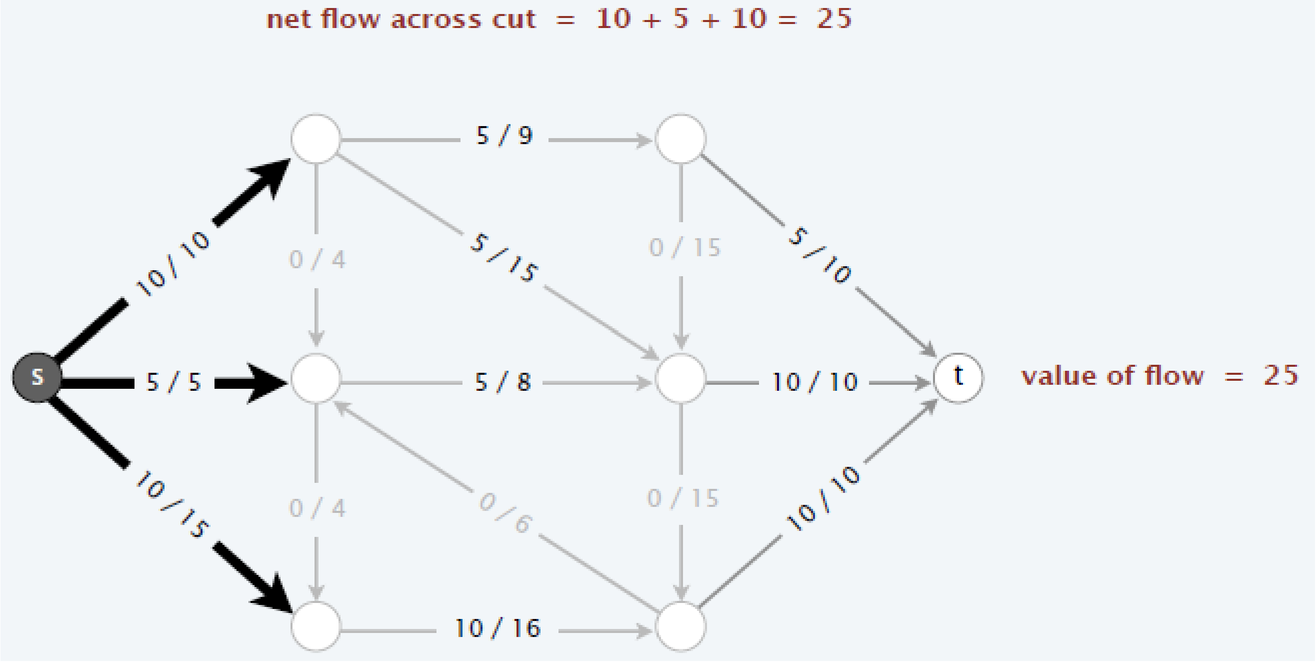

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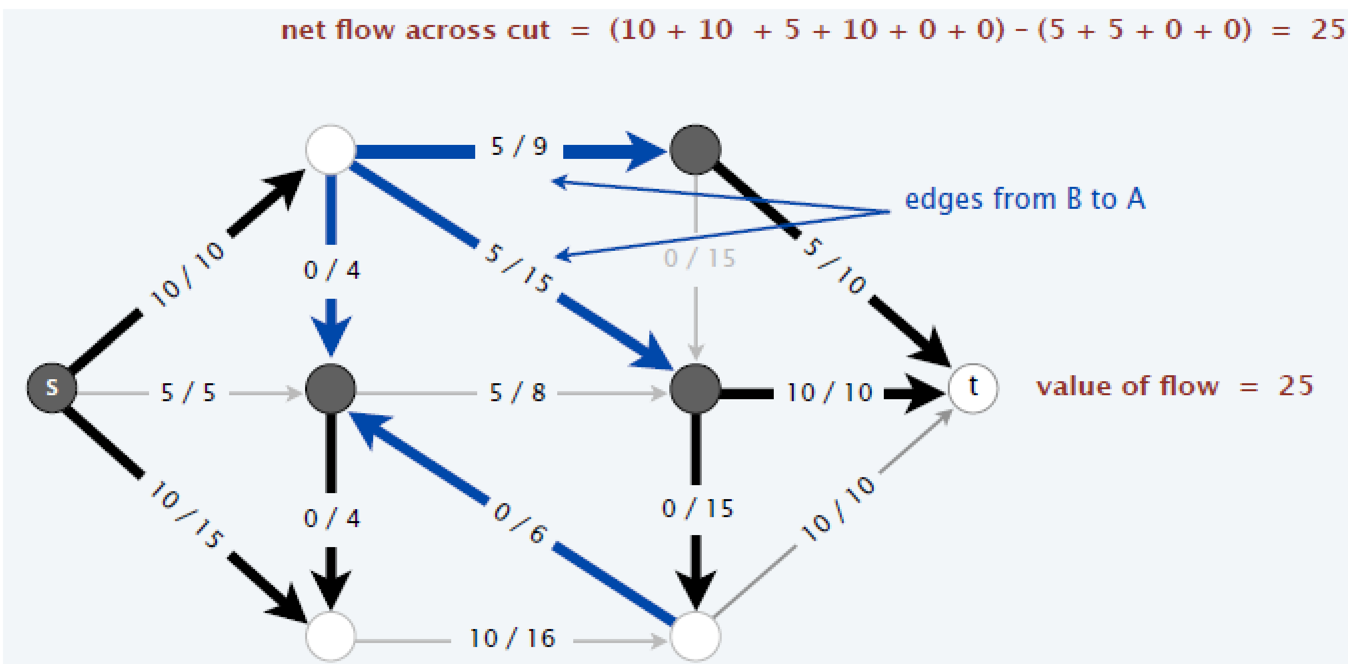
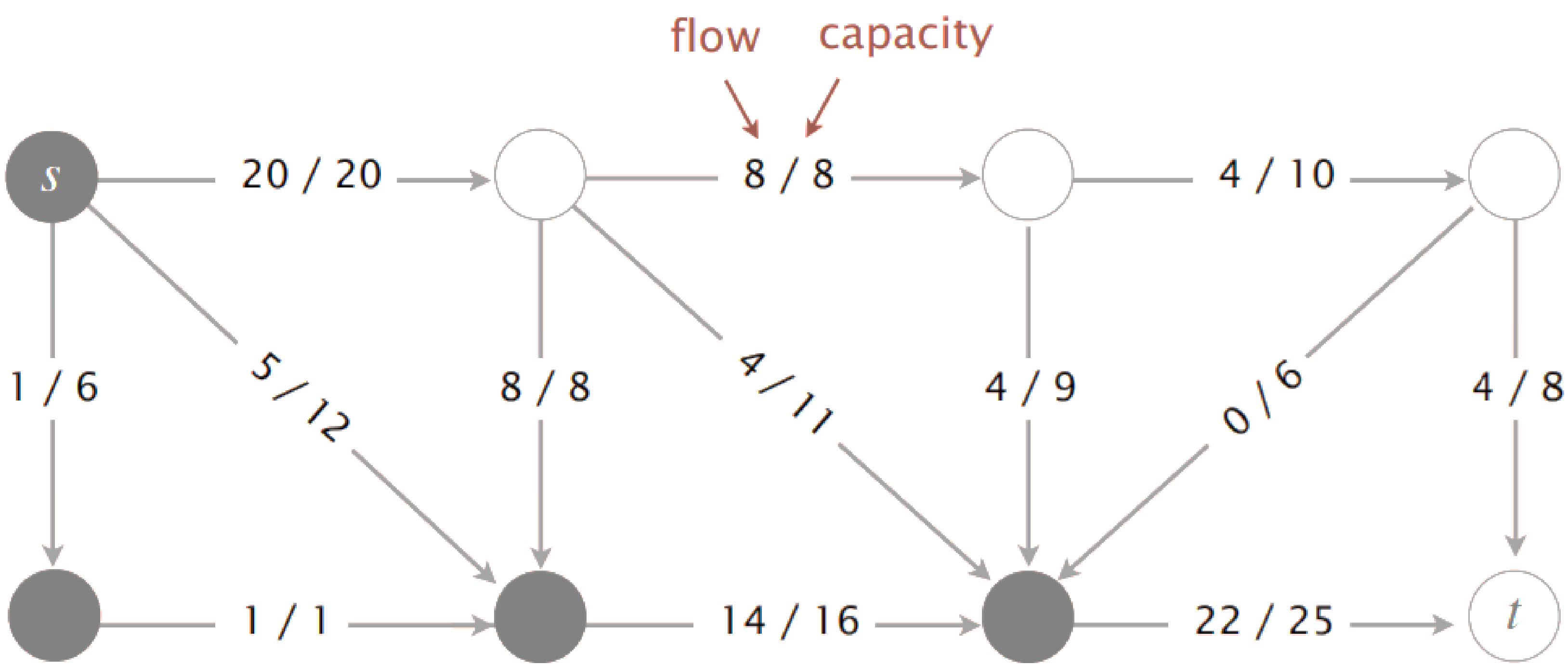


Relationship between flows and cuts

What is the net flow across the given *st*-cut?

$$val(f) = 26\ (20 + 22 - 8 - 4 - 4)$$



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$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

Pf. $$val(f) = \sum_{e \text{ out of } s} f(e) - \sum_{e \text{ in to } s} f(e)$$

$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to } v} f(e) \right)$$

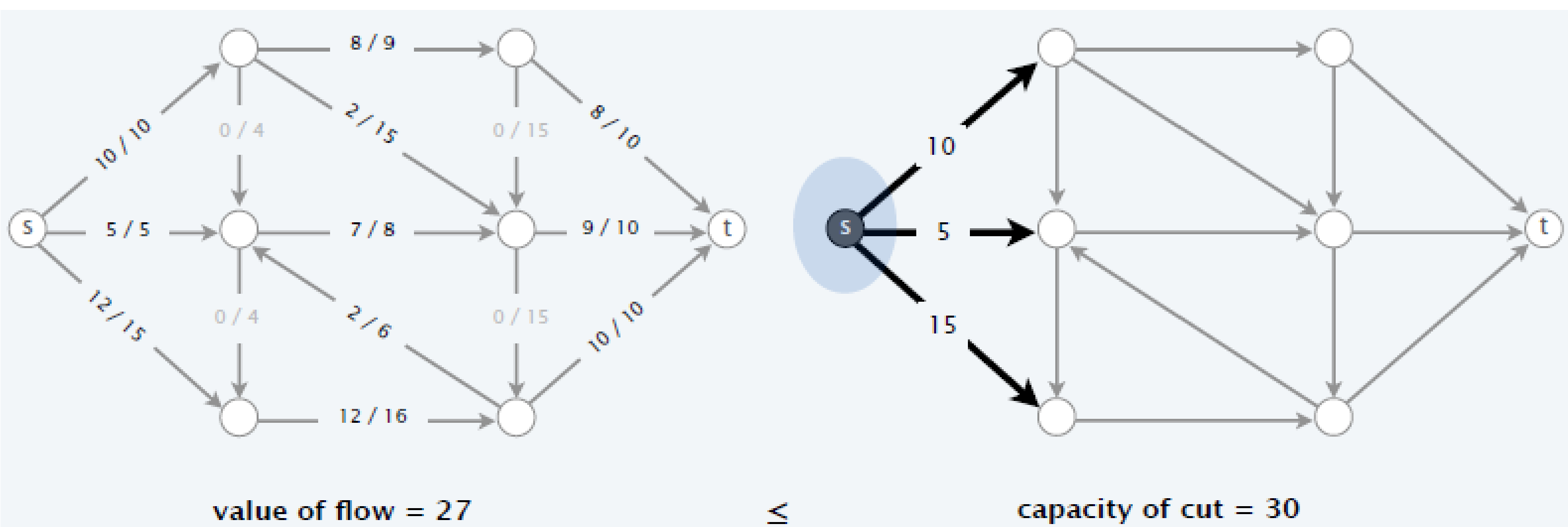
By flow conservation, all terms except for v=s are 0

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

Relationship between flows and cuts

Weak duality. Let f be any flow and (A, B) be any cut.
Then, $val(f) \leq cap(A, B)$.

Pf.
$$
\begin{aligned}
val(f) &= \textstyle\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&\leq \textstyle\sum_{e \text{ out of } A} f(e) \\
&\leq \textstyle\sum_{e \text{ out of } A} c(e) \\
&= cap(A, B)
\end{aligned}
$$

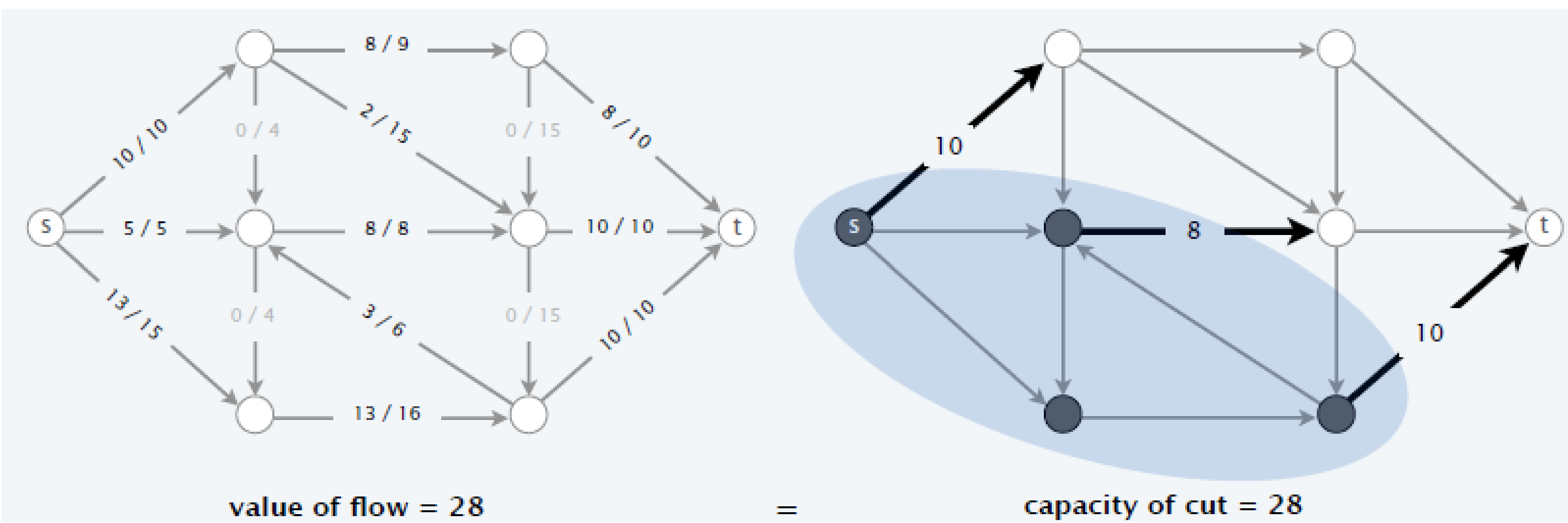


Certificate of Optimality

Corollary. Let f be a flow and let (A, B) be any cut. If $val(f) = cap(A, B)$, then f is a max flow and (A, B) is a min cut.

Pf.

- For any flow f': $val(f') \leq cap(A, B) = val(f)$.
- For any cut (A', B'): $cap(A', B') \geq val(f) = cap(A, B)$.



Max-Flow Min-Cut Theorem

Augmenting path theorem. A flow f is a max flow iff no augmenting paths.
Max-flow min-cut theorem. Value of a max flow = capacity of a min cut.

Pf. The following three conditions are equivalent for any flow f:

I. There exists a cut (A, B) such that $cap(A, B) = val(f)$.
II. f is a max flow.
III. There is no augmenting path with respect to f.

[I $\Longrightarrow$ II]

- Suppose that (A, B) is a cut such that $cap(A, B) = val(f)$.
- Then, for any flow f': $val(f') \leq cap(A, B) = val(f)$.
- Thus, f is a max flow.

Max-Flow Min-Cut Theorem

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I. There exists a cut (A, B) such that $cap(A, B) = val(f)$.

II. f is a max flow.

III. There is no augmenting path with respect to f.

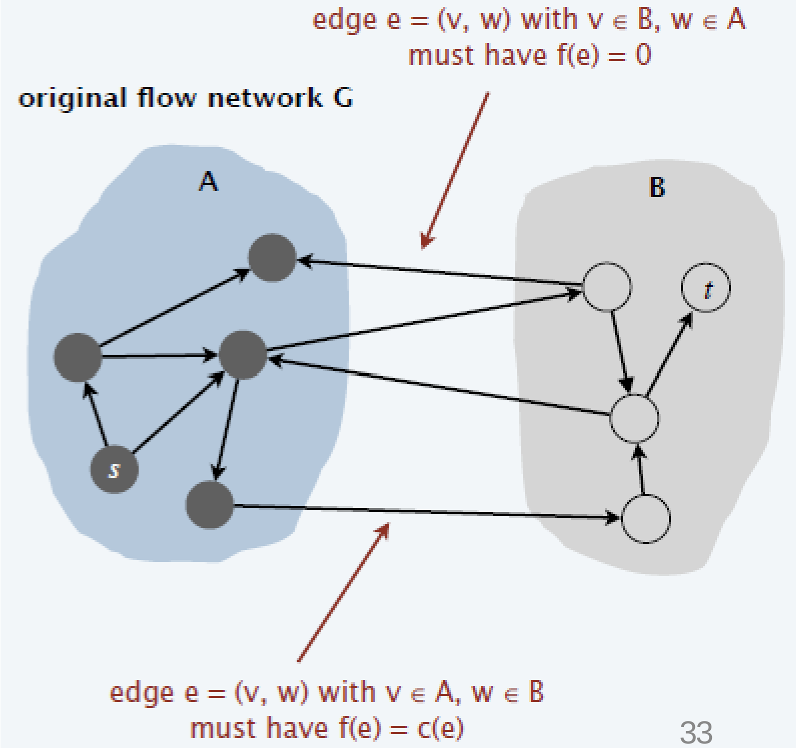
[II $\Longrightarrow$ III] We prove contrapositive: ~III $\Longrightarrow$ ~II.

- Suppose that there is an augmenting path with respect to f.
- Can improve flow f by sending flow along this path.
- Thus, f is not a max flow.

Max-Flow Min-Cut Theorem

[III ⟹ I]

- Let f be a flow with no augmenting paths.
- Let A be set of nodes reachable from s in residual network G_f.
- By definition of cut A: $s \in A$.
- By definition of flow f: $t \notin A$.



$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

$$= \sum_{e \text{ out of } A} c(e)$$

$$= cap(A, B)$$

Analysis of Ford-Fulkerson Algorithm (when capacities are integral)

Assumption. Capacities are integers between 1 and C.

Integrality invariant. Throughout the algorithm, the flows $f(e)$ and the residual capacities $c_f(e)$ are integers.

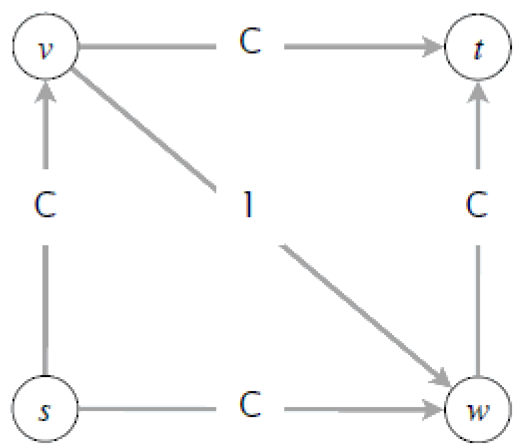
Corollary. The running time of Ford-Fulkerson is $O(mnC)$.
Corollary. If C=1, the running time of Ford-Fulkerson is $O(mn)$.

Bad Case for Ford-Fulkerson

Q. Is generic Ford-Fulkerson algorithm poly-time in input size?
A. No. If max capacity is C, then algorithm can take $\geq C$ iterations.

- $s \to v \to w \to t$
- $s \to w \to v \to t$
- $s \to v \to w \to t$
- $s \to w \to v \to t$
- ...
- $s \to v \to w \to t$
- $s \to w \to v \to t$

each augmenting path
sends only 1 unit of flow
(# augmenting paths = 2C)



Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.

Pathology. If capacities are irrational, algorithm not guaranteed to terminate (or converge to correct answer)!

Goal. Choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

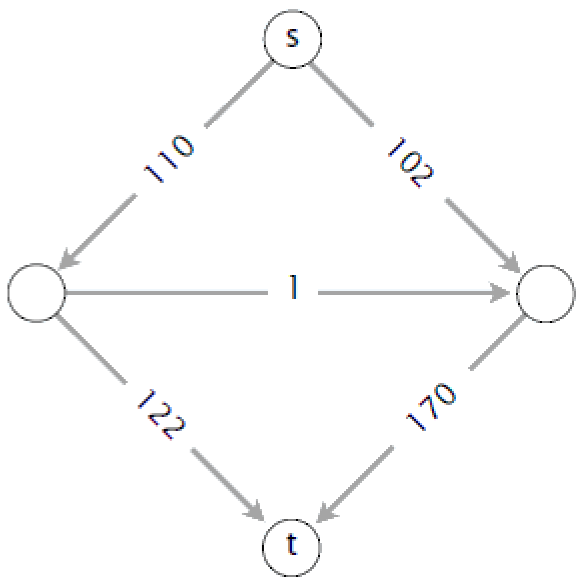
Choose augmenting paths with:

- Max bottleneck capacity (“fattest”).
- Sufficiency large bottleneck capacity.
- Fewest edges.

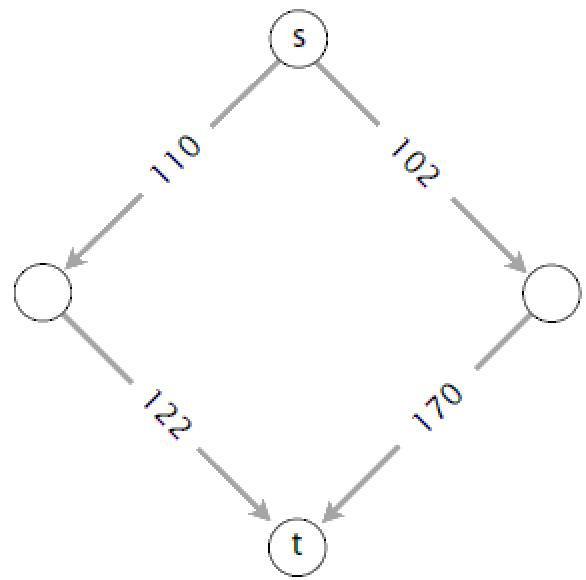
Capacity-Scaling Algorithm

Intuition. Choose augmenting path with highest bottleneck capacity: it increases flow by max possible amount in given iteration.

- Don't worry about finding exact highest bottleneck path.
- Maintain scaling parameter Δ.
- Let $G_f(\Delta)$ be the part of the residual network consisting of only those arcs with capacity $\geq \Delta$.



G_f



$G_f(\Delta)$, $\Delta = 100$

Capacity-Scaling Algorithm

Capacity-Scaling (G)

For each edge $e \in E$: $f[e] \leftarrow 0.$
$\Delta \leftarrow$ largest power of 2 $\leq C$.

While $(\Delta \geq 1)$
  $G_f(\Delta) \leftarrow \Delta$ -residual network of G with respect to flow f.
  While (there exists an $s \rightarrow t$ path P in $G_f(\Delta)$)
    $f \leftarrow$ Augment(f, c, P).
    Update $G_f(\Delta)$.
  $\Delta \leftarrow \Delta/2.$

Return f.

Capacity-Scaling Algorithm: Proof of Correctness

Assumption: All edge capacities are integers between 1 and C.

Integrality invariant. All flows and residual capacities are integral.

Theorem. If capacity-scaling algorithm terminates, then f is a max flow.

Pf.

- By integrality invariant, when $\Delta = 1 \Longrightarrow G_f(\Delta) = G_f$.
- Upon termination of $\Delta = 1$ phase, there are no augmenting paths.

Capacity-Scaling Algorithm: Analysis of Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.

Pf. Initially $\frac{C}{2} < \Delta \leq C$; Δ decreases by a factor of 2 in each iteration.

Lemma 2. Let f be the flow at the end of a Δ-scaling phase.
Then, the max-flow value $\leq val(f) + m\Delta$.

Capacity-Scaling Algorithm: Analysis of Running Time

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then, the max-flow value $\leq val(f) + m\Delta$.

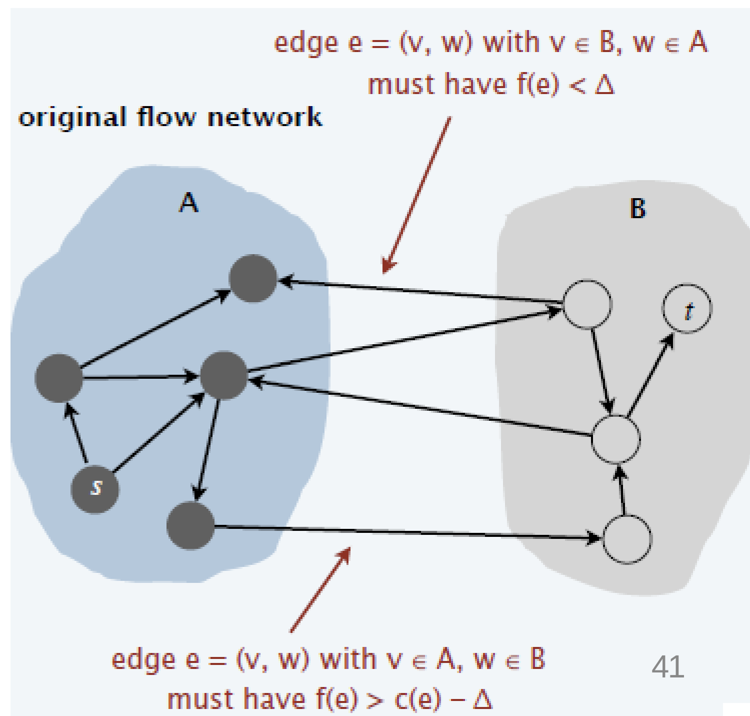
Pf.

- We show there exists a cut (A, B) such that $cap(A, B) \leq val(f) + m\Delta$.
- Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
- By definition of cut A: $s \in A$.
- By definition of flow f: $t \notin A$.

$$val(f) = \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e)$$

$$\geq \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta$$

$$\geq cap(A, B) - m\Delta$$



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Theorem. The scaling max-flow algorithm finds a max flow in $O(mlogC)$ augmentations. It can be implemented to run in $O(m^2 logC)$ time.

Shortest Augmenting Path

Q. Which augmenting path?

A. The one with the fewest edges (can find via BFS).

Shortest-Augmenting-Path (G)

For each edge $e \in E$: $f[e] \leftarrow 0$.

$G_f \leftarrow$ residual network of G with respect to flow f.

While (there exists an $s \rightarrow t$ path in G_f)

$P \leftarrow$ Breadth-First-Search (G_f).

$f \leftarrow$ Augment (f, c, P)

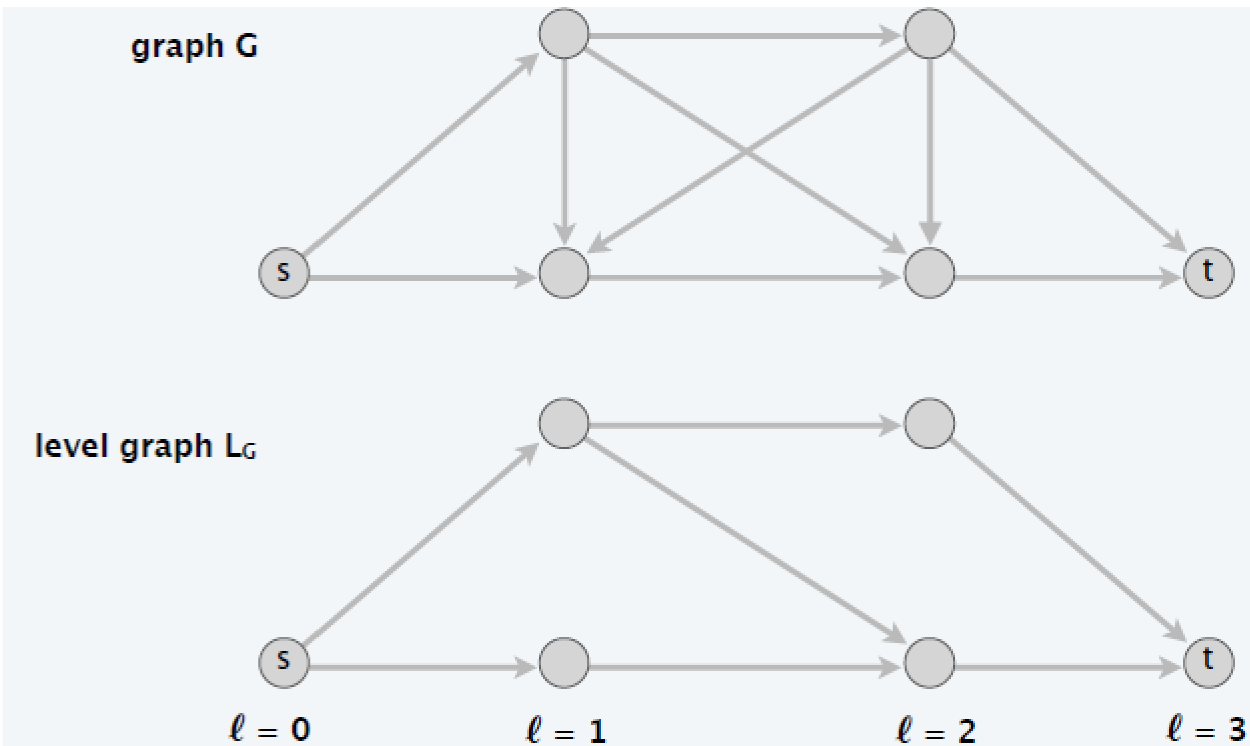
Update G_f.

Return f.

Shortest Augmenting Path: Analysis

Def. Given a digraph $G = (V, E)$ with source s, its **level graph** is defined by:

- $l(v) =$ number of edges in shortest path from s to v.
- $L_G = (V, E_G)$ is the subgraph of G that contains only those edge $(v, w) \in E$ with $l(w) = l(v) + 1$.



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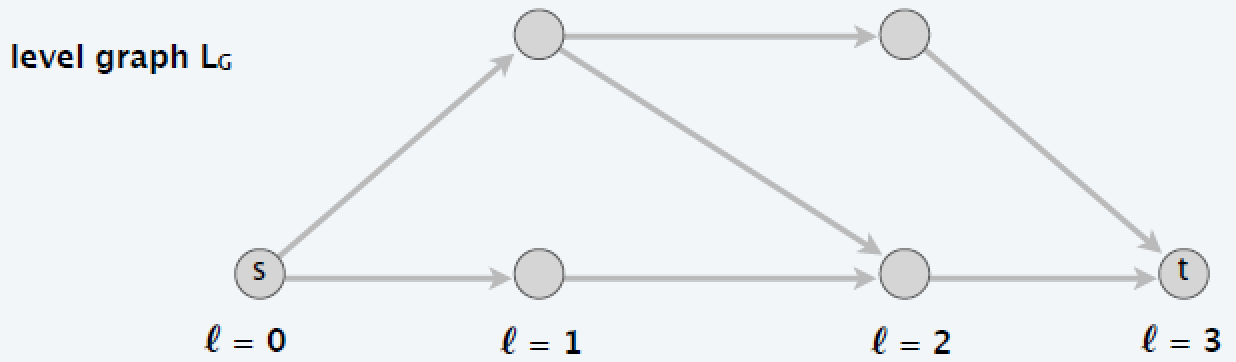
Key property. P is a shortest path $s \to v$ path in G iff P is an $s \to v$ path in L_G.



Shortest Augmenting Path: Analysis

Lemma 1. The length of a shortest augmenting path never decreases.

- Let f and f' be flow before and after a shortest-path augmentation.
- Let L and L' be level graphs of G_f and $G_{f'}$.
- Only back edges added to $G_{f'}$.

(any path with a back edge is longer than previous length)