习题：2.4

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习题2.4. A. P1.

1. (2). 不一定. 如 $1<1+\frac{1}{n}$. 但 $\lim\limits_{n\to\infty}1=1=\lim\limits_{n\to\infty}(1+\frac{1}{n})$ #.

(4). 不可以. $\lim\limits_{n\to\infty}z_n$ 不一定存在. 如 $z_n=(-1)^n,\ n\in N$.

故x_n 要同时给出 $\lim\limits_{n\to\infty}z_n$ 存在性的证明.

3. 设 $a>0$. $b>0$. 证明: $\lim\limits_{n\to\infty}\sqrt[n]{a^n+b^n}=\max(a,b)=L$.

Pf: $\sqrt[n]{L^n}=L\le\sqrt[n]{a^n+b^n}\le\sqrt[n]{2L^n}$

$\Rightarrow L\le\sqrt[n]{a^n+b^n}\le L\sqrt[n]{2}$. 又 $\lim\limits_{n\to\infty}\sqrt[n]{2}=1$.

$\therefore \lim\limits_{n\to\infty}\sqrt[n]{a^n+b^n}=1$ #.

4. 求 $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)$.

解: $\because \frac{n}{\sqrt{n^2+n}}\le\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\le\frac{n}{\sqrt{n^2}}=1$.

又 $\lim\limits_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=1$

$\therefore \lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)=1$ #.

5. 求下列极限.

(1). $\lim\limits_{x\to0}\frac{\sin x}{2x}=\frac{1}{2}$. (2). $\lim\limits_{x\to0}\frac{\sin 2x}{2x}\cdot2=2$.

(3) $\lim\limits_{x\to0}\frac{\sin 3x}{5x}=\lim\limits_{x\to0}\frac{\sin 3x}{3x}\cdot\frac{3}{5}=\frac{3}{5}$

(4). $\lim\frac{2x}{\sin 3x}=\lim\frac{3x}{\sin 3x}\cdot\frac{2}{3}=\frac{2}{3}$

(4). $\lim\limits_{x\to 0}\frac{2x}{\sin 3x}=\lim\limits_{x\to 0}\frac{3x}{\sin 3x}\cdot\frac{2}{3}=\frac{2}{3}$

(5). $\lim\limits_{\theta\to 0}\frac{\sin^2\theta}{\theta}=\lim\limits_{\theta\to 0}\frac{\sin^2\theta}{\theta^2}\cdot\theta=0.$

(6). $\lim\limits_{h\to 0}\frac{\tan^2 h}{h}=\lim\limits_{h\to 0}\frac{\sin^2 h}{h}\cdot\frac{1}{\cos^2 h}=0.$

习题 2.4. A. P2.

(7). $\lim\limits_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}=\lim\limits_{\theta\to 0}\frac{2\sin^2\frac{\theta}{2}}{\theta^2}=\lim\limits_{\theta\to 0}\frac{\sin^2\frac{\theta}{2}}{2(\frac{\theta}{2})^2}=\frac{1}{2}.$

(8). $\lim\limits_{x\to 0}\frac{\sin^2 x}{x^2}=1.$

(9). $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}=\lim\limits_{x\to 0}\frac{\frac{1}{\cos x}-1}{\sin^2 x}=\lim\limits_{x\to 0}\frac{1-\cos x}{\cos x(1-\cos^2 x)}$

$=\lim\limits_{x\to 0}\frac{1}{\cos x(1+\cos x)}=\frac{1}{2}.$

(10). $\lim\limits_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}=\lim\limits_{x\to 0}\frac{2\cos 4x\cdot\sin x}{\sin x}=\lim\limits_{x\to 0}2\cos 4x=2.$

(11). $\lim\limits_{x\to 0}\frac{\cos x-\cos 3x}{x^2}=\lim\limits_{x\to 0}\frac{\cos x-[\cos x\cos 2x-\sin x\sin 2x]}{x^2}$

$=\lim\limits_{x\to 0}\frac{\cos x-\cos x(1-4\sin^2 x)}{x^2}=4\lim\limits_{x\to 0}\frac{\sin^2 x}{x^2}=4.$

(12). $\lim\limits_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim\limits_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}$

$=\lim\limits_{x\to a}\frac{\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{\frac{x-a}{2}}=\cos a.$

(13). $\lim\ \frac{\cos x-\cos a}{\ }=\lim\ \frac{-2\sin\frac{x+a}{2}\sin\frac{x-a}{2}}{\ }$

(3). $\lim\limits_{x\to a}\frac{\cos x-\cos a}{x-a}=\lim\limits_{x\to a}\frac{-2\sin\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}$

$=-\lim\limits_{x\to a}\sin\frac{x+a}{2}\lim\limits_{x\to a}\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}=-\sin a.$

(4). $\lim\limits_{x\to a}\frac{\tan x-\tan a}{x-a}=\lim\limits_{x\to a}\frac{\sin x\cos a-\sin a\cos x}{\cos x\cos a\,(x-a)}$

$=\lim\limits_{x\to a}\frac{\sin(x-a)}{x-a}\lim\limits_{x\to a}\frac{1}{\cos a\cos x}$

$=\frac{1}{\cos^2 a}=\sec^2 a.\quad a\neq(2k+1)\frac{\pi}{2},\ k=0,\ \pm1,\ \pm2,\cdots$

习题2.4. A. P3.

6. 求极限.

(1). $\lim\limits_{x\to\infty}(1+\frac{3}{x})^{\frac{x}{2}}=\lim\limits_{x\to\infty}[(1+\frac{3}{x})^{\frac{x}{3}}]^{\frac{3}{2}}=e^{\frac{3}{2}}.$

(2). $\lim\limits_{x\to0}(1-x)^{\frac{1}{x}}\overset{t=-x}{=\!=\!=}\lim\limits_{t\to0}(1+t)^{-\frac{1}{t}}=\lim\limits_{t\to0}[(1+t)^{\frac{1}{t}}]^{-1}=e^{-1}$

(3). $\lim\limits_{\Delta x\to0}(1+\frac{\Delta x}{x})^{\frac{x}{\Delta x}}=e.$

(4). $\lim\limits_{x\to0}(1+\alpha x)^{\frac{1}{x}}=\lim\limits_{x\to0}[(1+\alpha x)^{\frac{1}{\alpha x}}]^{\alpha}=e^{\alpha}$

(5). $\lim\limits_{x\to\infty}(1-\frac{1}{x})^{4x}\overset{t=-x}{=\!=\!=}\lim\limits_{x\to\infty}(1+\frac{1}{t})^{-4t}=e^{-4}.$

B

1. 2. 略.

3. 证明 $\lim\frac{2^n}{\ }=0$

3. 证明 $\lim\limits_{n\to\infty}\frac{2^n}{n!}=0$.

Pf: $\because 0<\frac{2\cdot 2\cdot 2\cdot 2\cdots 2}{1\cdot 2\cdot 3\cdot 4\cdots n}<\frac{4}{n}\to 0$

$\therefore \lim\limits_{n\to\infty}\frac{2^n}{n!}=0$.

4. 求极限.

(1). $\lim\limits_{x\to\infty}\frac{3x-5}{x^2\sin\frac{1}{x^2}}=\lim\limits_{x\to\infty}\frac{3x-5}{x}\lim\limits_{x\to\infty}\frac{\frac{1}{x^2}}{\sin\frac{1}{x^2}}=3$

~~(2). $\lim\limits_{x\to 0}(1-x)^{\frac{1}{x}} \overset{t=-x}{=\!=\!=} \lim\limits_{t\to 0}(1+t)^{-\frac{1}{t}}=e^{-1}$~~

(2). $\lim\limits_{x\to\frac{\pi}{6}}\sin(\frac{\pi}{6}-x)\tan 3x$

令 $\frac{\pi}{6}-x=t$. 则 $x\to\frac{\pi}{6}\Leftrightarrow t\to 0$.

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$\because \sin(\frac{\pi}{6}-x)\tan 3x=\sin t\tan(\frac{\pi}{2}-3t)=\sin t\,\text{ctg}\,3t$

$\therefore \lim\limits_{x\to\frac{\pi}{6}}\sin(\frac{\pi}{6}-x)\tan 3x=\lim\limits_{t\to 0}\sin t\cot 3t$

$=\lim\limits_{t\to 0}\sin t\cdot\cos 3t\cdot\frac{1}{\sin 3t}=\lim\limits_{t\to 0}\frac{\sin t}{t}\cdot\frac{3t}{\sin 3t}\cdot\frac{\cos 3t}{3}$

$=\frac{1}{3}$.

(4). $\lim\limits_{x\to\frac{\pi}{4}}\frac{\tan x-1}{x-\frac{\pi}{4}} \overset{x-\frac{\pi}{4}=t}{=\!=\!=} \lim \tan(t+\frac{\pi}{\ldots}$

(4). $\lim\limits_{x\to\frac{\pi}{4}} \frac{\tan x-1}{x-\frac{\pi}{4}} \xlongequal{x-\frac{\pi}{4}=t} \lim\limits_{t\to 0}\frac{\tan(t+\frac{\pi}{4})-1}{t}$

$=\lim\limits_{t\to 0}\frac{1}{t}\left[\frac{1+tgt}{1-tgt}-1\right]=\lim\limits_{t\to 0}\frac{1}{t}\cdot\frac{2\tan t}{1-tgt}=2\lim\limits_{t\to 0}\frac{\tan t}{t}\lim\limits_{t\to 0}\frac{1}{1-tgt}$

$=2.$

(5). $\lim\limits_{x\to 1} x^{\frac{1}{1-x}}=\lim\limits_{x\to 1}[1+(x-1)]^{-\frac{1}{x-1}}=\lim\limits_{x\to 1}[1+(x-1)]^{\frac{-1}{x-1}}=\frac{1}{e}.$

5. 设 $\lim\limits_{x\to a}f(x)=A$. $\lim\limits_{x\to a}g(x)=B$.

(1). 问若在 $\mathring{U}_0(a,\delta)$ 内有 $f(x)<g(x)$. 是否一定有 $A<B$.?

(2). 证明若 $A>B$, 则存在空心邻域 $\mathring{U}_0(a,\delta)$. s.t 当 $x\in \mathring{U}_0(a,\delta)$ 时. 有 $f(x)>g(x)$.

(1). 解: 不一定. 反例: $f(x)=x^2$. $g(x)=|x|$.

则有 $\mathring{U}_0(0.\delta)$. s.t. $f(x)<g(x)$. 但

$\lim\limits_{x\to 0}f(x)=0=\lim\limits_{x\to 0}g(x)$.

(2). Pf: $\because A>B$ $\therefore$ 令 $\varepsilon_0=\frac{A-B}{2}>0$. 则由定义

$\exists\delta$. s.t $0<|x-a|<\delta$ 时有.

$|f(x)-A|<\varepsilon_0$. $|g(x)-B|<\varepsilon_0$

$\Rightarrow A-\varepsilon_0<f(x)<A+\varepsilon_0$ $\quad B-\varepsilon_0<g(x)<B+\varepsilon_0$

$\Rightarrow A-\frac{A-B}{2}<f(x)<A+\frac{A-B}{2}$ $\quad B-\frac{A-B}{2}<g(x)<B+\frac{A-B}{2}$

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$\therefore \frac{A+B}{2}<f(x)$

P_4.

$\therefore \frac{A+B}{2} < f(x)$, $g(x) < \frac{A+B}{2}$

$\Rightarrow \quad f(x) > \frac{A+B}{2} > g(x)$. #.

6. 试就$a=+\infty$的情形叙述并证明Th2.4.4.

定理：设 $\lim\limits_{x\to+\infty} f(x)$, $\lim\limits_{x\to+\infty} g(x)$ 存在. 且有某个 X_0 . s.t $x > X_0$时. $f(x) \le g(x)$ 成立.

则 $\lim\limits_{x\to+\infty} f(x) \le \lim\limits_{x\to a} g(x)$. #.