Written Test – February 6, 2023
Numerical Methods for Partial Differential Equations
max 26 pt (over 30) – duration 1h 30'

Students entitled to take the test reduced by 30% according to Law 170/2010 (**Multichance** team indications) DO NOT complete the questions marked with **(***)**

Answer to questions on the paper (handwritten). Upload files on WeBeeP.

Exercise 1 (15 pt)

Let us consider the domain $\Omega = (0,1)^2$, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N = \cup_{i=0}^{3}\Gamma_i$, where $\Gamma_D = \Gamma_2 \cup \Gamma_3$ and $\Gamma_N = \Gamma_0 \cup \Gamma_1 = \partial\Omega \backslash \Gamma_D$. In particular, $\Gamma_0 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega \, : \, x_1 = 0\}$, $\Gamma_1 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega \, : \, x_1 = 1\}$, $\Gamma_2 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega \, : \, x_2 = 0\}$, and $\Gamma_3 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega \, : \, x_2 = 1\}$; $\mathbf{n}$ indicates the unit vector normal to $\partial\Omega$ and outward directed. See Fig. 1.

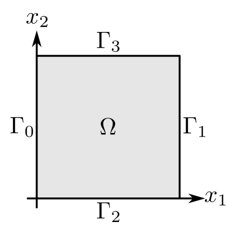

Figure 1: Domain Ω and boundary $\partial\Omega = \cup_{i=0}^{3}\Gamma_i$. Each boundary subset Γ_i corresponds to the tag i in the mesh files

Let us consider the following strong problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\mu\Delta u + \mathbf{b} \cdot \nabla u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \mu\nabla u \cdot \mathbf{n} = q & \text{on } \Gamma_N. \end{cases}$$

We have: $\mu \in \mathbb{R}$, with $\mu > 0$; $\mathbf{b} \in \mathbb{R}^2$; $f : \Omega \to \mathbb{R}$ and $q : \Gamma_N \to \mathbb{R}$ are given functions.

1.1 [*3 pt*] Write the weak formulation of the problem by including the definitions, choice of the function spaces, and the derivation of the formulation.

1.2 [*2 pt*] Write the Galerkin–Finite Element approximation of the problem with Finite Elements of degree $r \geq 1$. Include the definition of the function spaces, basis functions, and the approximate solution.

1.3 [*5 pt*] Set $\mu = 1$, $\mathbf{b} = (0,2)^T$, $f(x_1, x_2) = \pi^2 \, \sin(\pi \, x_2) \, e^{(x_1 + x_2)}$,

$$q(x_1, x_2) = \begin{cases} -\sin(\pi \, x_2) \, e^{x_2} & \text{if } x_1 = 0, \ x_2 \in (0,1) \ \ (\text{on } \Gamma_0), \\ \sin(\pi \, x_2) \, e^{(x_2+1)} & \text{if } x_1 = 1, \ x_2 \in (0,1) \ \ (\text{on } \Gamma_1). \end{cases}$$

Use the mesh $\mathcal{T}_h$ of triangular Finite Elements of size $h = 0.1$ provided at the following link:
`https://github.com/michelebucelli/nmpde-labs/tree/main/examples/gmsh`

See Fig. 1 for boundary tags. Use Finite Elements built over the polynomial space $\mathbb{P}_2$. Implement the Finite Element approximation in *deal.II*. Use the GMRES solver with suitable preconditioner and stopping criterion. Upload the necessary files.

1.4 [*1 pt*] Following the answer provided at Point 1.3), suitably visualize the Finite Element solution u_h in *Paraview* and upload the corresponding file with the picture.

1.5 [*2 pt*, *******] By knowing that the exact solution is $u(x_1, x_2) = \sin(\pi\, x_2)\, e^{(x_1+x_2)}$, compute the errors $\|u - u_h\|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$ for different values of the mesh size $h = 0.1$, 0.05, 0.025, and 0.0125, still using $\mathbb{P}_2$ FE. Upload the file and report the values of the errors.

1.6 [*2 pt*, *******] Following Point 1.5), estimate the convergence orders of the errors with respect to h. Report the procedure used, compare the results with the theory, and critically discuss them. What are the expected theoretical results by using $\mathbb{P}_3$ Finite Elements?

Exercise 2 (11 pt)

Let us consider the domain $\Omega = (0,1)^2$ and the following Stokes problem: find $\mathbf{u} : \Omega \to \mathbb{R}^2$ and $p : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\mu\, \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ (\mu \nabla \mathbf{u})\, \mathbf{n} - p\mathbf{n} = \mathbf{0} & \text{on } \Gamma_1, \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma_D = \partial\Omega \backslash \Gamma_1, \end{cases}$$

where $\mu \in \mathbb{R}$ and $\mu > 0$, while $\mathbf{g} : \Omega \to \mathbb{R}^2$. See Fig. 1 for the boundary subsets of $\partial\Omega = \cup_{i=0}^{3} \Gamma_i$; here, $\Gamma_D = \partial\Omega \backslash \Gamma_1 = \Gamma_0 \cup \Gamma_2 \cup \Gamma_3$.

2.1 [*2 pt*] Write the weak formulation of the problem by including the definitions, choice of the function spaces, and the derivation of the formulation.

2.2 [*2 pt*] Write the Galerkin–Finite Element approximation of the problem with Finite Elements built over the space $\mathbb{P}_r$ for $\mathbf{u}$, while $\mathbb{P}_q$ for $\mathbf{q}$. Include the definitions of the function spaces, basis functions, and the approximate solutions.

2.3 [*3 pt*] Set $\mu = 1$, $\mathbf{f} = \mathbf{0}$,

$$\mathbf{g}(x_1, x_2) = \begin{cases} \mathbf{0} & \text{if } x_1 \in (0,1),\ x_2 = 0 \text{or } x_2 = 1 \ \ (\text{on } \Gamma_2 \cup \Gamma_3), \\ \left(\dfrac{27}{4}\, x_2 (1 - x_2)^2,\, 0 \right)^T & \text{if } x_1 = 0,\ x_2 \in (0,1) \ \ (\text{on } \Gamma_0). \end{cases}$$

Use the mesh $\mathcal{T}_h$ of triangular Finite Elements of size $h = 0.1$ provided at the following link: `https://github.com/michelebucelli/nmpde-labs/tree/main/examples/gmsh`

See Fig. 1 for boundary tags. Use Finite Elements built on the pair of spaces $\mathbb{P}_2$–$\mathbb{P}_1$. Implement the Finite Element approximation in *deal.II* and upload the necessary files.

2.4 [*2 pt*, *******] Following Point 2.3), suitably visualize the Finite Element solutions $\mathbf{u}_h$ and p_h in *Paraview* and upload the corresponding files with the pictures. Then, still using *Paraview*, visualize the field $\nabla \cdot \mathbf{u}_h$, upload the corresponding image, and motivate the result obtained.

2.5 [*2 pt*, *******] Repeat Points 2.3)–2.4) by means of the Finite Element pair $\mathbb{P}_1$–$\mathbb{P}_1$. Visualize the result on the pressure field, upload the image, and motivate it on the basis of the theory.