Written Test – September 11, 2023
Numerical Methods for Partial Differential Equations
max 26 pt (over 30) – duration 1h 30'

Students entitled to take the test reduced by 30% according to Law 170/2010 (**Multichance** team indications) DO NOT complete the questions marked with **(***)**

Answer to questions on the paper (handwritten). Upload files on WeBeeP.

Exercise 1 (15 pt)

Let us consider the domain $\Omega = (0,1)^2$, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N = \cup_{i=0}^{3}\Gamma_i$, where $\Gamma_D = \Gamma_0 \cup \Gamma_2$ and $\Gamma_N = \Gamma_1 \cup \Gamma_3 = \partial\Omega \backslash \Gamma_D$. In particular, $\Gamma_0 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_1 = 0\}$, $\Gamma_1 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_1 = 1\}$, $\Gamma_2 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_2 = 0\}$, and $\Gamma_3 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_2 = 1\}$; $\mathbf{n}$ indicates the unit vector normal to $\partial\Omega$ and outward directed. See Fig. 1.

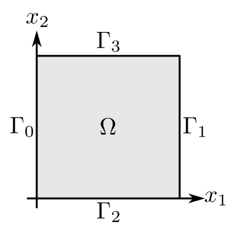

Figure 1: Domain Ω and boundary $\partial\Omega = \cup_{i=0}^{3}\Gamma_i$. Each boundary subset Γ_i corresponds to the tag i in the mesh files

Let us consider the following strong problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\mu\Delta u + \mathbf{b}\cdot\nabla u + \sigma\, u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \mu\nabla u\cdot\mathbf{n} = q & \text{on } \Gamma_N. \end{cases}$$

We have: $\mu \in \mathbb{R}$, with $\mu > 0$; $\mathbf{b} \in \mathbb{R}^2$; $\sigma \in \mathbb{R}$ with $\sigma \geq 0$; $f : \Omega \to \mathbb{R}$ and $q : \Gamma_N \to \mathbb{R}$ are given functions.

1.1 [*3 pt*] Write the weak formulation of the problem by including the definitions, choice of the function spaces, and the derivation of the formulation.

1.2 [*2 pt*] Write the Galerkin–Finite Element approximation of the problem with Finite Elements of degree $r \geq 1$. Include the definition of the function spaces, basis functions, and the approximate solution.

1.3 [*5 pt*] Set $\mu = 1$, $\mathbf{b} = (1,1)^T$, $\sigma = 1$, $f(x_1,x_2) = (e^{x_1} - 1)\,(e^{x_2} - 1)$,

$$q(x_1,x_2) = \begin{cases} e\,(e^{x_2} - 1) & \text{if } x_1 = 1,\ x_2 \in (0,1) \ \ (\text{on } \Gamma_1), \\ e\,(e^{x_1} - 1) & \text{if } x_1 \in (0,1),\ x_2 = 1 \ \ (\text{on } \Gamma_3). \end{cases}$$

Use the mesh $\mathcal{T}_h$ of triangular Finite Elements of size $h = 0.1$ provided at the following link: `https://github.com/michelebucelli/nmpde-labs/tree/main/examples/gmsh`

See Fig. 1 for boundary tags. Use Finite Elements built over the polynomial space $\mathbb{P}_1$. Implement the Finite Element approximation in *deal.II*. Use the GMRES solver with suitable preconditioner and stopping criterion. Upload the necessary files.

1.4 [*1 pt*] Following the answer provided at Point 1.3), suitably visualize the Finite Element solution u_h in *Paraview* and upload the corresponding file with the picture.

1.5 [*2 pt*, *******] By knowing that the exact solution is $u(x_1, x_2) = (e^{x_1} - 1)\,(e^{x_2} - 1)$, compute the errors $\|u - u_h\|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$ for different values of the mesh size $h = 0.1$, 0.05, 0.025, and 0.0125, still using $\mathbb{P}_1$ FE. Upload the file and report the values of the errors.

1.6 [*2 pt*, *******] Following Point 1.5), estimate the convergence orders of the errors with respect to h. Report the procedure used, compare the results with the theory, and critically discuss them. What are the expected theoretical results by using $\mathbb{P}_4$ Finite Elements?

Exercise 2 (11 pt)

Let us consider again the domain $\Omega = (0,1)^2$ and the following parabolic PDE: find $u : \Omega \times (0,T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu\,\Delta u = f & \text{in } \Omega \times (0,T], \\ u = 0 & \text{on } \Gamma_D \times (0,T], \\ \mu \nabla u \cdot \mathbf{n} = 0 & \text{on } \Gamma_N \times (0,T], \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $u = u(\mathbf{x}, t)$, $\mu \in \mathbb{R}$ with $\mu > 0$, $f : \Omega \times (0,T) \to \mathbb{R}$, and $u_0 : \Omega \to \mathbb{R}$. See Exercise 1 and Fig. 1 for the definitions of Γ_D and Γ_N.

2.1 [*4 pt*] Write the weak formulation of the problem. Write its semi–discrete formulation by means of the Galerkin–Finite Element approximation built over the space $\mathbb{P}_r$, for $r \geq 1$. Then, write the fully discrete problem by considering the θ–method. Define the fully discrete solution.

2.2 [*3 pt*] Set $\mu = 1$, $T = 3$, $f(x_1, x_2, t) = e^{-t-(x_1-0.75)^2-(x_2-0.75)^2}$, and $u_0(x_1, x_2) = \sin\left(\frac{\pi}{2}\,x_1\right)\,\sin\left(\frac{\pi}{2}\,x_2\right)$.

Use the mesh $\mathcal{T}_h$ of triangular Finite Elements of size $h = 0.1$ provided at the following link:
`https://github.com/michelebucelli/nmpde-labs/tree/main/examples/gmsh`

See Fig. 1 for boundary tags. Use Finite Elements built over the space $\mathbb{P}_2$ and the Crank–Nicolson method with time step size $\Delta t = 0.1$. Implement the Finite Element approximation in *deal.II* and upload the necessary files.

2.3 [*2 pt*, *******] Following the answer provided at Point 2.2), visualize the Finite Element solution at $t = 1$ in *Paraview*. Also, plot the Finite Element solution u_h vs. time $t \in [0,T]$ evaluated in the point $(0.7, 0.7) \in \Omega$ (use the filter `PlotSelectionOverTime` in *Paraview*). Upload the corresponding files with the pictures.

2.4 [*2 pt*, *******] Critically discuss the theoretical accuracy properties of the fully discretized problem for different values of the discretization parameters $h > 0$ and $\Delta t > 0$, also sustaining the answer with numerical tests obtained by repeating the simulations with the data at Point 2.2).