

Written Test – July 3, 2023
Numerical Methods for Partial Differential Equations
max 26 pt (over 30) – duration 1h 30'

Students entitled to take the test reduced by 30% according to Law 170/2010 (**Multichance** team indications) DO NOT complete the questions marked with (***)

Answer to questions on the paper (handwritten). Upload files on WeBeeP.

Exercise 1 (15 pt)

Let us consider the domain $\Omega = (0, 1)^2$, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N = \cup_{i=0}^3 \Gamma_i$, where $\Gamma_D = \Gamma_0 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_N = \Gamma_1 = \partial\Omega \setminus \Gamma_D$. In particular:
 $\Gamma_0 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_1 = 0\}$, $\Gamma_1 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_1 = 1\}$,
 $\Gamma_2 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_2 = 0\}$, and $\Gamma_3 = \{\mathbf{x} = (x_1, x_2) \in \partial\Omega : x_2 = 1\}$;
 \mathbf{n} indicates the unit vector normal to $\partial\Omega$ and outward directed. See Fig. 1.

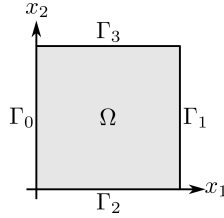


Figure 1: Domain Ω and boundary $\partial\Omega = \cup_{i=0}^3 \Gamma_i$. Each boundary subset Γ_i corresponds to the tag i in the mesh files

Let us consider the following strong problem: find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{cases} -\mu \Delta u + \sigma u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \\ \mu \nabla u \cdot \mathbf{n} = 0 & \text{on } \Gamma_N. \end{cases}$$

We have: $\mu \in \mathbb{R}$, with $\mu > 0$; $\sigma \in \mathbb{R}$, with $\sigma \geq 0$; $f : \Omega \rightarrow \mathbb{R}$ and $g : \Gamma_D \rightarrow \mathbb{R}$ are given functions.

- 1.1** [3 pt] Write the weak formulation of the problem by including the definitions, choice of the function spaces, and the derivation of the formulation.
- 1.2** [2 pt] Write the Galerkin–Finite Element approximation of the problem with Finite Elements of degree $r \geq 1$. Include the definition of the function spaces, basis functions, and the approximate solution.

- 1.3** [5 pt] Set $\mu = 1$, $\sigma = 1$, $f(x_1, x_2) = \left(1 + \frac{\pi^2}{4}\right) \sin\left(\frac{\pi}{2} x_1\right) x_2$,

$$g(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = 0, x_2 \in (0, 1) \text{ or } x_1 \in (0, 1), x_2 = 0 \text{ (on } \Gamma_0 \cup \Gamma_2), \\ \sin\left(\frac{\pi}{2} x_1\right) & \text{if } x_1 \in (0, 1), x_2 = 1 \text{ (on } \Gamma_3). \end{cases}$$

Use the mesh \mathcal{T}_h of triangular Finite Elements of size $h = 0.1$ provided at the following link:
<https://github.com/michelebucelli/nmpde-labs/tree/main/examples/gmsh>

See Fig. 1 for boundary tags. Use Finite Elements built over the polynomial space \mathbb{P}_2 . Implement the Finite Element approximation in *deal.II*. Use the GMRES solver with suitable preconditioner and stopping criterion. Comment and upload the necessary files.

- 1.4** [1 pt] Following the answer provided at Point 1.3), suitably visualize the Finite Element solution u_h in *Paraview* and upload the corresponding file with the picture.
- 1.5** [2 pt, ***] By knowing that the exact solution is $u(x_1, x_2) = \sin\left(\frac{\pi}{2} x_1\right) x_2$, compute the errors $\|u - u_h\|_{H^1(\Omega)}$ and $\|u - u_h\|_{L^2(\Omega)}$ for different values of the mesh size $h = 0.1, 0.05, 0.025$, and 0.0125 , still using \mathbb{P}_2 FE. Upload the file and report the values of the errors.
- 1.6** [2 pt, ***] Following Point 1.5), estimate the convergence orders of the errors with respect to h . Report the procedure used, compare the results with the theory, and critically discuss them. What are the *expected* theoretical results by using \mathbb{P}_r Finite Elements for $r \geq 1$?

Exercise 2 (11 pt)

Let us consider again the domain $\Omega = (0, 1)^2$ and the following parabolic PDE: find $u : \Omega \times (0, T] \rightarrow \mathbb{R}$ such that

$$\begin{cases} \frac{\partial u}{\partial t} - \mu \Delta u = f & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \Gamma_D \times (0, T], \\ \mu \nabla u \cdot \mathbf{n} = 0 & \text{on } \Gamma_N \times (0, T], \\ u = u_0 & \text{in } \Omega \times \{0\}, \end{cases}$$

where $u = u(\mathbf{x}, t)$, $\mu \in \mathbb{R}$ with $\mu > 0$, $f : \Omega \times (0, T) \rightarrow \mathbb{R}$, and $u_0 : \Omega \rightarrow \mathbb{R}$. See Exercise 1 and Fig. 1 for the definitions of Γ_D and Γ_N .

- 2.1** [4 pt] Write the weak formulation of the problem. Write its semi-discrete formulation by means of the Galerkin-Finite Element approximation built over the space \mathbb{P}_r , for $r \geq 1$. Then, write the fully discrete problem by considering the θ -method. Define the fully discrete solution.
- 2.2** [3 pt] Set $\mu = 1$, $T = 2$, $f(x_1, x_2, t) = e^{-t-(x_1-0.5)^2-(x_2-0.5)^2}$, and $u_0(x_1, x_2) = \sin\left(\frac{\pi}{2} x_1\right) \sin(\pi x_2)$.
Use the mesh \mathcal{T}_h of triangular Finite Elements of size $h = 0.1$ provided at the following link: <https://github.com/michelebucelli/nmpde-labs/tree/main/examples/gmsh>
See Fig. 1 for boundary tags. Use Finite Elements built over the space \mathbb{P}_1 and the Backward Euler method with time step size $\Delta t = 0.1$. Implement the Finite Element approximation in *deal.II* and upload the necessary files.
- 2.3** [2 pt, ***] Following the answer provided at Point 2.2), visualize the Finite Element solution at $t = 1$ in *Paraview*. Also, plot the Finite Element solution u_h vs. time $t \in [0, T]$ evaluated in the point $(0.5, 0.5) \in \Omega$ (use the filter **PlotSelectionOverTime** in *Paraview*). Upload the corresponding files with the pictures.
- 2.4** [2 pt, ***] Critically discuss the theoretical accuracy properties of the fully discretized problem for different values of the discretization parameters $h > 0$ and $\Delta t > 0$, also sustaining the answer with numerical tests obtained by repeating the simulations with the data at Point 2.2).