Relations and Functions

Assylbek Issakhov,
Ph.D., professor

RELATIONS

- Formally, we define a relation in terms of "ordered pairs."
- An *ordered pair* of elements a and b, where a is designated as the first element and b as the second element, is denoted by (a, b).
- In particular,

$$(a, b) = (c, d)$$

if and only if $a = c$ and $b = d$.

RELATIONS: Product Sets

- Thus $(a, b) \neq (b, a)$ unless $a = b$. This contrasts with sets where the order of elements is irrelevant; for example, $\{3, 5\} = \{5, 3\}$.

- Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B." By definition,

$$A \times B = \{(a, b) \mid a \in A \; and \; b \in B\}$$

- One frequently writes A^2 instead of $A \times A$.

RELATIONS

- **Definition 2.1:** Let A and B be sets. A *binary relation* or, simply, *relation* from A to B is a subset of $A \times B$.
- Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B.
- If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation *on* A.

RELATIONS

- The *domain* of a relation R is the set of all first elements of the ordered pairs which belong to R,

$$dom(R) = \{x | there\ is\ y\ for\ which\ (x, y) \in R\}$$

- and the *range* is the set of all second elements.

$$rng(R) = \{y | there\ is\ x\ for\ which\ (x, y) \in R\}$$

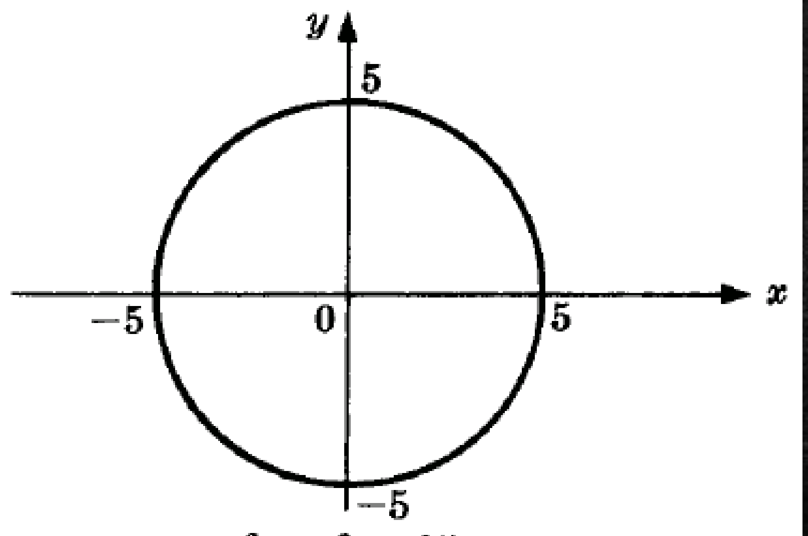
Inverse Relation

- Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R^{-1}, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R

$$R^{-1} = \{(b, a) | (a, b) \in R\}$$

PICTORIAL REPRESENTATIVES OF RELATIONS

- Let S be a relation on the set $\mathbb{R}$ of real numbers; that is, S is a subset of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Frequently, S consists of all ordered pairs of real numbers which satisfy some given equation $E(x, y) = 0$ (for example, $x^2 + y^2 = 25$).



$x^2 + y^2 = 25$

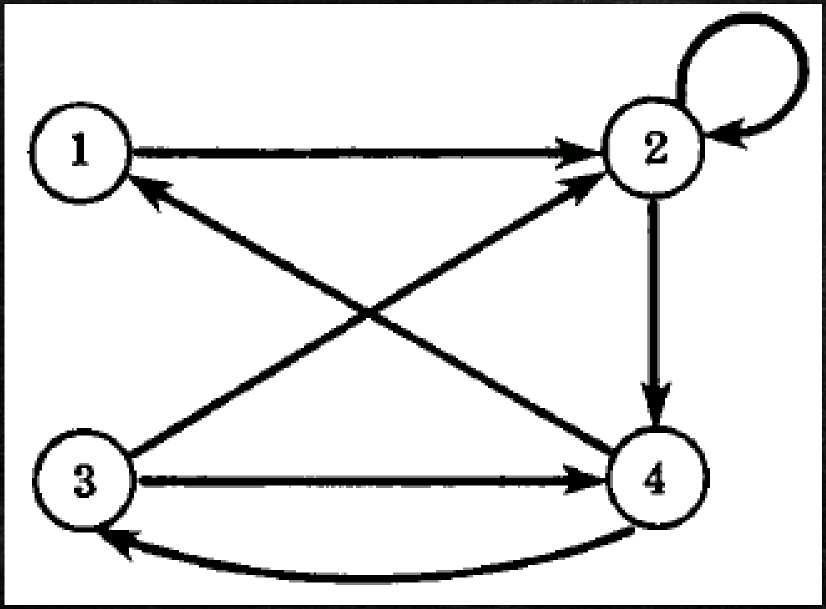
PICTORIAL REPRESENTATIVES OF RELATIONS

- There is an important way of picturing a relation R on a finite set.
- First we write down the elements of the set, and then we draw an arrow from each element x to each element y whenever x is related to y. This diagram is called the *directed graph* of the relation

PICTORIAL REPRESENTATIVES OF RELATIONS

- For example, shows the directed graph of the following relation R on the set $A = \{1, 2, 3, 4\}$:

$$R = \{(1,2), (2,2), (2,4), (3,2), (3,4), (4,1), (4,3)\}$$



Pictures of Relations on Finite Sets

- Suppose A and B are finite sets. There are two ways of picturing a relation R from A to B.

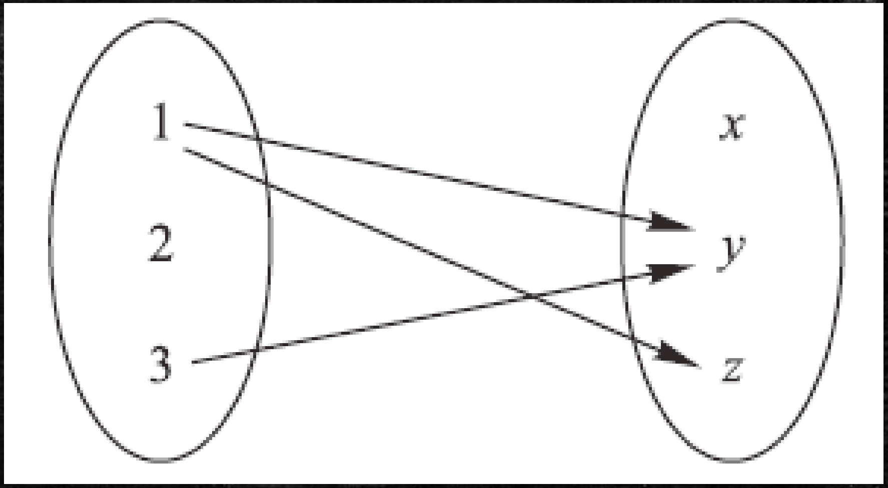
- Form a rectangular array (matrix) whose rows are labeled by the elements of A and whose columns are labeled by the elements of B. Put a **1** or **0** in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix of the relation*.

- $R = \{(1, y), (1, z), (3, y)\}$

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

Pictures of Relations on Finite Sets

- Suppose A and B are finite sets. There are two ways of picturing a relation R from A to B.
- Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from $a \in A$ to $b \in B$ whenever a is related to b. This picture will be called the *arrow diagram* of the relation.
- $R = \{(1, y), (1, z), (3, y)\}$



COMPOSITION OF RELATIONS

- Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of $A \times B$ and S is a subset of $B \times C$. Then R and S give rise to a relation from A to C denoted by $R \circ S$ and defined by: $a(R \circ S)c$ if for some $b \in B$ we have aRb and bSc. That is,

$$R \circ S = \{(a, c) | there\ exists\ b \in B$$

$$for\ which\ (a, b) \in R\ and\ (b, c) \in S\}$$

COMPOSITION OF RELATIONS

- The relation $R \circ S$ is called the *composition* of R and S; it is sometimes denoted simply by RS.

- **Theorem 2.1:** Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then
$$(RS)T = R(ST)$$

Composition of Relations and Matrices

- Let $A = \{1, 2, 3, 4\}, B = \{a, b, c, d\}, C = \{x, y, z\}$ and let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$
- There is another way of finding $R \circ S$. Let M_R and M_S denote respectively the matrix representations of the relations R and S. Then

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad M_S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Composition of Relations and Matrices

- Multiplying M_R and M_S we obtain the matrix

$$M = M_R M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

- The nonzero entries in this matrix tell us which elements are related by $R \circ S$. Thus $M = M_R M_S$ and $M_{R \circ S}$ have the same nonzero entries.

TYPES OF RELATIONS

- A relation R on a set A is *reflexive* if aRa for every $a \in A$, that is, if $(a, a) \in R$ for every $a \in A$.
- A relation R on a set A is *symmetric* if whenever aRb then bRa, that is, if whenever

  $$(a, b) \in R \text{ then } (b, a) \in R.$$

- A relation R on a set A is *antisymmetric* if whenever aRb and bRa then $a = b$. Thus R is not antisymmetric if there exist distinct elements a and b in A such that aRb and bRa.

Examples

- Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:
- $R_1 = \{(1,1), (1,2), (2,3), (1,3), (4,4)\}$
- $R_2 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$
- $R_3 = \{(1,3), (2,1)\}$
- $R_4 = \emptyset$, the *empty relation*
- $R_5 = A \times A$, *the universal relation*

Determine which of the relations are reflexive symmetric, antisymmetric.

Examples

- (1) Relation $\leq$ (less than or equal) on the set $\mathbb{Z}$ of integers;
- (2) Set inclusion $\subseteq$ on a collection C of sets;
- (3) Relation $\perp$ (perpendicular) on the set L of all lines in the plane;
- (4) Relation $\parallel$ (parallel) on the set L of all lines in the plane.
- (5) Relation $|$ of divisibility on the set $\mathbb{N}$ of positive integers.

Determine which of the relations are reflexive symmetric, antisymmetric.

TYPES OF RELATIONS

- A relation R on a set A is *transitive* if whenever aRb and bRc then aRc, that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$.

- **Theorem 2.2:** A relation R is transitive if and only if, for every $n \geq 1$, we have

$$R^n \subseteq R.$$

EQUIVALENCE RELATIONS

- Consider a nonempty set S. A relation R on S is an *equivalence relation* if R is reflexive, symmetric, and transitive. That is, R is an equivalence relation on S if it has the following three properties:

  (1) For every $a \in S$, aRa.

  (2) If aRb, then bRa.

  (3) If aRb and bRc, then aRc.

Equivalence Relations and Partitions

- The general idea behind an equivalence relation is that it is a classification of objects which are in some way "alike."
- Suppose R is an equivalence relation on a set S. For each $a \in S$, let $[a]$ denote the set of elements of S to which a is related under R; that is:

$$[a] = \{x \mid (a, x) \in R\}$$

Equivalence Relations and Partitions

- We call $[a]$ the *equivalence class* of a in S; any $b \in [a]$ is called a *representative* of the equivalence class.
- The collection of all equivalence classes of elements of S under an equivalence relation R is denoted by S/R, that is,

$$S/R = \{[a] \mid a \in S\}$$

It is called the *quotient set* of S by R.

Equivalence Relations and Partitions

- The fundamental property of a quotient set is contained in the following theorem.
- **Theorem 2.6:** Let R be an equivalence relation on a set S. Then S/R is a partition of S. Specifically:

  (i) For each a in S, we have $a \in [a]$.

  (ii) $[a] = [b]$ if and only if $(a, b) \in R$.

  (iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

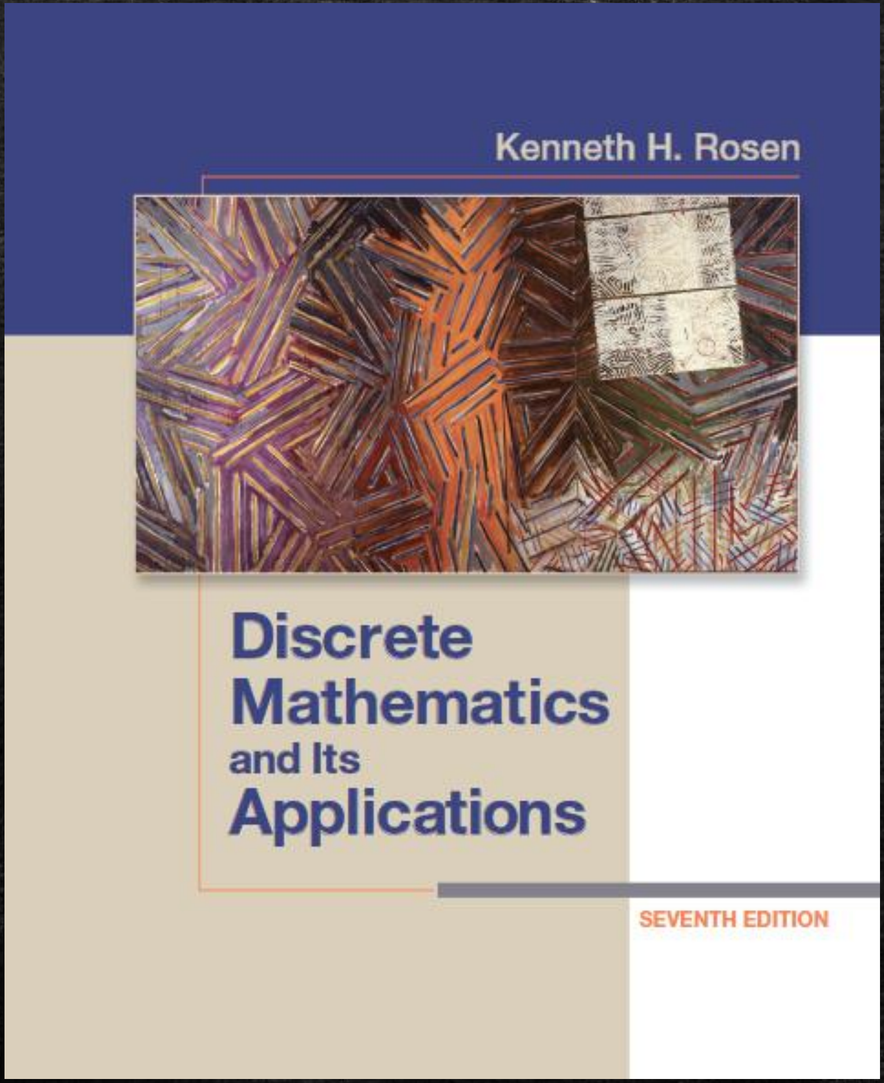
PARTIAL ORDERING RELATIONS

- A relation R on a set S is called a *partial ordering* or a *partial order* of S if R is reflexive, antisymmetric, and transitive.
- A set S together with a partial ordering R is called a *partially ordered set* or *poset.*

n-ARY RELATIONS

- All the relations discussed above were binary relations.
- By an *n-ary relation*, we mean a set of ordered n-tuples. For any set S, a subset of the product set S^n is called an n-ary relation on S.
- In particular, a subset of S^3 is called a *ternary relation* on S.

HOMEWORK: Exercises 2, 4, 6, 8, 10 on p. 581; Ex. 2, 4, 10 on p.596; Ex. 2, 10, 16, 22 on pp. 615-616



FUNCTIONS

- Suppose that to each element of a set A we assign a unique element of a set B; the collection of such assignments is called *a function* from A into B.
- The set A is called the *domain* of the function, and the set B is called the *target set* or *codomain*.

FUNCTIONS

- Functions are ordinarily denoted by symbols. For example, let f denote a function from A into B. Then we write

$$f: A \rightarrow B$$

which is read: "f is a function from A into B," or "f takes (or maps) A into B."

FUNCTIONS

- If $a \in A$, then $f(a)$ (read: "f of a") denotes the unique element of B which f assigns to a; it is called the *image* of a under f, or the *value* of f at a.
- The set of all image values is called the *range* or *image* of f. The image of $f : A \to B$ is denoted by

$$Ran(f), Im(f) \text{ or } f(A).$$

Functions as Relations

- There is another point of view from which functions may be considered. First of all, every function $f: A \to B$ gives rise to a relation from A to B called the *graph of f* and defined by

$$Graph(f) = \{(a, b) \mid a \in A, b = f(a)\}.$$

- **Definition:** A function $f: A \to B$ is a relation from A to B (i.e., a subset of $A \times B$) such that each $a \in A$ belongs to a unique ordered pair (a, b) in f.

Composition of Functions

- Consider functions $f: A \to B$ and $g: B \to C$; that is, where the codomain of f is the domain of g. Then we may define a new function from A to C, called the *composition* of f and g and written $g \circ f$, as follows:

$$(g \circ f)(\boldsymbol{a}) \equiv g(f(\boldsymbol{a})).$$

That is, we find the image of a under f and then find the image of $f(a)$ under g.

ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

- A function $f\colon A \to B$ is said to be *one-to-one* (written 1-1) if different elements in the domain A have distinct images. Another way of saying the same thing is that f is *one-to-one* if $f(a) = f(a')$ implies $a = a'$.
- A function $f\colon A \to B$ is said to be an *onto* function if each element of B is the image of some element of A. In other words, $f\colon A \to B$ is onto if the image of f is the entire codomain, i.e., if $f(A) = B$. In such a case we say that f is a function from A onto B or that f maps A onto B.

ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

- A function $f: A \to B$ is *invertible* if its inverse relation f^{-1} is a function from B to A. In general, the inverse relation f^{-1} may not be a function.
- The following theorem gives simple criteria which tells us when it is.
- **Theorem 3.1:** A function $f: A \to B$ is invertible if and only if f is both one-to-one and onto.

ONE-TO-ONE, ONTO, AND INVERTIBLE FUNCTIONS

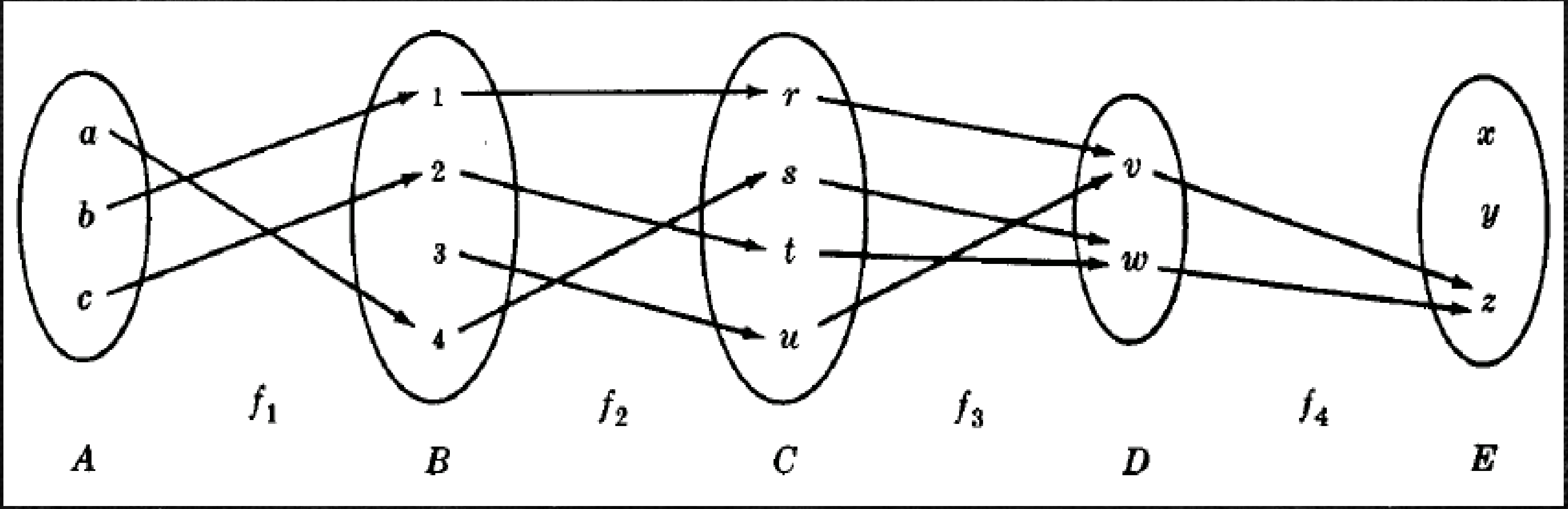
- If $f : A \rightarrow B$ is one-to-one and onto, then f is called a *one-to-one correspondence* between A and B. This terminology comes from the fact that each element of A will then correspond to a unique element of B and vice versa.
- Sometimes we use the terms *injective* for a one-to-one function, *surjective* for an onto function, and *bijective* for a one-to-one correspondence.

Example

- Consider the functions

$$f_1\colon A \to B, f_2\colon B \to C, f_3\colon C \to D, f_4\colon D \to E$$

defined by the diagram



- Determine which of the functions are one-to-one, which ones are onto.

HOMEWORK: Exercises 4, 6, 10, 12, 14, 16, 20, 22, 30, 36, 42 on pp. 152-154;