

Chapter 4

Determinants

Eigenvalue Problem: given $A_{n \times n}$, solve (λ, \vec{x}) such that

$$A\vec{x} = \lambda\vec{x}, \quad \text{where } \lambda \in \mathbb{R}, \vec{x} \in \mathbb{R}^n.$$

This helps to understand the effect of matrix A / the matrix transformation defined by A via decomposition:

Learning Objectives

Determinants: their computations and properties.

- determinant of a square matrix A is a number $\det(A)$, i.e., determinant is a function:

$$\det : \text{square matrix } A \mapsto \det(A) \in \mathbb{R}.$$

- it is computational expensive: $\det(A_{10 \times 10})$ can take 10^7 summand.

4.1 Determinants: Definition + 4.3 Determinants and Volumes

Learning Objectives

- learn definition of the determinant;
- understand properties of determinants;
- compute determinants using row and **column** operations.
- geometric interpretation as volumes.

Definition 4.1. Determinant, denoted as \det , is a function:

$$\begin{aligned}\det : \{\text{square matrices}\} &\rightarrow \mathbb{R} \\ A_{n \times n} &\mapsto \det(A)\end{aligned}$$

with the following properties

- (1) if $A \rightarrow B$ by row replacement, then $\det(B) = \det(A)$;
- (2) if $A \rightarrow B$ by scaling a row by c , then $\det(B) = c \det(A)$;
- (3) if $A \rightarrow B$ by swapping two rows, then $\det(B) = -\det(A)$;
- (4) $\det(I_n) = 1$.

Example

Compute $\det(A)$ using the definition of determinants.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 5 & 1 & 0 \end{pmatrix}.$$

Example

If $A_{n \times n}$ has zero rows/columns, then $\det(A) = 0$.

Example

If $A_{n \times n}$ is non-invertible, then $\det(A) = 0$.

Example

If $A_{n \times n}$ is lower-triangular/upper triangular, i.e.,

$$A = \begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix},$$

then

$$\det(A_{n \times n}) = \text{product of diagonal entries} = a_{11}a_{22} \cdots a_{nn}.$$

General Recipe for $\det(A)$:

First apply row operations to A ,

$$A_{n \times n} \longrightarrow B_{n \times n} \quad (\text{in REF, hence upper triangular}).$$

Then

$$\det(A) = (-1)^r \cdot \frac{\text{product of diagonals in } B}{\text{product of scaling parameters used in row operations}}$$

with r being the number of row swaps applied in the row operations.

Example

Determinants of $A_{2 \times 2} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\det(A) = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

(1) if $a = 0$, then $\det(A) = -bc$.

(2) if $a \neq 0$, then $\det(A) = ad - bc$.

More properties of $\det(A)$:

- (1) (Existence property) According to the definition, $\det(A)$ is unique.
- (2) (Invertibility property) A is invertible $\iff \det(A) \neq 0$,
- (3) (Multiplicativity property) $\det(AB) = \det(A) \det(B)$.

- (4) (Transpose property) $\det(A^\top) = \det(A)$.

Geometric interpretation as volumes:

Let $A_{n \times n}$ has row/column vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$. Absolute value of $\det(A)$, $|\det(A)|$, computes the volume of an object P , parallelepiped:

$$P = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_n\vec{v}_n \mid 0 \leq a_1, a_2, \dots, a_n \leq 1\}$$

Relation to Transition of Volumes:

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear transformation with standard matrix $A_{n \times n}$. Then

$$\text{Vol}(T(S)) = |\det(A)| \text{Vol}(S), \quad \text{for any subset } S \text{ in } \mathbb{R}^n.$$

Example

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation with standard matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

4.2 Cofactor Expansion

Learning Objectives

- learn to compute determinants using **cofactor expansions**
- Recipes: the determinants of 3×3 matrices via cofactor expansion
- learn to compute determinants using combination of existing methods

What is cofactor expansion ?

It is a recursive formula of computing determinants of $n \times n$ matrices:

$$n \times n \rightarrow (n - 1) \times (n - 1) \rightarrow \cdots \rightarrow 2 \times 2 \rightarrow 1 \times 1$$

Example

$(2 \times 2 \rightarrow 1 \times 1)$ Cofactor expansion of $\det(A)$ with $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Example

$(3 \times 3 \rightarrow 2 \times 2)$ Cofactor expansion of $\det(A)$ with $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Recipe for $\det(A_{3 \times 3})$:

For $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$

$$\det(A) = \underbrace{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}}_{\text{diagonal direction}} - \underbrace{a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}}_{\text{anti-diagonal direction}}$$

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Example

$$\det \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

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=

Recipe for cofactor expansion: $n \times n \rightarrow (n-1) \times (n-1)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Define

(1) (i, j) **minor** of A , denoted as A_{ij} , is the $(n-1) \times (n-1)$ matrix from deleting the i^{th} row and j^{th} column in A .

(2) (i, j) **cofactor** of A , denoted as C_{ij} is defined as

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

sign pattern:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Then the cofactor expansion of $\det(A)$ can be written as

$$\begin{aligned} & \det(A) \\ &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}, && \text{(expansion along the 1st row)} \\ &= a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}, && \text{(expansion along the 1st column)} \\ &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}, && \text{(expansion along the } i\text{th row)} \\ &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}, && \text{(expansion along the } j\text{th column).} \end{aligned}$$

Trick: expand row/column with most zeros.

Example

Compute

$$\det \begin{pmatrix} 3 & 8 & 7 & 9 & 6 \\ 0 & 5 & 2 & 7 & 3 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Pick/combine existing methods:

- (1) Row/column operations
- (2) Recipe for $\det(A_{2 \times 2})$, $\det(A_{3 \times 3})$, upper/lower triangular matrix or matrix with zero rows/columns
- (3) Cofactor expansion

Example

Compute

$$\det \begin{pmatrix} 1 & -1 & 1 & 2 \\ -2 & 2 & -1 & -4 \\ 1 & 1 & 3 & 5 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

