Chapter 4

Determinants

Eigenvalue Problem: given $A_{n\times n}$, solve $(\lambda, \vec{x})$ such that

$$A\vec{x} = \lambda\vec{x}, \quad \text{where } \lambda \in \mathbb{R}, \vec{x} \in \mathbb{R}^n.$$

This helps to understand the effect of matrix A/ the matrix transformation defined by A via decomposition:

$\Rightarrow$ If $\vec{v} = c_1\vec{x_1} + c_2\vec{x_2} + \cdots + c_k\vec{x_k}$, and

$A\vec{x_1} = \lambda_1\vec{x_1}$, $A\vec{x_2} = \lambda_2\vec{x_2}$, $\cdots$, $A\vec{x_k} = \lambda_k\vec{x_k}$,

$\Rightarrow A\vec{v} = c_1\lambda_1\vec{x_1} + c_2\lambda_2\vec{x_2} + \cdots + c_k\lambda_k\vec{x_k}$.

Learning Objectives

Determinants: their computations and properties.

- determinant of a square matrix A is a number $\det(A)$, i.e., determinant is a function:

$$\det : \text{square matrix } A \mapsto \det(A) \in \mathbb{R}.$$

- it is computational expensive: $\det(A_{10\times 10})$ can take 10^7 summand.

4.1 Determinants: Definition + 4.3 Determinants and Volumes

Learning Objectives

- learn definition of the determinant;
- understand properties of determinants;
- compute determinants using row and **column** operations.
- geometric interpretation as volumes.

Definition 4.1. Determinant, denoted as det, is a function:

$$\begin{aligned} \det : \{\text{square matrices}\} &\to \mathbb{R} \\ A_{n\times n} &\mapsto \det(A) \end{aligned}$$

with the following properties

(1) if $A \to B$ by row replacement, then $\det(B) = \det(A)$;
(2) if $A \to B$ by scaling a row by c, then $\det(B) = c\det(A)$; ← $c = 0$
(3) if $A \to B$ by swapping two rows, then $\det(B) = -\det(A)$;
(4) $\det(I_n) = 1$.

Computation based def^{-n}:

$$A \xrightarrow{(1)} B_1 \xrightarrow{(2)} B_2 \xrightarrow{(3)} B_3 \xrightarrow{\cdots} RREF(A).$$

$$\det(A) \xrightarrow{(1)} \underset{=\det(A)}{\det(B_1)} \xrightarrow{(2)} \underset{= c\cdot\det(B_1)}{\det(B_2)} \xrightarrow{(3)} \underset{=-\det(B_2)}{\det(B_3)} \xrightarrow{\cdots} \begin{cases} \det(I_n) = 1. \\ \text{zero rows} \end{cases}$$

$\det(RREF(A)) = 0$
$\Rightarrow \det(A) = 0.$

Example

Compute $\det(A)$ using the definition of determinants.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 5 & 1 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & 2 \\ 5 & 1 & 0 \end{pmatrix} \xrightarrow[(1)]{R_3 = R_3 - 5R_1} \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & 2 \\ 0 & \boxed{1} & 0 \end{pmatrix}$$

$\det(A)$ $\qquad$ $\det(\cdots) = \det(A)$

$$\xrightarrow[(3)]{R_2 \leftrightarrow R_3} \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{2} \end{pmatrix}$$

$\det(\cdots) = -\det(A)$

$$\xrightarrow[(2)]{R_3 = \frac{1}{2} R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

$\det(\cdots) = \frac{1}{2} \cdot -\det(A) = \det(I_3) \overset{(4)}{=} 1.$

Eq^{-n}: $\quad -\frac{1}{2}\det(A) = 1.$

$\Rightarrow \det(A) = -2.$

Example

If $A_{n\times n}$ has zero rows/columns, then $\det(A) = 0$.

Eg. $\det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{R_2 = 0 \cdot R_2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$\det(I_2) = 1 \xrightarrow{(2)\ \text{"}c=0\text{"}} \det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0 \cdot \det(I_2) = 0.$

Example

If $A_{n\times n}$ is non-invertible, then $\det(A) = 0$.

$\det\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow[(1)]{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = B.$

$\Rightarrow \det(A) = \det(B) = 0.$

Example

If $A_{n\times n}$ is lower-triangular/upper triangular, i.e.,

$$A = \begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ * & * & 0 & \cdots & 0 \\ * & * & * & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix},$$

then

$$\det(A_{n\times n}) = \text{product of diagonal entries} = a_{11}a_{22}\cdots a_{nn}.$$

$\hookrightarrow \quad A_{n\times n} \xrightarrow{(1)} \xrightarrow{(1)} \xrightarrow{(1)} \underbrace{\begin{pmatrix} * & & 0 \\ & * & \\ 0 & & \ddots \; * \end{pmatrix}}_{}$

$\det(\cdot) =$ product of diagonals.

Eg. $\det\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc.$

① if one of a, b, c is zero, then $\det(\cdots) = 0$;

② if $a, b, c \neq 0$, then

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \xrightarrow{R_1 = \frac{1}{a}R_1} \xrightarrow{R_2 = \frac{1}{b}R_2} \xrightarrow{R_3 = \frac{1}{c}R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det \cdot \frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c} = \det(I_3) = 1.$

Eg. $\det\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = 1 \cdot 2 = 2.$

General Recipe for $\det(A)$**:**

First apply row operations to A,

$$A_{n\times n} \quad\longrightarrow\quad B_{n\times n} \quad \text{(in REF, hence upper triangular).}$$

Then

$$\det(A) = (-1)^r \cdot \frac{\text{product of diagonals in } B}{\text{product of scaling parameters used in row operations}}$$

with r being the number of row swaps applied in the row operations.

Example

Determinants of $A_{2\times 2} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\det(A) = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

(1) if $a = 0$, then $\det(A) = -bc$.

$$\det\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{=} -\det\begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc.$$

(2) if $a \neq 0$, then $\det(A) = ad - bc$.

$$\begin{pmatrix} \boxed{a} & b \\ c & d \end{pmatrix} \xrightarrow{R_1 = \frac{1}{a}R_1} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{b}{a}\cdot c \end{pmatrix} \rightarrow \text{REF, upper triangular}$$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (-1)^0 \cdot \frac{1\cdot\left(d - \frac{b}{a}\cdot c\right)}{\frac{1}{a}} = ad - bc.$$

More properties of $\det(A)$**:**

(1) (Existence property) According to the definition, $\det(A)$ is unique.
(2) (Invertibility property) A is invertible $\iff$ $\det(A) \neq 0$,
(3) (Multiplicativity property) $\det(AB) = \det(A)\det(B)$.

$\Rightarrow$ Compute $\det(A^{-1})$:

$$\det(AA^{-1}) = \det(A)\cdot\det(A^{-1}) \Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}.$$
$$= \det(I_n) = 1$$

(4) (Transpose property) $\det(A^{\intercal}) = \det(A)$.

$$A = \begin{pmatrix} - \vec{v}_1 - \\ - \vec{v}_2 - \\ \vdots \\ - \vec{v}_n - \end{pmatrix}, \quad A^T = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix}$$

If we want to compute $\det(A)$, we can do it using column operations.

(1)' column replacement : $\det(B) = \det(A)$

(2)' column scaling by c : $\det(B) = c\cdot\det(A)$.

(3)' column swap : $\det(B) = -\det(A)$.

Geometric interpretation as volumes:

Let $A_{n\times n}$ has row/column vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$. Absolute value of $\det(A)$, $|\det(A)|$, computes the volume of an object P, parallelepiped:

$$P = \{a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots \vec{v}_n \mid 0 \leq a_1, a_2, \cdots, a_n \leq 1\}$$

Relation to Transition of Volumes:

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation with standard matrix $A_{n\times n}$. Then

$$\mathrm{Vol}(T(S)) = |\det(A)|\ \mathrm{Vol}(S), \quad \text{for any subset } S \text{ in } \mathbb{R}^n.$$

Example

$T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

4.2 Cofactor Expansion

Learning Objectives

- learn to compute determinants using **cofactor expansions**
- Recipes: the determinants of 3×3 matrices via cofactor expansion
- learn to compute determinants using combination of existing methods

What is cofactor expansion ?

It is a recursive formula of computing determinants of $n \times n$ matrices:

$$n \times n \to (n-1) \times (n-1) \to \cdots \to 2 \times 2 \to 1 \times 1$$

Example

($2 \times 2 \to 1 \times 1$) Cofactor expansion of $\det(A)$ with $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Example

($3 \times 3 \to 2 \times 2$) Cofactor expansion of $\det(A)$ with $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

Recipe for $\det(A_{3\times 3})$:

For $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$,

$$\det(A) = \underbrace{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}}_{\text{diagonal direction}} \underbrace{-a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}}_{\text{anti-diagonal direction}}$$

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \qquad \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Example

$$\det\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$=$$

$$=$$

Recipe for cofactor expansion: $n \times n \to (n-1) \times (n-1)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Define

(1) (i, j) **minor** of A, denoted as A_{ij}, is the $(n-1) \times (n-1)$ matrix from deleting the i^{th} row and j^{th} column in A.

(2) (i, j) **cofactor** of A, denoted as C_{ij} is defined as

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

sign pattern:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Then the cofactor expansion of $\det(A)$ can be written as

$$\begin{aligned} & \det(A) \\ &= a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}, && \text{(expansion along the 1st row)} \\ &= a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1}, && \text{(expansion along the 1st column)} \\ &= a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}, && \text{(expansion along the ith row)} \\ &= a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}, && \text{(expansion along the jth column).} \end{aligned}$$

Trick: expand row/column with most zeros.

Example

Compute

$$\det \begin{pmatrix} 3 & 8 & 7 & 9 & 6 \\ 0 & 5 & 2 & 7 & 3 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

Pick/combine existing methods:

(1) Row/column operations
(2) Recipe for $\det(A_{2\times 2})$, $\det(A_{3\times 3})$, upper/lower triangular matrix or matrix with zero rows/columns
(3) Cofactor expansion

Example

Compute
$$\det\begin{pmatrix} 1 & -1 & 1 & 2 \\ -2 & 2 & -1 & -4 \\ 1 & 1 & 3 & 5 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$