The arithmetic of pairing-based proof systems

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Overview

1. Motivation
2. zk-SNARK
3. SNARK-friendly curves
4. SNARK-friendly 2-chains
5. Pairings in R1CS
6. Multi-scalar multiplication
7. Conclusion

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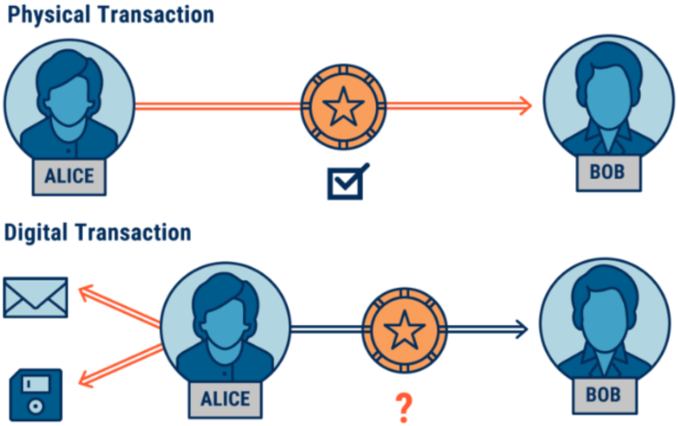
The story of Alice and Bob



(Courtesy of CBINSIGHTS)

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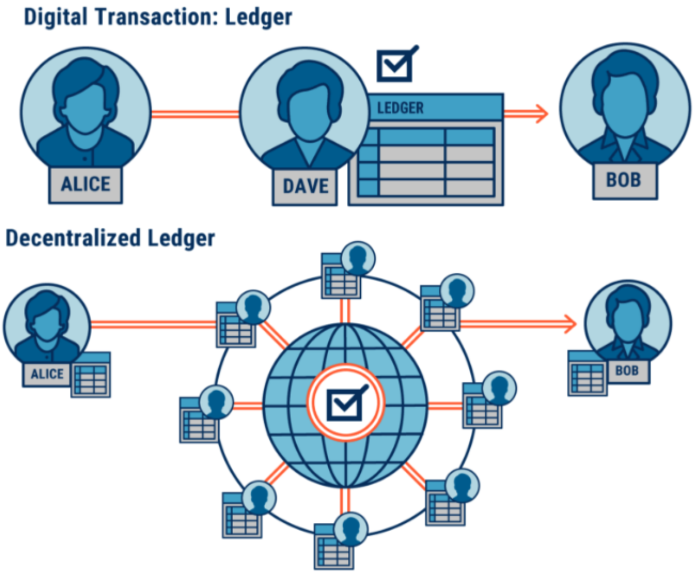

Blockchains

A blockchain is a public peer-to-peer *decentralized*, *transparent, immutable, paying* ledger.

- *Transparent*: everything is visible to everyone
- *Immutable*: nothing can be removed once written
- *Paying*: everyone should pay a fee to use

$$\text{Transparent} \xrightarrow[\text{Problem}]{} \text{confidentiality} \qquad \xrightarrow[\text{Solution}]{} ?$$

$$\text{Immutable} \xrightarrow[\text{Problem}]{} \text{scalability} \qquad \xrightarrow[\text{Solution}]{} ?$$

$$\text{Paying} \xrightarrow[\text{Problem}]{} \text{cost} \qquad \xrightarrow[\text{Solution}]{} ?$$

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Zero-knowledge proofs (ZKP)

Alice

I know the solution to this complex equation

Bob

No idea what the solution is but Alice claims to know it

Challenge
$\longleftarrow$

Response
$\longrightarrow$

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- **Sound**: **Alice** has a wrong solution $\implies$ **Bob** is not convinced.
- **Complete**: **Alice** has the solution $\implies$ **Bob** is convinced.
- **Zero-knowledge**: **Bob** does NOT learn the solution.

Example: Sigma protocol

Alice

Bob

I know x such that $g^x = y$

Example: Sigma protocol

Alice **Bob**

I know x such that $g^x = y$

$n \xleftarrow{\$} \mathbb{Z}_r$ $\xrightarrow{A = g^n}$

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$$\xleftarrow{c}$$

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$s = n + c \cdot x$ $\xrightarrow{s}$

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$n \xleftarrow{\$} \mathbb{Z}_r$ $\xrightarrow{A = g^n}$

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$s = n + c \cdot x$ $\xrightarrow{s}$ $g^s \stackrel{?}{=} A \cdot y^c$

with $A \cdot y^c = g^n \cdot g^{x \cdot c}$

then $g^n \cdot g^{x \cdot c} = g^{n + x \cdot c}$

Non-Interactive Zero-Knowledge (NIZK) Sigma protocol

Alice

Bob

I know x such that $g^x = y$

$$n \xleftarrow{\$} \mathbb{Z}_r$$
$$A = g^n$$
$$c = H(A, y)$$
$$s = n + c \cdot x$$

$$\xrightarrow{\pi = (A, c, s)}$$

$$g^s \stackrel{?}{=} A \cdot y^c$$
$$c \stackrel{?}{=} H(A, y)$$

Non-Interactive Zero-Knowledge (NIZK) Sigma protocol

Alice

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I know x such that $g^x = y$

$$n \xleftarrow{\$} \mathbb{Z}_r$$

$$\underbrace{g}_{\text{Setup}} ;\ A = g^n$$

$$\underbrace{\pi = (A, c, s)}_{\text{proof}} \longrightarrow$$

$$c = H(A, y)$$

$$\underbrace{s = n + c \cdot x}_{\text{Prove}}$$

$$g^s \stackrel{?}{=} A \cdot y^c$$

$$\underbrace{c \stackrel{?}{=} H(A, y)}_{\text{Verify}}$$

ZKP families

Expressivity

- *specific* statement vs. *general* statement

ZKP families

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Deployability

- *interactive* vs. $non - interactive$ protocol
- *trapdoored* setup vs. *transparent* setup
- *Designated* verifier vs. *any* verifier

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Complexity

- prover complexity (Alice)
- verifier complexity (Bob)
- communication complexity (size of the proof and the setup)

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Security

- Cryptographic assumptions
- Cryptographic primitives

Blockchains and ZKP

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Transparent $\xrightarrow[\text{Problem}]{}$ confidentiality $\xrightarrow[\text{Solution}]{}$ *ZKP*

setup, prover?, verifier?

Immutable $\xrightarrow[\text{Problem}]{}$ scalability $\xrightarrow[\text{Solution}]{}$ *ZKP*

Communication complexity

Paying $\xrightarrow[\text{Problem}]{}$ cost $\xrightarrow[\text{Solution}]{}$ *ZKP*

Verifier complexity, *prover*?

ZKP literature landmarks

- First ZKP work [GMR85]
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- Pairing-based SNARK with shortest proof and verifier time [Groth16]
- SNARK with universal and updatable setup [GKMMM18, BKMM19 (Sonic), GWC19 (PlonK), CHMMVW19 (Marlin),...]

What is a zero-knowledge proof?

"I have a *sound*, *complete* and *zero-knowledge* proof that a statement is true". [GMR85]

Sound

False statement $\implies$ cheating prover cannot convince honest verifier.

Complete

True statement $\implies$ honest prover convinces honest verifier.

Zero-knowledge

True statement $\implies$ verifier learns nothing other than statement is true.

zk-SNARK: Zero-Knowledge Succinct Non-interactive ARgument of Knowledge

"I have a *computationally sound*, *complete*, *zero-knowledge*, **succinct**, **non-interactive** proof that a statement is true and that I know a related secret".

Succinct

A proof is very "short" and "easy" to verify.

Non-interactive

No interaction between the prover and verifier for proof generation and verification (except the proof message).

ARgument of Knowledge

Honest verifier is convinced that a computationally bounded prover knows a secret information.

Preprocessing zk-SNARK for NP language

F: public NP program, x, z: public inputs, w: private input

$$z := F(x, w)$$

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$$\textit{Setup}: \quad (pk, vk) \quad \leftarrow \quad S(F, 1^\lambda)$$

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Verify :	`false/true`	$\leftarrow$	$V(x, z, \pi, vk)$

Anyone

$(pk, vk) \leftarrow S(F, 1^\lambda)$

pk $\qquad\qquad$ vk

Alice (prover) $\qquad\qquad$ Bob (verifier)

$\pi \leftarrow P(x, z, w, pk) \xrightarrow{\quad\pi\quad} V(x, z, \pi, vk)$?

(Trapdoored) preprocessing zk-SNARK for NP language

F: public $\mathtt{NP}$ program, x, z: public inputs, w: private input

$$z := F(x, w)$$

A zk-SNARK consists of algorithms S, P, V s.t. for a security parameter λ:

Setup :	(pk, vk)	$\leftarrow$	$S(F, \tau, 1^\lambda)$
Prove :	π	$\leftarrow$	$P(x, z, w, pk)$
Verify :	$\mathtt{false/true}$	$\leftarrow$	$V(x, z, \pi, vk)$

$$\begin{tikzcd}
 & \begin{array}{c}\text{TTP (secret trapdoor } \tau\text{)}\\ (pk, vk) \leftarrow S(F, \tau, 1^\lambda)\end{array} \arrow[dl, dotted, "pk"'] \arrow[dr, dotted, "vk"] & \\
\begin{array}{c}\text{Alice (prover)}\\ \pi \leftarrow P(x, z, w, pk)\end{array} \arrow[rr, "\pi"] & & \begin{array}{c}\text{Bob (verifier)}\\ V(x, z, \pi, vk)?\end{array}
\end{tikzcd}$$

(Trapdoored) preprocessing zk-SNARK for NP language

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\end{array}$$

MPC (secret trapdoor τ)

$(pk, vk) \leftarrow S(F, \tau, 1^{\lambda})$

pk vk

Alice (prover) Bob (verifier)

$\pi \leftarrow P(x, z, w, pk) \xrightarrow{\pi} V(x, z, \pi, vk)$?

zk-SNARK

Succinctness: A proof is very "short" and "easy" to verify.

Definition [BCTV14b]

A succinct proof π has size $O_\lambda(1)$ and can be verified in time $O_\lambda(|F| + |x| + |z|)$, where $O_\lambda(.)$ is some polynomial in the security parameter λ.

zk-SNARKs in a nutshell

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1. Reduce a "general statement" satisfiability to a polynomial equation satisfiability.

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3. Use homomorphic hiding cryptography to blindly verify the polynomial equation.

zk-SNARKs in a nutshell

Main ideas:

1. Reduce a "general statement" satisfiability to a polynomial equation satisfiability.
2. Use Schwartz-Zippel lemma to succinctly verify the polynomial equation with high probability.
3. Use homomorphic hiding cryptography to blindly verify the polynomial equation.
4. Make the protocol non-interactive.

Arithmetization

Statement $\rightarrow$ **Arithmetic circuit** $\rightarrow$ Intermediate representation $\rightarrow$ Polynomial identities $\rightarrow$ zk-SNARK proof

$$x^3 + x + 5 = 35 \qquad ({\color{red}x = 3})$$

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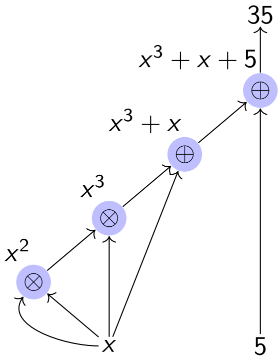

Statement $\rightarrow$ Arithmetic circuit $\rightarrow$ **Intermediate representation** $\rightarrow$ Polynomial identities $\rightarrow$ zk-SNARK proof

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$O = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

witness:

$$\begin{aligned} \vec{w} &= \begin{pmatrix} \text{one} & x & d & a & b & c \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 & 35 & 9 & 27 & 30 \end{pmatrix} \end{aligned}$$

$$O \bullet \vec{w} = L \bullet \vec{w} \cdot R \bullet \vec{w}$$

Arithmetization

e.g. Quadratic Arithmetic Program

Statement → Arithmetic circuit → Intermediate representation → **Polynomial identities** → zk-SNARK proof

$$L(X)R(X) - O(X) = H(X)T(X) \qquad (QAP \in \mathbb{F}[X])$$

Arithmetization

e.g. Quadratic Arithmetic Program

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$$L(X)R(X) - O(X) = H(X)T(X) \qquad (QAP \in \mathbb{F}[X])$$

$$L(\tau)R(\tau) - O(\tau) = H(\tau)T(\tau) \qquad (\textit{trapdoor } \tau \xleftarrow{\$} \mathbb{F})$$

Arithmetization

e.g. Quadratic Arithmetic Program

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$$
\begin{aligned}
L(X)R(X) - O(X) &= H(X)T(X) && (QAP \in \mathbb{F}[X]) \\
L(\tau)R(\tau) - O(\tau) &= H(\tau)T(\tau) && (\textit{trapdoor } \tau \xleftarrow{\$} \mathbb{F}) \\
C(L(\tau)R(\tau) - O(\tau)) &= C(H(\tau)T(\tau)) && (\textit{Homomorphic commitment})
\end{aligned}
$$

Succinct evaluation of polynomials

Instead of verifying the QAP on the whole domain $\mathbb{F}$ $\rightarrow$ verify it in a single random point $\tau \in \mathbb{F}$.

Schwartz–Zippel lemma

Any two distinct polynomials of degree d over a field $\mathbb{F}$ can agree on at most a $d/|\mathbb{F}|$ fraction of the points in $\mathbb{F}$.

Blind evaluation of polynomials

Statement $\rightarrow$ Arithmetic circuit $\rightarrow$ Intermediate representation $\rightarrow$ Polynomial identities $\rightarrow$ **zk-SNARK proof**

Let's take the example of polynomial L:

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Let's take the example of polynomial L:

- Alice can send L to Bob and he computes $L(\tau)$ $\rightarrow$ breaks the zero-knowledge.

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- Alice can send L to Bob and he computes $L(\tau)$ $\rightarrow$ breaks the zero-knowledge.
- Bob can send τ to Alice and she computes $L(\tau)$ $\rightarrow$ breaks the soundness.

Blind evaluation of polynomials

Statement $\rightarrow$ Arithmetic circuit $\rightarrow$ Intermediate representation $\rightarrow$ Polynomial identities $\rightarrow$ **zk-SNARK proof**

Let's take the example of polynomial L:

- Alice can send L to Bob and he computes $L(\tau)$ $\rightarrow$ breaks the zero-knowledge.
- Bob can send τ to Alice and she computes $L(\tau)$ $\rightarrow$ breaks the soundness.

$\Longrightarrow$ homomorphic cryptography to evaluate $L(X)$ at τ without Bob learning L nor Alice learning τ.

Blind evaluation of polynomials

$$L(\tau) = l_0 + l_1\tau + l_2\tau^2 + \cdots + l_d\tau^d \in \mathbb{F}$$
$$C(L(\tau)) = l_0 C(1) + l_1 C(\tau) + l_2 C(\tau^2) + \cdots + l_d C(\tau^d)$$

Somewhat homomorphic commitment w.r.t.:

- depth-d **additions** (arbitrary d)
- depth-1 **multiplications** (for $L(\tau) \cdot R(\tau)$ and $H(\tau) \cdot T(\tau)$).

Blind evaluation of polynomials

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$$C(\tau) = \tau G \qquad (DL)$$
$$L(\tau)G = l_0 G + l_1 \tau G + l_2 \tau^2 G + \cdots + l_d \tau^d G$$

Blind evaluation of polynomials

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- depth-1 **multiplications** (for $L(\tau) \cdot R(\tau)$ and $H(\tau) \cdot T(\tau)$).

$$C(\tau_1) = \tau_1 G;\ C(\tau_2) = \tau_2 G$$
$$C(\tau_1) \cdot C(\tau_2) = C(\tau_1 \cdot \tau_2) \qquad (?)$$

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$$C(\tau_1) \cdot C(\tau_2) = C(\tau_1 \cdot \tau_2) \qquad (?)$$

$$\underbrace{e(C(\tau_1), C(\tau_2))}_{\text{product of commitments}} = \underbrace{Z^{\tau_1 \cdot \tau_2}}_{\substack{\text{new commitment to } \tau_1 \cdot \tau_2 \\ \text{(where } Z = e(G, G))}} \qquad \text{(bilinear pairing)}$$

Blind evaluation of QAP

Blind evaluation can be achieved with *black-box* pairings:

$$
\begin{aligned}
e(C(H(\tau)), C(T(\tau)) \cdot e(C(O(\tau)), C(1)) &= e(C(L(\tau)), C(R(\tau))) \\
e(H(\tau)G, T(\tau)G) \cdot e(O(\tau)G, G) &= e(L(\tau)G, R(\tau)G) \\
e(G, G)^{H(\tau)T(\tau)} \cdot e(G, G)^{O(\tau)} &= e(G, G)^{L(\tau)R(\tau)} \\
Z^{H(\tau)T(\tau)+O(\tau)} &= Z^{L(\tau)R(\tau)}
\end{aligned}
$$

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- Proof composition for better confidentiality and scalability $\rightarrow$ 2-chains and 2-cycles **[CANS 2020, EuroCrypt 2022, DCC 2022]**

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- Pairings in R1CS for fast generation of the composed proof **[ACNS 2023]**

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- Multi-scalar multiplication for fast generation of proofs **[(in submission), zprize winner]**

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Instantiation

DL:

- $E\colon y^2 = x^3 + ax + b$ elliptic curve defined over $\mathbb{F}_q$, q a prime power.
- r prime divisor of $\#E(\mathbb{F}_q) = q + 1 - t$, t Frobenius trace.

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Pairing-friendly:

- small embedding degree k (smallest integer $k \in \mathbb{N}^*$ s.t. $r \mid q^k - 1$).
- $\mathbb{G}_1 \subset E(\mathbb{F}_q)$ and $\mathbb{G}_2 \subset E(\mathbb{F}_{q^k})$ two groups of order r.
- $\mathbb{G}_T \subset \mathbb{F}_{q^k}^*$ group of r-th roots of unity.
- pairing $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$.

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SNARK-friendly:

- $r - 1 \equiv 0 \mod 2^L$ for some large $L \in \mathbb{N}^*$ ($\mathbb{F}_r$ FFT-friendly)

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- r prime divisor of $\#E(\mathbb{F}_q) = q + 1 - t$, t Frobenius trace.

Pairing-friendly:

- small embedding degree k (smallest integer $k \in \mathbb{N}^*$ s.t. $r \mid q^k - 1$).
- $\mathbb{G}_1 \subset E(\mathbb{F}_q)$ and $\mathbb{G}_2 \subset E(\mathbb{F}_{q^k})$ two groups of order r.
- $\mathbb{G}_T \subset \mathbb{F}_{q^k}^*$ group of r-th roots of unity.
- pairing $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$.

SNARK-friendly:

- $r - 1 \equiv 0 \mod 2^L$ for some large $L \in \mathbb{N}^*$ ($\mathbb{F}_r$ FFT-friendly)

BLS12-381: q-bit=381, r-bit=255, $k = 12$, $L = 32$

SNARK-friendly curves from the literature

[D. F. Aranha, Y.EH, A. Guillevic - DCC 2022]

Curve	seed x	L	$r = \#\mathbb{G}_1$ (bits)	p, $\mathbb{G}_1$ (bits)	$p^{k/d}$, $\mathbb{G}_2$ (bits)	$p \equiv 3 \bmod 4$	security (bits) $\mathbb{G}_1$	$\mathbb{F}^*_{p^k}$
BN-256 [PHGR13]	1868033^3 $\mathrm{HW}_{\text{2-NAF}}(6x+2) = 19$	5	256	256	512	✓	128	103
BN-254 [BFR+13]	$2^{62} - 2^{54} + 2^{44}$ $\mathrm{HW}_{\text{2-NAF}}(6x+2) = 7$	45	254	254	508	×	127	102
GMV6-183 [BCG+13]	`0x8eed757d90615e40000000` $\mathrm{HW}(-26x-2) = 16$	31	181	183	549	NA	90	71
BN-254 [BCTV14b]	`0x44e992b44a6909f1` $\mathrm{HW}_{\text{2-NAF}}(6x+2) = 22$	28	254	254	508	✓	127	103
BLS12-381 [Bow17]	`-0xd201000000010000` $\mathrm{HW}(x) = 6$	32	255	381	762	✓	127	126

Families of SNARK-friendly curves [D. F. Aranha, Y.EH, A. Guillevic - DCC 2022]

Family, $r, p \in \mathbb{N},\ t \in \mathbb{Z}$	$r \equiv 1 \bmod 2^L$	$p \equiv 3 \bmod 4$	$\mathbb{G}_2$ coord. in
BN	$x \equiv 2570880382155688433 \bmod 2^{63} \Rightarrow 2^{64} \mid r-1$	✓	$\mathbb{F}_{p^2}$
any x	$x \equiv 0 \bmod 2^{L-1} \Rightarrow 2^L \mid r-1,\ 2^L \mid p-1$	✗	
BLS12	$x \equiv 1 \bmod 3 \cdot 2^{L-1} \Rightarrow 2^L \mid r-1,\ 2^{L-1} \mid p-1$	✗	
$x \equiv 1$	$x \equiv 2^{L-1}-1 \bmod 3 \cdot 2^{L-1} \Rightarrow 2^L \mid r-1,\ 6 \mid p-1$	✓	$\mathbb{F}_{p^2}$
mod 3	$x \equiv 2^{L/2} \bmod 3 \cdot 2^{L/2} \Rightarrow 2^L \mid r-1,\ 6 \mid p-1$	✓	
BLS24	$x \equiv 1 \bmod 3 \cdot 2^{L-2} \Rightarrow 2^L \mid r-1,\ 2^{L-2} \mid p-1$	✗	
$x \equiv 1$	$x \equiv 2^{L-1}-1 \bmod 3 \cdot 2^{L-2} \Rightarrow 2^L \mid r-1,\ 6 \mid p-1$	✓	$\mathbb{F}_{p^4}$
mod 3	$x \equiv 2^{L/4} \bmod 3 \cdot 2^{L/4} \Rightarrow 2^L \mid r-1,\ 6 \mid p-1$	✓	
MNT4, $t = x+1$	$x \equiv 0 \bmod 2^{L/2} \Rightarrow 2^L \mid r-1, 2^{L/2} \mid p-1$	✗	$\mathbb{F}_{p^2}$
MNT6	$x \equiv 0 \bmod 2^{L-1} \Rightarrow 2^L \mid r-1, 2^{2L} \mid p-1$	✗	$\mathbb{F}_{p^3}$
GMV6($h=4$) any x	$x \equiv 0 \bmod 2^{L-1} \Rightarrow 2^L \mid r-1,\ 2^{L-1} \mid p-1$	NA	$\mathbb{F}_{p^3}$
KSS16 ($x \equiv \pm 25 \bmod 70$)	$\pm 1439818652098642188 5, \pm 3745661661312336140 5$ mod $35 \cdot 2^{62} \Rightarrow 2^{64} \mid r-1,\ p \equiv 1 \bmod 4$	✗	$\mathbb{F}_{p^4}$
KSS18 ($x \equiv 14 \bmod 42$)	$x = 14 \cdot 2^{L/3} \bmod 42 \cdot 2^{L/3} \Rightarrow 2^L \mid r-1,\ 12 \mid p-1$	NA	$\mathbb{F}_{p^3}$

New SNARK-friendly curves

[D. F. Aranha, Y.EH, A. Guillevic - DCC 2022]

Curve	x	L	$r = \#\mathbb{G}_1$ (bits)	p, $\mathbb{G}_1$ (bits)	$p^{k/d}$, $\mathbb{G}_2$ (bits)	$p \equiv 3 \bmod 4$	security (bits) $\mathbb{G}_1$	security (bits) $\mathbb{F}^*_{p^k}$
BN383	`0x49e69d16fdc80216226909f1` $\mathsf{HW}_{2\text{-NAF}}(6x+2) = 30$	44	383	383	766	✓	191	123
BLS24-317	`0xd9018000` $\mathsf{HW}_{2\text{-NAF}}(x) = 6$	60	255	317	1268	✓	127	160
KSS16-329	`0x38fab7583` $\mathsf{HW}(x) = 12$	19	255	329	1316	✓	127	140
KSS18-345	`0xc0c44000000` $\mathsf{HW}(x) = 6$	78	254	345	690	NA	127	150

https://github.com/yelhousni/gnark-crypto

Co-factor clearing and subgroup membership

[Y.EH, A. Guillevic, T. Piellard - AfricaCrypt 2022]

$$e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$$

- Pairing groups: $\mathbb{G}_1, \mathbb{G}_2$ and $\mathbb{G}_T$ are sub-groups of some prime order r.
- They are defined over some larger groups of composite orders $\underbrace{c_{1,2,T}}_{\text{co-factors}} \times r$

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Let P be a random element of order $c_1 \times r$

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Let Q be a random element of order $c_{1,2,T} \times r$

- Subgroup membership testing: $[r]Q \stackrel{?}{=} \mathcal{O}$

Co-factor clearing and subgroup membership

[Y.EH, A. Guillevic, T. Piellard - AfricaCrypt 2022]

Proposition ($\mathbb{G}_1$ co-factor clearing)

Many curve families have the $\mathbb{G}_1$ cofactor of the form $c_1 = 3\ell^2$. To clear this cofactor, the map $P \mapsto [\ell]P$ is sufficient for all curves in [FST10] except KSS and 6.6 where $k \equiv 2, 3 \mod 6$.

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Theorem ($\mathbb{G}_1$ and $\mathbb{G}_2$ membership testing)

Let $q' = q$ or resp. q^k and $c' = c_1$ or resp. c_2. If ψ acts as the multiplication by λ on $E(\mathbb{F}_{q'})[r]$ and $\gcd(\chi(\lambda), c') = 1$ *then*

$$\psi(Q) = [\lambda]Q \iff Q \in E(\mathbb{F}_{q'})[r]$$

with χ the characteristic polynomial of ψ.

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Proposition ($\mathbb{G}_T$ membership testing)

For $z \in \mathbb{F}_{p^k}^*$ and Φ_k the k-th cyclotomic polynomial, we have:

$$z^{\Phi_k(p)} = 1 \text{ and } z^p = z^{t-1} \text{ and } \gcd(p + 1 - t, \Phi_k(p)) = r \implies z^r = 1 \ .$$

Overview

1. Motivation
2. zk-SNARK
3. SNARK-friendly curves
4. SNARK-friendly 2-chains
5. Pairings in R1CS
6. Multi-scalar multiplication
7. Conclusion

A pairing-based SNARK

Example: Groth16 [Gro16]

Given an instance $\Phi = (a_0, \ldots, a_\ell) \in \mathbb{F}_r^\ell$ of a public `NP` program F

A pairing-based SNARK

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Given an instance $\Phi = (a_0, \ldots, a_\ell) \in \mathbb{F}_r^\ell$ of a public $\mathtt{NP}$ program F

- Setup: $(pk, vk) \leftarrow S(F, \tau, 1^\lambda)$ where

$$vk = (vk_{\alpha,\beta}, \{vk_{\pi_i}\}_{i=0}^{\ell}, vk_\gamma, vk_\delta) \in \mathbb{G}_T \times \mathbb{G}_1^{\ell+1} \times \mathbb{G}_2 \times \mathbb{G}_2$$

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$$\pi = (A, B, C) \in \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_1 \qquad (O_\lambda(1))$$

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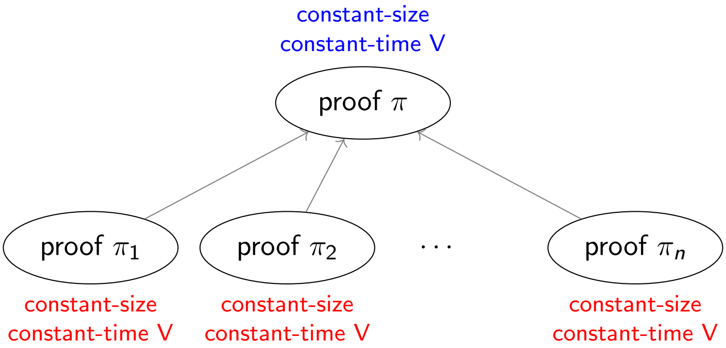
- Verify: $0/1 \leftarrow V(\Phi, \pi, vk)$ where V is

$$e(A, B) = vk_{\alpha,\beta} \cdot e(vk_x, vk_\gamma) \cdot e(C, vk_\delta) \qquad (O_\lambda(|\Phi|)) \tag{1}$$

  and $vk_x = \sum_{i=0}^{\ell} [a_i] vk_{\pi_i}$ depends only on the instance Φ and $vk_{\alpha,\beta} = e(vk_\alpha, vk_\beta)$ can be computed in the trusted setup for $(vk_\alpha, vk_\beta) \in \mathbb{G}_1 \times \mathbb{G}_2$.

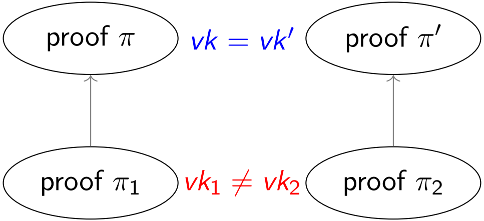
Proof composition: why?

Aggregation:

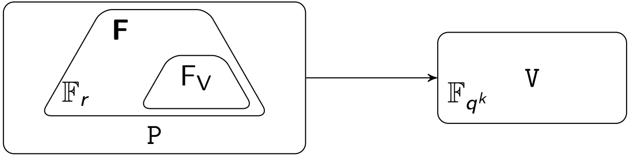


Proof composition: why?

Decentralized private computation (DPC):



Proof composition: how?



- **F** any program is expressed in $\mathbb{F}_r$
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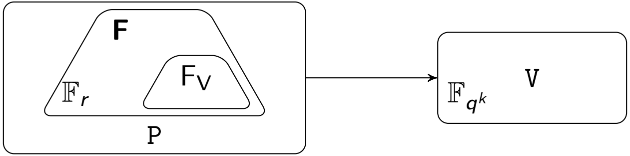

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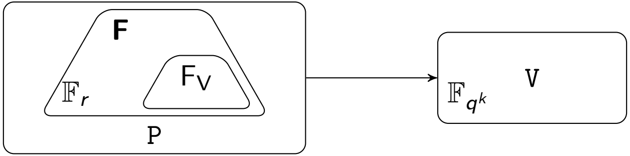

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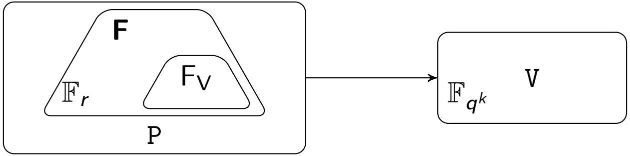

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- 2nd attempt: simulate $\mathbb{F}_q$ operations via $\mathbb{F}_r$ operations ($\times \log q$ blowup)
- 3rd attempt: use a cycle/chain of pairing-friendly elliptic curves [CFH+15, BCTV14a, BCG+20]

2-cycles and 2-chains

A 2-cycle of elliptic curves:

$$\begin{tikzcd} E_1(\mathbb{F}_q) \arrow[d, bend left, "\#E_0(\mathbb{F}_p) = q"] \\ E_0(\mathbb{F}_p) \arrow[u, bend left, "\#E_1(\mathbb{F}_q) = p"] \end{tikzcd}$$

A 2-chain of elliptic curves:

$$\begin{tikzcd} E_1(\mathbb{F}_q) \\ E_0(\mathbb{F}_p) \arrow[u, "\#E_1(\mathbb{F}_q) = h \cdot p"'] \end{tikzcd}$$

2-chains of elliptic curves

SNARK-0 with a pairing e: $\mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_T$ $\#\mathbb{G}_i = r_0$

pairing statement

computation in $\mathbb{F}_p[X]$

proof of statement

SNARK-1 with a pairing e: $\mathbb{G}'_1 \times \mathbb{G}'_2 \to \mathbb{G}'_T$ $\#\mathbb{G}'_i = p$

inner curve:
curve $E_0(\mathbb{F}_p)$
of prime order r_0

outer curve:
curve $E_1(\mathbb{F}_q)$
of order $h \times p$

Given p, search for a pairing-friendly curve E_1 of order $h \cdot p$ over a field $\mathbb{F}_q$

SNARK-friendly curves, 2-cycles and 2-chains

- SNARK
  - $E/\mathbb{F}_q$ — BN, BLS12, BW12?, KSS16? ... [FST10]
    - pairing-friendly
    - $2^L \mid r - 1$
- Recursive SNARK (2-cycle)
  - $E_0/\mathbb{F}_p$ and $E_1/\mathbb{F}_q$ — MNT4/MNT6 [FST10, Sec.5], ? [CCW19]
    - both pairing-friendly
    - $\#E_1(\mathbb{F}_q) = p$ and $\#E_0(\mathbb{F}_p) = q$
    - $2^L \mid p - 1$
    - $2^L \mid q - 1$
- Recursive SNARK (2-chain)
  - $E_0/\mathbb{F}_p$ — BLS12 ($x \equiv 1 \bmod 3 \cdot 2^L$) [BCG+20], ?
    - pairing-friendly
    - $2^L \mid r_0 - 1$ ($r_0 \mid \#E_0(\mathbb{F}_p)$)
    - $2^L \mid p - 1$
  - $E_1/\mathbb{F}_q$ — Cocks–Pinch algorithm [ZEXE]
    - pairing-friendly
    - $p \mid \#E_1(\mathbb{F}_q)$

2-chains: outer curve $E_1/\mathbb{F}_q$

- q is a prime or a prime power
- t is relatively prime to q
- ~~r is prime~~
- $\cancel{r \mid q^k - 1}$
- $\cancel{r \mid q + 1 - t}$

r is a **fixed** chosen prime
s.t. $r \mid q + 1 - t$
and $r \mid q^k - 1$

- $4q - t^2 = Dy^2$ (for $D < 10^{12}$) and some integer y

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Algorithm: Cocks–Pinch method

Fix k and D and choose a prime r s.t. $k|r-1$ and $\left(\frac{-D}{r}\right) = 1$;
Compute $t = 1 + x^{(r-1)/k}$ for x a generator of $(\mathbb{Z}/r\mathbb{Z})^\times$;
Compute $y = (t-2)/\sqrt{-D} \bmod r$;
Lift t and y in $\mathbb{Z}$;
Compute $q = (t^2 + Dy^2)/4$ (in $\mathbb{Q}$);

2-chains: outer curve $E_1/\mathbb{F}_q$

- $\rho = \log_2 q / \log_2 r \approx 2$ (because $q = f(t^2, y^2)$ and $t, y \xleftarrow{\$} \bmod r$).
- The curve parameters (q, r, t) are not expressed as polynomials.

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- The curve parameters (q, r, t) are not expressed as polynomials.

Algorithm: Brezing–Weng method

Fix k and D and choose an irreducible polynomial $r(x) \in \mathbb{Z}[x]$ with positive leading coefficient s.t. $\sqrt{-D}$ and the primitive k-th root of unity ζ_k are in $K = \mathbb{Q}[x]/r(x)$;
Choose $t(x) \in \mathbb{Q}[x]$ be a polynomial representing $\zeta_k + 1$ in K;
Set $y(x) \in \mathbb{Q}[x]$ be a polynomial mapping to $(\zeta_k - 1)/\sqrt{-D}$ in K;
Compute $q(x) = (t^2(x) + Dy^2(x))/4$ in $\mathbb{Q}[x]$;

2-chains: outer curve $E_1/\mathbb{F}_q$

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- $\rho = 2 \max(\deg t(x), \deg y(x)) / \deg r(x) < 2$
- $r(x), q(x), t(x)$ but does $\exists\ x_0 \in \mathbb{Z}^*, r(x_0) = r_{\text{fixed}}$ and $q(x_0)$ is prime ?

2-chains: outer curve $E_1/\mathbb{F}_q$

[Y.EH, A. Guillevic - CANS 2020]

1. Cocks–Pinch method
   - $k = 6$ and $-D = -3$ $\implies$ 128-bit security, $\mathbb{G}_2$ coordinates in $\mathbb{F}_q$ (pairing over $\mathbb{F}_q$ instead if $\mathbb{F}_{q^3}$), GLV multiplication over $\mathbb{G}_1$ and $\mathbb{G}_2$
   - restrict search to size$(q) \leq 768$ bits $\implies$ smallest machine-word size

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   - restrict search to size$(q) \leq 768$ bits $\implies$ smallest machine-word size
2. Brezing–Weng method
   - choose $r(x) = q_{\mathrm{BLS12}}(x)$
   - $q(x) = (t^2(x) + 3y^2(x))/4$ factors $\implies$ $q(x_0)$ cannot be prime
   - lift in $\mathbb{Z}$ $t = r \times h_t + t(x_0)$ and $y = r \times h_y + y(x_0)$ [FK19, GMT20]

2-chains: outer curve $E_1/\mathbb{F}_q$

[Y.EH, A. Guillevic - CANS 2020]

$E: y^2 = x^3 - 1$ over $\mathbb{F}_q$ of 761-bit with seed $x_0 = \mathtt{0x8508c00000000}$ and polynomials:

Our curve, $k = 6$, $D = 3$, $r = q_{\mathrm{BLS12}}$
$r(x) = (x^6 - 2x^5 + 2x^3 + x + 1)/3 = q_{\mathrm{BLS12-377}}(x)$ $t(x) = x^5 - 3x^4 + 3x^3 - x + 3 + h_t r(x)$ $y(x) = (x^5 - 3x^4 + 3x^3 - x + 3)/3 + h_y r(x)$ $q(x) = (t^2 + 3y^2)/4$ $q_{h_t=13,h_y=9}(x) = (103x^{12} - 379x^{11} + 250x^{10} + 691x^9 - 911x^8$ $-79x^7 + 623x^6 - 640x^5 + 274x^4 + 763x^3 + 73x^2 + 254x + 229)/9$

SNARK-0: inner curves

[Y.EH, A. Guillevic - EuroCrypt 2022]

Groth16 SNARK

- 128-bit security
- pairing-friendly
- efficient $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_T$ and pairing
- $p - 1 \equiv r - 1 \equiv 0 \mod 2^L$ for large input $L \in \mathbb{N}^*$ (FFTs)

$\rightarrow$ BLS ($k = 12$) family of ≈ 384 bits with seed $x \equiv 1 \mod 3 \cdot 2^L$

SNARK-0: inner curves

[Y.EH, A. Guillevic - EuroCrypt 2022]

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- pairing-friendly
- efficient $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_T$ and pairing
- $p - 1 \equiv r - 1 \equiv 0 \mod 2^L$ for large input $L \in \mathbb{N}^*$ (FFTs)

$\to$ BLS ($k = 12$) family of ≈ 384 bits with seed $x \equiv 1 \mod 3 \cdot 2^L$

Universal SNARK

- 128-bit security
- pairing-friendly
- efficient $\mathbb{G}_1$, $\cancel{\mathbb{G}_2}$, $\cancel{\mathbb{G}_T}$ and pairing
- $p - 1 \equiv r - 1 \equiv 0 \mod 2^L$ for large $L \in \mathbb{N}^*$ (FFTs)

$\to$ BLS ($k = 24$) family of ≈ 320 bits with seed $x \equiv 1 \mod 3 \cdot 2^L$

SNARK-1: outer curves

[Y.EH, A. Guillevic - EuroCrypt 2022]

Groth16 SNARK

- 128-bit security
- pairing-friendly
- efficient $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_T$ and pairing
- $r = p \ (r - 1 \equiv 0 \mod 2^L)$

$\rightarrow$ BW ($k = 6$) family of ≈ 768 bits with $(t \mod x) \mod r \equiv 0$ or 3

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$\rightarrow$ BW ($k = 6$) family of $\approx$ 704 bits with $(t \mod x) \mod r \equiv 0$ or 3
$\rightarrow$ CP ($k = 8$) family of $\approx$ 640 bits
$\rightarrow$ CP ($k = 12$) family of $\approx$ 640 bits

All $\mathbb{G}_i$ formulae and pairings are given in terms of x and some $h_t, h_y \in \mathbb{N}$.

Implementation and benchmark

[Y.EH, A. Guillevic - EuroCrypt 2022]

Short list of 2-chains with some additional nice engineering properties:

- Groth16: BLS12-377 and BW6-761
- Universal: BLS24-315 and BW6-633 (or BW6-672)

Table: Groth16 (ms)

	S	P	V
BLS12-377	387	34	1
BLS24-315	501	54	4
BW6-761	1226	114	9
BW6-633	710	69	6
BW6-672	840	74	7

Table: Universal (ms)

	S	P	V
BLS12-377	87	215	4
BLS24-315	76	173	1
BW6-761	294	634	9
BW6-633	170	428	6
BW6-672	190	459	7

(on aAMD EPYC 7R32 AWS (c5a.24xlarge) machine)

`https://github.com/ConsenSys/gnark-crypto`

Overview

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4 SNARK-friendly 2-chains

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Cost of pairing-based SNARKs

Table: Cost of $\mathtt{S}$, $\mathtt{P}$ and $\mathtt{V}$ algorithms for Groth16 and Universal. n =number of multiplication gates, a =number of addition gates and ℓ =number of public inputs. $\mathtt{M}_{\mathbb{G}}$ =multiplication in $\mathbb{G}$ and $\mathtt{P}$=pairing.

	Setup	Prove	Verify
Groth16	$3n\ \mathtt{M}_{\mathbb{G}_1},\ n\ \mathtt{M}_{\mathbb{G}_2}$	$(4n-\ell)\ \mathtt{M}_{\mathbb{G}_1},\ n\ \mathtt{M}_{\mathbb{G}_2}$	$3\ \mathtt{P},\ \ell\ \mathtt{M}_{\mathbb{G}_1}$
Universal (PLONK-KZG)	$d_{\geq n+a}\ \mathtt{M}_{\mathbb{G}_1},\ 1\ \mathtt{M}_{\mathbb{G}_2}$	$9(n+a)\ \mathtt{M}_{\mathbb{G}_1}$	$2\ \mathtt{P},\ 18\ \mathtt{M}_{\mathbb{G}_1}$

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F_V: program that checks $\mathtt{V}$ (eq. 1) ($\ell = 1$, $n = 90000$)

Pairings out-circuit

ate pairing

$$
\begin{aligned}
e : \mathbb{G}_1 \times \mathbb{G}_2 &\to \mathbb{G}_T \\
(P, Q) &\mapsto f_{t-1,Q}(P)^{(q^k-1)/r}
\end{aligned}
$$

- $f_{t-1,Q}(P)$ is the Miller function
- $f \mapsto f^{(q^k-1)/r}$ is the final exponentiation

Examples: For polynomial families in the seed x,

BLS12 $e(P, Q) = f_{x,Q}(P)^{(q^{12}-1)/r}$

BLS24 $e(P, Q) = f_{x,Q}(P)^{(q^{24}-1)/r}$

[BN06, AKL+11, ABLR14, ABLR14, Sco19] [HHT20, AFK+13, GF16, GS10, Kar13]

Pairings out-circuit: Miller algorithm

Algorithm: MillerLoop(s, P, Q)

Output: $m = f_{s,Q}(P)$

$m \leftarrow 1$; $R \leftarrow Q$

for *b from the second most significant bit of s to the least* **do**

  $\ell \leftarrow \ell_{R,R}(P)$; $R \leftarrow [2]R$; $v \leftarrow v_{[2]R}(P)$ Doubling Step

  $m \leftarrow m^2 \cdot \ell / v$

  if $b = 1$ **then**

    $\ell \leftarrow \ell_{R,Q}(P)$; $R \leftarrow R + Q$; $v \leftarrow v_{R+Q}(P)$ Addition Step

    $m \leftarrow m \cdot \ell / v$

return m

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Pairings in-circuit (R1CS)

[Y.EH - ACNS 2023]

	Time	Constraints
BLS12-377	< 1 ms	$\approx$ **80 000**

Inverses, in R1CS, cost (almost) as much as multiplications !

- Miller loop:
  - Affine coordinates $\rightarrow\approx$ 19k (`arkworks`)
  - Division in extension fields
  - Double-and-Add in affine
  - lines evaluations (1/y, x/y)
  - Loop with short addition chains
  - Torus-based arithmetic
- Final Exponentiation:
  - Karatsuba cyclotomic squarings
  - Torus-based arithmetic
  - Exp. with short addition chains

19k $\rightarrow\approx$ 11k (`gnark`)

Pairings in-circuit (R1CS)

[Y.EH - ACNS 2023]

e.g. For BLS12-377,

https://github.com/ConsenSys/gnark

	Constraints
Pairing	**11535**
Groth16 verifier	19378
BLS sig. verifier	14888
KZG verifier	20679

For BLS24-315, a pairing is **27608** contraints .

More optimizations in mind:

- Quadruple-and-Add Miller loop [CBGW10]
- Fixed argument Miller loop (KZG, BLS sig) [CS10]
- ~~Longa's sums of products Mul [Lon22]~~

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Multi-Scalar-Multiplication (MSM)

[Y.EH and G. Botrel - *In submission*]

$a_1P_1 + a_2P_2 + \cdots + a_nP_n$ with $P_i \in \mathbb{G}_1$ (or $\mathbb{G}_2$) and $a_i \in \mathbb{F}_r(|r| = \text{b}-bit)$

- Step 1: reduce the b-bit MSM to several c-bit MSMs for some chosen fixed $c \leq b$
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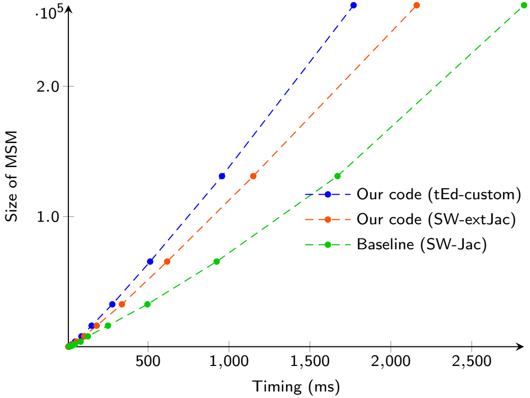
$\rightarrow$ Overall cost is: $b/c(n + 2^{c-1}) + (b - c - b/c - 1)$

- Mixed re-additions: to accumulate P_i in the c-bit MSM buckets with cost $b/c(n - 2^{c-1} + 1)$
- Additions: to combine the bucket sums with cost $b/c(2^c - 3)$
- Additions and doublings: to combine the c-bit MSMs into the b-bit MSM with cost $b - c + b/c - 1$
  - $b/c - 1$ additions and
  - $b - c$ doublings

Our MSM code vs. the ZPrize baseline (BLS12-377 $\mathbb{G}_1$)

[Y.EH and G. Botrel - *In submission*]

- All inner curves have a twisted Edwards form $-y^2 + x^2 = 1 + dx^2y^2$
- We use a custom coordinates system $(y - x : y + x : 2dxy) \rightarrow (7\mathtt{m}$ per addition)
- 2-NAF buckets, Parallelism, software optimizations...



Samsung Galaxy A13 5G (Model SM-A136ULGDXAA with SoC MediaTek Dimensity 700 (MT6833))

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Industrial Impact

gnark

CONSENSYS

celo

Aleo

EY

Go Ethereum

ESPRESSO SYSTEMS

Summary

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- Multi-scalar multiplication for fast generation of proofs **[(in submission), zprize winner]**

Perspectives

- Is it possible to find a silver bullet construction of elliptic curves that can address all the efficiency/security requirements?
- Are there more efficient cycles of pairing-friendly curves? How to generate them?

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- Are there more efficient cycles of pairing-friendly curves? How to generate them?
- Can we get rid of the FFT-friendliness?
  - Field-agnostic SNARKs [Brakedown, Orion]
  - FFT over non-smooth fields [ECFFT]

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