23. Fresnel Equations

- EM Waves at boundaries

- Fresnel Equations:
  Reflection and Transmission Coefficients

- Brewster’s Angle

- Total Internal Reflection (TIR)
- Evanescent Waves

- The Complex Refractive Index
- Reflection from Metals

We will derive the Fresnel equations

r : reflection coefficient

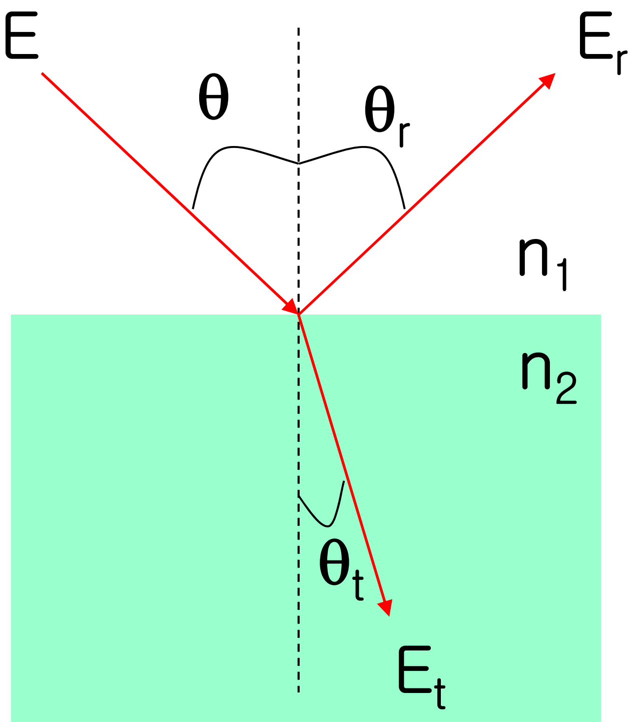
$$r_{TE} = \frac{E_r}{E} = \frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}$$

$$r_{TM} = \frac{E_r}{E} = \frac{-n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}$$

t : transmission coefficient

$$t_{TE} = \frac{E_t}{E} = \frac{2\cos\theta}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}$$

$$t_{TM} = \frac{E_t}{E} = \frac{2n\cos\theta}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}$$



$$n \equiv \frac{n_{transmitted}}{n_{incident}} = \frac{n_2}{n_1}$$

EM Waves at an Interface

$$\textit{Incident beam:} \quad \vec{E}_i = \vec{E}_{oi} \exp\left[i\left(\vec{k}_i \cdot \vec{r} - \omega_i t \right) \right]$$

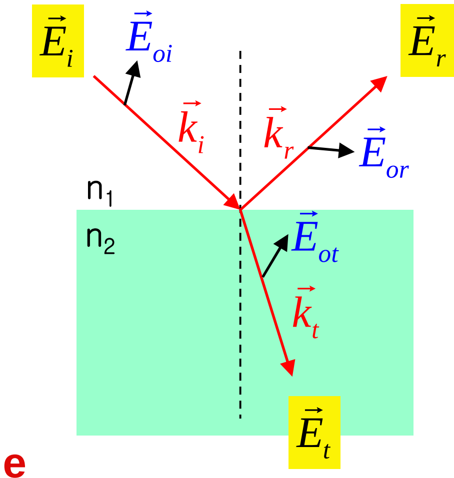
$$\textit{Reflected beam:} \quad \vec{E}_r = \vec{E}_{or} \exp\left[i\left(\vec{k}_r \cdot \vec{r} - \omega_r t \right) \right]$$

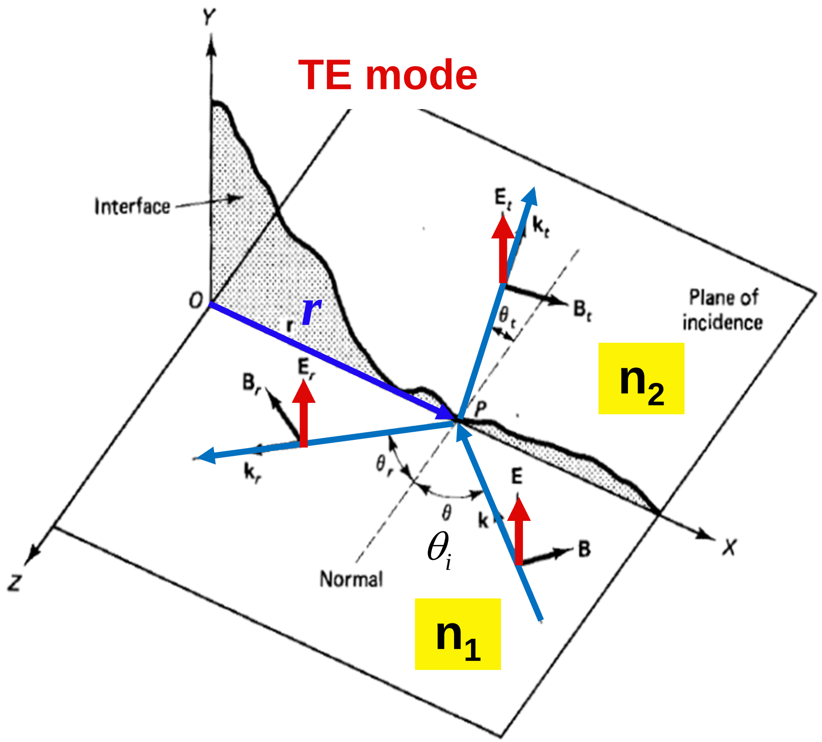
$$\textit{Transmitted beam:} \quad \vec{E}_t = \vec{E}_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega_t t \right) \right]$$

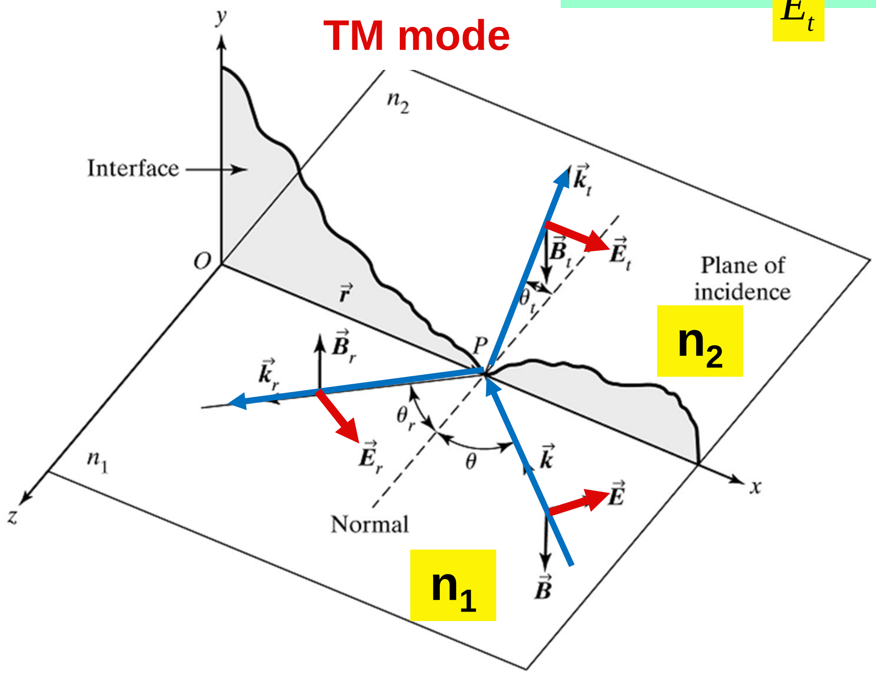
$$\left|\vec{k}_i\right| = n_1 k_0$$

$$\left|\vec{k}_r\right| = n_1 k_0$$

$$\left|\vec{k}_t\right| = n_2 k_0$$







Note the definition of the positive E-field directions in both cases.

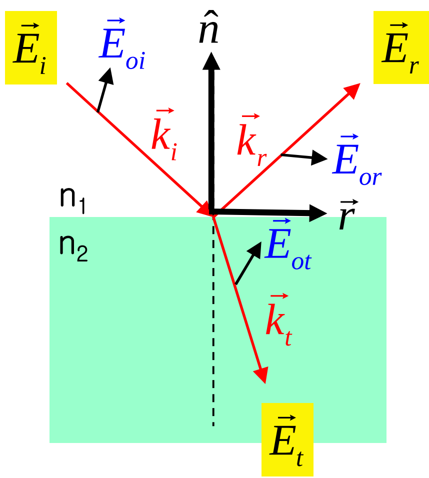
EM Waves at an Interface

$$Incident\ beam: \quad \vec{E}_i = \vec{E}_{oi} \exp\left[i\left(\vec{k}_i \cdot \vec{r} - \omega_i t\right)\right]$$

$$Reflected\ beam: \quad \vec{E}_r = \vec{E}_{or} \exp\left[i\left(\vec{k}_r \cdot \vec{r} - \omega_r t\right)\right]$$

$$Transmitted\ beam: \quad \vec{E}_t = \vec{E}_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega_t t\right)\right]$$

At the boundary between the two media (the $x-y$ plane), all waves must exist simultaneously, and the tangential component must be equal on both sides of the interface. Therefore, for all time t and for all boundary points $\vec{r}$ on the interface,



$$\hat{n} \times \vec{E}_i + \hat{n} \times \vec{E}_r = \hat{n} \times \vec{E}_t$$

$$\hat{n} \times \vec{E}_{oi} \exp\left[i\left(\vec{k}_i \cdot \vec{r} - \omega_i t\right)\right] + \hat{n} \times \vec{E}_{or} \exp\left[i\left(\vec{k}_r \cdot \vec{r} - \omega_r t\right)\right] = \hat{n} \times \vec{E}_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega_t t\right)\right]$$

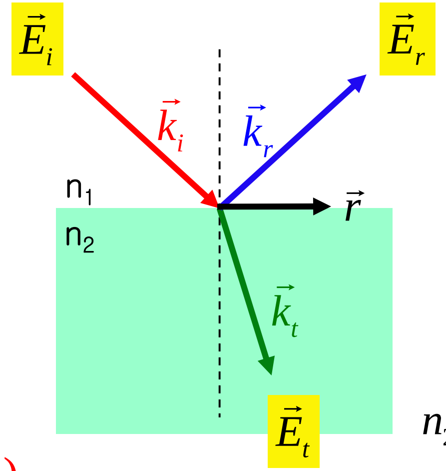
Assuming that the wave amplitudes are constant, the only way that this can be true over the entire interface and for all t is if :

$$\Rightarrow \quad \left(\vec{k}_i \cdot \vec{r} - \omega_i t\right) = \left(\vec{k}_r \cdot \vec{r} - \omega_r t\right) = \left(\vec{k}_t \cdot \vec{r} - \omega_t t\right) \quad : \ \text{Phase matching at the boundary!}$$

EM Waves at an Interface

Phase matching condition:

$$\left(\vec{k}_i\cdot\vec{r}-\omega_i t\right)=\left(\vec{k}_r\cdot\vec{r}-\omega_r t\right)=\left(\vec{k}_t\cdot\vec{r}-\omega_t t\right)$$



$$\left|\vec{k}_i\right|=n_1k_0$$
$$\left|\vec{k}_r\right|=n_1k_0$$
$$\left|\vec{k}_t\right|=n_2k_0$$

At $\vec{r}=0$, *this results in*

$$\omega_i t=\omega_r t=\omega_t t$$

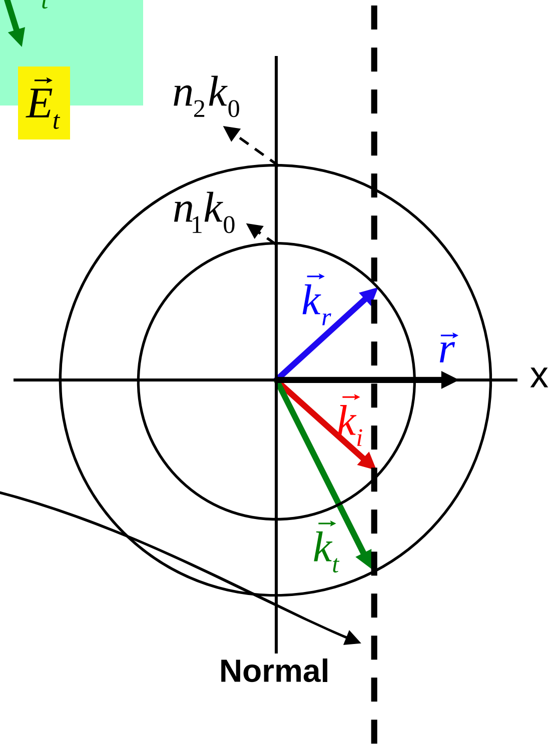
$$\Rightarrow \quad \omega_i=\omega_r=\omega_t$$

(Frequency does not change at the boundary!)

At $t=0$, *this results in*

$$\Rightarrow \quad \vec{k}_i\cdot\vec{r}=\vec{k}_r\cdot\vec{r}=\vec{k}_t\cdot\vec{r}$$

(Phases on the boundary does not change!)

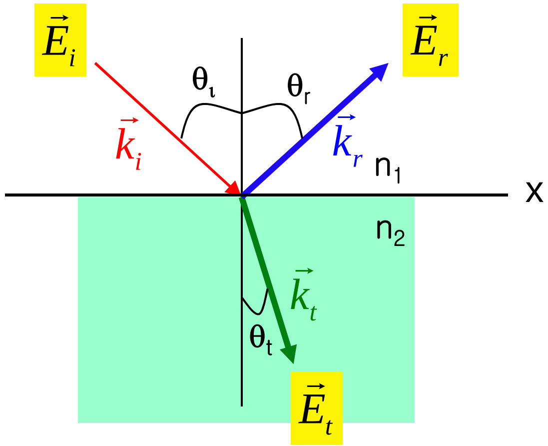


$$\Rightarrow \vec{k}_{i,r,t}\cdot\vec{r}=constant$$

$\rightarrow$ *the equation for a plane perpendicular to* $\vec{k}_{i,r,t}$ and $\vec{r}$.

$\Rightarrow \vec{k}_i,\ \vec{k}_r,$ *and* $\vec{k}_t$ *are coplanar in the plane of incidence.*

EM Waves at an Interface



At $t = 0$,

$\vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} = \vec{k}_t \cdot \vec{r} = \text{constant}$

Considering the relation for the incident and reflected beams,

$$\vec{k}_i \cdot \vec{r} = \vec{k}_r \cdot \vec{r} \quad \Rightarrow \quad k_i\, r \sin\theta_i = k_r\, r \sin\theta_r$$

Since the incident and reflected beams are in the same medium,

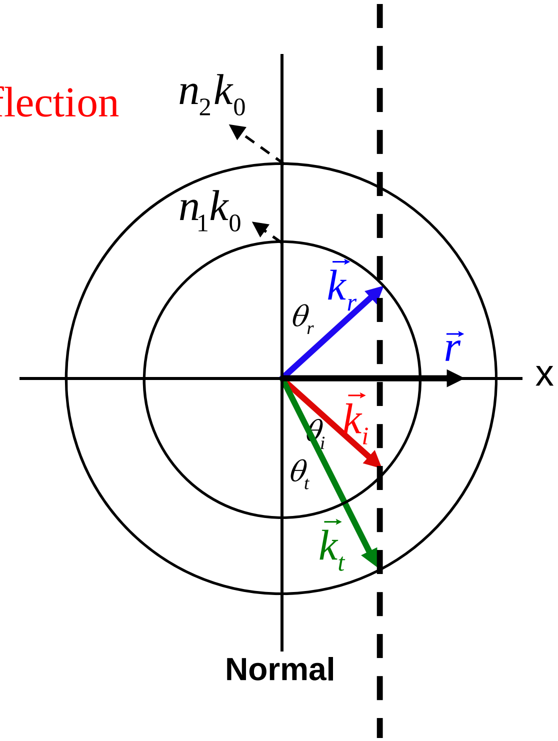
$$k_i = k_r = \frac{n_i\, \omega}{c} \quad \Rightarrow \quad \sin\theta_i = \sin\theta_r \quad \Rightarrow \quad \boxed{\theta_i = \theta_r} : \text{law of reflection}$$

Considering the relation for the incident and transmitted beams,

$$\vec{k}_i \cdot \vec{r} = \vec{k}_t \cdot \vec{r} \quad \Rightarrow \quad k_i\, r \sin\theta_i = k_t\, r \sin\theta_t$$

But the incident and transmitted beams are in different media,

$$k_i = \frac{n_i\, \omega}{c} \qquad k_t = \frac{n_t\, \omega}{c} \quad \Rightarrow \quad \boxed{n_i \sin\theta_i = n_t \sin\theta_t} : \text{law of refraction}$$



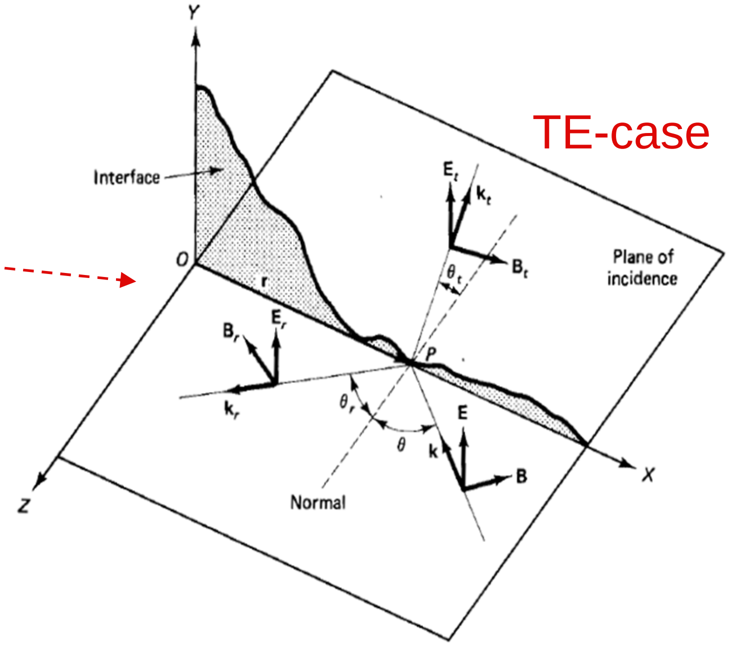
Development of the Fresnel Equations

From Maxwell's EM field theory,
we have the boundary conditions at the interface *for the TE case*:

$$E_i + E_r = E_t$$
$$B_i \cos\theta_i - B_r \cos\theta_r = B_t \cos\theta_t$$

The above conditions imply that the tangential components of both $\vec{E}$ and $\vec{B}$ are equal on both sides of the interface. We have also assumed that $\mu_i \cong \mu_t \cong \mu_0$, as is true for most dielectric materials.

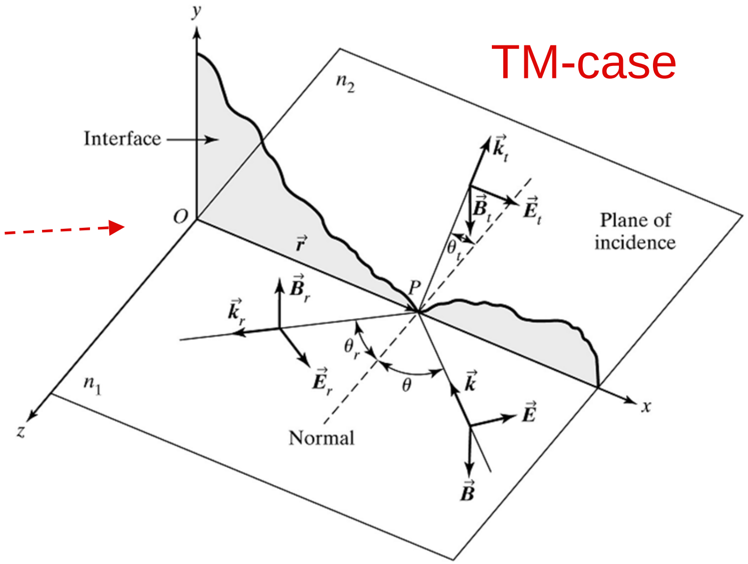
TE-case



For the TM mode:

$$E_i \cos\theta_i + E_r \cos\theta_r = E_t \cos\theta_t$$
$$-B_i + B_r = -B_t$$

TM-case

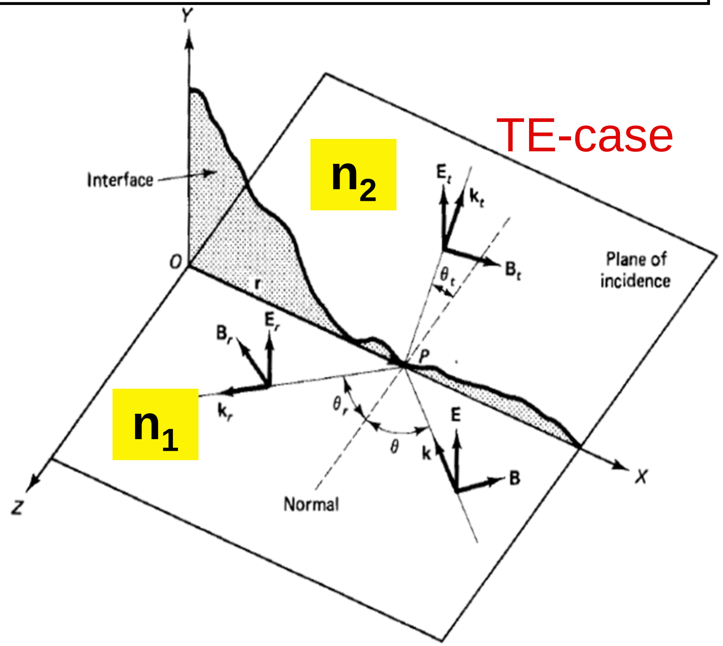


Development of the Fresnel Equations

$$Recall\ that\ \ E = \mathrm{v}\,B = \left(\frac{c}{n}\right)B \ \Rightarrow \quad B = \frac{nE}{c}$$

$Let\ n_1 = refractive\ index\ of\ incident\ medium$

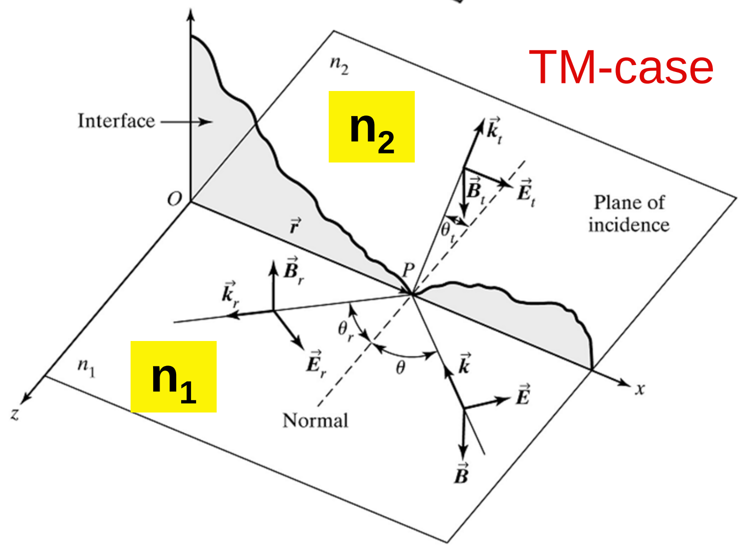
$n_2 = refractive\ index\ of\ refracting\ medium$



For the TE mode:

$$E_i + E_r = E_t$$

$$n_1 E_i \cos\theta_i - n_1 E_r \cos\theta_r = n_2 E_t \cos\theta_t$$



For the TM mode:

$$E_i \cos\theta_i + E_r \cos\theta_r = E_t \cos\theta_t$$

$$-n_1 E_i + n_1 E_r = -n_2 E_t$$

Development of the Fresnel Equations

Eliminating E_t *from each set of equations and solving for the reflection coefficient we obtain:*

$$TE\ case:\ r_{TE} = \frac{E_r}{E_i} = \frac{\cos\theta_i - n\cos\theta_t}{\cos\theta_i + n\cos\theta_t}$$

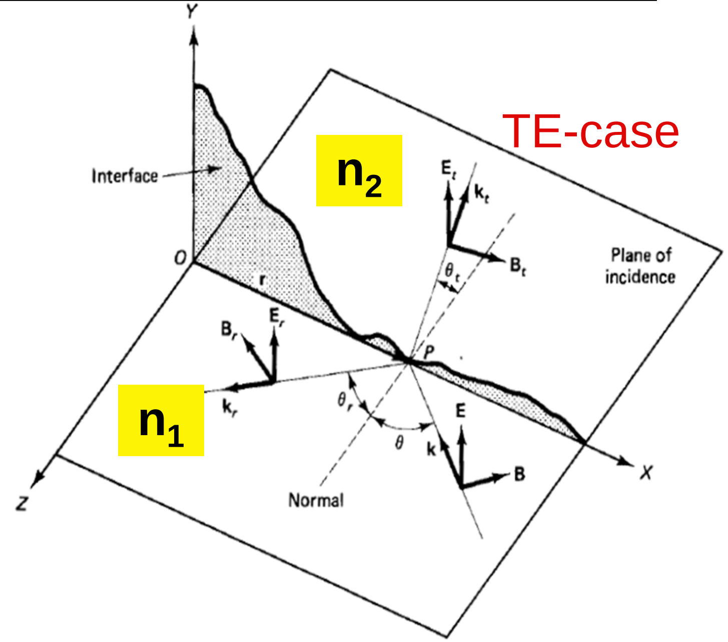
$$TM\ case:\ r_{TM} = \frac{E_r}{E_i} = \frac{-n\cos\theta_i + \cos\theta_t}{n\cos\theta_i + \cos\theta_t}$$

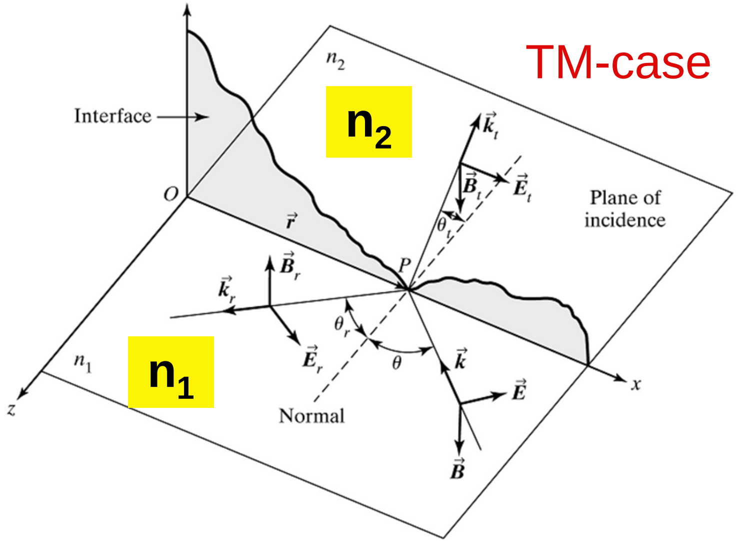
where $n = \frac{n_2}{n_1}$

We know that

$$\sin\theta_i = n\sin\theta_t$$

$$n\cos\theta_t = n\sqrt{1-\sin^2\theta_t} = n\sqrt{1-\frac{\sin^2\theta_i}{n^2}} = \sqrt{n^2-\sin^2\theta_i}$$





Now we have derived the Fresnel Equations

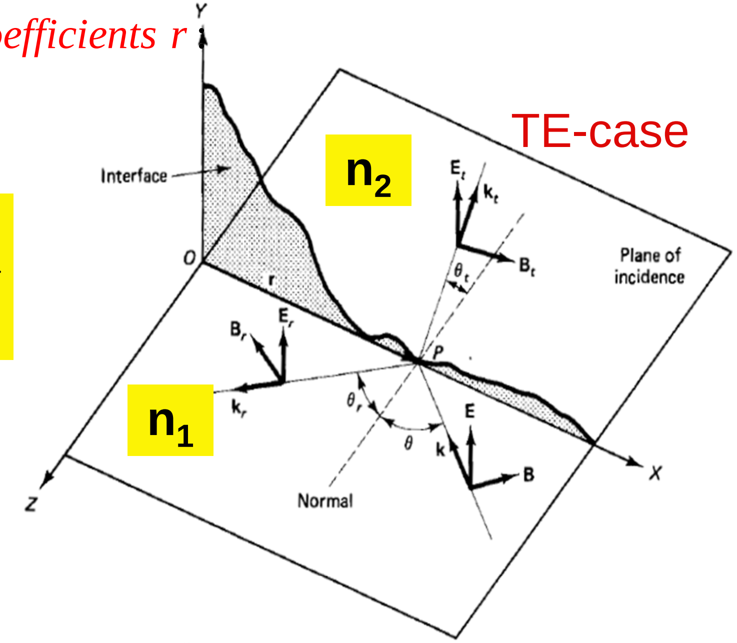
Substituting we obtain the Fresnel equations for *reflection coefficients r*

$$TE\ case:\ r_{TE} = \frac{E_r}{E_i} = \frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$

$$TM\ case:\ r_{TM} = \frac{E_r}{E_i} = \frac{-n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$

$$n \equiv \frac{n_2}{n_1}$$

TE-case

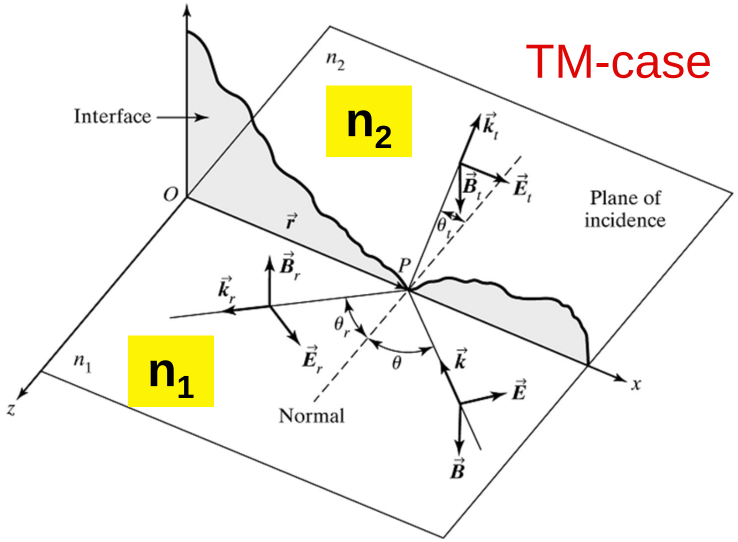


For the *transmission coefficient t*:

$$TE\ case:\ t_{TE} = \frac{E_t}{E_i} = \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$

$$TM\ case:\ t_{TM} = \frac{E_t}{E_i} = \frac{2n\cos\theta_i}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$

TM-case



$$TE:\quad t_{TE} = r_{TE} + 1$$

$$TM:\quad nt_{TM} = 1 - r_{TM}$$

These just mean the boundary conditions.

$$For\ the\ TE\ case: E_i + E_r = E_t$$

$$For\ the\ TM\ mode: -B_i + B_r = -B_t$$

Power : Reflectance (R) and Transmittance (T)

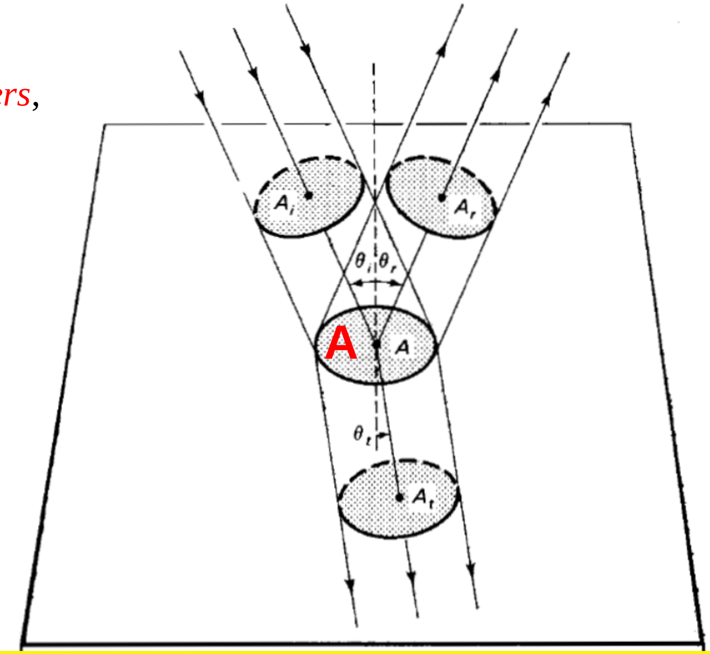
The quantities r and t are ratios of electric field amplitudes.

The ratios R and T are the ratios of reflected and transmitted powers, respectively, to the incident power :

$$R = \frac{P_r}{P_i} \qquad T = \frac{P_t}{P_i}$$

From conservation of energy :

$$P_i = P_r + P_t \quad \Rightarrow \quad 1 = R + T$$



We can express the power in each of the fields in terms of the product of an irradiance and area :

$$P_i = I_i A_i \qquad P_r = I_r A_r \qquad P_t = I_t A_t$$

$$\Rightarrow \quad I_i A_i = I_r A_r + I_t A_t$$

$$I_i A\cos\theta_i = I_r A\cos\theta_r + I_t A\cos\theta_t$$

$$I_i \cos\theta_i = I_r \cos\theta_r + I_t \cos\theta_t$$

$$Power\_ratio = \frac{I_{out}\cos\theta_{out}}{I_{in}\cos\theta_{in}} = \left(\frac{n_{out}\left|E_{out}\right|^2\cos\theta_{out}}{n_{in}\left|E_{in}\right|^2\cos\theta_{in}}\right)$$

But $I = \frac{1}{2}n\varepsilon_0 c E_0^2 \quad \Rightarrow \quad \frac{1}{2}n_1\varepsilon_0 c E_{0i}^2\cos\theta_i = \frac{1}{2}n_1\varepsilon_0 c E_{0r}^2\cos\theta_r + \frac{1}{2}n_2\varepsilon_0 c E_{0t}^2\cos\theta_t$

$$\Rightarrow \quad 1 = \frac{E_{0r}^2}{E_{0i}^2} + \frac{n_2 E_{0t}^2\cos\theta_t}{n_1 E_{0i}^2\cos\theta_i} = \frac{E_{0r}^2}{E_{0i}^2} + n\left(\frac{\cos\theta_t}{\cos\theta_i}\right)\frac{E_{0t}^2}{E_{0i}^2} = R + T$$

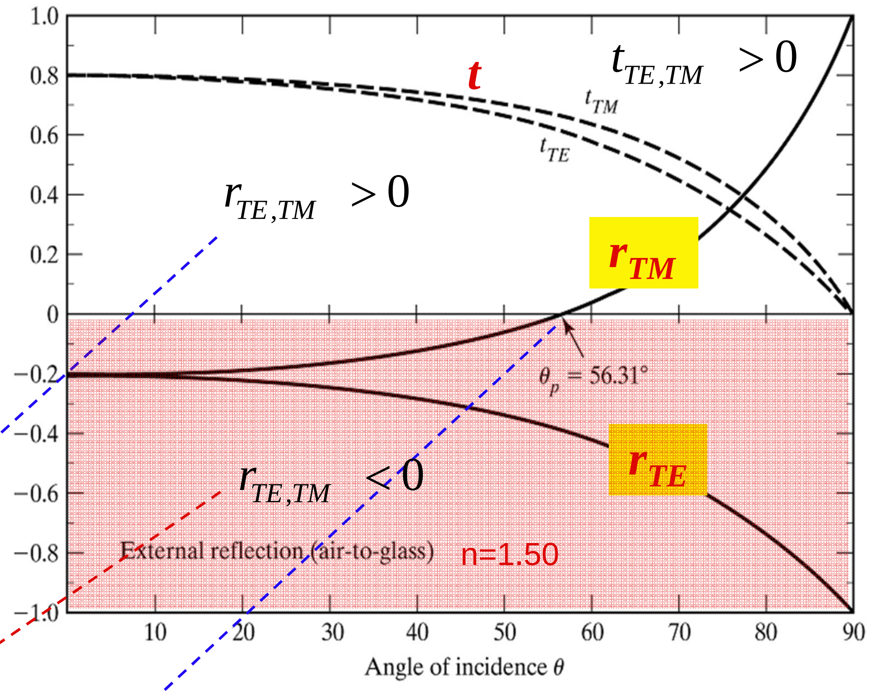
$$\Rightarrow \quad \boxed{R = \frac{E_{0r}^2}{E_{0i}^2} = r^2 \qquad T = n\left(\frac{\cos\theta_t}{\cos\theta_i}\right)\frac{E_{0t}^2}{E_{0i}^2} = n\left(\frac{\cos\theta_t}{\cos\theta_i}\right)t^2}$$

$$R = rr^* = \left|r\right|^2$$

$$T = \left(n\frac{\cos\theta_t}{\cos\theta_i}\right)tt^* = \left(n\frac{\cos\theta_t}{\cos\theta_i}\right)\left|t\right|^2$$

23-2. External and Internal Reflection

$$r_{TE} = \frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \qquad r_{TM} = \frac{-n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$



External Reflection $[n = n_2/n_1 > 1]$

$\Rightarrow n_2 > n_1$

$\Rightarrow n = n_2/n_1 > 1 \Rightarrow (n^2 - \sin^2\theta) \geq 0$

$\Rightarrow r_{TE,TM}$ are always real

$\Rightarrow$ If $r_{TE,TM} > 0$ then there are no phase changes after reflection.

$\Rightarrow$ If $r_{TE,TM} < 0$ then there are always $\pi (= 180^\circ)$ phase changes.

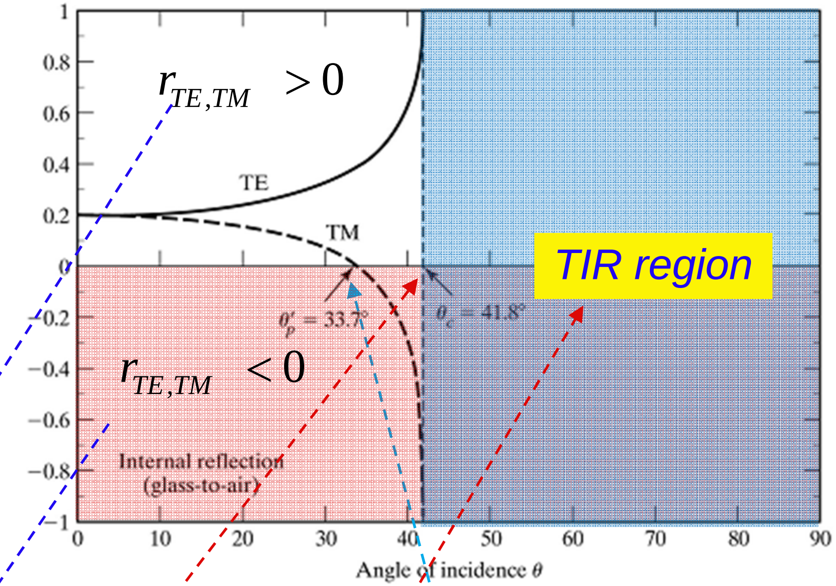
$$\rightarrow r_{TE,TM} = -|r_{TE,TM}| = e^{i\pi}|r_{TE,TM}|$$

Note for the TM case:

$\Rightarrow r_{TM}(\theta = \theta_p) = 0 \quad when \quad \theta_p = \tan^{-1} n$ $\Longrightarrow$ *Brewster's angle (or, polarizing angle)* (No reflection of TM mode)

Internal Reflection $[n = n_2 / n_1 < 1]$

$$r_{TE} = \frac{\cos\theta_i - \sqrt{n^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}} \qquad r_{TM} = \frac{-n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}{n^2\cos\theta_i + \sqrt{n^2 - \sin^2\theta_i}}$$



$n_1 > n_2 \Rightarrow n = n_2 / n_1 < 1$

$\Rightarrow (n^2 - \sin^2\theta) > 0,\ or,\ (n^2 - \sin^2\theta) < 0$

$\Rightarrow$ If $(n^2 - \sin^2\theta) > 0$, $r_{TE,TM}$ are always real

$\rightarrow$ If $r_{TE,TM} > 0$ then there are no phase changes after reflection.

$\rightarrow$ If $r_{TE,TM} < 0$ then there are $\pi(= 180^\circ)$ phase changes.

$\Rightarrow$ If $(n^2 - \sin^2\theta) = 0$, $|r_{TE,TM}| = 1$

$\rightarrow \sin\theta_c = n = (n_2 / n_1)$ ➡ *critical angle*

Note Brewster's angle $(\theta_p = \tan^{-1} n)$ *for the TM case*: $r_{TM} = 0$

$\Rightarrow$ If $(n^2 - \sin^2\theta) < 0$, $|r_{TE,TM}| = 1$, *BUT* $r_{TE,TM}$ *are complex!*

$\rightarrow |r_{TE,TM}| = 1$ ➡ *Total internal reflection (TIR) when* $\theta > \theta_c$

$\rightarrow r_{TE,TM} = |r_{TE,TM}| e^{i\phi} = e^{i\phi}$

$\rightarrow \phi\ (-\pi \sim +\pi)$ phase change may occur after reflection

Derivation of Brewster's Angle

Brewster's angle θ_p *(for polarizing angle):*

$$r_{TM}\left(\theta_p\right) = \frac{-n^2\cos\theta_p + \sqrt{n^2-\sin^2\theta_p}}{n^2\cos\theta_p + \sqrt{n^2-\sin^2\theta_p}} = 0$$

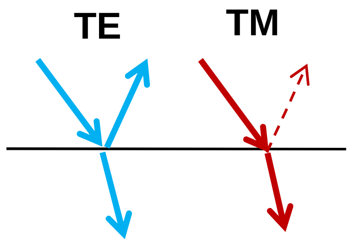
$$\Rightarrow \quad n^4\cos^2\theta_p = n^2-\sin^2\theta_p$$

$$n^4\cos^2\theta_p - n^2+\sin^2\theta_p$$

$$=(n^2-1)\left[n^2\cos^2\theta_p-\sin^2\theta_p\right]=0$$

$$\Rightarrow \quad \theta_p = \tan^{-1} n$$

$$For\ n=1.50,\quad \theta_p = 56.31°$$



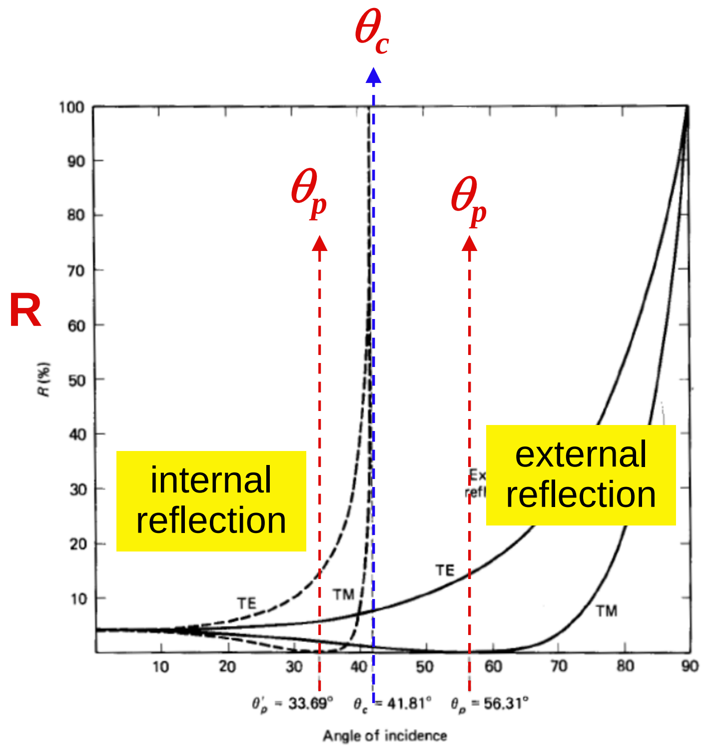


Figure 20-4 Reflectance for both external and internal reflection when $n_1 = 1$ and $n_2 = 1.50$.

Brewster 's angle : $\tan\theta_p = n \quad : n > 1 \quad or \quad n < 1$

→ External & Internal reflections, but TM-polarization only

Critical angle : $\sin\theta_c = n \quad : n < 1$

→ TE & TM polarizations, but Internal reflection only

Total Internal Reflection (TIR)

$$\textit{Internal reflection}: \quad n = \frac{n_2}{n_1} < 1$$

$$\textit{For } \theta \ge \theta_c = \sin^{-1} n, \quad \textit{called } \text{total internal reflection}(TIR),$$

$$\Rightarrow r = 1 \text{ and } R = rr^* = 1 \text{ for both (TE and TM) cases.}$$

$$\Rightarrow r \textit{ is a complex number}$$

$$r_{TE} = \frac{E_r}{E_i} = \frac{\cos\theta_i - i\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}}$$

$$r_{TM} = \frac{E_r}{E_i} = \frac{-n^2\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}}$$

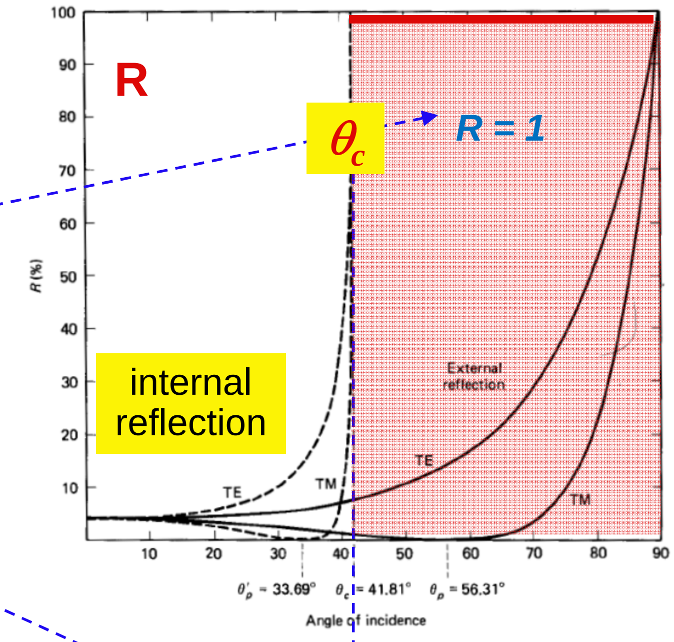
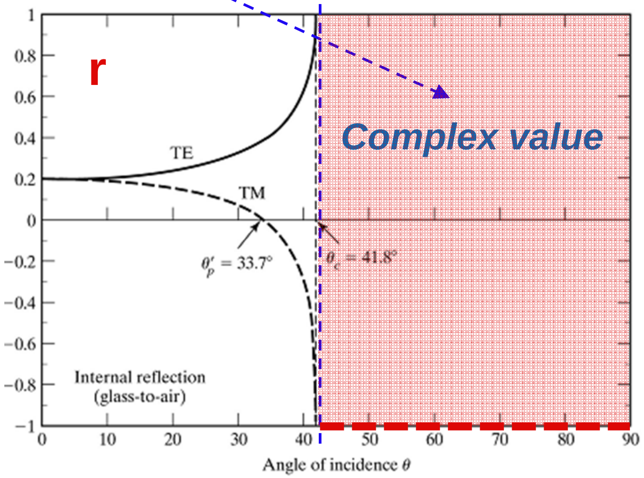


Figure 20-4 Reflectance for both external and internal reflection when $n_1 = 1$ and $n_2 = 1.50$.



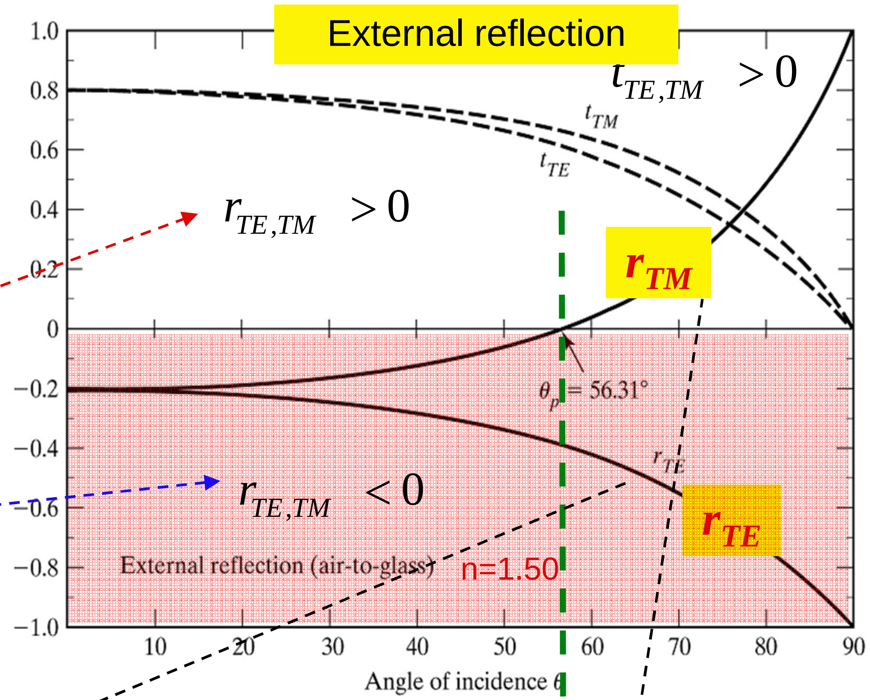
23-3. Phase changes on reflection

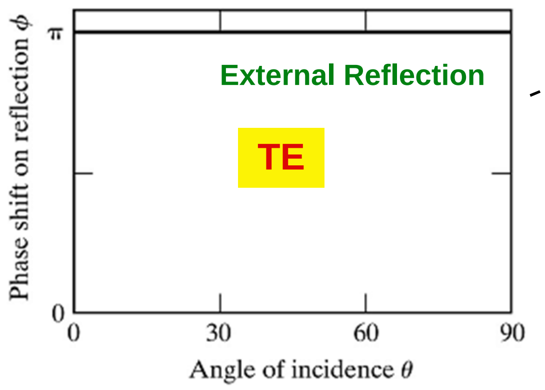
Phase shift after External Reflection

$r_{TE,TM}$ is always a real number for external reflection,

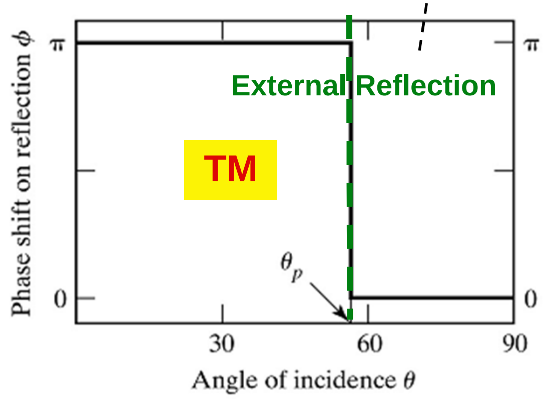
then the phase shift is $0°$ for $r_{TE,TM} > 0$,

and the phase shift is $180°(=\pi)$ for $r_{TE,TM} < 0$.





(a) TE mode, external reflection (air to glass)



(c) TM mode, external reflection (air-to-glass)

For TE case, π phase shift for all incident angles

For TM case, π phase shift for $\theta < \theta_p$
No phase shift for $\theta > \theta_p$

Phase shift after Internal Reflection

$\Rightarrow \; r_{TE} > 0$ for $\theta < \theta_c = \sin^{-1} n$

$\Rightarrow \; r_{TE}$ is complex in TIR region where $\theta > \theta_c$

$\rightarrow \; r_{TE} = |r_{TE}| e^{i\phi_{TE}} = e^{i\phi_{TE}}$

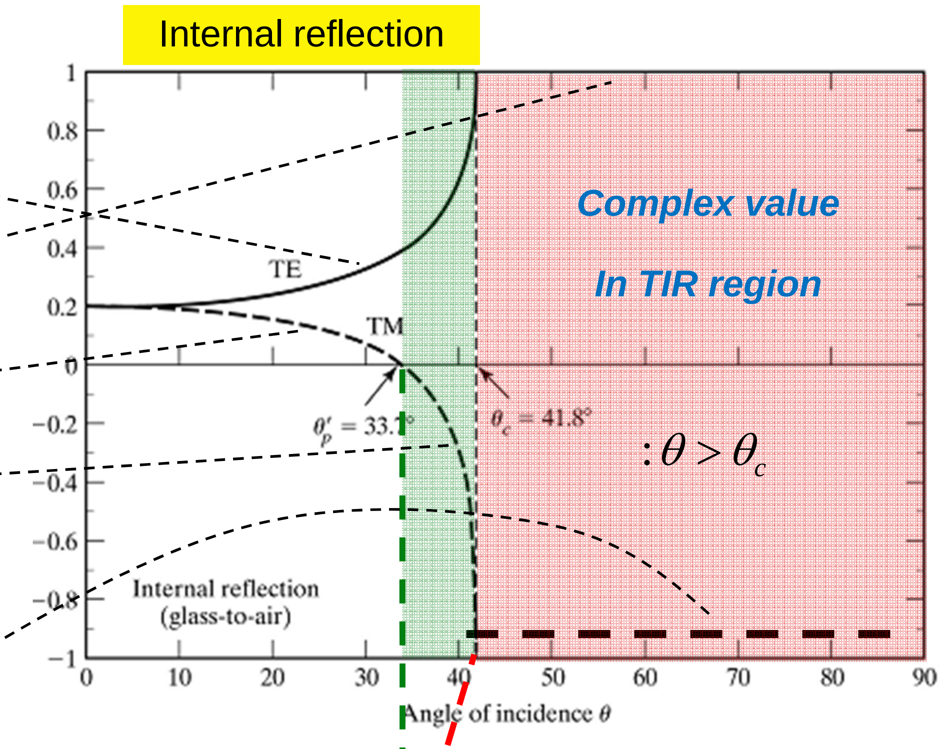
$\Rightarrow \; r_{TM} > 0$ for $\theta < \theta_p = \tan^{-1} n$

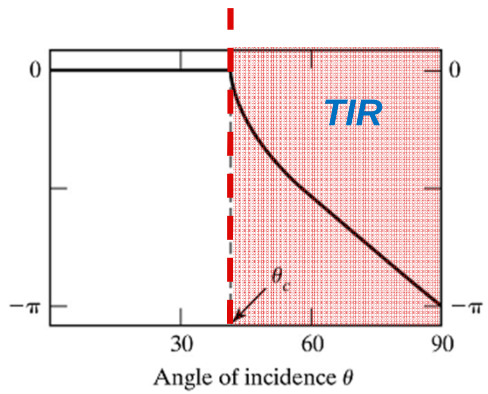
$\Rightarrow \; r_{TM} < 0$ for $\theta_p < \theta < \theta_c$

$\rightarrow \; r_{TM} = -|r_{TM}| = e^{i\phi_{TM}} |r_{TM}| \rightarrow \phi_{TM} = \pi$

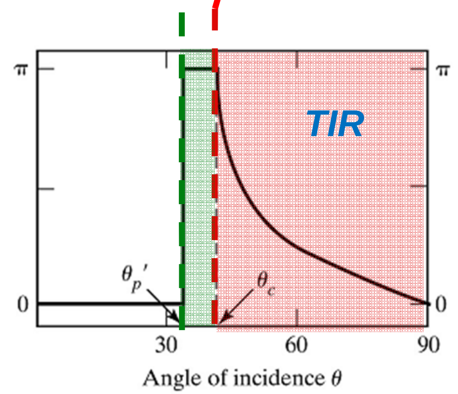
$\Rightarrow \; r_{TM}$ is complex in TIR region where $\theta > \theta_c$

$\rightarrow \; r_{TM} = |r_{TM}| e^{i\phi_{TM}} = e^{i\phi_{TM}}$





(b) TE mode, internal reflection (glass to air)



(d) TM mode, internal reflection (glass-to-air)

For TE case, no phase shift for $\theta < \theta_c$

$\phi_{TE}(\theta)$ phase shift for $\theta > \theta_c$

For TM case, no phase shift for $\theta < \theta_p$

π phase shift for $\theta_p < \theta < \theta_c$

$\phi_{TM}(\theta)$ phase shift for $\theta > \theta_c$

Phase shifts on total Internal Reflection for both TE- and TM-cases

When $\theta \geq \theta_c$ *(TIR case) then r is complex and for both the TE and TM cases has the form:*

$$r = \frac{a-ib}{a+ib} = \frac{\cos\alpha - i\sin\alpha}{\cos\alpha + i\sin\alpha} = \frac{e^{-i\alpha}}{e^{+i\alpha}} = e^{-i2\alpha} = e^{i\phi} \quad \Rightarrow \quad \tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{b}{a} \quad \phi = -2\alpha$$

ϕ *is the phase shift on total internal reflection(TIR).*

$$TE\ case: \quad r_{TE} = \frac{E_r}{E_i} = \frac{\cos\theta_i - i\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}}$$

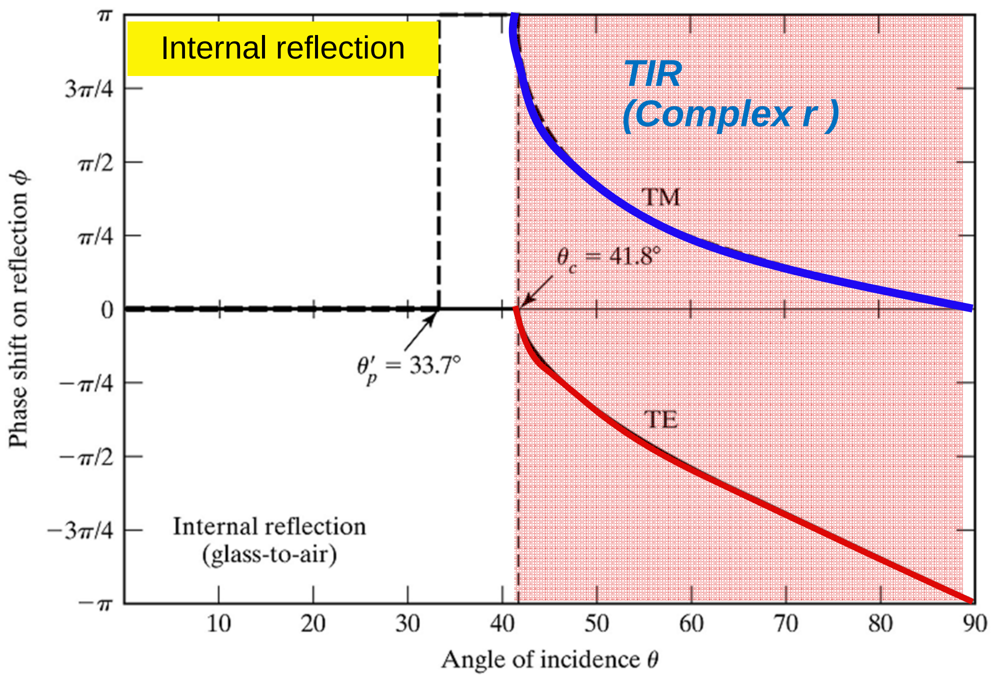
$$a = \cos\theta_i \qquad b = \sqrt{\sin^2\theta_i - n^2}$$

$$\Rightarrow \quad \tan\alpha = \tan\left(-\frac{\phi_{TE}}{2}\right) = \frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta}$$

$$\boxed{\phi_{TE} = -2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i}\right)} \quad : \theta_i > \theta_c$$

A similar analysis for the TM case gives:

$$\boxed{\phi_{TM} = \pi - 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i}\right)} \quad : \theta_i > \theta_c$$



Therefore, $r_{TE,TM}$ after TIR is ....

For TIR case $\left(\theta_{incident} > \theta_c\right)$

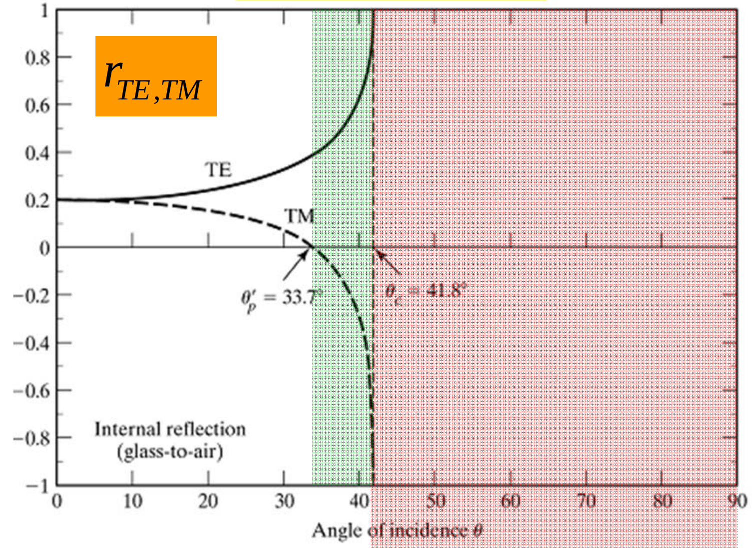
$$r_{TE} = \frac{E_r}{E_i} = \frac{\cos\theta_i - i\sqrt{\sin^2\theta_i - n^2}}{\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}}$$

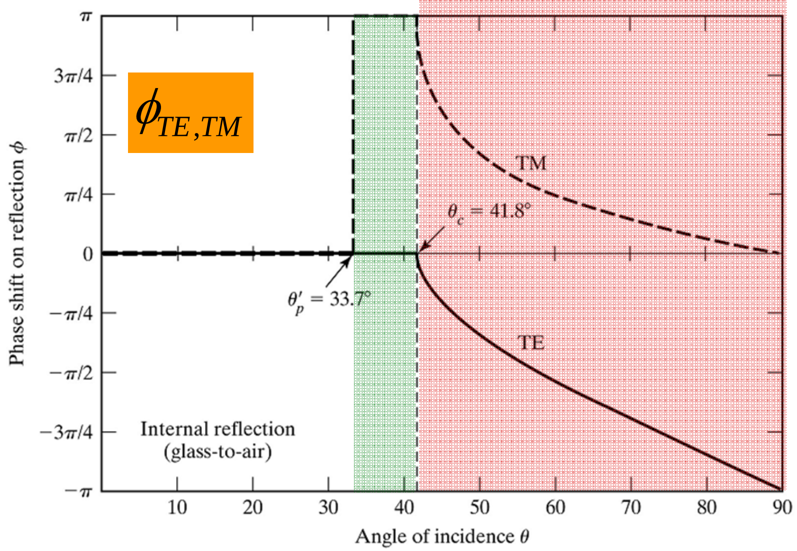
$$r_{TM} = \frac{E_r}{E_i} = \frac{-n^2\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta_i + i\sqrt{\sin^2\theta_i - n^2}}$$

$$\phi_{TM} = \pi - 2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta}\right)$$

$$\phi_{TE} = -2\tan^{-1}\left(\frac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta}\right)$$

Internal reflection



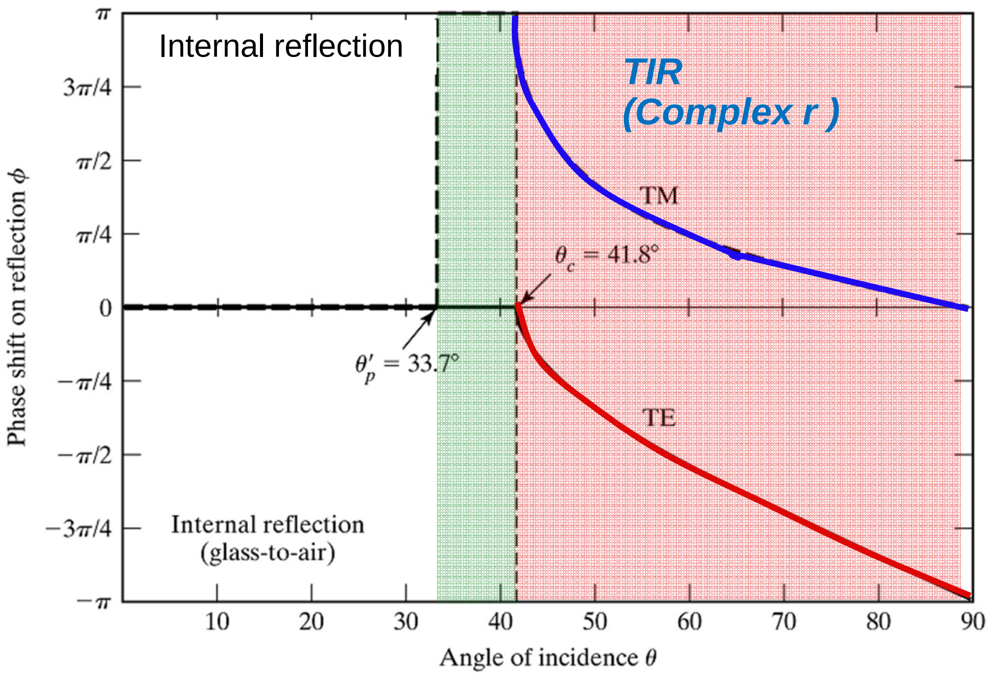


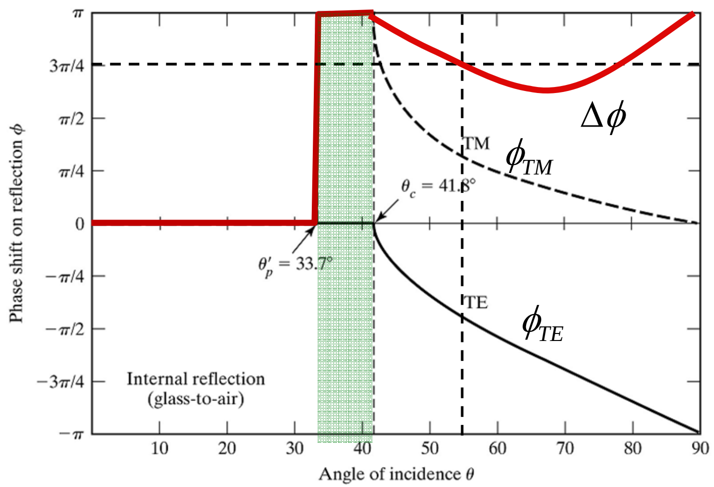
Summary of Phase Shifts on Internal Reflection

$$\phi_{TM} = \begin{cases} 0^\circ & \theta < \theta_p' \\ \pi (=180^\circ) & \theta_p' < \theta < \theta_c \\ \pi - 2\tan^{-1}\left(\dfrac{\sqrt{\sin^2\theta_i - n^2}}{n^2\cos\theta}\right) & \theta < \theta_c \end{cases}$$

$$\phi_{TE} = \begin{cases} 0^\circ & \theta < \theta_c \\ -2\tan^{-1}\left(\dfrac{\sqrt{\sin^2\theta_i - n^2}}{\cos\theta}\right) & \theta > \theta_c \end{cases}$$

$$\Delta\phi = \phi_{TM} - \phi_{TE} \begin{cases} = 0^\circ & \theta < \theta_p \\ = \pi & \theta_p < \theta < \theta_c \\ > 0^\circ & \theta_c < \theta \end{cases}$$





Fresnel Rhomb

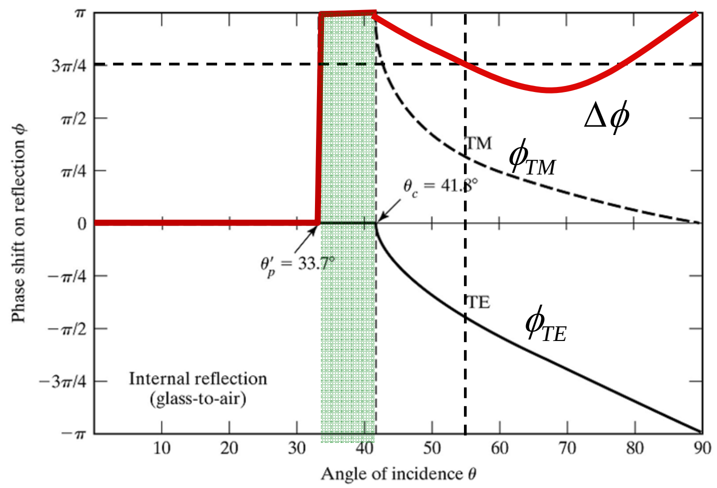
$$Note \;\; \phi_{TM} - \phi_{TE} = \frac{3\pi}{4} \;\; near \;\; \theta_i = 53^\circ \;\; when \; n = 1.5$$

$\rightarrow$ After two consequentive TIRs,

$$\rightarrow \;\; \phi_{TM} - \phi_{TE} = \frac{3\pi}{2}$$

$$\rightarrow \;\; |\Delta\phi| = |\phi_{TM} - \phi_{TE}| = \frac{\pi}{2}$$

$\rightarrow$ *Quarter – wave retarder*



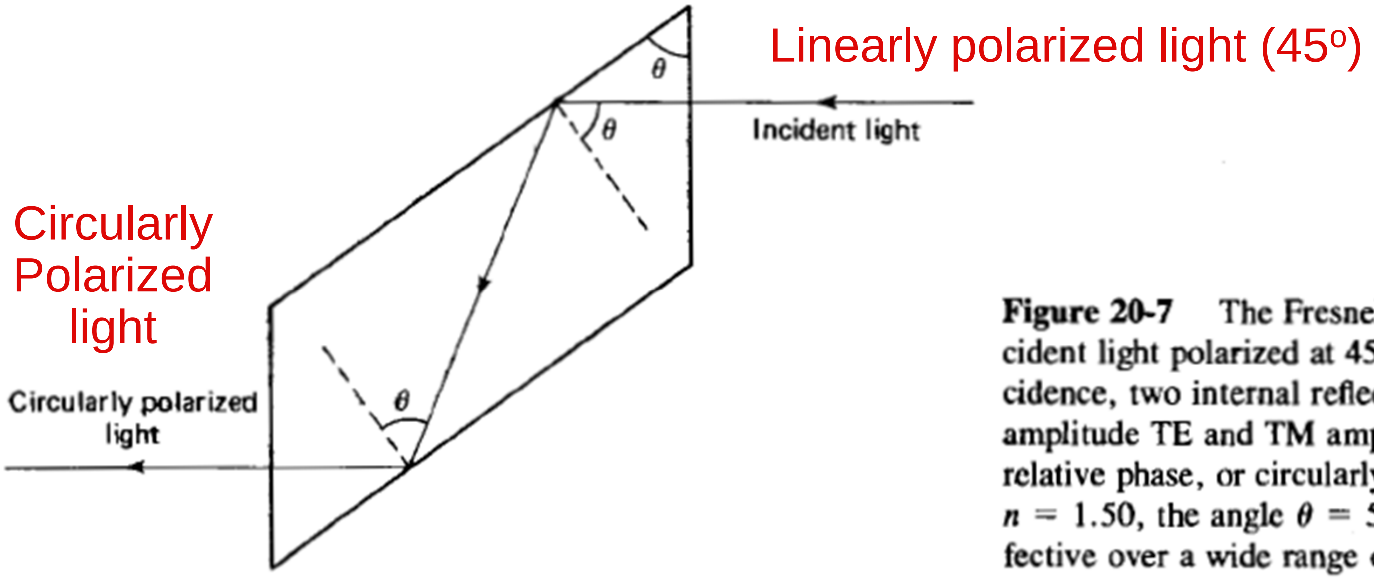


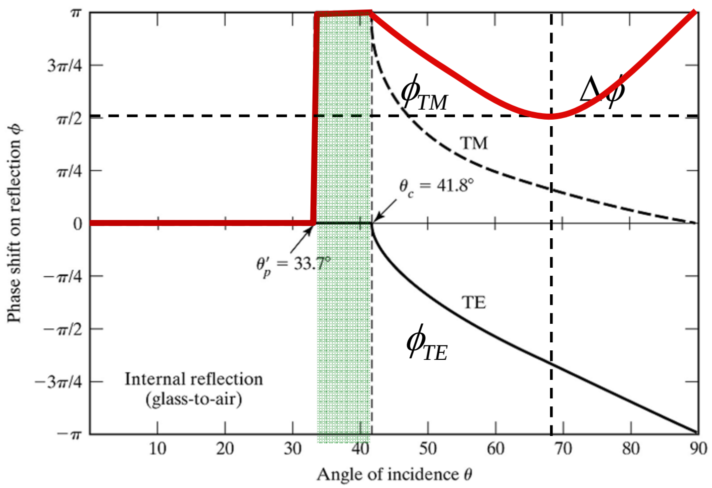
Figure 20-7 The Fresnel rhomb. With the incident light polarized at 45° to the plane of incidence, two internal reflections produce equal amplitude TE and TM amplitudes with a 90° relative phase, or circularly polarized light. For $n = 1.50$, the angle $\theta = 53°$. The device is effective over a wide range of wavelengths.

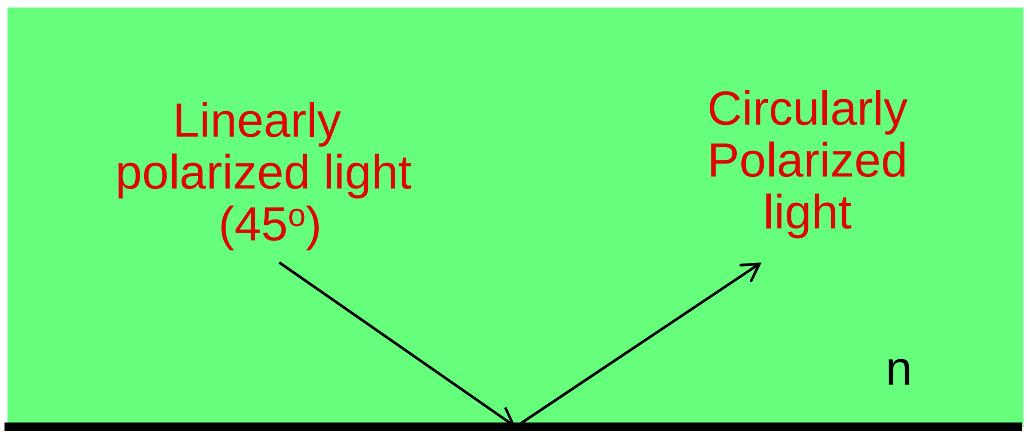
Quarter-wave retardation after TIR

$$Note \;\; \phi_{TM} - \phi_{TE} = \frac{\pi}{2} \;\; near \;\; \theta_i = 69^\circ \;\; when \; n = ???$$

$$\rightarrow \;\; |\Delta\phi| = |\phi_{TM} - \phi_{TE}| = \frac{\pi}{2}$$

$$\rightarrow \;\; Quarter - wave \; retarder$$



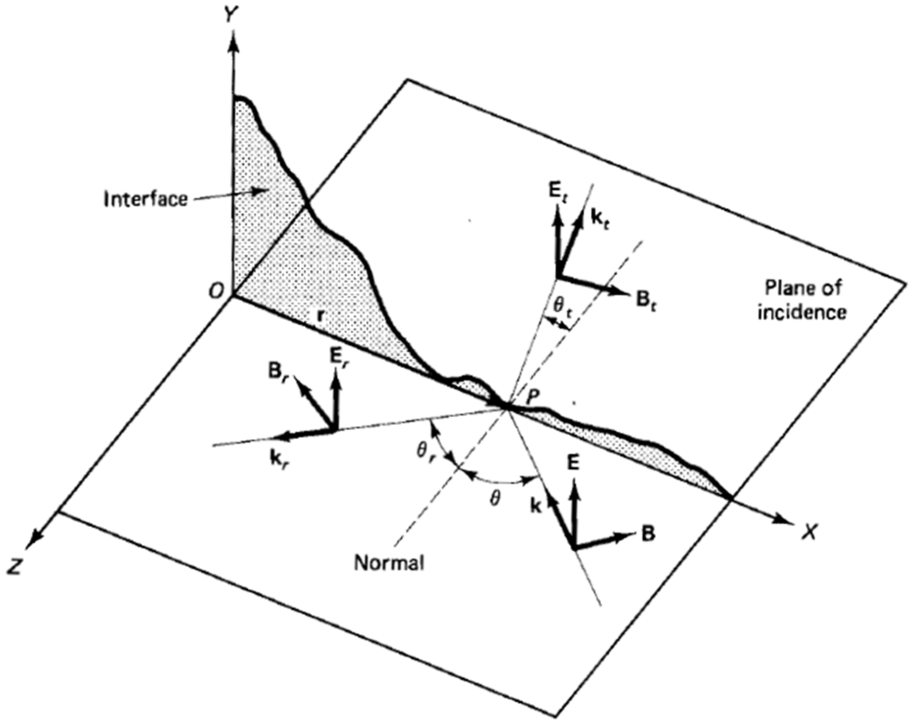


23-5. Evanescent Waves at an Interface

$$Incident\ beam: \quad \vec{E}_i = \vec{E}_{oi} \exp\left[i\left(\vec{k}_i \cdot \vec{r} - \omega_i t \right) \right]$$

$$Reflected\ beam: \quad \vec{E}_r = \vec{E}_{or} \exp\left[i\left(\vec{k}_r \cdot \vec{r} - \omega_r t \right) \right]$$

$$Transmitted\ beam: \quad \vec{E}_t = \vec{E}_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega_t t \right) \right]$$



For the transmitted beam:

$$E_t = E_{ot} \exp\left[i\left(\vec{k}_t \cdot \vec{r} - \omega_t t \right) \right]$$

$$\begin{aligned} \vec{k}_t \cdot \vec{r} &= \left(k_t \sin\theta_t\, \hat{x} + k_t \cos\theta_t\, \hat{z} \right) \cdot \left(x\, \hat{x} + z \hat{z} \right) \\ &= k_t \left(x \sin\theta_t + z \cos\theta_t \right) \end{aligned}$$

$$But, \quad \cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{\sin^2\theta_i}{n}}$$

When $\sin\theta_i > n$ *(total internal reflection), then:*

$$\cos\theta_t = i\sqrt{\frac{\sin^2\theta_i}{n} - 1} \quad \Rightarrow \quad a\ purely\ imaginary\ number$$

Evanescent Waves at an Interface

For the transmitted beam with an TIR condition $(\sin\theta_i > n)$,

we can write the phase factor as:

$$\vec{k}_t \cdot \vec{r} = k_t\left(x\,\frac{\sin\theta_t}{n} + i\,z\sqrt{\frac{\sin^2\theta_i}{n} - 1}\right)$$

Defining the coefficient α:

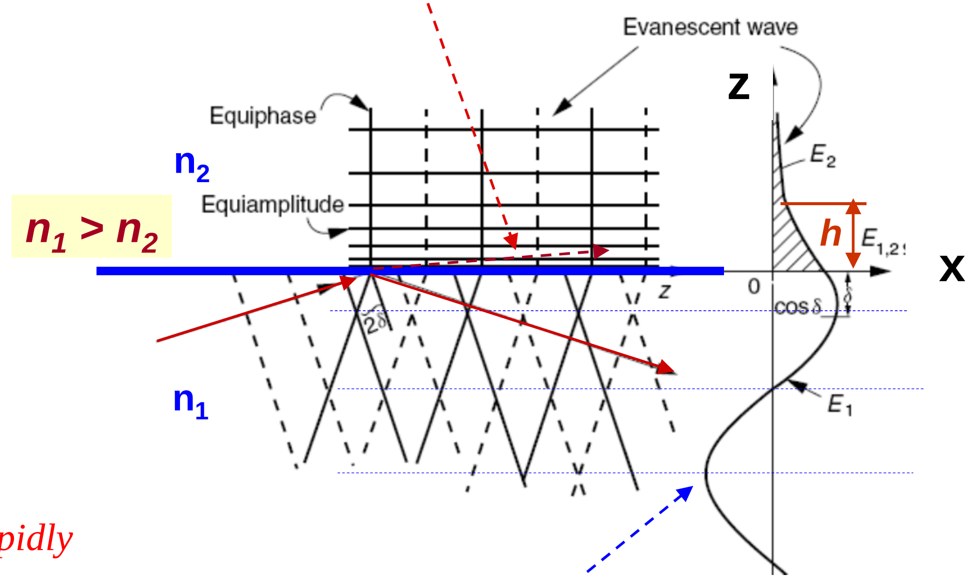
$$\alpha = k_t\sqrt{\frac{\sin^2\theta_i}{n} - 1} = \frac{2\pi}{\lambda_t}\sqrt{\frac{\sin^2\theta_i}{n} - 1}$$

We can write the transmitted wave as:

$$E_t = E_{0t}\exp\left[i\left(\frac{k_t x\sin\theta_t}{n} - \omega t\right)\right]\exp(-\alpha z)$$

The evanescent wave amplitude will decay rapidly as it penetrates into the lower refractive index medium.

$$E_t = E_{0t}\exp\left[i\left(\frac{k_t x\sin\theta_t}{n} - \omega t\right)\right]\exp(-\alpha z)$$



Note that the incident and reflection waves form a standing wave in x direction

Penetration depth: $E_t = \left(\frac{1}{e}\right)E_{ot} \;\Rightarrow\; h = \frac{1}{\alpha} = \frac{\lambda}{2\pi\sqrt{\frac{\sin^2\theta_i}{n^2} - 1}}$

Frustrated TIR

T_p = fraction of intensity
transmitted across gap

$$T_p = [1/(\alpha \sinh^2 y + 1)],$$

$$\alpha = \left(\frac{n^2-1}{2n}\right)^2 \frac{[(n^2+1)\sin^2\theta_i - 1)]^2}{\cos^2\theta_i(n^2\sin^2\theta_i - 1)},$$

$$y = 2\pi\left(\frac{d}{\lambda}\right)(n^2\sin^2\theta_i - 1)^{1/2}.$$

Zhu et al., "Variable Transmission Output Coupler and Tuner for Ring Laser Systems," Appl. Opt. ***24****, 3610-3614 (1985).*

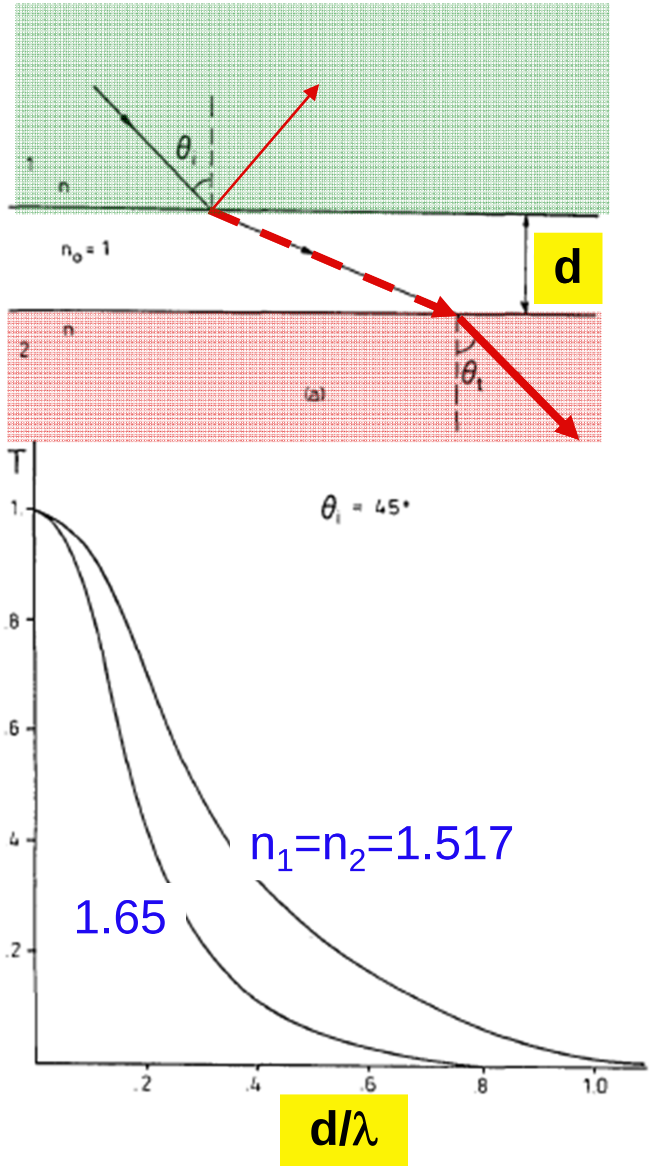
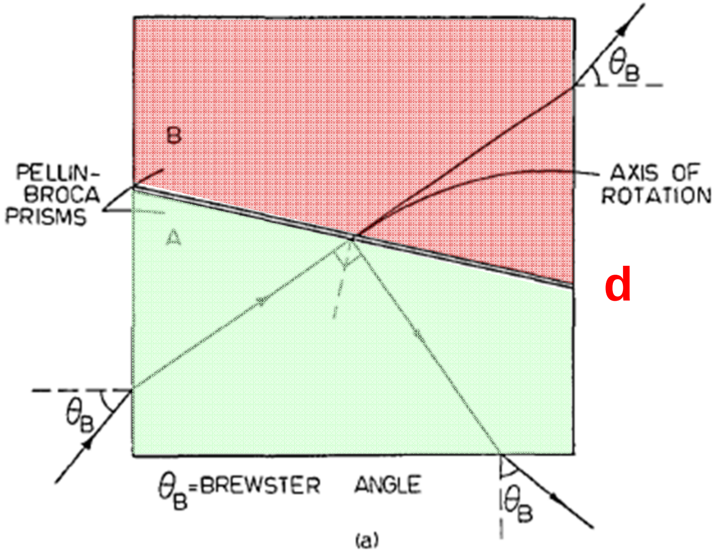


Fig. 2. (a) Tunneling of light through the gap between the regions 1 and 2; frustrated total internal reflection. (b) The fraction of transmitted light vs (d/λ) plotted for two different values of the refractive index n.

Frustrated Total Internal Reflectance



Pellin-Broca prism

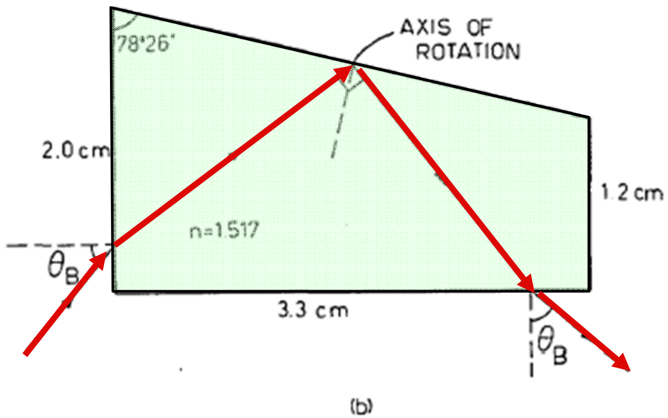


Fig. 3. (a) Configuration of two Pellin-Broca prisms for frustrated total internal reflection. All beams enter and exit at Brewster angle θ_B. (b) Dimensions of the Pellin-Broca prism.

Zhu et al., “Variable Transmission Output Coupler and Tuner for Ring Laser Systems,” Appl. Opt. **24**, 3610-3614 (1985).

d = 1 ~ λ: changing the reflectance
Rotation: changing the wavelength resonant at θ_B

23-6. Complex Refractive Index

$$\text{For a material with conductivity}(\sigma): \quad \tilde{n} = \sqrt{1 + i\left(\frac{\sigma}{\varepsilon_0\,\omega}\right)} = n_R + i\,n_I$$

$$\tilde{n}^2 = 1 + i\left(\frac{\sigma}{\varepsilon_0\,\omega}\right) = n_R^2 - n_I^2 + i\;2n_R\,n_I$$

Solving for the real and imaginary components we obtain:

$$n_R^2 - n_I^2 = 1 \qquad 2n_R\,n_I = \frac{\sigma}{\varepsilon_0\,\omega} \quad \Rightarrow \quad n_R = \frac{\sigma}{2\,n_I\,\varepsilon_0\,\omega}$$

$$\Rightarrow \quad \left(\frac{\sigma}{2\,n_I\,\varepsilon_0\,\omega}\right)^2 - n_I^2 = 1 \quad \Rightarrow \quad n_I^4 - n_I^2 - \left(\frac{\sigma}{2\,\varepsilon_0\,\omega}\right)^2 = 0$$

From the quadratic solution we obtain:

$$n_I^2 = \frac{1 \pm \sqrt{1 + 4\left(\frac{\sigma}{2\,\varepsilon_0\,\omega}\right)^2}}{2} \quad \Rightarrow \quad n_I^2 = \frac{1 + \sqrt{1 + 4\left(\frac{\sigma}{2\,\varepsilon_0\,\omega}\right)^2}}{2}$$

We need to take the positive root because n_I is a real number.

Complex Refractive Index

Substituting our expression for the complex refractive index back into our expression for the electric field we obtain

$$\vec{E} = \vec{E}_0 \exp\left[i\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]$$

$$= \vec{E}_0 \exp\left\{ i\left[\left(n_R + i\, n_I\right)\frac{\omega}{c}\left(\hat{u}_k \cdot \vec{r}\right) - \omega t \right]\right\}$$

$$= \vec{E}_0 \exp\left\{ i\,\omega\left[\frac{n_R}{c}\left(\hat{u}_k \cdot \vec{r}\right) - t \right]\right\} \exp\left[-\frac{n_I\,\omega}{c}\left(\hat{u}_k \cdot \vec{r}\right)\right]$$

The first exponential term is oscillatory.

The EM wave propagates with a velocity of n_R / c.

The second exponential has a real argument (absorbed).

Complex Refractive Index

$$\vec{E} = \vec{E}_0 \exp\left\{ i\,\omega \left[\frac{n_R}{c}\left(\hat{u}_k \cdot \vec{r}\right) - t \right] \right\} \exp\left[-\frac{n_I\,\omega}{c}\left(\hat{u}_k \cdot \vec{r}\right) \right]$$

The second term leads to absorption of the beam in metals due to inducing a current in the medium. This causes the irradiance to decrease as the wave propagates through the medium.

$$I \equiv \vec{E}\vec{E}^* = \vec{E}_0\vec{E}_0^{\,*} \exp\left[-\frac{2\,n_I\,\omega\left(\hat{u}_k \cdot \vec{r}\right)}{c} \right]$$

$$I = I_0 \exp\left[-\frac{2\,n_I\,\omega\left(\hat{u}_k \cdot \vec{r}\right)}{c} \right] = I_0 \exp\left[-\alpha\left(\hat{u}_k \cdot \vec{r}\right) \right]$$

The absorption coefficient is defined : $\alpha = \dfrac{2\,n_I\,\omega}{c} = \dfrac{4\pi\,n_I}{\lambda}$

23-7. Reflection from Metals

Reflection from metals is analyzed

by substituting the complex refractive index ñ in the Fresnel equations:

$$TE\ case:\ r_{TE} = \frac{E_r}{E_i} = \frac{\cos\theta_i - \sqrt{\tilde{n}^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}}$$

$$TM\ case:\ r_{TM} = \frac{E_r}{E_i} = \frac{-\tilde{n}^2\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}}{\tilde{n}^2\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}}$$

Substituting $\tilde{n} = n_R + i\,n_I$ *we obtain:*

$$TE\ case:\ r = \frac{E_r}{E_i} = \frac{\cos\theta_i - \sqrt{\left(n_R^2 - n_I^2 - \sin^2\theta_i\right) + i\left(2n_R\,n_I\right)}}{\cos\theta_i + \sqrt{\left(n_R^2 - n_I^2 - \sin^2\theta_i\right) + i\left(2n_R\,n_I\right)}}$$

$$TM\ case:\ r = \frac{E_r}{E_i} = \frac{-\left[\left(n_R^2 - n_I^2\right) + i\left(2n_R\,n_I\right)\right]\cos\theta_i + \sqrt{\left(n_R^2 - n_I^2 - \sin^2\theta_i\right) + i\left(2n_R\,n_I\right)}}{\left[\left(n_R^2 - n_I^2\right) + i\left(2n_R\,n_I\right)\right]\cos\theta_i + \sqrt{\left(n_R^2 - n_I^2 - \sin^2\theta_i\right) + i\left(2n_R\,n_I\right)}}$$

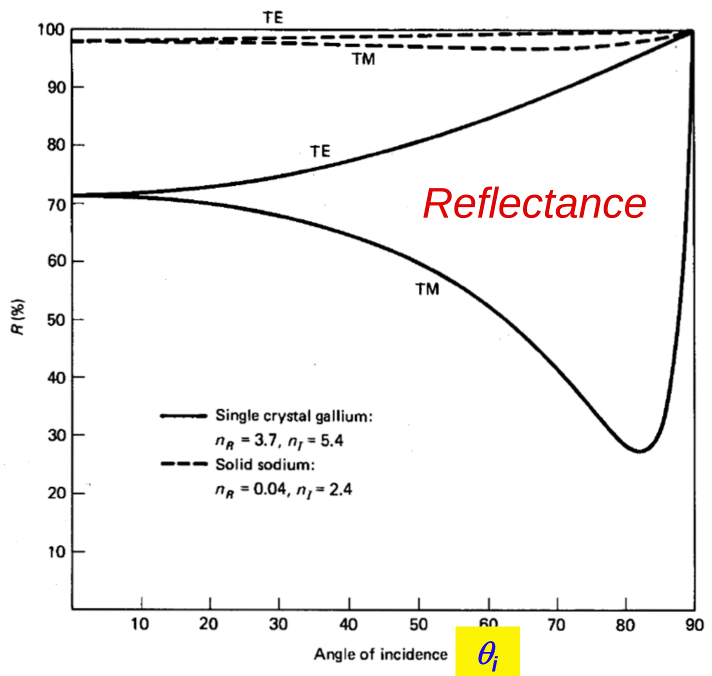


Figure 20-10 Reflectance from metal surfaces by using Fresnel's equations. The values of n_R and n_I are given for sodium light of λ = 589.3 nm.

Reflection from Metals at normal incidence (θ_i=0)

$$\text{At normal incidence, } \theta_i = 0° :$$

$$r_{TE} = \frac{\cos\theta_i - \sqrt{\tilde{n}^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}} = \frac{1-\tilde{n}}{1+\tilde{n}}$$

$$r_{TM} = \frac{-\tilde{n}^2\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}}{\tilde{n}^2\cos\theta_i + \sqrt{\tilde{n}^2 - \sin^2\theta_i}} = \frac{1-\tilde{n}}{1+\tilde{n}}$$

$$\therefore \quad r = \frac{1-(n_R - i\,n_I)}{1+(n_R - i\,n_I)}$$

The *power reflectance* R *is given by*

$$R = r\,r^*$$

$$= \left[\frac{1-(n_R - i\,n_I)}{1+(n_R - i\,n_I)}\right]\left[\frac{1-(n_R + i\,n_I)}{1+(n_R + i\,n_I)}\right] = \left(\frac{1-\ 2n_R + n_R^2 + n_I^2}{1+\ 2n_R + n_R^2 + n_I^2}\right)$$

$$\boxed{R = \frac{(n_R - 1)^2 + n_I^2}{(n_R + 1)^2 + n_I^2}}$$

At normal incidence
(from Hecht, page 113)

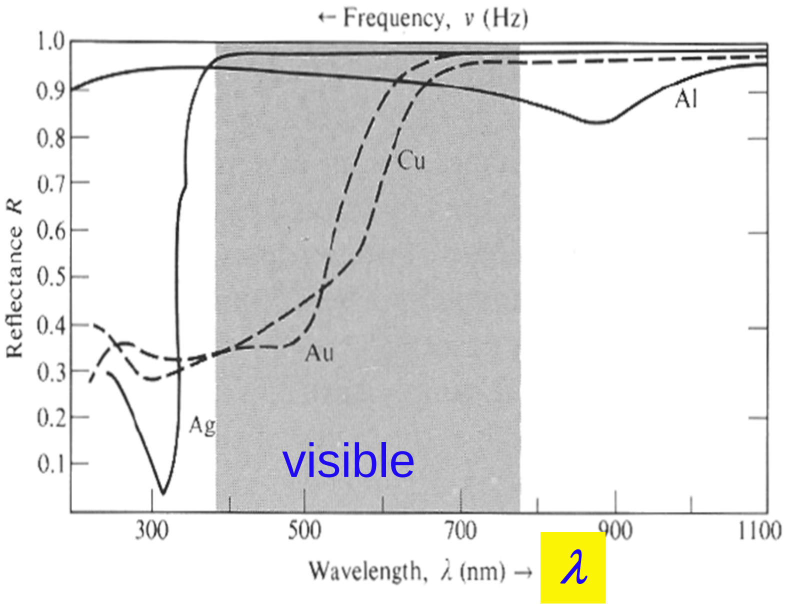


Figure 4.42 Reflectance versus wavelength for silver, gold, copper, and aluminum.