Pricing Mortgage-backed Securities and Collateralized Mortgage Obligations

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Introduction

Mortgage-backed securities (MBS) and *collateralized mortgage obligations* (CMO) are an increasingly popular and important class of financial instruments. As of June 2000, the MBS market, also known as the pass-through market, amounted to almost $2 trillion. Only U.S. Treasury issuance, with about $3.5 trillion outstanding, is larger[1]. The MBS market is one of the fastest growing, as well as one of the largest financial markets in the United States. To facilitate the flow of mortgage capital and to make it easier for potential home buyers to obtain mortgages, the U.S. government created three housing finance agencies – Ginnie Mae, Fannie Mae, and Freddie Mac – to develop this secondary mortgage market. These agencies support the secondary market by issuing an MBS in exchange for pools of mortgages from lenders. MBSs provide lenders with a liquid asset that they can hold or sell. This ensures that lenders have funds to make additional loans.

Ginnie Mae (formally known as the Government National Mortgage Association, or GNMA) is part of the U.S. government, and any MBS guaranteed by Ginnie Mae carry the full faith and credit of the government. Hence, MBSs issued through Ginnie Mae are generally considered to have no credit risk. Although Fannie Mae (formally the Federal National Mortgage Association, or FNMA) and Freddie Mac (formally the Federal Home Loan Mortgage Corporation, or FHLMC) are now private entities and do not have explicit U.S. government guarantees, they continue to maintain close ties to the U.S. government and are considered to have negligible credit risk. These three agencies constitute the majority of the MBS market but there exist private-label MBSs which are issued by S&Ls, savings and commercial banks. These private-label MBSs do not carry the same guarantee as do the three agencies and bear credit risk.

Ownership of a unit of an MBS entitles the owner to a cash flow from the principal and/or interest of the mortgage payments. All mortgages have a given payment schedule and often permit prepayment without penalty. This results in the owners of the pool units to bear interest rate and prepayment risk. All owners are treated equally and bear equivalent risks.

Despite the dramatic growth of the mortgage pass-through market, the cash-flow characteristics of pass-throughs did not meet the needs of many investors. Investors wanted a new instrument that provided choices on a variety of maturity and prepayment profiles. In June 1983, Freddie Mac issued the first CMO. In CMOs, the total pool is divided into classes or tranches, each having different prepayment and interest rate risk characteristics. Instead of the traditional 30-year fixed-rate mortgage, investors could now pick from a seemingly unlimited range of bond characteristics. Within the CMO, the various tranches differ according to the priority of cash flows received and according to the degree to which the tranches have claims against principal, interest, or both. Today, the vast majority of outstanding securitized mortgage instruments have been transformed from MBSs into CMOs.

[1] Statistics from Fabozzi (2001)

In this paper, we will discuss different types of pay rules, which describe how principal is allocated among the various CMO bonds and how interest is allocated among the bonds. We will also describe the prepayment model of Richard and Roll (1989) and examine different term structure models for our use in pricing MBS and CMO structures. We will perform simulations to price various MBS and CMO structures to determine the amount of risk each type of structure bears. Option adjusted spread (OAS) analysis will be used to compare bonds with different structures.

Prepayment Models

Mortgages have the unique characteristic of giving the borrower the right to prepay a mortgage at any time during the life of the loan. This right means the investor in a mortgage cannot know the maturity of the loan or the amount of interest that will be received with absolute certainty. There are three basic reasons for prepayment: moving, refinancing, and default. Defaults are not actually prepayments but most MBSs have credit guarantees that transform borrower defaults into prepayments to investors. One of the difficulties in pricing MBSs and CMOs is being able to quantify this prepayment risk.

Over the past 20 years, there has been much research done in the field of prepayment modeling. Many prepayment models have been developed using one of two techniques. Prepayment models developed by, among others, Dunn and McConnell (1981a, 1981b), Stanton (1995), and Downing, Stanton and Wallace (2002) use a "rational" prepayment model where mortgagors are assumed to exercise their call option whenever the value of the mortgage would exceed the remaining principal balance of the loan plus the transaction costs associated with refinancing. This class of models relies on individual mortgagors following a constrained utility-maximizing call policy. Although this approach is robust to structural changes in the economic environment, it does not perform well in the valuation of CMO tranches. These models rely upon a finite difference backward solution procedure that begins with the maturity date of the mortgage. Since CMO tranches require knowledge of prior mortgage prepayments, this procedure does not accommodate the "memory" required to determine the allocation of cash flows among the CMO tranches.

The second approach uses an estimation technique. Prepayment models are developed by using mathematical equations that relate environmental assumptions to prepayment rates. This approach allows for an easier adaptation to the analysis of CMOs. This empirically estimated prepayment model is used with a "forward-looking" Monte Carlo simulation to value the CMO tranches. The prepayment model developed by Richard and Roll (1989) uses this technique and will be implemented in our simulation.

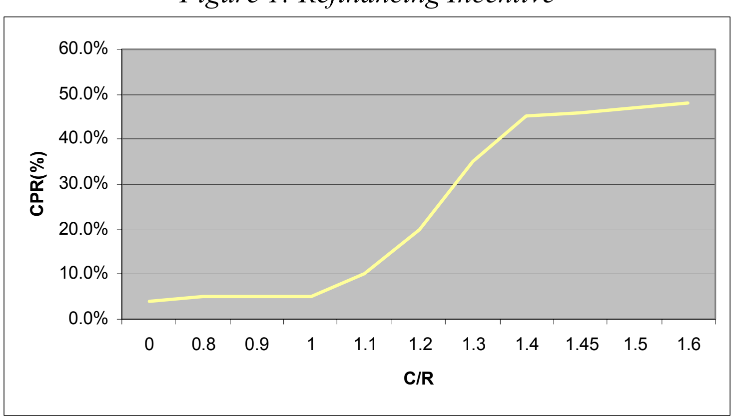
Prepayment Model of Richard and Roll (1989)

The prepayment model of Richard and Roll is an empirical estimation of the mortgagor's financing decision. It tries to explain prepayments by observing actual prepayments and relating them to the measurable factors suggested by their economic theory of prepayments. Richard and Roll suggest since mortgagors have heterogeneous refinancing costs, not all mortgagors follow a utility-maximizing call policy. Their model identifies four factors that must be included in a prepayment model. These factors are: Refinancing Incentive, Seasoning or age of the mortgage, the month of the year (seasonality) and Premium burnout. The conditional prepayment rate, CPR[2], is computed as the product of these four factors. We will now examine each of these four factors.

The Refinancing Incentive

The most crucial component in the valuation of MBSs. This component is the most volatile and contributes the most to the value of the prepayment option. It is generally expressed as the percentage of monthly savings that can be realized by refinancing. The refinancing incentive is measured by either taking the difference or the ratio between the mortgage coupon rate and the mortgage-refinancing rate. As the interest rates decrease, the ratio or spread becomes larger. A ratio greater than 1, or a positive spread will motivate the mortgage holder to refinance. A ratio smaller than 1, or a negative spread, means that current mortgage rates are higher than the rate of the mortgage held, and the prepayment option is out of the money. The relationship between prepayment and C/R (or C-R), where C is the coupon rate and R is the mortgage-refinancing rate, is described by using a curve-fitting technique. The arctangent function was a convenient nonlinear representation for the shape of the prepayment curve. At one point, many analysts used this function. Today, this function has been replaced by more complex variations. In our simulation, we will use a slight variation of the arctangent function. The curve in Figure 1 displays the pure refinancing incentive for our prepayment model.

Figure 1: Refinancing Incentive

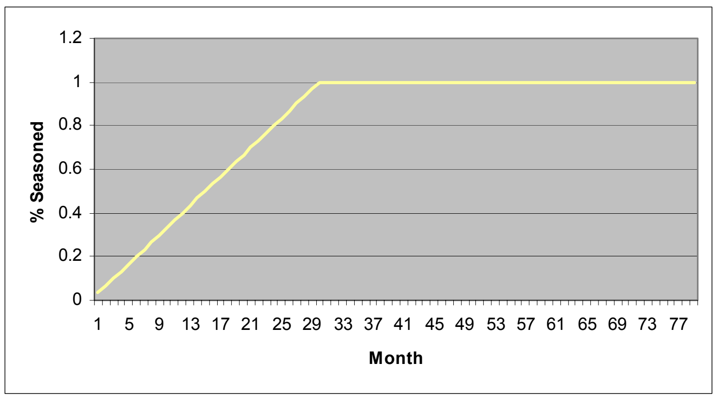


[2] See Appendix for a more detailed definition of CPR

Seasoning

Seasoning or aging reflects the observation that newer loans tend to prepay slower than older or "seasoned" loans. This factor follows the rationale behind the PSA[3] standard prepayment model. This industry convention adopted by the Public Securities Association models mortgage prepayment rates as increasing linearly from 0.2% CPR at issue to 6% CPR at thirty months and then remaining constant. Figure 2 shows the seasoning percentage in terms of months. In our simulation, the formula for Seasoning is as follows: $Age(t) = \min(\frac{t}{30}, 1)$, where t is in terms of months.

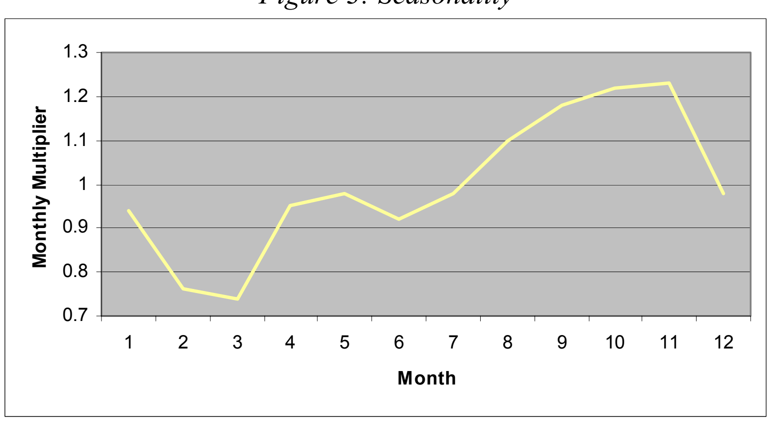
Figure 2: Seasoning



Month of the Year (Seasonality)

Seasonality takes into consideration the time of year. It is believed that prepayments peak in the summer and decrease in the winter. One of the sources of prepayments due to seasonality is housing turnover. This could be due to weather and school schedules. Figure 3 shows the monthly multipliers for each month. In our simulation, the monthly parameters were taken from figure 3 in Richard and Roll (1989).

Figure 3: Seasonality

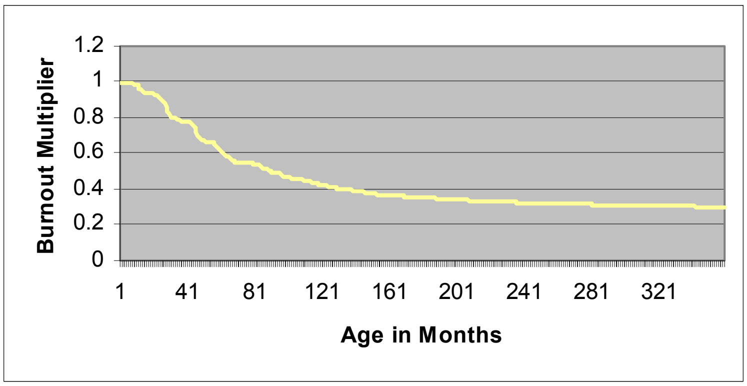


[3] See Appendix for a more detailed definition of PSA

Premium Burnout

Premium Burnout takes into account the tendency for prepayment to diminish over time, even when refinancing incentives are favorable. Factors leading to this effect include inadequate home equity to qualify for refinancing, and/or a negative change in the credit status of the borrower. Richard and Roll try to quantify burnout by measuring how much the option to prepay has been deep in-the-money since the pool was issued. They suggest the more the prepayment option has been deep in-the-money, the more burned out the pool is, and the smaller prepayments are, all other things being equal. In our simulation, the Burn% is calculated as a function of the pool factor[4]. Figure 4 shows the Burnout Multiplier in terms of age in months.

Figure 4: Premium Burnout



Multiplicative Model

The four effects are combined in a multiplicative formula to determine the prepayment rates. The effects of these factors interact proportionately to produce the CPR prepayment model. The formula is ***CPR = (Refinancing Incentive) x (Age Multiplier) x (Month Multiplier) x (Burnout Multiplier).*** The details of the actual formulas used in our simulation can be found in the Appendix.

[4] The pool factor is the decimal value that represents the proportion of the original principal amount outstanding at a given time.

Interest Rate Process

For the purpose of valuating MBSs and CMOs, any arbitrage-free model of the term structure of interest rates can be used. Equilibrium interest rate models are based on the assumption that bond prices, and yields, are determined by the market's assessment of the evolution of the short-term interest rate. For the following models, the short rate is assumed to follow a diffusion (a continuous time stochastic) process. The general form of these models is described in terms of changes in the short rate, as follows:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\, r^{\alpha}\, dB_t\,,\;\; r(0) = r_0 \qquad (1)$$

where dr_t represents an infinitesimal change in r_t over an infinitesimal time period, dt, and dB_t is a standard Wiener process. κ is the speed of mean-reversion, θ is the long-run mean of the interest rate process, α is the proportional conditional volatility exponent, and σ is the instantaneous standard deviation of changes in r_t.

The three models we will examine are the Vasicek (1977), Dothan (1978), and Cox-Ingersoll-Ross (CIR) (1985) models. The difference between these models primarily revolves around the parameter α. Vasicek assumes it to be 0, CIR assumes it to be 0.5 and Dothan assumes it to be 1.0.

Vasicek Model

The Vasicek (1977) model is a popular choice for simple interest rate processes. It is based on the Ornstein-Uhlenbeck (OU) stochastic process for the spot interest rate r_t. It is a one-factor model, where all rates ultimately depend on the shortest-term interest rate. The interest rate is normally distributed and follows the process:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\, dB_t\,,\;\; r(0) = r_0 \qquad (2)$$

The solution to this SDE is as follows:

$$r_t = r_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-s)} dB_s\,. \qquad (3)$$

In order to simulate interest rates using this model, we discretize the basic Vasicek equation by considering changes in the interest rate over a short period Δt:

$$\Delta r = \kappa(\theta - r)\Delta t + \sigma B\sqrt{\Delta t}\,. \qquad (4)$$

Term structure models are additive, which implies that the spot rate at time $t + \Delta t$ can be expressed as follows:

$$r_{t+\Delta} = r_t + \kappa(\theta - r_t)\Delta t + \sigma B\sqrt{\Delta t} \text{ , where } B \sim N(0,1). \tag{5}$$

A drawback of this process is that negative interest rates can be simulated. In most simulations, absolute values of the Vasicek model are used.

Cox, Ingersoll, and Ross (CIR) Model

The CIR model (CIR,1985) considers an interest rate process of the type:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\, dB_t,\ \ r(0) = r_0. \tag{6}$$

The only change is in the volatility function. Volatility is now dependent on the level of interest rates. Interest rates will never reach zero in this model if $\sigma^2 < \kappa\theta$. Although in continuous-time the process never reaches zero, it may still do so during simulation in discrete-time. This was first pointed out by Beaglehold and Tenney (1992). Like the Vasicek model, during the Monte Carlo step, the absolute value of the interest rate is also taken.

Dothan Model

The model of Dothan (1978) increases the volatility exponent to 1.0. It considers an interest rate process of the type:

$$dr_t = \kappa\, r_t dt + \sigma\, r\, dB_t,\ \ r(0) = r_0 \tag{7}$$

Since the volatility exponent is higher in this model, the model relates the volatility of interest movements more strongly to the level of interest rates. Dothan's model does not include a mean reverting term in the drift. Courtadon (1982) extends Dothan's model to include mean reversion. Courtadon's model is as follows:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\, r\, dB_t,\ \ r(0) = r_0. \tag{8}$$

Mean Reversion and the Volatility Assumption

All of our interest rate models have some form of reversion, reverting the generated interest rate paths to some “normal” level. Without reversion, interest rates can obtain unreasonably high and low levels. Volatility, over time, would theoretically approach infinity. Similarly, a large percentage assumption of volatility will result in greater fluctuations in yield. This results in a greater probability of the opportunity to refinance. The outcome of refinancing is a greater value attributed to the implied call option, and a higher resulting option cost.

The speed of reversion, κ, ultimately affects the shape of the yield curve. If κ is high, the yield curve quickly tends toward the long-run yield rate θ. If κ is low, the yield curve slowly tends toward θ.

The parameters chosen for our interest rate process are described in the “*MBS Price Simulations*” section.

The Structure of CMO's

A CMO structure, also known as Real Estate Mortgage Investment Conduits (REMICS), is a mechanism for reallocating cash flows from one or more mortgage pass-through or a pool of mortgages into multiple classes with different priority claims. Typically, pools of mortgages that have coupon rates and maturities that lie within a narrow range collateralize a CMO. The CMO structure assigns the principal and interest cash flows from the underlying collateral to various tranches. It must be structured in a way that even in the most adverse circumstances there will be adequate cash flows to satisfy the entire principal and interest due on the bonds. This protection can be accomplished by structuring the CMO to a worst-case scenario, where we assume zero prepayment.

CMO structuring is premised on two rules:[5]

1. The total principal of the collateral must always equal the sum of all principal payments scheduled for the bonds.

2. The total yield spread provided by the CMO bonds must equal not more than the purchase yield spread of the collateral, plus issuance expenses and any profit and/or residual interest to the issuer.

CMO structures come in two major types: One type redirects only principal payments to various tranches, and the other redirects both interest and principal payments. This paper will focus on Sequential-pay, PAC-companion, and TAC-companion structures that only redirect principal payments.

[5] From "The Valuation of Mortgage-Backed Securities", (Bartlett, 1994)

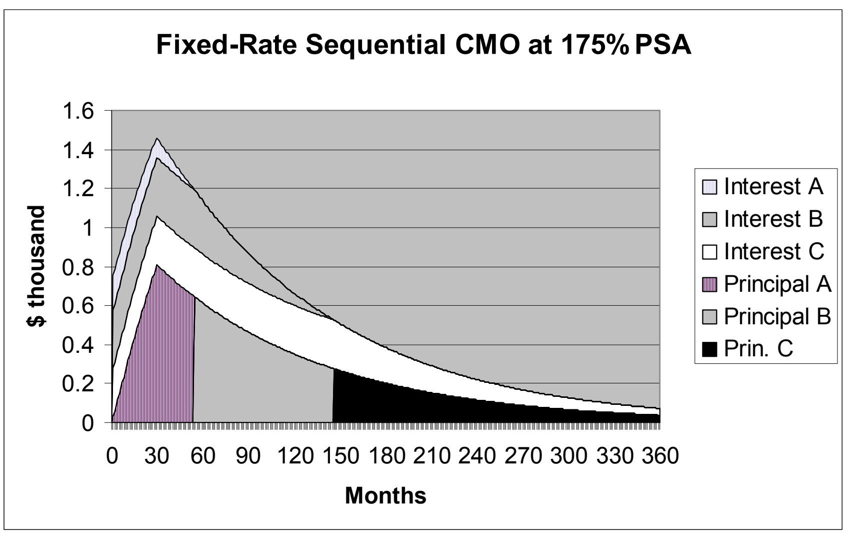
Sequential-Pay (SEQ) classes

Sequential-pay classes are the most basic classes within a CMO structure. They are also called *Plain Vanilla, Clean Pay,* or *Current Pay* classes. The primary purpose of this class is to bring a broader range of maturity choices to the MBS market. This class reallocates collateral principal payments sequentially to a series of bonds. The sequential class with the shortest maturity receives principal payments, including prepayments, until it is fully retired; then principal payments are redirected to the next sequential class with the shortest maturity until it is retired. This process continues until the last sequential class is fully retired. While one class is receiving principal payments, the other existing classes receive monthly interest payments at their coupon rate on their principal.

Sequential-pay structures enable capital market participants with short investment horizons to enter the MBS market. They could purchase bonds that more closely match their desired terms. Also, investors with long-term horizons benefit since they are insulated from prepayment during the early years of a pool's life.

Figure 4 demonstrates how the principal cash flows of a $100,000 MBS is distributed in a sequential-pay structure containing three bonds, with balances of $30,000, $40,000, and $30,000 respectively. Also the prepayment speed of the collateral was set to 175% PSA.

Figure 4: Principal and Interest Flows from a Three-Tranche Sequential-Pay Structure[*]

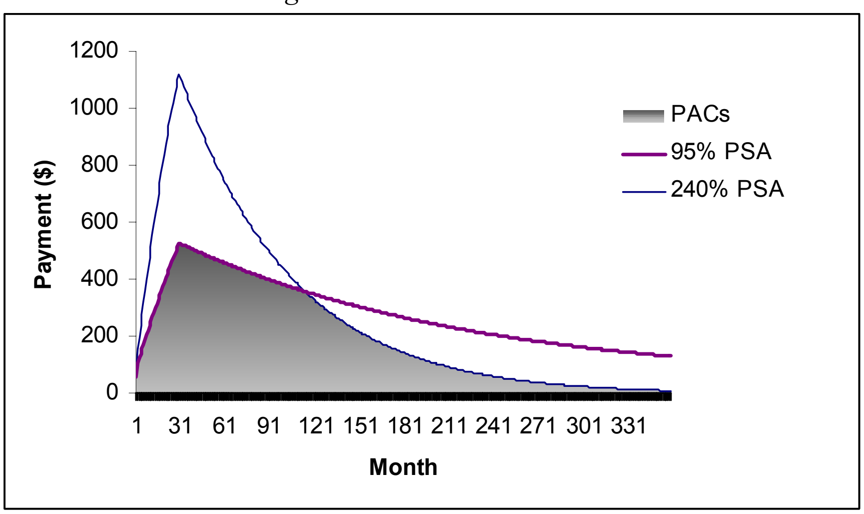


[*] Based on a $100,000 10% pool at 175% PSA

Planned Amortization Class

Introduced in August 1986, a new type of structure called *planned amortization classes* (PACs) was issued. These structures were developed to help reduce the effects of prepayment risk. PAC bonds are designed to produce more stable cash flows by redirecting prepayments from the underlying securities to other classes called companion or support classes. The PAC investor is scheduled to receive fixed principal payments (the PAC "schedule") over a predetermined period of time. This schedule is based on the minimum amount of principal cash flow produced by the collateral at two prepayment rates known as the PAC "bands". The schedule will be met only if the underlying securities prepay at a constant rate within the range assumed for the structuring of the PAC. In times of fast prepayments, companions support PACs by absorbing principal payments in excess of the PAC schedule. A sustained period of fast prepayments may completely eliminate a PAC's outstanding support class. When this occurs, the PAC is called a "busted" or "broken" PAC. A busted PAC behaves like a sequential-pay class and the investor is subject to the same yield fluctuations as a sequential-pay class investor. Alternatively, in times of slow prepayments, amortization of the companions is delayed if there is not enough principal for the currently paying PAC. This results in an extension of the average life of the class. The total PAC and companion principal cash flows can be further divided sequentially, much like a sequential-pay structure, or using another PAC structure. The shaded area in Figure 5 represents the PAC principal payment schedule, based on a 95% PSA and 240% PSA PAC band.

Figure 5: The PAC Schedule[*]



[*] Based on a $100,000 10% pool

Targeted Amortization Class

Targeted amortization classes (TACs) were introduced to offer investors a prepayment-protected class at wider spreads than PACs. TACs pay a "targeted" principal payment schedule at a single, constant prepayment speed. As long as the underlying securities do not prepay at a rate slower than this speed, the schedule will be met. TACs may provide protection against increasing prepayments and early retirement of the investment ("call" risk). If the principal cash flow from the collateral exceeds the TAC schedule, the excess is allocated to TAC companion classes. Alternatively, if prepayments fall below the speed necessary to maintain the TAC schedule, the weighted average life of the TAC will be extended. They do not protect against low prepayment rates.

TAC bonds are appealing because they offer higher yields than comparable PAC bonds. The unaddressed risk from low prepayment rates generally does not concern investors as much as risk from high prepayment rates.

The typical TAC can be viewed as a PAC with a lower band equal to the CMO pricing speed and an upper band similar to that of PACs backed by comparable collateral.

Accrual (Z) Class

A Z-bond or accrual bond is a type of interest and principal pay rule. The Z-bond does not receive any principal or interest cash flow until the prior bonds are completely paid down. Like any other bond, it generates coupon cash flows. As long as the Z-bond is not paying out principal, this coupon flow is used to pay down other classes. The interest generated is added (accrues) to the principal amount due to the Z-bond. Once the classes preceding the Z-bond are fully paid down, it begins to receive principal and interest. The Z-bond is used in a simple sequential-pay structure to accelerate the principal repayments of the sequential-pay bonds. Since the portion of the principal payments of these sequential-pay bonds is coming from the Z-coupon flows, the average life volatility is decreased in the sequential-pay classes.

In our simulation, we will show that the price of the Z-bond is highly sensitive to interest rate movements and the resulting changes in prepayment rates because its ultimate principal balance depends on total accretions credited by the time it begins to pay down.

Figure 6, on page 16, shows the principal and interest cash flows for a three-tranche sequential pay structure that includes a Z-bond. Comparing Figure 4 with Figure 6, there is a noticeable difference in the weighted average life (WAL) of the bonds. The inclusion of a Z-bond decreases the WAL for both bonds A and B, while increases for the Z-bond. Table 1 shows the WAL in months for each bond and the difference caused by the inclusion of the Z-bond.

Figure 6: Principal and Interest Flows from a Three-Tranche Sequential-Pay Structure with a Z-Bond[*]

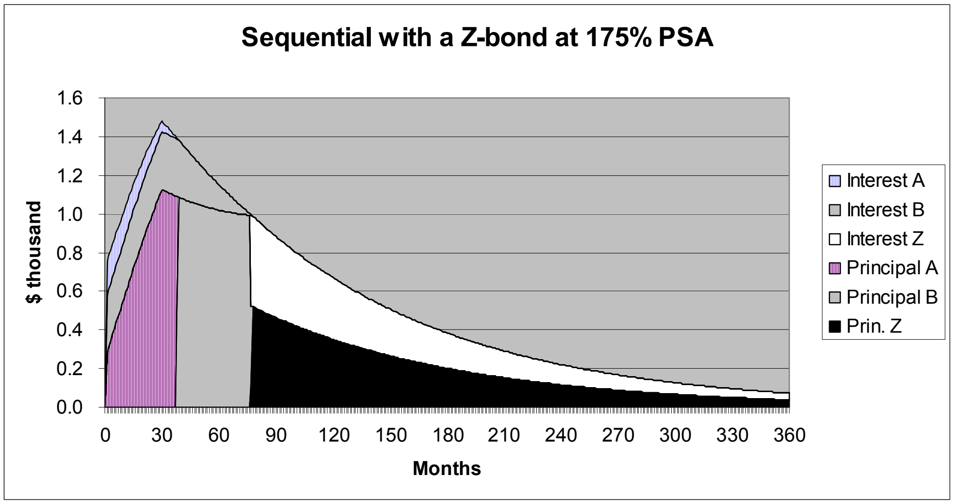


*Table 1: Average Life Comparison**

	Average Life (in months)		
Bond	**No Z**	**W/Z**	**Difference**
A	32.3	23.0	9.3
B	93.3	57.2	36.1
C	219.5	286.1	-66.6

Other Classes and Pay Rules

In this paper, and in our simulation, we only touch on a few types of CMO classes. There exist many other types of classes, including Pro-Rata CMOs, which are designed to assign different coupons to bonds with the same principal payment characteristics and Type II or Type III PACs, which are structured from the companion cash flows in a PAC or companion structure.

Interest pay rules reallocate interest cash flows into multiple classes. Floaters and Inverse Floaters are such rules. They are extremely sensitive to interest rate fluctuations. Floating rate coupons and Inverse Floaters are both tied together and act as companion classes. The Floater is set at a margin above an index, where a variety of indices can be used. The most common are LIBOR and Constant Maturity Treasury (CMT). For the Inverse Floater, the coupon rate is periodically adjusted in the opposite direction of the

[*] Based on a $100,000 10% pool at 175% PSA

index. Floaters are appealing to investors since they can be used to hedge against changes in the direction of interest rates.

Stripped securities have also proved to be a valuable tool in the MBS market. Altering the distribution of principal and interest from a pro-rata distribution to an unequal distribution create these securities. Principal and interest payments are “stripped apart” from the underlying mortgages and divided into two classes. One class receives the interest cash flow (IO, or Interest Only); the other receives the principal cash flow (PO, or Principal Only). There also exist strips, which allocate specified percentages of interest and principal to each new strip. For example, a class can be stripped to produce two classes, one with a 5% coupon and the other with a 8% coupon, simply by directing more of the interest from the underlying class to the higher coupon and less of the interest to the lower coupon. These tools are generally used as hedging instruments against interest rate and prepayment risk. Investors who stand to lose future cash flows from other investments due to falling interest rates may use PO classes, while IO classes may be used by fixed-income investors whose portfolios decrease in value as rates increase.

Within any class, another class can further reallocate the cash flows, and that class can be further broken down using another class, conceivably producing an unlimited number of scenarios. Since, in our simulation, our main objective is to determine the price volatility of each class, our CMO structures will be fairly simple. The section, “*MBS Price Simulations*” provides the details on the exact structures used in our simulation.

MBS Valuation

Generally the price of any security can be written in terms of the net present value (NPV) of its discounted cash flows under the risk neutral probability measure. The price of any fixed income security can be described as follows:

$$P = E^Q\left[\sum_{t=0}^{M} PV(t)\right] = E^Q\left[\sum_{t=0}^{M} df(t)cf(t)\right] \tag{9}$$

where:

P is the price of the security;
Q is the risk neutral probability measure;
M is the maturity of the security;
PV(t) is the present value for cash flow at time *t*;
df(t) is the discounting factor for time *t*;
cf(t) is the cash flow at time *t*.

The discounting factor is found from the short-term (risk-free) interest rate process:

$$df(t) = \prod_{k=1}^{t} \frac{1}{(1+r_k)} \quad , \tag{10}$$

where r_k is the short-term rate generated from one of the interest rate models discussed.

Since we are generating cash flows for an MBS and not just a risk-free zero coupon bond, generating *cf*(t) is more complicated as the cash flows depend not only on interest rates but also on prepayment behavior. The cash flows can be calculated using the following formulas from Fabozzi (2001):

$$\begin{aligned} cf(t) &= MP(t) + PP(t) = TPP(t) + IP(t); \\ MP(t) &= SP(t) + IP(t); \\ TPP(t) &= SP(t) + PP(t); \end{aligned} \tag{11}$$

where,

MP(t) is the scheduled mortgage payment for period *t;*
TPP(t) is the total principal payment for period *t;*
IP(t) is the Interest payment for period *t;*
SP(t) is the scheduled principal payment for period *t;*
PP(t) is the principal prepayment for period *t;*

The formulas for *MP(t), TPP(t), IP(t), SP(t),* and *PP(t)* are provided in the Appendix.

To evaluate cash flows, the only uncertainty lies with calculating *PP(t).* The principal prepayment is calculated using *CPR(t),* the conditional prepayment rate, which was derived from our prepayment model. Once *CPR(t)* is known, everything else can be calculated.

Monte-Carlo Simulation

Monte-Carlo simulation is a numerical integration technique based on random numbers. It has become an important tool in the pricing of complex financial instruments. In pricing MBSs or CMOs, Monte-Carlo simulation is used to simulate many independent interest rate paths using a risk-neutral interest rate process. Once the interest rate path is generated, it is mapped into the prepayment model to generate a set of anticipated payments to the mortgage security holders. Along each simulated path, mortgage security cash flows are projected, and the present value of these anticipated cash flows are calculated. Many simulations are run and the average of the present values is calculated. The law of large numbers guarantees the consistency of the estimate.

Option Adjusted Spread (OAS)

Option-adjusted spread (OAS) analysis is designed to measure the yield spread of a fixed income security that is <u>not</u> attributable to imbedded options. The OAS measures such factors as the securities credit risk and liquidity. It is used to evaluate the relative cheapness or richness of securities with imbedded options such as MBS. OAS uses Monte-Carlo simulation to value a security under thousands of possible interest rate scenarios. The value of the security is found for each scenario by discounting the cash flows at the projected risk-free rates plus a spread. This generates a distribution of values for the security. For a given price, the OAS is the spread such that the average of the distribution of values equals the price. Since this process nets out the impact of prepayments of MBS, OAS allows direct comparisons among MBS and other callable and non-callable fixed income securities which have similar characteristics, but trade at significantly different yields because of imbedded options. If two had comparable credit risk and liquidity, an investor might purchase whichever one had the higher OAS since it would offer higher compensation for the risks being taken.

OAS is considered a more advanced method then making relative value comparisons based on static measures since it captures the effect of yield-curve shape on the valuation of prepayments. However, there are drawbacks of relying solely on OAS as a relative tool. The estimate of option cost derived by the OAS model is highly dependent on the robustness and predictive power of the underlying prepayment model. It is important that the derived prepayment model is acceptable before making value judgments based on OAS.

Equation (12) represents the discounting factor modified for OAS. By using this discounting factor for *df(t)* and the current market price as "*P",* equation (9) can be solved for OAS by iteration.

$$df_{OAS}(t) = \prod_{k=1}^{t} \frac{1}{(1 + r_k + OAS)} \qquad (12)$$

Effective Duration

In general, duration measures the price sensitivity of a bond to a small change in interest rates. For securities such as an MBS, duration measures such as "*Modified duration*", and another well-known formula, the "*Macaulay duration*" have little meaning when it comes to interest rate risk management. The formulas are limited since they assume cash flows do not change if interest rates change. In the case of MBS, where cash flows change according to interest rates and prepayments, these formulas are inappropriate. Effective duration takes into account the cash flow changes when interest rates change. It is computed using an OAS model. While holding OAS constant, the term structure is parallel shifted in both directions by Δy, creating two prices, P_- and P_+. Effective duration is then calculated by using equation (13).

$$Effective\,Duration = \frac{100}{P_0} x \frac{P_- - P_+}{2\,x\,\Delta y} \qquad ,^6 \qquad (13)$$

where

P_- = price if yield is decreased by a small amount Δy

P_+ = price if yield is increased by a small amount Δy

P_0 = intial price or full price

Δy = yield curve change (stated as an annualized percent)

Duration can be interpreted as the approximate percentage change in price for a 100-basis point (bp) parallel shift in the yield curve. If a bond's duration is 5, a 100-bp increase in interest rates will result in a price decrease of approximately 5%.

Effective Convexity

Effective convexity describes the rate at which duration changes in response to changes in interest rates. A pass through security can exhibit either a positive or negative convexity, depending on the current mortgage refinance rate and the mortgage coupon rate. In most cases, MBSs have negative convexity, which means that, other things being equal, effective duration will over-project price increases when rates move down and under-project price declines when rates move up. Positive convexity describes securities that become less price sensitive when interest rates rise (duration shortens) and become much more price sensitive when interest rates decline (duration lengthens).
Equation (14) represents the formula used to approximate convexity.

$$Convexity = \frac{100}{P_0} x \frac{P_+ + P_- - 2P_0}{(\Delta y)^2} \qquad ,^7 \qquad (14)$$

---

[6] Equation (13) is a numerical approximation to the exact formula (-100/P) x (dP/dy).
[7] Equation (14) is a numerical approximation to the exact formula (100/P) x (d^2P/dy^2).

Simulated Average Life

The *average life* of an MBS is the weighted average time that $1 of principal remains outstanding. The formula for the average life (in years) is

$$Average\,Life = \frac{\sum_{i=1}^{WAM} T_i \, x \, Principal_i}{\sum_{i=1}^{WAM} Principal_i}, \qquad (15)$$

where

$Principal_i$ = principal payment (scheduled payment and projected payment)
T_i = time until the principal payment
WAM = weighted average maturity

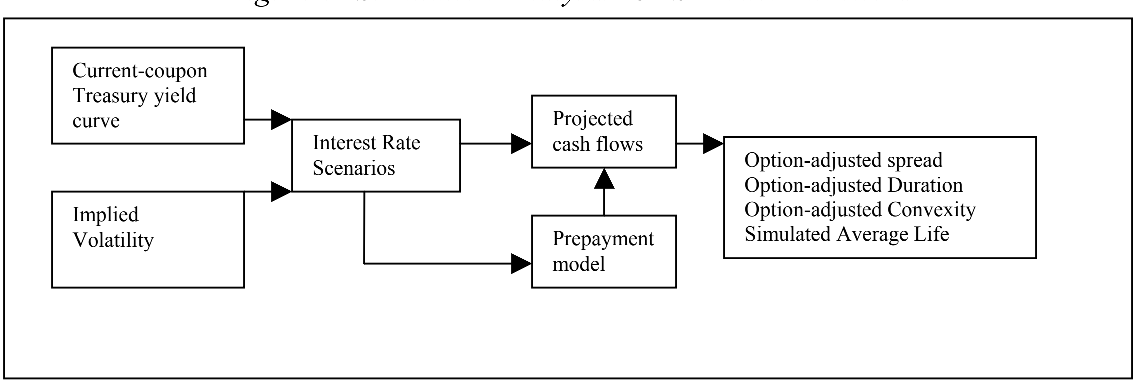
Under the OAS model, the *Simulated Average Life* is the average of the average lives along the interest rate paths. The standard deviation is also calculated for *Simulated Average Life* to gather further information about the certainty of a tranche's average life.

OAS Simulation

The objective is to use OAS analysis in our simulation to estimate the sensitivity of price of MBS against changes in our parameters of interest. Also, when testing CMO structures, the objective is to figure out how the OAS of the collateral gets transmitted to the different CMO tranches. This will help determine where the value goes and where the risk goes so that the tranches with low risk and high value can be identified. Simulated average life will allow us to observe the stability of a tranche or tranches in the CMO structure.

To summarize, Figure 5 is a schematic of the functional steps related to the electronic derivation of OAS.

Figure 5: Simulation Analysis: OAS Model Functions [8]



[8] Source: (Bartlett, 1994), pg106, slightly modified.

MBS Price Simulations

Three separate scenarios were analyzed using the OAS methodology: a plain vanilla structure, a sequential bond structure with a Z-bond, and a PAC-support structure. The cash flows of the mortgage pool were generated using FINCAD's MBS cash flow function. For this simulation, a 30-year time horizon was chosen using a weighted average coupon (WAC) of 8.75% and a coupon of 8.50%. The difference among the coupons accounts for the servicing spread. The compounding frequency was set to monthly and the original face value of the pool was set to $1,000,000. For our interest rate model, the Courtadon (1982) model was chosen since interest rates are always positive and do not exhibit reflected rates around 0% due to the restriction of setting absolute values on the other models. The long-run average, θ, was set to 8.00%, κ was set to 0.29368 and the volatility term, σ, was set to 11%. The starting interest rate was set to 7.15%. The Appendix provides the actual formulas of the prepayment model used in this simulation.

Figure 6 plots one path of the interest rate process used in our simulation. As can be seen, the path always reverts back to the long-run average of 8.00%.

Figure 6: Interest Rates simulated using the Courtadon interest rate process

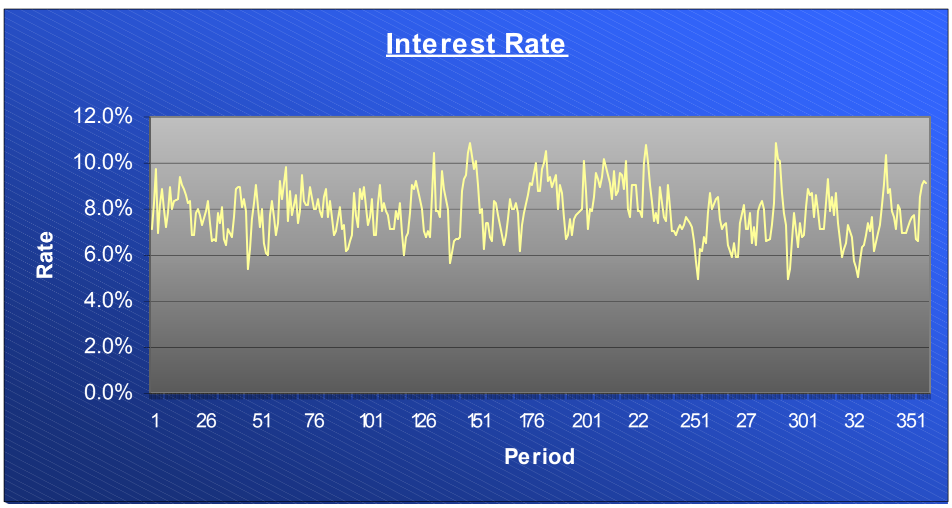


Figure 7 plots the CPR values for the interest rate path described in Figure 6. The CPR values do exhibit somewhat of a pattern. As interest rates fall in Figure 6, CPR values rise. This can be viewed, among other periods, during period 45 and 251 where interest rates are low and the corresponding CPR values are high. This is mainly attributed to the Refinancing Incentive.

Figure 7: CPR values simulated using the prepayment model

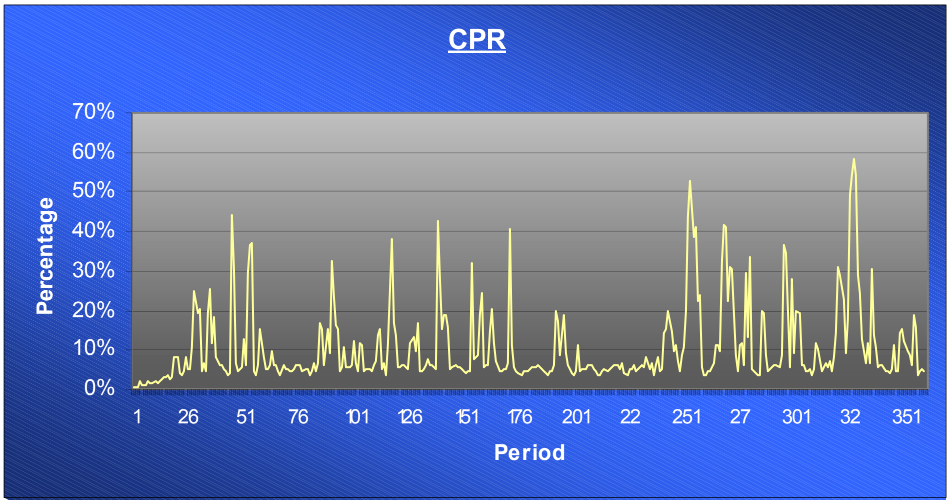
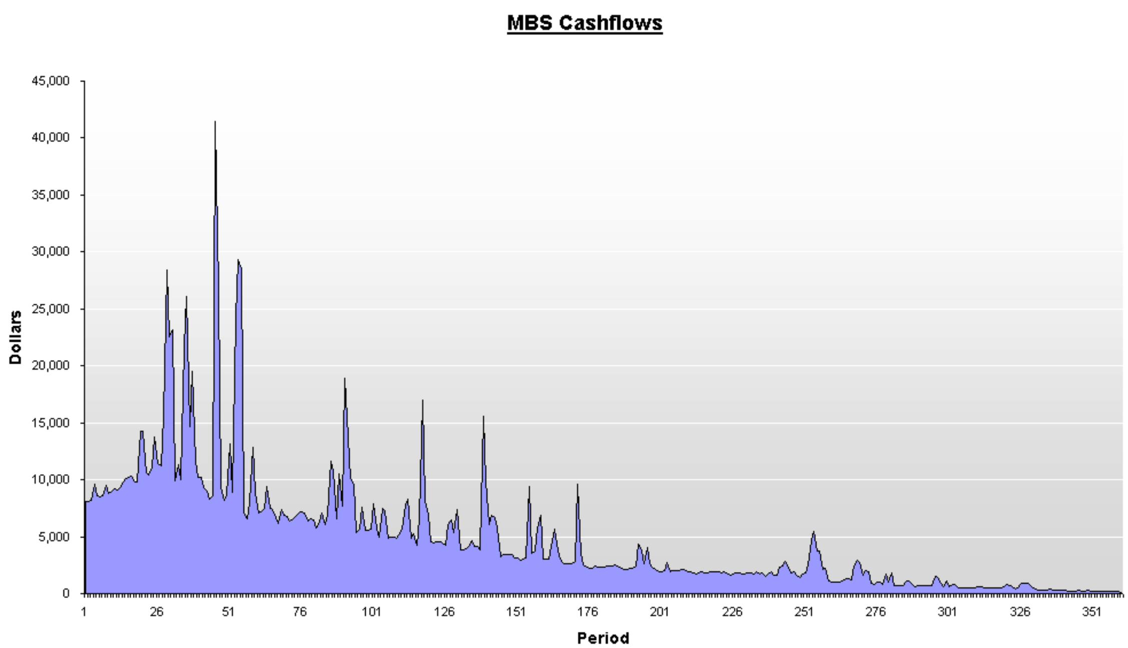


Figure 8 describes the MBS cash flows for this one path of simulated CPR values.

Figure 8: MBS Cash flows for a 30-year, 8.5% coupon, $1,000,000 pool



Plain Vanilla Structure

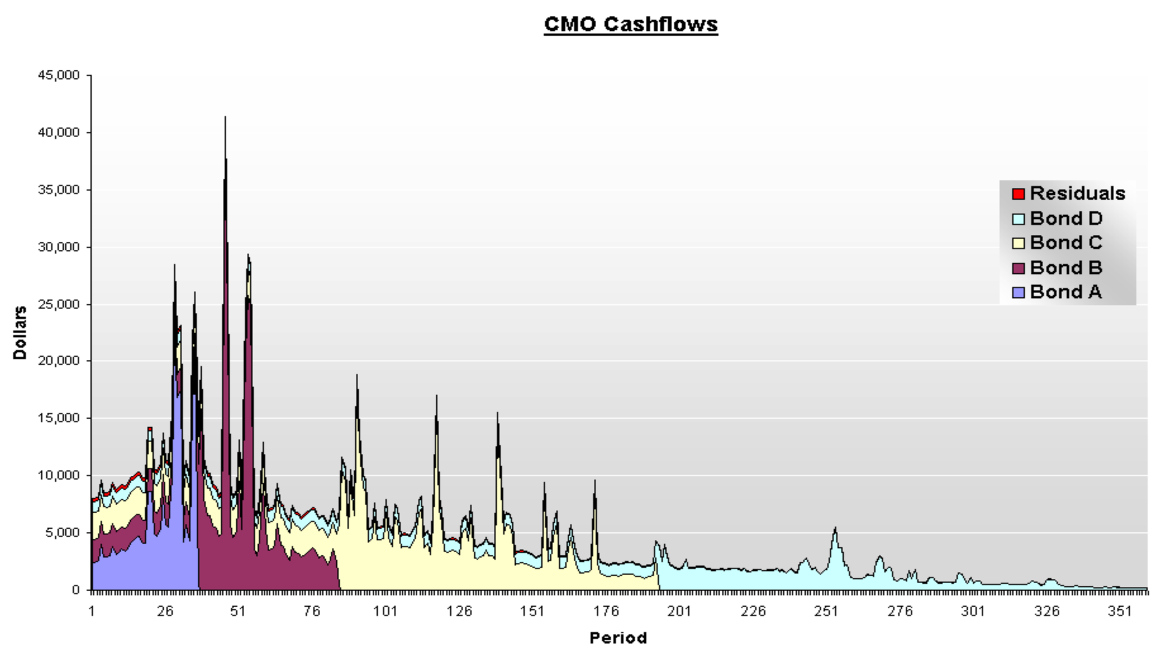
The plain vanilla sequential pay CMO bond structure used in our simulation includes four tranches, A, B, C, and D. Table 2 describes the principal values of each of the bonds and their respective coupon rates.

Table 2: CMO Bond Structure

	Principal	Coupon
Collateral	$1,000,000	8.5%
Bond A	$200,000	8.5%
Bond B	$300,000	8.0%
Bond C	$350,000	8.2%
Bond D	$150,000	7.8%

There also exists a residual class, which is essentially an Interest-Only strip, collecting the residual of interest from the differences of the coupons on the bonds and the coupon on the pool. Figure 9 shows the cash flows of the CMO plain vanilla sequential pay structure from the path taken in Figure 6.

Figure 9: CMO Sequential Structure from the cash flows of Figure 6.



A Monte Carlo simulation was run to produce model prices for different OAS values. 1024 simulations were run. Since our main objective is to determine how risk is divided among the tranches, the market price of the collateral and the bonds were set to their respective present values, assuming OAS to equal 0. Exhibit 1 shows the OAS analysis for the collateral and the four classes of the CMO structure. The base case scenario was run generating interest rate paths for a scenario that set prepayments to 100% of the prepayment model and the interest rate volatility to 11%. Once the interest rate paths were generated, the second scenario set the prepayment rates at 80% and 120% of the original prepayment model. The third scenario set prepayments to 100% of the prepayment model and set interest rate volatility at 6% and 17%. Simulated Average Life, Effective Duration, and Effective Convexity were calculated during simulation.

Exhibit 1: OAS Analysis of a Plain Vanilla Structure

Base Case (assumes 11% interest rate volatility)										
	Market Price		OAS (in bps)		Simulated Average Life		Effective Duration		Effective Convexity	
Collateral	$1,034,110.94		0		7.91		4.20		-3.09	
Class										
A	$201,801.06		0		2.01		1.54		-0.80	
B	$301,813.64		0		3.98		3.54		-2.87	
C	$358,685.77		0		8.28		8.18		-2.86	
D	$147,997.52		0		14.83		9.43		-1.03	
Prepayments at 80% and 120% of Prepayment Model (assumes 11% interest rate volatility)										
	New OAS (in bps)		Change in Price per $1000 Par (holding OAS constant)		Simulated Average Life		Effective Duration		Effective Convexity	
	80%	120%	80%	120%	80%	120%	80%	120%	80%	120%
Collateral	0.42	-0.44	$3.16	-$2.76	8.96	7.02	4.80	3.70	-2.87	-3.19
Class										
A	0.60	-0.51	$1.51	-$1.13	2.16	1.89	1.67	1.43	-0.92	-0.72
B	0.13	-0.13	$0.67	-$0.52	4.70	3.52	4.03	3.19	-3.03	-2.67
C	0.17	-0.31	$1.60	-$2.41	9.65	7.25	6.61	5.10	-4.16	-4.64
D	-0.03	0.20	-$0.33	$2.32	15.98	13.56	9.91	8.98	-0.16	-2.23
Interest Rate Volatility of 6% and 17%										
	New OAS (in bps)		Change in Price per $1000 Par (holding OAS constant)		Simulated Average Life		Effective Duration		Effective Convexity	
	6%	17%	6%	17%	6%	17%	6%	17%	6%	17%
Collateral	0.29	-0.32	$2.35	-$1.80	9.80	5.97	4.79	3.64	-3.18	-2.04
Class										
A	1.04	-1.35	$2.74	-$2.84	2.27	1.79	1.72	1.36	-1.03	-0.35
B	-0.29	0.03	-$1.77	$0.12	5.46	3.07	4.06	2.82	-2.91	-1.63
C	-0.28	0.33	-$2.88	$2.18	10.83	6.05	6.43	4.94	-4.08	-3.34
D	-0.28	1.36	-$3.57	$14.41	16.71	11.95	9.83	9.29	-0.57	-2.60

The first panel of Exhibit 1 shows the base case scenario. The other two panels help to understand the interaction of the tranches in the structure and which tranches are bearing the risk. The second panel of Exhibit 1 examines the sensitivity of the tranches to changes in prepayments while the third panel examines the sensitivity of the tranches to changes in interest rate volatility.

Slowing down prepayments to 80% of the initial model shows an increase in OAS and the price of the collateral (if OAS is held at 0), as can be seen in panel two of Exhibit 1. This is due to the fact that the collateral is trading above par. Since the prepayment speed has slowed down to 80% of the original speed, the weighted average life and duration of the collateral and the tranches have increased. Since the collateral is trading at a premium, the collateral coupon rate is greater than and more desirable then current mortgage refinancing rates for MBS investors. As a result, an extension in average life will push the price even higher, holding OAS at 0. In addition, holding the price constant at the same market price as in the 100% prepayment scenario, OAS must increase. The tranches created by the collateral also behave similarly. Classes A, B, and C all have an increased OAS and price since they also were traded above par. Class D has a decreased OAS and price since it was sold below par.

The change in OAS for both classes B and C are quite similar in the prepayment scenario. However, the change in price per $1000 par value for these two classes is quite different. This is due to shorter tranches having less duration. The Effective Duration for class B was 4.03 while for class C was 6.61. Therefore, prices for shorter tranches do not move as much from a change in OAS as do longer tranches. Overall, this suggests, the slowdown in prepayments would benefit the longer tranches that were sold above par.

Increasing prepayments to 120% of the initial model shows results behaving in the opposite direction to the 80% prepayment model. Here the collateral and classes A, B, and C have a reduction in price, while class D has an increased price due to it selling below par.

The third panel of Exhibit 1 examines the sensitivity of the tranches to changes in the interest rate volatility. Interest rate volatility was changed to 6% and 17% from the base case of 11%. Here, the reduction of volatility to 6% resulted in the increase in OAS and price of the collateral. When volatility decreases, the imbedded short options in the MBSs depreciate in value. This is true since call options on a low-volatility stock have a lesser chance of being in the money at expiration than a highly volatile stock. This depreciation in value of the imbedded option causes the price of the MBS to outperform that of a comparable Treasury resulting in an increased price of the collateral. The imbedded options are short since it is the mortgage holder holding the imbedded option and not the investor.

Examining the tranches, we see that the increase in the price of the collateral is not distributed equally among the tranches. Class A seems to be the only class benefiting from a reduction in volatility. This is due to how the tranches were initially created. Bond A was the only class in this CMO structure that had a coupon greater than or equal to the coupon of the collateral, and a coupon much greater than the long-run mean of the interest rate path of 8%. The other classes had coupons very close to the long-run mean of the interest rate path. When MBSs or bonds are priced at a slight-premium, as were Bonds B, C and D, they will experience the greatest correlation to implied volatility. As volatility decreases, so will the prices of these Bonds, and vice versa. When options are trading near its strike price, its premium is maximized by the probability that it can move into or out of the market. Higher premium MBSs are the least volatile since their options are already well in the money. As a result, Bond A experienced an increase in price when volatility fell to 6%, while Bonds B, C and D experienced a decrease in price. Similar results follow for an increase in volatility to 17%. The collateral and class A decreased in OAS and price, while classes B, C, and D all had an increase in price. Also note, when examining classes B, C and D, the OAS gain/loss for each of the tranches follows more or less the durations of those tranches. The longer the tranches, the greater the interest rate risk, and when volatility increased, the reward was greater for the accepted risk.

The final column in Exhibit 1 gives the values for Effective Convexity. In all of our cases, Effective Convexity was negative. For MBSs, convexity is often negative, because rising prepayment rates dampen the price increase in declining rate scenarios. This means, even though MBS may have significantly higher returns than a comparable Treasury under unchanged rates, it may under perform the Treasury if rates drop.

Even though OAS incorporates the impact of prepayment volatility and, hence, of negative convexity, it is still important to examine convexity. Extreme negative convexity of the collateral or the bonds amplifies its downside risk and limits its upside potential. So a bond with an extreme negative convexity and a lower duration can still be considered more risky than a comparable bond with a longer duration and not as extreme negative convexity.

Higher negative convexity does imply a greater degree of uncertainty about the OAS. As a result, it is important to consider other analytical tools along with OAS in order to determine an investor's risk/reward profile for the MBS. In our analysis, the results were stress-tested by changing some of the assumptions (e.g. volatility, prepayment speed) in the model.

Sequential Bond Structure with a Z-Bond

In order to see how Effective Duration, Simulated Average Life, OAS and price are affected with the introduction of a Z-Bond, this simulation uses the exact parameters of the plain vanilla structure. The only adjustment was changing class D to a Z-Bond. Figure 10, shows the cash flows of this CMO structure from the same path taken as the plain vanilla structure. It is already noticeable how the Z-Bond has affected the cash flows. Comparing Figure 9 with Figure 10, the reduction of average life for classes A, B, and C can be easily viewed, as well as the increased average life for class D.

Figure 10: CMO Sequential Structure with a Z-Bond [9]

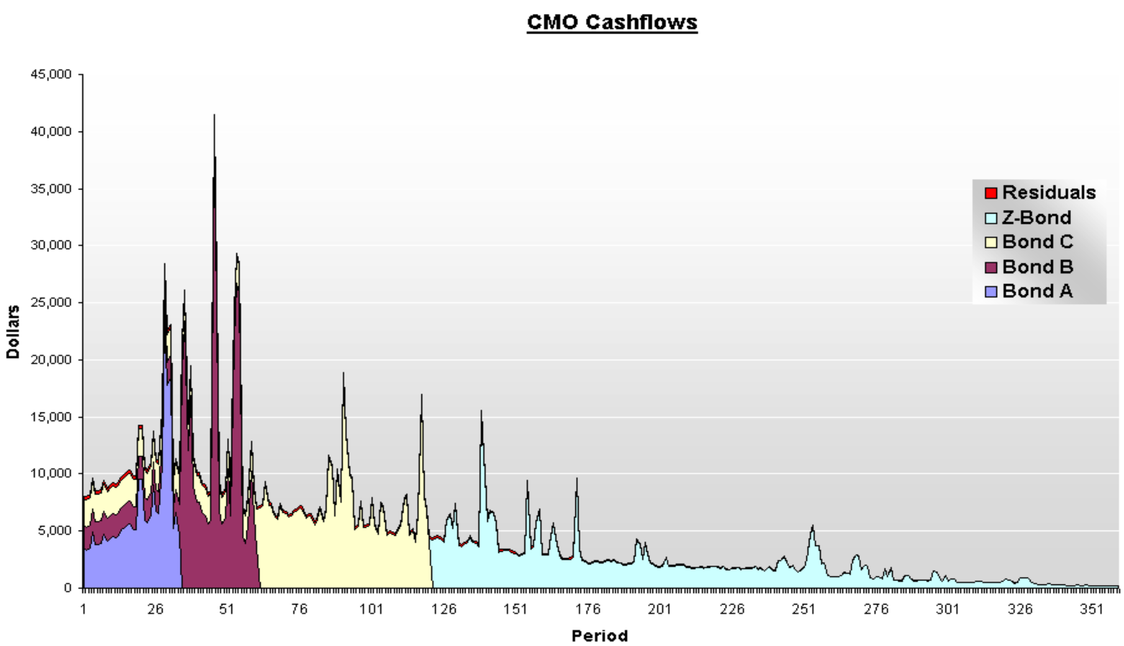


Exhibit 2 follows a similar structure as Exhibit 1. In all three scenarios: the Base Case, and the adjustments of prepayments and interest rate volatility, the simulated average life and effective duration decreased for Bonds A, B, and C and increased for the Z-Bond compared to the plain vanilla structure. The ▲ and ▼ arrows display the up shift and down shift in Simulated Average Life and Effective Duration in the base case compared to the Simulated Average Life and Effective Duration of the plain vanilla structure.

In all three scenarios, Effective Convexity reduced for Bonds A, B and C, and increased for Bond D. This suggests that the percentage increase in price over the percentage decrease in price will be greater for bonds A, B, and C, with the inclusion of a Z-Bond, but makes the Z-Bond even more sensitive to decreases in price. This can be seen in both the prepayment and interest rate volatility scenarios. The change in price per $1000

[9] This CMO structure assumes the scenario as in Figure 9 where Bond D is now a Z-Bond.

decreased from -$0.33 to -$4.67 in the 80% prepayment scenario but only increased from $2.32 to $3.72 in the 120% prepayment scenario. Also, in the interest rate volatility scenario, the price decreased from -$3.57 to -$14.43 in the 6% volatility scenario and increased from $14.41 to $21.11 in the 17% interest rate scenario.

Exhibit 2: OAS Analysis of a Sequential Bond Structure with a Z-Bond

Base Case (assumes 11% interest rate volatility)					
	Market Price	OAS (in bps)	Simulated Average Life	Effective Duration	Effective Convexity
Collateral	$1,034,110.94	0	7.91	4.20	-3.09
Class					
A	$201,417.21	0	1.77 ▼ (2.01)*	1.40 ▼ (1.54)*	-0.56
B	$301,622.24	0	3.49 ▼ (3.98)*	3.09 ▼ (3.54)*	-2.19
C	$357,013.48	0	6.30 ▼ (8.28)*	4.72 ▼ (8.18)*	-2.95
Z	$147,809.47	0	16.15 ▲ (14.83)*	14.05 ▲ (9.43)*	-6.53

Prepayments at 80% and 120% of Prepayment Model (assumes 11% interest rate volatility)										
	New OAS (in bps)		Change in Price per $1000 Par (holding OAS constant)		Simulated Average Life		Effective Duration		Effective Convexity	
	80%	120%	80%	120%	80%	120%	80%	120%	80%	120%
Collateral	0.42	-0.44	$3.16	-$2.76	8.96	7.02	4.80	3.70	-2.87	-3.19
Class										
A	0.49	-0.36	$1.07	-$0.71	1.88	1.69	1.51	1.32	-0.63	-0.53
B	0.16	-0.31	$0.70	-$1.16	3.89	3.24	3.43	2.80	-2.26	-1.96
C	0.29	-2.6	$2.22	-$1.63	7.04	5.64	5.29	4.26	-2.76	-3.10
Z	-0.25	0.24	-$4.67Ψ	$3.72Ψ	17.70	14.72	15.39	12.92	-5.17	-7.48

Interest Rate Volatility of 6% and 17%										
	New OAS (in bps)		Change in Price per $1000 Par (holding OAS constant)		Simulated Average Life		Effective Duration		Effective Convexity	
	6%	17%	6%	17%	6%	17%	6%	17%	6%	17%
Collateral	0.29	-0.32	$2.35	-$1.80	9.80	5.97	4.79	3.64	-3.18	-2.04
Class										
A	1.11	-1.15	$2.48	-$2.18	1.92	1.63	1.52	1.29	-0.71	-0.32
B	-0.34	-0.19	-$1.71	-$0.65	4.41	2.84	3.48	2.48	-2.08	-1.11
C	-0.12	0.37	-$0.95	$2.07	7.63	4.89	5.28	4.20	-2.96	-2.17
Z	-0.71	1.53	-$14.43Ψ	$21.11Ψ	18.82	12.92	14.84	12.77	-4.51	-6.93

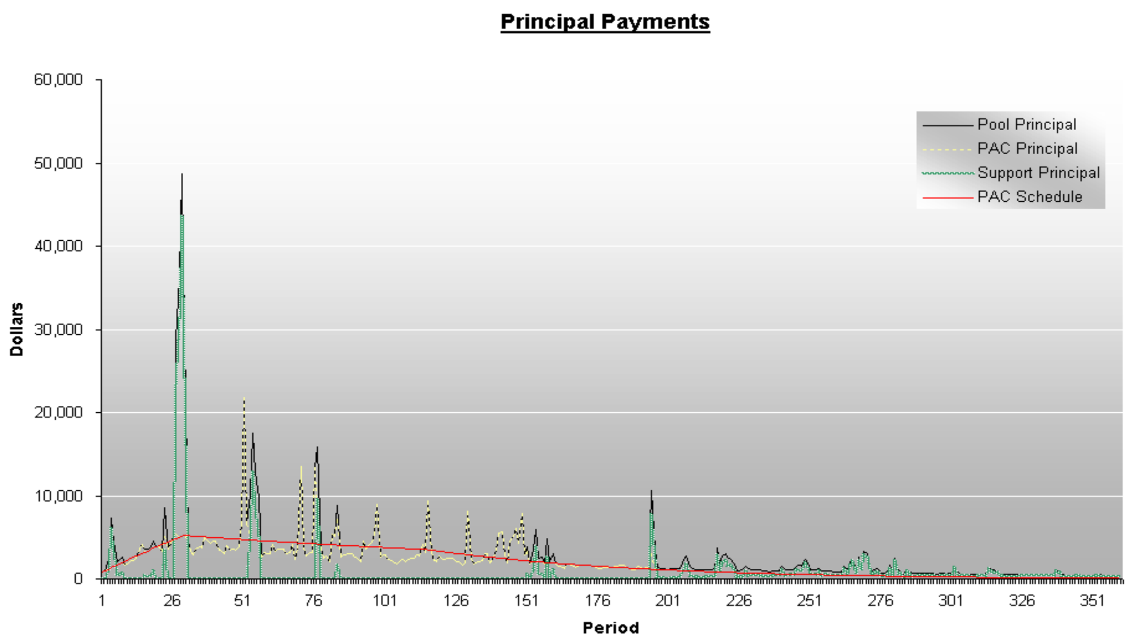
* The arrows represent the up or down shift of average life or duration compared to the plain vanilla structure (Exhibit 1). The values in brackets represent the values of the plain vanilla structure in Exhibit 1.
Ψ The introduction of a Z-class has caused Bond D to become more sensitive to price changes compared to the plain vanilla structure.

Overall, the inclusion of a Z-Bond in a CMO does help stabilize the price changes for the sequential bonds and does help reduce average life and duration of the classes. However, in exchange for these benefits, the Z-class takes on more risk of price changes and does increase its average life and duration in doing so.

PAC-Support Bond Structure

PAC-Support Bond structures were designed to help reduce the effects of prepayment risk. In this simulation, we will analyze the effects of changes to the prepayment speeds and interest rate volatility to this structure. Figure 11 shows the principal payment cash flows for a PAC-Support structure using the cash flows derived from the interest rate path taken in Figure 6. A 95% lower band and a 240% upper band were chosen to produce the PAC schedule. The PAC structure retains $700,291.92 of the collateral principal while the Support retains $299,708.08. Notice in times of high prepayment (around period 26) the Support class collects the residual principal after the PAC meets its PAC schedule. In periods of slow prepayment (periods 30 – 48) the PAC doesn't meet its PAC schedule, but makes up the amount during times of high prepayment (periods 49 – 57).

Figure 11: Principal Payment cash flows for a PAC-Support Structure derived from the interest rate path of Figure 6.[10]



[10] The PAC structure retains $700,291.92 of the original MBS principal cash flow, while the support contains $299.708.08 of the principal. The PAC bands for this model are 95% PSA and 240% PSA.

Exhibit 3 provides a summary of the OAS analysis of the PAC-Support Bond structure. The stability of the PAC structure is very clear in the 80% and 120% prepayment scenario. The changes in the OAS values were minimal resulting in a minimal change in price per $1000. Also, the Simulated Average Life for both the 80% and 120% scenarios were almost identical to the base case scenario. The PAC structure provided very good protection against average life extension if prepayments fell below the speed necessary to maintain the PAC schedule. The Support Class did bear much of the prepayment risk. The price change for the Support class fluctuated from $9.92 for the 80% prepayment scenario to -$9.18 for the 120% prepayment scenario. Also the Simulated Average Life did fluctuate drastically according to prepayment speed, ranging from 11.61 years for 80% prepayment to 5.47 years in the 120% prepayment scenario.

Under the Interest Rate Volatility scenario, the Support class continued to provide stability to the PAC structure. The PAC structure's Simulated Average Life was very well protected under the 6% and 17% volatility scenario. The total shift in average life was about half a year under the 6% scenario and just under a year for the 17% scenario. However, there was a noticeable change in price for the PAC class under this scenario. Although the PAC was not completely stable, the Support class did continue to provide some stability. Again, the Support class did bear much of the prepayment risk, as can be seen in the fluctuation of OAS and price, as well as in the change of Simulated Average Life under both volatility scenarios.

Exhibit 3: OAS Analysis of a PAC-Support Bond Structure

Base Case (assumes 11% interest rate volatility)										
	Market Price		OAS (in bps)		Simulated Average Life		Effective Duration		Effective Convexity	
Collateral Class	$1,034,110.94		0		7.91		4.20		-3.09	
PAC	$724,592.95		0		7.52		4.58		-1.45	
Support	$309,517.98		0		8.81		3.31		-6.92	
Prepayments at 80% and 120% of Prepayment Model (assumes 11% interest rate volatility)										
	New OAS (in bps)		Change in Price per $1000 Par (holding OAS constant)		Simulated Average Life		Effective Duration		Effective Convexity	
	80%	120%	80%	120%	80%	120%	80%	120%	80%	120%
Collateral Class	0.42	-0.44	$3.16	-$2.76	8.96	7.02	4.80	3.70	-2.87	-3.19
PAC	0.04	-0.00	$0.26	-$0.01	7.54	7.58	5.16	4.12	-0.50	-2.33
Support	1.12	-1.93	$9.92	-$9.18	11.61	5.47	4.22	2.26	-8.20	-4.60
Interest Rate Volatility of 6% and 17%										
	New OAS (in bps)		Change in Price per $1000 Par (holding OAS constant)		Simulated Average Life		Effective Duration		Effective Convexity	
	6%	17%	6%	17%	6%	17%	6%	17%	6%	17%
Collateral Class	0.29	-0.32	$2.35	-$1.80	9.80	5.97	4.79	3.64	-3.18	-2.04
PAC	-0.47	0.68	-$3.19	$4.46	7.49	7.02	5.12	4.27	-0.44	-1.82
Support	1.42	-5.15	$15.29	-$16.43	13.50	2.69	4.45	1.01	-8.85	-0.35

Conclusion

This paper provided an introduction into the world of mortgage-backed securities and collateralized mortgage obligations. An empirical analysis of MBS and CMO pricing was done using Option-Adjusted spread valuation. Interest rate movements were derived using Courtadon's interest rate model, which was then used to derive the prepayment model developed by Richard and Roll. The prepayment model was used to generate a set of cash flows, which in turn were reallocated into the different CMO structures.

OAS analysis helped determine how risk was reallocated under each structure and how the different structures performed under different scenarios. The analysis only focused on simple structures to demonstrate the effectiveness of each type of tranche or class. Typically CMOs have four or more classes (or tranches), with the last class often being a Z-bond. However, some CMOs have been issued with only one class (plus a residual) and others with more than 20 classes.

In our analysis, the inclusion of a Z-bond decreased the WAL for all classes, while increased for the Z-bond itself. Also, as predicted, the price of the Z-bond became highly sensitive to interest rate movements since its ultimate principal balance depends on total accretions credited by the time it begins to pay down.

The PAC structure did provide more stable cash flows and provided a very stable structure in changes to prepayment speeds. It also performed well under the change in volatility of interest rates. In turn, the support class had to bear much of the prepayment risk and its price greatly fluctuated according to these changes in prepayment speeds and volatility of interest rates.

The approach taken in evaluating the different classes in our simulations can be easily implemented to any type of CMO structure. Using OAS analysis helps money managers to understand where the risks are in a CMO deal and helps identify which tranches are cheap, rich and fairly priced.

Appendix

In this Appendix we summarize the mathematical formulas used to conduct the simulation experiments for our mortgage-backed security model.

In our simulation, the index *t* denotes the month for all variables. Let *B(t)* be the mortgage balance in the beginning of month *t.* The initial mortgage balance, *B(0)*, is equal to $1,000,000. WAC is the weighted average coupon rate for the MBS and WAM is the weighted average maturity for the MBS. r_t is the interest rate at time *t.*

SMM(*t*) is the single monthly mortality for month *t*, observed at the end of month *t.* It measures the percentage of dollars prepaid in any month, expressed as a percentage of the expected mortgage balance. The formula for *SMM*(*t*) is as follows:

$$SMM = 100x \frac{(Scheduled\ Balance - Actual\ Balance)}{Scheduled\ Balance}$$

CPR(*t*) is the conditional prepayment path for month *t,* observed at the end of month *t.* *CPR* reflects the percentage prepayment rate resulting from converting the *SMM* to an annual rate. It measures the percentage of dollars prepaid on an annual basis. The formula for *CPR*(*t*) is as follows:

$$CPR = 100x(1-(1-\frac{SMM}{100})^{12})$$

PSA(*t*) is an industry convention adopted by the Public Securities Association in which prepayment rates, expressed in *CPR*, are assumed to follow a standard path over time. The prepayment rate starts at 0.2% *CPR* in the first month and then rises 0.2% *CPR* per month until month 30 when the prepayment rate levels out at 6% *CPR*.
The formula for *PSA*(*t*) is as follows:

$$PSA = 100x \frac{CPR}{\min(age, 30)\, x\, 0.2},$$

where *age* is expressed in months.

Prepayment Model

Assumptions:

Maximum CPR is 50%
Minimum CPR is 0%
The midpoint 25% CPR occurs at diff = 200 basis points
At midpoint, max slope is 6% CPR for a 10 basis point rate shift

i. **Refinancing incentive:** $RI(t) = a + b \text{ x } (\arctan(c + d \text{ x } (\text{WAC} - r_t))),$

where: [11]

a = (max CPR + min CPR) / 2;
b = 100 x (max CPR – a) / (π /2);
d = max slope / b;
c = -d x midpoint diff

ii. **Seasoning multiplier:** $Age(t) = \min(\frac{t}{30}, 1).$

iii. **Monthly multiplier:** $MM(t)$, for the ith month is assumed to be given by $MM(i)$ = (0.94, 0.76, 0.74, 0.95, 0.98, 0.92, 0.98, 1.10, 1.18, 1.22, 1.23, 0.98). [12]

iv. **Burnout multiplier:** $BM(t) = 0.3 + 0.7\frac{B(t)}{B(0)}.$

The annualized prepayment rate, CPR(t) , is equal to:

CPR(*t*) = *RI(t) x Age(t) x MM(t) x BM(t).*

[11] Formulas from Davidson/Herskovitz (1996).
[12] Monthly parameters were taken from figure 3 in Richard and Roll (1989).

Cash flow Functions for the MBS

- *MP(t)* is the scheduled mortgage payment for period *t;*

$$MP(t) = B(t)\left(\frac{WAC/12}{1-(1+WAC/12)^{-WAM+t}}\right)$$

- *IP(t)* is the Interest payment for period *t;*

$$IP(t) = B(t)\frac{WAC}{12}$$

- *PP(t)* is the principal prepayment for period *t;*

$$PP(t) = SMM(t)\ (\ B(t)\text{-}SP(t)\ ),$$

where

$$SMM(t) = 1 - \sqrt[12]{1-CPR(t)}\ ,\ and\ SP(t) = MP(t) - IP(t).$$

- The reduction in the mortgage balance for each month is given by:

$$B(t+1) = B(t) - TPP(t).$$

Interest disbursements for the CMOs

Let $B_i(t)$ denote the balance for tranche *i,* where *i* = {1,2,3,4}, at the beginning of month *t.* The initial balances for the simulation are $B_1(t)$ = \$200,000, $B_2(t)$ = \$300,000, $B_3(t)$ = \$350,000, and $B_4(t)$ = \$150,000. Similar to the cash flow functions of the MBS, let $IP_i(t)$ denote the interest payments for tranche *i,* in month *t.* $IP_i(t) = B_i(t)\frac{cpn_i}{12}$. The coupon for each tranche is as follows: cpn_1 = 8.5%, cpn_2 = 8.0%, cpn_3 = 8.2%, and cpn_4 = 7.8%.

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