MANDATORY HOMEWORK 8 - STAT 211 - V19

DUE AT THE END OF APRIL 02

March 27, 2019

PROBLEM 8.1

Sunspots bd, Brockwell et al., 2016, p. 117, exercise 4.7 . Let $\{X_t\}$ denote the sunspots data.

a) Load and plot the sunspots data. Let $Y_t = X_t - \overline{X}_n$.

b) Plot the ACF and PACF of $\{Y_t\}$. Notice that the ACF tails off while the PACF cuts off after two lags.

c) Fit an AR(2) model to $\{Y_t, t = 1, \ldots, n\}$ and report $\widehat{\phi}_1$, $\widehat{\phi}_2$ and $\widehat{\sigma}^2$.

d) Plot the periodogram.

e) Plot a smoothed periodogram.

f) Plot the spectral density of the fitted model, find the frequency which achieves its maximum value and mark on the plot. What is the corresponding period?

g) Calulate the residuals and plot their ACF and PACF.

h) Plot the periodogram and the spectral density for the residuals.

i) Do an independence test of the residuals, i.e. a Ljung-Box test.

j) Plot the marginal density for the residuals and compare it to a normal density.

k) Simulate the estimated model. You may use the a normal approximation of the residual distribution. Alternatively it is possible to draw from the empirical residual distribution.

l) Plot yearly, monthly and daily sunspot data for the last 13 years.

PROBLEM 8.2

The spectral density for an AR(2) model is given by

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{1}{\left|\phi(\exp(-i\omega))\right|^2}, \quad \omega \in (-\pi, \pi], \qquad \phi(z) = 1 - \phi_1 z - \phi_2 z^2$$

In an example we use `phi<-c(1.4, -0.90)` and $\sigma^2 = 1$.

a) Plot the spectral density for this AR(2) model.

b) What happen with spectral density if ϕ_2 is changed to 0.95?

c) Simulate this AR(2) model `N<-100`. Plot the time series. Plot the periodogram, a smoothed periodogram and a estimated spectral density from the model.

d) Can you see a periodic structure of the plotted time series. Calculate the apparent period length from the true and from the estimated spectral density,

e) Repeat the two previous point with `N<-1000`.

PROBLEM 8.3
Explain that the spectral density for an invertible and causal ARMA(p,q) process is continuous on $[-\pi, \pi]$ and with a minimum value strictly greather than zero.

PROBLEM 8.4
Let $\{X_t,\, t = 1, \ldots, n\}$ be data from a time series. Suppose that $\{X_t\}$ is an invertible MA(q) procecss with respect to $\{Z_t\} \sim \mathrm{IID}(0, \sigma^2)$. Argue that estimated $\widehat{\gamma}_{\mathrm{ML}}$ does not fit $\widehat{\gamma}$, i.e. $\widehat{\gamma}_{\mathrm{ML}}(h) \not\equiv \widehat{\gamma}_n(h)$ for $|h| \leq q$ in general.

PROBLEM 8.5
Let $\{X_t,\, t = 1, \ldots, n\}$ be data from a time series. Let $1 \leq p < n$ and suppose that we fit an AR(p) to the data using the Yule Walker estimates of ϕ and σ^2. Explain that $\widehat{\gamma}_{\mathrm{YW}}(h) \equiv \widehat{\gamma}_n(h)$ for $|h| \leq p$.

PROBLEM 8.6
Let $\{X_t\}$ be an MA(2) process;

$$X_t = \theta(B)Z_t = (1 - B\xi_1^{-1})(1 - B\xi_2^{-1})Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

with $|\xi_j| < 1$, $\xi_j \in \mathbb{R}$ for $j = 1, 2$ and $\xi_1 \neq \xi_2$.

a) Find $\{\theta_j,\, j = 1, 2\}$ for the MA(2) process, i.e. $X_t = \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + Z_t$.

Let $\widetilde{\theta}(B) = (1 - z\xi_1)(1 - z\xi_2)$. Suppose that we can find $\{\widetilde{Z}_t\}$ so that

$$X_t = \widetilde{\theta}(B)\widetilde{Z}_t, \qquad \{\widetilde{Z}_t\} \sim \mathrm{WN}(0, \widetilde{\sigma}^2). \tag{1}$$

b) Find $\{\widetilde{\theta}_j,\, j = 1, 2\}$ for $X_t = \widetilde{\theta}_1 Z_{t-1} + \widetilde{\theta}_2 Z_{t-2} + \widetilde{Z}_t$.

c) Find the filter $\widetilde{\psi}$ that fits (1) and show that

$$\widetilde{Z}_t = \widetilde{\psi}(B)Z_t = \sum_{j=0}^{\infty} \widetilde{\psi}_j Z_{t-j}, \qquad \widetilde{\sigma}^2 = \Big[\prod_{j=1}^{2} \xi_j^{-2}\Big]\sigma^2 > \sigma^2.$$

d) Calculate $\{\widetilde{\theta}_j,\, j = 1, 2\}$, $\{\widetilde{\psi}_j,\, j \geq 0\}$ and $\widetilde{\sigma}^2$ when $\xi_1 = 1/2$, $\xi_2 = 1/3$ and $\sigma^2 = 1$.

REFERENCES

Peter J. Brockwell and Richard A. Davis. *Introduction to time series and forecasting.* Springer Texts in Statistics. Springer, [Cham], third edition, 2016. ISBN 978-3-319-29852-8; 978-3-319-29854-2. doi: 10.1007/978-3-319-29854-2. URL https://doi.org/10.1007/978-3-319-29854-2.