# STAT211 Mandatory Homework 9

Teaching Assistant: Sondre Hølleland
10:15-12:00, 10.04 and 24.04.2019
Auditorium 5, Realfagsbygget

**Deadline: 23:59 28.04.2019**

### Problem 9.1

On mitt.uib you find a dataset called `ma_q.txt`. This is a simulated MA($q$) process with Gaussian noise where $q$ unknown. In this exercise you should determine the order of the model.

**a)** Load the data and plot the time series.

**b)** Successively fit an MA($k$) model to the data, for $k = 1, \ldots, 20$ using the `arima` function. For each of the fitted models, calculate AIC and BIC, using the `AIC` and `BIC` functions on the model objects. Which model would you prefer based on these information criterions?

**c)** Plot the ACF and PACF. Are these consitent with the preferred MA(q) model from (b)?

We now leave this dataset and will compare two specific models.

**d)** Simulate an ARMA(2,2) of length $n = 50$ with $\phi_1 = -0.7$, $\phi_2 = 0.2$, $\theta_1 = 0.3$, $\theta_2 = -0.2$ with $N(0, 4^2)$ iid innovations.

**e)** Fit an ARMA(2,2) and a MA(5) model to the simulated data from (d). Combare the two models in terms of AIC and BIC. Which model is preferred?

**f)** Repeat d-e) 500 times and present the frequencies of which each model is selected in a table.

### Problem 9.2

Here we consider the real spectral basis vectors when $n = 21$,

$$\begin{aligned} q &= [(n-1)/2] = 10 \\ F_n &= \{-q, -q+1, \ldots, -1, 0, 1, \ldots, q\} = \{-10, -9, \ldots, -1, 0, 1, \ldots, 9, 10\} \\ \omega_j &= \frac{2\pi \cdot j}{n} = \frac{2\pi \cdot j}{21}, \ \ j \in F_n. \end{aligned}$$

Note that the q strictly positive Fourier frequencies are $\omega_j = (2\pi/n) \cdot j$ for $j = 1, \ldots, q$.

The real basis vectors are for $n = 21$,

$$\mathbf{e}_0 = \frac{1}{\sqrt{21}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \ \ \mathbf{c}_j = \sqrt{\frac{2}{21}} \begin{bmatrix} \cos(\omega_j 1) \\ \vdots \\ \cos(\omega_j 21) \end{bmatrix} \ \ \mathbf{s}_j = \sqrt{\frac{2}{21}} \begin{bmatrix} \sin(\omega_j 1) \\ \vdots \\ \sin(\omega_j 21) \end{bmatrix}, \ \ j = 1, \ldots, q.$$

which in total are $1 + 2q = n$ orthonormal vectors as it should be. Now, we can also view them as functions defined on integers in the interval $[1, n]$, for instance $\mathbf{c}_j(t) = \cos(\omega_j t)$. Moreover, as functions of $t$ nothing prevent us from having them defined for any $t$ in the interval $[1, n]$.

Plot $\mathbf{e}_0$ and $\mathbf{c}_j$, $\mathbf{s}_j$ for $j = 1, 5, 10$.

### Problem 9.3

What is meant by the name *white noise*?

### Problem 9.4

**a)** Simulate an Gaussian AR(2) with $n = 1\,000$, $\phi = (1.4, -0.8)$ and $\sigma^2 = 1$.

From Brockwell et al. (2016, p. 107), we have that, for $k = -\left[\frac{n-1}{2}\right], \dots, \left[\frac{n}{2}\right]$,

$$\begin{aligned}
\omega_k &= \frac{2\pi k}{n} \\
\mathbf{e}_k &= \frac{1}{\sqrt{n}}\left(e^{i\omega_k},\, e^{2i\omega_k}, \dots, e^{ni\omega_k}\right)' \\
a_k &= \frac{1}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-it\omega_k}
\end{aligned}$$

and

$$\mathbf{x} = (x_1, \dots, x_n)' = \mathbb{G}\mathbf{a} = \sum_k \mathbf{e}_k a_k, \tag{1}$$

with $\mathbb{G}$ is a matrix with column vectors $\{\mathbf{e}_k\}$ and $\mathbf{a}$ is a vector with elements $\{a_k\}$. This corresponds to (4.2.4) of Brockwell et al. (2016).

Some useful R-syntax dealing with complex numbers for this exercise.

```
1i # Imaginary i = sqrt(-1) in R:
Re(5 + 3i) # Real component of complex number: 5
Im(5 + 3i) # Imaginary component: 3
Mod(5 + 3i) # Calculate Modulus of complex number: 5.83
z <- 5 + 0i
class(z) # "complex"
class(Re(z)) # "numeric"
```

**b)** Calculate $\mathbf{a}$ and $\mathbb{G}$ for the simualted series.

**c)** Calculate the sum in (1) for every $t$ and compare this to the original signal.

**d)** Truncate the sum in (1) for every $t$ by only including the 10% largest values of $|a_k|$ and compare this to the original signal.

**e)** We also have that

$$\widehat{\gamma}(h) = n^{-1} \sum_k |a_k|^2 e^{ih\omega_k}.$$

Estimate $\widehat{\gamma}$ using this formula and compare it to the output of the `acf` function in R.

**f)** Repeat (e) using the same truncation of the sum as was used in (d).

**Problem 9.5**
Suppose that

i) $g(\lambda) = K|a(\exp(-i\lambda))|^2$ where $a(z) = 1 + a_1 z + \cdots a_q z^q$ and $K > 0$ is a constant.

ii) $a(z) = 0 \;\; \Rightarrow \;\; |z| \geq 1$.

iii) $g$ is a strictly positive function on $[-\pi, \pi]$.

**a)** Explain that the polynomial $a$ cannot have a root on the unit circle.

**b)** Let $Z_t \sim \mathrm{WN}(0, 2\pi K)$ and $X_t = a(B)Z_t$. Explain that $\{X_t\}$ is an invertible MA($q$) process. Find its spectral density.

## References

*Brockwell Peter J, Davis Richard A, Calder Matthew V.* Introduction to time series and forecasting. 3. 2016.