STAT211 Mandatory Homework 4

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1 Problem 4.1

Consider an ARMA(p,q) model

$$X_t - \sum_{k=1}^{p} \phi_k X_{t-k} = Z_t + \sum_{k=1}^{p} \theta_k Z_{t-k} \tag{1}$$

1.1 Part a: Invertibility

An ARMA(p,q) process $\{X_t\}$ is invertible if there exist constant $\{\pi_j\}$ such that

$$\sum_{j=0}^{\infty} |\pi_j| \leq \infty \tag{2}$$

and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad \text{for all t.} \tag{3}$$

In other word $\{X_t\}$ is invertible if Z_t can be written as a linear combination of X_{t-j}, $j = 0, 1, \dots, \infty$, [1].

Invertibility is equivalent to

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0 \quad \text{for all} \quad |z| \leq 1 \tag{4}$$

where $\theta(z)$ is the moving average polynomial.

The process X_t is invertible if and only if the zeros of the moving average polynomial $\theta(z)$ lie outside the unit circle.

1.2 Part b: Linear filter π_j

The sequence $\{\pi_j\}$ in (3) is determined by the relation

$$(1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q)(\pi_0 + \pi_1 z + \cdots) = (1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^q). \tag{5}$$

Multiplying the left hand side together gives

$$\begin{aligned}(1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q)(\pi_0 + \pi_1 z + \cdots) &= \pi_0 + \pi_1 z + \pi_2 z^2 + \cdots + \theta_1 \pi_0 z + \theta_1 \pi_1 z^2 + \cdots + \theta_2 \pi_0 z^2 \\ &= \pi_0 + (\pi_1 + \theta_1 \pi_0) z + (\pi_2 + \theta_1 \pi_1 + \theta_2 \pi_0) z^2 + \cdots\end{aligned}$$

and equation (5) can be rewritten as

$$\pi_0 + (\pi_1 + \theta_1 \pi_0) z + (\pi_2 + \theta_1 \pi_1 + \theta_2 \pi_0) z^2 + \cdots = (1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^q). \tag{6}$$

And equating the coefficients of $z^j, j = 0, 1, \cdots$, we obtain

$$\begin{aligned}
\pi_0 &= 1 \\
\pi_1 + \theta_1\pi_0 &= -\phi_1 \\
\pi_2 + \theta_1\pi_1 + \theta_2\pi_0 &= -\phi_2 \\
&\vdots
\end{aligned}$$

or equivalently

$$\pi_j + \sum_{k=1}^{q} \theta_k \pi_{j-1} = -\phi_j, \quad j = 0, 1, \cdots \tag{7}$$

2 Problem 4.2

Consider a causal ARMA(2,3) given by

$$X_t - \sum_{k=1}^{p} \phi_k X_{t-k} = Z_t + \sum_{k=1}^{p} \theta_k Z_{t-k} \tag{8}$$

where the linear representation satisfies

$$\psi_j = \sum_{k=1}^{p} \phi_k \psi_{j-k} + \theta_j, \quad j \geq 0, \quad \theta_0 = 1 \tag{9}$$

2.1 Part a: Finding $\{\psi_j, j = 0, 1, 2\}$

From (9) we get

$$\begin{aligned}
&\psi_0 = 1 \\
&\psi_1 = \theta_1 + \psi_0\phi_1 = \theta_1 + \phi_1 \\
&\psi_2 = \theta_2 + \psi_1\phi_1 + \psi_0\phi_2 = \theta_2 + (\theta_1 + \phi_1)\phi_1 + \phi_2
\end{aligned} \tag{10}$$

Expanding (9), for $p = 2$

$$\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2} + \theta_j \tag{11}$$

or equivalently

$$\psi_{j+2} - \phi_1\psi_{j+1} - \phi_2\psi_j = \theta_{j+2}, \quad j = 0, 1, \cdots \tag{12}$$

which is the second order difference equation with

$$\theta_j \equiv 0, \quad \text{for j} \notin [0, 3]. \tag{13}$$

The second order homogeneous difference equation is defined for $j = 2, 5, \cdots$, because then the right hand side of (12) is zeros, and we have

$$\psi_{j+2} - \phi_1\psi_{j+1} - \phi_2\psi_j = 0, \quad j = 2, 3, \cdots \tag{14}$$

2.2 Part b: Check causality and invertibility

The auto regressive polynomial $\phi(z)$ and the moving average polynomial $\theta(z)$ are given respectively by

$$\begin{aligned} \phi(z) &= 1 - \phi_1 z - \phi_2 z^2 \\ &= 1 - 1.7z + 0.9z^2 \end{aligned} \tag{15}$$

$$\begin{aligned} \theta(z) &= 1 + \theta_1 z + \theta_2 z^2 + \theta_3 z^3 \\ &= 1 - 1.4z + 0.8z^2 + 0.1z^3 \end{aligned} \tag{16}$$

The ARMA process is causal and invertible if the zeros of the auto regressive polynomial and the zeros of the moving average polynomial are located outside the unit circle respectively. A complex number $z = a + bi$ is located outside the unit circle if its magnitude is greater than 1, that is

$$|z| = |a + bi| = \sqrt{a^2 + b^2} > 1.$$

By solving

$$\phi(z) = 1 - 1.7z + 0.9z^2 = 0$$

we get

$$z_1 = 0.94 - 0.47i, \quad z_2 = 0.94 + 0.47i$$

The magnitude of z_1 and z_2 are

$$|z_i| = \sqrt{0.94^2 + 0.47^2} = 1.05 > 1, \quad i = 1, 2$$

Therefore we conclude that all the roots of the auto regressive polynomial are outside the unit circle, thus the ARMA(2,3) process is causal.

In the same fashion, by solving

$$\theta(z) = 1 - 1.4z + 0.8z^2 + 0.1z^3 = 0$$

we get

$$\begin{aligned} z_1 &= -9.57178 \\ z_2 &= 0.78589 + 0.65354i \\ z_3 &= 0.78589 - 0.65354i \end{aligned}$$

and

$$\begin{aligned} |z_1| &= \sqrt{(-9.57178)^2} = 9.57178 > 1 \\ |z_2| &= \sqrt{0.78589^2 + 0.65354^2} = 1.02 > 1 \\ |z_3| &= \sqrt{0.78589^2 + (-0.65354)^2} = 1.02 > 1 \end{aligned}$$

and since all the roots of the moving average polynomial are located outside the unit circle, the ARMA(2,3) process is invertible

2.3 Part c: Plot $\{\psi_j, j = 0, \cdots, 50\}$

Recall that

$$\psi_j = \sum_{k=1}^{p} \phi_k \psi_{j-k} + \theta_j, \quad j \geq 0, \quad \theta_0 = 1 \tag{17}$$

With

$$\boldsymbol{\phi} = (\phi_1, \phi_2) = (1.7, -0.9), \quad \boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) = (-1.4, 0.8, 0.1), \quad \sigma^2 = 1. \tag{18}$$

Expanding (17), for $p = 2$ and using $(\phi_1, \phi_2) = (1.7, -0.9)$ we get

$$\psi_j = 1.7\psi_{j-1} - 0.9\psi_{j-2} + \theta_j \tag{19}$$

or equivalently

$$\psi_{j+2} - 1.7\psi_{j+1} + 0.9\psi_j = \theta_{j+2}. \tag{20}$$

From part a) we know that

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \theta_1 + \psi_0\phi_1 = \theta_1 + \phi_1 = -1.4 + 1.7 = 0.3 \\
\psi_2 &= \theta_2 + \psi_1\phi_1 + \psi_0\phi_2 \\
&= \theta_2 + (\theta_1 + \phi_1)\phi_1 + \phi_2 \\
&= 0.8 + (-1.4 + 1.7) \times 1.7 - 0.9 \\
&= 0.41
\end{aligned} \tag{21}$$

So we have the final difference equation

$$\psi_{j+2} - 1.7\psi_{j+1} + 0.9\psi_j = \theta_{j+2}, \quad j = 1, \cdots, 50 \tag{22}$$

with initial conditions

$$\psi_0 = 1, \quad \psi_1 = 0.3, \quad \psi_2 = 0.41 \tag{23}$$

and

$$\theta_j \equiv 0, \quad \text{for j} \notin [0, 3] \tag{24}$$

```r
#initialization
psi0 = 1
psi1 = 0.3
psi2 = 0.41
theta3 = 0.1
psi3 = 1.7*psi2 - 0.9*psi1 + theta3
psi <- c(psi0,psi1,psi2,psi3)
#compute the rest
for (j in 2:48)
  psi[j+2] = 1.7*psi[j+1] - 0.9*psi[j]

#plot
plot(psi)
```

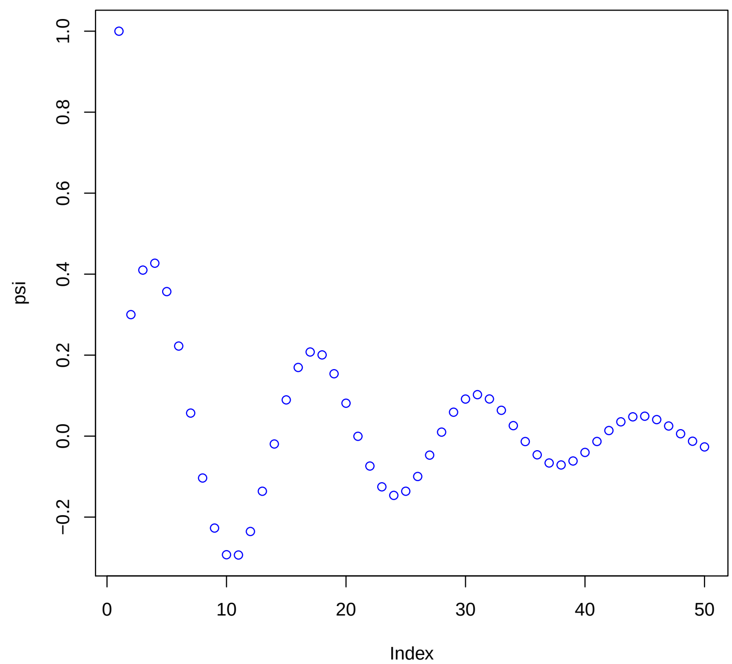

Figure 1: Plot of ψ for $j = 0, 50$

3 Problem 4.3

Consider a causal ARMA(p,q) process. Then

$$\gamma(h) = \sum_{k=1}^{p} \phi_k \gamma(h-k) + \sigma^2 \sum_{j=0}^{q} \theta_{j+h}\psi_j, \quad h \geq 0 \tag{25}$$

3.1 Part a: Finding $\{\gamma(h), h = 0, \cdots, 4\}$ for ARMA(2,3)

Expanding (25) for $p = 2, q = 3$, we get

$$\begin{aligned}\gamma(h) &= \sum_{k=1}^{2} \phi_k \gamma(h-k) + \sigma^2 \sum_{j=0}^{3} \theta_{j+h}\psi_j \\ &= \phi_1\gamma(h-1) + \phi_2\gamma(h-2) + \sigma^2(\theta_h\psi_0 + \theta_{1+h}\psi_1 + \theta_{2+h}\psi_2 + \theta_{3+h}\psi_3)\end{aligned} \tag{26}$$

which can also be written as

$$\gamma(h+2) = \phi_1\gamma(h+1) + \phi_2\gamma(h) + \sigma^2(\theta_{h+2}\psi_0 + \theta_{3+h}\psi_1 + \theta_{4+h}\psi_2 + \theta_{5+h}\psi_3). \tag{27}$$

From [1], page 88, equation (3.2.3) given by

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \phi_j \psi_{j+|h|} \tag{28}$$

holds true for an ARMA(p,q) process. So we can compute $\gamma(0), \gamma(1)$ as

$$\gamma(0) = \sigma^2 \sum_{j=0}^{3} \psi_j \psi_j \tag{29}$$

$$\gamma(1) = \sigma^2 \sum_{j=0}^{3} \psi_j \psi_{j+1} \tag{30}$$

And from (27) we have

$$\begin{aligned}\gamma(0) &= 0 + \sigma^2 \sum_{j=0}^{3} \psi_j \psi_j \\ \gamma(1) &= 0 + \sigma^2 \sum_{j=0}^{3} \psi_j \psi_{j+1} \\ \gamma(2) &= \phi_1\gamma(1) + \phi_2\gamma(0) + \sigma^2(\theta_2\psi_0 + \theta_3\psi_1 + \theta_4\psi_2 + \theta_5\psi_3) \\ \gamma(3) &= \phi_1\gamma(2) + \phi_2\gamma(1) + \sigma^2(\theta_3\psi_0 + \theta_4\psi_1 + \theta_5\psi_2 + \theta_6\psi_3) \\ \gamma(4) &= \phi_1\gamma(3) + \phi_2\gamma(2) + \sigma^2(\theta_4\psi_0 + \theta_5\psi_1 + \theta_6\psi_2 + \theta_7\psi_3)\end{aligned} \tag{31}$$

or in a matrix equation

$$\begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \\ \gamma(3) \\ \gamma(4) \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \gamma(1) & \gamma(0) \\ \gamma(2) & \gamma(1) \\ \gamma(3) & \gamma(2) \end{bmatrix} + \sigma^2 \begin{bmatrix} \sum_{j=0}^{3} \psi_j \psi_j \\ \sum_{j=0}^{3} \psi_j \psi_{j+1} \\ \sum_{j=0}^{3} \theta_{j+2} \psi_j \\ \sum_{j=0}^{3} \theta_{j+3} \psi_j \\ \sum_{j=0}^{3} \theta_{j+4} \psi_j \end{bmatrix} \tag{32}$$

3.2 Part b: Homogeneous difference equation $\phi(B)\gamma(h) = 0$

From part a) equation (27) we had

$$\gamma(h+2) = \phi_1 \gamma(h+1) + \phi_2 \gamma(h) + \sigma^2 (\theta_{h+2}\psi_0 + \theta_{3+h}\psi_1 + \theta_{4+h}\psi_2 + \theta_{5+h}\psi_3). \tag{33}$$

For an ARAM(p=2,q=3), for $h \geq 4$, the right hand side of (33) is 0, resulting in

$$\gamma(h+2) = \phi_1 \gamma(h+1) + \phi_2 \gamma(h) \tag{34}$$

or

$$\gamma(h+2) - \phi_1 \gamma(h+1) - \phi_2 \gamma(h) = 0 \tag{35}$$

3.3 Part c: Plot of $\{\gamma(h), h = 0, \cdots, 50\}$

The parameter are given by

$$\boldsymbol{\phi} = (\phi_1, \phi_2) = (1.7, -0.9), \quad \boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3) = (-1.4, 0.8, 0.1), \quad \sigma^2 = 1. \tag{36}$$

and

$$\gamma(h+2) = 1.7\gamma(h+1) - 0.9\gamma(h) \tag{37}$$

R code

```
#initialization for psi
psi0 = 1
psi1 = 0.3
psi2 = 0.41
theta3 = 0.1
psi3 = 1.7*psi2 - 0.9*psi1 + theta3
psi <- c(psi0,psi1,psi2,psi3)
#compute the rest
for (j in 2:48)
  psi[j+2] = 1.7*psi[j+1] - 0.9*psi[j]

#Initialize gamma with gamma0 and gamma1
gamma0 = 0
for (k in 1:5)
  gamma0 = gamma0 + psi[k]*psi[k]

gamma1 = 0
for (k in 1:5)
  gamma1 = gamma1 + psi[k]*psi[k+1]

gamma = c(gamma0,gamma1)
# comute the rest of the gamma's
for (k in 1:48)
  gamma[k+2] = 1.7*gamma[k+1] - 0.9*gamma[k]
print(length(gamma))

#plot
plot(gamma, col='blue')
```

We use the R function ARMAacf to comopute γ with the following code

```
Rfunction<-ARMAacf( c(1.7,-0.9),c(-1.4,0.8,0.1),50)
plot(Rfunction, col='green')
```

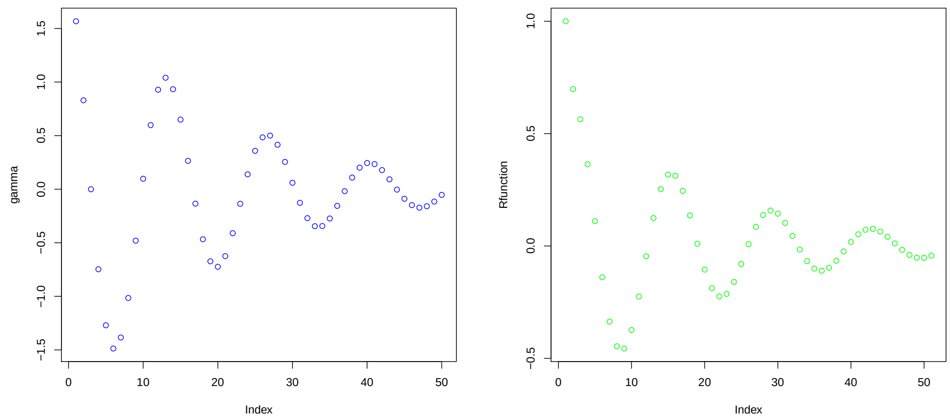

Figure 2: Plot of γ for $j = 0, 50$. Computed in blue vs R function (green)

Figure 2 shows the computed γ versus the γ computed with the R function ARMAacf.

4 Problem 4.4

Let $\{X_t\}$ be a causal AR(2) process with white noise process $WN(0, \sigma^2)$,

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = Z_t, \quad X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \tag{38}$$

5 All R code Code

References

[1] Petter J. Brockwell. Richard A. Davis *Introduction to Time Series and Forecasting.* Springer. Second edition. 2001