STAT211 Mandatory Homework 4
Solutions

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Auditorium 5, Realfagsbygget

Problem 4.1

a) Ivertibility means that you can write $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$, i.e. express Z_t in terms of X_s, $s \leq t$. Brockwell et al. (2016, p.76) states that an ARMA(p,q) is invertible if $\theta(z) = 1 + \theta_1 z + \cdots \theta_q z^q \neq 0$ for all $|z| \leq 1$.

b) Interchanging the roles of the MA and AR parts, we get

$$Z_t - \sum_{j=1}^{q} (-\theta_j) Z_{t-j} = X_t + \sum_{k=1}^{p} (-\phi_k) X_{t-k}$$

Replacing $\phi_j \to -\theta_j$ and $\theta_j \to -\phi_j$ in (2), we get that the sequence $\{\pi_j\}$ is determind by the equations

$$\pi_j = \sum_{k=1}^{q} (-\theta_k) \pi_{j-k} - \phi_j, \quad \phi_0 = -1.$$

You can read more on p. 76 of Brockwell et al. (2016).

Problem 4.2

a) ARMA$(2,3)$:

$$\begin{aligned}
\psi_0 &= \theta_0 = 1 \\
\psi_1 &= \phi_1 \psi_0 + \theta_1 = \phi_1 + \theta_1 \\
\psi_2 &= \phi_1 \psi_1 + \phi_2 \psi_0 + \theta_2 = \phi_1^2 + \phi_1 \theta_1 + \phi_2 + \theta_2 \\
\psi_k &= \phi_1 \psi_{k-1} + \phi_2 \psi_{k-2}, \quad k \geq 3.
\end{aligned}$$

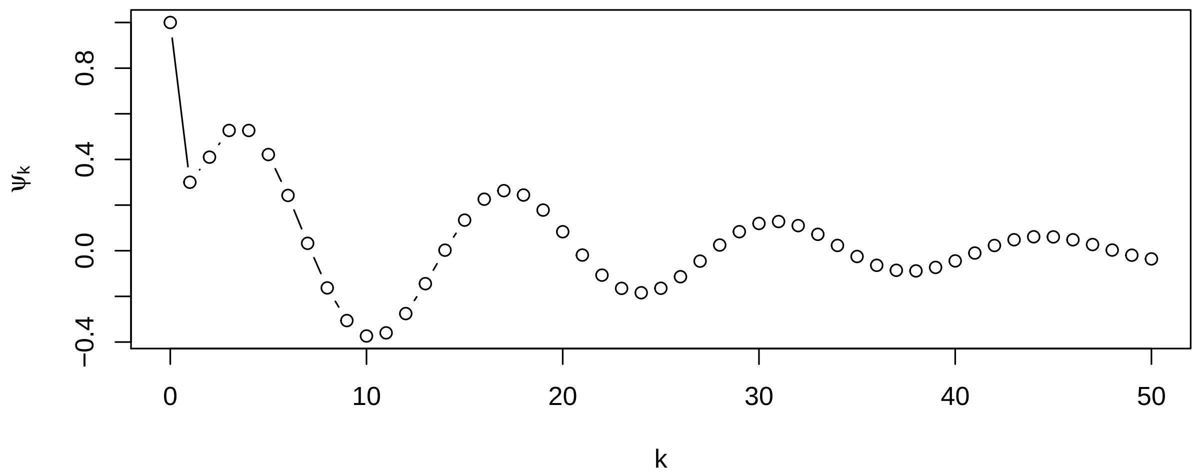
b)

```r
library(itsmr)
phi <- c(1.7,-.9)
theta <- c(-1.4,.8,.1)
sigma2 <- 1
#Using a function to check for invertibility and causality
check(list(phi=phi, theta=theta, sigma2 = sigma2))

## Causal
## Invertible

#Checking roots of AR-polynomial
Mod(polyroot(c(1,-phi)))

## [1] 1.054093 1.054093

#Checking roots of MA-polynomial
Mod(polyroot(c(1,theta)))

## [1] 1.022124 1.022124 9.571780
```

Since the roots of the AR polynomial are outside the unit circle, the model is causal. Since the roots of the MA polynomial are outside the unit circle, the model is invertible.

c)

```
psi <- numeric(51)
psi[1]<-1
psi[2]<-phi[1]+theta[1]
psi[3]<-phi[1]*psi[2]+phi[2]*psi[1]+theta[2]
psi[4]<-phi[1]*psi[3]+phi[2]*psi[2]+theta[3]
for(k in 5:51){
  psi[k]<-phi[1]*psi[k-1]+phi[2]*psi[k-2]
}
plot(0:50,psi, xlab = "k", ylab = expression(psi[k]), type= "b")
```



Problem 4.3

a)

$$\gamma(h) = \sum_{k=1}^{2} \phi_k \gamma(h-k) + \sigma^2 \sum_{j=0}^{3} \theta_{j+h}\psi_j, \quad h \geq 0.$$

$$\begin{aligned}
\gamma(0) - \phi_1\gamma(1) - \phi_2\gamma(2) &= \sigma^2(1 + \theta_1\psi_1 + \theta_2\psi_2 + \theta_3\psi_3) = \sigma^2 s_0 \\
\gamma(1) - \phi_1\gamma(0) - \phi_2\gamma(1) &= \sigma^2(\theta_1 + \theta_2\psi_1 + \theta_3\psi_2) = \sigma^2 s_1 \\
\gamma(2) - \phi_1\gamma(1) - \phi_2\gamma(0) &= \sigma^2(\theta_2 + \theta_3\psi_1) = \sigma^2 s_2 \\
\gamma(3) - \phi_1\gamma(2) - \phi_2\gamma(1) &= \sigma^2\theta_3 = \sigma^2 s_3
\end{aligned}$$

On matrix form, we get

$$\begin{bmatrix} 1 & -\phi_1 & -\phi_2 & 0 \\ -\phi_1 & 1-\phi_2 & 0 & 0 \\ -\phi_2 & -\phi_1 & 1 & 0 \\ 0 & -\phi_2 & -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \\ \gamma(3) \end{bmatrix} = \sigma^2 \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \sigma^2 \begin{bmatrix} 1 + \theta_1\psi_1 + \theta_2\psi_2 + \theta_3\psi_3 \\ \theta_1 + \theta_2\psi_1 + \theta_3\psi_2 \\ \theta_2 + \theta_3\psi_1 \\ \theta_3 \end{bmatrix}$$

b) We have that

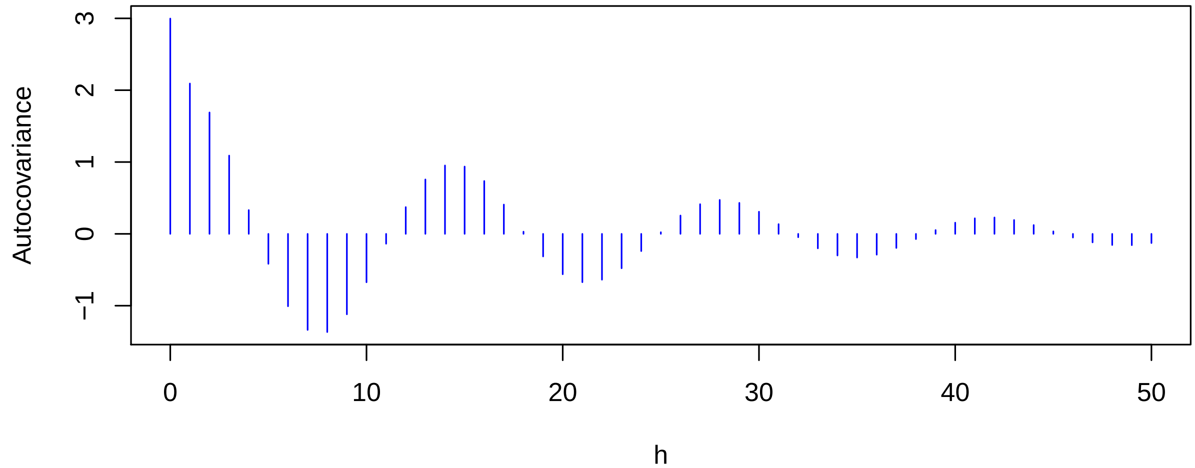
$$\phi(B)\gamma(h) = \gamma(h) - \phi_1\gamma(h-1) - \phi_2\gamma(h-2) = 0, \quad h \geq 4.$$

Thus, for $h = 0, \dots, m$ we get

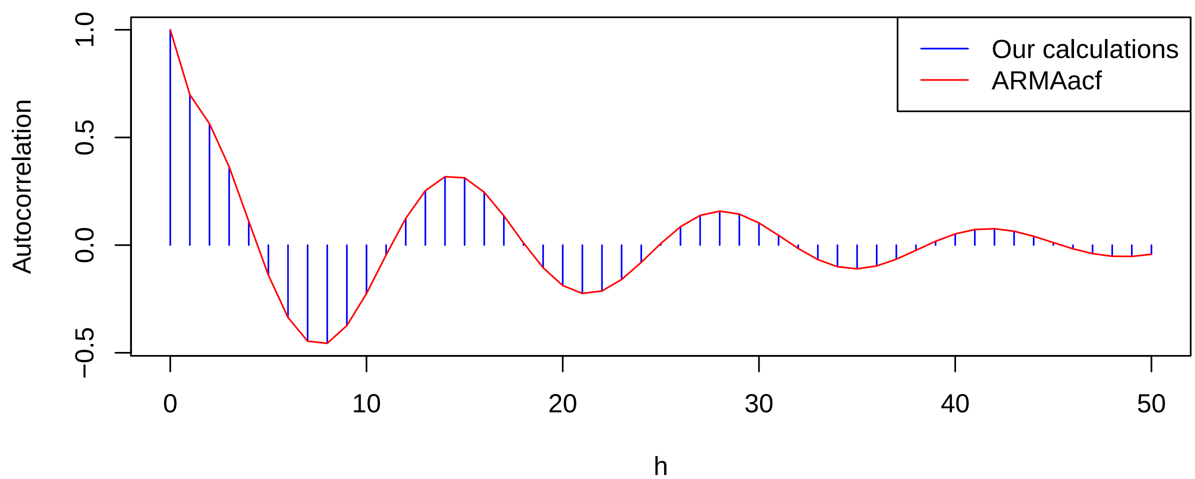
$$\begin{bmatrix} 1 & -\phi_1 & -\phi_2 & 0 & 0 & 0 \\ -\phi_1 & 1-\phi_2 & 0 & 0 & 0 & 0 \\ -\phi_2 & -\phi_1 & 1 & 0 & 0 & 0 \\ 0 & -\phi_2 & -\phi_1 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -\phi_2 & -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \\ \gamma(3) \\ \vdots \\ \gamma(m) \end{bmatrix} = \sigma^2 \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

c)

```
m <- 50
M <- matrix(0, ncol=m+1,nrow=m+1)
diag(M)<-1
M[2,2]<-1-phi[2]; M[1,2:3]<--phi; M[m+1,m]<--phi[1]
for(i in 1:(m-1)){
  M[seq(i+1,i+2),i]<--phi
}
M[1:5,1:5]

##      [,1] [,2] [,3] [,4] [,5]
## [1,]  1.0 -1.7  0.9  0.0    0
## [2,] -1.7  1.9  0.0  0.0    0
## [3,]  0.9 -1.7  1.0  0.0    0
## [4,]  0.0  0.9 -1.7  1.0    0
## [5,]  0.0  0.0  0.9 -1.7    1

M[(m-3):(m+1),(m-3):(m+1)]

##      [,1] [,2] [,3] [,4] [,5]
## [1,]  1.0  0.0  0.0  0.0    0
## [2,] -1.7  1.0  0.0  0.0    0
## [3,]  0.9 -1.7  1.0  0.0    0
## [4,]  0.0  0.9 -1.7  1.0    0
## [5,]  0.0  0.0  0.9 -1.7    1

s <- c(1+theta[1]*psi[2]+theta[2]*psi[3]+theta[3]*psi[4],
       theta[1]+theta[2]*psi[2]+theta[3]*psi[3],
       theta[2]+theta[3]*psi[2],theta[3],rep(0,nrow(M)-4))
gamma.h <- solve(M,s)
plot(0:m,gamma.h, col = 4, type = "h", xlab= "h", ylab = "Autocovariance")
```




```
plot(0:m,gamma.h/gamma.h[1], col = 4, type = "h", xlab= "h", ylab = "Autocorrelation")
lines(0:m, acf.th <- ARMAacf(ar=phi, ma = theta, lag.max = m), col = 2)
legend("topright", col = c(4,2), legend = c("Our calculations","ARMAacf"), lty=1)
```




```
sqrt(sum((acf.th-gamma.h/gamma.h[1])^2))

## [1] 1.746377e-15
```

Perfect match with the result of the `ARMAacf` function.

Problem 4.4

a)

$$\begin{aligned}
X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} &= Z_t \\
X_t X_{t-h} - \phi_1 X_{t-1} X_{t-h} - \phi_2 X_{t-2} X_{t-h} &= Z_t X_{t-h} \\
\mathbb{E}\, X_t X_{t-h} - \phi_1 \mathbb{E}\, X_{t-1} X_{t-h} - \phi_2 \mathbb{E}\, X_{t-2} X_{t-h} &= \mathbb{E}\, Z_t X_{t-h} \\
\gamma(h) - \phi_1 \gamma(h-1) - \phi_2 \gamma(h-2) &= \delta_{0,h} \sigma^2
\end{aligned}$$

b) The equations are:

$$\begin{aligned}
\gamma(0) &= \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2, \\
\gamma(1) &= \phi_1 \gamma(0) + \phi_2 \gamma(1), \\
\gamma(2) &= \phi_1 \gamma(1) + \phi_2 \gamma(0).
\end{aligned}$$

Dividing by $\gamma(0)$ we get

$$\begin{aligned}
\gamma(0)(1 - \phi_1 \rho(1) - \phi_2 \rho(2)) &= \sigma^2, \\
\rho(1) &= \phi_1 + \phi_2 \rho(1), \\
\rho(2) &= \phi_1 \rho(1) + \phi_2,
\end{aligned}$$

or

$$\begin{aligned}
\gamma(0)(1 - \phi_1 \rho(1) - \phi_2 \rho(2)) &= \sigma^2, \\
(1 - \phi_2)\rho(1) &= \phi_1, \\
-\phi_1 \rho(1) + \rho(2) &= \phi_2.
\end{aligned}$$

c)

$$\begin{aligned}
\rho(1) &= \phi_1/(1-\phi_2), \\
\rho(2) &= \phi_2 + \phi_1 \rho(1) = \phi_2 + \phi_1^2/(1-\phi_2), \\
\gamma(0) &= \frac{\sigma^2}{1 - \phi_1 \rho(1) - \phi_2 \rho(2)} = \frac{\sigma^2}{1 - \phi_1^2/(1-\phi_2) - \phi_2^2 - \phi_2 \phi_1^2/(1-\phi_2)} \\
&= \frac{(1-\phi_2)\sigma^2}{(1-\phi_2)(1-\phi_2^2) - \phi_1^2(1+\phi_2)}
\end{aligned}$$

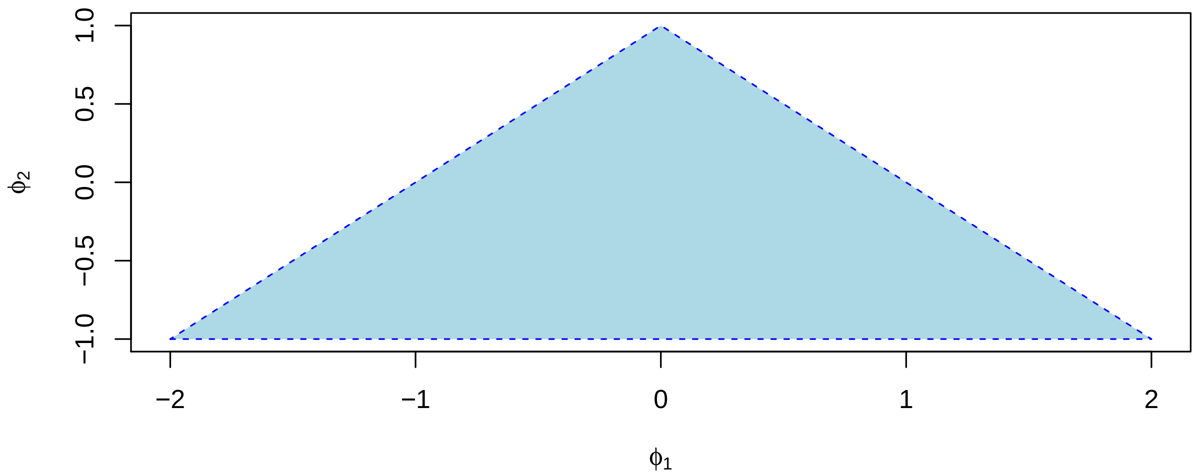
d) We must have that $|\rho(h)| \leq 1$ and in particular

$$-1 \leq \rho(1) = \frac{\phi_1}{1-\phi_2} \leq 1.$$

Thus

$$\begin{aligned}
\phi_1 \leq 1 - \phi_2 &\Leftrightarrow \phi_2 + \phi_1 \leq 1 \\
\phi_1 \leq -1 + \phi_2 &\Leftrightarrow \phi_2 - \phi_1 \leq 1.
\end{aligned}$$

The case $\phi_2 = 1$ is problematic, because then neither $\rho(1)$ nor $\rho(2)$ is defined. If $\phi_2 = -1$, then $\gamma(0)$ is infinite. Therefore $|\phi_2| < 1$ is also a constraint. With these three constraints we get the following area:



e) $\mathbb{E}\left[X_3|X_1\right] = \rho(2)X_1 = (\phi_2 - \phi_1^2/(1-\phi_2))X_1$.

f) First of all

$$\Gamma_2 = \begin{bmatrix}\gamma(0) & \gamma(1)\\ \gamma(1) & \gamma(0)\end{bmatrix} \quad \text{and} \quad \Gamma_2^{-1} = \frac{1}{\gamma^2(0)-\gamma^2(1)}\begin{bmatrix}\gamma(0) & -\gamma(1)\\ -\gamma(1) & \gamma(0)\end{bmatrix} = \frac{\gamma(0)}{\gamma^2(0)-\gamma^2(1)}\begin{bmatrix}1 & -\rho(1)\\ -\rho(1) & 1\end{bmatrix}.$$

We have that

$$\begin{aligned}
\gamma^2(0)-\gamma^2(1) &= \gamma(0)(1-\rho(1)) = \gamma^2(0)(1-\frac{\phi_1^2}{(1-\phi_2)^2}) = \gamma^2(0)\frac{(1-\phi_2)^2-\phi_1^2}{(1-\phi_2)^2},\\
\frac{\gamma(0)\sigma^2}{\gamma^2(0)-\gamma^2(1)} &= \frac{\sigma^2}{\gamma(0)}\frac{(1-\phi_2)^2}{(1-\phi_2)^2-\phi_1^2}\\
&= \frac{(1-\phi_2)(1-\phi_2^2)-\phi_1^2(1+\phi_2)}{(1-\phi_2)\sigma^2}\frac{\sigma^2(1-\phi_2)^2}{(1-\phi_2)^2-\phi_1^2}\\
&= \frac{(1-\phi_2)^2(1-\phi_2^2)-\phi_1^2(1+\phi_2)(1-\phi_2)}{(1-\phi_2)^2-\phi_1^2}\\
&= \frac{(1-\phi_2)^2(1-\phi_2^2)-\phi_1^2(1-\phi_2^2)}{(1-\phi_2)^2-\phi_1^2}\\
&= \frac{(1-\phi_2)^2-\phi_1^2}{(1-\phi_2)^2-\phi_1^2}(1-\phi_2^2) = 1-\phi_2^2
\end{aligned}$$

Thus,

$$\sigma^2\Gamma_2^{-1} = \begin{bmatrix}1-\phi_2^2 & -\frac{\phi_1}{1-\phi_2}(1-\phi_2^2)\\ -\frac{\phi_1}{1-\phi_2}(1-\phi_2^2) & 1-\phi_2^2\end{bmatrix}$$

Hence, the asymptotic variance of $\widehat{\phi}_1$ in an AR(2) is $1-\phi_2^2$. If $\phi_2 = 0$, this variance is 1. In an AR(1), the asymptotic variance is $\sigma^2\Gamma_1^{-1} = \sigma^2\gamma^{-1}(0) = \sigma^2(1-\phi_1^2)/\sigma^2 = 1-\phi_1^2 \leq 1$, $|\phi_1| < 1$. The asymptotic variance of $\widehat{\phi}_1$ is clearly smaller for an AR(1) than an AR(2) with $\phi_2 = 0$.

References

Brockwell Peter J, Davis Richard A, Calder Matthew V. Introduction to time series and forecasting. 3. 2016.