STAT211 Mandatory Homework 6

Yapi Donatien Achou

March 4, 2019

Contents

1 Problem 6.6

Let $\mathbf{P}_k$ be the linear projection onto

$$\mathbf{S}_k = span\{X_1, \cdots, X_k\} \tag{1}$$

and

$$e_k = \frac{X_k - \hat{X}_k}{\nu_{k-1}}. \tag{2}$$

$\{e_1, \cdots, e_n\}$ is orthonormal basis for $\mathbf{S}_n$ if $\{e_1, \cdots, e_n\}$ is a linearly independent subset of $\mathbf{S}_n$ that span $\mathbf{S}_n$, and for any e_j, e_i in $\{e_1, \cdots, e_n\}$ the inner product of e_j and e_i is zero and any e_i as norm 1.

Proof.

- Linearly independence. Assume that

  $$a_1 e_1 + \cdots + a_n e_n = 0 \tag{3}$$

  where a_i are real numbers. The we have

  $$\begin{aligned} a_1 e_1 + \cdots + a_n e_n &= 0 \\ a_1 \frac{X_1 - \hat{X}_1}{\nu_0} + \cdots + a_n \frac{X_n - \hat{X}_n}{\nu_{n-1}} &= 0 \\ \frac{a_1}{\nu_0}(X_1 - \hat{X}_1) + \cdots + \frac{a_n}{\nu_{n-1}}(X_n - \hat{X}_n) &= 0 \end{aligned} \tag{4}$$

  The last expression in (4) holds true if

  $$X_1 - \hat{X}_1 = 0, \cdots, X_n - \hat{X}_n = 0 \tag{5}$$

  But from (2) we know that

  $$X_k - \hat{X}_k = e_k \nu_{k-1}. \tag{6}$$

  Therefor equation (4) is true if

  $$\frac{a_1}{\nu_1} = \cdots = \frac{a_n}{\nu_{n-1}} = 0 \tag{7}$$

  equivalently

  $$a_1 = \cdots = a_n = 0 \tag{8}$$

  This means that $\{e_1, \cdots, e_n\}$ is a linearly independent

- $\{e_1, \cdots, e_n\}$ span $\mathbf{S}_n$. We want to show that any vector in $\mathbf{S}_n$ can be written as a linear combination of $\{e_1, \cdots, e_n\}$. Let $Z \in \mathbf{S}_n$. Since $\mathbf{S}_n = span\{X_1, \cdots, X_n\}$, we have

  $$Z = b_1 X_1 + \cdots + b_n X_n \tag{9}$$

where b_i are real numbers. Then we have

$$\begin{aligned} Z &= b_1 X_1 + \cdots + b_n X_n \\ Z &= b_1(\nu_0 e_1 + \hat{X}_1) + \cdots + b_n(\nu_{n-1} e_n + \hat{X}_n) \\ Z &= b_1 \nu_0 e_1 + \cdots + b_n \nu_{n-1} e_n + \underbrace{b_1 \hat{X}_1 + \cdots + b_n \hat{X}_n}_{Z'} \end{aligned} \tag{10}$$

$$\underbrace{Z - Z'}_{Z''} = \underbrace{b_1 \nu_0}_{\alpha_1} e_1 + \cdots + \underbrace{b_n \nu_{n-1}}_{\alpha_n} e_n$$

Since $Z'' \in \mathbf{S}_n$ We have

$$Z'' = \alpha_1 e_1 + \cdots + \alpha_n e_n \tag{11}$$

- Horthogonality Let Z_1, Z_2 be two arbitrarily vectors in $\mathbf{S}_n$

$$\begin{aligned} \langle Z_1, Z_2 \rangle &= \langle \alpha_1 e_1 + \cdots + \alpha_n e_n, \beta_1 e_1 + \cdots + \beta_n e_n \rangle \\ &= \alpha_1 \beta_1 + \cdots + \alpha_n \beta_n \end{aligned} \tag{12}$$

□

References

[1] Petter J. Brockwell. Richard A. Davis *Introduction to Time Series and Forecasting.* Springer. Second edition. 2001