Homework stat210

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Contents

1 HOMEWORK 1 - January 30

Problem 1.1

[CB, E3.17, p. 130]

Establish a formula similar to (3.3.18) , p 107, for the gamma distribution. If $X \sim \text{Gamma}(\alpha, \beta)$, then for any positive constant ν [1],

$$\mathbb{E}\, X^\nu = \frac{\beta^\nu \Gamma(\nu + \alpha)}{\Gamma(\alpha)}.$$

Problem 1.2

The moment generating function for the geometric distribution, X_p, and for the exponential distribution Y_β are respectively,

$$M_{X_p}(t) = \frac{p \exp\bigl(t\bigr)}{1 - q\exp\bigl(t\bigr)}, \quad M_{Y_\beta}(t) = \frac{1}{1 - t\beta}.$$

i) Show that $M_{pX_p}(t)$ approaches $M_{Y_1}(t)$ for each fixed small enough t as p goes to zero. Conclude that the distribution of pX_p can be approximated by the standard exponential distribution.

ii) Compare the expectation and the variance of X_p and $Y_{1/p}$ for p small.

Problem 1.3

[CB, E3.20, p. 131]

Let Z be standard normally distributed and $X = |Z|$. Then X has density

$$f(x) = \frac{2}{\sqrt{2\pi}} \exp\bigl(-x^2/2\bigr), \quad x \geq 0.$$

i) Find the mean and the variance of X [2].

ii) If X has the folded normal distribution, find the transformation g and the value of (α, β) so that $g(X) \sim \text{Gamma}(\alpha, \beta)$.

Problem 1.4

[CB, E3.21, p. 131]

The standard Cauchy distribution has density, $f(x) = \dfrac{1}{\pi} \dfrac{1}{1 + x^2}$.

Show that $\mathbb{E}\,|X|^p < \infty$ for $p > 0$ iff $p < 1$.

Problem 1.5

[1] Same calculation as the evaluation of (3.3.7), p 100.

[2] This distribution is soemtimes called a folded normal.

[CB, E3.23, p. 131]

The Pareto(α, β) distribution has pdf

$$f(x) = \frac{\beta\alpha^\beta}{x^{1+\beta}}\, 1(\alpha < x < \infty), \quad \alpha, \beta > 0.$$

i) Verify that f is a pdf.
ii) Derive the mean and the variance of this distribution.
iii) Prove that the variance does exist if $\beta \leq 2$ and more general $\mathbb{E}\, X^p < \infty$ iff $p < \beta$.

PROBLEM 1.6
[CB, E3.24bc, p. 131]
Many named distributions are special cases of the more common distributions already discussed. For each of the following named distributions derive the form of the pdf[3], verify that it is a pdf, and calculate the mean and variance.

i) If X is Exp(β) then $\sqrt{2X/\beta}$ has the Rayleigh distribution.
ii) If $X \sim$ Gamma(a, b), then $Y = 1/X$ has the inverted gammaIG(a, b) distribution[4].

PROBLEM 1.7
Let the real function$f$ be strictly positive on the interval (a, b) and zero elsewhere. Assume that $\int_R f(x)dx = 1$ and in additon we also assume that f is continuous.

i) Explain that $F(x) = \int_{-\infty}^{x} f(t)dt$ is a distribution function.
ii) Let U be standard uniformly distributed. Define X by the equation $U = F(X)$ and prove by the transformation formulae that X has density f_X.
iii) Suppose that X is given and $F_X = F$. Define U by the equation $U = F(X)$ and prove by the transformation formulae that U is standard uniformly distributed.
iv) Is the continuity of f necessary? A short answer is enough.

[3] Use the transformation formulae.
[4] This distribution is useful in Bayesian estimation of variance, see Exercise 7.23.

PROBLEM 1.8

Let X be discrete distributed,

x	x_1	x_2	$\cdots$	x_{m-1}	x_m
$\mathbb{P}(X=x)$	p_1	p_2	$\cdots$	p_{m-1}	p_m

where $x_i < x_{i+1}$ for $i = 1, \ldots, m-1$. The indicator function means that $1[a,b)(x) = 1$ for $x \in [a,b)$ and otherwise $1[a,b)(x) = 0$. We have learned that in the discrete case the quantile function and the distribution function have formulas

$$Q(u) = \sum_{j=1}^{m} x_j 1(s_{j-1}, s_j](u), \quad u \in (0,1), \quad F(x) = \sum_{j=1}^{m} s_j 1[x_j, x_{j+1})(x), \quad x \in \Re,$$

respectively, where $x_{m+1} = \infty$ and $s_j = \sum_{\ell=1}^{j} p_j$ for $j = 1, \ldots, m$ with $s_0 = 0$.

Suppose that $m = 4$. Consider the following numeric example

x	0	1	2	3
$\mathbb{P}(X=x)$	0.216	0.432	0.288	0.064

i) Plot the function F and the function Q. You may use the program R. The lines below are part of a possible source file for this problem.

```
m<-4
x<-0:(m-1)
p<-c(0.216,0.432,0.288,0.064)
s<-rep(0,(m+1))
for(i in 1:m) s[i+1]<-sum(p[1:i])
#
# The quantile function; Finv =Q
Finv <-function(u){
j<- length(s[s <=u]) # j>=1
j<-min(j,m)
x[j]
}
#
....
....
```

ii) Plot F(Q(u)).

iii) Plot Q(F(x)).

2 HOMEWORK 2 - FEBRUARY 06

PROBLEM 2.1

[CB, E3.28c, p. 132]
Show that each of the following families is an exponential family.

i) Normal family with either parameter μ or σ^2 known.
ii) Gamma family with either parameter α or β or both known.
iv) Poisson family.
v) Negative binomial family where $p \in (0, 1)$.

PROBLEM 2.2

[CB, E3.29, p. 132]
For each family in Exercise 3.28, describe the natural parameter space [$= \mathcal{H}$, p 114 CB].

PROBLEM 2.3

In R there is a library of common parametric distributions. You will get an overview by typing *help(distribution)* in the command window.

a) Use the R-function *dbeta* and make a plot of the beta density for $(\alpha, \beta) = (1.5, 2.5)$.
b) Plot the beta density with parameter $(\alpha, \beta) = (15, 25)$ together with the normal density with matching expectation and variance.
c) Make a plot of the gamma density with parameters $(\alpha = 10, \beta = 1)$. Does it look like a normal density?.
d) Finally, plot the beta density for $(\alpha, \beta) = (0.01, 0.03)$. Generate a sample from this density with $n = 100$ and look at data with only 2 decimals. Is it close to a Bernoulli sequence? You get the sample by $X \leftarrow rbeta(n, \alpha, \beta)$.

PROBLEM 2.4

[CB, E3.31, p. 132]
In this exercise we will prove Theorem 3.4.2 (page 112).

i) Start from the equality

$$\int f(x|\boldsymbol{\theta})dx = \int h(x)c(\boldsymbol{\theta})\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})T_i(x)\Big)dx = 1.$$

ii) Differentiate the above equality a second time; then rearrange to establish (3.4.5) [5]

[5]The fact that $\ln'' g = -(\ln' g)^2 + g^{-1}g''$ may be helpful.

Problem 2.5
[CB, E3.32, p. 133]

i) If an exponential family can be written in the form (3.4.7), show that the identities of Theorem 3.4.2 simplify to

$$\begin{aligned} \mathbb{E}\, T_i &= -\frac{\partial}{\partial \eta_i} \log c(\boldsymbol{\eta}), \\ \operatorname{Var}\big(T_i\big) &= -\frac{\partial^2}{\partial^2 \eta_i} \log c(\boldsymbol{\eta}). \end{aligned}$$

ii) Use this identify to calculate the mean and variance of a $\text{Gamma}(a, b)$ random variable.

Problem 2.6
[CB, E3.33ac, p. 133]
For each of the following families:

i) Verify that it is an exponential family:
iii) Sketch a graph of the curved parameter space.

a) $\mathcal{N}(\theta, \theta)$.
b) $\mathcal{N}(\theta, a\theta^2)$, a is known.
c) $\text{Gamma}(\alpha, \alpha^{-1})$.

Problem 2.7
The density of the bivariate normal distribution is

$$f(\mathbf{x}|\boldsymbol{\mu}, \Sigma) = (2\pi)^{-1}\ |\Sigma|^{-1/2} \exp\big\{ - 2^{-1}(\mathbf{x}-\boldsymbol{\mu})^t \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\big\}$$

where

$$\mathbf{x} = (x_1, x_2)', \quad \boldsymbol{\mu} = (\mu_1, \mu_2)', \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, \quad \Sigma^{-1} = \begin{bmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{bmatrix}.$$

Show that the bivariate normal distribution belongs to the exponential family with $k = 5$.

Hint: Do first vector multiplication in the exponent giving 3 terms which you then must further dissect and reorganise.

Problem 2.8
We want to find eventually stationary points for the density in the beta distribution. Let $g(x) = \log f(x|\alpha, \beta)$ and

$$g'(x) = (\alpha - 1)\frac{1}{x} - (\beta - 1)\frac{1}{1-x}, \qquad x_0 \stackrel{\text{def}}{=} \frac{\alpha - 1}{\alpha + \beta - 2}.$$

a) Show that the density has at most one stationary point and the only candidate is x_0.

b) Explain that if $\min(\alpha, \beta) > 1$ then x_0 is the unique modal value for the density.

c) What is the situation if $\alpha = \beta = 1$?

d) What about $\alpha < 1$ and $\beta > 2 - \alpha$?

e) Prove that x_0 is a global minimum if $\max(\alpha, \beta) < 1$.

PROBLEM 2.9

In the beta distribution we have that

$$\mathbb{E}\, X^n = \frac{\alpha^{(n)}}{(\alpha+\beta)^{(n)}} = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{(\alpha+\beta)(\alpha+\beta+1)\cdots(\alpha+\beta+n-1)}, \quad n \geq 1.$$

Assume that $\beta = \tau\alpha$ for a positive constant τ.

a) Let $p \in (0,1)$ and compute the mgf for the indicator I with success probability p. You should get $M_I(t) = 1 + p\big(\exp(t) - 1\big)$.

Let $p = 1/(1+\tau)$.

b) Show that $\mathbb{E}\, X^n \to p$ when $\alpha \to 0$ for any fixed $n \geq 1$.

c) Explain that $M_X(t) \approx M_I(t)$ when α is small enough for all fixed t. What is your conclusion?

3 HOMEWORK 3 - February 13

PROBLEM 3.1

[CB, E4.1, p. 192]

A random point is distributed uniformly on the square with vertices $(1, 1)$, $(1-1)$, $(-1, 1)$, and $(-1, -1)$. That is the joint pdf is $f(x, y) = 4^{-1}$ on the square. [6] Determine the probabilities of the following events,

a) i) $X^2 + Y^2 < 1$.
   ii) $2X - Y > 0$.
   iii) $|X + Y| < 2$.

b) Simulate $\{(X_i, Y_i),\ i = 1, \ldots, n\}$ and plot all data as points. Use different symbols or color for the points that fall into the region as define by i). Repeat the plot but this time you mark the points that are inside the region defined by ii). For both regions use the sample to estimate the probability you have calculated in a). Choose $n = 100000$.

PROBLEM 3.2

[CB, E4.10, p. 193]

The random pair (X, Y) has distribution:

	$X = 1$	$X = 2$	$X = 3$
$Y = 2$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
$Y = 3$	$\frac{1}{6}$	0	$\frac{1}{6}$
$Y = 4$	0	$\frac{1}{3}$	0

i) Show that X and Y are dependent.

ii) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

PROBLEM 3.3

[CB, E4.11, p. 193]

Let U be the number of trials needed to get the first head and V be the number of trials needed to get the two heads in repeated tosses of a fair coin. Are U and V independent random variables?

[6] $f(x, y) = 4^{-1}1\big((x, y) \in \text{square}\big)$, the last term is an indicator.

PROBLEM 3.4
[CB, E4.22, p. 195]
Let (X, Y) be a bivariate random vector with joint pdf $f(x, y)$[7]. Let $U = aX + b$ and $V = cY + d$, where a, b, c, and d are fixed constants with $a > 0$ and $c > 0$. Show that the joint pdf of (U, V) is

$$f_{U,V}(u, v) = a^{-1}c^{-1}f_{X,Y}\big(a^{-1}(u - b), c^{-1}(v - d)\big).$$

Hint: Use the transformation formula.

PROBLEM 3.5
[CB, E4.27, p. 195]
Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\gamma, \sigma^2)$. Suppose that X and Y are independent normal random variables. Define $U = X+Y$ and $V = X-Y$. Show that U and V are independent normal random variables. Find the distribution of each of them.

PROBLEM 3.6
[CB, E4.32, p. 196]

i) For the hierarchical model

$$Y|\Lambda \sim \text{Poisson}(\Lambda), \quad \Lambda \sim \text{Gamma}(\alpha, \beta).$$

Find the marginal distribution, mean and variance of Y. Show that the marginal distribution of Y is negative binomial if α is an integer.

ii) Show that the three stage model

$$Y|N \sim \text{binomial}(N, p), \quad N|\Lambda \sim \text{Poisson}(\Lambda), \quad \Lambda \sim \text{Gamma}(\alpha, \beta),$$

leads to the same marginal distribution of Y.

iii) Generate sample from Y of size n based on recipe above ii) with $(p, \alpha, \beta) = (0.4, 3, 2)$ and compare the empirical distribution with the negative binomial distribution with $r = \alpha$ and an appropriate p. Choose $n = 1000$. You may also try to estimate p. For plotting a discrete probability you may use

```
hist(y,probability=TRUE,breaks="fd",ylim=c(0,1))
```

where `y` is the sample vector of size n.

PROBLEM 3.7
Calculate the variance in the geometric distribution from the exponential family representation with the natural parameter.
Hint: Use the formula $\sigma_T^2 = -\log'' \widetilde{c}$.

[7] The alternative way of writing this simultaneous density is $f_{X,Y}$.

4 HOMEWORK 4 - February 20

Problem 4.1

[CB, E4.45, p. 199]

The bivariate normal distribution is treated in textbook at [CB, Def. 4.5.10, pp. 175-177] and [CB, Exercises 4.45, 4.46, pp. 199-200] . This is done in principle in the same way as we did in the lecture February 15. In the notation of textbook [CB, Exer 4.46, p. 199], the variables (X, Y) are defined by a linear transformation of two independent variables standard normally distributed Z_1 and Z_2, i.e. ,

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a_X & b_X \\ a_Y & b_Y \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} c_X \\ c_Y \end{bmatrix}. \tag{1}$$

Then in [CB, Exer 4.46 c), p. 199] the density as given in [CB, Def. 4.5.10, p. 175],

$$\begin{aligned} f(x, y) = \Big(2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}\Big)^{-1} \times \exp\Big(- \frac{1}{2(1-\rho^2)}\Big(\Big(\frac{x-\mu_X}{six}\Big)^2 \\ - 2\rho\Big(\frac{x-\mu_X}{six}\Big)\Big(\frac{y-\mu_Y}{\sigma_Y}\Big) + \Big(\frac{y-\mu_Y}{\sigma_Y}\Big)^2\Big)\Big). \end{aligned} \tag{2}$$

is obtained by the transformation formula. However, our notation[8] is more[9] suitable, but in this exercise you are guided by the textbook. This means that you must stick to (2) while we have in the aforementioned lecture used vector notation for describing and handling the density. The notation for the parameters are compatible, though.

Let $(X, Y) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ with density given by (2) and

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{bmatrix}.$$

Show that the following are true:

c) i) The marginal distribution of X is $\mathcal{N}(\mu_X, \sigma_X^2)$ and the marginal distribution of Y is $\mathcal{N}(\mu_Y, \sigma_Y^2)$.
Hint: Start with transformation to standardised variables.

ii) The conditional distribution of Y given $X = x$ is

$$\mathcal{N}\Big(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X),\ (1-\rho^2)\sigma_Y^2\Big). \tag{3}$$

Hint: Begin with transformation to standardised variables. We have that:

$$\begin{aligned} f_{Y'|X'}(y|x) &= \sigma_Y f_{Y|X}\big(\sigma_Y y + \mu_Y \big| \sigma_Y x + \mu_X\big) \\ f_{Y|X}(y|x) &= \sigma_Y^{-1} f_{Y'|X'}\big(\sigma_Y^{-1}(y-\mu_Y)\big|\sigma_X^{-1}(x-\mu_X)\big) \end{aligned}$$

[8]

$$f_{\mathbf{X}}(\mathbf{x} \mid \boldsymbol{\mu}, \Sigma) = (2\pi)^{-n/2}|\Sigma|^{-1/2}\exp\{-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})\}.$$

[9] much more

iii) For any constants a and b, the distribution of $aX + bY$ is

$$\mathcal{N}\big(a\mu_X + b\mu_Y, a^2\sigma_X^2 + 2ab\rho\sigma_X\sigma_Y + b^2\sigma_Y^2\big).$$

Hint: The least painful way is to take advantage of (3) and use conditional mgf. Note that the mgf for a normal distributed variable is $\exp\big(\mu t + 2^{-1}\sigma^2 t^2\big)$.

d) Let $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho) =(1, 2, 1, 4, 0.7)$ and draw $n = 100$ observations $\{(X_1, Y_1), \ldots, (X_n, Y_n)\}$ from this bivariate normal distribution in the following way for $i = 1, \ldots, n$:

i) Draw $X_i \sim \mathcal{N}(\mu_X, \sigma_X^2)$.

ii) Draw $Y_i \sim \mathcal{N}\big(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X_i - \mu_X), (1-\rho^2)\sigma_Y^2\big)$.

e) Calculate the empirical version of each of your input parameters with $\widehat{\rho}$ as the empirical correlation coefficient. Make a scatter plot including the empirical Y-regression line: $\big(x, \widehat{\mathbb{E}}(Y|X = x)\big)$. You may also add the theoretical regression line: $\big(x, \mathbb{E}(Y|X = x)\big)$ where this conditional expectation is given by (3).

PROBLEM 4.2

[CB, E4.52, p. 199]

Bullets are fired at the origin at an (x, y) coordinate system, and the point hit, say (X, Y), is a random variable. The variables X and Y are taken to be independent $\mathcal{N}(0, 1)$ random variables. If the two bullets are fired independently, what is the distribution of the distance between them?

PROBLEM 4.3

[CB, E5.2, p. 255]

Suppose that the sequence $\{X_n\}$ is iid with pdf f where each X_n represents the annual rainfall at a given location.

i) Find the distribution of the number of years, N, until the first rainfall, X_1, is exceeded for the first time.

Hint: Start with $\mathbb{P}(N = n|X_1 = x_1)$.

ii) Show that the mean number of years until X_1 is exceeded for the first time is infinite.

PROBLEM 4.4

[CB, E5.15, p. 258]

Establish the following recursion relations for means and variances. Let $\overline{X}_n$ and S_n^2 be the mean and the empirical variance, respectively, of $X_1, \ldots, X_n$. Then suppose another observation, X_{n+1}, becomes available. Show that

a) i) $\overline{X}_{n+1} = \frac{n}{n+1}\overline{X}_n + \frac{1}{n+1}X_{n+1}$.

ii) $S_{n+1}^2 = \frac{n-1}{n}S_n^2 + \frac{1}{n+1}(X_{n+1} - \overline{X}_n)^2$.

b) Simulate $N = 1000$ from $\mathcal{N}(2, 2)$ and draw the graphs $\{(n, \overline{X}_n), n = 1, N\}$ and $\{(n, S_n^2), n = 1, N\}$.

c) Do the same as in the previous point but this time you draw from the standard Pareto distribution defined on $[1,\infty]$ with $F(x) = 1 - x^{-\alpha}$ and here with $\alpha = 3/2$. Do this simulation by first finding an explicit expression for the quantile function, i.e., solve $F(x) = u$ with respect to x so that $x = F^{-1}(u)$. Then generate the U's and subsequently calculate $X_i = F^{-1}(U_i)$.

PROBLEM 4.5

Let $X_1, \ldots, X_n$ be random sample with n as time index. Assume that the parent distribution X is continuous and let $\mathbf{X}_{(n)} = \mathbf{X}_{(n,1:n)}$ be the order statistic vector of the first n variables so that $X_{(n,k)}$ is the kth smallest variable in the sample. We want to update the order statistic vector when the sample is increased by adding the variable X_{n+1}.

a) Explain that

$$
\begin{aligned}
k &\stackrel{\text{def}}{=} \sum_{j=1}^{n} 1(X_j < X_{n+1}) + 1, \\
\mathbf{X}_{(n+1)} &= \big(\mathbf{X}_{(n,1:k-1)}, X_{n+1}, \mathbf{X}_{(n,k:n)}\big).
\end{aligned} \tag{4}
$$

You can assume that there are no ties. Why?

b) The integer k defined above is stochastic. Find $\mathbb{E}\, k$.

c) Let $N = 200$, X be the standard uniformly distribution, choose $p = 0.9$ and let $q_p = p$ be the $p \times 100$-percentile. Use the recursion defined above and calculate for each n:

   i) $\mathbf{X}_{(n,1:n)}$.

   ii) $\widehat{q}_p(n) = X_{(n,[np])}$[10]

d) Let $\mathbb{A}$ be a lower triangular 5×5 matrix given by $\mathbb{A} = \big\{X_{(i,j)},\ 1 \le j \le i \le 5\big\}$. Print $\mathbb{A}$ and see if the algorithm is doing correctly.

e) Plot $\big\{(n, \widehat{q}_p(n)),\ \ n = 1, \ldots, N\big\}$.

PROBLEM 4.6

An important variant of the second law of conditional expectation is

$$
\mathbb{E}\big[g(X,Y)|X = x\big] = \mathbb{E}\big[g(x,Y)|X = x\big]. \tag{5}
$$

a) Suppose that (X,Y) has density $f_{X,Y}$. Use (5) and find an expression for $\mathbb{E}\big(g(X,Y)|X\big)$[11].

b) Prove (5) when (X,Y) is discrete. Start with $Z = g(X,Y)$ and the definition of $\mathbb{E}(Z|X = x)$.

[10] By $[np]$ we mean the smallest integer not exceeding p when $n \ge 1/p$. Otherwise we choose $[np] = 1$.

[11] You must assume that the formula $\mathbb{E}\, h(Y)|X = x) = \int h(y) f_{Y|X}(y|x)dy$ holds for any reasonable function h whenever the conditional distribution of Y given X has a density.

5 HOMEWORK 5 - February 27

Problem 5.1

[CB, E5.17, p. 258]

Let X be a random variable with an $F_{p,q}$ distribution.

i) Derive the pdf of X.

ii) Derive the mean and the variance of X.

iii) Show that $\dfrac{1}{X}$ has an $F_{q,p}$ distribution.

iv) Show that $\dfrac{\frac{p}{q}X}{1+\frac{p}{q}X}$ has a beta$\left(\frac{p}{2}, \frac{q}{2}\right)$ distribution.

Problem 5.2

[CB, E5.18, p. 258]

Let X be a random variable with a t-distribution with p degrees of freedom.

i) Derive the mean and the variance of X.

ii) Show that X^2 has an $F_{1,p}$ distribution.

Problem 5.3

[CB, E5.21, p. 259]

What is the probability that the large of two continuous iid random variables will exceed the population median. Generalize this result to samples of size n.

Problem 5.4

[CB, E5.22, p. 259]

Let X and Y be iid $\mathcal{N}(0,1)$ random variables and $Z \stackrel{\text{def}}{=} \min(X,Y)$. Prove that $Z^2 \sim \chi^2(1)$.

Problem 5.5

[CB, E5.23, p. 259]

Let $\{U_i\}$ be iid $\mathcal{U}(0,1)$ random variables and let X have the distribution

$$P(X = x) = \frac{c}{x!}, \qquad x \geq 1, \qquad c = \frac{1}{e-1}.$$

Find the distribution of $Z = \bigwedge_{i=1}^{x} U_i = \min\left\{U_1, \ldots, U_x\right\}$.

Hint: $(Z \mid X = x) \sim U_{(1)}$, $n = x$.

6 HOMEWORK 6 - March 06

PROBLEM 6.1

[CB, E5.30, p. 258]
If $\overline{X}_1$ and $\overline{X}_2$ are means of two independent samples of size n from a population with variance σ^2, find a value of n so that

$$\mathbb{P}\Big(|\overline{X}_1 - \overline{X}_2| < \frac{\sigma}{5}\Big) \approx 0.99.$$

Justify your calculations.

PROBLEM 6.2

[CB, E5.34, p. 258]
Let $\{X_1, \ldots, X_n\}$ be a random sample from a population with mean μ and variance σ^2. Let $\overline{X}_n$ be the empirical mean and Z_n the corresponding standardised variable. Explain that $Z_n = \sigma^{-1}n^{1/2}(\overline{X}_n - \mu)$.

PROBLEM 6.3

[CB, E5.36a, p. 258]
Suppose that the conditional distribution of Y is $\chi^2(n)$ given that $N = n$. Assume that $N \sim \text{Poisson}(\theta)$. Find the expectation and the variance of Y. Explain that $Y \sim \sum_{j=1}^{N} Z_i^2$ where the Z's are iid standard normally distributed and independent of N.

PROBLEM 6.4

[CB, E5.49, p. 258]
Let U be standard uniformly distributed.

i) Explain that $1-U$ is standard uniformly distributed. Show that both $-\log(U)$ and $-\log(1-U)$ are $\text{Exp}(1)$.

ii) Verify that

$$X \stackrel{\text{def}}{=} \log\Big(\frac{U}{1-U}\Big) \sim \text{logistic}(0,1).$$

iii) Show how to generate a $\text{logistic}(\mu,\beta)$ variable. Generate $n = 500$ with $(\mu,\beta) = (1,2)$ and make a plot the empirical density. You may also estimate the parameters and plot the empirical parametric density.
Hint: The density of the standard logistic distribution, logistic(0, 1), is $f(x|0,1) = f_0(x) = \dfrac{\exp(-x)}{\big(1+\exp(-x)\big)^2}$ and in general $f(x \mid \mu,\beta) = \dfrac{1}{\beta} f_0\Big(\dfrac{x-\mu}{\beta}\Big)$.

PROBLEM 6.5

The multinomial pmf with m categories and in total n trials is given by

$$\mathbb{P}(X_1 = x_1, \ldots, X_m = x_m) = \frac{n!}{x_1! \cdots x_m!} p_1^{x_1} \cdots p_m^{x_m} = \binom{n}{\mathbf{x}} \mathbf{p}^{\mathbf{x}}$$

with $\sum_{i=1}^{m} x_i = n$ and $x_i \in \mathbb{N}$. In each trial one of m possible categories is observed. The category k has success probability p_k and the indicator I_{jk} is equal to 1 iff trial number j results in category k. Otherwise I_{jk} is zero. The variable $\mathbf{X}$ counts the total number of successes that are observed in each category. Thus $\mathbf{X} = \sum_{j=1}^{n} \mathbf{I}_j$ with $\mathbf{I}_j = (I_{j1}, \ldots, I_{jm})^T$ and $\mathbf{X} = (X_1, \ldots, X_m)^T$. In particular $X_k = \sum_{j=1}^{n} I_{jk}$. Suppose that we are mainly interested in a subset of the m categories. That corresponds to divide $\mathbf{X}$ in two subvectors $(\mathbf{X}_{(1)}, \mathbf{X}_{(2)})$ and also $\mathbf{p} = (\mathbf{p}_{(1)}, \mathbf{p}_{(2)})$. The marginal pmf for $\mathbf{X}_{(1)}$ is

$$\mathbb{P}(\mathbf{X}_{(1)} = \mathbf{x}_{(1)}) = \mathbb{P}(\mathbf{X}_{(1)} = \mathbf{x}_{(1)}, Y = y) = \binom{n}{\mathbf{x}_{(1)}\, y} \mathbf{p}_{(1)}{}^{\mathbf{x}_{(1)}} |\mathbf{p}_{(2)}|^{y}$$

with $Y = n - |\mathbf{X}_{(1)}| = n - \sum_{k \in (1)} X_k$ and $y = n - |\mathbf{x}_{(1)}|$.

In the following $m = 4$ and $\mathbf{X}_{(1)} = (X_1, X_2)$.

a) Explain that $\mathbf{X}_{(1)}$ is multinomial $\Big(n, 3, (\mathbf{p}_{(1)}, |\mathbf{p}_{(2)}|)\Big)$.

b) Explain thay $I_{j1} I_{j2} \equiv 0$ and use this for computing the covariance between (I_{j1}, I_{j2}), and the covariance and correlation matrix for (X_1, X_2).

c) Find the conditional pmf of X_3 given (X_1, X_2).

A classical example of a multinomial distribution is the distribution of the four basic blood types, $\{\mathrm{O}, \mathrm{A}, \mathrm{B}, \mathrm{AB}\}$, in a population. For Norway the actual probabilities are $p_{\mathrm{O}} = 0.40$, $p_{\mathrm{A}} = 0.48$, $p_{\mathrm{B}} = 0.08$ and $p_{\mathrm{AB}} = 0.04$.

d) Write down the covariance and the correlation matrix for $(X_{\mathrm{O}}, X_{\mathrm{A}})$.

e) Let $n = 100$. What is the probability that $X_{\mathrm{AB}} = X_{\mathrm{B}}$ given that $X_{\mathrm{O}} = 40$ and $X_{\mathrm{A}} = 48$?

f) Simulate a full experiment with $n = 1000$ trials from the blood type distribution given above. Estimate the success probabilities, $\{\widehat{p}_{\mathrm{O}}, \widehat{p}_{\mathrm{A}}, \widehat{p}_{\mathrm{B}}, \widehat{p}_{\mathrm{AB}}\}$, from the sample.

g) Assume that we are only interested in the blood types B and AB . Reorganize the total sample for this situation and reestimate $\{\widehat{p}_{\mathrm{B}}, \widehat{p}_{\mathrm{AB}}\}$.

h) Suppose we wish estimate the conditional probability of AB given A and B . Reorganise the sample for this situation and estimate the conditional probability.

7 HOMEWORK 7 - March 13

Problem 7.1

[CB, E6.1, p. 258]

Let X be one observation from a $\mathcal{N}(0, \sigma^2)$ population. Is $|X|$ sufficient?

Problem 7.2

[CB, E6.2, p. 258]

Let $\{X_1, \ldots, X_n\}$ be independent random variables with densities[12]

$$f_{X_i}(x|\theta) = \exp\big(i\theta - x\big)1(x \geq i\theta).$$

Prove that $T = \bigwedge_i \left(\frac{X_i}{i}\right) = \min_i \left(\frac{X_i}{i}\right)$ is sufficient.

Hint: Write the product of indicators as a new indicator.

Problem 7.3

[CB, E6.3, p. 258]

Let $\{X_1, \ldots, X_n\}$ be a random sample from the exponential$(\sigma) + \mu$ distribution

$$f_{X_i}(x|\mu, \sigma) = \sigma^{-1} \exp\big(-\sigma^{-1}(x-\mu)\big)1(x > \mu).$$

Find a two-dimensional sufficient statistics for (μ, σ).

Problem 7.4

[CB, E6.4, p. 258]

Prove [Theorem 6.2.10, p 229 CB]; Let $\{X_1, \ldots, X_n\}$ be iid where X belongs to an exponential family,

$$f(x \mid \boldsymbol{\theta}) = h(x)c(\boldsymbol{\theta})\exp\Big(\sum_{i=1}^{k} w_i(\boldsymbol{\theta})T_{1i}(x)\Big)$$

where $\boldsymbol{\theta} \in \Theta \subset \Re^d$ and $d \leq k$. Prove that $T = (T_1, \ldots, T_k)$[13] is sufficient where $T_i = T_{ni} = \sum_{j=1}^{n} T_{1i}(X_j)$.

Hint: Just organize the simultaneous density.

Problem 7.5

[CB, E6.5, p. 258]

Let $\{X_1, \ldots, X_n\}$ be independent random variables with pdfs [14]

$$f_{X_i}(x|\theta) = \frac{1}{2i\theta}\, 1\big(-i(\theta-1) \leq x \leq i(\theta+1)\big), \quad \theta > 0.$$

[12] $X_i = \text{Exp}(1) + i\theta$

[13] Note the notation here, T_i is both a function and and a variable.

[14] $X_i = \text{uniform}(-i(\theta-1), i(\theta+1))$

Find a two-dimensional sufficient statistics for θ.

PROBLEM 7.6

[CB, E6.15, p. 258]

Let $\{X_1, \dots, X_n\}$ be a random sample from $\mathcal{N}(\theta, a\theta^2)$ where $a > 0$ is a constant.

i) Show that the parameter space does not contain a two-dimensional open set.
Hint: The parameter space is $\left\{(\mu, \sigma^2) = (\theta, a\theta^2), \theta \in \Re\right\}$.

ii) Show that the statistic $T = (\overline{X}_n, S^2)$ is a sufficient statistics for θ, but the family is not complete.
Hint: Use the definition of completeness and find a non-trivial function g so that $\mathbb{E}_\theta\, g(\overline{X}_n, S^2) \equiv 0$ and remember that a is known constant.

PROBLEM 7.7

[CB, E6.17, p. 258]

Let $\{X_1, \dots, X_n\}$ be iid with geometric distribution.

$$\mathbb{P}(X = x\,|\,\theta) = \theta(1-\theta)^{x-1}, \quad x = 1, 2, \dots, \theta \in (0, 1).$$

Show that $T = \sum_{i=1}^n X_i$ is sufficient for θ and find the family of distributions of T. Is this family complete?

Hint: [(3.2.9), p 95 CB], with $r = n$.

PROBLEM 7.8

[CB, E6.18, p. 258]

Let $\{X_1, \dots, X_n\}$ be iid Poisson(λ). Show that the family of distributions of $T = \sum X_i$ is complete. Prove completeness without using [Theorem 6.2.25, p 288 CB].

PROBLEM 7.9

[CB, E6.19, p. 258]

The random variable takes values $\{0, 1, 2\}$ according to one of the following distributions

	$P(X=0)$	$P(X=1)$	$P(X=2)$	Θ
Distribution 1	p	$3p$	$1-4p$	$(0, \frac{1}{4})$
Distribution 2	p	p^2	$1-\ p-p^2$	$(0, \frac{1}{2})$

In each case determine whether the family of distributions of X is complete.

Hint: A polynomial of degree r has at most r distinct roots.

PROBLEM 7.10

[CB, E6.20ab, p. 258]

For each of the following pdfs let $\{X_1, \dots, X_n\}$ be iid observations. Find a complete sufficient statistic, or show that one does not exists.

i) $f(x|\theta) = \dfrac{2x}{\theta^2}\, 1(0 < x < \theta)\,,\ \theta > 0$[15].

ii) $f(x|\theta) = \dfrac{\theta}{(1+x)^{1+\theta}}\quad 1(0 < x < \infty)\,,\ \theta > 0$[16].

[15] beta$(2, 1)$

[16] Pareto$(0, \theta)$

8 HOMEWORK 8 - March 20

Problem 8.1

[CB, E7.1, p. 258]

One observation is taken on a discrete random variable X with pmf $f(x|\theta)$ where $\theta \in \{1, 2, 3\}$. Find the MLE of θ.

x	$f(x\|1)$	$f(x\|2)$	$f(x\|3)$
0	$\frac{1}{3}$	$\frac{1}{4}$	0
1	$\frac{1}{3}$	$\frac{1}{4}$	0
2	0	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
4	$\frac{1}{6}$	0	$\frac{1}{4}$

Problem 8.2

[CB, E7.2, p. 258]

Let $X_1, \ldots, X_n$ be a random sample from Gamma(α, β) population. Find the MLE for β assuming α is known.

Problem 8.3

[CB, E7.6, p. 258]

Let $X_1, \ldots, X_n$ be a random sample from the pdf[17]

$$f(x|\theta) = \theta x^{-2} 1(\theta < x < \infty), \theta > 0.$$

i) What is a sufficient statistics for θ?

ii) Find the MLE of θ.

iii) Find the moment estimator for θ.

Problem 8.4

[CB, E7.7, p. 258]

Let $X_1, \ldots, X_n$ be iid from

$$f(x|\theta) = \begin{cases} 1(0 < x < 1), & \text{when } \theta = 0; \\ \dfrac{1}{2\sqrt{x}} 1(0 < x < 1), & \text{when } \theta = 1. \end{cases}$$

[17] Pareto$(\theta, 1)$

Find the MLE of θ.

PROBLEM 8.5

[CB, E7.10, p. 258]

The independent random variables $X_1, \dots, X_n$ have common distribution[18]

$$F(x \mid \alpha, \beta) = \left(\frac{x}{\beta}\right)^{\alpha} 1(0 < x < \beta)\,, \alpha, \beta > 0.$$

i) Find a two-dimensional sufficient statistics for (α, β).

ii) Find the MLE of (α, β).

PROBLEM 8.6

[CB, E7.14, p. 258]

Let X and Y be independent exponential random variables $\mathrm{Exp}(\lambda)$ and $\mathrm{Exp}(\mu)$, respectively. We observe Z and W where

$$Z = X \wedge Y\,, W = 1(Z = X).$$

Now assume that $\{(Z_i, W_i)\}$ is random sample from (Z, W). Find the MLE of (λ, μ). In Exercise 4.26 the joint distribution of Z and W was obtained:

iii) Z and W are independent.

iv) $Z \sim \mathrm{Exp}(\tau)$ where $\tau \stackrel{\text{def}}{=} \dfrac{\lambda\mu}{\mu + \lambda}$.

v) $p \stackrel{\text{def}}{=} P(W = 1) = P(X \le Y) = \dfrac{\mu}{\lambda + \mu}$.

Find the MLEs of λ and μ.

[18] A scaled beta distribution, e.g. $X = \beta\,\mathrm{beta}(\alpha, 1)$.

9 HOMEWORK 9 - April 03

Problem 9.1

[CB, E7.19, p. 358]

Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = x_i \beta + \epsilon_i \,, i = 1, \ldots, n,$$

where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are iid $\mathcal{N}(0, \sigma^2)$, σ^2 is unknown.

a) Find a two-dimensional sufficient statistics for (β, σ^2).
b) Find the MLE of β, and show that is is an unbiased estimator of β.
c) Find the distribution of β.

Problem 9.2

Exam 2015, Problem 1.

Let X be geometric(p) distributed,

$$f(x|p) = p(1-p)^{x-1}, \qquad x \geq 1, \quad p \in (0,1), \qquad \mu_X = p^{-1}, \;\; \sigma_X^2 = p^{-2}(1-p). \tag{6}$$

and suppose that we have a random sample $X_1, \ldots, X_n$ from (6). In the following you are supposed to perform standard estimation theory and results for this one-parameter population model.

a) Write down the simultaneous pmf, $f(\mathbf{x}|p)$, for $\mathbf{X} = (X_1, \ldots, X_n)$.
b) Show by the factorisation criterium that $S_n = \sum_{j=1}^n X_j$ is sufficient for p.
c) Derive the log likelihood equation and show that $\widehat{p} = \widehat{p}_{\mathrm{ML}} = n/S_n = 1/\overline{S}_n$ is the MLE for p.
d) Compute the population information matrix, $\mathbb{I}$, which is 1×1 and therefore a scalar here.
   Hint: You are done when you get $\mathbb{I} = p^{-2}(1-p)^{-1}$. Let $b(p) = \mathbb{E}\,\widehat{p} - p$ denote the bias. Formulate a lower bound for $\operatorname{Var}(\widehat{p})$ in terms of $b' = db/dp$ and $\mathbb{I}$.
   Hint: No calculations.
e) Explain with reference to general ML-theory that $\sqrt{n}(\widehat{p}_{\mathrm{ML}} - p) \xrightarrow[n]{d} \mathcal{N}(0, p(1-p))$ and find $p(1-p)$.
   Hint: Take a look at the previous point.
   We can write $\widehat{p}_{\mathrm{ML}} = g(\overline{S}_n)$ with $g(x) = x^{-1}$, Prove the convergence in distribution more directly by the delta method. Verify that σ_X^2 given by (6) holds by combining both metods.

Problem 9.3

[CB, E7.25, p. 358]
We examine generalisation of the hierarchical (Bayes) model considered in Example 7.2.16 and Exercise 7.22. Suppose that we observe $X_1, \dots, X_n$ where

$$X_i \mid \theta_i \sim \mathcal{N}(\theta_i, \sigma^2), i = 1, \dots, n, \text{independent},$$
$$\theta_i \sim \mathcal{N}(\mu, \tau^2), i = 1, \dots, n, \text{independent}.$$

a) Show that the marginal distribution of X_i is $\mathcal{N}(\mu, \sigma^2 + \tau^2)$ and that, marginally, $X_1, \dots, X_n$ are iid. [19] [20]

b) Show, in general, that if

$$X_i \mid \theta_i \sim f(x|\theta_i), i = 1, \dots, n, \text{independent},$$
$$\theta_i \sim \pi(\theta|\tau), i = 1, \dots, n, \text{independent}.$$

then marginally $X_1, \dots, X_n$ are iid.

Problem 9.4
[CB, E7.39, p. 362]
For each of the following distributions, let $X_1, \dots, X_n$ be a random sample. Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao lower bound? If so find it. If not, show why not.

a) $f(x|\theta) = \theta x^{\theta-1} 1(0 < x < 1), \theta > 0.$

b) $f(x|\theta) = \dfrac{\log(\theta)}{\theta - 1}\theta^x 1(0 < x < 1), \theta > 1.$

Problem 9.5
[CB, E7.40, p. 362]
Let $X_1, \dots, X_n$ be iid Bernoulli(p). Show that variance of $\overline{X}_n$ attains the Cramér Rao lower bound, and hence $\overline{X}_n$ is an UMVE of p.

Problem 9.6
[CB, E7.41, p. 363]
Let $X_1, \dots, X_n$ be a random sample from a population with mean μ and variance σ^2.

a) Show that the estimator $\sum_i a_i X_i$ is an unbiased estimator of μ if $\sum_i a_i = 1$.

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[19] Empirical Bayes analysis would use the marginal distribution of the X_is to estimate the prior parameters μ and σ^2. See Miscellanea.

[20] A frequentist interpretation of this situation is that the θ_is are latent, i.e. unobserved, variables and the model could be written as

$$X_i = \theta_i + \sigma Z_i,$$
$$\theta_i = \mu + \tau V_i$$

where $\{Z_i\}$ and $\{V_i\}$ are independent $\mathcal{N}(0,1)$.

b) Among all unbiased estimators of this form (called the linear unbiased estimators) find the one with minimum variance and calculate the variance.

PROBLEM 9.7
[CB, E7.49, p. 363]
Let $X_1, \dots, X_n$ be iid $\text{Exp}(\lambda)$.

a) Find a unbiased estimator of λ based only on $Y = \bigwedge X_i$[21].

b) Find a better estimator than the one in a). Prove that it is better.

.

[21]You can solve this by a direct calculation based on $\{\wedge X_i > x\} = \cap_i \{X_i > x\}$.

[21]Empirical Bayes analysis would use the marginal distribution of the X_i's to estimate the prior parameters μ and σ^2. See Miscellanea.

[21]A frequentist interpretation of this situation is that the θ_is are latent, i.e. unobserved, variables and the model could be written as $X_i = \theta_i + \sigma Z_i$, $\theta_i = \mu + \tau V_i$ where $\{Z_i\}$ and $\{V_i\}$ are independent $\mathcal{N}(0, 1)$.

[21]You can solve this by a direct calculation based on $\{\wedge X_i > x\} = \cap_i \{X_i > x\}$.

10 HOMEWORK 10 - April 10

Problem 10.1

[CB, E7.55ab, p. 365]

For each of the following pdfs, let $X_1, \ldots, X_n$ be a sample from that distribution. In each case find the best unbiased estimator of θ^r [Guenther 1978].

i) $f(x\,|\,\theta) = \dfrac{1}{\theta}\,1(0 < x < \theta)$ and $r < n$.

ii) $f(x\,|\,\theta) = \exp\big(-(x-\theta)\big)\,1(\theta < x < \infty)$.

Problem 10.2

[CB, E7.57, p. 365]

Let $X_1, \ldots, X_{n+1}$ be sample from $B(1,p)$ and $S_n = \sum_{i=1}^n X_i$. Define the function $h(p) = P\big(S_n > X_{n+1}\big)$ which is the probability that the sum of the first n variables exceeds the $(n+1)$st.

i) Show that $T = 1\big(S_n > X_{n+1}\big)$ is an unbiased estimator of $h(p)$.

ii) Find the UMVUE of $h(p)$.

Problem 10.3

[CB, E7.59, p. 365]

Let $X_1, \ldots, X_n$ be a random sample form $\mathcal{N}(\mu, \sigma^2)$. Find the best unbiased estimator of σ^p where $p > 0$ is a known real number.

Problem 10.4

[CB, E7.60, p. 365]

Let $X_1, \ldots, X_n$ be a random sample form $\text{Gamma}(\alpha, \beta)$ where α is known.

i) Find a unbiased estimator for β.

ii) Let $\alpha = 1$ and find the UMVUE for $\tau(\beta) \stackrel{\text{def}}{=} P(X > x_0) = \exp\big(-\dfrac{x_0}{\beta}\big)$.

11 HOMEWORK 11 - April 17

Problem 11.1

[CB, E10.3, p. 365]

A random sample $X_1, \ldots, X_n$ is drawn from a population that is $\mathcal{N}(\theta, \theta)$ where $\theta > 0$.

i) Show that the MLE of θ is the root of the quadratic equation

$$\theta^2 + \theta - W = 0$$

where $W = n^{-1} \sum_{j=1}^{n} X_i^2$. Determine which root equals the MLE.

ii) Find the asymptotic variance of $\widehat{\theta}_{\text{ML}}$ using the information matrix.

Problem 11.2

[CB, E10.9, p. 505]

Suppose that $X_1, \ldots, X_n$ are iid $\text{Poisson}(\lambda)$. Find the best unbiased estimator of

ii) $\tau \stackrel{\text{def}}{=} P(X = 1) = \lambda e^{-\lambda}$.

Problem 11.3

[CB, E8.1, p. 402]

In 1000 tosses of a coin, 560 heads and 440 tails appear. Is it reasonable to assume that the coin is fair? Justify your answer.

Problem 11.4

[CB, E8.3, p. 402]

Here the LRT alluded in Example 8. 2.9 will be derived. Suppose that we observe iid $\text{Bernoulli}(\theta)$ random variables, denoted by $I_1, \ldots, I_n$. Show that the LRT of

$$H_0 \colon \theta \le \theta_0 \text{ versus } H_1 \colon \theta > \theta_0$$

will reject if $X = \sum_{j=1}^{n} I_j > b$.

Problem 11.5

[CB, E8.5, p. 402]

A random sample, $X_1, \ldots, X_n$, is drawn form the Pareto population with pdf

$$f(x \,|\, \theta, \nu) = \frac{\theta \nu^{\theta}}{x^{\theta+1}} 1(x \ge \nu), \theta > 0, \nu > 0.$$

i) Find the MLE of (θ, ν).

ii) Show that the LRT of

$$H_0 \colon \theta = 1 \text{ versus } H_1 \colon \theta \ne 1$$

has critical region of the form $\{T \le c_1\} \cup \{T \ge c_2\}$ where

$$T = \sum_i \log X_i - n \log \big(\bigwedge_i X_i \big).$$

for some constants $0 < c_1 < c_2 < \infty$.

12 HOMEWORK 12 - April 24

Problem 12.1

[CB, E8,18, p. 405]

Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{N}(\theta, \sigma^2)$ population, σ^2 known. An LRT of $H_0\colon \theta = \theta_0$ versus $H_1\colon \theta \neq \theta_0$ is a test that rejects H_0 if $\left|\overline{X}_n - \theta_0\right|/(\sigma/\sqrt{n}) > c$.

i) Find an expression, in terms of standard normal probabilities, for the power function of this test.

ii) The experimenter desires a Type I Error probability of 5% and a maximum Type II Error probability of 25% at $\theta = \theta_0 + \sigma$. Find the values of n and c that will achieve this.

Problem 12.2

[CB, E8.19, p. 405]

The random variable X has pdf $\exp\bigl(-x\bigr)1(x > 0)$. One observation is obtained on the random variable $Y = X^\theta$, and a test $H_0\colon \theta = 1$ versus $H_1\colon \theta = 2$ needs to be constructed. Find the UMP level $\alpha = 10\%$ test an compute the Type II Error probability.

Problem 12.3

[CB, E8.25, p. 405]

Show that each of the following families has an MLR.

i) $\mathcal{N}(\theta, \sigma^2)$ family with σ^2 known.

ii) Poisson(θ) family.

iii) binomial(n, θ) family with n known.

Problem 12.4

[CB, E8.27, p. 405]

Suppose that $g(t|\theta) = h(t)c(\theta)\exp\bigl(w(\theta)t\bigr)$ is a one-parameter exponential family for the random variabel T. Show that this family has an MLR if $w(\theta)$ is an increasing function of θ. Give three examples of such a family.

Problem 12.5

[CB, E8.28, p. 405]

Let $f(x|\theta)$ be the logistic location pdf

$$f(x|\theta) = \frac{\exp\bigl(x-\theta\bigr)}{\Bigl(1+\exp\bigl(x-\theta\bigr)\Bigr)^2}\,, x \in R, \quad \theta \in R.$$

i) Show that this family has a MLR.

ii) Based on one observation, X, find the most powerful size α test of $H_0\colon \theta = 0$ versus $H_1\colon \theta = 1$. For α =20%, find the size of the Type II Error.

iii) Show that the test in b) is UMP for testing $H_0\colon \theta \leq 0$ versus $H_1\colon \theta > 0$. What can be said about UMP tests in general for the logistic location family?

13 HOMEWORK 13 - May 08

PROBLEM 13.1

[CB, E8.6, p. 505]

Suppose that we have two independent random samples; $X_1, \ldots, X_n$ are $\text{Exp}(\lambda)$ and $Y_1, \ldots, Y_n$ are $\text{Exp}(\mu)$.

i) Find the LRT of $H_0\colon \theta = \mu$ versus $H_1\colon \theta \neq \mu$.

ii) Show that a) can be based on

$$T = \frac{\sum_i X_i}{\sum_i X_i + \sum_i Y_i}.$$

iii) Find the distribution of T under the null hypothesis.

PROBLEM 13.2

[CB, E8.7, p. 505]

We have already seen the usefulness of the LRT in dealing with problems with nuisance parameters. We now look at some other nuisance problems.

i) Find the LRT of $H_0\colon \theta \le 0$ versus $H_1\colon \theta > 0$ based on a sample $X_1, \ldots, X_n$ from a population with pdf[22]

$$f(x|\theta) = \lambda^{-1}\exp\bigl(-\lambda^{-1}(x-\theta)\bigr)1(x>\theta),$$
$$\lambda > 0, \theta \in \Re$$

where both λ and θ are unknown parameters.

ii) We have already seen that $\text{Exp}(\lambda) = \text{Gamma}(1,\lambda)$. Generalizating in another way, $\text{Exp}(\lambda) = \text{Weibull}(1,\lambda)$. Suppose that $X_1, \ldots, X_n$ is a random sample from $\text{Weibull}(\gamma,\lambda)$. Find the LRT of $H_0\colon \gamma = 1$ versus $H_1\colon \gamma \neq 1$.

PROBLEM 13.3

[CB, E8.8, p. 505]

A special case of a normal family is one in which the mean and the variance are related, the $\mathcal{N}(\theta, a\theta)$ family. If we are interested in testing this relationship regardless of the value of θ, we are again faced with a nuisance parameter problem.

i) Find the LRT of $H_0\colon a = 1$ versus $H_1\colon a \neq 1$ based on a sample $X_1, \ldots, X_n$ from $\mathcal{N}(\theta, a\theta)$ family where θ is unknown.

[22] $X \sim \theta + \text{Exp}(\lambda)$

14 HOMEWORK 14 - MAY 15

PROBLEM 14.1

[CB, E8.31, p. 505]

Let $X_1, \ldots, X_n$ be iid Poisson(λ).

i) Find a UMP for $H_0\colon \lambda \leq \lambda_0$ versus $H_1\colon \lambda > \lambda_0$.

ii) Consider the specific case $H_0\colon \lambda \leq 1$ versus $H_1\colon \lambda > 1$. Use the CLT to determine the sample size n so that a UMP test satisfies $P(\text{reject } H_0 \,|\, \lambda = 1) = 5\%$ and $P(\text{reject } H_0 \,|\, \lambda = 2) = 90\%$.

PROBLEM 14.2

[CB, E8.37, p. 505]

Let $X_1, \ldots, X_n$ be a random sample from $\mathcal{N}(\theta, \sigma^2)$ population. Consider

$$H_0\colon \theta \leq \theta_0 \text{ versus } H_1\colon \theta > \theta_0.$$

iii) If σ^2 is unknown, show that the that rejects H_0 when[23]

$$\overline{X}_n > \theta_0 + t_{n.1,\alpha} \frac{S}{\sqrt{n}}$$

is a test of size α. Show that the test can be derived as an LRT.

PROBLEM 14.3

[CB, E10.34, p. 505]

For testing $H_0\colon p = p_0$ versus $H_1\colon p \neq p_0$, suppose that we observe $I_1, \ldots, I_n$ iid Bernoulli(p). Derive an expression for $-2 \ln \lambda$ where λ is the LRT statistics.

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[23] This is the one-sided one-sample t-test.