Mandatory homework 4 - Stat 221 - V19

February 07, 2019

Due at the end of Tuesday February 19

Problem 4.1

Consider an ARMA(p,q) model,

$$X_t - \sum_{k=1}^{p} \phi_k X_{t-k} = Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j} \tag{1}$$

where the autoregressive- and moving average polynomial have no common roots.

The model is causal iff all the roots of the autoregressive polynomial $\phi(z)$ has modulus strictly larger than one. In that case $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$ where

$$\psi_j = \sum_{k=1}^{p} \phi_k \psi_{j-k} + \theta_j, \quad j \geq 0, \quad \theta_0 = 1 \tag{2}$$

with $\phi_k \equiv 0$ for $k \notin [1, p]$, and $\theta_j \equiv 0$ for $j \notin [0, q]$,

a) What is meant by invertibility for this model? Formulate a necessary and sufficient condition for the model to be invertible.

b) Find an analoguous structure to (2) for the $\{\pi_j\}$ when the model is invertible.

Problem 4.2

Consider a causal ARMA(2,3) given by (1) where the linear representation satisfies (2).

a) Find $\{\psi_j, \, j = 0, 1, 2\}$ in terms of the model parameters and the general second order homogenous difference equation so that all ψ_j's can be calculated.

Let

$$\boldsymbol{\phi} = (1.7, -0.9), \quad \boldsymbol{\theta} = (-1.4, 0.8, 0.1), \quad \sigma^2 = 1. \tag{3}$$

b) Check that (3) defines a causal and invertible ARMA(p,q) model.

c) Use R and plot $\{\psi_j, \, j = 0, \ldots, 50\}$ when the paramteres are given by (3).

Problem 4.3

Consider a causal ARMA(p,q). Then

$$\gamma(h) = \sum_{k=1}^{p} \phi_k \gamma(h-k) + \sigma^2 \sum_{j=0}^{q} \theta_{j+h} \psi_j, \quad h \geq 0. \tag{4}$$

a) For an ARMA(2,3), find explicit expressions for $\{\gamma(h),\ h = 0, \dots, 4\}$ in terms of the model parameters and formulate them in matrix notation.

b) Complete the discription with the homogenuous difference equation $\phi(B)\gamma(h) = 0$ for $h \geq 4$.

c) Implement the results in R compute and plot$\{\gamma(h),\ h = 0, \dots, 50\}$ with parameter values given by (3). Check your computations with help of an R-function.

PROBLEM 4.4

Let $\{X_t\}$ be a causal AR(2) process with white noise process $\mathrm{WN}(0, \sigma^2)$,

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-1} = Z_t, \qquad X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}. \tag{5}$$

Then by (4),

$$\gamma(h) = \sum_{k=1}^{p} \phi_k \gamma(h-k) + \delta_{0,h}\sigma^2, \quad h \geq 0, \quad \delta_{h,0} = \begin{cases} 1, & \text{h=0;} \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

a) Multiply (5) with X_{t-h} for $h = 0, 1, 2$, take the expectation of equations and deduce (6) without reference to (4).

b) Devide the equations by $\gamma(0)$ and verfiy that for this model,

$$\begin{aligned} (1-\phi_2)\rho(1) &= \phi_1, \\ -\phi_1\rho(1) + \rho(2) &= \phi_2, \\ \gamma(0)(1 - \phi_1\rho(1) - \phi_2\rho(2)) &= \sigma^2. \end{aligned} \tag{7}$$

c) Solve the two first equations above with respect to $\rho(1), \rho(2)$ and then find a formula for $\gamma(0)$.

d) Argue from (7) that the following boundaries on the parameters are necessary for a causal model:

$$\begin{aligned} \phi_2 &= 1, \\ \phi_2 - \phi_1 &= 1, \\ \phi_2 + \phi_1 &= 1. \end{aligned}$$

e) Find $\mathbb{E}[X_3|X_1]$.

f) The asymptotic covariance matrix for the least square estimator of $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$ is $\sigma^2\Gamma_2^{-1}$. Compare the asymptotic variance for $\widehat{\phi}_1$, when the estimated and true model is an AR(1), with the corresponding asymptotic variance for the estimator of $\widehat{\phi}_1$ in the AR(2) model when $\phi_2 = 0$.