STAT211 Mandatory Homework 6

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Contents

1 Problem 6.2

1.1 Part a: Polynomial roots of moving average model

The model

$$X_t = Z_t + Z_{t-2} \tag{1}$$

can be rewritten as

$$X_t = Z_t + 0Z_{t-1} + 1Z_{t-2}, \tag{2}$$

and the corresponding moving average polynomial is

$$\theta(z) = 1 + 0z + 1.z^2 = 1 + z^2, \tag{3}$$

whose roots are

$$z_1 = i, \quad z_2 = -i \tag{4}$$

1.2 Part b: Polynomial roots of moving average model

The corresponding moving average polynomial for the model

$$X_t = Z_t - 2\cos(w)Z_{t-1} + Z_{t-2} \tag{5}$$

is

$$\begin{aligned}
\theta(z) &= 1 - 2\cos(w)z + z^2 \\
&= (z - \cos(w))^2 - \cos(w)^2 + 1 \\
&= (z - \cos(w))^2 - (\cos(w)^2 - 1) \\
&(z - \cos(w))^2 - (-\sin(w)^2) \\
&(z - \cos(w))^2 - (i^2\sin(w)^2) \\
&(z - \cos(w))^2 - (i\sin(w))^2 \\
&(z - \cos(w) + i\sin(w))(z - \cos(w) - i\sin(w))
\end{aligned} \tag{6}$$

whose roots are

$$z_1 = \cos(w) - i\sin(w), \quad z_2 = \cos(w) + i\sin(w) \tag{7}$$

2 Problem 6.3

Consider a causal AR(2) model with

$$\{Z_t\} \sim WN(0, \sigma^2) \tag{8}$$

The two step predictor $\widehat{X}_{n+2}$ is defined by $\mathcal{P}_n(X_{n+2})$. From [1], page 65 property 1, we have

$$\mathcal{P}_n(X_{n+h}) = \sum_{i=1}^{n} a_i X_{n+1-i} \tag{9}$$

where the a_i satisfy

$$\Gamma_n \mathbf{a}_n = \gamma_n(h), \quad \text{equation 2.5.7 from [1],} \tag{10}$$

where

$$\mathbf{a}_n = (a_1, \cdots, a_n) \tag{11}$$

$$\Gamma_n = [\gamma(i-j)]_{i,j=0}^{n} \tag{12}$$

and

$$\gamma_n(h) = (\gamma(h), \gamma(h+1), \cdots, \gamma(h+n-1) \tag{13}$$

$$\gamma(h) = \mathrm{Cov}(X_{t+h}, X_t). \tag{14}$$

To compute $\mathcal{P}_n(X_{n+2})$, we set $h = 2$ in equation (9) and compute the coefficient a_i by solving

$$\Gamma_n \mathbf{a}_n = \gamma_n(2) \tag{15}$$

or more generally

$$\underbrace{\begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n) & \gamma(n-1) & \cdots & \gamma(0) \end{pmatrix}}_{\Gamma_n} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{\mathbf{a}_n} = \underbrace{\begin{pmatrix} \gamma(2) \\ \gamma(3) \\ \vdots \\ \gamma(n+1) \end{pmatrix}}_{\gamma_n(2)} \tag{16}$$

To evaluate $\gamma(h)$, we note that the process is causal, which leads to

$$X_t = Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2} \tag{17}$$

and

$$\begin{aligned} \gamma(h) &= \mathrm{Cov}(X_{t+h}, X_t) \\ &= \mathrm{Cov}(Z_{t+h} + \psi_1 Z_{t+h-1} + \psi_2 Z_{t+h-2}, Z_t + \psi_1 Z_{t-1} + \psi_2 Z_{t-2}) \\ &= \mathrm{Cov}(Z_{t+h}, Z_t) + \psi_1 \mathrm{Cov}(Z_{t+h}, Z_{t-1}) + \psi_2 \mathrm{Cov}(Z_{t+h}, Z_{t-2}) \\ &+ \psi_1 \mathrm{Cov}(Z_{t+h-1}, Z_t) + \psi_1^2 \mathrm{Cov}(Z_{t+h-1}, Z_{t-1}) + \psi_1\psi_2 \mathrm{Cov}(Z_{t+h-1}, Z_{t-2}) \\ &+ \psi_2 \mathrm{Cov}(Z_{t+h-2}, Z_t) + \psi_2\psi_1 \mathrm{Cov}(Z_{t+h-2}, Z_{t-1}) + \psi_2^2 \mathrm{Cov}(Z_{t+h-1}, Z_{t-2}) \\ &= \sigma^2(\delta_{h,0} + \psi_1\delta_{h,-1} + \psi_2\delta_{h,-2} + \psi_1\delta_{h,1} + \psi_1^2\delta_{h,0} + \psi_1\psi_2\delta_{h,-1} + \psi_2\delta_{h,2} + \psi_2\psi_1\delta_{h,1} + \psi_2^2\delta_{h,-1}) \end{aligned} \tag{18}$$

From which we get

$$\begin{aligned}
\gamma(0) &= \sigma^2(1+\psi_1^2) \\
\gamma(1) &= \sigma^2(\psi_1+\psi_1\psi_2) \\
\gamma(2) &= \sigma^2\psi_2 \\
\gamma(n) &= 0, \quad \text{for n} \geq 3
\end{aligned} \tag{19}$$

Now the variance of $\mathcal{P}_n(X_{n+2})$ is given by

$$\begin{aligned}
\mathrm{Var}(\mathcal{P}_n(X_{n+2})) &= \mathrm{Var}\left(\sum_{i=1}^{n} a_i X_{n+1-i}\right) \\
&= \sum_{i,j=1}^{n} a_i a_j \,\mathrm{Cov}(X_{n+1-i}, X_{n+1-j})
\end{aligned} \tag{20}$$

3 Problem 6.4

Let $\{X_t\}$ be a stationary and linear causal time series with white noise process $\{Z_t\} \sim WN(0,\sigma^2)$. Let $\mathcal{P}_n$ be the projection onto $\{X_1,\cdots,X_n\}$. Let compute $\widehat{Z}_{n+1} = \mathcal{P}_n(Z_{n+1})$ and $\widehat{Z}_n = \mathcal{P}_n(Z_n)$ Since $\{X_t\}$ is linear we can write

$$X_t = \sum_{j=0}^{n} \psi_j Z_{t-j}. \tag{21}$$

From Problem 6.3 equation (9) we have

$$\mathcal{P}_n(Z_{n+1}) = \sum_{i=1}^{n} a_i Z_{n+1-i} \tag{22}$$

and

$$\mathcal{P}_n(Z_n) = \sum_{i=1}^{n} b_i Z_{n-i} \tag{23}$$

where the a_i and b_i are solution of

$$\Gamma_n \mathbf{a}_n = \gamma_n(1) \tag{24}$$

$$\Gamma_n \mathbf{b}_n = \gamma_n(0) \tag{25}$$

respectively. Or

$$\mathbf{a}_n = \Gamma_n^{-1}\gamma_n(1) \tag{26}$$

$$\mathbf{b}_n = \Gamma_n^{-1}\gamma_n(0) \tag{27}$$

$$\underbrace{\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}}_{\mathbf{a}_n} = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n) & \gamma(n-1) & \cdots & \gamma(0) \end{pmatrix}^{-1} \underbrace{\begin{pmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(n+1) \end{pmatrix}}_{\gamma_n(1)} \tag{28}$$

$$\underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\mathbf{b}_n} = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n) & \gamma(n-1) & \cdots & \gamma(0) \end{pmatrix}^{-1} \underbrace{\begin{pmatrix} \gamma(1) \\ \gamma(2) \\ \vdots \\ \gamma(n+1) \end{pmatrix}}_{\gamma_n(0)} \tag{29}$$

4 Problem 6.5

let $\{Z_n\} \sim WN(0,1)$ and

$$X_t = Z_t - Z_{t-1} \tag{30}$$

4.1 Part a

The corresponding polynomial of model (35) is

$$\theta(z) = 1 - z, \tag{31}$$

whose root is

$$z = 1 \tag{32}$$

The root is on the unit circle. Since the root is not outside the unit circle, the process is not invertible.

4.2 Part b

Let try to create different representation of the model in equation (35). (35) we have

$$X_{t-1} = Z_{t-1} - Z_{t-2} \Rightarrow Z_{t-1} = X_{t-1} + Z_{t-2} \tag{33}$$

and inserting the last expression back into (35) we get

$$X_t + x_{t-1} = Z_t - Z_{t-2}. \tag{34}$$

If we repeat the same process we get the following representation.

$$X_t + X_{t-1} + X_{t-2} = Z_t - Z_{t-3}. \tag{35}$$

which is an ARMA(2,3), and the autoregressive polynomial

$$1 + z + z^2 = \left(z + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(z + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \tag{36}$$

has root on the unit circle. So It seams like through this process, it is not possible to create a representation which is invertible. But We also know that for each covariance function of a MA(q) process, there exists one set of coefficients $d_1, \cdots, d_q$ such that the process is invertible. If we find this coefficients then we can have an invertible representation.

4.3 Part c

using DL to find $\widehat{X}_{n+1}$ for $n = 1, 2, 3..$ The Durbin-Levinson recursion gives the coefficients of $X_n, \cdots, X_1$ in the following representation [1],

$$\widehat{X}_{n+1} = \sum_{j=1}^{n} \phi_{nj} X_{n+1-j}. \tag{37}$$

We compute $\gamma(h)$ as follow:

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}(X_{t+h}, X_t) \\
&= \mathrm{Cov}(Z_{t+h} - Z_{t+h-1}, Z_t - Z_{t-1}) \\
&= \mathrm{Cov}(Z_{t+h}, Z_t) - \mathrm{Cov}(Z_{t+h}, Z_{t-1}) - \mathrm{Cov}(Z_{t+h-1}, Z_t) + \mathrm{Cov}(Z_{t+h-1}, Z_{t-1}) \\
&= \sigma^2(\delta_{h,0} - \delta_{h,-1} - \delta_{h,1} + \delta_{h,0}) \\
&= (\delta_{h,0} - \delta_{h,-1} - \delta_{h,1} + \delta_{h,0})
\end{aligned} \tag{38}$$

and

$$\phi_{11} = \frac{\gamma(1)}{\gamma(0)} = -\frac{1}{2} \tag{39}$$

For $n = 1$

$$\widehat{X}_2 = \phi_{11} X_1 = -\frac{1}{2} X_1 \tag{40}$$

For n = 2

$$\begin{aligned}
\widehat{X}_3 &= \sum_{j=1}^{2} \phi_{2j} X_{3-j} \\
&= \phi_{21} X_2 + \phi_{22} X_1
\end{aligned} \tag{41}$$

where

$$\begin{aligned}\phi_{22} &= \frac{\gamma(2) - \phi_{11}\gamma(1)}{\nu_1} \\ &= \frac{-\phi_{11}\gamma(1)}{\nu_1}\end{aligned} \tag{42}$$

and

$$\begin{aligned}\nu_1 &= \nu_0(1 - \phi_{11}^2) \\ &= \gamma(0)\left(1 - \left(\frac{\gamma(1)}{\gamma(0)}\right)^2\right) \\ &= 2\left(1 - \frac{1}{4}\right) \\ &= \frac{3}{2}\end{aligned} \tag{43}$$

so

$$\begin{aligned}\phi_{22} &= \frac{-\phi_{11}\gamma(1)}{\nu_1} \\ &= -\frac{1}{3}\end{aligned} \tag{44}$$

$$\begin{aligned}\phi_{21} &= \phi_{11} - \phi_{22}\phi_{11} \\ &= -\frac{1}{2} - \frac{1}{3}\frac{1}{2} \\ &= -\frac{2}{3}\end{aligned} \tag{45}$$

$$\begin{aligned}\widehat{X}_3 &= \phi_{21}X_2 + \phi_{22}X_1 \\ &= -\frac{2}{3}X_2 - \frac{1}{3}X_1\end{aligned} \tag{46}$$

for n = 3

4.4 Part d

let prove that

$$\widehat{X}_{n+1} = -\sum_{j=1}^{n} \frac{n+1-j}{n+1} X_{n+1-j}. \tag{47}$$

We use induction. Thats is we show that its true for $n = 1$, then we show that it is true for $n + 1$. Let go: for $n = 1$

$$\begin{aligned}
\widehat{X}_2 &= -\frac{1}{2}X_1 \\
&= -\frac{1+1-1}{1+1}X_{1+1-1} \\
&= -\sum_{j=1}^{n=1}\frac{n+1-j}{n+1}X_{n+1-j}
\end{aligned} \tag{48}$$

Now let j run from 1 to $n + 1$. Then we have

$$\begin{aligned}
\widehat{X}_{n+1} &= -\sum_{j=1}^{n+1}\frac{n+1-j}{n+1}X_{n+1-j} \\
&= -\sum_{j=1}^{n}\frac{n+1-j}{n+1}X_{n+1-j} - \sum_{j=n+1}^{n+1}\frac{n+1-j}{n+1}X_{n+1-j} \\
&= -\sum_{j=1}^{n}\frac{n+1-j}{n+1}X_{n+1-j} - \underbrace{\frac{n+1-(n+1)}{n+1}}_{=0}X_{n+1-(n+1)} \\
&= -\sum_{j=1}^{n}\frac{n+1-j}{n+1}X_{n+1-j}
\end{aligned} \tag{49}$$

4.5 Part e

Let prove that

$$||Z_n - (-\widehat{X}_{n+1})|| = O(1). \tag{50}$$

Now

$$\left|\left|Z_n - \sum_{j=1}^{n}\frac{n+1-j}{n+1}X_{n+1-j}\right|\right| = \left|\left|Z_n - \left(\frac{n}{n+1}X_n + \cdots + \frac{2}{n+1}X_2 + \frac{1}{n+1}X_1\right)\right|\right| \tag{51}$$

Now as n goes to ∞, $\left(\frac{n}{n+1}X_n + \cdots + \frac{1}{n+1}X_1\right) = \left(\frac{1}{1+\frac{1}{n}}X_n + \cdots + \frac{1}{n+1}X_1\right)$ goe to X_n so $\left|\left|Z_n - \sum_{j=1}^{n}\frac{n+1-j}{n+1}X_{n+1-j}\right|\right|$ goes to $||Z_n - X_n|| = ||Z_{n-1}||$ which goes to 0

so that Z_n can be written as a linear combination of X_s so we can conclude that $Z_t \in spanX_s$

5 Problem 6.6

Let $\mathcal{P}_k$ be the linear projection onto

$$\mathbf{S}_k = span\{X_1, \cdots, X_k\} \tag{52}$$

and

$$e_k = \frac{X_k - \hat{X}_k}{\nu_{k-1}}. \tag{53}$$

$\{e_1, \cdots, e_n\}$ is orthonormal basis for $\mathbf{S}_n$ if $\{e_1, \cdots, e_n\}$ is a linearly independent subset of $\mathbf{S}_n$ that span $\mathbf{S}_n$, and for any e_j, e_i in $\{e_1, \cdots, e_n\}$ the inner product of e_j and e_i is zero and any e_i as norm 1.

Proof.

- Linearly independence. Assume that

$$a_1 e_1 + \cdots + a_n e_n = 0 \tag{54}$$

where a_i are real numbers. The we have

$$\begin{aligned} a_1 e_1 + \cdots + a_n e_n &= 0 \\ a_1 \frac{X_1 - \hat{X}_1}{\nu_0} + \cdots + a_n \frac{X_n - \hat{X}_n}{\nu_{n-1}} &= 0 \\ \frac{a_1}{\nu_0}(X_1 - \hat{X}_1) + \cdots + \frac{a_n}{\nu_{n-1}}(X_n - \hat{X}_n) &= 0 \end{aligned} \tag{55}$$

From (53) we know that

$$X_k - \hat{X}_k = e_k \nu_{k-1}. \tag{56}$$

Thus

$$X_1 - \hat{X}_1 \neq 0, \cdots, X_n - \hat{X}_n \neq 0 \tag{57}$$

Therefor the last expression in equation (55) is true if

$$\frac{a_1}{\nu_0} = \cdots = \frac{a_n}{\nu_{n-1}} = 0 \tag{58}$$

equivalently

$$a_1 = \cdots = a_n = 0 \tag{59}$$

This means that $\{e_1, \cdots, e_n\}$ is a linearly independent

- $\{e_1, \cdots, e_n\}$ span $\mathbf{S}_n$. We want to show that any vector in $\mathbf{S}_n$ can be written as a linear combination of $\{e_1, \cdots, e_n\}$. Let $Z \in \mathbf{S}_n$. Since $\mathbf{S}_n = span\{X_1, \cdots, X_n\}$, we have

$$Z = b_1 X_1 + \cdots + b_n X_n \tag{60}$$

where b_i are real numbers. Then we have

$$\begin{aligned}
Z &= b_1X_1 + \cdots + b_nX_n \\
Z &= b_1(\nu_0 e_1 + \hat{X}_1) + \cdots + b_n(\nu_{n-1}e_n + \hat{X}_n) \\
Z &= b_1\nu_0 e_1 + \cdots + b_n\nu_{n-1}e_n + \underbrace{b_1\hat{X}_1 + \cdots + b_n\hat{X}_n}_{Z'} \\
\underbrace{Z - Z'}_{Z''} &= \underbrace{b_1\nu_0}_{\alpha_1} e_1 + \cdots + \underbrace{b_n\nu_{n-1}}_{\alpha_n} e_n
\end{aligned} \tag{61}$$

Since $Z'' \in \mathbf{S}_n$ We have

$$Z'' = \alpha_1 e_1 + \cdots + \alpha_n e_n \tag{62}$$

- Horthogonality Let e_r, e_r be two arbitrarily vectors in $\{e_1, \cdots, e_n\}$ such that $r \neq s$.

$$\begin{aligned}
\langle\, e_r, e_s\rangle &= \left\langle \frac{X_r - \hat{X}_r}{\nu_{r-1}}, \frac{X_s - \hat{X}_s}{\nu_{s-1}} \right\rangle \\
&= \frac{1}{\nu_{r-1}\nu_{i-s}} \left\langle X_r - \hat{X}_r, X_s - \hat{X}_s \right\rangle
\end{aligned} \tag{63}$$

From the innovation algorithm [1], the coefficient of $X_n - \hat{X}_n, \cdots, X_1 - \hat{X}_1$ are of the form

$$\theta_{n,n-k}, \quad k = 0, \cdots, n. \tag{64}$$

so that

$$\begin{aligned}
\langle\, e_r, e_s\rangle &= \left\langle \frac{X_r - \hat{X}_r}{\nu_{r-1}}, \frac{X_s - \hat{X}_s}{\nu_{s-1}} \right\rangle \\
&= \frac{1}{\nu_{r-1}\nu_{i-s}} \left\langle X_r - \hat{X}_r, X_s - \hat{X}_s \right\rangle \\
&= \frac{\theta_{r,r-k}\theta_{s,s-k}}{\nu_{r-1}\nu_{s-1}}
\end{aligned} \tag{65}$$

And from [1] equation 2.5.26, for $r \neq s$ we get

$$\theta_{r,r-k}\theta_{s,s-k} = 0, \Rightarrow \langle\, e_r, e_s\rangle = 0 \tag{66}$$

- Normality The

$\square$

6 Problem6.7

Let $\{X_t\}$ be a stationary time series. Suppose that $\gamma(n) = O(1)$. Prove that this assumption is sufficient for Γ_n to be non singular.

Proof. We know that

$$\Gamma_n = \begin{pmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(n) & \gamma(n-1) & \cdots & \gamma(0) \end{pmatrix} \tag{67}$$

$\gamma(n) = O(1)$ means that as as n goes to infinity, $\gamma(n)$ goes to zero. This means that the $n \times n$ matrix Γ_n is well defined. Since $\{X_t\}$ is a stationary time series, the best linear predictor in terms of $\{1, X_n, \cdots, X_1\}$ is [1],

$$P_n X_{n+h} = a_0 + a_1 X_n + \cdots + a_n X_1 \tag{68}$$

where the a_i are solution of [1],

$$\Gamma_n \mathbf{a}_n = \gamma_n(h) \tag{69}$$

Since $P_n X_{n+h}$ in (68) exists, then the a_i also are well defined, which means that the solution of (69) exists, thus Γ_n is invertible, and

$$\mathbf{a}_n = \Gamma_n^{-1} \gamma_n(h) \tag{70}$$

□

References

[1] Petter J. Brockwell. Richard A. Davis *Introduction to Time Series and Forecasting.* Springer. Second edition. 2001