Mandatory homework 6 - Stat 211 - V19

Due at the end of March 7

February 26, 2019

Theorem. Let $\{X_t\}$ be a stationary time series with autocovariance function γ. Define $\mathcal{M}_t = \text{span}\{X_s,\, s \le t\}$. Let $q \ge 1$ and assume that $\gamma(h) \equiv 0$ for $|h| \ge q+1$ and $\gamma(q) \ne 0$. Define $W_t = X_t - \mathcal{P}_{\mathcal{M}_{t-1}}(X_t)$. Then $\{W_t\} \sim \text{WN}(0, \sigma_W^2)$, $\{X_t\}$ is an MA(q)-process; $X_t = \sum_{j=0}^{q} \theta_j W_{t-j}$ and the MA(q) polynomial has all its roots outside or on the unit circle.

[bd, Proposition 2.1.1, p. 44]

Corollary. Let the situation be as in the theorem. Then for $n \ge 1$,

$$\widehat{X}_{n+1} = \sum_{j=1}^{n\wedge q} \theta_{n,j}(X_{n+1-j} - \widehat{X}_{n+1-j}), \tag{1}$$

$\theta_{n,j} = \theta_j + o(1)$ for $j = 1, \dots, n$ and $\nu_n = \sigma_W^2 + o(1)$.

Remark. The Corollary does not depend on the innovation algorithm, but IA is an efficient way to find the coefficients in (1); the innovation representation for the one step predictor.

Problem 6.1
Let $q = 5$. Let $\{X_t\}$ be a MA(q) process;

$$X_t = \sum_{j=0}^{q} \theta_j Z_{t-j}, \quad \theta_0 = 1, \tag{2}$$

where $\{Z_t\}$ is iid from a Laplace (λ) with λ is chosen so that the variance $\sigma_Z^2 = 2$.

a) Draw θ_j from the uniform distribution on $[-2, 2]$ for $j = 1, \dots, q$.
b) For $N = 1000$, generate $\{X_t,\, t = 1, \dots, N\}$ according to (2).
c) Calculate and plot the empirical autocovariance function.

Consider the moment estimators, $\{\widehat{\theta}_j^{\text{M}}\}$ equations,

$$\widehat{\gamma}(h) = \widehat{\sigma}^2 \sum_{j=0}^{n-h} \widehat{\theta}_{j+h}^{\text{M}} \widehat{\theta}_j^{\text{M}} = f_h(\widehat{\sigma}^2, \widehat{\boldsymbol{\theta}}^{\text{M}}), \text{ say, } h = 0, \dots, q \tag{3}$$

where $\widehat{\gamma}$, the sample autocovariance function, is the input. Let $g_h(\sigma^2, \boldsymbol{\theta}) = f_h(\sigma^2, \boldsymbol{\theta}) - \widehat{\gamma}(h)$ for $h = 0, \dots, q$ and $\mathbf{G} = \{g_h\}$ so that $\mathbf{G}\colon R^{q+1} \rightsquigarrow R^{q+1}$.

d) Is there a unique solution of (3)?

e) Draw an initial value $\boldsymbol{\theta}_0$ as in a) until you get an invertible parameter. Then proceed by solving $\mathbf{G} = \mathbf{0}$ numerically. The solution is your moment estimator. Check if your solution is invertible.

f) Find $\{(\theta_{\infty,j},\, j = 1, \ldots, q),\, \nu_\infty\}$ by using IA with input $\{\widehat{\gamma}(h),\, h = 0, \ldots, q\}$.

Define

$$\widehat{\theta}_j = \theta_{\infty,j} \quad \text{for} \quad j = 1, \ldots, q \quad \text{and} \quad \widehat{\sigma}^2 = \nu_\infty \tag{4}$$

g) Compute and plot the autocovariance function defined (4). Compare it with the sample autocovariance function from c).

h) Explain that the IA- estimators defined by (4) satisfies (3).

i) Can you express $\{W_t\}$ by $\{X_t\}$?

j) Can you generate data from estimated invertible model? Be careful with the choice of the white noise process. Look to the previous point.

PROBLEM 6.2

Find the roots of the moving average polynomial for the following models.

a) $X_t = Z_t + Z_{t-2}$.

b) $X_t = Z_t - 2\cos(\omega) Z_{t-1} + Z_{t-2}$.

PROBLEM 6.3

Consider a causal AR(2) model with $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Find an expression for the two step predictor $\widehat{X}_{n+2}$ for $n \geq 2$ and the corresponding two step predictor variance.

PROBLEM 6.4

Let $\{X_t\}$ be a stationary and linear causal time series with white noise process $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$. Let $\mathcal{P}_n$ be the projection onto $\{X_1, \ldots, X_n\}$.
Find $\widehat{Z}_{n+1} = \mathcal{P}_n(Z_{n+1})$ and $\widehat{Z}_n = \mathcal{P}_n(Z_n)$.

PROBLEM 6.5

Let $\{Z_n\} \sim \mathrm{WN}(0, 1)$ and $X_t = Z_t - Z_{t-1}$.

a) Is $\{X_t\}$ invertible with respect to $\{Z_t\}$?

b) Can we find an invertible representation for $\{X_t\}$?

c) Use DL and find $\widehat{X}_{n+1}$ for $n = 1, 2, 3$.

d) Prove general n that $\widehat{X}_{n+1} = -\sum_{j=1}^{n} \frac{n+1-j}{n+1} X_{n+1-j}$.

Hint: You can use the prediction equations.

e) Prove that $\|Z_n - (-\widehat{X}_{n+1})\| = o(1)$. Conclude that $Z_t \in \mathcal{M}_t^X = \mathrm{span}\{X_s,\, s \leq t\}$.

PROBLEM 6.6

Let $\{X_t\}$ be a stationary time series with $\gamma(h) = o(1)$, let $\mathcal{P}_k$ be the linear projection onto $\mathcal{S}_k \overset{\text{def}}{=} \text{span}\{X_1, \ldots, X_k\}$ and define $e_k = (X_k - \widehat{X}_k)/\nu_{k-1}$ for $k \geq 1$. Prove that $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $\mathcal{S}_n$.

PROBLEM 6.7

Let $\{X_t\}$ be stationary time series with covariance function γ. Suppose that $\gamma(n) = o(1)$. Prove that this assumption is sufficient for $\mathbb{\Gamma}_n$ to be nonsingular for all n. *Hint: Assume that* $\mathbb{\Gamma}_n$ *is singular for some fixed* n *and then argue that* $X_{n+m} \in \mathcal{S}_n \overset{\text{def}}{=} \text{span}\{X_1, \ldots, X_n\}$ *for all* $m \geq 1$.

A. IA

The stationary innovation algorithm [IA] can be written as

for $n = 0, 1$

$$\nu_0 = \gamma(0), \qquad \theta_{11} = \frac{\gamma(1)}{\gamma(0)}, \quad \nu_1 = (1 - \theta_{11}^2)\nu_0$$

for $n \geq 2$

$$\theta_{n,n} = \nu_0^{-1}\gamma(n)$$

$$\theta_{n,n-k} = \nu_k^{-1}\Big\{\gamma(n-k) - \Big\langle \boldsymbol{\theta}_k^{\text{rev}}, \boldsymbol{\theta}_n^{\text{rev}} * \boldsymbol{\nu}_k \Big\rangle\Big\} \quad \text{for } k = 1, \ldots, n-1$$

$$\nu_n = \gamma(0) - \Big\langle \boldsymbol{\theta}_n^{\text{rev}}, \boldsymbol{\theta}_n^{\text{rev}} * \boldsymbol{\nu}_n \Big\rangle$$

where

$$\Big\langle \boldsymbol{\theta}_k^{\text{rev}}, \boldsymbol{\theta}_n^{\text{rev}} * \boldsymbol{\nu}_k \Big\rangle = \sum_{j=0}^{k-1} \theta_{k,k-j}\, \theta_{n,n-j}\, \nu_j \quad \text{for} \quad k = 1, \ldots, n.$$

B. Proofs

Proof of Theorem .
The proof use basic projection theory. Let t be fixed. By the assumptions we have that $\langle X_t, Y\rangle \equiv 0$ for $Y \in \mathcal{M}_{t-q-1}$. Let $\mathcal{S} = \text{span}\{W_{t-q}, \ldots, W_t\}$ then $\mathcal{P}_{\mathcal{M}_t} = \mathcal{P}_{\mathcal{M}_{t-q-1}} + \mathcal{P}_{\mathcal{S}}$ by the definition of $\{W_t\}$. We write $\mathcal{P}_{W_s}$ for $\mathcal{P}_{\{W_s\}}$ which is the projection onto the variable W_s.

Now,

$$\begin{aligned} X_t = \mathcal{P}_{\mathcal{M}_t}(X_t) &= \mathcal{P}_{\mathcal{M}_{t-q-1}}(X_t) + \mathcal{P}_{\mathcal{S}}(X_t) = \mathcal{P}_{\mathcal{S}}(X_t) \\ &= \sum_{j=0}^{q} \mathcal{P}_{W_{t-j}}(X_t) = \sum_{j=0}^{q} \frac{\langle X_t, W_{t-j}\rangle}{\|W_{t-j}\|^2} W_{t-j} \\ &= W_t + \sum_{j=1}^{q} \frac{\langle X_t, W_{t-j}\rangle}{\|W_{t-j}\|^2} W_{t-j} = \sum_{j=0}^{q} \theta_j W_{t-j} \end{aligned}$$

Suppose that $\theta(z)$ has a root ξ strictly inside the unit circle. Then ξ is real or $\bar{\xi}$ is also a root. Define $\zeta(z) = (1 - z/\xi)$ in the real case and $\zeta(z) = (1 - z/\xi)(1 - z/\bar{\xi})$ in the complex case. For the rest of the roots we use $\theta_2(z)$ so that $\theta(z) = \zeta(z)\theta_2(z)$. Let $U_t = \theta_2(B)W_t$ and $\pi(z) = 1/\zeta(z)$. Then we have $X_t = \zeta(B)U_t$ and

$$U_t = \pi(B)X_t = \psi(B)X_t + \tau(B)X_t$$

$$\psi(z) \stackrel{\text{def}}{=} \sum_{k=1}^{q} \alpha_k z^{-k} \text{ and } \tau(z) \stackrel{\text{def}}{=} \sum_{k=q+1}^{\infty} \alpha_k z^{-k}, \ \alpha_k = \begin{cases} -\xi^k, & \text{real case;} \\ \sum_{\ell=1}^{k} \xi^\ell\, (\bar{\xi})^{(\ell-k)} & \text{complex case.} \end{cases}$$

Since $U_t \in \text{span}\{W_s,\, s \le t\}$ and $X_{t+q+k} \in \text{span}\{W_s,\, s > t\}$ for $k \ge 1$ we must have $\tau(1) \equiv 0$. In the real case $\tau(1) = -\xi^{q+1}(1-\xi)^{-1}$. But $\tau(1) = 0$ requires that $\xi = 0$. which is impossible. The complex case is more reluctant and appear to resist the last argument used above. We therefore need a different strategy that also includes complex roots. By what we already have done,

$$U_t = \psi(B)X_t = \theta_2(B)W_t \quad \Rightarrow \quad \psi(B)\theta(B)W_t = \theta_2(B)W_t$$

which means that,

$$\sum_{k=1}^{q}\sum_{j=0}^{q} \alpha_k\, B^{-k}\, \theta_j\, B^j\, W_t = \sum_{\ell=0}^{m} \beta_\ell W_{t-\ell}, \ \text{ say}$$
$$\Downarrow$$
$$\sum_{u=1-q}^{q} \Big\{\sum_{j=0}^{q-u} \theta_j \alpha_{u+j}\Big\} W_{t+u} = \sum_{\ell=0}^{m} \beta_\ell W_{t-\ell}.$$

Starting with $u = q$, we see that

$$\theta_0\alpha_q = 0,$$
$$\theta_0\alpha_{q-1} + \theta_1\alpha_q = 0,$$
$$\Downarrow$$
$$\theta_0\alpha_u + \theta_1\alpha_{u+1} + \cdots + \theta_{q-u}\alpha_q = 0,, \quad u = q, \ldots, 1.$$

This is a triangular non-singular lineary system with the unique solution $\alpha_k \equiv 0$. Hence we have contradiction and we can conlude that the MA(q) polynomial cannot have any root inside the unit circle. But note that we cannot eliminate an eventually root on the unit circle.

$\square$

Proof of Corollary .

We use $\{W_t\}$ as in the theorem. Let $Y_{n+1} \overset{\text{def}}{=} X_{n+1} - W_{n+1} = \mathcal{P}_{\mathcal{M}_n}(X_{n+1}) \in \mathcal{M}_n$. By definition of $\mathcal{M}_n$ there exists a sequence $\{S_n(m) \overset{\text{def}}{=} \sum_{j=1}^m a_{mj} X_{n+1-j},\, m \ge 1\}$ so that $\|Y_{n+1} - S_n(m)\| = o(1)$ with respect to m and independent of n due to stationarity. Now $S_n(n) \in \text{span}\{X_1, \ldots, X_n\}$ and therefore,

$$\|X_{n+1} - W_{n+1} - \widehat{X}_{n+1}\| \le \|Y_{n+1} - S_n(n)\| = o(1).$$

This means that

$$\|U_n - W_n\| = o(1) \quad \text{and} \quad \nu_n = \|U_{n+1}\| = \sigma_W^2 + o(1)$$

where $U_n = X_n - \widehat{X}_n$.

It remains to show that the coefficients converges. By (1),

$\widehat{X}_{n+1} = \sum_{j=1}^{q} \theta_{n,j} U_{n+1-j}$ for $n \geq q$ and for such n's,

$$\begin{aligned}
\nu_n - \sigma_W^2 &= \Big\| X_{n+1} - \sum_{j=1}^{q} \theta_{n,j} U_{n+1-j} \Big\| - \sigma_W^2 \\
&= \Big\| \sum_{j=1}^{q} \theta_j W_{n+1-j} - \sum_{j=1}^{q} \theta_{n,j} U_{n+1-j} \Big\| \\
&= \Big\| \sum_{j=1}^{q} (\theta_j - \theta_{n,j}) U_{n+1-j} - \sum_{j=1}^{q} \theta_j \big(W_{n+1-j} - U_{n+1-j} \big) \Big\| \\
&\geq \Big| \, \Big\| \sum_{j=1}^{q} (\theta_j - \theta_{n,j}) U_{n+1-j} \Big\| - \Big\| \sum_{j=1}^{q} \theta_j \big(W_{n+1-j} - U_{n+1-j} \big) \Big\| \, \Big|,
\end{aligned}$$

which means that

$$\Big\| \sum_{j=1}^{q} (\theta_j - \theta_{n,j}) U_{n+1-j} \Big\| \leq \Big\| \sum_{j=1}^{q} \theta_j \big(W_{n+1-j} - U_{n+1-j} \big) \Big\| + \nu_n - \sigma_W^2 = o(1) + o(1) = o(1).$$

Hence, for any $k \in \{1, \ldots, q\}$

$$|\theta_k - \theta_{n,k}|^2 \sigma_W^2 \leq \sum_{j=1}^{q} \big(\theta_j - \theta_{n,j} \big)^2 \nu_{n-j} = \Big\| \sum_{j=1}^{q} (\theta_j - \theta_{n,j}) U_{n+1-j} \Big\|^2 = o(1).$$

$\square$

Remark. We have used that $\|x - y\| \geq \big| \, \|x\| - \|y\| \, \big|$.

References

Peter J. Brockwell and Richard A. Davis. *Introduction to time series and forecasting.* Springer Texts in Statistics. Springer, [Cham], third edition, 2016. ISBN 978-3-319-29852-8; 978-3-319-29854-2. doi: 10.1007/978-3-319-29854-2. URL https://doi.org/10.1007/978-3-319-29854-2.