STAT211 Mandatory Homework 4

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February 14, 2019

1 Problem 4.1

Considere an ARMA(p,q) model

$$X_t - \sum_{k=1}^{p} \phi_k X_{t-k} = Z_t + \sum_{k=1}^{p} \theta_k Z_{t-k} \tag{1}$$

1.1 Part a: invertibility

An ARMA(p,q) process $\{X_t\}$ is invertible if there exist constant $\{\pi_j\}$ such that

$$\sum_{j=0}^{\infty} |\pi_j| \leq \infty \tag{2}$$

and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad \text{for all t.} \tag{3}$$

In other word $\{X_t\}$ is invertible if Z_t can be written as a linear combination of X_{t-j}, $j = 0, 1, \ldots, \infty$, [1].

Invertibility is equivalent to

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0 \quad \text{for all} \quad |z| \leq 1 \tag{4}$$

where $\theta(z)$ is the moving average polynomial.

The process X_t is invertible if and only if the zeros of the moving average polynomial $\theta(B)$ lie outside the unit circle.

1.2 Part b

The sequence in (3) is determined by the relation

$$(1+\theta_1 z+\theta_2 z^2+\cdots+\theta_q z^q)(\pi_0+\pi_1 z+\cdots)=(1-\phi_1 z-\phi_2 z^2-\cdots-\phi_p z^q). \qquad (5)$$

Multiplying the left hand side together gives

$$\begin{aligned}(1+\theta_1 z+\theta_2 z^2+\cdots+\theta_q z^q)(\pi_0+\pi_1 z+\cdots) &= \pi_0+\pi_1 z+\pi_2 z^2+\cdots+\theta_1\pi_0 z+\theta_1\pi_1 z^2+\cdots+\theta_2\pi_0 z^2\\ &= \pi_0+(\pi_1+\theta_1\pi_0)z+(\pi_2+\theta_1\pi_1+\theta_2\pi_0)z^2+\cdots\end{aligned}$$

and equation (5) can be rewritten as

$$\pi_0+(\pi_1+\theta_1\pi_0)z+(\pi_2+\theta_1\pi_1+\theta_2\pi_0)z^2+\cdots=(1-\phi_1 z-\phi_2 z^2-\cdots-\phi_p z^q). \qquad (6)$$

And equating the coefficients of $z^j, j=0,1,\cdots$, we obtain

$$\begin{aligned}\pi_0 &= 1\\ \pi_1+\theta_1\pi_0 &= -\phi_1\\ \pi_2+\theta_1\pi_1+\theta_2\pi_0 &= -\phi_2\\ &\vdots\end{aligned}$$

or equivalently

$$\pi_j+\sum_{k=1}^{q}\theta_k\pi_{j-1}=-\phi_j, \quad j=0,1,\cdots \qquad (7)$$

2 All R code Code

References

[1] Petter J. Brockwell. Richard A. Davis *Introduction to Time Series and Forecasting.* Springer. Second edition. 2001