STAT211 Mandatory Homework 8

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1 Problem 8.2

In this exercise we study the spectral representation of a time series. The spectral representation of a time series is used to decompose a time series into sums of sinusoidal components [1]. By doing so, we can identify the dominant periods or frequencies of a time series by using its periodogram [2]. The periodogram is a sample based function that gives an estimate of the spectral density [1]. If $f(\cdot)$ is the spectral density and $I_n(\cdot)$ is the periodogram of n observations, then $I_n(\cdot)$ can be view as an estimation of $2\pi f(\cdot)$ [1].

The spectral density for an AR(2) model is given by

$$\begin{aligned} f(w) &= \frac{\sigma^2}{2\pi}\frac{1}{|\phi(\exp(-iw))|^2}, \quad w \in (-\pi, \pi], \quad \phi(z) = 1 - \phi_1 z - \phi_2 z^2. \\ &= \frac{\sigma^2}{2\pi}\frac{1}{|1-\phi_1 e^{-iw} - \phi_2 e^{-2iw}|^2} \end{aligned} \tag{1}$$

The spectral density of a stationary process $\{X_t\}$ specifies the frequency decomposition of the autocovariance function (ACF) [1].

1.1 Part a: Spectral density

```
f <- function(w){
  phi1 <- 1.4
  phi2 <- -0.9
  phi <- 1-phi1*w-phi2*w**2
  sigma <- 1
  denominator <- abs(1-phi1*exp(-(0+1i)*w) - phi2*exp(-(0+2i)*w))**2
  factor <- (sigma*sigma)/(2*pi)
  value <- factor*(1./denominator)
  return(value)

}
set.seed(10)
w <- seq(-pi, pi)
png("theoreticalDensity.png")
plot(f(w), col="blue",type="b",xlab="frequency")
```

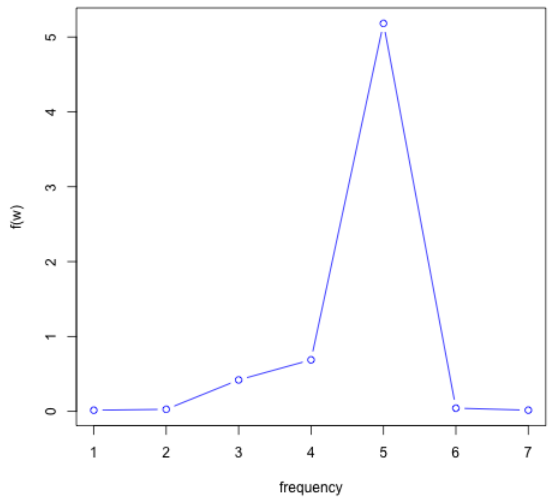

Figure 1: Spectral density of the AR(2) process

1.2 Part b: Comparing spectral density

For spectral for $\phi_2 = 0.95$

```
f <- function(w){
  phi1 <- 1.4
  phi2 <- 0.95
  phi <- 1-phi1*w-phi2*w**2
  sigma <- 1
  denominator <- abs(1-phi1*exp(-(0+1i)*w) - phi2*exp(-(0+2i)*w))**2
  factor <- (sigma*sigma)/(2*pi)
  value <- factor*(1./denominator)
  return(value)

}
set.seed(10)
w <- seq(-pi, pi)
png("theoreticalDensity.png")
plot(f(w), col="blue",type="b",xlab="frequency")
```

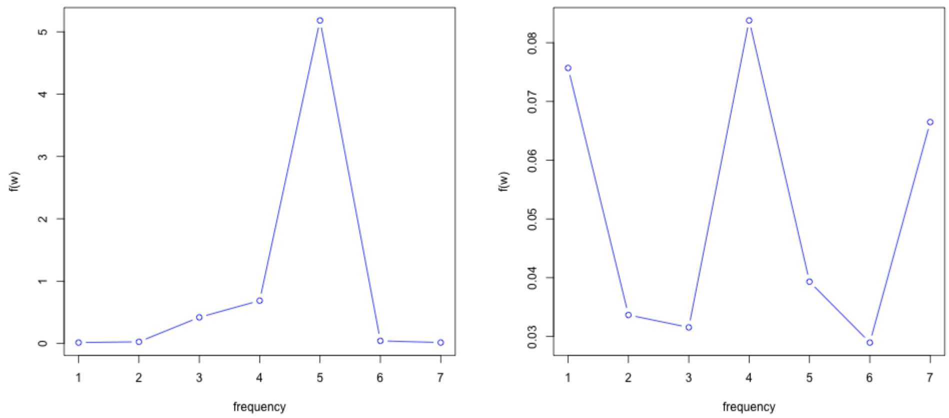

Figure 2: Spectral density of the AR(2) process. Left $\phi_2 = -0.90$, right $\phi_2 = 0.95$

We can observe from Figure (2) that the spectral density for $\phi_2 = 0.95$ oscillate more compared to the one with $\phi_2 = -0.9$.

1.3 Part c: Simulation of the AR(2) process

Simlating the AR(2) process with 100 obervations

```
# Plot of AR(2) model
ar.sim <- arima.sim(model=list(ar=c(1.4, -0.9)), n=100)
ts.plot(ar.sim)
```

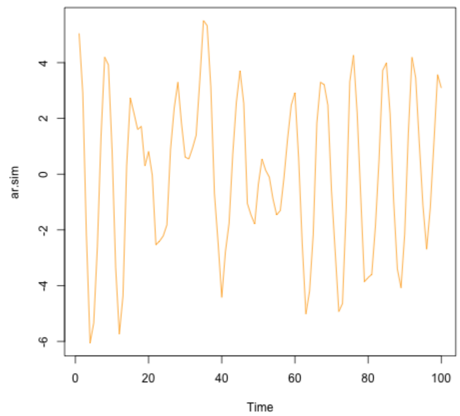

Figure 3: AR(2) simulation for $\phi_1 = 1.4, \phi_2 = -0.9$

```
#Periodogram  of AR(2) model
ar.sim <- arima.sim(model=list(ar=c(1.4, -0.9)), n=100)
per <- periodogram(ar.sim)
plot(per)
```

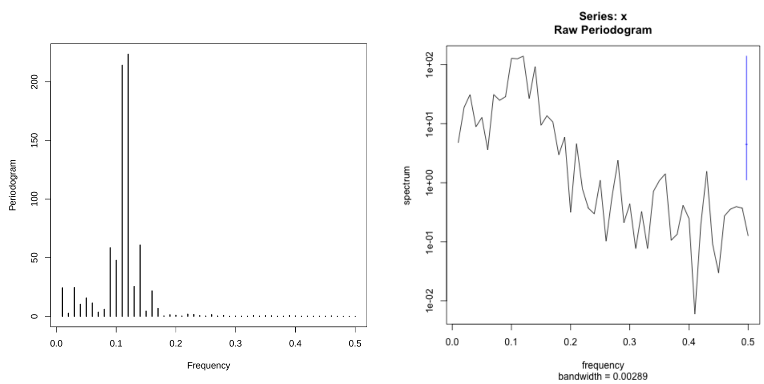

Figure 4: Periodogram of the AR(2) process

```
#Moving average smoothening of periodogram
set.seed(10)
ar.sim <- arima.sim(model=list(ar=c(1.4, -0.9)), n=100)
periodogram <- periodogram(ar.sim)
spectrum <- periodogram$spec
trendpattern = filter(spectrum, filter = c(1/3,3), sides=2)
```

```
plot(trendpattern)
```

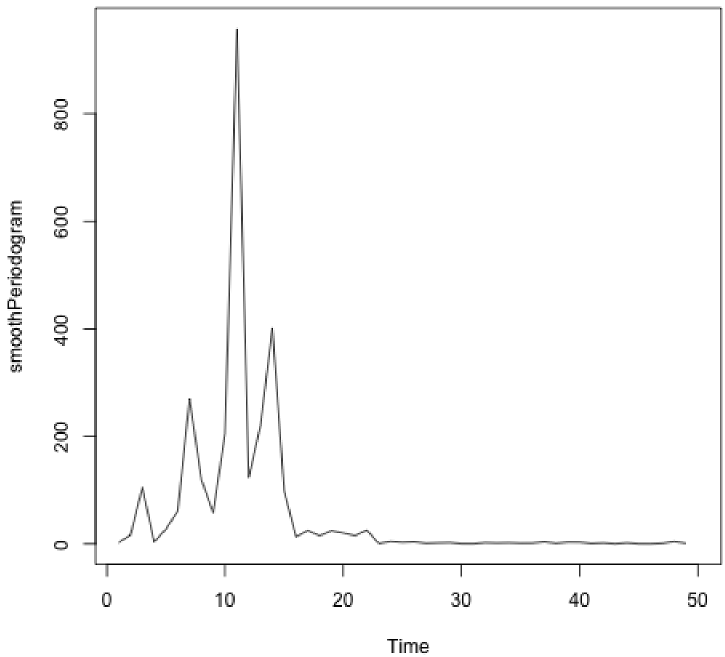

Figure 5: Smooth periodogram

```
#Estimated spectral density
set.seed(10)
ar.sim <- arima.sim(model=list(ar=c(1.4, -0.9)), n=100)
estimatedSpecralDensity <- spectrum(ar.sim,method="ar")
png("EstimatedSpectralDensity.png")
plot(estimatedSpecralDensity)
```

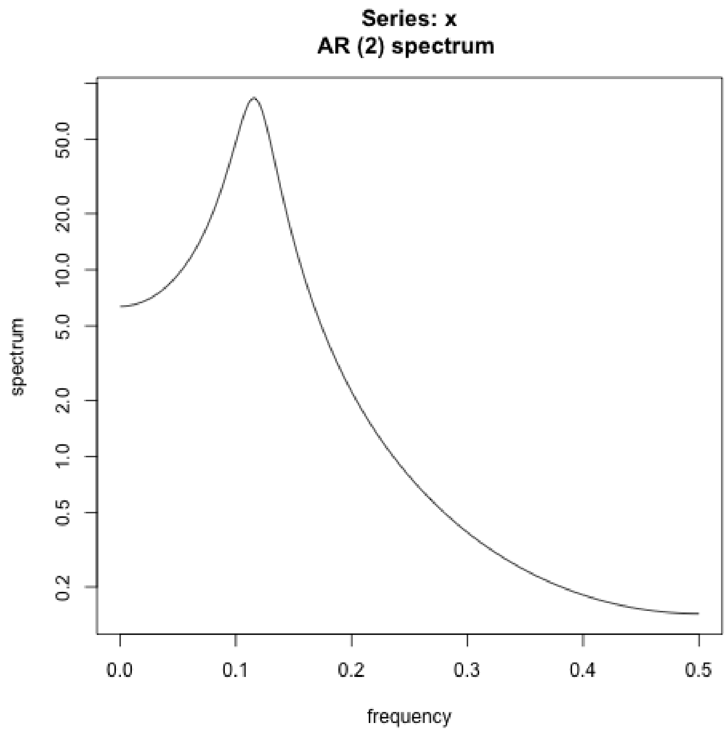

Figure 6: Estimated spectral density from R function spectrum

Figure 3 shows the simulated AR(2) process, Figure 4 shows the periodogram of the AR(2) process, Figure 5 shows the smoothen periodogram and Figure 6 shows the estimated spectral density from the AR(2) model.

1.4 Part d

Recall the plot of the AR(2) model

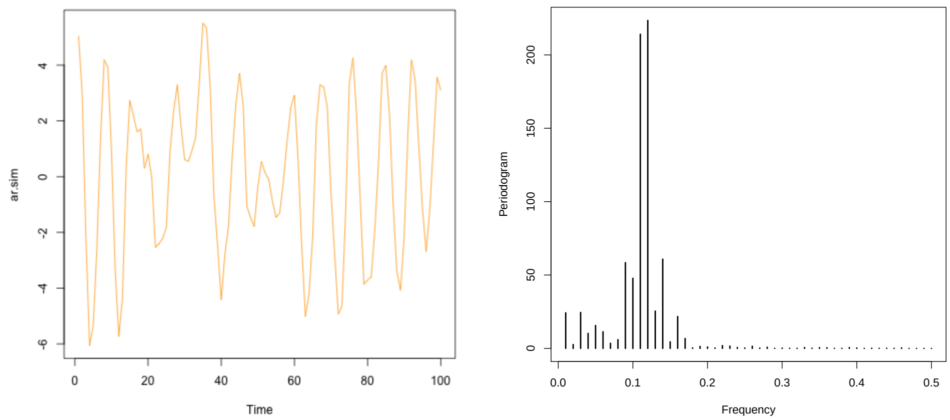

Figure 7: AR(2) simulation for $\phi_1 = 1.4, \phi_2 = -0.9$ left and periodogram right

From Figure 7 we can clearly see a periodic structure of the time series modelled by the AR(2) process. From the periodogram, we can see a pick. We get the peak and corresponding frequency as follow

```
ar.sim <- arima.sim(model=list(ar=c(1.4, -0.9)), n=100)
periodogram <- periodogram(ar.sim)
spectrum <- periodogram$spec
frequency <- periodogram$freq
pick <- max(spectrum)
pickIndex = match(pick, spectrum)
freq <- frequency[pickIndex]
print(pick)
print(freq)

>>[1] 315.5908
>>[1] 0.11
```

We observe a peak at a frequency of 0.11 which corresponds to period

$$T = \frac{1}{0.11} = 9 \tag{2}$$

which means that a cycle is completed in 9 time periods.

1.5 Part e

Simulation with 1000 observation

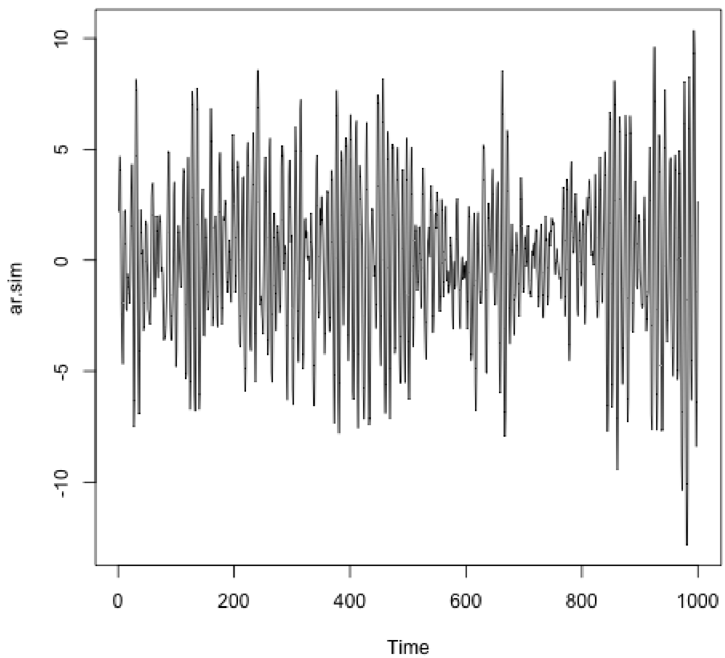

Figure 8: Simulation with 1000 obervartion

Periodogram for 1000 observations

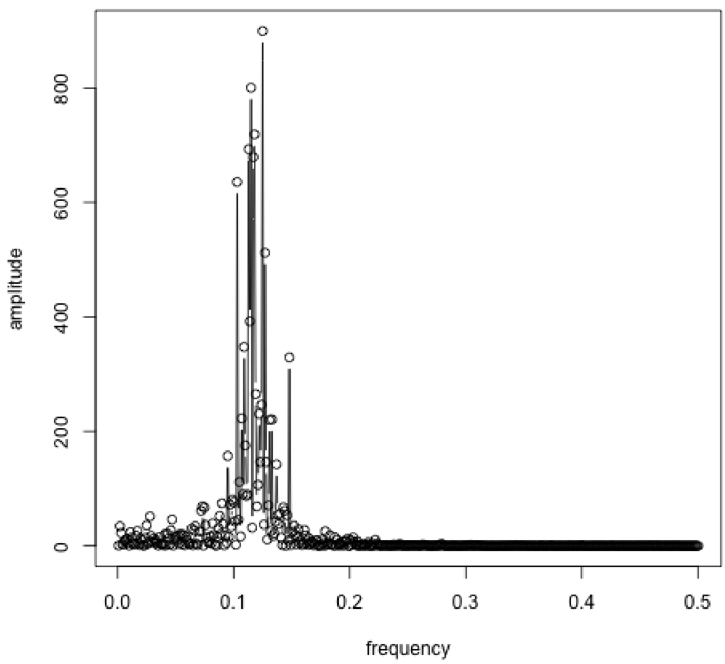

Figure 9: periodogram for 1000 obervation

Smooth peridogram

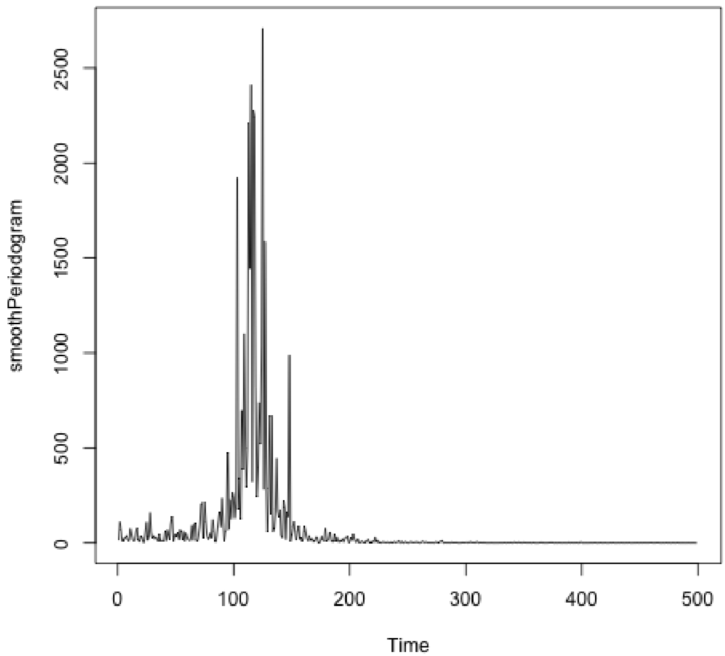

Figure 10: Smoother periodogram

Estimated spectral density

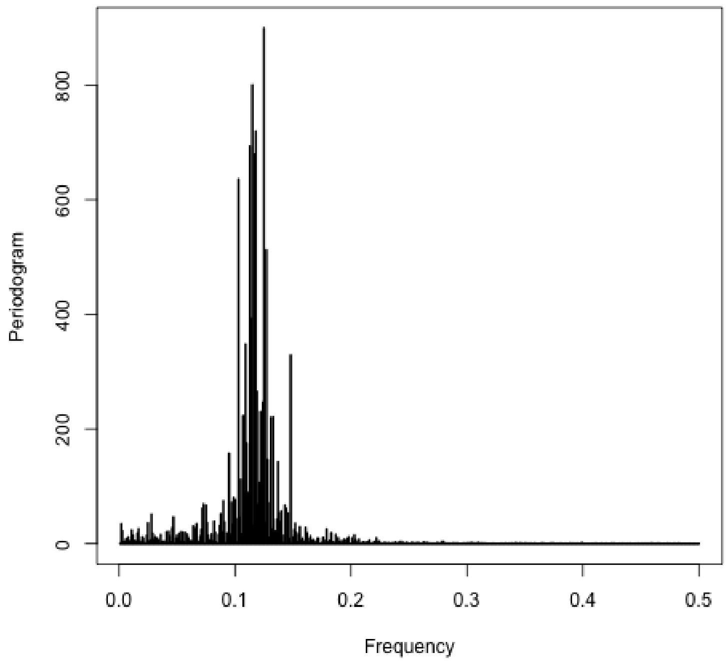

Figure 11: Estimated spectral density

For 1000 observations, the the peak is at at frequency of 0.125 corresponding to 8 time periods

2 Problem 8.3

If the process $\{X_t\}$ is a causal ARMA(p,q), its spectral density is given by [1]

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}, \quad -\pi \leq \lambda \leq \pi. \tag{3}$$

Furthermore,

$$\begin{aligned} \theta(e^{-i\lambda}) &= 1 + \theta_1 e^{-i\lambda} + \theta_2 e^{-i2\lambda} + \cdots + \theta_n e^{-in\lambda} \\ &= 1 + \sum_{k=0}^{n} \theta_n e^{-ik\lambda}, \end{aligned} \tag{4}$$

and

$$\phi(e^{-i\lambda}) = 1 - \phi_1 e^{-i\lambda} - \phi_2 e^{-i2\lambda} - \cdots - \phi_n e^{-in\lambda}$$
$$= 1 - \sum_{k=0}^{n} \phi_n e^{-ik\lambda} \tag{5}$$

where

$$e^{-ik\lambda} = \cos(k\lambda) - i\sin(k\lambda). \tag{6}$$

Now let

$$f(\lambda) = e^{-ik\lambda}$$
$$= \cos(k\lambda) - i\sin(k\lambda), \tag{7}$$

and let λ_0 be arbitrary taken from the interval $[-\pi, \pi]$. Then

$$\lim_{\lambda\to\lambda_0} f(\lambda) = \lim_{\lambda\to\lambda_0} \cos(k\lambda) - i\sin(k\lambda)$$
$$= cos(k\lambda_0) - i\sin(k\lambda_0) \tag{8}$$
$$= f(\lambda_0)$$

and this prove that $e^{-ik\lambda}$ is continuous and so is its linear combination

$$\theta(e^{-ik\lambda}) = 1 + \sum_{k=0}^{n} \theta_n e^{-ik\lambda}$$

and

$$\phi(e^{-ik\lambda}) = 1 - \sum_{k=0}^{n} \phi_n e^{-ik\lambda}.$$

Since we have a causal process, the roots of $\phi(e^{-ik\lambda})$ lies outside the unit circle, and $\phi(e^{-ik\lambda}) \neq 0$, and since the quotient of two continuous functions is continuous we have that

$$\frac{\theta(e^{-i\lambda})}{\phi(e^{-i\lambda})}$$

is continuous and

$$\frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}$$

is also continuous. Therefore

$$f_X(\lambda) = \frac{\sigma^2}{2\pi} \frac{|\theta(e^{-i\lambda})|^2}{|\phi(e^{-i\lambda})|^2}, \quad -\pi \leq \lambda \leq \pi.$$

is continuous.

Assume now that the minimum of the spectral density is attained at some value, say λ_0. Then the minimum of the spectral density is

$$\begin{aligned} f_X(\lambda_0) &= \frac{\sigma^2}{2\pi}\frac{|\theta(e^{-i\lambda_0})|^2}{|\phi(e^{-i\lambda_0})|^2}, \quad -\pi \le \lambda \le \pi. \\ &= \frac{\sigma^2}{2\pi}\left|\frac{\theta(e^{-i\lambda_0})}{\phi(e^{-i\lambda_0})}\right|^2 \\ &= \frac{\sigma^2}{2\pi}\left|\frac{1+\sum_{k=0}^{n}\theta_k e^{-ik\lambda}}{1-\sum_{k=0}^{n}\phi_k e^{-ik\lambda}}\right|^2 \ge 0. \end{aligned}$$

For $f_X(\lambda_0)$ to be equal to zero, we need

$$1+\sum_{k=0}^{n}\theta_k e^{-ik\lambda} = 1+\sum_{k=0}^{n}\theta_k\cos(k\lambda) + i\sum_{k=0}^{n}\theta_k\sin(k\lambda) = 0.$$

This mean that we need to have simultaneously

$$\sum_{k=0}^{n}\theta_k\sin(k\lambda) = 0.$$

and

$$\sum_{k=0}^{n}\theta_k\cos(k\lambda) = -1.$$

Furthermore,

$$\sum_{k=0}^{n}\theta_k\sin(k\lambda) = 0,$$

means that either $\theta_k = 0$ or $\sin(k\lambda) = 0$, $\forall k = 1, \cdots, n$. Since all θ_k can not be equal to zero, the latter must hold, that is

$$\begin{aligned} \sin(\lambda) &= 0 \\ \sin(2\lambda) &= 0 \\ &\vdots \\ \sin(n\lambda) &= 0 \end{aligned}$$

which means that

$$n\lambda = \pm\frac{\pi}{2} \Rightarrow \lambda = \pm\frac{\pi}{2n}. \tag{9}$$

Similarely,

$$\sum_{k=0}^{n} \theta_k \cos(k\lambda) = -1.$$

means that at $\lambda = \pm\frac{\pi}{2n}$ we must have

$$\theta_1 \cos\left(\frac{\pi}{2}\right) + \theta_2 \cos\left(2\frac{\pi}{4}\right) + \cdots + \theta_n \cos\left(n\frac{\pi}{2n}\right) = -1,$$

but

$$\theta_1 \cos\left(\frac{\pi}{2}\right) + \theta_2 \cos\left(2\frac{\pi}{4}\right) + \cdots + \theta_n \cos\left(n\frac{\pi}{2n}\right) = 0$$

so

$$\theta_1 \cos\left(\frac{\pi}{2}\right) + \theta_2 \cos\left(2\frac{\pi}{4}\right) + \cdots + \theta_n \cos\left(n\frac{\pi}{2n}\right) \neq -1$$

Therefore

$$f_X(\lambda_0) = \frac{\sigma^2}{2\pi} \left| \frac{1 + \sum_{k=0}^{n} \theta_k e^{-ik\lambda}}{1 - \sum_{k=0}^{n} \phi_k e^{-ik\lambda}} \right|^2 > 0.$$

3 Problem 8.4

Let $\{X_t, t = 1, \cdots, n\}$ be a data from a time series The likelihood estimator of the auto-covariance function Γ_{LM} is given by [1]

$$\Gamma_{LM} = (2\pi)^{-n/2}(det(\Gamma_n))^{-1/2} \exp\left(-\frac{1}{2}(X'\Gamma_n^{-1}(X_n))\right) \tag{10}$$

and the sample autocovariance is given by

$$\hat{\Gamma}_n = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) & \dots & \hat{\gamma}(k-1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) & \dots & \hat{\gamma}(k-2) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{\gamma}(k-1) & \hat{\gamma}(k-2) & \dots & \hat{\gamma}(0) \end{bmatrix} \tag{11}$$

where

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (X_{t+|h|} - \bar{X})(X_t - \bar{X}) \tag{12}$$

References

[1] Petter J. Brockwell. Richard A. Davis *Introduction to Time Series and Forecasting.* Springer. Second edition. 2001

[2] PennState Eberly College of science, *STAT 510. Applied Time Series Analysis.* https://newonlinecourses.science.psu.edu/stat510/node/71/