STAT211 Mandatory Homework 6

Yapi Donatien Achou

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1 Problem 6.2

1.1 Part a

Root of

$$X_t = Z_t + Z_{t-2} \tag{1}$$

We can rewrite this as

$$X_t = Z_t + 0Z_{t-1} + 1Z_{t-2} \tag{2}$$

and the corresponding moving average polynomial is then

$$\theta(z) = 1 + 0z + 1.z^2 = 1 + z^2. \tag{3}$$

whose roots are

$$z_1 = i, \quad z_2 = -i \tag{4}$$

1.2 Part b

Root of

$$X_t = Z_t - 2\cos(w)Z_{t-1} + Z_{t-2} \tag{5}$$

and the corresponding moving average polynomial is

$$\begin{aligned} \theta(z) &= 1 - 2\cos(w)z + z^2 \\ &= (z - \cos(w))^2 - \cos(w)^2 + 1 \\ &= (z - \cos(w))^2 - (\cos(w)^2 - 1) \\ &(z - \cos(w))^2 - (-\sin(w)^2) \\ &(z - \cos(w))^2 - (i^2 \sin(w)^2) \\ &(z - \cos(w))^2 - (i \sin(w))^2 \\ &(z - \cos(w) + i\sin(w))(z - \cos(w) - i\sin(w)) \end{aligned} \tag{6}$$

and the roots are

$$z_1 = \cos(w) - i\sin(w), \quad z_2 = \cos(w) + i\sin(w) \tag{7}$$

2 Problem 6.6

Let $\mathbf{P}_k$ be the linear projection onto

$$\mathbf{S}_k = span\{X_1, \cdots, X_k\} \tag{8}$$

and

$$e_k = \frac{X_k - \hat{X}_k}{\nu_{k-1}}. \tag{9}$$

$\{e_1, \cdots, e_n\}$ is orthonormal basis for $\mathbf{S}_n$ if $\{e_1, \cdots, e_n\}$ is a linearly independent subset of $\mathbf{S}_n$ that span $\mathbf{S}_n$, and for any e_j, e_i in $\{e_1, \cdots, e_n\}$ the inner product of e_j and e_i is zero and any e_i as norm 1.

Proof.

- Linearly independence. Assume that

$$a_1 e_1 + \cdots + a_n e_n = 0 \tag{10}$$

where a_i are real numbers. The we have

$$\begin{aligned} a_1 e_1 + \cdots + a_n e_n &= 0 \\ a_1 \frac{X_1 - \hat{X}_1}{\nu_0} + \cdots + a_n \frac{X_n - \hat{X}_n}{\nu_{n-1}} &= 0 \\ \frac{a_1}{\nu_0}(X_1 - \hat{X}_1) + \cdots + \frac{a_n}{\nu_{n-1}}(X_n - \hat{X}_n) &= 0 \end{aligned} \tag{11}$$

From (9) we know that

$$X_k - \hat{X}_k = e_k \nu_{k-1}. \tag{12}$$

Thus

$$X_1 - \hat{X}_1 \neq 0, \cdots, X_n - \hat{X}_n \neq 0 \tag{13}$$

Therefor the last expression in equation (11) is true if

$$\frac{a_1}{\nu_0} = \cdots = \frac{a_n}{\nu_{n-1}} = 0 \tag{14}$$

equivalently

$$a_1 = \cdots = a_n = 0 \tag{15}$$

This means that $\{e_1, \cdots, e_n\}$ is a linearly independent

- $\{e_1, \cdots, e_n\}$ span $\mathbf{S}_n$. We want to show that any vector in $\mathbf{S}_n$ can be written as a linear combination of $\{e_1, \cdots, e_n\}$. Let $Z \in \mathbf{S}_n$. Since $\mathbf{S}_n = span\{X_1, \cdots, X_n\}$, we have

$$Z = b_1 X_1 + \cdots + b_n X_n \tag{16}$$

where b_i are real numbers. Then we have

$$
\begin{aligned}
Z &= b_1X_1 + \cdots + b_nX_n \\
Z &= b_1(\nu_0e_1 + \hat{X}_1) + \cdots + b_n(\nu_{n-1}e_n + \hat{X}_n) \\
Z &= b_1\nu_0e_1 + \cdots + b_n\nu_{n-1}e_n + \underbrace{b_1\hat{X}_1 + \cdots + b_n\hat{X}_n}_{Z'}
\end{aligned} \tag{17}
$$

$$
\underbrace{Z - Z'}_{Z''} = \underbrace{b_1\nu_0}_{\alpha_1} e_1 + \cdots + \underbrace{b_n\nu_{n-1}}_{\alpha_n} e_n
$$

Since $Z'' \in \mathbf{S}_n$ We have

$$
Z'' = \alpha_1e_1 + \cdots + \alpha_ne_n \tag{18}
$$

- Horthogonality Let e_r, e_r be two arbitrarily vectors in $\{e_1, \cdots, e_n\}$ such that $r \neq s$.

$$
\begin{aligned}
\langle\, e_r, e_s\rangle &= \left\langle \frac{X_r - \hat{X}_r}{\nu_{r-1}}, \frac{X_s - \hat{X}_s}{\nu_{s-1}} \right\rangle \\
&= \frac{1}{\nu_{r-1}\nu_{i-s}} \left\langle X_r - \hat{X}_r, X_s - \hat{X}_s \right\rangle
\end{aligned} \tag{19}
$$

From the innovation algorithm [1], the coefficient of $X_n - \hat{X}_n, \cdots, X_1 - \hat{X}_1$ are of the form

$$
\theta_{n,n-k}, \quad k = 0, \cdots, n. \tag{20}
$$

so that

$$
\begin{aligned}
\langle\, e_r, e_s\rangle &= \left\langle \frac{X_r - \hat{X}_r}{\nu_{r-1}}, \frac{X_s - \hat{X}_s}{\nu_{s-1}} \right\rangle \\
&= \frac{1}{\nu_{r-1}\nu_{i-s}} \left\langle X_r - \hat{X}_r, X_s - \hat{X}_s \right\rangle \\
&= \frac{\theta_{r,r-k}\theta_{s,s-k}}{\nu_{r-1}\nu_{s-1}}
\end{aligned} \tag{21}
$$

And from [1] equation 2.5.26, for $r \neq s$ we get

$$
\theta_{r,r-k}\theta_{s,s-k} = 0, \Rightarrow \langle\, e_r, e_s\rangle = 0 \tag{22}
$$

- Normality The

$\square$

References

[1] Petter J. Brockwell. Richard A. Davis *Introduction to Time Series and Forecasting.* Springer. Second edition. 2001