The proof is in the pudding.

Opening a copy of The Mathematical Intelligencer *you may ask yourself uneasily, "What is this anyway—a mathematical journal, or what?" Or you may ask, "Where am I?" Or even "Who am I?" This sense of disorientation is at its most acute when you open to Colin Adams's column.*

Relax. Breathe regularly. It's mathematical, it's a humor column, and it may even be harmless.

➢ Submissions should be uploaded to http://tmin.edmgr.com
or sent directly to **Colin Adams,**
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Zombies & Calculus: An Excerpt

Colin Adams

This is an excerpt from the book Zombies & Calculus, *written by Colin Adams and published by Princeton University Press, 2014. At this point in the book, the narrator, math professor Craig Williams of Roberts College in Westbridge, Massachusetts, and his small band have managed so far to survive the zombie apocalypse using mathematics. They are hoping to rescue Professor Raphael Ortiz from the chemistry department, who has smashed his car into a tree at the center of the quad. Angus is a student, Marsha an administrative assistant, Gunderson an applied mathematician, and Jessie Sullivan a professor of biology.*

It was hard for me to believe I had agreed to the plan, but Raphael Ortiz was someone I had known and respected since my arrival on campus, and I couldn't bring myself to leave him to die. I waited next to Angus and Jessie inside the door through which I had previously let Jessie into the building. I had the Chief's revolver in my hand. Up above, on the third floor, Gunderson and Marsha watched from the open window. Angus nodded, and I carefully opened the door, the revolver pointed out. There were no zombies on the other side, so I slipped through the door, concealing myself as well as possible against the building. Angus slipped out, too and then he sprinted toward the bike rack. Several zombies spotted the quick motion and they turned in pursuit.

Angus reached the rack and grabbed the first bike.

"Not that one," Marsha yelled as Angus jerked at it, realizing only then it was chained to the rack. The first zombie was closing fast. I took aim, but doubted my ability to hit anything further than a few feet away. Angus grabbed the second bike and was relieved to see there was no lock. He pulled it from the rack, threw a leg over the crossbar and started pedaling.

By now, quite a few zombies were in pursuit, but Angus could pedal a lot faster than they could move. He swerved around several that were headed straight for him and then started pedaling around the walkway that encircled the interior of the quad. As he did so, he started yelling.

"Come on you screwed-up assholes," he screamed. "Come and get me."

He swerved around Karen Holm, who reached for him.

"Sorry for the language, Professor Holm, but come and get me," he yelled back at her. A pack of a dozen zombies was now trailing the bike. As Angus came around the far side of the quad, more joined the chase.

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Just as predicted, they didn't have the sense to cut across the quad and go to where he would be by the time they got there. They simply headed toward where he was at the instant, making it easy to stay ahead of them. They seemed to be settling into a circular path on a circle of a slightly smaller radius than the one Angus was taking.

"Slow down," yelled Marsha. "You don't want to lap them."

Angus now had all of the zombies in the quad lurching after him, some a few yards behind and some others quite a bit farther behind. But all were essentially in a pack that was following a circle inside his own circular path. As he passed my hiding place, he motioned for me to go. I waited until the parade of zombies passed, and then crept across the quad to Ortiz's car. I knew that any fast motion might trigger a reaction, so I moved slowly, crouching the entire time. When I reached the car, I could see Ortiz in the passenger compartment. His eyes were open, but there was blood dripping from a large cut on his forehead. I tried the door but it wouldn't open.

Tapping lightly on the window, I said quietly, "Unlock the door, Raphael." He turned to stare at me with an uncomprehending look.

"Come on Raphael. We don't have a lot of time. Unlock the door!" Out of the corner of my eye. I could see Angus coming around the quad again. It appeared that the mob following him had grown.

"How do I know you're not one of them?" asked Ortiz.

"Do you really think they know your name?" I responded. "Open the god damn door. I'm trying to rescue you."

Ortiz seemed to finally grasp the situation, and he leaned over and hit the button to unlock the door. I pulled on the handle and the door swung open.

"Can you walk?" I asked.

"I don't know," said Ortiz. "What's going on? I don't understand any of this."

"Just get out of the car."

He went to climb out, but his seatbelt jerked him back into the seat. I saw Angus and the zombies approaching.

"Get out of there!" yelled Angus.

I reached across Ortiz and unlatched the seat belt. Then I pulled him from the car. I could see several of the zombies peeling off from the pack and making a beeline for the car. I grabbed Ortiz's arm and swung it over my shoulder.

"Time to get a move on," I said, as I half dragged him across the quad.

Angus continued to yell at the top of his lungs. "Come on, you stupid evolutionary throwbacks. Even if you used to be smart, you aren't anymore."

Ortiz and I reached the Science Center and I banged on the door.

"Open up," I yelled.

Jessie pushed the door open, and we fell through as several zombies arrived. Jessie pulled the door shut. I lowered Ortiz to the floor and helped him prop himself up against the wall.

I could hear Angus on the other side of the door still yelling to attract the zombies in his direction.

"Come on, you demented faculty. You used to be so much smarter than me. But not anymore. Look who's leading the parade now."

"Angus, they're safe," yelled Marsha from her window. "Get in the building."

I couldn't hear Angus's response but the banging noises outside the door had stopped so I cracked it open. I could see Angus at the far side of the quad. All of the zombies were again following him in a pack. He was laughing when he spotted me.

"Look at me," he yelled. "I'm the pied piper of Westbridge." It was at the instant when he was looking back at me rather than ahead that the bike ran into the curb and he flipped over the handlebars.

"Oh, crap," I said. Angus landed in a heap. The horde of zombies closed on him. I knew I couldn't get there in time. Pulling the gun from my pocket, I aimed in the general direction of the mob, and fired several shots.

The sound of the gun was enough to attract the attention of a few. I started screaming. "Hey, over here, over here!" But they weren't about to give up on the meal right in front of them.

Angus stood wobbly. "I'm okay," he yelled, waving to me.

"Look behind you," I yelled back, motioning frantically.

He turned just in time to see the first zombie reaching for him. Leaping backward, over the bicycle, he then lifted it up and heaved it at the zombies. Several fell over, tangled in the bike. Then he turned, and sprinted toward us across the quad. There were now lots of zombies between him and us. Here was the true test of the theory. He was faster than they were and they always headed toward where he was. So as long as he didn't get within an arm's reach of them, he could get by before they cut him off. But upwards of thirty zombies were in pursuit.

"Move it, Angus!" I yelled. I aimed the gun toward the mob of zombies behind him, but didn't dare take a shot for fear of hitting him.

There was a row of zombies between him and us, moving toward him on either side of a picnic table. Angus never slowed. He just hit the bench with one foot and the top of the table with the other and then launched himself into the air, sailing over the head of one zombie who futilely tried to grab at him. Angus rolled on landing and then was back up on his feet, sprinting hard.

Just as he was closing on the Science Center, with a clear path between him and the door I was holding open, three zombies rounded the corner of the building, the first grabbing my left arm, and pulling it toward its gnashing teeth. Without even thinking about it, I lifted the gun and fired into its face, blowing the back of its head off. It fell instantly. The other two tumbled over the body as they reached for me. I pushed Jessie back into the building just as Angus dove through the door. I pulled it shut as the zombies scrabbled at us.

"Holy crap," said Angus, collapsed on the floor breathing heavily. "I haven't run that hard since I quit ultimate frisbee."

"Angus," said Jessie. "You were magnificent out there." He smiled at her as he pulled himself to a sitting position.

"Thanks, Professor Sullivan. The plan worked, didn't it? Hi, Professor Ortiz."

Ortiz looked at Angus for a moment and then said, "Angus? What are you doing here?"

"He just saved you," I said. "It was his plan."

"What plan?" asked Ortiz.

"He rode a bike in a circle and got all the zombies to follow him," said Jessie.

"The circular pursuit problem goes a long way back," I added.

"Yeah?" said Angus, still breathing hard. "How long?"

"Actually, it dates back to at least 1748."

"That's over 250 years ago," said Angus, visibly pleased. "Was there even calculus then?"

"Oh, yeah. Calculus had been around almost 100 years by then."

"Why were they interested in circular pursuit problems? There weren't zombies then."

"No, there weren't. The problem was first posed in terms of a spider trying to catch a fly walking along the edge of a semi-circular pane of glass."

"Really? And somebody cared about that?"

"Angus," said Jessie. "They didn't have TV and YouTube. Entertainment was hard to come by."

"That's true," I added. "There was a lot of general interest in recreational math back then. The problem first appeared in a British journal called the *Ladies' Diary*. Not exactly where you'd expect to find math problems. But they published a variety of math puzzlers." Ortiz seemed to be listening, which I took to be a good sign.

"Eventually the problem was rephrased in terms of a duck swimming around the edge of a circular pond and a dog swimming after it."

"Oh," said Angus. "A classic dog-and-duck problem." He was grinning.

I smiled. "If you say so. But it turns out that even though you can find the differential equations that need to be satisfied, you can't solve them analytically."

"How do you figure out the differential equations that give the zombie's path?" asked Angus.

"Look at it this way," I said. "Angus, you were going in a circular path at a constant speed. So we can say that your path was given by

$$\langle x_A(t), y_A(t)\rangle = \langle R\cos(\omega t), R\sin(\omega t)\rangle$$

where R is your radius and ω determines how fast you ride around the circle."

"How'd you get that?" asked Angus.

"Look," I said as I drew Figure 1.

"Since the angle is ωt, your x-coordinate is $R\cos(\omega t)$ and your y-coordinate is $R\sin(\omega t)$."

"Okay," said Angus.

"Then let $\langle x_Z(t), y_Z(t)\rangle$ be the position of the zombie at time t. Let s_Z be the speed of the zombie. All we know is that the zombie's velocity vector $\mathbf{v_Z}(t)$ always points toward you, Angus." I drew Figure 2. "So at a given time t, we know that $\mathbf{v_Z}(t) = \langle x'_Z(t), y'_Z(t)\rangle$ points at $\langle x_A, y_A\rangle$."

"So take the vector that goes from $\langle x_Z, y_Z\rangle$ to $\langle x_A, y_A\rangle$. That's the vector $\langle x_A - x_Z, y_A - y_Z\rangle$. Divide it by its length to get a unit vector in the same direction. So we have

$$\left\langle \frac{x_A - x_Z}{\sqrt{(x_A - x_Z)^2 + (y_A - y_Z)^2}}, \frac{y_A - y_Z}{\sqrt{(x_A - x_Z)^2 + (y_A - y_Z)^2}} \right\rangle$$

If we multiply this by s_Z, we have our velocity vector, since it has the right direction and it has the right length.

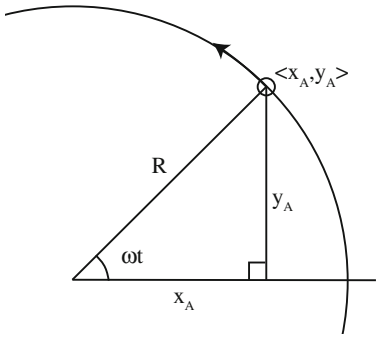

Figure 1. Angus's path in circular pursuit.

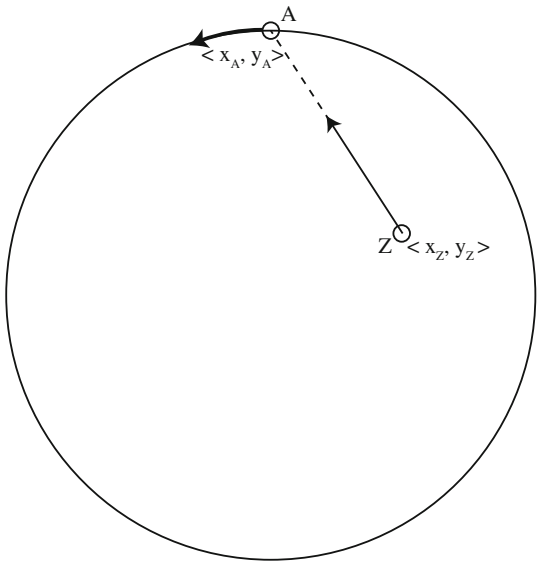

Figure 2. Determining the zombie's path in circle pursuit.

"Then we get

$$\mathbf{v_Z}(t) = \langle x'_Z, y'_Z\rangle = s_Z\left\langle \frac{x_A - x_Z}{\sqrt{(x_A - x_Z)^2 + (y_A - y_Z)^2}}, \frac{y_A - y_Z}{\sqrt{(x_A - x_Z)^2 + (y_A - y_Z)^2}} \right\rangle$$

By considering each component separately, we obtain two coupled differential equations:

$$\frac{dx_Z}{dt} = s_Z \frac{x_A - x_Z}{\sqrt{(x_A - x_Z)^2 + (y_A - y_Z)^2}}$$
$$\frac{dy_Z}{dt} = s_Z \frac{y_A - y_Z}{\sqrt{(x_A - x_Z)^2 + (y_A - y_Z)^2}}$$

Adding in the expressions for x_A and y_A, we obtain:

$$\frac{dx_Z}{dt} = s_Z \frac{R\cos(\omega t) - x_Z}{\sqrt{(R\cos(\omega t) - x_Z)^2 + (R\sin(\omega t) - y_Z)^2}}$$
$$\frac{dy_Z}{dt} = s_Z \frac{R\sin(\omega t) - y_Z}{\sqrt{R\cos(\omega t) - x_Z)^2 + (R\sin(\omega t) - y_Z)^2}}$$

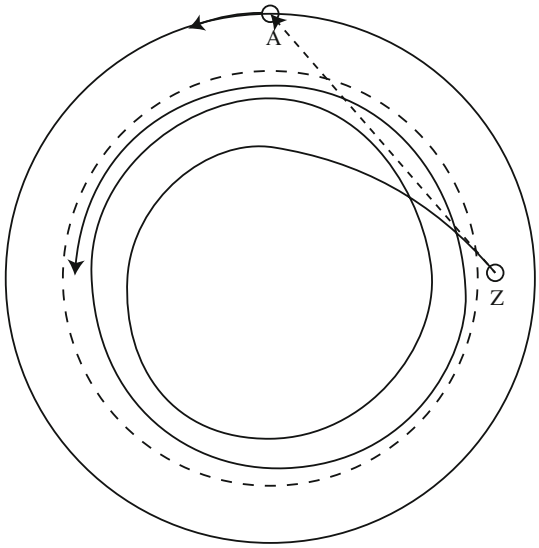

Figure 3. Zombie pursuing Angus, who rides his bike in a circle.

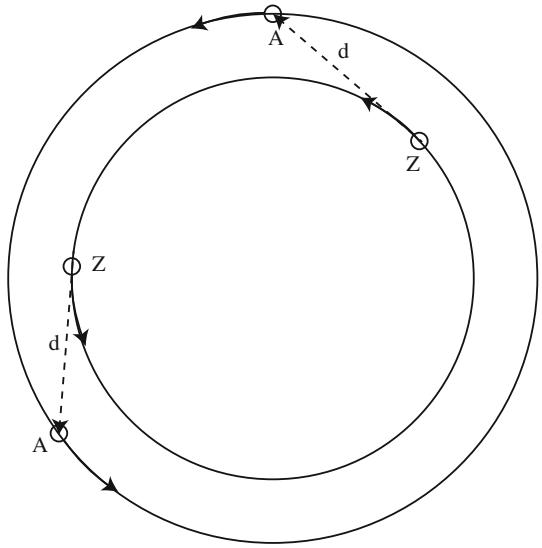

Figure 4. On the limit cycle, the zombie's distance to Angus never changes.

So these are the differential equations that need to be satisfied by the zombie's path. But they're too complicated to get an analytic solution."

'What do you mean when you say you can't solve them analytically?" asked Angus.

"It means you can't write down a solution in terms of the usual functions we work with, polynomials, trig functions, radicals, et cetera."

"Then how do you figure out what'll happen?"

"Well, you take the differential equations, and solve them numerically. The computer draws the resulting path for you."

"So why did the zombies follow me in a circle that was smaller than the circular path I was following?"

"That's interesting. When you do computer simulations, always assuming that you pedal faster than the speed of the zombie who is chasing you, and still assuming that the zombie is always headed straight for you, then no matter where the zombie starts, it eventually travels in a circle. I pulled a piece of paper out of my back pocket, and drew Figure 3.

"So here, where the zombie starts near the right side of your path, it eventually settles into a path that is closer and closer to a circle. That circle is called the *limit cycle*."

"I thought the zombie's tangent vector was supposed to always point at me."

"It does. See look." I drew Figure 4 on the paper.

"Once the zombie gets to a position like this, then it just follows you around. See its tangent vector is always pointed at you, and the distance between you and the zombie never changes. The size of the circle that the zombie follows is determined by the relative speed of the zombie to you."

"What do you mean?"

"Say the zombie moves at half your speed. Once the zombie settles into what is essentially the limit cycle circle, both you and the zombie travel around your respective circles in the same amount of time, call it t_0. But you travel the circumference of your circle, which is $2\pi R$, where R is the radius of your circle, while the zombie travels $2\pi r$, where r is the radius of its circle. So your speed is $\frac{2\pi R}{t_0}$, and its speed is $\frac{2\pi r}{t_0}$. If it travels half as fast as you, then

$$\frac{1}{2} = \frac{\text{zombie's speed}}{\text{your speed}} = \frac{\frac{2\pi r}{t_0}}{\frac{2\pi R}{t_0}} = \frac{r}{R}$$

So we get $r = \frac{R}{2}$. The radius of the zombie's circle is exactly half the radius of your circle. And in general, the radius of the zombie's circle will be the fraction of the radius of your circle corresponding to the fraction of your speed that is its speed."

"Then the zombie will never catch me, no matter where it starts. And since the zombies all seem to move at the same speed, it doesn't matter where they start. They all end up with the same limit cycle. Which is why they were all following me on that same circle," said Angus, pleased by his own comprehension.

"Exactly," I said. "And it looked like the circle they were on had a radius of $\frac{3}{4}$ of your circle. So their speed was $\frac{3}{4}$ of your speed."

"And what would've happened if I had slowed down?"

"Then the radius of their circle would have grown. It's radius would shift to be the fraction of the radius of your circle that their speed was as a fraction of your speed."

"But even though they came from different directions, after a while they were all bunched up on the circular path."

"Exactly," I said. "They can start from anywhere, but since they all move at the same speed, they all end up on the same radius circle. And when you were at a point on the outer circle, there was only one point on the inner circle that was behind you and that had its tangent vector pointed at you, So all the zombies ended up in a pack at that point."

"Too cool!" said Angus. "I want to try it again."

Jessie and I traded smiles. "Maybe some other time, Angus," I said.

At that point, a low moan echoed down the hall. "Time to go," I said.

REFERENCES

1. Adams, Colin, *Zombies & Calculus*, Princeton University Press, 2014.
2. Nahin, Paul, *Chases and Escapes*, Princeton University Press, 2007.