Solution exercise 2.

Problem 1 a) The first order condition reads $\sum_{m=0}^{s} a_m = 0$

Thus $P(1) = \sum_{m=0}^{s} a_m = 0$

b) $P(w) = w^3 - 1 = (w-1)(w^2+w+1) = (w-1)\left(w - \frac{-1+i\sqrt{3}}{2}\right)\left(w + \frac{1+i\sqrt{3}}{2}\right)$

All roots are distinct roots on the unit circle $\Rightarrow$

The root criterion is satisfied and the method is convergent.

$$\sigma(w) = \frac{3}{8}w^3 + \frac{9}{8}w^2 + \frac{9}{8}w + \frac{3}{8}$$

As usual set $w - 1 = \xi$ (or $w = 1+\xi$) (See example p. 23)

then:

$$P(w) = P(\xi) = \xi^3 + 3\xi^2 + 3\xi$$

$$\sigma(w) = \sigma(\xi) = \frac{1}{8}(3^3 + 18\xi^2 + 36\xi + 24)$$

$$\ln w = \ln(1+\xi) = \xi - \frac{\xi^2}{2} + \frac{\xi^3}{3} - \frac{\xi^4}{4} + \dots$$

giving:

$$P(w) - \sigma(w)\ln w =$$

$$(\xi^3 + 3\xi^2 + 3\xi) - \left(\frac{3}{8}\xi^3 + \frac{9}{4}\xi^2 + \frac{9}{2}\xi + 3\right)\left(\xi - \frac{\xi^2}{2} + \frac{\xi^3}{3} + \frac{\xi^4}{4} + \dots\right)$$

$$= \underset{\overset{\|}{0}}{(3 - 3)}\xi + \underset{\overset{\|}{0}}{\left(3 - \left(\frac{9}{2} - \frac{3}{2}\right)\right)}\xi^2 + \underset{\overset{\|}{0}}{\left(1 - \left(\frac{9}{4} - \frac{9}{4} + 3\cdot\frac{1}{3}\right)\right)}\xi^3 - \underset{\overset{\|}{0}}{\left(\frac{3}{8} - \frac{9}{8} + \frac{3}{2} - \frac{3}{4}\right)}\xi^4$$

$$- \underset{\overset{\|}{3/16}}{\left(\frac{3}{8}\left(-\frac{1}{2}\right) + \frac{9}{4}\cdot\frac{1}{3} + \frac{9}{2}\left(-\frac{1}{4}\right) + 3\cdot\frac{1}{5}\right)}\xi^5 + O(\xi^6)$$

We have $P(w) - \sigma(w)\ln w = \frac{3}{16}\left((w-1)^5\right) + O\left((w-1)^6\right)$

Thus the method is order 4

Problem 3

$y_{n+2} = y_n + 2hf(t_{n+1}, y_{n+1})$ applied to $y' = -y$ gives:

$$y_{n+2} + 2hy_{n+1} - y_n = 0$$

Which is a linear homogeneous difference equation with solutions

$$y_n = c_1 r_1^n + c_2 r_2^n$$

where $r_{1,2}$ solves

$$r^2 + 2hr - 1 = 0$$

$$r_{1,2} = -h \pm i\sqrt{1-h^2} \qquad (\text{Assuming } 0 < h < 1)$$

Complex roots give trouble. we rewrite i polar coordinates

$$r_{1,2} = \rho e^{\pm\theta}, \quad \text{Here } \rho = \sqrt{1^2 + (\sqrt{1-h^2})^2} = 1$$

$$\sin\theta = \frac{\sqrt{1-h^2}}{\rho} = \sqrt{1-h^2}$$

We replace the to (independent) basic solutions with

$$q_1 = \frac{r_1 + r_2}{2} = \frac{e^{i\theta} + e^{i\theta}}{2} = \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\sqrt{1-h^2}^2} = h$$

$$q_2 = \frac{r_1 - r_2}{2i} = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta = \sqrt{1-h^2}$$

New general solution:

$$y_n = c_1 h^n + c_2 (1-h^2)^{n/2}$$

$c_{1,2}$ determined by: $y_0 = 1$, $y_1 = 1-h$

$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ hc_1 + \sqrt{1-h^2}\,c_2 &= 1-h \end{aligned}\right\} \Rightarrow \quad c_1 = \frac{\sqrt{1-h^2}}{\sqrt{1-h^2}-h}; \quad c_2 = \frac{h}{h-\sqrt{1-h^2}}$$

True solution $y(t) = e^{-t}$; For $t = nh$ $y(nh) = e^{-nh} \to 1$ as $h \to 0$

Does $y_n \to 1$ as $h \to 0$?

$$\left.\begin{aligned} c_1 \to 1 \quad h^n \to 0 \quad & c_1 h^2 \to 0 \\ c_2 \to 0 \quad (1-h^2)^{n/2} \to 1 \quad & c_2 (1-h^2)^{n/2} \to 0 \end{aligned}\right\} \Rightarrow y_n = 0$$

Not the right way to compare; for $y_n \to y(T)$ as $h \to 0$ we must have $n = \frac{T - t_0}{h} \to \infty$