Problem 1

a) ① $u''' = t^2 u u'' - u v'$
$v'' = t v v' + 4u'$

set: $y_1 = u$
$y_2 = u'$
$y_3 = u''$
$y_4 = v$
$y_5 = v'$

② $$\underline{y}' = f(t,\underline{y}) \iff \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}' = \begin{pmatrix} y_2 \\ y_3 \\ t^2 y_1 y_3 - y_1 y_5 \\ y_5 \\ t\, y_4 y_5 + 4 y_2 \end{pmatrix}$$

b) $$f_y = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_5} \\ \vdots & & \\ \frac{\partial f_5}{\partial y_1} & \cdots & \frac{\partial f_5}{\partial y_5} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ t^2 y_3 - y_5 & 0 & t^2 y_1 & 0 & -y_1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & t y_5 & t y_4 \end{pmatrix}$$

c) $\underline{f}$ Lipschitz iff $\|\underline{f}(t,\underline{x}) - \underline{f}(t,\underline{y})\| \leq \lambda \|\underline{x} - \underline{y}\| \quad \forall x,y \in \mathbb{R}^d,\ t \geq t_0$

If $f, x, y \in \mathbb{R}^1$ the mean value theorem gives:
$$f(t,x) - f(t,y) = \frac{\partial f}{\partial x}(t, \xi)\,(x-y)$$
Thus if $\left|\frac{\partial f}{\partial x}\right| \leq \lambda \ \forall x \in \mathbb{R}^1$ we have $|f(t,x) - f(t,y)| \leq \lambda |x-y|$
and the function is Lipschitz. The mean value theorem does not extend straightforward to x, y & f in higher dimensions.
But let us do a multi-dim Taylor expansion*:
$$\underline{f}(t,\underline{y}) = f(t, x + (\underline{y} - x)) = \underline{f}(t,\underline{x}) + f_{\underline{y}}(\xi)\,(\underline{y} - \underline{x})$$
$$\Rightarrow \quad \underline{f}(t,\underline{x}) - f(t,\underline{y}) = \underline{f}_{\underline{y}}(\xi)\,(\underline{x} - \underline{y}) \qquad \|$$

$$\|f(t,\underline{x}) - f(t,\underline{y})\| = \|f_{\underline{y}}(\underline{\xi})\cdot(\underline{x}-\underline{y})\| \leq \|f_{\underline{y}}(\xi)\|\,\|x-y\|$$

Thus if we can bound $\|f_{\underline{y}}(\xi)\|$ for any ξ we have a Lipschitz constant. Here: t: $0 \leq t \leq 1$

$\|y\|_1 \leq 1 \Rightarrow |y_i| \leq 1$, Thus all components in $f_{\underline{y}}$ is bounded and so is $\|f_{\underline{y}}\|$. The smallest possible λ depends on the norm we choose.

Problem 2

a) $\underline{y}' = A\underline{y}$; $A \in \mathbb{R}^{s\times s}$ & $A = A^T$ $\Rightarrow$ $\exists Q$ orthogonal $(Q^TQ = \mathbb{I})$

$\underline{y}(0) = \underline{y}_0$ $\qquad A = Q^TDQ$, set $u = Q\underline{y} =$

Exact solution: $Q\underline{y}' = QA\underline{y}$

$$\underline{u}' = D\underline{u} \Rightarrow \underline{u}(t) = \overbrace{\begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_s t}\end{pmatrix}}^{=E} \underline{u}_0 \Rightarrow \underline{y}(t) = E(t)\,\underline{y}_0$$

Euler:

$$y_{n+1} = y_n + hAy_n = (\mathbb{I} + hA)y_n \Rightarrow y_n = (\mathbb{I}+hA)^n y_0$$
$$= [Q^T(I+hD)Q]^n y_0$$

Error:

$$e_n = y_n - y(t_n) = \left(-E(t) + (\mathbb{I}+hA)^n\right) y_0$$

$$\|e_n\|_2 \leq \|(\mathbb{I}+hA)^n - E(t)\|_2 \|y_0\|_2$$

$$= \|Q^T[(\mathbb{I}+hD)^n - E(t)]Q\|_2 \|y_0\|_2$$

$$= \|(\mathbb{I}+hD)^n - E(t)\|_2 \|y_0\|$$

$$= \max_{\lambda\in\sigma(A)} |(1+h\lambda)^n - e^{nh\lambda}|\,\|y_0\|$$

Note 1: $(I+hA) = Q^T(\mathbb{I}+hD)Q$
then $(\mathbb{I}+hA)^n =$
$Q^T(\mathbb{I}+hD)\underbrace{QQ^T}_{=I}(\mathbb{I}+hD)\cdots QQ^T(I+hD)Q$
$= Q^T(1+hD)^nQ$

Note 2: Since $E(t)$ is a diagonal matrix
$E(t) = Q^TE(t)Q$

Note 3: $\|Q^TBQ\|_2 = \|B\|$ whenever Q is ortogonal

Note 4. $\|D\|_2$; D diagonal $= \max|d_i|$

Remark: Contrary to the estimate in proof of Th. 1.1 this is a good estimate for the error.

b

$$e^x = 1 + x + \frac{\xi^2}{2} \; ; \; \xi \in [-1, 0] \Rightarrow e^x \geq (1+x) \Rightarrow e^{nx} \geq (1+x)^n$$

proves the right inequality

$$e^x = 1 + x + \frac{x^2}{2} + \frac{\xi^3}{3!} \quad \xi \in [-1, 0] \Rightarrow e^x \leq 1 + x + \frac{x^2}{2}$$

same even

$$(a-b)^n = a^n - n a^{n-1} b + \binom{n}{2} a^{n-2} b^2 - \binom{n}{3} a^{n-3} b^3 + \dots$$

$$= a^n - n a^{n-1} b + \sum_{k=1}^{n/2} a^{n-2k-1} b^k \left(\binom{n}{2k} a - \binom{n}{2k+1} b \right)$$

$|a-1| \ll 1$, $|b| \ll 1$

$\Rightarrow a \gg |b|$

and $\left(\binom{n}{2k} a - \binom{n}{2k+1} b \right) > 0$

$$\geq a^n - n a^{n-1} b$$

Set $a = 1 + x + \frac{x^2}{2}$; $b = \frac{x^2}{2}$ then:

$$(1+x)^n = (a-b)^n \geq a^n - n a^{n-1} b = \left(1 + x + \frac{x^2}{2}\right)^n - n \left(1 + x + \frac{x^2}{2}\right)^{n-1} \frac{x^2}{2}$$

$$\geq e^{xn} - \frac{n x^2}{2} e^{x(n-1)} \quad (**)$$

c) From a

$nh = t$

$$\|e_n\|_2 \leq \left| (1 + h\lambda_{max})^n - e^{nh\lambda_{max}} \right| \|y_0\|$$

$$\leq \left| \cancel{e^{n\lambda_{max} h}} - \frac{n}{2} (h\lambda_{max})^2 e^{h\lambda_{max}(n-1)} - \cancel{e^{nh\lambda_{max}}} \right| \|y_0\|$$

$$= \frac{1}{2} \left| t \lambda_{max}^2 \cdot e^{(t-h)\lambda_{max}} \right| h \|y_0\|$$

$$\leq \frac{1}{2} t^* h \lambda_{max}^2 \|y_0\| \quad \left(\text{since } \lambda_{max} < 0 \quad e^{t\lambda_{max}} < 1\right)$$

d) eigen values of $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ are $(-3, -1)$

From formula p. 7 (Here the λ is the Lipschitz constant which will be equall to $\|A\|_2 = |\lambda_{max}| = 3$

$$|e_{n,h}\|_2 = \frac{C}{3} (e^{3t^*} - 1) h \; ;$$

From above

$$\|e_n\|_2 \leq \frac{1}{2} t^* h \, 3^2 \|y_0\|$$

linear vs. exponential in t^*