Examiners' commentaries 2016

MT105b Mathematics 2

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2015–16. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2011). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements — if none are available, please use the contents list and index of the new edition to find the relevant section.

General remarks

Learning outcomes

At the end of this half course and having completed the Essential reading and activities, you should have:

- used the concepts, terminology, methods and conventions covered in the half course to solve mathematical problems in this subject;
- the ability to solve unseen mathematical problems involving the understanding of these concepts and application of these methods;
- seen how mathematical techniques can be used to solve problems in economics and related subjects.

Showing your working

We start by emphasising that you should **always** include your working. This means two things. Firstly, you should not simply write down the answer in the examination script, but you should explain the method by which it is obtained. Secondly, you should include rough working (even if it is messy!). The examiners want you to get the right answers, of course, but it is more important that you demonstrate that you know what you are doing: this is what is really being examined.

We also stress that if you have not completely solved a problem, you may still be awarded marks for a partial, incomplete, or slightly wrong, solution; but, if you have written down a wrong answer and nothing else, no marks can be awarded. So it is certainly in your interests to include all your workings.

Covering the syllabus and choosing questions

You should ensure that you have covered the bulk of the syllabus in order to perform well in the examination: it is bad practice to concentrate only on a small range of major topics in the expectation that there will be lots of marks obtainable for questions on these topics. There are no formal options in this course: you should study the full extent of the topics described in the syllabus and subject guide. In particular, since the whole syllabus is examinable, **any** topic could potentially appear in the examination questions.

Expectations of the examination paper

Every examination paper is different. You should not assume that your examination will be almost identical to the previous year's: for instance, just because there was a question, or a part of a question, on a certain topic last year, you should not assume that there will be one on the same topic this year. Each year, the examiners want to test that candidates know and understand a number of mathematical methods and, in setting an examination paper, they try to test whether the candidate does indeed know the methods, understands them and is able to use them, and not merely whether he or she vaguely remembers them. Because of this, every year there are some questions which are likely to seem unfamiliar, or different from previous years' questions. You should **expect** to be surprised by some of the questions. Of course, you will only be examined on material in the syllabus, so all questions can be answered using the material of the course. There will be enough routine, familiar content in the examination so that a candidate who has achieved competence in the course will pass, but, of course, for a high mark, more is expected: you will have to demonstrate an ability to solve new and unfamiliar problems.

Answer the question

Please do read the questions carefully. You might be asked to use specific methods, even when other methods could be used. The purpose of the examination is to test that you know certain methods, so the examiners might ask you to use a specific technique. In such circumstances, very little (or no) credit can be given if you do not do what the question asks. It is also worth reading the question carefully so that you do not do more than is required. For instance, if a question asked you only to find the critical points of a function, but not their natures, then you should not determine their natures. Look out for such things! Even though a question may look like one from a previous examination at first glance, there can be subtle differences!

Calculators

You are reminded that calculators are **not** permitted in the examination for this course, under any circumstances. The Examiners know this, and so they set questions that do not require a calculator. It is a good idea to prepare for this by attempting not to use your calculator as you study and revise for this course.

Examination revision strategy

Many candidates are disappointed to find that their examination performance is poorer than they expected. This may be due to a number of reasons. The Examiners commentaries suggest ways of addressing common problems and improving your performance. One particular failing is **'question spotting'**, that is, confining your examination preparation to a few questions and/or topics which have come up in past papers for the course. This can have serious consequences.

We recognise that candidates may not cover all topics in the syllabus in the same depth, but you need to be aware that Examiners are free to set questions on **any aspect** of the syllabus. This means that you need to study enough of the syllabus to enable you to answer the required number of examination questions.

The syllabus can be found in the Course information sheet in the section of the VLE dedicated to each course. You should read the syllabus carefully and ensure that you cover sufficient material in preparation for the examination. Examiners will vary the topics and questions from year to year and may well set questions that have not appeared in past papers. Examination papers may legitimately include questions on any topic in the syllabus. So, although past papers can be helpful during your revision, you cannot assume that topics or specific questions that have come up in past examinations will occur again.

If you rely on a question spotting strategy, it is likely you will find yourself in difficulties when you sit the examination paper. We strongly advise you not to adopt this strategy.

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Comments on specific questions — Zone A

Question 1

The inverse supply function for the market is $p^S(q) = aq + b$ for some constants $a, b > 0$ and the equilibrium price and quantity are 7 and 2 respectively. In particular, this means that a and b must satisfy the equation

$$7 = 2a + b,$$

as the equilibrium point must be on the supply curve. We are also told that the producer surplus is four and, following Section 2.9 of the subject guide, we see that as the producer surplus is given by

$$\text{PS} = p^*q^* - \int_0^{q^*} p^S(q)\,\mathrm{d}q,$$

we have

$$\int_0^2 (aq + b)\,\mathrm{d}q = \left[a\frac{q^2}{2} + bq\right]_0^2 = 2a + 2b - (0 + 0) = 2a + 2b,$$

so that

$$4 = (7)(2) - (2a + 2b) \quad\Longrightarrow\quad 2a + 2b = 10,$$

is another equation that a and b must satisfy. Consequently, solving these two equations simultaneously, we find that $a = 2$ and $b = 3$.[1]

[1] An alternative method here would be to observe that the supply curve is a straight line and so the producer surplus is the area of a triangular region whose height is $7 - b$ and whose base is 2. This means that, if we find the area of this triangle, we have

$$4 = \tfrac{1}{2}(7 - b)(2) \quad\Longrightarrow\quad 7 - b = 4 \quad\Longrightarrow\quad b = 3.$$

Then, again using the equation $7 = 2a + b$ that we found above, we use $b = 3$ to see that $a = 2$. Of course, if you wished to tackle the question this way, the Examiners would expect your answer to be justified by means of a sketch of the supply function and an identification of this clearly indicated triangular region with the producer surplus!

Question 2

As in Section 2.4 of the subject guide, Taylor's theorem about $x = 0$ states that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \cdots$$

and so, bearing in mind that we are only interested in terms up to x^4, we have

$$\begin{aligned}
f(x) &= (4+x)^{-1} &\implies&\quad f(0) = 1/4\\
f'(x) &= -(4+x)^{-2} &\implies&\quad f'(0) = -1/16\\
f''(x) &= 2(4+x)^{-3} &\implies&\quad f''(0) = 2/64 = 1/32\\
f'''(x) &= -6(4+x)^{-4} &\implies&\quad f'''(0) = -6/256 = -3/128\\
f''''(x) &= 24(4+x)^{-5} &\implies&\quad f''''(0) = 24/1024 = 3/128
\end{aligned}$$

and this gives us

$$(4+x)^{-1} = \frac{1}{4} + \left(-\frac{1}{16}\right)x + \left(\frac{1}{32}\right)\frac{x^2}{2} + \left(-\frac{3}{128}\right)\frac{x^3}{6} + \left(\frac{3}{128}\right)\frac{x^4}{24} + \cdots = \frac{1}{4} - \frac{x}{16} + \frac{x^2}{64} - \frac{x^3}{256} + \frac{x^4}{1024} + \cdots$$

as required.

To find the sum of the series

$$1 - \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots,$$

we have two methods available to us. Following 'hence', we want to rewrite this series so that it looks like the one we have just found. That is, we can note that

$$\begin{aligned}
1 - \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots &= 1 - \frac{3}{4} + \frac{3^2}{16} - \frac{3^3}{64} + \frac{3^4}{256} + \cdots\\
&= 4\left[\frac{1}{4} - \frac{3}{16} + \frac{3^2}{64} - \frac{3^3}{256} + \frac{3^4}{1024} + \cdots\right],
\end{aligned}$$

where the series in the square brackets is just our Taylor series for $(4+x)^{-1}$ with $x = 3$. As such, we see that

$$1 - \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots = 4(4+3)^{-1} = \frac{4}{7},$$

is the sought after sum. Alternatively, following 'otherwise', we note that we are being asked to sum a geometric series with first term 1, common ratio $-3/4$ and an infinite number of terms. As such, using the formula, we see that

$$1 - \frac{3}{4} + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots = \frac{1}{1-(-3/4)} = \frac{4}{7},$$

which is, once again, the sought after sum.[2]

[2] Observe that, of these two methods, although the latter may seem more straightforward, the former is actually more powerful as it allows us to use Taylor series to sum series which are not geometric!

Question 3

We need to use the method of Lagrange multipliers to find the minimum value of

$$\frac{1}{x+1}+\frac{1}{4(y+2)}+\frac{1}{9(z+3)},$$

subject to the constraint $x+y+z=16$ where x, y and z are positive numbers. To do this, as in Section 3.4.4 of the subject guide, we use the Lagrangean

$$L(x,y,z,\lambda)=\frac{1}{x+1}+\frac{1}{4(y+2)}+\frac{1}{9(z+3)}-\lambda(x+y+z-16),$$

and solve the equations

$$\begin{aligned} L_x(x,y,z,\lambda) &= -\frac{1}{(x+1)^2}-\lambda=0,\\ L_y(x,y,z,\lambda) &= -\frac{1/4}{(y+2)^2}-\lambda=0,\\ L_z(x,y,z,\lambda) &= -\frac{1/9}{(z+3)^2}-\lambda=0,\\ L_\lambda(x,y,z,\lambda) &= -(x+y+z-16)=0 \end{aligned}$$

simultaneously. Eliminating λ from the first three equations yields

$$-\lambda=\frac{1}{(x+1)^2}=\frac{1/4}{(y+2)^2}=\frac{1/9}{(z+3)^2} \qquad \Longrightarrow \qquad x+1=2(y+2)=3(z+3),$$

as x, y and z must be positive here. Thus, in terms of z (say) we have,

$$x+1=3(z+3) \qquad \Longrightarrow \qquad x=3z+8,$$

and

$$y+2=\tfrac{3}{2}(z+3) \qquad \Longrightarrow \qquad y=\tfrac{3}{2}z+\tfrac{5}{2}.$$

Substituting these into the fourth equation (i.e. the constraint) we then find that

$$(3z+8)+\left(\tfrac{3}{2}z+\tfrac{5}{2}\right)+z=16 \quad \Longrightarrow \quad \frac{11}{2}z=\frac{11}{2} \quad \Longrightarrow \quad z=1,$$

which means that we also have

$$x=3(1)+8=11 \qquad \text{and} \qquad y=\tfrac{3}{2}(1)+\tfrac{5}{2}=4$$

if we use $x=3z+8$ and $y=\frac{3}{2}z+\frac{5}{2}$ respectively. Consequently, we find that the minimum value of

$$\frac{1}{x+1}+\frac{1}{4(y+2)}+\frac{1}{9(z+3)} \quad \text{is} \quad \frac{1}{11+1}+\frac{1}{4(4+2)}+\frac{1}{9(1+3)}=\frac{1}{12}+\frac{1}{24}+\frac{1}{36}=\frac{11}{72},$$

if we use the values of x, y and z that we have just found.

Question 4

As in Section 2.5 of the subject guide, the elasticity of demand is given by

$$\varepsilon=-\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p},$$

where $q=q^D(p)$ is the demand function. In this case, as in Section 5.11 of the subject guide, we are given that

$$\varepsilon=\frac{3p^2+5p}{p^2+3p+2},$$

and so we need to solve the differential equation

$$-\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{3p^2+5p}{p^2+3p+2},$$

to find $q^D(p)$ given that we know that $q^D(1) = 1$. Indeed, this differential equation is separable and so using the method in Section 5.2 of the subject guide, we rewrite it as

$$-\int \frac{\mathrm{d}q}{q} = \int \frac{3p+5}{p^2+3p+2}\,\mathrm{d}p,$$

and determine the integrals. So, noting that partial fractions gives us

$$\frac{3p+5}{p^2+3p+2} = \frac{3p+5}{(p+1)(p+2)} = \frac{2}{p+1} + \frac{1}{p+2},$$

and that prices and quantities are always positive, we get

$$-\ln q = 2\ln(p+1) + \ln(p+2) + c,$$

where c is an arbitrary constant. So, to proceed, we could set $k = \mathrm{e}^c$ so that $c = \ln k$, and this then gives us

$$\ln\left(\frac{1}{q}\right) = \ln(p+1)^2 + \ln(p+2) + \ln k \implies \ln\left(\frac{1}{q}\right) = \ln[k(p+1)^2(p+2)] \implies \frac{1}{q} = k(p+1)^2(p+2),$$

which means that the general solution to our differential equation is

$$q = \frac{1}{k(p+1)^2(p+2)}.$$

Now, as we know that $q^D(1) = 1$, we get

$$1 = \frac{1}{12k} \implies k = \frac{1}{12},$$

and so the particular solution to our differential equation is

$$q = \frac{12}{(p+1)^2(p+2)} \quad \text{which means that} \quad q^D(p) = \frac{12}{(p+1)^2(p+2)},$$

is the required demand function as we have $q = q^D(p)$.

Question 5

The supply and demand functions for this market are

$$q^S(p) = p - 6 \qquad \text{and} \qquad q^D(p) = 18 - 2p,$$

and so, since the equilibrium price occurs when $q^S(p) = q^D(p)$, we see that

$$p - 6 = 18 - 2p \implies 3p = 24 \implies p = 8,$$

is the equilibrium price and this gives an equilibrium quantity of $q^D(8) = 18 - 2(8) = 2$.

Now, as in Section 6.6 of the subject guide, the cobweb model dictates that

$$q_t = q^S(p_{t-1}) \qquad \text{and} \qquad p_t = p^D(q_t).$$

and, to apply these, we need to find the inverse demand function by rearranging $q = q^D(p)$ to get $p = p^D(q)$, i.e.

$$q = 18 - 2p \implies p = \frac{18-q}{2} \quad \text{and so} \quad p^D(q) = \frac{18-q}{2}.$$

Then, using the given $q^S(p)$ and this $p^D(q)$, we see that the cobweb model equations give us

$$q_t = p_{t-1} - 6 \qquad \text{and} \qquad p_t = \frac{18 - q_t}{2},$$

respectively. Thus, eliminating q_t from these two equations, the recurrence equation we seek is

$$p_t = \frac{18 - (p_{t-1} - 6)}{2} = \frac{24 - p_{t-1}}{2} = 12 - \tfrac{1}{2}p_{t-1}.$$

To solve this, we use the standard method from Section 6.4 of the subject guide which involves comparing it to the standard form $p_t = ap_{t-1} + b$ to see that $a = -1/2$ and $b = 12$. In this case, $a \neq 1$ and so we have a time-independent solution, p^*, which is given by

$$p^* = \frac{b}{1-a} \qquad \Longrightarrow \qquad p^* = \frac{12}{1 - (-1/2)} = 8,$$

so that, using the formula $p_t = p^* + (p_0 - p^*)a^t$, we have the solution

$$p_t = 8 + (4 - 8)\left(-\frac{1}{2}\right)^t = 8 - 4\left(-\frac{1}{2}\right)^t,$$

as $p_0 = 4$. Indeed, as in Section 6.5 of the subject guide, we can see that as t increases, p_t oscillates decreasingly towards 8.[3]

Question 6

To solve the second-order difference equation

$$y_t + 5y_{t-1} + 6y_{t-2} = 24,$$

we follow the method in Sections 6.8 and 6.9 of the subject guide. Here the auxiliary equation is

$$m^2 + 5m + 6 = 0 \qquad \Longrightarrow \qquad (m+2)(m+3) = 0 \qquad \Longrightarrow \qquad m = -2, -3.$$

In this case, as there are two distinct real solutions, the complementary sequence is

$$y_t = A(-2)^t + B(-3)^t,$$

for some arbitrary constants A and B. To find a particular sequence, as the right-hand side is a constant, we try something of the form $y_t = \alpha$ where α is a constant to be determined. Thus, $y_{t-1} = \alpha$ and $y_{t-2} = \alpha$ and, substituting these in to the given difference equation, we get

$$\alpha + 5\alpha + 6\alpha = 24 \qquad \Longrightarrow \qquad 12\alpha = 24 \qquad \Longrightarrow \qquad \alpha = 2.$$

Consequently, the particular solution is $y_t = 2$ and, adding this to the complementary sequence, we get

$$y_t = A(-2)^t + B(-3)^t + 2,$$

as the general solution of the given difference equation. Indeed, since $y_0 = 0$, we have

$$0 = A + B + 2 \qquad \Longrightarrow \qquad A + B = -2,$$

and, since $y_1 = 4$ as well, we also have

$$4 = -2A - 3B + 2 \qquad \Longrightarrow \qquad 2A + 3B = -2.$$

Solving these equations simultaneously, we can then find that $A = -4$ and $B = 2$, which allows us to see that

$$y_t = -4(-2)^t + 2(-3)^t + 2,$$

is the required particular solution.

[3] Observe, in particular, that the *behaviour* of p_t as t increases is that it is *oscillating decreasingly* to 8, its limit as $t \to \infty$. You could also add that this limit is the equilibrium price (which we found at the start of the question) and this means that we have a *stable* cobweb here.

Question 7

(a) As in Section 4.8 of the subject guide, we use row operations to deal with the given system of equations. Indeed, from the equations we get the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 1 & -1 & a & b \end{array}\right),$$

and, performing the obvious row operations, we then get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 3 \\ 1 & -1 & a & b \end{array}\right) \xrightarrow[R_3 \to R_1 - R_3]{R_2 \to 2R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 1-a & 2-b \end{array}\right) \xrightarrow{R_3 \to 2R_2 - R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 5+a & b \end{array}\right).$$

Now, at this point, some care must be taken because row operations require that we multiply by *non-zero* constants and, as we don't know the value of a, we can't be sure that $5 + a$ *is* non-zero. So, at this point we stop doing row operations and look at the third row of our augmented matrix and to see that:

(i) If $a = -5$ and $b \neq 0$, there are no solutions.

(ii) If $a \neq -5$ and b is any real number, there is a unique solution given by the equations

$$x + y + z = 2, \quad y + 3z = 1 \quad \text{and} \quad (5 + a)z = b.$$

Now, using back-substitution, we can see that the

- third equation gives us $z = \dfrac{b}{5+a}$, so the
- second equation gives us $y = 1 - \dfrac{3b}{5+a} = \dfrac{5+a-3b}{5+a}$, and then the
- first equation gives us $x = 1 + \dfrac{2b}{5+a} = \dfrac{5+a+2b}{5+a}$.

So, writing this in vector form we find that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{5+a} \begin{pmatrix} b \\ 5+a-3b \\ 5+a+2b \end{pmatrix},$$

is the required unique solution.

(iii) If $a = -5$ and $b = 0$, there are an infinite number of solutions given by the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and so we have the equations

$$x + y + z = 2 \quad \text{and} \quad y + 3z = 1.$$

Now, using back-substitution, we set $z = t \in \mathbb{R}$ so that the second equation gives us

$$y = 1 - 3t,$$

and then the first equation gives us

$$x = 2 - (1 - 3t) - t = 1 + 2t.$$

So, writing these in vector form we find that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2t \\ 1-3t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix},$$

for $t \in \mathbb{R}$ is the required infinite number of solutions.

Note: In part **(iii)**, the positions of the 'leading ones' (in the first and second rows of the echelon form) mean that, when we do the back-substitution, we can easily find y in terms of z and then x in terms of z. This means that, in this case, we **should** pick z to be the parameter t which can be any real number. This holds more generally: Any variable associated with a 'leading one' in the echelon form **should** be determined (via back-substitution) in terms of the variables that are not associated with a 'leading one' and it is these latter variables that should be assigned parameters which can be any real number. Up until now, the Examiners have not penalised poor choices when it comes to choosing which variable(s) should be 'any real number' but, from next year, for full credit in questions on this topic, they will expect you to do this properly.

(b) As in Section 3.3 of the subject guide, for the function q given by

$$q(k,l) = \frac{k^\alpha + l^4}{k^\beta l^5} \ln(k^\gamma \sqrt{l}),$$

to have increasing returns to scale it must be homogeneous and its degree of homogeneity must be greater than one. In particular, for it to be homogeneous,

- the term within the logarithm must be homogeneous of degree zero, i.e. we must have $\gamma = -1/2$,
- the numerator must be homogeneous, i.e. we must have $\alpha = 4$.

Then, as the degree of homogeneity of q is $4 - (\beta + 5)$, for increasing returns to scale we need

$$4 - (\beta + 5) > 1 \quad \Longrightarrow \quad -\beta > 2 \quad \Longrightarrow \quad \beta < -2.$$

Thus, all the possible values of α, β and γ are given by $\alpha = 4$, $\beta < -2$ and $\gamma = -1/2$.

Question 8

(a) As in Section 5.6.1 of the subject guide, it is easy to reduce the given coupled system of differential equations to the given second-order differential equation by noting that the first equation gives us

$$v(t) = u'(t) + u(t) - 4, \tag{$\star$}$$

and so differentiating this with respect to t we get

$$v'(t) = u''(t) + u'(t).$$

Then if we substitute these expressions for $v(t)$ and $v'(t)$ into the second equation we get

$$u''(t) + u'(t) = -u(t) - 3\big[u'(t) + u(t) - 4\big],$$

and this can be simplified to give

$$u''(t) + 4u'(t) + 4u(t) = 12,$$

as required. We are also given $u(0) = 2$ and $v(0) = 0$ in the question and so, using the first equation with $t = 0$, we have

$$u'(0) = 4 - u(0) + v(0) = 4 - 2 + 0 = 2.$$

That is, we also have $u(0) = 2$ and $u'(0) = 2$ as required.

(b) We then **use this result** to find $u(t)$ by noting that this is a second-order differential equation with constant coefficients and, as such, it can be solved using the ideas in Section 5.4 of the subject guide. Here, the auxiliary equation is

$$m^2 + 4m + 4 = 0 \implies (m+2)^2 = 0 \implies m = -2, -2.$$

In this case, as there is one repeated real solution, the complementary function is

$$u_c(t) = (At + B)e^{-2t},$$

for arbitrary constants A and B. To find a particular solution, we notice that the right-hand-side of our differential equation is a constant and so we seek a particular solution of the form $u_p(t) = \alpha$ where α is a constant that has to be determined. This gives us

$$u_p'(t) = 0 \qquad \text{and} \qquad u_p''(t) = 0,$$

which, when substituted into the differential equation, gives us

$$0 + 4(0) + 4\alpha = 12 \implies 4\alpha = 12 \implies \alpha = 3,$$

and so $u_p(t) = 3$. Lastly, adding these together, we get the general solution of our differential equation, which is

$$u(t) = u_c(t) + u_p(t) = (At + B)e^{-2t} + 3,$$

for arbitrary constants A and B.

We now want to find the function, $u(t)$, of this form that also satisfies our initial conditions. Indeed, since $u(0) = 2$, we have

$$2 = B + 3 \implies B = -1,$$

which means that

$$u(t) = (At - 1)e^{-2t} + 3,$$

for some arbitrary constant A. Then, using the product rule, we can see that

$$u'(t) = Ae^{-2t} - 2(At - 1)e^{-2t},$$

so that, since $u'(0) = 2$, we have

$$2 = A + 2 \implies A = 0.$$

Thus, we have found that

$$u(t) = 3 - e^{-2t},$$

is the particular solution for $u(t)$. We can now see that, as t increases, this function will increase to 3.[4]

(c) We are then asked to find $v(t)$ and, to do this, it makes sense to use ($\star$) from part **(a)** and our answer to part **(b)**. That is, from part **(b)** we have

$$u(t) = 3 - e^{-2t} \implies u'(t) = 2e^{-2t},$$

which, when substituted into ($\star$) from part **(a)**, gives us

$$v(t) = u'(t) + u(t) - 4 = 2e^{-2t} + \left[3 - e^{-2t}\right] - 4 = -1 + e^{-2t},$$

as the particular solution for $v(t)$. We can now see that, as t increases, this function will decrease to -1.[5]

[4]Observe, in particular, that the *behaviour* of $u(t)$ as t increases is that it is *increasing* to 3, its limit as $t \to \infty$. Indeed, if you can't see why it is *increasing*, here are two ways to think about it:

- e^{-2t} is decreasing to zero and so $-e^{-2t}$ must be increasing to zero. Thus, $u(t) = 3 - e^{-2t}$ must be increasing to 3.
- $u'(t) = 2e^{-2t} > 0$ for all $t \geq 0$ and so $u(t)$ is increasing.

[5]Observe, in particular, that the *behaviour* of $v(t)$ as t increases is that it is *decreasing* to -1, its limit as $t \to \infty$. Indeed, if you can't see why it is *decreasing*, here are two ways to think about it:

- e^{-2t} is decreasing to zero and so $v(t) = -1 + e^{-2t}$ must be decreasing to -1.
- $v'(t) = -2e^{-2t} < 0$ for all $t \geq 0$ and so $v(t)$ is decreasing.

Examiners' commentaries 2016

MT105b Mathematics 2

Important note

This commentary reflects the examination and assessment arrangements for this course in the academic year 2015–16. The format and structure of the examination may change in future years, and any such changes will be publicised on the virtual learning environment (VLE).

Information about the subject guide and the Essential reading references

Unless otherwise stated, all cross-references will be to the latest version of the subject guide (2011). You should always attempt to use the most recent edition of any Essential reading textbook, even if the commentary and/or online reading list and/or subject guide refer to an earlier edition. If different editions of Essential reading are listed, please check the VLE for reading supplements — if none are available, please use the contents list and index of the new edition to find the relevant section.

Comments on specific questions — Zone B

Question 1

The inverse demand function for the market is $p^D(q) = a - bq$ for some constants $a, b > 0$ and the equilibrium price and quantity are 4 and 1 respectively. In particular, this means that a and b must satisfy the equation

$$4 = a - b,$$

as the equilibrium point must be on the demand curve. We are also told that the consumer surplus is one and, following Section 2.9 of the subject guide, we see that as the consumer surplus is given by

$$\text{PS} = \int_0^{q^*} p^D(q)\,\mathrm{d}q - p^* q^*,$$

we have

$$\int_0^1 (a - bq)\,\mathrm{d}q = \left[aq - b\frac{q^2}{2}\right]_0^1 = a - \frac{b}{2} - (0 - 0) = a - \frac{b}{2},$$

so that

$$1 = \left(a - \frac{b}{2}\right) - (4)(1) \quad \Longrightarrow \quad a - \frac{b}{2} = 5,$$

is another equation that a and b must satisfy. Consequently, solving these two equations simultaneously, we find that $a = 6$ and $b = 2$.[1]

[1] An alternative method here would be to observe that the demand curve is a straight line and so the consumer surplus is the area of a triangular region whose height is $a - 4$ and whose base is 1. This means that, if we find the area of this triangle, we have

$$1 = \tfrac{1}{2}(a-4)(1) \quad \Longrightarrow \quad a - 4 = 2 \quad \Longrightarrow \quad a = 6.$$

Then, again using the equation $4 = a - b$ that we found above, we use $a = 6$ to see that $b = 2$. Of course, if you wished to tackle the question this way, the Examiners would expect your answer to be justified by means of a sketch of the demand function and an identification of this clearly indicated triangular region with the consumer surplus!

Question 2

As in Section 2.4 of the subject guide, Taylor's theorem about $x = 0$ states that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \cdots$$

and so, bearing in mind that we are only interested in terms up to x^4, we have

$$\begin{aligned}
f(x) &= (2+x)^{-1} &&\implies& f(0) &= 1/2 \\
f'(x) &= -(2+x)^{-2} &&\implies& f'(0) &= -1/4 \\
f''(x) &= 2(2+x)^{-3} &&\implies& f''(0) &= 2/8 = 1/4 \\
f'''(x) &= -6(2+x)^{-4} &&\implies& f'''(0) &= -6/16 = -3/8 \\
f''''(x) &= 24(2+x)^{-5} &&\implies& f''''(0) &= 24/32 = 3/4
\end{aligned}$$

and this gives us

$$(2+x)^{-1} = \frac{1}{2} + \left(-\frac{1}{4}\right)x + \left(\frac{1}{4}\right)\frac{x^2}{2} + \left(-\frac{3}{8}\right)\frac{x^3}{6} + \left(\frac{3}{4}\right)\frac{x^4}{24} + \cdots = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \frac{x^4}{32} + \cdots$$

as required.

To find the sum of the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots,$$

we have two methods available to us. Following 'hence', we want to rewrite this series so that it looks like the one we have just found. That is, we can note that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2\left[\frac{1}{2} - \frac{(-1)}{4} + \frac{(-1)^2}{8} - \frac{(-1)^3}{16} + \frac{(-1)^4}{32} + \cdots\right],$$

where the series in the square brackets is just our Taylor series for $(2+x)^{-1}$ with $x = -1$. As such, we see that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 2(2-1)^{-1} = 2,$$

is the sought after sum. Alternatively, following 'otherwise', we note that we are being asked to sum a geometric series with first term 1, common ratio 1/2 and an infinite number of terms. As such, using the formula, we see that

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1}{1-(1/2)} = 2,$$

which is, once again, the sought after sum.[2]

Question 3

We need to use the method of Lagrange multipliers to find the minimum value of

$$(x+1)^2 + 2(y+2)^2 + 3(z+3)^2,$$

subject to the constraint $x + y + z = 16$ where x, y and z are positive numbers. To do this, as in Section 3.4.4 of the subject guide, we use the Lagrangean

$$L(x, y, z, \lambda) = (x+1)^2 + 2(y+2)^2 + 3(z+3)^2 - \lambda(x + y + z - 16),$$

[2] Observe that, of these two methods, although the latter may seem more straightforward, the former is actually more powerful as it allows us to use Taylor series to sum series which are not geometric!

and solve the equations

$$\begin{aligned}
L_x(x,y,z,\lambda) &= 2(x+1)-\lambda = 0,\\
L_y(x,y,z,\lambda) &= 4(y+2)-\lambda = 0,\\
L_z(x,y,z,\lambda) &= 6(z+3)-\lambda = 0,\\
L_\lambda(x,y,z,\lambda) &= -(x+y+z-16) = 0
\end{aligned}$$

simultaneously. Eliminating λ from the first three equations yields

$$\lambda = 2(x+1) = 4(y+2) = 6(z+3),$$

as x, y and z must be positive here. Thus, in terms of z (say) we have,

$$x+1 = 3(z+3) \qquad \Longrightarrow \qquad x = 3z+8,$$

and

$$y+2 = \tfrac{3}{2}(z+3) \qquad \Longrightarrow \qquad y = \tfrac{3}{2}z+\tfrac{5}{2}.$$

Substituting these into the fourth equation (i.e. the constraint) we then find that

$$\big(3z+8\big)+\big(\tfrac{3}{2}z+\tfrac{5}{2}\big)+z = 16 \quad \Longrightarrow \quad \frac{11}{2}z = \frac{11}{2} \quad \Longrightarrow \quad z = 1,$$

which means that we also have

$$x = 3(1)+8 = 11 \qquad \text{and} \qquad y = \tfrac{3}{2}(1)+\tfrac{5}{2} = 4$$

if we use $x = 3z+8$ and $y = \frac{3}{2}z+\frac{5}{2}$ respectively. Consequently, we find that the minimum value of

$$(x+1)^2+2(y+2)^2+3(z+3)^2 \quad \text{is} \quad (11+1)^2+2(4+2)^2+3(1+3)^2 = 144+72+48 = 264,$$

if we use the values of x, y and z that we have just found.

Question 4

As in Section 2.5 of the subject guide, the elasticity of demand is given by

$$\varepsilon = -\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p},$$

where $q = q^D(p)$ is the demand function. In this case, as in Section 5.11 of the subject guide, we are given that

$$\varepsilon = \frac{3p^2+7p}{p^2+4p+3},$$

and so we need to solve the differential equation

$$-\frac{p}{q}\frac{\mathrm{d}q}{\mathrm{d}p} = \frac{3p^2+7p}{p^2+4p+3},$$

to find $q^D(p)$ given that we know that $q^D(1) = 1$. Indeed, this differential equation is separable and so using the method in Section 5.2 of the subject guide, we rewrite it as

$$-\int \frac{\mathrm{d}q}{q} = \int \frac{3p+7}{p^2+4p+3}\,\mathrm{d}p,$$

and determine the integrals. So, noting that partial fractions gives us

$$\frac{3p+7}{p^2+4p+3} = \frac{3p+7}{(p+1)(p+3)} = \frac{2}{p+1}+\frac{1}{p+3},$$

and that prices and quantities are always positive, we get

$$-\ln q = 2\ln(p+1) + \ln(p+3) + c,$$

where c is an arbitrary constant. So, to proceed, we could set $k = \mathrm{e}^c$ so that $c = \ln k$, and this then gives us

$$\ln\left(\frac{1}{q}\right) = \ln(p+1)^2 + \ln(p+3) + \ln k \quad \Longrightarrow \quad \ln\left(\frac{1}{q}\right) = \ln[k(p+1)^2(p+3)] \quad \Longrightarrow \quad \frac{1}{q} = k(p+1)^2(p+3),$$

which means that the general solution to our differential equation is

$$q = \frac{1}{k(p+1)^2(p+3)}.$$

Now, as we know that $q^D(1) = 1$, we get

$$1 = \frac{1}{16k} \quad \Longrightarrow \quad k = \frac{1}{16},$$

and so the particular solution to our differential equation is

$$q = \frac{16}{(p+1)^2(p+3)} \quad \text{which means that} \quad q^D(p) = \frac{16}{(p+1)^2(p+3)},$$

is the required demand function as we have $q = q^D(p)$.

Question 5

The supply and demand functions for this market are

$$q^S(p) = p - 6 \qquad \text{and} \qquad q^D(p) = 8 - p,$$

and so, since the equilibrium price occurs when $q^S(p) = q^D(p)$, we see that

$$p - 6 = 8 - p \quad \Longrightarrow \quad 2p = 14 \quad \Longrightarrow \quad p = 7,$$

is the equilibrium price and this gives an equilibrium quantity of $q^D(7) = 8 - 7 = 1$.

Now, as in Section 6.6 of the subject guide, the cobweb model dictates that

$$q_t = q^S(p_{t-1}) \qquad \text{and} \qquad p_t = p^D(q_t).$$

and, to apply these, we need to find the inverse demand function by rearranging $q = q^D(p)$ to get $p = p^D(q)$, i.e.

$$q = 8 - p \quad \Longrightarrow \quad p = 8 - q \quad \text{and so} \quad p^D(q) = 8 - q.$$

Then, using the given $q^S(p)$ and this $p^D(q)$, we see that the cobweb model equations give us

$$q_t = p_{t-1} - 6 \qquad \text{and} \qquad p_t = 8 - q_t,$$

respectively. Thus, eliminating q_t from these two equations, the recurrence equation we seek is

$$p_t = 8 - (p_{t-1} - 6) = 14 - p_{t-1}.$$

To solve this, we use the standard method from Section 6.4 of the subject guide which involves comparing it to the standard form $p_t = ap_{t-1} + b$ to see that $a = -1$ and $b = 14$. In this case, $a \neq 1$ and so we have a time-independent solution, p^*, which is given by

$$p^* = \frac{b}{1-a} \qquad \Longrightarrow \qquad p^* = \frac{14}{1-(-1)} = 7,$$

so that, using the formula $p_t = p^* + (p_0 - p^*)a^t$, we have the solution

$$p_t = 7 + (9-7)(-1)^t = 7 + 2(-1)^t,$$

as $p_0 = 9$. Indeed, as in Section 6.5 of the subject guide, we can see that as t increases, p_t oscillates between the values 9 and 5.[3]

Question 6

To solve the second-order differential equation

$$y''(t) - y'(t) - 2y(t) = 4,$$

we use the ideas in Section 5.4 of the subject guide. Here the auxiliary equation is

$$m^2 - m - 2 = 0 \quad \Longrightarrow \quad (m+1)(m-2) = 0 \quad \Longrightarrow \quad m = -1, 2.$$

In this case, as there are two distinct real solutions, the complementary function is

$$y(t) = A\mathrm{e}^{-t} + B\mathrm{e}^{2t},$$

for some arbitrary constants A and B. To find a particular solution, as the right-hand side is a constant, we try something of the form $y(t) = \alpha$ where α is a constant to be determined. Thus, $y'(t) = 0$ and $y''(t) = 0$ and, substituting these in to the given differential equation, we get

$$0 - 0 - 2\alpha = 4 \quad \Longrightarrow \quad -2\alpha = 4 \quad \Longrightarrow \quad \alpha = -2.$$

Consequently, the particular solution is $y(t) = -2$ and, adding this to the complementary function, we get

$$y(t) = A\mathrm{e}^{-t} + B\mathrm{e}^{2t} - 2,$$

as the general solution of the given differential equation. Indeed, since $y(0) = 0$, we have

$$0 = A + B - 2 \quad \Longrightarrow \quad A + B = 2,$$

and, since $y'(0) = 7$ as well, we find that

$$y'(t) = -A\mathrm{e}^{-t} + 2B\mathrm{e}^{2t},$$

so that we get

$$7 = -A + 2B \quad \Longrightarrow \quad A - 2B = -7.$$

Solving these equations simultaneously, we can then find that $A = -1$ and $B = 3$, which allows us to see that

$$y(t) = -\mathrm{e}^{-t} + 3\mathrm{e}^{2t} - 2,$$

is the required particular solution.

[3] Observe, in particular, that the *behaviour* of p_t as t increases is that it is *oscillating* between two specific values, i.e. 9 and 5. You could also add that p_t is oscillating *constantly* around 7, the equilibrium price of the market (which we found at the start of the question), and this means that we have a *cyclic* cobweb here.

Question 7

(a) As in Section 4.8 of the subject guide, we use row operations to deal with the given system of equations. Indeed, from the equations we get the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 3 \\ 1 & 1 & a & b \end{array}\right),$$

and, performing the obvious row operations, we then get

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 3 \\ 1 & 1 & a & b \end{array}\right) \xrightarrow[R_3 \to R_1 - R_3]{R_2 \to 2R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 3 & 3 & 3 \\ 0 & 1 & 1-a & 3-b \end{array}\right) \xrightarrow[R_3 \to R_2 - R_3]{R_2 \to \frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a & b-2 \end{array}\right).$$

Now, at this point, some care must be taken because row operations require that we multiply by *non-zero* constants and, as we don't know the value of a, we can't be sure that it *is* non-zero. So, at this point we stop doing row operations and look at the third row of our augmented matrix and to see that:

(i) If $a = 0$ and $b \neq 2$, there are no solutions.

(ii) If $a \neq 0$ and b is any real number, there is a unique solution given by the equations

$$x + 2y + z = 3, \quad y + z = 1 \quad \text{and} \quad az = b - 2.$$

Now, using back-substitution, we can see that the

- third equation gives us $z = \dfrac{b-2}{a}$, so the
- second equation gives us $y = 1 - \dfrac{b-2}{a} = \dfrac{a-b+2}{a}$, and then the
- first equation gives us $x = 1 + \dfrac{b-2}{a} = \dfrac{a+b-2}{a}$.

So, the required unique solution is $x = \dfrac{a+b-2}{a}$, $y = \dfrac{a-b+2}{a}$ and $z = \dfrac{b-2}{a}$.

(iii) If $a = 0$ and $b = 2$, there are an infinite number of solutions given by the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and so we have the equations

$$x + 2y + z = 3 \quad \text{and} \quad y + z = 1.$$

Now, using back-substitution, we set $z = t \in \mathbb{R}$ so that the second equation gives us

$$y = 1 - t,$$

and then the first equation gives us

$$x = 3 - 2(1-t) - t = 1 + t.$$

So, the required infinite number of solutions is $x = 1 + t$, $y = 1 - t$ and $z = t$ for $t \in \mathbb{R}$.

Note: In part **(iii)**, the positions of the 'leading ones' (in the first and second rows of the echelon form) mean that, when we do the back-substitution, we can easily find y in terms of z and then x in terms of z. This means that, in this case, we **should** pick z to be the parameter t which can be any real number. This holds more generally: Any variable associated with a 'leading one' in the echelon form **should** be determined (via back-substitution) in terms of the variables that are not associated with a 'leading one' and it is these latter variables that should be assigned parameters which can be any real number. Up until now, the Examiners have not penalised poor choices when it comes to choosing which variable(s) should be 'any real number' but, from next year, for full credit in questions on this topic, they will expect you to do this properly.

(b) For $t > 0$, we are given the differential equation

$$t\frac{\mathrm{d}y}{\mathrm{d}t} + y = 3t^2,$$

and this is linear. Thus, as in Section 5.3 of the subject guide, we rewrite it in standard form and note that comparing

$$\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{y}{t} = 3t \quad \text{to} \quad \frac{\mathrm{d}y}{\mathrm{d}t} + P(t)y = Q(t),$$

we have $P(t) = 1/t$ and $Q(t) = 1$ so that the integrating factor is

$$\mathrm{e}^{\int P(t)\,\mathrm{d}t} = \mathrm{e}^{\int t^{-1}\,\mathrm{d}t} = \mathrm{e}^{\ln t} = t,$$

and, multiplying both sides of our differential equation by this, we get

$$t\frac{\mathrm{d}y}{\mathrm{d}t} + y = 3t^2 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big[ty\Big] = 3t^2,$$

if we use the product rule to simplify the left-hand side. We then integrate both sides with respect to t to get

$$ty = \int 3t^2\,\mathrm{d}t = t^3 + c,$$

for some arbitrary constant c and so

$$y(t) = t^2 + \frac{c}{t},$$

is the general solution for $t > 0$. However, given that $y(1) = 2$, we can use this to get $c = 1$ so that

$$y(t) = t^2 + \frac{1}{t} = \frac{t^3 + 1}{t},$$

is the sought after particular solution.

Question 8

(a) To find the eigenvalues of this matrix, as in Sections 4.12–4.15 of the subject guide, we solve the equation

$$|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} \frac{5}{2} - \lambda & 1 \\ -3 & -1 - \lambda \end{vmatrix} = 0 \quad \Longrightarrow \quad \left(\tfrac{5}{2} - \lambda\right)\left(-1 - \lambda\right) + 1 = 0,$$

which, multiplying out the brackets, gives us the quadratic equation

$$\lambda^2 - \tfrac{3}{2}\lambda + \tfrac{1}{2} = 0 \quad \Longrightarrow \quad \left(\lambda - \tfrac{1}{2}\right)\left(\lambda - 1\right) = 0,$$

and so the eigenvalues are $1/2$ and 1. To find the corresponding eigenvectors we seek a non-zero vector, $\boldsymbol{x}$, which is a solution to the equation $(A - \lambda I)\boldsymbol{x} = \mathbf{0}$, i.e.

- For $\lambda = 1/2$, we solve

$$\begin{pmatrix} 2 & 1 \\ -3 & -\frac{3}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad 2x + y = 0 \quad \Longrightarrow \quad y = -2x \quad \Longrightarrow \quad \boldsymbol{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

(or any non-zero multiple of this) is an eigenvector.

- For $\lambda = 1$, we solve

$$\begin{pmatrix} \frac{3}{2} & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \quad \Longrightarrow \quad 3x + 2y = 0 \quad \Longrightarrow \quad y = -\frac{3}{2}x \quad \Longrightarrow \quad \boldsymbol{x} = \begin{pmatrix} 2 \\ -3 \end{pmatrix},$$

(or any non-zero multiple of this) is an eigenvector.

Consequently, if we take

$$P = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix},$$

we have an invertible matrix, P, and a diagonal matrix, D, such that $P^{-1}AP = D$.[4]

(b) As in Section 5.6.2 of the subject guide, we can now use our answer to part **(a)** to solve the system of difference equations given by $\boldsymbol{x}_{t+1} = A\boldsymbol{x}_t$ with

$$\boldsymbol{x}_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix}. \quad \text{We let} \quad \boldsymbol{u}_t = \begin{pmatrix} u_t \\ v_t \end{pmatrix},$$

where $\boldsymbol{x}_t$ and $\boldsymbol{u}_t$ are related by $\boldsymbol{x}_t = P\boldsymbol{u}_t$. This means that $\boldsymbol{x}_{t+1} = P\boldsymbol{u}_{t+1}$ and, substituting this into $\boldsymbol{x}_{t+1} = A\boldsymbol{x}_t$, we get

$$P\boldsymbol{u}_{t+1} = AP\boldsymbol{u}_t \quad \Longrightarrow \quad \boldsymbol{u}_{t+1} = P^{-1}AP\boldsymbol{u}_t \quad \Longrightarrow \quad \boldsymbol{u}_{t+1} = D\boldsymbol{u}_t,$$

as $P^{-1}AP = D$. Using this, we have

$$\begin{pmatrix} u_{t+1} \\ v_{t+1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_t \\ v_t \end{pmatrix} \quad \Longrightarrow \quad u_{t+1} = \tfrac{1}{2}u_t \quad \text{and} \quad v_{t+1} = v_t,$$

and this pair of difference equations can easily be solved to get

$$u_t = A\left(\tfrac{1}{2}\right)^t \qquad \text{and} \qquad v_t = B,$$

for arbitrary constants A and B. This means that, using $\boldsymbol{x}_t = P\boldsymbol{u}_t$, we have

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} A\left(\frac{1}{2}\right)^t \\ B \end{pmatrix} = \begin{pmatrix} A\left(\frac{1}{2}\right)^t + 2B \\ -2A\left(\frac{1}{2}\right)^t - 3B \end{pmatrix},$$

which means that

$$x_t = A\left(\tfrac{1}{2}\right)^t + 2B \quad \text{and} \quad y_t = -2A\left(\tfrac{1}{2}\right)^t - 3B,$$

is the general solution to our coupled system of difference equations.

Now, using the initial conditions $x_0 = 1$ and $y_0 = 1$, our general solution gives us the equations

$$1 = A + 2B \qquad \text{and} \qquad 1 = -2A - 3B,$$

which are easily solved to get $A = -5$ and $B = 3$. Consequently, we find that

$$x_t = 6 - 5\left(\tfrac{1}{2}\right)^t \qquad \text{and} \qquad y_t = 10\left(\tfrac{1}{2}\right)^t - 9,$$

is the required particular solution to our system of difference equations. Indeed, as t increases, x_t will increase to 6 and y_t will decrease to -9.[5]

[4] Of course, this is only one of the many possible pairs of matrices that we could choose for P and D: others are possible depending on which eigenvectors we choose when we form the columns of P and the order in which we choose to place them in P. For instance, choosing the other order for the eigenvectors we found above, we can see that

$$P = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

is another possible answer here.

[5] Observe, in particular, that the *behaviour* of x_t as t increases is that it is *increasing* to 6, its limit as $t \to \infty$. Indeed, if you can't see why it is *increasing*, notice that $5(1/2)^t$ is decreasing to zero and so $-5(1/2)^t$ must be increasing to zero. Thus, $x_t = 6 - 5(1/2)^t$ must be increasing to 6.

Similarly, the *behaviour* of y_t as t increases is that it is *decreasing* to -9, its limit as $t \to \infty$. Indeed, if you can't see why it is *decreasing*, notice that $10(1/2)^t$ is decreasing to zero and so $y_t = 10(1/2)^t - 9$ must be decreasing to -9.