Error Bounds for Linear ODEs

A1) First Order linear ODE with constant coefficients.

first order linear ODE, $u'(t) + cu(t) = f(t)$

$$c = \lambda + i\omega \in \mathbb{C}, \quad \lambda, \omega \in \mathbb{R} \quad \text{----- (1)}$$

Derivation

$$\left| \int f(x)\,dx \right| \leq \int |f(x)|\,dx$$

(Triangle inequality sorts of for integrals)

Multiply by integrating factor, e^{ct} for $u' + cu = f$.

$$\int_{t_0}^{t} e^{\lambda\tau + i\omega\tau} \left| u'(\tau) + (\lambda + i\omega)u(\tau) - f(\tau) \right| d\tau \leq \varepsilon \int_{t_0}^{t} e^{\lambda\tau + i\omega\tau}\, d\tau$$

$$\geq \left| \int e^{\lambda\tau + i\omega\tau} \left\{ u'(\tau) + (\lambda + i\omega)u(\tau) - f(\tau) \right\} d\tau \right|$$

skipping the middle term in the inequality

$$\left| \int_{t_0}^{t} d\left(e^{\lambda\tau + i\omega\tau} \cdot u(\tau)\right) - \int_{t_0}^{t} f(\tau) e^{\lambda\tau + i\omega\tau}\, d\tau \right|$$

$$\left| e^{\lambda t + i\omega t} u(t) - e^{\lambda t_0 + i\omega t_0} u(t_0) - \int_{t_0}^{t} e^{\lambda \tau + i\omega \tau} f(\tau)\, d\tau \right|$$

$$\leq \varepsilon \int_{t_0}^{t} e^{\lambda \tau}\, d\tau$$

$$\left| e^{\lambda\tau + i\omega\tau} \right| = \left| e^{\lambda\tau} \cdot e^{i\omega\tau} \right|$$

$$= \left| e^{\lambda\tau} \right| \cdot \left| e^{i\omega\tau} \right| = e^{\lambda\tau}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\left| e^{i\theta} \right| = 1$$

$$\therefore \left| e^{\lambda\tau + i\omega\tau} \right| = e^{\lambda\tau}$$ divide by

Divide by $e^{\lambda t + i\omega t}$, $\boxed{u(t_0) = t_0^*}$ Initial condition

$$u(t) = e^{\lambda(t_0 - t) + i\omega(t_0 - t)} u_0^* + e^{-\lambda t - i\omega t}$$

Analytical solution for $u' + cu = f$ with $IF = e^{ct}$

is $u^*(t) = e^{c(t_0 - t)} u_0^* + e^{-ct} \int e^{ct} f\, dt$

initial condition, $u(t_0) = u_0^*$

Alternative solution, replacing u_0^* with $u(t_0)$ i.e different initial condition,

$$\tilde{u}(t) = e^{\lambda(t_0-t)+i\omega(t_0-t)}u(t_0) + e^{-\lambda t-i\omega t}\int_{t_0}^{t} e^{\lambda\tau+i\omega\tau}f(\tau)d\tau$$

$$|u(t)-\tilde{u}(t)| \leq \varepsilon e^{-\lambda t}\int_{t_0}^{t} e^{\lambda\tau}d\tau$$

multiplying with $|e^{\lambda t+i\omega t}|$ and rearranging.

We want to bound $|u-u^*|$

$$|u-u^*| \leq |u-\tilde{u}| + |\tilde{u}-u^*| \leq \varepsilon e^{-\lambda t}\int_{t_0}^{t} e^{\lambda\tau}d\tau + |\tilde{u}-u^*|$$

0 , $\tilde{u}=u^*$ when $u(t_0)=u_0^*$

$$|u-u^*| \leq \varepsilon e^{-\lambda t}\int_{t_0}^{t} e^{\lambda\tau}d\tau$$

$\lambda>0$, $$|u-u^*| \leq \varepsilon \frac{e^{-\lambda t}(e^{\lambda t}-e^{\lambda t_0})}{\lambda}$$

$$\leq \varepsilon \frac{1-e^{\lambda(t_0-t)}}{\lambda}$$

$t>t_0$, $\lambda(t_0-t)<0$

$$0<1-e^{\lambda(t_0-t)}<1$$

Deriving natural response, $u_n^*(t_0)=u_0^*$ and $u_n^{*\prime}(t)+(\lambda+i\omega)u_n^*(t)=0$

$$u^*(t) = e^{\lambda(t_0-t)+i\omega(t_0-t)}u_0^* + e^{-\lambda t-i\omega t}\int_{t_0} e^{\lambda\tau+i\omega\tau}f(\tau)d\tau$$

$$\cancel{u_n^*(t) = -(\lambda+i\omega)\, e^{\lambda(t_0-t)+i\omega(t_0-t)}\, u_0^*}$$

$$\cancel{+\, e^{\lambda t - i\omega t}\left(e^{\lambda\tau + i\omega\tau}(0)\, d\tau\right)}$$

for natural response $f(\tau) = 0$

$$\therefore\ u_n^*(t) = e^{\lambda(t_0-t)+i\omega(t_0-t)}\, u_0^* + 0$$

$$|u_n^*(t)| = e^{\lambda(t_0-t)}\, |u_0^*|$$

$$(t_0 - t) < 0,\ \lambda < 0$$

$\lambda < 0$ $\Longrightarrow$ $e^{\lambda(t_0-t)}$ is diverging

$$\left|\frac{u-u^*}{u_n^*}\right| \le \varepsilon\, \frac{1 - e^{-|\lambda||t-t_0|}}{-|\lambda|\,|u_0^*|} = O(\varepsilon)$$

$\lambda = 0$

$$|u - u^*| \le \varepsilon e^{-0t} \int_{t_0}^{t} e^{0}\, d\tau = O(\varepsilon t)$$

$$\left|\frac{u-u^*}{u_n^*}\right| \le \frac{\varepsilon e^{-0} \int_{t_0}^{t} e^{0}\, d\tau}{|u_0^*|}$$

12) Higher Order linear ODE with constant coefficients

$$u^{(n)} + a_{n-1} u^{(n-1)} + \cdots + a_0 u = f$$

Initial conditions $u^k(t_0) = u_0^k$

By fundamental theorem of algebra

$$p_c(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_0 = \prod_{k=0}^{n-1} (x + \lambda_k + i\omega_k)$$

$-(\lambda_k + i\omega_k)$ will be the n complex roots.

Exact solution, $u^*(t) = u_p^*(t) \sum_{k=0}^{n-1} c_k \exp(\lambda_k t + i\omega_k t)$

k is count of purely imaginary roots.

Derivation (Second order case)

$$\left| u'' + ((\lambda_1 + i\omega_1) + (\lambda_2 + i\omega_2)) u' + (\lambda_1 + i\omega_1)(\lambda_2 + i\omega_2) u - f \right| \le \varepsilon$$

WLOG order such that $\lambda_1 \ge \lambda_2$

rearranging,

$$\left| (u' + (\lambda_1 + i\omega_1) u)' + (\lambda_2 + i\omega_2)(u' + (\lambda_1 + i\omega_1) u) - f \right| \le \varepsilon$$

$$v = u' + (\lambda_1 + i\omega_1) u$$

$$\left|v' + (\lambda_2 + i\omega_2)v - f\right| \le \varepsilon,$$

We have transformed second order PDE to First order PDE

by using previous results,

$$\left|v(t) - v^*(t)\right| \le \varepsilon e^{-\lambda_2 t} \int_{t_0}^{t} e^{-\lambda_2 \tau} d\tau$$

$$v^*(t) = u^{*\prime}(t) + (\lambda_1 + i\omega_1)\, u^*(t)$$

$$v = u' + (\lambda_1 + i\omega_1)\, u$$

Analytical solution

$$v(t) - v^*(t) = (u'(t) - u^{*\prime}(t)) + (\lambda_1 + i\omega_1)(u^*(t) - u(t))$$

From results of first order

$$\left|v(t) - v^*(t)\right| \le \varepsilon e^{-\lambda_2 t} \int_{t_0}^{t} e^{\lambda_2 \tau} d\tau$$

$$\cancel{\le \frac{\varepsilon e^{-\lambda_2 t}\left(e^{\lambda_2 t} - e^{\lambda_2 t_0}\right)}{\lambda_2}}$$

$$\cancel{= \varepsilon\left(1 - e^{-\lambda_2 t_0}\right)}$$

$$\le \frac{\varepsilon e^{-\lambda_2 t}}{\lambda_2}\left[e^{\lambda_2 t} - e^{\lambda_2 t_0}\right]$$

$$\le \frac{\varepsilon\left[1 - e^{\lambda_2 (t_0 - t)}\right]}{\lambda_2}$$

Similar to the first order case

$$\left|u(t) - u^*(t)\right| \le \varepsilon' e^{-\lambda_1 t}\int_{t_0}^{t} e^{\lambda_1 \tau}\, d\tau$$

$$\varepsilon' = \frac{\varepsilon\left[1 - e^{\lambda_2(t_0 - t)}\right]}{\lambda_2}$$

$$\cancel{\left|u(t) - u^*(t)\right| \le \varepsilon \frac{\left[1-e^{\lambda_1(t_0-t)}\right]\left[1-e^{-\lambda_2(t_0-t)}\right]}{\lambda_1\lambda_2}}$$

$$\cancel{\le \frac{\varepsilon}{\lambda_1\lambda_2}\left(1-e\right.}$$

$$\cancel{\left|u(t)-u^*(t)\right| \le \varepsilon' e^{-\lambda_1 t}\int_{t_0}^{t}}$$

multiply, take modulus and then divide

$$\int_{t_0}^{t}\left| e^{\lambda_1 \tau + i\omega_1 \tau}\left(v'(\tau) + (\lambda_1 + i\omega_1)u(\tau) - v^*(\tau)\right)\right| d\tau$$

$$\le \varepsilon \int_{t_0}^{t} \frac{\left(1-e^{\lambda_2(t_0-t)}\right)}{\lambda_2}\left(e^{\lambda_1+i\omega_1}\right)$$

$$\le \varepsilon\int_{t_0}^{t}\frac{\left(e^{\lambda_1+i\omega_1} - e^{\lambda_2(t_0-t)(\lambda_1+i\omega_1)}\right)}{\lambda_2}$$

Lets simplify

$$\int_{t_0}^{t}\left|\frac{e^{\lambda_1+i\omega_1} - e^{\lambda_2(t_0-t)(\lambda_1+i\omega_1)}}{\lambda_2}\right| d$$

$$\left| e^{\lambda\tau + i\omega\tau} \right| = e^{\lambda\tau}$$

$$\int |ab| = \int |a||b| \quad \text{if } |a| = c \text{ (may not be a constant)}$$

$$= \int c|b|$$

$$\int_{t_0}^{t} \left|\left(\quad \right)\right| d\tau = \int_{t_0}^{t} \frac{e^{\lambda_1\tau}\left(1 - e^{\lambda_2(t_0-t)}\right)}{\lambda_2} d\tau$$

$$= \int_{t_0}^{t} \frac{e^{\lambda_1\tau} - e^{\lambda_1\lambda_2\tau(t_0-t)}}{\lambda_2} d\tau$$

$$= \frac{e^{\lambda_1 t} - e^{\lambda_1 t_0}}{\lambda_1\lambda_2}$$

$$\left| \int_{t_0}^{t} e^{\lambda_1 t + i\omega t} \left(u'(t) + (\lambda_1 + i\omega_1)\, u(t) - v^*(t)\right) d\tau \right|$$

$$\leq \varepsilon e^{-\lambda_2 t} \int_{t_0}^{t} e^{\lambda_2\tau} e^{\lambda_1\tau} d\tau$$

removing modulus and simplifying RHS

$$\leq \varepsilon e^{-\lambda_2 t} e^{(\lambda_1+\lambda_2)(}$$

$$\varepsilon \int \frac{1 - e^{\lambda_2(t_0-t)}\, e^{\lambda_1\tau}}{\lambda_2} d\tau$$

$$\cancel{\varepsilon \int_{t}^{t_0} \frac{1-e^{\lambda_2(t_0-t)}}{\lambda_2} e^{\lambda_1 \tau}\, d\tau}$$

$$\cancel{= \frac{\varepsilon}{\lambda_2 \lambda_1}\left[e^{\lambda_1(t_0-t)} - e^{\lambda_2(t_0-t)} e^{\lambda_1(t_0-t)}\right]}$$

$$\left| v'(t) + (\lambda_1 + i\omega_1) u(t) - v^*(t) \right| \leq \varepsilon e^{-\lambda_2 t} \int_{t_0}^{t} e^{\lambda_2 \tau} d\tau$$

$$= \frac{\varepsilon\left(1 - e^{\lambda_2(t_0-t)}\right)}{\lambda_2}$$

Multiply both sides by $\left| e^{\lambda_1 + i\omega_1} \right|$, $\quad |a||b| = |ab|$

$$\left| e^{\lambda_1 t + i\omega_1 t}\left(u'(t) + (\lambda_1 + i\omega_1) u(t)\right) - v^*(t) e^{\lambda_1 t + i\omega_1 t} \right|$$

$$\leq \frac{\varepsilon\left(1 - e^{\lambda_2(t_0-t)}\right)}{\lambda_2} \left| e^{\lambda_1 t + i\omega_1 t} \right|$$

Integrate on both sides,

$$\int_{t_0}^{t_0} \left| e^{\lambda_1 t + i\omega_1 t}\left(v(t) + (\lambda_1 + i\omega_1) u(t)\right) - v^*(t) e^{\lambda_1 t + i\omega_1 t} \right| dt$$

$$\leq \int_{t_0}^{t_0} \frac{\varepsilon\left(1 - e^{\lambda_2(t_0-t)}\right)}{\lambda_2} e^{\lambda_1 t} dt$$

$$= \frac{\varepsilon}{\lambda_2} \int_{t_1}^{t_0} \left(e^{\lambda_1 t} - e^{\lambda_2(t_0-t) + \lambda_1 t} \right) dt$$

$$\int | \, | \geq | \int \, |$$

integral of modulus is greater than modulus of integrals.

$$\left| \int_t^{t_0} e^{\lambda_1 t + i\omega_1 t} \left(u'(t) + (\lambda_1 + i\omega_1) u(t) \right) - \int v^*(t) e^{\lambda_1 t + i\omega_1 t} dt \right|$$

$$\leq \frac{\varepsilon}{\lambda_2} \left[\frac{e^{\lambda_1 (t_0 - t)}}{\lambda_1} - \frac{e^{\lambda_2 t_0 + t(\lambda_1 - \lambda_2)}}{\lambda_1 - \lambda_2} \right]$$

$$\left| e^{\lambda_1 t + i\omega_1 t} \left(u(t) - u(t_0) \right) - \int v^*(t) e^{\lambda_1 t + i\omega_1 t} dt \right| \leq$$

$$\frac{\varepsilon}{\lambda_2} \left[\frac{e^{\lambda_1 (t_0 - t)}}{\lambda_1} - \frac{e^{\lambda_2 t_0 + t(\lambda_1 - \lambda_2)}}{\lambda_1 - \lambda_2} \right]$$

Dividing by $\left| e^{\lambda_1 t + i\omega_1 t} \right| = e^{\lambda_1 t}$

$$\left| u(t) - e^{-(\lambda_1 t + i\omega_1 t)} \int v^*(t) e^{\lambda_1 t + i\omega_1 t} dt \right|$$

$$\leq \frac{\varepsilon}{\lambda_2} \left[\frac{e^{\lambda_1 (t - t_0)}}{\lambda_1} - \frac{e^{\lambda_2 (t_0 - t) + \lambda_1 (t - t_0)}}{\lambda_1 - \lambda_2} \right] e^{-\lambda_1 t}$$

$$= \frac{\varepsilon}{\lambda_2} \left[\frac{e^{-\lambda_1 t_0}}{\lambda_1} - \frac{e^{\lambda_2 (t_0 - t) - \lambda_1 t_0}}{\lambda_1 - \lambda_2} \right]$$

simplifying using particular solution method

$$= \frac{\varepsilon}{\lambda_1 \lambda_2} \left[\lambda_1 e^{-\lambda_1 t_0} - \frac{\lambda_1 e^{\lambda_2 (t_0 - t) - \lambda_1 t_0}}{\lambda_1 - \lambda_2} \right]$$

$|u(t) - u^*(t)|$

$\boxed{\text{Different result in paper}}$

Correct Calculation ?:

$$\frac{1}{|e^{\lambda_1 t + i\omega_1 t}|}\int_{t_0}^{t} \varepsilon\frac{(1-e^{\lambda_2(t_0-t)})}{\lambda_2}\,|e^{\lambda_1 t+i\omega_1 t}|\,dt$$

$$\frac{\varepsilon}{\lambda_2}e^{-\lambda_1 t}\left[\int_{t_0}^{t} e^{\lambda_1 t}\,dt - \int_{t_0}^{t} e^{\lambda_2(t_0-t)+\lambda_1 t}\,dt\right]$$

$$\cancel{\frac{\varepsilon}{\lambda_2}e^{-\lambda_1 t}\left[\frac{e^{\lambda_1(t-t_0)}}{\lambda_1} - \frac{\quad}{\lambda_1-\lambda_2}\right]}$$

$$\frac{\varepsilon}{\lambda_2}e^{-\lambda_1 t}\left[\frac{(e^{\lambda_1 t}-e^{\lambda_1 t_0})}{\lambda_1} + \frac{(e^{\lambda_1 t_0}-e^{\lambda_2(t_0-t)+\lambda_1 t})}{\lambda_1-\lambda_2}\right]$$

$$\frac{\varepsilon}{\lambda_2}\left[\frac{1-e^{\lambda_1(t_0-t)}}{\lambda_1}+\frac{e^{\lambda_1(t_0-t)}-e^{\lambda_2(t_0-t)}}{\lambda_1-\lambda_2}\right]$$

$$\frac{\varepsilon}{\lambda_2\lambda_1}\left[\left[1-e^{\lambda_1(t_0-t)}\right]+\frac{\lambda_1(e^{\lambda_1(t_0-t)}-e^{\lambda_2(t_0-t)})}{\lambda_1-\lambda_2}\right]$$

at $t=t_0$, $\frac{\varepsilon}{\lambda_2\lambda_1}[0]=0$

at $t=\infty$, $\frac{\varepsilon}{\lambda_2\lambda_1}[1-0]=\frac{\varepsilon}{\lambda_1\lambda_2}$

Got different (but similar) expression but the correct limits (range)

~~For $\lambda_2 \to 0$~~

$$\cancel{\frac{\varepsilon}{\lambda_1}\left[\frac{1-e^{\lambda_2(t_0-t)}}{\lambda_2} - \frac{\lambda_1 e^{\lambda_2(t_0-t)}}{\lambda_2\lambda_1}\right]}$$

~~ignoring time terms~~

$$\cancel{\frac{\varepsilon}{\lambda_1}\left[\frac{1-e^{\lambda_2}}{\lambda_2} - \frac{\lambda_1 e^{\lambda_2}}{\lambda_1\lambda_2}\right] = \frac{\varepsilon}{\lambda_1}\left[\frac{1-2e^{\lambda_2}}{\lambda_2}\right]}$$

For $\lambda_2 \to 0$

$$\lim_{\lambda_2\to 0}\frac{\varepsilon}{\lambda_1}\left[\frac{1-e^{\lambda_1}}{\lambda_2} + \frac{\lambda_1(e^{\lambda_1}-e^{\lambda_2})}{\lambda_2(\lambda_1-\lambda_2)}\right]$$

$$= \lim_{\lambda_2\to 0}\frac{\varepsilon}{\lambda_1}\left[\frac{1-e^{\lambda_1}}{\lambda_2} + \frac{e^{\lambda_1}-e^{\lambda_2}}{\lambda_2}\right]$$

$$= \lim_{\lambda_2\to 0}\frac{\varepsilon}{\lambda_1}\left[\frac{1-e^{\lambda_2}}{\lambda_2}\right]$$

$$= \boxed{O\left(\frac{\varepsilon}{\lambda_1}\right)}$$ Incorrect

Result in Paper is $O\left(\frac{\varepsilon}{\lambda_1\lambda_2}\right) = O(\varepsilon)$.

A4) First Order Linear ODE with Nonconstant Coefficients.

$$u'(t) + (p(t) + i\,q(t))\,u = f(t)$$

Derivation : For integrating factor,

$$P(t) = \int_{t_0}^{t} p(\tau)\,d\tau \quad ; \quad Q(t) = \int_{t_0}^{t} Q(\tau)\,d\tau$$

$$\left| \int_{t_0}^{t} e^{P(t)+iQ(t)}\, u'(t) + (p(t)+iq(t))\,u(t) - \int_{t_0}^{t} e^{P(t)+iQ(t)} f(t)\,dt \right| \le \varepsilon \int_{t_0}^{t} e^{P(t)}\,dt$$

Again, modulus of integral smaller than integral of modulus

Dividing by $\left| e^{P(t)+iQ(t)} \right|$

$$\left| u(t) - e^{-P(t)+iQ(t)}\,u_0^* - e^{-P(t)-iQ(t)} \int_{t_0}^{t} e^{P(t)+iQ(t)} f(t)\,dt \right|$$

Analytical solution, u^* $|u(t) - u^*(t)| \le \varepsilon\, e^{-P(t)} \int_{t_0}^{t} e^{P(t)}\,dt$

Rearrange and define,

$$\phi(t) = \int_{t_0}^{t} e^{P(\tau)}\,d\tau - t e^{P(t)}$$

$$|u(t)-u^*(t)| \le \varepsilon t\left(1+\frac{\phi(t)}{te^{P(t)}}\right).$$

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$\therefore$ $p(t) \ge 0$ for $t > t'$ (sufficiently large t, stated in problem).

$$P(t) = \int_{t_0}^{t} p(\tau)\,d\tau$$ which is increasing for $t > t'$

$$\phi(t) \le \phi(t'), \quad t > t'$$

$$\frac{\phi(t)}{te^{P(t)}}$$ $\phi(t)$ ← decreasing, $te^{P(t)}$ ← increasing

bounded by a constant value c.

$$\therefore \quad |u(t)-u^*(t)| \le \varepsilon t(1+c) = O(\varepsilon t).$$

A3) System of First Order Linear ODEs with Constant Coefficients

$$u' + Au = f \quad ; \quad u(t_0) = u_0^*$$

$$J = MA^{-1}M \quad (\text{exists as } A \in \mathbb{C}^{n \times n})$$

Derivations:

$$v = M^{-1}u, \quad g = M^{-1}f$$

$$\|v' + Jv - g\| = \|M^{-1}u' + M^{-1}A(MM^{-1}u) - M^{-1}f\|$$

p-Norm of residual

$$\leq \|M^{-1}\| \, \|u' + Au - f\|$$

$$\leq \|M^{-1}\| \varepsilon$$

Product of norms > Norm of Products

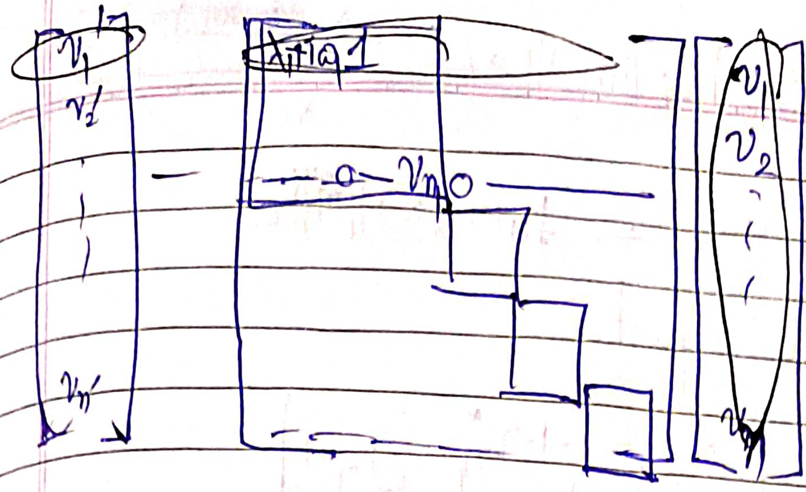
(Chebyshev's inequality sorts of).

$$\|v' + Jv - g\| \leq \|M^{-1}\| \varepsilon$$

each individual component of vector must hold condition

blocks (J_k)

$$J = \begin{bmatrix} \lambda_1 + i\omega_1 & & & & & & \\ & \lambda_1 + i\omega_1 & & & & & \\ & & \lambda_1 + i\omega_1 & & & & \\ & & & \lambda_2 + i\omega_2 & & & \\ & & & & \lambda_3 + i\omega_3 & & \\ & & & & & \lambda_3 + i\omega_3 & \\ & & & & & & \lambda_3 + i\omega_3 \end{bmatrix}$$



For first Jordan chain,

$$\left|v_1' + (\lambda_1 + i\omega_1)v_1 + v_2 - g_1\right| \le \|m^{-1}\|\,\varepsilon$$

$$\vdots$$

$$\left|v_{n_1}' + (\lambda_1 + i\omega_1)\,v_{n_1} - g_{n_1}\right| \le \|m^{-1}\|\,\varepsilon.$$

n_1 conditions

from first order linear ODE

$$\left|v_{n_1} - v_{n_1}^*\right| \le \|m^{-1}\|\,\varepsilon\,\frac{\left(1 - e^{\lambda_1(t_0 - t)}\right)}{\lambda_1}$$

Auxilliary function

$$h_k(t;\lambda) = \frac{1}{\lambda^k}\left(1 - \sum_{j=0}^{k-1} \frac{\lambda^j (t-t_0)^j}{j!}\, e^{\lambda(t_0 - t)}\right)$$

Taylor series expansion of soln

$$H_k(t;\lambda) = \sum_{j=1}^{k} h_k(t;\lambda)$$

$$h_1(t,\lambda) = \frac{1}{\lambda}\left(1 - \frac{\lambda^0 (t-t_0)^0}{0!}\, e^{\lambda(t_0-t)}\right)$$

$$= \frac{1}{\lambda}\left(1 - e^{\lambda(t_0-t)}\right)$$

$$H_1(t,\lambda) = h_1(t,\lambda)$$

$$|v_n - v_{n_1}^*| \le \|m-1\| \varepsilon \left(\frac{1 - e^{\lambda_1 (t_0 - t)}}{\lambda_1} \right)$$

$$= H_1(t;\lambda) \|m-1\| \varepsilon$$

Using last two equations,

$$|v_{n_1}' - 1 + (\lambda_1 + i\omega_1) v_{n_1-1} + v_{n_1}^* - g_{m-1}|$$

$$\le \|m-1\| \varepsilon + H_1(t;\lambda) \|m-1\| \varepsilon$$

$$= \|m-1\| \varepsilon (1 + H_1(t;\lambda)).$$

Integrating factor $e^{\lambda_1 + i\omega_1}$

RHS term becomes

$$\frac{1}{e^{\lambda_1 t}} \int_{t_0}^{t} (1 + h_1(t;\lambda)) e^{\lambda_1 t} \, dt$$

$$= \frac{1}{e^{\lambda_1 t}} \int_{t_0}^{t} \left(1 + \frac{(1 - e^{\lambda_1 (t_0 - t)})}{\lambda_1} \right) e^{\lambda_1 t} \, dt$$

$$= \frac{1}{e^{\lambda_1 t}} \left\{ \int_{t_0}^{t} e^{\lambda_1 t} dt + \left[\frac{e^{\lambda_1 t}}{\lambda_1^2} \right]_{t_0}^{t} - \frac{1}{\lambda_1^2} \int_{t_0}^{t} e^{\lambda_1 t_0} dt \right\}$$

$$= \frac{1}{e^{\lambda_1 t}} \left[\frac{e^{\lambda_1 t} - e^{\lambda_1 t_0}}{\lambda_1} + \frac{e^{\lambda_1 t} - e^{\lambda_1 t_0}}{\lambda_1^2} - \frac{e^{\lambda_1 t_0} (t - t_0)}{\lambda_1^2} \right]$$

$$= \underbrace{\left(\frac{1-e^{\lambda_1(t_0-t)}}{\lambda_1}\right)}_{h_1(t;\lambda)} + \underbrace{\frac{1-e^{\lambda_1(t_0-t)}}{\lambda_1^2} - \frac{e^{\lambda_1(t_0-t)}(t-t_0)}{\lambda_1^2}}_{h_2(t;\lambda)}$$

$\cancel{h_1(\lambda;t)}$

$$\therefore \ |v_{n_1-1} - v^*_{n_1-1}| \le \left(h_1(t;\lambda) + h_2(t;\lambda)\right) \|M^{-1}\| \varepsilon$$

$$\le H_2(t;\lambda) \|M^{-1}\| \varepsilon$$

Similarly, $|v_1 - v_1^*| \le H_{n_1}(t;\lambda) \|M^{-1}\| \varepsilon$

$$|v_2 - v_2^*| \le H_{n_1-1}(t;\lambda_1) \|M^{-1}\| \varepsilon$$

$$\vdots$$

$$|v_{n_1} - v_{n_1}^*| \le H_1(t;\lambda_1) \|M^{-1}\| \varepsilon$$

Limits for $\lambda \to 0$

$$h_k(t;0) = \lim_{\lambda\to 0} \frac{1}{\lambda^k}\left(1 - \sum_{j=0}^{k-1} \frac{\lambda^j (t-t_0)^j}{j!} e^{\lambda(t_0-t)}\right)$$

Taylor expansion of $e^{\lambda(t_0-t)}$

$\cancel{e^{\lambda(t-t_0)}}$

$$= \lim_{\lambda\to 0}\left(\frac{e^{\lambda(t_0-t)}}{\lambda^k}\right)\left(e^{\lambda(t-t_0)} - \sum_{j=0}^{k-1} \frac{\lambda^j (t-t_0)^j}{j!}\right)$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda^k}\left(\sum_{l=0}^{\infty} \frac{\lambda^l (t_0-t)^l}{l!}\right)\left(\sum_{j=k}^{\infty} \frac{\lambda^j (t-t_0)^j}{j!}\right)$$

since $\lambda \to 0$, for the lowest power d we would have

$$h_k(t;\lambda) = O(\lambda^d)$$

other terms will become smaller and negligible,

lowest term will λ^k $\begin{cases} 0 \text{ to } \infty \\ k \text{ to } \infty \end{cases}$

minimum degree $k+0$.

$$\therefore \; h_k(t;0) = \frac{(t-t_0)^k}{k!}$$

$$H_k(t;0) = \sum_{j=1}^{k} \frac{(t-t_0)^j}{j!}$$

Using this result in the equations derived earlier,

$$|v_1 - v_1^*| \le \|M^{-1}\| \epsilon \sum_{j=1}^{n_1} \frac{(t-t_0)^j}{j!}$$

$$\vdots$$

$$|v_{n_1} - v_{n_1}^*| \le \|M^{-1}\| \epsilon (t-t_0).$$

~~substituting back $v = \|M\| y$, we get~~

for $\lambda = 0$,

$$\cancel{\|v - v^*\| \le \|M\|\varepsilon\left(\sum_{i=1}^{n_1}\sum_{j=1}^{n_1}\frac{(t-t_0)^j}{j}\right)}$$

$$\|v - v^*\| \le \max_j \left\{ \sum_{j=1}^{n_1} \frac{(t-t_0)^j}{j!} \right\} \|M\|\varepsilon$$

for $\lambda > 0$

$$\|v - v^*\| \le \|M^{-1}\|\varepsilon \cdot \left(\max_k H(t; \lambda_k)\right)$$

$$0 \le h_k(t;\lambda) < \frac{1}{\lambda^k}$$

$$0 \le H_k(t;\lambda) < \sum_{j=1}^{k} \frac{1}{\lambda^j}$$

$$u = Mv$$

$$\therefore \|u - u^*\| = \|M\|\,\|v - v^*\| \le \left(\max_k \sum_{j=1}^{n_k} \frac{1}{\lambda_k^j}\right) \mathrm{cond}(M)\varepsilon$$

$$\boxed{\|u - u^*\| = O(\varepsilon)}$$

Got same result in paper, but difference in a constant; didn't understand the $(\sqrt{n})$ term.

A2) Higher Order Linear ODE with constant coefficients

General Case Derivation

Define auxiliary functions $\{\phi_n\}_{n=1}^{\infty}$

$$\phi_n(t; \lambda_{1:n}) = \frac{1}{\prod_{j=1}^{n} \lambda_j} - \sum_{k=1}^{n} \frac{e^{-\lambda_k (t - t_0)}}{\lambda_k \prod_{j=1, j\neq k}^{n} (\lambda_j - \lambda_k)}$$

Consider, the recurrence holds, with $\phi_0(t) = 1$.

$$\phi_1(t) = \frac{1}{\lambda_1} - \frac{e^{-\lambda_1 (t - t_0)}}{\lambda_1 (1)} = \frac{1 - e^{-\lambda_1 (t - t_0)}}{\lambda_1}$$

~~From the recurrence, n=0~~

$$\cancel{\phi_1(t) = e^{-\lambda_1 t} \int_{t_0}^{t} e^{\lambda_1 t}}$$

Induction

Base Case

$$\phi_1(t) = e^{-\lambda_1 t} \int_{t_0}^{t} e^{\lambda_1 \tau} (1)\, d\tau \quad , \quad n \geq 0 \quad n = 0 \quad n = 1$$

$$= \frac{e^{-\lambda_1 t} \left(e^{\lambda_1 t} - e^{\lambda_1 t_0} \right)}{\lambda_1}$$

$$= \frac{1 - e^{-\lambda_1 (t - t_0)}}{\lambda_1}$$

Thus, we have base case proven and the induction hypothesis set up.

Consider we have, from our hypothesis

$$\phi_n(t) = e^{-\lambda_n t}\int_{t_0}^{t} e^{\lambda_n \tau}\phi_{n-1}(\tau)\,d\tau$$

Consider the expression,

$$e^{-\lambda_{n+1}t}\int_{t_0}^{t} e^{\lambda_{n+1}\tau}\left(e^{-\lambda_n\tau}\int_{t_0}^{t} e^{\lambda_n\tau}\phi_{n-1}(\tau)\right)d\tau$$

Induction doesn't work here, not helpful,

$$\int_{t_0}^{t} e^{\lambda_{n+1}\tau}\left(\frac{1}{\prod_j \lambda_j} - \sum_k \frac{e^{-\lambda_k(\tau - t_0)}}{\prod_{j, j\neq k}(\lambda_j - \lambda_k)}\right)d\tau$$

$$\frac{\left(e^{\lambda_{n+1}t} - e^{\lambda_{n+1}t_0}\right)}{\lambda_{n+1}\prod_j p_j} - \sum_{k=1}^{n}\int_{t_0}^{t}\frac{e^{+\tau(\lambda_{n+1}-\lambda_k)+\lambda_k t_0}\,d\tau}{\prod_{j,j\neq k}(\lambda_j-\lambda_k)}$$

$$- \sum_{k=1}^{n}\frac{\left[e^{t(\lambda_{n+1}-\lambda_k)+\lambda_k t_0} - e^{t\lambda_{n+1}}\right]}{\prod_{j,j\neq k}(\lambda_j-\lambda_k)(\lambda_{n+1}-\lambda_k)}$$

multiplying both terms by $e^{-\lambda_{n+1}t}$

$$\frac{1 - e^{\lambda_{n+1}(t_0 - t)}}{\lambda_{n+1}\prod_j p_j} - \sum_{k=1}^{n}\frac{\left(e^{+\lambda_k(t_0-t)} - 1\right)}{\prod_{j,j\neq k}(\lambda_j-\lambda_k)(\lambda_{n+1}-\lambda_k)}$$

$$= \boxed{\phi_{n+1}(t; \lambda_{1:n})}$$

The recurrence relation holds true

Define,

$$u_0(t) = u(t)$$
$$u_1(t) = u_0'(t) + (\lambda_n + i\omega_n) u_0(t)$$
$$u_2(t) = u_1'(t) + (\lambda_{n-1} + i\omega_{n-1}) u_1(t)$$
$$\vdots \qquad \vdots$$
$$u_{(n-1)}(t) = u_{(n-2)}'(t) + (\lambda_2 + i\omega_2) u_{n-2}(t)$$
$$u_{(n)}(t) = u_{(n-1)}'(t) + (\lambda_1 + i\omega_1) u_{n-1}(t)$$

Given condition i.e. bound on residual,

$$|u^{(n)} + a_{n-1} u^{(n-1)} + \dots + a_0 u - f| \le \varepsilon$$

we have,

$$x^n + a_{n-1} x^{n-1} + \dots + a_0 = \prod_{k=0}^{n-1} (x + \lambda_k + i\omega_k)$$

$$\therefore \quad a_{n-1} = -\sum (\lambda_k + i\omega_k)$$
$$a_{n-2} = \sum_{k<j} (\lambda_k + i\omega_k)(\lambda_j + i\omega_j)$$
$$\vdots$$
$$a_0 = (-1)^n \prod_k (\lambda_k + i\omega_k).$$

we can see,

$$u_k(t) = u^{(k)} + a_{k-1} u^{(k-1)} + \dots + a_0 u.$$

$$\therefore \quad |u^{(n)} + a_{n-1} u^{(n-1)} + \dots + a_0 u - f|$$
$$= |u_n(t) - f|$$
$$= |u_{n-1}' + (\lambda_1 + i\omega_1) u_{n-1} - f| \le \varepsilon$$

By the integrating factor technique,

$$|u_{n-1} - u_{n-1}^{*}| \le \varepsilon e^{-\lambda_1 t} \int_{t_0}^{t} e^{\lambda_1 \tau} d\tau = \varepsilon \phi_1(t, \lambda_1)$$

repeating this we get,

$$|u_0 - u_0^{*}| = |u - u^{*}| \le \varepsilon \phi_n(t; \lambda_{1:n})$$