2/3/2018

Poisson R.V.

X is a Poisson R.V. if PMF of X has the form

$$P_X(x) = \begin{cases} \alpha^x \cdot e^{-\alpha}/x! , & x = 0, 1, 2, \ldots \\ 0 , & \text{otherwise} \end{cases}$$

λ = Number of successes per unit time / length

T : length / duration

$$\alpha = \lambda \cdot T$$

Ex

The number of database queries processed by a computer in any 10-second interval is a Poisson random variable, K, with $\alpha = 5$ queries. What is the probability that there will be no queries processed in a 10-second interval? What is the probability that at least two queries will be processed in a 2-second interval?

The PMF of K $\quad T = 10$ sec.

$\alpha = 5$ q.

$$P_K(k) = 5^k \cdot e^{-5}/k! , \quad k = 0, 1, 2, \ldots$$

$$P(K=0) = P_K(0) = \frac{5^0 \cdot e^{-5}}{0!} = 0.0067$$

- $\lambda = \frac{\alpha}{T} = \frac{5}{10} = 0.5$ q./sec.

N: # of queries processed in 2 seconds

$$\alpha_N = 2 \cdot \lambda = 1 \text{ q.}$$

$$P_N(n) = \frac{e^{-1} \cdot 1^n}{n!}$$

$$P(N \geq 2) = 1 - P(N < 2) = 1 - \left(P(N=0) + P(N=1)\right)$$

$$= 1 - \left[\frac{e^{-1}}{0!} + \frac{e^{-1}}{1}\right] = 0.264.$$

$$/* \; P(N \geq 2) = \sum_{n=2}^{\infty} e^{-1} \cdot \frac{1^n}{n!} = \sim \; */$$

CDF (Cumulative Distribution Function)

X is a Random variable. The CDF of X is

$$F_X(x) = P(X \leq x)$$

Theorem

For any discrete R.V. with range $S_X = \{x_1, x_2, x_3, \dots\}$ such that $x_1 \leq x_2 \leq \dots$

(a) $F_X(-\infty) = 0$ and $F_X(\infty) = 1$

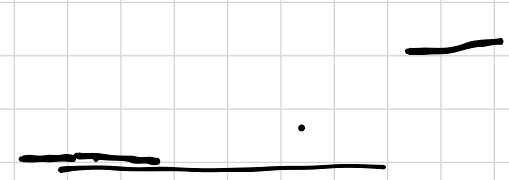
(b) For all $\underline{\underline{x'}} \geq \underline{\underline{x}}$, $F_X(x') \geq F_X(x)$

(c) For $x_i \in S_X$ and ε, an arbitrary small positive number

$$F_X(x_i) - F_X(x_i - \varepsilon) = P_X(x_i)$$

(d) $F_X(x) = F_X(x_i)$ for all x such that

$$x_i \leq x \leq x_i + 1$$



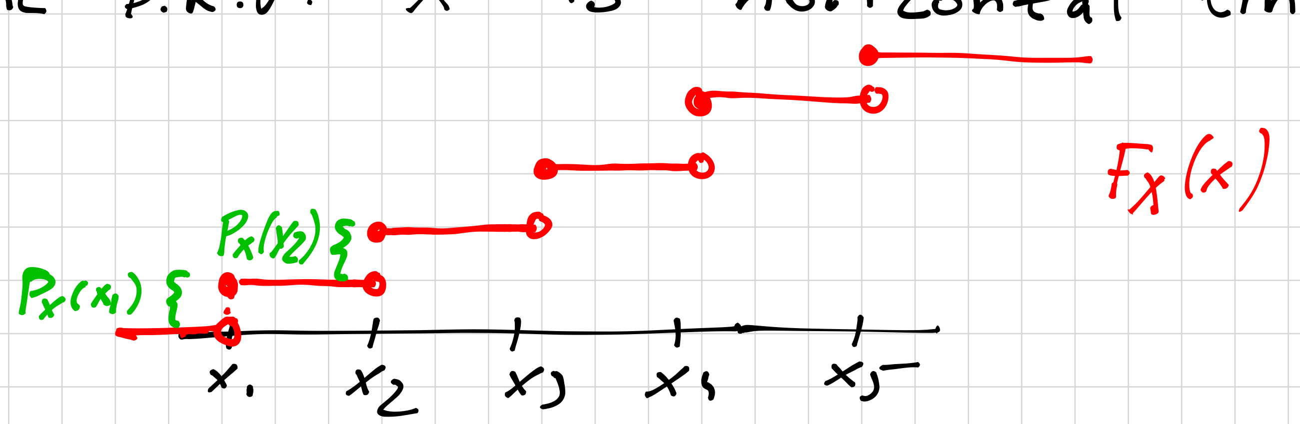
Comments on this theorem

(a) Going from left to right on the x-axis $F_X(x)$ starts from zero and ends at one

(b) The CDF never decreases from left to right.

(c) For a D.R.V., X there is a jump (discontinuity) at each values of $x_i \in S_X$. The height of the jump at x_i is $P_X(x_i)$.

(d) Between jumps the graph of the CDF ~~is~~ of the D.R.V. X is horizontal line.



Theorem

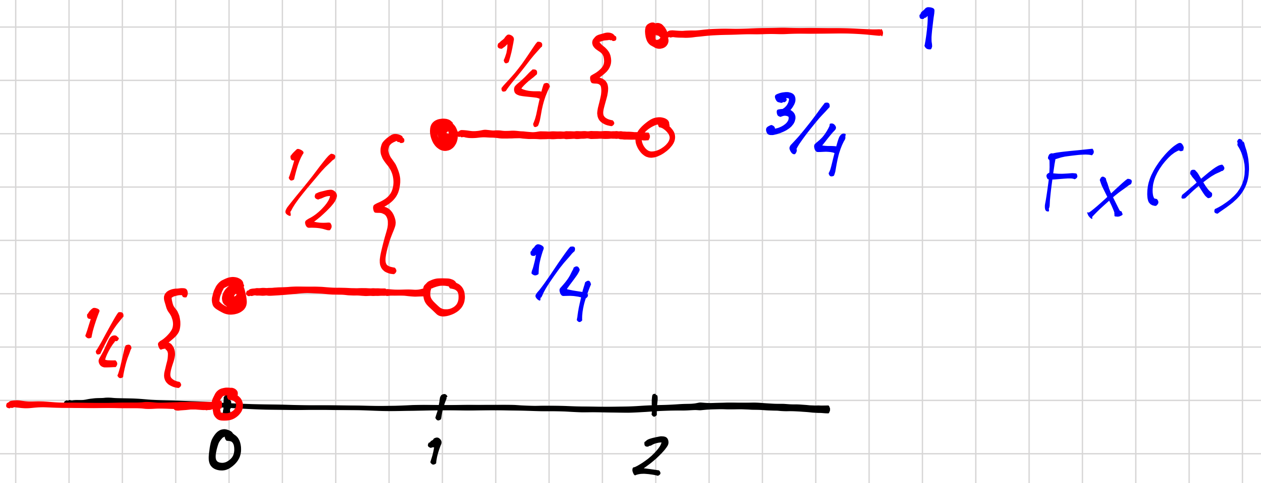
For all $b \geq a$

$$F_X(b) - F_X(a) = P(a < X_1 \leq b)$$

Ex X is a D.R.V. with PMF

$$P_X(x) = \begin{cases} 1/4, & x=0 \\ 1/2, & x=1 \\ 1/4, & x=2 \\ 0, & \text{otherwise} \end{cases}$$

Find and sketch the C.D.F.

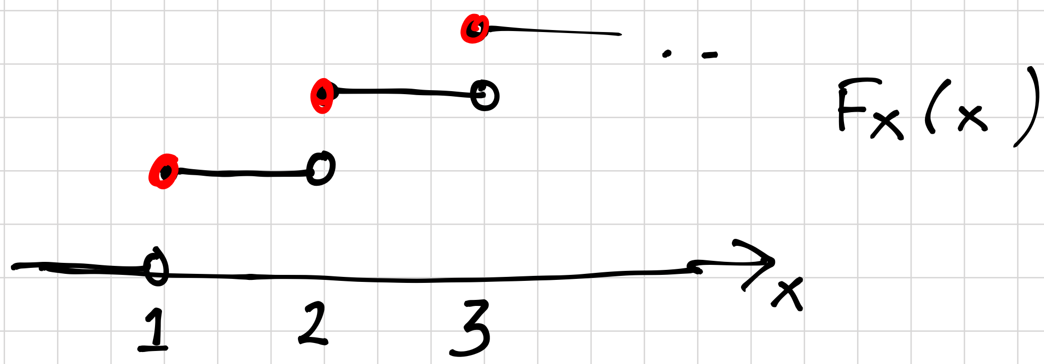


$$F_X(x) = P(X \leq x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \leq x < 1 \\ 3/4, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

Ex Y is D.R.V with PMF

$$P_Y(y) = \begin{cases} (1/4)(3/4)^{y-1}, & y = 1, 2, \ldots \\ 0, & \text{otherwise} \end{cases}$$

what is the CDF of Y.



. $y < 1 \Rightarrow F_X(y) = 0$

For any integer $n \geq 1$ the CDF would be

$$F_Y(n) = \sum_{j=1}^{n} P_Y(j) = \sum_{j=1}^{n} \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{j-1}$$

/* Appendix $\rightarrow$ $(1-x)\sum_{j=1}^{n} x^{j-1} = 1 - x^n$ */

$$F_Y(n) = 1 - \left(\frac{3}{4}\right)^n$$

floor function $\lfloor y \rfloor$: the largest integer less than or equal to y.

$$\lfloor 3.5 \rfloor = 3 \qquad \lfloor 3.9 \rfloor = 3 \qquad \lfloor 4 \rfloor = 4$$

If $n \leq y < n+1$ for some integer n

then $\lfloor y \rfloor = n$

$$F_Y(y) = P(Y \leq y) = P[Y \leq n] = F_Y(n) = F_Y(\lfloor y \rfloor)$$

$$F_Y(y) = \begin{cases} 0, & y < 1 \\ 1 - \left(\frac{3}{4}\right)^{\lfloor y \rfloor}, & y \geq 1 \end{cases}$$

Averages and Expected Values

X is a D.R.V. The expected value (or average) of X is

$$E[X] = \mu_X = \sum_{x \in S_X} x \cdot P_X(x)$$

Ex X is a D.R.V.

$$P_X(x) = \begin{cases} 1/4, & x = 0 \\ 1/2, & x = 1 \\ 1/4, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4}$$

$$= \underline{1}$$

Theorem

X is a Bernoulli R.V. with parameter p

then $E[X] = p$

$$P_X(x) = \begin{cases} p, & x = 1 \\ 1-p, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = 0 \cdot (1-p) + 1 \cdot p = p.$$

Theorem

if X is a Geometric R.V. with parameter p, then $E[X] = 1/p$

Let $q = 1-p$

$$P_X(x) = \begin{cases} p \cdot q^{x-1}, & x = 1, 2, 3, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{x=1}^{\infty} x \cdot p \cdot q^{x-1}$$

/* Appendix $\sum_{i=1}^{\infty} i \cdot q^{i} = \frac{q}{(1-q)^2}$ */
$|q| < 1$

$$E[X] = \frac{p}{q} \sum_{x=1}^{\infty} x \cdot q^{x} = \frac{p}{\cancel{q}} \cdot \frac{\cancel{q}}{\underbrace{(1-q)^2}_{p}} = \frac{p}{p^2} = \frac{1}{p}$$

Theorem

If ~~The~~ X is a Poisson R.V. with parameter α then $E[X] = \alpha$

$$P_X(x) = e^{-\alpha} \cdot \frac{\alpha^x}{x!}, \quad x \geq 0$$

Proof

$$E[X] = \sum_{x=0}^{\infty} x \cdot P_X(x) = \sum_{x=0}^{\infty} x \cdot \frac{\alpha^x}{x!} \cdot e^{-\alpha}$$

$$= \sum_{x=1}^{\infty} x \cdot \frac{\alpha^x}{x!} e^{-\alpha}$$

$$\frac{x}{x!} = \frac{1}{(x-1)!}$$

$$\alpha^x = \alpha \cdot \alpha^{x-1}$$

$$= \alpha \sum_{x=1}^{\infty} \frac{\alpha^{x-1}}{(x-1)!} \cdot e^{-\alpha}$$

$\ell = x - 1$

$$E[X] = \alpha \cdot \underbrace{\sum_{\ell=1-1=0}^{\infty} \frac{\alpha^{\ell}}{\ell!} \cdot e^{-\alpha}}_{1}$$

$$E[X] = \alpha$$

Homework

Prove the following

(a) X is a Binomial R.V. with (n,p)

$$E[X] = n \cdot p$$

(b) X is a Pascal (Negative Binomial) R.V with (k,p)

$$E[X] = \frac{k}{p}$$

(c) X is a Discrete Uniform R.V. Between (k, ℓ)

$$E[X] = \frac{k+\ell}{2}$$

(Functions of Discrete R.V.s)

Derived Random Variable

X is a random variable. $Y = g(X)$ is a function of X and it is itself a R.V.

Each sample value y of a derived R.V. Y is mathematical function $g(x)$ of sample value x of R.V. X.

$$Y = g(X)$$

Ex

A parcel shipping company offers a charging plan: \$1.00 for the first pound, \$0.90 for the second pound, etc., down to \$0.60 for the fifth pound, with rounding up for a fraction of a pound. For all packages between 6 and 10 pounds, the shipper will charge \$5.00 per package. (It will not accept shipments over 10 pounds.) Find a function $\boxed{Y = g(X)}$ for the charge in cents for sending one package.

$X = \{1, 2, 3, \ldots, 10\}$ X: weight in pounds

Y: charging plan in cents

1. \$1
2. \$0.9

$$Y = g(X) = \begin{cases} 105X - 5X^2, & x = 1,2,3,4,5 \\ 500, & x = 6,7,8,9,10 \end{cases}$$

Theorem

For a discrete R.V X, the P.M.F of $Y = g(X)$ is

$$P_Y(y) = \sum_{x:\, g(x)=y} P_X(x)$$

Ex. In example 3.25 suppose that all packages weigh 1, 2, 3 or 4 pounds with equal probability. Find the PMF and the expected value of Y, the shipping charge.

$$P_X(x) = \begin{cases} 1/4, & x = 1,2,3,4 \\ 0, & \text{otherwise} \end{cases}$$

$Y = g(X)$

$Y = 105X - 5X^2$

$x=1 \rightarrow y = 100 \qquad P_X(1) = 1/4$

$x=2 \rightarrow y = 190 \qquad P_X(2) = 1/4$

$x=3 \rightarrow y = 270 \qquad P_X(3) = 1/4$

$x=4 \rightarrow y = 340 \qquad P_X(4) = 14$

$S_Y = \{100, 190, 270, 340\}$

$$P_Y(y) = \begin{cases} 1/4, & y = 100, 190, 270, 340 \\ 0, & \text{otherwise} \end{cases}$$

$$E[Y] = 100 \times \frac{1}{4} + 190 \times \frac{1}{4} + 270 \times \frac{1}{4} + 340 \times \frac{1}{4}$$

$= 225$ cents.

Ex

$$g(X) = Y = \begin{cases} 105X - 5X^2, & X = 1,2,3,4,5 \\ 500, & X = 6,7,8,9,10 \end{cases}$$

$$P_X(x) = \begin{cases} 0.15 & x = 1,2,3,4 \\ 0.10 & x = 5,6,7,8 \\ 0, & \text{otherwise.} \end{cases}$$

$X=1 \rightarrow y=100 \qquad P_X(1) = 0.15$

$X=2 \rightarrow y=190 \qquad P_Y(190) = 0.15$

$X=3 \rightarrow Y=270 \qquad P_Y(270) = 0.15$

$X=4 \rightarrow Y=340 \qquad P_Y(340) = 0.15$

$X=5 \rightarrow Y=400 \qquad P_Y(400) = 0.10$

$X=6, \rightarrow Y=500$

$X=7 \rightarrow Y=500 \qquad P_Y(500) = 0.10+0.10+0.10$

$X=8 \rightarrow Y=500 \qquad = 0.30$

$$P_Y(y) = \begin{cases} 0.15, & y = 100,190,270,340 \\ 0.10, & y = 400 \\ 0.30, & y = 500 \\ 0, & \text{otherwise.} \end{cases}$$

$$E[Y] = 100 \times 0.15 + 190 \times 0.15 + \cdots$$

$$= 325 \text{ cents.}$$

Theorem

Given a D.R.V. with PMF $P_X(x)$

$Y = g(X)$ is a function of X

$$E[Y] = \mu_Y = \sum_{x \in S_X} g(x) \cdot P_X(x)$$

Variance

The variance of R.V. X is

$$\sigma_X^2 = Var(X) = E[(X-\mu_X)^2]$$

Standard Deviation of X is

$$\sigma_X = \sqrt{Var(X)}$$

Theorem

$$Var(X) = E[X^2] - \mu_X^2$$

$$Var(X) = \sum_X x^2 . P_X(x) - \sum_X 2.\mu_X.x\, P_X(x) + \sum_X \mu_X^2 . P_X(x)$$

$$= E[X^2] - 2\mu_X \underbrace{\sum_X x . P_X(x)}_{\mu_X} + \mu_X^2 \underbrace{\sum_X P_X(x)}_{1}$$

$$= E(X^2) - \mu_X^2 \qquad \blacksquare$$

Moments

For a random variable X

(a) The nth moment of X is $E(X^n)$

(b) The nth central moment is $E[(X-\mu)^n]$

- $E(X) = \mu_X$ = 1st moment
- $E(X^2) \Rightarrow$ 2nd moment.
- $Var(X) \Rightarrow$ 2nd central moment.

Theorem

$$Var[aX+b] = a^2 Var[X]$$

$$Y = aX+b \qquad Var[Y] = E[Y^2] - \mu_Y^2$$

$$E[Y^2] = E[a^2X^2 + 2abX + b^2]$$

$$= a^2 E[X^2] + 2ab\,E[X] + E(b^2)$$

$$= a^2 E[X^2] + 2ab\mu_X + b^2$$

$$E[Y] = \mu_Y = a\,E[X] + b$$

$$= a \cdot \mu_X + b$$

$$\mu_Y^2 = a^2\mu_X^2 + 2ab\mu_X + b^2$$

$$E[Y^2] - \mu_Y^2 = a^2 E[X^2] - a^2\mu_X^2$$

$$= a^2 Var[X] \quad /\!\!\!\square$$

Homework

Prove the following

(a) If X is a Bernoulli R.V., p
then $Var(X) = p\cdot(1-p)$

(b) If X is geometric (p)

$$Var(X) = \frac{1-p}{p^2}$$

(c) If X is Binomial (n,p)

$$Var(X) = np(1-p)$$

(d) If X is Pascal (Negative Binomial) with (k,p) then

$$Var(X) = \frac{k(1-p)}{p^2}$$

(e) X is Poisson, α
$Var(X) = \alpha$

(f) If X is discrete uniform (k, ℓ)

$$Var(X) = \frac{(\ell-k)(\ell-k+2)}{12}$$