Ref Book
Thomas calculus 12th edition

1) - limits

- sandwich Theorem (sıkıştırma teo)
- continuity
- intermediate value Theorem (Ara Değer teo)
- Asymptotes
  - oblique asymp
  - Horizontal asymp (yatay asimp)
  - vertical asymp (Dikey asimptot)

1-38

Preliminaries
Basic concepts

limits :

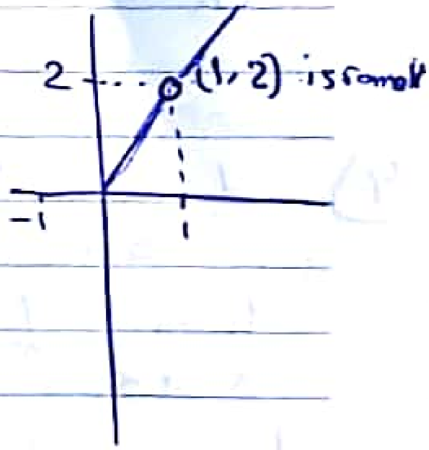
P/39 if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$

Then $\lim_{x\to a} f(x) = L$

otherwise if (1) ≠ (2) Then $\lim_{x\to a} f(x)$ does not exist

i.e

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2$$



$x=1 \notin$ Domain of $f(x)$ set

2 Sandwich Theorem

on an interval which contains the point c

if $\boxed{g(x) \le f(x) \le h(x)}$

& if

$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$

Then $\boxed{\lim_{x\to c} f(x) = L}$

Sandwich Theorem also works as $\boxed{x \to \infty}$

3) example
Does given limit exist?
if so find limit value

$\lim_{x \to \infty} \frac{\cos x}{x}$ → we don't have in determind for $\frac{\infty}{\infty}$

$\frac{-1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$

1) $\lim_{x \to \infty} -\frac{1}{x} = 0$

$\lim_{x \to \infty} \frac{1}{x} = 0$

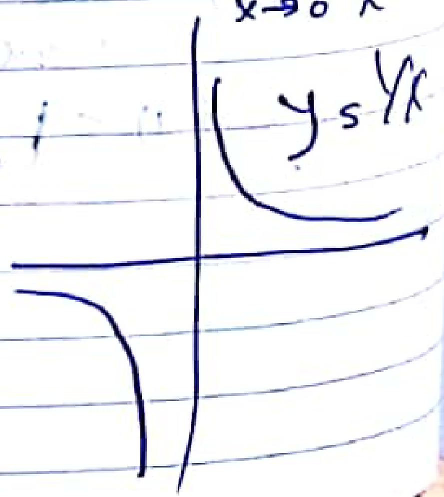
$\Rightarrow$ By sandwich Theo

$\lim_{x \to \infty} = \frac{\cos x}{x} = 0$

! $\infty + 2 \neq \infty$ because ∞ is not a number

! $\lim_{x \to \infty} (x+2) = \infty$

$\lim_{x \to 0} \frac{1}{x} = -\infty$



$\frac{1}{0}$ = undfined

$\lim_{x \to 0^-} \frac{1}{x}$: limit doesn't exist

1.
2.
3.

! Average ≠ difference

ates cünes

$$\lim_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

even Though it seems That x^{1000} grows faster Than e^x as $x \to \infty$

~~[illegible]~~ we applay hopital's Rule repeadetly 1000 times

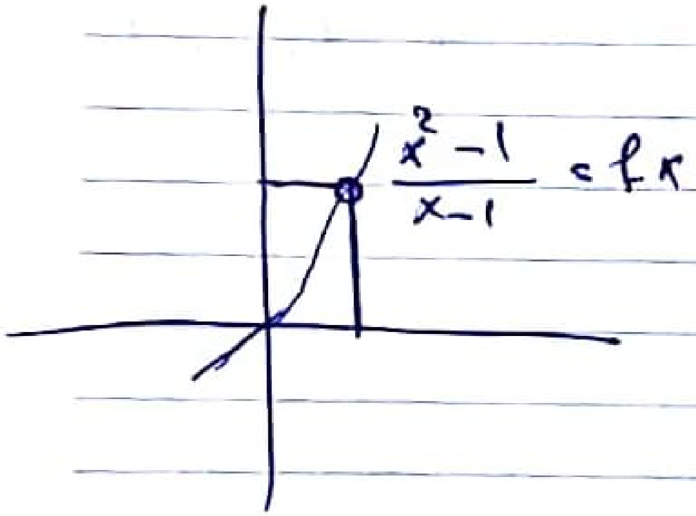
$$\lim_{x\to\infty} \frac{1000999}{e^x} = 0$$

Continuity

if all Three conditions given below are satisfied Then $f(x)$ is continuous at $x = a$

if one (or more) of The conditions fail Then $f(x)$ is discontinuous.

1. $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$
2. $f(x)$ is defined at $x = a$
3. $L = f(a)$



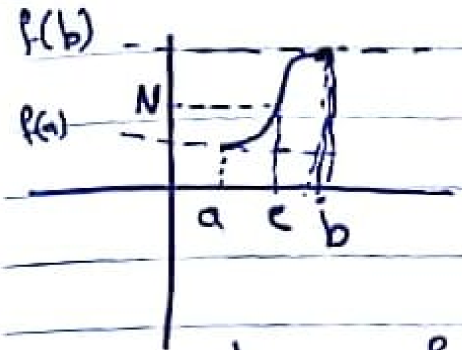
f(x) is undefined
at x = 1
f(x) is discontinuous
at x = 1

1) Intermediate Value Theorem (IVT) :

if $f(x)$ is continuous over $[a,b]$

and if $f(a) \leq N \leq f(b)$

Then at lest one real number $c \in (a,b)$
exactly (kesinlikle) exists for wich $f(c) = N$



example: $f(x) = x^3 - x - 1$
show That over $[2\ 5]$

$\exists c \in (2,5)$ for which $f(c) = 70.2$

2] (7p)
1^{st} step
9 $f(x)$ is a polynomial
function Therefor.
9 $f(x)$ is continuous on $(-\infty, \infty)$
and naturaly $f(x)$ is also continuous on
$[2, 5]$ we may apply IVT

2^{nd} step evaluate → Değerini belirlemek

respectively → sırası ile

~~f(25)~~

$f(2) = 5$

$f(5) = 119$

find step

since $5 < 70.2 < 119$

by IVT $\exists\, c \in (2,5)$

for which $f(c) = 70.2$

example

root → c ⇒ f(c) = 0

kök

$x^3 - x = 1$

show That given equ. has at lest one real root (kök) over $[0,4]$

1st way

$\underbrace{x^3 - x - 1}_{f(x)} = 0$

to be denoted by ______ ile gösterilmek

$f(0) = -1$

$f(4) = 59$

since we have opposite signs, by IVT $\exists\, c \in (0,4)$ for which $f(c) = 0$

2^{nd} way

$x^3 - x = 1$ $\qquad N$

$\underbrace{x^3 - 1}_{g(x)} = 1 \qquad a\,g(x)\,1$

$g(0) = 0 \leq N < g(4) = 60$

by IVT $\exists\, c \in (0,4)$

for which $g(c) = 1$

Matematik

*Asymptots:

① Vertical asymptote:

∀ For $f(x) = \frac{\sim}{\sim}$ if $x = a$ makes denominator zero then $x = a$ is the vertical asymptote
The given exprission is not correct

i.e: $f(x) = \frac{\sin x}{x}$ for $x = 0$ denominator is equal to zero(payda) But $f(x)$ doesn't have a vertical asymptote.

$\left.\begin{array}{l} \sin x \longrightarrow \text{odd fun.} \\ x \longrightarrow \text{odd fun.} \end{array}\right\}$ $f(x)$ is even func. ⟹ graph of $f(x)$ is symmetric with respect to y axis

If $\lim_{x \to a^+} f(x) = \infty$ or
$\lim_{x \to a^+} f(x) = -\infty$ or
$\lim_{x \to a^-} f(x) = \infty$ or
$\lim_{x \to a^-} f(x) = -\infty$

} if one or more of them is satisfied then $x = a$ is the vertical asymptote line of $f(x)$

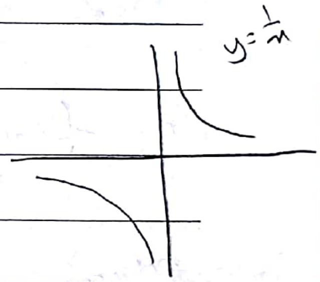
$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{0}{0} \text{ (indeterminate form)}$$

⇒ By L'hopital's Rule

$$\lim_{x \to 0} \frac{\cos x}{1} = 1 \Rightarrow \text{Real number (not } \mp\infty)$$

⇒ $f(x)$ has no vertical asymptote

<u>Example:</u>



$f(x) = \frac{1}{x}$ Domain set $\mathbb{R} - \{0\}$

$\lim_{x \to 0^+} \frac{1}{x} = \infty \longrightarrow$ as $x \to 0^+$ $x = 0$ is vertical asymptote of $f(x)$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty \longrightarrow$ as $x \to 0^-$ $x = 0$ is vertical asymptote of $f(x)$

& graph of $f(x)$ and its vertical asymptote does not intersect (kesişmez)

* Horizontal Asymptote:

If $\lim_{x\to\infty} f(x) = L_1$ (Real Number)

then as $x\to\infty$ $y=L_1$ is horizontal asymptote of $f(x)$

If $\lim_{x\to-\infty} f(x) = L_2$ (Real num)

then as $x\to-\infty$ $y=L_2$ is the horizontal asymptote of $f(x)$

- For some types of $f(x)$ L_1 may be equal to L_2

Example: $f(x) = 2 + \frac{\sin x}{x}$ → even func.

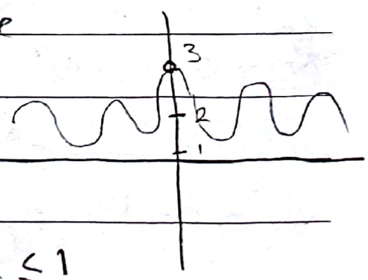
Determine asymptote(s), if any.

① Domain set $R - \{0\}$

Recall that: $\lim_{x\to 0} \frac{\sin(kx)}{mx} = \frac{k}{m}$, k, m : Real num. $k \neq 0, m \neq 0$

② $\lim_{x\to 0} 2 + \frac{\sin x}{x} = 3$ (Real num)

$\Rightarrow$ No vertical asymptote



$\lim_{x\to\infty} 2 + \frac{\sin x}{x}$

By sandwich Theorem $\frac{-1}{|x|} \le \frac{\sin x}{|x|} \le \frac{1}{x}$

$$\lim_{x \to \infty} \frac{1}{x} = 0 \quad \Big\} \quad \lim_{x \to \infty} \frac{\sin x}{x} = 0$$

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\Rightarrow \lim_{x \to \infty} f(x) = 2 \quad \text{(Real num)}$$

(stanted)

③ Oblique Asymptote

if $f(x) = \dfrac{\text{Polynomial func}}{\text{Polymomial func}}$

if degree of numerator is one greater than degree of denonumerator

$\Rightarrow$ $f(x)$ has an oblique asymptote

By long divison (polynomial divison) we determine ~~t~~ equation of oblique ~~s~~asymptote

Example: $f(x) = \dfrac{x^3+1}{x^2}$

Determine ~~Oblique~~ asymptote(s), if any

Domain set $R - \{0\}$

$$\lim_{x \to 0^-} \frac{x^3+1}{x^2} = \infty \qquad \lim_{x \to 0^+} \frac{x^3+1}{x^2} = \infty$$

as $x \to 0^+$ and as $x \to 0^-$ $x = 0$ (y axis) is the vertical asymptote of $f(x)$

$$\lim_{x \to \infty} \frac{x^3+1}{x^2} = \infty$$

$$\lim_{x \to -\infty} \frac{x^3+1}{x^2} = -\infty$$

$\Rightarrow$ No horizontal asymptote

③ $f(x)$ has an oblique asymptote

$$\begin{array}{r|l} x^3+1 & x^2 \\ \hline x^3 & x \\ \hline 1 & \end{array}$$

Then $f(x) = x + \frac{1}{x^2}$

※ For asymptote(s): Graph of $f(x)$ approaches Asymptote line as $x \to \infty$ and/or $x \to -\infty$

In other words: distance between $f(x)$ and asymptote line goes to zero

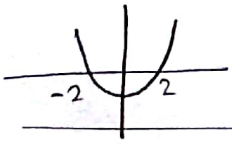
It means that $f(x)$ $\boxed{\text{behaves}}$ similar with its asymptote line.

$f(x) \approx$ asymptote line

oblique asym.

$$\Rightarrow f(x) = x + \frac{1}{x^2}$$

<u>H.W</u> Find a general formula of oblique asymptotes.



slope

Derivatives

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

H.W. Differentiation Rules

*Rolle's Theorem:

If $f(x)$ is continous on $[a,b]$ and
if $f(x)$ is differentiable on (a,b) and
if $f(a) = f(b)$ then by Rolle's Theorem

$$\boxed{\exists\, c \in (a,b) \quad f'(c) = 0}$$

$f(a) \neq f(b)$

*Mean Value Theorem: (ortalama değer t.)

Rolle's Theo. says that slope of tangent line at $x = c$ is equal to slope of the line passing through a and b

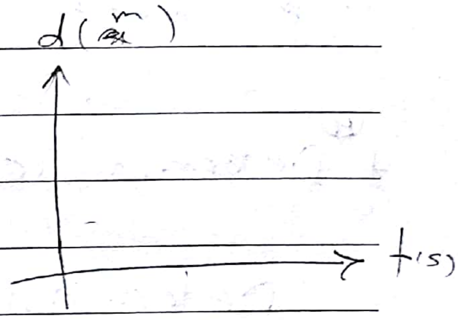
if $f(x)$ is cont. on $[a,b]$
and if $f(x)$ is - - - - diff. - - (a,b)

$$\Rightarrow \text{Slope of } d = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \exists\, c \in (a,b) \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$

Two runners start a race from C (at the same time) and finish the race at D (together).
Show that during the race theirs speeds are equal at least once.

$f_1(s_1) = f_1(s_2)$



$f(s_1') = f(s_2')$

$f(x)$ position func. of (A) } $[0, a]$
$g(x)$ " " " (B) }

$h(x) = f(x) - g(x)$

By MVT $\exists\, c \in (0, a) \Rightarrow h'(c) = \dfrac{\overbrace{h(a)}^{=0} - \overbrace{h(0)}^{=0}}{\underbrace{a - 0}_{\neq 0}} = 0$

speed of A, speed of B

$h'(x) = f'(x) - g'(x)$
$h'(c) = f'(c) - g'(c)$
$0 = f'(c) - g'(c) \Rightarrow f'(c) = g'(c)$

Calculus

* Exercises:

$x^3 + x^2 - 5 = 0$ show that given equation has only one real root over $[1, \infty)$

a) By using LVT and or MVT

b) By using properties of derivatives

1st step: $f(x) = x^3 + x^2 - 5 \to$ polynomial func, which is continueus on $(-\infty, \infty)$

Then on $[1, \infty)$ we may apply LVT

$f(1) = -3 < 0$

$f(2) = 7 > 0$ we may also choose any other point over $[1, \infty)$

Then by IVT $\exists\, c \in (1, 2)$ for which $f(c) = 0$ root

$\Rightarrow -3 < f(c) = \overset{0}{N} < 7$ By IVT we verify the existence of the root over $(1, 2)$

2nd step: We have to show that there are no more real roots over $[1, \infty) \Rightarrow$ we have to show that c is unique over the given interval

We assume that there are two more roots (c_2, c_3) over $[1,\infty)$, now we have to prove that our assumption fails.
Let $c_2 < c_3$

3rd step: $f(x) = x^3 + x^2 - 5$ is cont. over $[1,\infty)$
and it is differentiable over $(1,\infty)$
$\Rightarrow$ we may apply MVT $\quad (c_2, c_3) \subset (1,\infty)$

$\Rightarrow \exists d \in (c_2, c_3)$

for which $f'(d) = \dfrac{\overbrace{f(c_3) - f(c_2)}^{=0}}{\underbrace{c_3 - c_2}_{\neq 0}} = 0$

$f(x) = x^3 + x^2 - 5 \Rightarrow f'(x) = 3x^2 + 2x$
$\Rightarrow f'(d) = 3d^2 + 2d = 0 \Rightarrow d_1 = 0 \quad d_2 = -2/3$

at this step we face a problem
$d_1 = 0$ and $d_2 = -2/3 \notin (c_2, c_3)$

Consequently, since our assumption fails; it implies that c is unique

if $c_1 = c_2 = c_3 = m$
then $f(x) = (x-m)^3 = 0 \Rightarrow \cancel{c_1 = c_2 = c_3}$

2nd way: 1'st step is the same

$f'(x) = 3x^2 + 2x = 0$

$x = 0 \qquad x = -2/3$

x	$-\infty$		$-2/3$		0		∞
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$		$\nearrow$		$\searrow$		$\nearrow$	

$\rightarrow$ c is unique

show that

2) $|\sin y - \sin x| \leq |y - x|$ for all x, y

We assume that $f(x) = \sin x$ continous and $f(x)$ is differentiable

$\Rightarrow$ We may use Mean value Th.

$\exists\, d \in (x,y) \quad f'(d) = \dfrac{f(y) - f(x)}{y - x}$

$\Rightarrow \cos(d) = \dfrac{\sin y - \sin x}{y - x}$

$\Rightarrow \left| \dfrac{\sin y - \sin x}{y - x} \right| \leq 1$

$-1 \leq \cos d \leq 1 \Rightarrow |\cos d| \leq 1 \quad x \neq y$

Final step for $x = y \quad 0 \leq 0$

* To draw graph of f(x) we follow the steps given in the example:

Ex: Draw graph of $f(x) = e^{\frac{1}{x}}$

2nd step: We find x-intercept and y-intercept if any.

1st step: Domain set of f(x): $\mathbb{R} - \{0\}$

for x intercept $y = 0$ but $y = e^{\frac{1}{x}}$ then no x intercept because $y \neq 0$

for y intercept $x = 0$ But $\notin$ Domain set of x $\Rightarrow$ no y intercept

3rd step: We find asymptote(s), if any:

① vertical asym. ⊛ $\lim_{x \to 0^+} e^{\frac{1}{x}} = \infty$

$\Rightarrow$ As $x \to 0^+$ the line $x = 0$ is a vertical asym. of f(x).

⊛ $\lim_{x \to 0^-} e^{\frac{1}{x}} = 0 \Rightarrow$ As $x \to 0^-$ there is no V. Asym.

② Horizontal: $\lim_{x \to \infty} e^{\frac{1}{x}} = 1 \Rightarrow$ As $x \to \infty$ and $x \to -\infty$ $y = 1$ is horizontal asym. of f(x)

③ Oblique asym? $\to$ No H.W

4.th step we find critical point(s) if any and determine local extreme value(s) if any.

⊛ if $f'(c)=0$ or $f'(c)=$ undefined ($c \in$ Domain set) then c is called critical point

⊛ if f' $\underset{\searrow \quad \nearrow}{-\overset{c}{\phi}+}$ $\Rightarrow$ at $c=0$ $f(x)$ has local min value $f(c)$

⊛ if f' $\underset{\nearrow \quad \searrow}{+\overset{c}{\phi}-}$ $\Rightarrow$ then at $x=c$ $f(x)$ has local max value $f(c)$

! Local extreme values occur at critical points if sign of f' changes at this point

$y=e^{\frac{1}{x}} \Rightarrow y'=e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)=0$

Recall that:
$y=e^{g(x)}$
$y'=e^{g(x)} \cdot g'(x)$

$\Rightarrow$ There is no Critical point
$\Rightarrow$ There is no local extreme value

5.th step We determine intervals over which f is increasing or decreasing

Over $(-\infty,0)\cup(0,\infty)$ f is decreasing

6th step: We search inflection point(s) and and also concavity

$y' = -\frac{e^{\frac{1}{x}}}{x^2}$ By quotient Rule of derivation

$y'' = -\frac{e^{\frac{1}{x}}\left(\frac{1}{-x^2}\right)x^2 - 2x\,e^{\frac{1}{x}}}{x^4}$

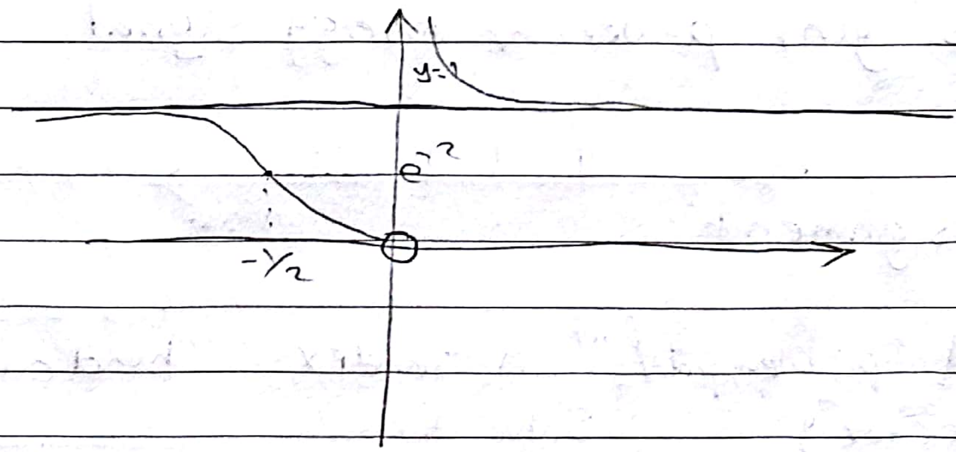
$\frac{e^{\frac{1}{x}}(1+2x)}{x^4}$ for $x = -\frac{1}{2}$ $y''_{(x)} = 0$

inflection point

	$-\infty$		$-\frac{1}{2}$		0		∞
f''		$-$	0	$+$	$\|\|$	$+$	

over $(-\infty, -\frac{1}{2})$ concave down
Over $(-\frac{1}{2}, 0)$ concave up
over $(0, \infty)$ concave up

final step: We draw ~~the~~ graph of $f(x)$



Calculus 10.10.2018

* Related Rates:

To find rate of change of a variable, at the given instant, we follow the given steps:

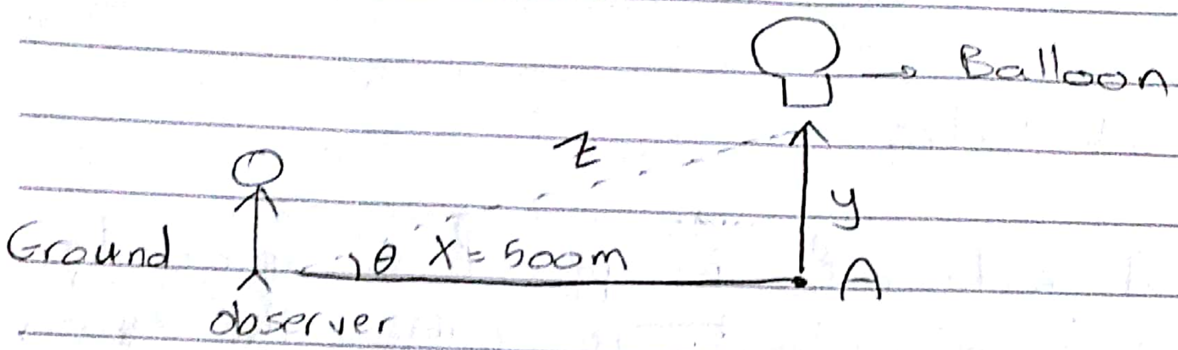
① We draw the graph and write given numerical information and we express, what is asked, by derivative notation.

② We determine constant(s) and variables.

③ We determine the equation(s) we will use and find the relation between variables by which we can find what is asked.

④ We differentiate with respect to tim(t) (or any other parameter)

⑤ We interpret the result



The balloon moves straightly up from the point A. The rate of change of θ is 0.14 rad/min
Find speed of the balloon when $\theta = \frac{\pi}{4}$
(During the movement of balloon observer does not move).

Speed of balloon: $\frac{dy}{dt} = ?$

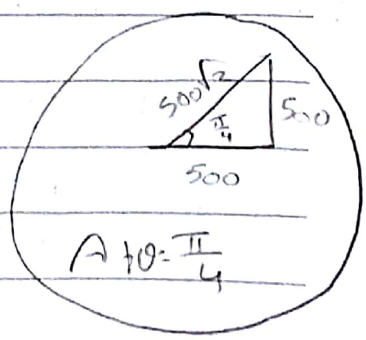
② Constant(s)   Variables

$X = 500m$ $z(t)$ $y(t)$ $\theta(t)$

③ ⑦ $z^2 = x^2 + y^2$

④ We differentiate both sides with respect to t

$$\Rightarrow 2z \underbrace{\frac{dz}{dt}}_{?} = 0 + 2y \frac{dy}{dt}$$



③ $\tan\theta = \frac{y}{x}$

Reminder
$y = \tan(f(t))$
$y' = \sec^2(f(t))\, f'(t)$

④ We differentiate both sides with respect to t

$$\Rightarrow \sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dy/dt}{500}$$

$\sec\theta = \frac{1}{\cos\theta}$

At the given instant $\theta = \frac{\pi}{4} \Rightarrow \frac{d\theta}{dt} = 0.14$

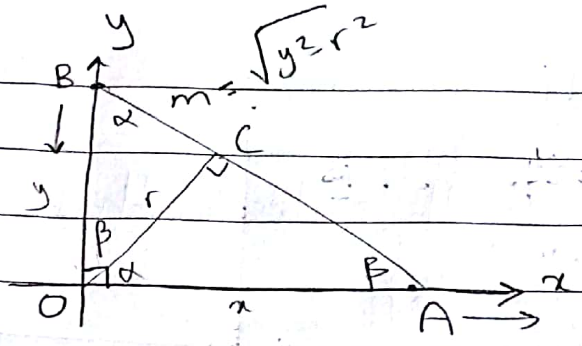
$$\Rightarrow \left(\frac{2}{\sqrt{2}}\right)^2 (0.14) = \frac{dy/dt}{500}$$

$$\Rightarrow \frac{dy}{dt} = 140 \text{ meter/min}$$

⑤

⊛ Final step: We interpret the result at the given instant $\theta = \frac{\pi}{4}$ balloon's speed is 140 m/min.

In other words y is increasing by the rate of 140 m/min



A and B move by the given directions B approaches to the point o by $0.3r$. And during the movement of A and B the length (r) of height from origin to hypotenuse remains constant.
Find speed of A when $y = 2r$

① When $y = 2r$ $\frac{dx}{dt} = ?$

There are similar tringles $\overset{\triangle}{BOC} \sim \overset{\triangle}{BAO}$ then we have:

$$\frac{|BC|}{y} = \frac{r}{x}$$

② Constants — variables

r — $x(t)$ $y(t)$ $m(t)$ $\alpha(t)$ $\beta(t)$

$$\Rightarrow \frac{\sqrt{y^2 - r^2}}{y} = \frac{r}{x} \Rightarrow \frac{y}{\sqrt{y^2 - r^2}} = \frac{x}{r}$$

We differentiate both sides with respect to t

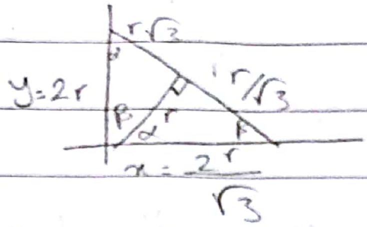
By quotient rule of derivative

$$\Rightarrow \frac{\frac{dy}{dt}\cdot\sqrt{y^2-r^2} - \frac{2y\frac{dy}{dt}}{2\sqrt{y^2-r^2}}\cdot y}{\left(\sqrt{y^2-r^2}\right)^2} = \frac{dx/dt}{r}$$

when $y = 2r$

$\Rightarrow \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$

$\frac{dy}{dt} = -0.3r$



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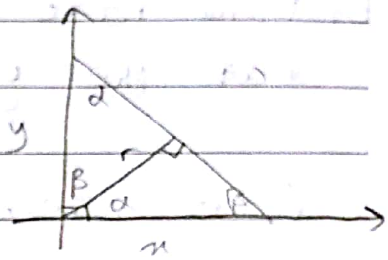
2. way:

$\sin\alpha = \frac{r}{y}$

$\cos\alpha = \frac{r}{x}$



$\sin^2\alpha + \cos^2\alpha = 1$

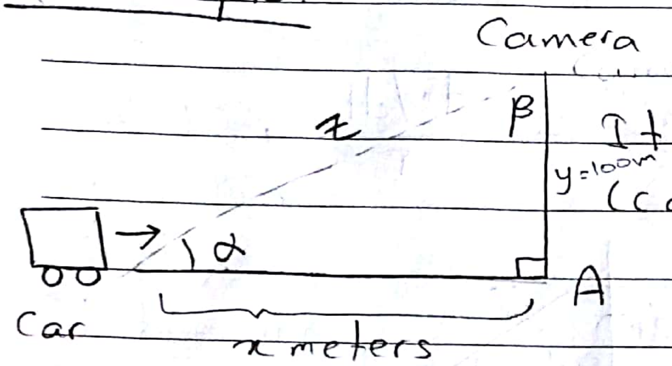
$\Rightarrow \frac{r^2}{y^2} + \frac{r^2}{x^2} = 1 \Rightarrow y^{-2} + x^{-2} = \frac{1}{r^2}$

$\Rightarrow -2y^{-3}\frac{dy}{dt} - 2x^{-3}\frac{dx}{dt} = 0$

At the given instant

$y = 2r, \quad x = \frac{2r}{\sqrt{3}}$

***Example:** The car is moving by the given direction. It's speed is 100 km/min (constant speed) when the car is in front of the camera (at the point A);



a) Find rate of change of distance between the car and camera

b) Find rate of change of α an β (at the given instant).

Constant(s)	Variables
speed of the car = 100 m/min	$x(t), z(t), \alpha(t), \beta(t)$
$y = 100$ m	

a) At the given instant $\frac{dz}{dt} = ?$

$$z^2 = y^2 + x^2 \Rightarrow z^2 = x^2 + 100^2$$

$$\Rightarrow 2z\frac{dz}{dt} = 2x\frac{dx}{dt}$$

At the given instant (when the car at A)

$$x = 0 \Rightarrow \frac{dz}{dt} = 0$$

b) When the car is at A, $\frac{d\alpha}{dt} = ?$ $\frac{d\beta}{dt} = ?$

At the given instant $\alpha = \frac{\pi}{2}$ $\beta = 0$

$\tan\beta = \frac{x}{y} \Rightarrow \underbrace{\sec^2\beta}_{1} \frac{d\beta}{dt} = \frac{\overset{-100}{dx/dt}}{100}$

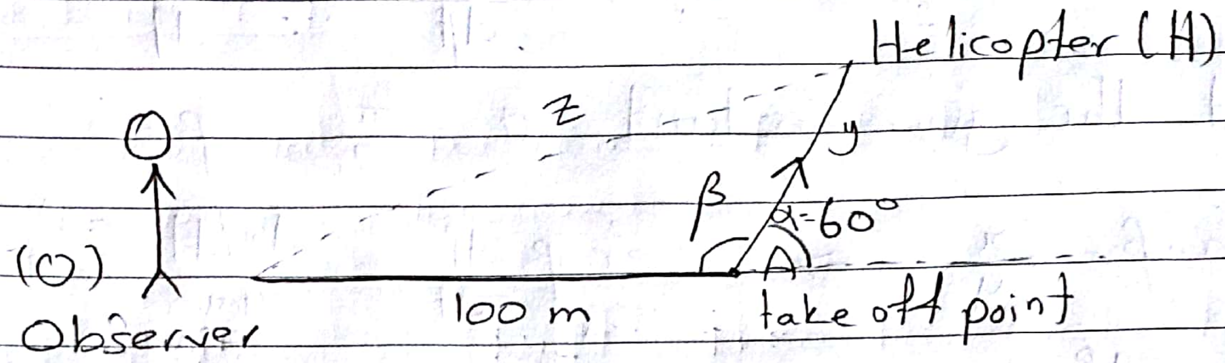
$\Rightarrow \frac{d\beta}{dt} = -1$ rad/min

$\alpha + \beta = \frac{\pi}{2}$ (constant)

We differentiate both sides with respect to time.

$\frac{d\alpha}{dt} + \frac{d\beta}{dt} = 0 \Rightarrow \frac{d\alpha}{dt} = 1$ rad/min

At the given instant; α is increasing by the rate of 1 rad/min and β is decreasing by the rate of 1 rad/min



Helicopter takes off from the point A.
During its movement α is constant.
Its speed is 20 m/s.
After 5 sec find rate of change of distance between O and H

Constants

$\alpha = 60^\circ$ $\beta = 120^\circ$

speed of helicopter = 20 m/s $\rightarrow \frac{dy}{dt}$

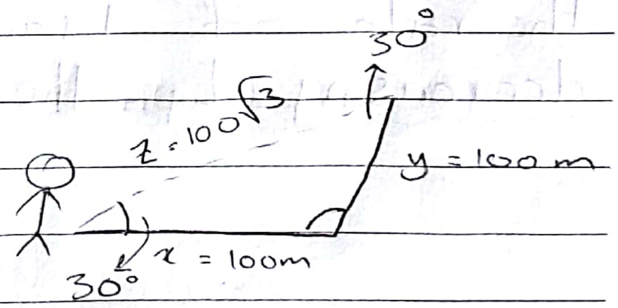
$x = 100$ m

Variables

$z(t), y(t)$

(*) When $t = 5$ sec $\frac{dz}{dt} = ?$

(*) After 5 sec $\Rightarrow$



By Cosine Rule:

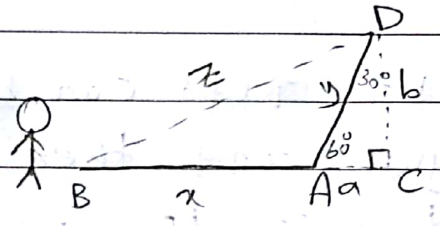
$$z^2 = \overset{100}{x}{}^2 + y^2 - 2\overset{100}{x}\,y\,\underbrace{\cos\beta}_{-\frac{1}{2}}$$

$$\Rightarrow 2z\frac{dz}{dt} = 2y\overbrace{\frac{dy}{dt}}^{20} + 100\overbrace{\frac{dy}{dt}}^{20}$$

$$\Rightarrow 2\sqrt{3}\frac{dz}{dt} = 60$$

$$\frac{dz}{dt} = 10\sqrt{3} \text{ m/sec}$$

When $t=5$ z is increasing by the rate $10\sqrt{3}$ m/sec



2.nd way:

$\frac{b}{a} = \sqrt{3}$ (constant) $\Rightarrow b = \sqrt{3}a$

$\frac{db}{dt} = \sqrt{3}\frac{da}{dt} \Rightarrow \frac{db}{dt} = 10\sqrt{3}$ m/sec

$\frac{y}{a} = 2$ (constant)

$\frac{dy}{dt} = 2\frac{da}{dt}$

By $\overset{\Delta}{BCD}$: $z^2 = (x+a)^2 + b^2$

$$\Rightarrow 2z\frac{dz}{dt} = 2(100+a)\frac{da}{dt} + 2b\frac{db}{dt}$$

At $t=5$, $y = 100$ m, $a = 50$ m, $b = 50\sqrt{3}$ m

*Integral

Fundamental Theorem of Calculus:
(Leibniz's Theorem)

if $F(x) = \int_{h(x)}^{g(x)} f(t)\,dt$ then to evaluate
integrand

$\frac{dF}{dx}$ we may follow the ways given below:

1st way: If $f(t)$ can be integrated, we solve the integral, and then by using bounds we find a result in terms of x
And then we differentiate $F(x)$ and find dF/dx

2nd way: By using Leibniz's Rule we shorten the solution and find $\frac{dF}{dx}$ directly

$$\frac{dF}{dx} = \underbrace{f(g(x)).g'(x) - f(h(x)).h'(x)}_{\text{By Leibniz's Theorem}}$$

Example: If $F(x) = \int_{x^2}^{\ln x} \cos t \, dt$ then $\frac{dF}{dx} = ?$

1st way $\sin t \Big|_{x^2}^{\ln x} \Rightarrow F(x) = \sin(\ln x) - \sin x^2$

$\Rightarrow \frac{dF}{dt} = \cos(\ln x) \frac{1}{x} - \cos(x^2) 2x$

2nd way Be Leibniz's Rule:

$\frac{dF}{dx} = \cos(\ln x) \cdot \frac{1}{x} - \cos(x^2) \cdot 2x$

Example: $\lim_{x \to 3} x \cdot \frac{\int_3^x \frac{\sin t}{t} dt}{x-3} = ?$

without using l'hopital

1st way: Since $\lim_{x \to 3} x$ exists we may separate given product

$\Rightarrow \lim_{x \to 3} x \cdot \underbrace{\lim_{x \to 3} \frac{\overbrace{\int_3^x \frac{\sin t}{t} dt}^{f(x)} - \overset{f(3)}{0}}{x-3}}_{f'(3)}$

$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}}$

By Leibniz's Rule: $f'(x) = \frac{\sin x}{x} \cdot 1 - \frac{\sin 3}{3}(0)$

$= \sin 3$

Calculus 17.10.2018

* Exercises:

1) $F = \dfrac{d}{dx}\displaystyle\int_5^{\ln x} \frac{\cos t}{t}\,dt$ Find $\dfrac{dF}{dx}$

! $\displaystyle\int \frac{\cos t}{t}\,dt$ The given integration can not be solved by elementary methods (substitution, integration by parts ...etc.

⇒ For the given example we have to apply Leibniz's Rule in order to find F.

By Leibniz's Rule:

$$F = \frac{\cos \ln x}{\ln x} \cdot \underbrace{\frac{1}{x}}_{\text{derivative of upper bound}} - \frac{\cos 5}{5} \cdot \underbrace{0}_{\text{derivative of lower bound}}$$

Reminder
If $y = \ln(g(x))$
$y' = \dfrac{g'(x)}{g(x)}$

Final step: By using Product Rule and Quotient Rule of derivative we will find $\dfrac{dF}{dx}$

$$\frac{dF}{dx} = \frac{-\sin(\ln x)\cdot\frac{1}{x}\cdot x\ln x - (\ln x + x\frac{1}{x})\cdot \cos \ln x}{(x\ln x)^2}$$

$$\#2)\quad \frac{d^2}{dx^2}\int_0^x \Big(\underbrace{\int_1^{\sin t}\sqrt{1+u^4}\,du}_{H(t)}\Big)\,dt = ?$$

! Solution of $\int \sqrt{1+u^4}\,du$ is so complex

$\frac{d^2F}{dx^2} \rightarrow$ Differentiate F repeadetly twice with respect to x

By Leibniz's Rule

$$\frac{dF}{dx} = H(x)\cdot 1 - \cancel{H(0)\cdot 0} = \int_1^{\sin x}\sqrt{1+u^4}\,du$$

Now we differentiate both sides with respect to x.

$$\frac{d^2F}{dx^2} = \sqrt{1+\sin^4 x}\cdot \cos x - 0 \quad \text{(By Leibniz's Rule)}$$

3) If $F = \int_0^x x^2 \sin(t^2)\,dt$

! x and t are independent variables

$\frac{dF}{dx} = ?$

$\int \sin(t^2)\,dt$ can't be solved.

FALSE $\left\{ \frac{dF}{dx} = x^2 \sin(x^2) \cdot 1 - 0 \right.$ (By Leibniz's Rule)

* TRUE:

$F = x^2 \int_0^x \sin(t^2)\,dt$ Now we differentiate both sides with respect to x / By Product Rule of derivative and by Leibniz's Rule.

$$\frac{dF}{dx} = 2x \int_0^x \sin(t^2)\,dt + x^2\left(\sin(x^2) \cdot 1 - 0\right)$$

4) $F=\int_0^x \overbrace{\sin(x-t)}^{\text{integrand}}\,dt \Rightarrow \frac{dF}{dx}=?$

! x and t are independent variables.

FALSE $\left\{ \frac{dF}{dx}=\sin(x-x).1=0 \right.$

Reminder

$\sin(x \pm t)=\sin x\cos t \pm \cos x\sin t$

* TRUE:

$F=\int_0^x \sin x\cos t-\cos x\sin t\,dt$

! Since the given integrand is continuous over $[0,x] \Rightarrow$ can be separated into two parts.

$\Rightarrow F=\int_0^x \sin x\cos t\,dt-\int_0^x \cos x\sin t\,dt$

$=\sin x\int_0^x \cos t\,dt-\cos x\int_0^x \sin t\,dt$

At this step we may follow two different ways:

$\frac{dF}{dx}=\cos x\int_0^x \cos t\,dt+\sin x\overbrace{(\cos x.1-0)}^{\text{By Leibniz's Rule}}$

(By Product Rule of derivative)

$-\left(-\sin x\int_0^x \sin t\,dt+(\sin x.1-0)\cos x\right)$

$=\cos x(\sin x-0)+\sin x(\cos x)-\left(\sin x(\cos x-\cos 0)+\sin x\cos x\right)$

2. way: $F = \sin x \left(\sin x \Big|_0^x \right) - \cos x \left(-\cos x \Big|_0^x \right)$

3. way $F = \int_0^x \sin(x - t)\, dt$

Since x and t are independent variables we may integrate by having x as a constant term.

$$F = \left(\frac{-\cos(x-t)}{-1} \right)\Bigg|_0^x = \cos(x-x) - \cos(x-0)$$

$$\frac{dF}{dx} = 0 + \sin x$$

Reminder:

$$\int \sin(\text{Linear term})\, dt = \frac{-\cos(\text{Linear term})}{\text{Derivative of Linear term}}$$

4. way We use substitution $x - t = u$ We differentiate both sides with respect to t

$$0 - 1 = \frac{du}{dt} \Rightarrow dt = -du$$

$$\Rightarrow F = -\int_x^0 \sin u\, du \quad \left(\begin{array}{ll} \text{For } t = 0 & u = x \\ \text{for } t = x & u = 0 \end{array} \right)$$

$$= \int_0^x \sin u\, du$$

Reminder

$$\int_a^b f(u)\, du = -\int_b^a f(u)\, du$$

By Leibniz's Rule

$$\frac{dF}{dx} = \sin x \cdot 1 - 0$$

$F(a,b) \to \int_a^b (2+x-x^2)\,dx$ (a and b are real numbers)

5) Determine values of a and b for which result of given definite integral is maximum

1st way: We solve the integral with respect to x

$$= 2x + \frac{x^2}{2} - \frac{x^3}{3} \Big|_a^b$$

$$= \underbrace{\left(2b + \frac{b^2}{2} - \frac{b^3}{3}\right)}_{B} - \underbrace{\left(2a + \frac{a^2}{2} - \frac{a^3}{3}\right)}_{A}$$

We will maximize B and minimize A

By this way B-A will be max.

Then we have to determine critical points for B and A $\Rightarrow B' = 2 + b - b^2 = 0 \Rightarrow b_1 = 2,\ b_2 = -1$

At $\boxed{b=2}$ B has max value

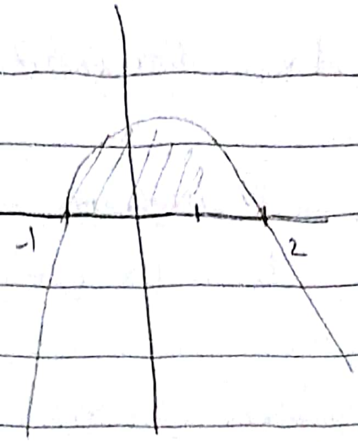
		-1		2	
B'	−		+		−
B	↘		↗		↘

$A' = 2 + a - a^2 = 0$

at $\boxed{a=-1}$ A has min value

		-1		2	
A'	−		+		−
A	↘		↗		↘

2.way



$a = -1$
$b = 2$

3.way We want to maximize F

for $F(a,b)$ we need to use partial derivatives
by this way we determine critical points

$\frac{\partial F}{\partial b} \rightarrow F_b \rightarrow$ Partial derivative of F with respect to b

By Leibniz's Rule:

$$\frac{\partial F}{\partial b} = (2+b-b^2).1 - 0 = 0 \Rightarrow b=2,\ b=-1$$

$$\frac{\partial F}{\partial a} = 0 - (2+a-a^2) = 0 \Rightarrow a=2,\ a=-1$$

$\Rightarrow$ Critical Points: $(2,2),\ (2,-1),\ (-1,2),\ (-1,-1)$

$\checkmark$

Calculus

24.10.2018

* Exercises

1) If u and x are independent variables; Prove that:

$$\int_0^x \left(\int_0^u f(t)\,dt \right) du = \int_0^x \underbrace{f(u)\,(x-u)}_{xf(u)-f(u).u}\,du$$

$$= x\int_0^x f(u)\,du - \int_0^x f(u)\,u\,du$$

Then we differentiate both sides with respect to x. For both sides we apply Leibniz's Rule. Then we have;

$$\underbrace{\int_0^x f(t)\,dt .\, 1 - 0}_{\text{By Leibniz's Rule}} = \underbrace{1\int_0^x f(u)\,du + x\,(f(x).1 - 0)}_{\text{By product rule of derivative}} - \underbrace{[f(x).x.1 - 0]}_{\text{By Leibniz's Rule}}$$

By Leibniz's Rule

$$\int_0^x f(t)\,dt = \int_0^x f(u)\,du$$

Results are equal

Real Numbers

Example: For $b > a > 0$, Prove the given inquality.

$$\frac{1}{b} < \frac{\ln b - \ln a}{b-a} < \frac{1}{a}$$

1st step: If we assume that $f(x) = \ln(x)$ $[a,b]$

$f(x)$ is continuous on $[a,b]$ and $f(x)$ is differentiable on (a,b) $\Rightarrow$ We can apply MVT

$\Rightarrow f'(x) = \frac{1}{x}$

Reminder
if $y = \ln(g(x))$
$y' = \frac{g'(x)}{g(x)}$

2nd step: MVT says that $\exists d \in (a,b)$ for which $f'(d) = \frac{f(b)-f(a)}{b-a}$

$$\Rightarrow * \boxed{\frac{1}{d} = \frac{\ln b - \ln a}{b-a}}$$

Then we have $\frac{1}{b} < \frac{1}{d} < \frac{1}{a}$

If we use reciprocals then we reverse direction of inquality

$\Rightarrow b > d > a$ ✓

3) a) $\log_a b^c \stackrel{?}{=} c \log_a |b|$

b) Determine domain set of $f(x) = \ln \frac{1}{x^2}$

Domain set $= \mathbb{R} - \{0\}$

c) Determine domain set of $g(x) = 2 \ln x$

Domain set is $x > 0$

d) Determine asymptote(s) of $f(x) = \ln \frac{1}{x^2}$

$\lim_{x \to 0^+} \ln \frac{1}{x^2} = \infty$ } As $x \to 0^+$ and as $x \to 0^-$ the line $\boxed{x = 0}$ is vertical asymptote of $f(x)$

$\lim_{x \to 0^-} \ln \frac{1}{x^2} = \infty$

$\lim_{x \to \infty} \ln \frac{1}{x^2} = -\infty$ } As $x \to \infty$ and as $x \to -\infty$ the result aren't real numbers therefor there is no horizontal asymptote.

$\lim_{x \to -\infty} \ln \frac{1}{x^2} = \infty$

Oblique Asymptote H.W

4) Prove that $\frac{15}{0}$ is undefined

$\frac{15}{5} = 3 \Rightarrow 15 - 5 - 5 - 5 = 0$

$\frac{15}{0} = ? \Rightarrow 15 - 0 - 0 - \dots \neq 0$

5) Find limit (if exists)

$\lim_{x \to 0} \underbrace{\frac{6x - \sin 2x}{2x + 3\sin 4x}}_{f(x)}$, (without using l'hopital's Rule)

For $\forall x \ (x \neq 0)$ $f(-x) = f(x) \Rightarrow f(x)$ is an even func.

$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) \Rightarrow \lim_{x \to 0} f(x)$ exists

$\lim_{x \to 0} \frac{x\left(6 - \frac{\sin 2x}{x}\right)}{x\left(2 + \frac{3\sin 4x}{x}\right)} = \lim_{x \to 0} \frac{6 - 2}{2 + 12} = \frac{4}{14}$

6) $\lim_{t \to \infty} \dfrac{t^{\frac{2}{3}} + t^{-1}}{t^{\frac{2}{3}} + \cos^2 t}$

! For the given example we can not apply L'Hopital's Rule Because we don't have indeterminate form $\frac{\infty}{\infty}$

1st way $\lim_{t \to \infty} \dfrac{t^{\frac{5}{3}} + 1}{t^{\frac{5}{3}} + t \cdot \cos^2 t}$

Reminder
$0 \leq \cos^2 t \leq 1$

$\Rightarrow \dfrac{t^{\frac{5}{3}} + 1}{t^{\frac{5}{3}} + t \cdot 1} \leq \dfrac{t^{\frac{5}{3}} + 1}{t^{\frac{5}{3}} + t \cos^2 t} \leq \dfrac{t^{\frac{5}{3}} + 1}{t^{\frac{5}{3}} + t \cdot 0}$

And we apply Sandwich Theorem

$\lim_{t \to \infty} \dfrac{t^{\frac{5}{3}} + 1}{t^{\frac{5}{3}} + t} = \lim_{t \to \infty} \dfrac{t^{\frac{5}{3}} (1 + t^{-\frac{5}{3}})}{t^{\frac{5}{3}} (1 + t^{-\frac{2}{3}})} = 1$

Since limit results are equal.

7) Sketch the graph of $f(x) = \frac{2x^2+5x+1}{x-1}$

1st step: Domain set of $f(x)$ is $\mathbb{R} - \{1\}$.

2nd step: We determine x-intercept and y-intercept (if any)

For $x=0 \rightarrow$ y-intercept $(0,-1)$.

For $y=0 \rightarrow 2x^2+5x+1=0 \Rightarrow \Delta = 17$

$\Rightarrow x_{1,2} = \frac{-5 \mp \sqrt{17}}{4}$ (x-intercepts)

3rd step: Asymptote(s) if any:

$$\left.\begin{array}{l} \lim\limits_{x \to 1^+} \frac{2x^2+5x+1}{x-1} = \infty \\ \lim\limits_{x \to 1^-} \frac{2x^2+5x+1}{x-1} = -\infty \end{array}\right\}$$

As $x \to 1^+$ and as $x \to 1^-$ the line $x=1$ is vertical asymptote of $f(x)$

$$\left.\begin{array}{l} \lim\limits_{x \to \infty} \frac{2x^2+5x+1}{x-1} = \infty \\ \lim\limits_{x \to -\infty} \frac{2x^2+5x+1}{x-1} = -\infty \end{array}\right\}$$

No horizontal asymptote.

Oblique asymptote: Degree of numerator is one greater than degree of denumerator.

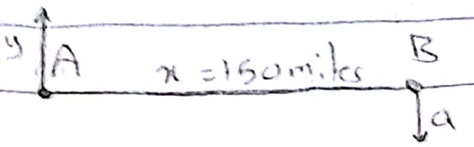
$\Rightarrow$ There is oblique asymptote

$$\begin{array}{r|l} 2x^2+5x+1 & x-1 \\ \hline - & \underbrace{2x+7} \\ 8 & \end{array}$$

$\downarrow$

oblique asymptote

8) At 1^{00} pm, ship B is 150 miles east of ship A, ship A is moving 30 miles/hour north and B is moving 20 miles/hour south. How fast is the distance between A and B changing at 3^{00} pm.



$\frac{dy}{dt} = +30$ miles/hour

$\frac{da}{dt} = +20$ miles/hour

Constants

$x = 150m$ $\frac{dy}{dt} = 30$

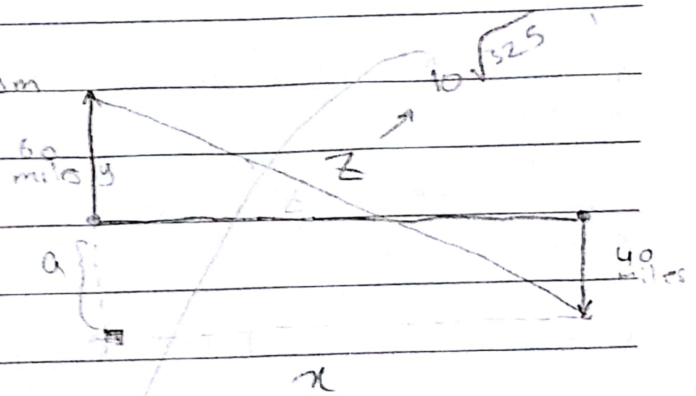
$\frac{da}{dt} = 20$

Variables

$Z(t)$ $y(t)$ $a(t)$

Find $\frac{dz}{dt}$ at 3^{00} pm

At 3^{00} pm



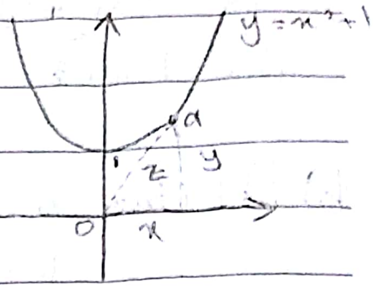
$(y+a)^2 + x^2 = z^2$

$\cancel{2}(y+a)\left(\frac{dy}{dt} + \frac{da}{dt}\right) = \cancel{2}z\frac{dz}{dt}$

$\underset{60}{\downarrow}$ $\underset{40}{\downarrow}$ $\underset{30}{\downarrow}$ $\underset{20}{\downarrow}$

9) Find the rate of change of the distance between the origin and a moving point on the graph of $y=x^2+1$ at $x=2$ if y coordinate is increasing at a rate of 4 Cm/sec

Variables	Constant
$x(t)$	$\frac{dy}{dt}=4$ Cm/sec
$y(t)$	
$z(t)$	



$z^2 = x^2 + y^2$

$z^2 = x^2 + (x^2+1)^2$

$z^2 = 3x^2 + x^4 + 1$

$2z \frac{dz}{dt} = 6x \frac{dx}{dt} + 4x^3 \frac{dx}{dt}$

$\sqrt{29}$ $\quad x=2 \quad x=2$

$y = x^2+1$

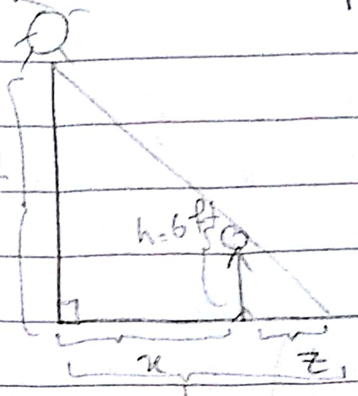
$\frac{dy}{dt} = 2x \frac{dx}{dt}$

$\Rightarrow \frac{dx}{dt} = 1$ Cm/sec

10) A 6 ft man walks at a rate of 3 ft/sec away from a street light that is 21 ft above the ground

a) At what rate is the length of his shadow is changing when he is 8 ft away from the light

b) At the instant what rate is the tip of his shadow?



Constants	variables
$h=6$	$x(t)$
$y=21$	$z(t)$

$$\frac{z}{z+x} = \frac{6}{21}$$

By similar triangles:

$$\frac{z}{z+x} = \frac{6}{21}$$

a) Find $\frac{dz}{dt}$ when $x=8$

b) Find $\frac{dL}{dt}$ when $x=8$

1st way: $\frac{z}{z+x} = \frac{6}{21}$ By Quoteint Rule of Derivative

$$\frac{\frac{dz}{dt}(z+x) - \left(\frac{dz}{dt} + \frac{dx}{dt}\right)z}{(z+x)^2} = 0$$

when $x=8$

$$\frac{z}{z+8} = \frac{6}{21}$$

$$z = \frac{16}{5}$$

2^{nd} way: $5z = 2x \Rightarrow z = \frac{2x}{5}$

$$\Rightarrow \frac{dz}{dt} = \frac{2}{5} \overset{3}{\frac{dx}{dt}} = \boxed{\frac{6}{5} \text{ ft/sec}}$$

b) $x + z = L$

$$\underset{\downarrow \atop 3}{\frac{dx}{dt}} + \underset{\downarrow \atop 6/5}{\frac{dz}{dt}} = \frac{dL}{dt} = \frac{21}{5} \text{ ft/sec}$$

Calculus 31.10.2018

* Exercises

1) If $y = mx + n$ is an oblique asymptote of $f(x)$ where m and n are real numbers and $m \neq 0$ then determine the formulas which give us m and n

Reminder:
Graph of function Does not intersect with its vertical asymptote and with its oblique asymptote.
The distance between $f(x)$ and its asymptotes (vertical and oblique, if any) approches to zero as $x \to \infty$ and/or $x \to -\infty$
Then,

$$\lim_{x \to \mp\infty} f(x) - (mx + n) = 0$$

$$\boxed{\lim_{x \to \mp\infty} f(x) - mx = n} \dots 1$$

If we write x as a common term, then we have a product given below

$$\lim_{x \to \infty_-} x \underbrace{\left(\frac{f(x)}{x} - m \right)}_{T} = n$$

Since $x \to \infty$ and n is a real number we will search the possibilities we have.

① $XT \neq \infty$? But n is a real number

② XT is a real number "

except zero

③ $T = 0$?

then we have indeterminate form $0 \cdot \infty$

which can be transformed into

$\frac{0}{0}$ or $\frac{\infty}{\infty}$ to which we may

apply L'hopital rule.

After that we may find the result a

real number(n).

If we find limit result $= \infty$ it implies

that (for final case) oblique asymptote

does not exist.

Since $T = 0$

$$\lim_{x \to \infty} \left(\frac{f(x)}{x} - m \right) = 0 \Rightarrow \boxed{m = \lim_{x \to \infty} \frac{f(x)}{x}} \quad ②$$

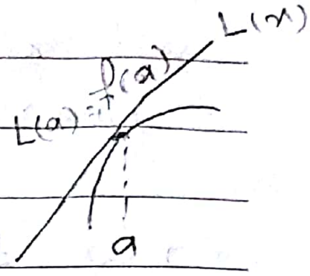
If limit result is not real number then

oblique asymptote does not exist.

If $m = 0$ and if n exists then $y = n$ is

the horizontal asymptote.

2) To find value of $f(x)$ near $x=a$ we may use its tangent line $L(x)$ (at $x=a$). This is called Linearization (Doğrusal yaklaşım). $f(x)$ is continuous and differentiable at $x=a$



Near $x=a$ $f(x)$ behaves similar with $L(x)$.

$$\boxed{f(x) \approx L(x)}$$

We need to determine equation of tangent line $L(x)$.

* Reminder:

$L(x)$ is passing through the point $(a, f(a))$.

Slope of $L(x)$ is $f'(a)$

$$L(x) - f(a) = f'(a)(x-a)$$

$$\Rightarrow L(x) = \underbrace{f(a)}_{y_0} + \underbrace{f'(a)}_{\text{slope}}(x-\underbrace{a}_{x_0})$$

2) If $f(x) = \sqrt{1+x}$ find Linearization of $f(x)$ near $x=0$

$f(x)$ is continuous and differentiabe at $x=0$

$f'(x) = \frac{1}{2\sqrt{1+x}}$ Then:

$f(0) = \sqrt{1} = 1$

$f'(0) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

$L(x) = 1 + \frac{1}{2}(x-0)$

$\boxed{L(x) = 1 + \frac{x}{2}}$

At $x=0$ $f(0) = L(0)$

Near $x=0$ $f(x) \approx L(x)$

Let's choose any point close to zero at $x=0.2$

$f(0.2) = \sqrt{1+0.2} = 1.095$ (TRUE value)

$L(0.2) = 1 + \frac{0.2}{2} = 1.1$

$\Rightarrow$ Error is less than $\frac{1}{100}$

If we choose any other point closer than the first one the percentage of the error will decrease / $x=0.05$

$f(0.05) = \sqrt{1.05} = 1.024\ldots$

$L(0.05) = 1 + \frac{0.05}{2} = 1.025$

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3) By the defition of derivative, we know that $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$

if A is the area of a circle with the radius r then we evaluate the change of area when r is increased from 10 cm to 10.1 compare the differance between ΔA and dA

$A = \pi r^2$

$A_1 = \pi(10)^2 = 100\pi$

$A_2 = 102.1\pi$

$\Delta A = A_2 - A_1 = 2.1\pi$ (True value)

$A = \pi r^2$ we differentiate both sides with respect to r

$\frac{dA}{dr} = 2\pi r \Rightarrow dA = 2\pi \underset{10}{r}\, \underset{0.1}{dr} = 2\pi$

$\Rightarrow dA \approx \Delta A$

4) $1-2x = \sin x$

Show that the given equation has <u>only</u> one root.

<u>1st step:</u> We have to show that the root exists

$$1-2x-\sin x = 0$$

We first rearrange the given equ.
But for some types of questions when we rearrange Domain set may change.
Therefor determine the domain set of f at first.

$\Rightarrow$ Domain set: $(-\infty, \infty)$

We choose any two points on the domain set Let's use 0 and π

$$\left.\begin{aligned} f(0) &= 1 > 0 \\ f(\pi) &= 1-2\pi < 0 \end{aligned}\right\} \text{ Then by IVT } \exists c \in (0,\pi) \quad 1-2\pi < f(c) = \underset{N}{\underset{\downarrow}{0}} < f(0)$$

2nd step: Now we will show that the root is <u>unique</u>
If we assume that there are one more root (a)
By MVT $\exists d \in (a,c)$ for which $f'(d) = \dfrac{f(c)-f(a)}{c-a}$
& $f(x)$ is differentiable on $(-\infty, \infty)$

$$\Rightarrow 0-2-\cos d = \frac{0-0}{c-a} = 0 \Rightarrow \cos d = -2$$

Here we face with a contradiction Because if our assumption is true by MVT d must exist

By MVT d must exist (at least once)

But we know that $-1 \le \cos d \le 1$

Then our assumption fails.

Calculus 14.11.2018

* Indefinite Integrals:

① Some basic formulas:

1) $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$ integration constant

$n \neq -1$, n real number

2) $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$

3) $\int e^{mx+n} \, dx = \frac{e^{mx+n}}{m} + C$

$m \neq 0$ m and n are real numbers

! $\int e^{x^2} \, dx \neq \frac{e^{x^2}}{2x} + C$

4) $\int \sin(mx+n) \, dx = -\frac{\cos(mx+n)}{m} + C$

5) $\int \cos(mx+n) \, dx = \frac{\sin(mx+n)}{m} + C$

② Substitution: (Değişken dönüşümü)

! Depending on the given integrand, different solution ways may be available. But for every integral (if it can be solved) the RESULT IS UNIQUE.

Example. Solve $\int 2x e^{x^2} dx$

By using substitution:

1st way: $x^2 = u$ We differentiate both sides with respect to x then we have:

$2x = \frac{du}{dx} \Rightarrow 2x\,dx = du$

$\Rightarrow \int 2x e^{x^2} dx = \int e^u du = e^u + C = e^{x^2} + C$

2nd way: We assume that $e^{x^2} = u$

$\Rightarrow 2x e^{x^2} dx = du$

$\Rightarrow \int 2x e^{x^2} dx = \int du = u + C$

$= e^{x^2} + C$

H.W: $\int \sin 2x\, dx$ ① Use basic formula $u = \sin x$, $u = \cos x$

② Use half angle formula

! Remember that the result must be UNIQUE

Compare your results for the given three different ways

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③ Integration by parts: $\left(uv - \int v\,du\right)$

Example: $\int \frac{x-1}{x^2} e^x \, dx = I$

1st way: without using integration by parts.

$I = \int \frac{e^x x - e^x}{x^2} dx$ (By quotient rule of derivative)

$\Rightarrow I = \int \frac{d}{dx}\left(\frac{e^x}{x} + c\right) dx = \frac{e^x}{x} + c$

2nd way: $I = \underbrace{\int \frac{x e^x}{x^2} dx}_{(1)} + \underbrace{\int \frac{e^x}{x^2} dx}_{(2)}$

But here we face with a problem (1) can not be solved by using elementary ways.
We apply integration by parts for (2)
We assume that:

$$\left.\begin{aligned} e^x = u &\Rightarrow e^x dx = du \\ \frac{1}{x^2} dx = dv &\Rightarrow -\frac{1}{x} = v \end{aligned}\right\} (2) = -\frac{e^x}{x} + \int \frac{e^x}{x} dx$$

Then (1) _ (2):

$$I = \cancel{\int \frac{e^x}{x} dx} + \frac{e^x}{x} - \cancel{\int \frac{e^x}{x} dx}$$

$$= \frac{e^x}{x} + C$$

3rd way: $\underbrace{\int \frac{e^x}{x}\,dx}_{(1)} - \underbrace{\int \frac{e^x}{x^2}\,dx}_{(2)}$

Eventhough (1) can't be solved we apply integration by parts for (1)

$\frac{1}{x} = u \qquad e^x dx = dv$

$\downarrow \qquad\qquad \downarrow$

$-\frac{1}{x^2}\,dx = du \qquad e^x = v$

Then we have:

$\underbrace{\frac{1}{x} e^x - \int - \cancel{\frac{e^x}{x^2}}\,dx}_{\text{Result of (1)}} - \int \cancel{\frac{e^x}{x^2}}\,dx$

$= \frac{e^x}{x} + C$

H.W: $\int \sec^3 x\,dx$

Calculus 21.11.2018

⊛ *Example: Evaluate $\int_0 3(x-1)^2\left(\underbrace{\int_0^x \sqrt{1+(t-1)^4}\,dt}_{\circledast}\right)dx$

$\because$ Solution of $\circledast$ is so complex

1st step: We apply integration by parts:

We assume that $\circledast = u$

and $3(x-1)^2\,dx = dv$

2nd step:

For $\circledast = u$ we differentiate both sides with respect to x (We use Leibniz's Rule):

Then we have:

$$\sqrt{1+(x-1)^4}\,(1) - 0 = \frac{du}{dx} \Rightarrow \sqrt{1+(x-1)^4}\,dx = du$$

For $3(x-1)^2\,dx = dv$ we integrate both sides:

$$(x-1)^3 = v$$

3rd step: We apply the formula of integration by parts:

$$\left[\underbrace{\int_0^x \sqrt{1+(t-1)^4}\,dt \cdot (x-1)^3}_{A} - \underbrace{\int (x-1)^3\sqrt{1+(x-1)^4}\,dx}\right]_{x=0}^{x=1}$$

Let's denote the result by B

4^{th} step: $[A(1) - B(1)] - [A(0) - B(0)]$

$$[0 - B(1)] - [0 - B(0)] \quad \text{(**)}$$

Reminder
If $f(x)$ is cont. at $x = a$ then $\int_a^a f(x)\,dx = 0$

5^{th} step: We solve second integral and find B the we evaluate B(1) and B(0) and replace those values in (**) in order to find the result of given question

$$\int (x-1)^3 \sqrt{1+(x-1)^4}\,dx$$

$$1 + (x-1)^4 = z \Rightarrow 4(x-1)^3\,dx = dz$$

$$= \frac{1}{4}\int \sqrt{z}\,dz = \frac{1}{4}\,\frac{2}{3}\,z^{\frac{3}{2}}$$

$$= \frac{1}{6}\left(1 + (x-1)^4\right)^{\frac{3}{2}} = B$$

$$B(1) = \frac{1}{6}, \quad B(0) = \frac{\sqrt{8}}{6}$$

Final step:

$$\text{(**)} = \left(0 - \frac{1}{6}\right) - \left(0 - \frac{\sqrt{8}}{6}\right) = \frac{\sqrt{8} - 1}{6}$$

⊛ Trigonometric Substitutions:

① $x = a\sin\theta$
② $x = a\tan\theta$
③ $x = a\sec\theta$
④ $z = \tan\left(\frac{x}{2}\right)$

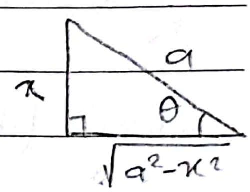
a Real number
$a \neq 0$

By using ①, ② and ③ we want to simplify (or eliminate) the term square root in the given integrand.

① If the integrand contains the term $\sqrt{a^2-x^2}$ then we use the substitution

⊛ $x = a\sin\theta \Rightarrow \frac{x}{a} = \sin\theta$

θ : Acute angle (Dar ac(1))



By using ⊛ we will write the integrand and also dx in terms of θ.
We will see the details in the given example.

*Example: Solve $\int \frac{dx}{\sqrt{1-x^2}}$

Reminder
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

$(a=1)$ By the given formula

$= \sin^{-1}\left(\frac{x}{1}\right) + C$

2nd way: $a=1 \Rightarrow x = 1\sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta$

① ②

We replace both ① and ② in the given question:

Reminder
$\sin^2\theta + \cos^2\theta = 1$

$$\int \frac{\cos\theta \, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\cos\theta \, d\theta}{\sqrt{\sin^2\theta + \cos^2\theta - \sin^2\theta}} = \int \frac{\cos\theta}{|\cos\theta|} \, d\theta$$

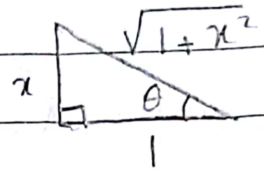
θ is acute angle $\Rightarrow = \int \frac{\cos\theta}{\cos\theta} \, d\theta = \int d\theta$

$= \theta + C$ $\quad\quad \sin\theta = x \Rightarrow \theta = \sin^{-1}x + C$

$= \sin^{-1}x + C$

(2) If the integrand contains the term $\sqrt{a^2+x^2}$ then we use the substitution $x = a\tan\theta$ and follow similar steps with previous substitution:

Example: Solve $\int \sqrt{1+x^2}\, dx$



$a=1 \Rightarrow x = \tan\theta$

$0 < \theta < \frac{\pi}{2}$

$\sec\theta > 0$

$dx = \sec^2\theta\, d\theta$

Then we have:

Reminder

$1+\tan^2\theta = \sec^2\theta$

$\int \sqrt{1+\tan^2\theta}\ \sec^2\theta\, d\theta$

$= \int \sqrt{\sec^2\theta}\ \sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta$

$\sec\theta = u \qquad \sec^2\theta\, d\theta = dv$

$\downarrow \qquad \downarrow$

$\sec\theta\tan\theta\, d\theta = du \qquad \tan\theta = v$

By integration by parts $\quad uv - \int v\, du$

$= \sec\theta\tan\theta - \int \tan^2\theta\,\sec\theta\, d\theta$

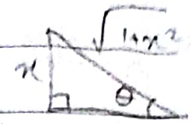
$\tan^2\theta \to \sec^2\theta - 1$

$= \sec\theta\tan\theta - \left[\int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta\right]$

$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$

Find step: Do not forget to write the result in terms of x

$x = \tan\theta$



$$\Rightarrow \int \sec^3\theta \, d\theta = \frac{\left(\sqrt{1+x^2}\right)x + \ln|\sec\theta + \tan\theta|}{2} + C$$

* Partial Fractions (Basit Kesirlere ayırma)

$$\int \frac{f(x)}{g(x)}\,dx$$

If both of $f(x)$ and $g(x)$ are polynomial functions and if degree of $f(x)$ is less than degree of $g(x)$ we may use partial fractions.

$\boxed{1}$ When we factor out $g(x)$ if roots are real numbers $r_1, r_2, \dots, r_k$ and if $r_1 \neq r_2 \neq \dots \neq r_k$ as given in the example.

*Example:

$$\int \frac{x}{\underbrace{x^2-7x+10}}\,dx \qquad = \frac{A}{x-2} + \frac{B}{x-5}$$

$$\Rightarrow x = A(x-5) + B(x-2) \Rightarrow \begin{cases} 0 = 5A - 2B \\ 1 = A + B \end{cases}$$

$$= \int \frac{-\frac{2}{3}}{x-2} + \frac{\frac{5}{3}}{x-5}\,dx = -\frac{2}{3}\ln|x-2| + \frac{5}{3}\ln|x-5| + C$$

$\boxed{2}$ If roots of $g(x)$ are as given $r_1 = r_2 = \cdots = r_k$

* Example: $\int \frac{x}{\underbrace{(x-5)^3}} dx$

$$= \frac{A}{x-5} + \frac{B}{(x-5)^2} + \frac{C}{(x-5)^3}$$

$\boxed{3}$ If some roots of $g(x)$ are complex numbers:

$$\int \frac{x+1}{\underbrace{x(x^2+1)}} dx$$

$$= \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

We reduce degree of denominator by 1 and determine numerator

If denominator is x^4+1 then numerator will be $Bx^3 + Cx^2 + Dx + E$

* Example: Solve: $\int \frac{dx}{x(x^4+1)}$ (*)

At first it seems as if we will use partial fractions directly.

$$= \int \frac{x^3}{x^4(x^4+1)}$$

$$\int \frac{x^3}{x^4(x^4+1)}\,dx$$

$x^4+1=u \Rightarrow x^3\,dx=du$

$$=\int \frac{du}{u(u-1)}$$

$\boxed{4}$ Example:

$$\int \frac{x+1}{x(x^2+1)^2}\,dx$$

$x_1=0 \quad x_{2,3}=\mp i \quad x_{4,5}=\mp i$

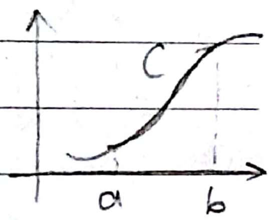
$$\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

* Length of Plane Curve

If $f(x)$ is a smooth curve,

How can we find length of C ?

$= L$



We may use 3 formulas given below:

① $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad a \le x \le b$

② $L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad c \le y \le d$

③ $L = \int_m^n \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad m \le t \le n$

*Example: Show that circumference of a circle with radius r is $2\pi r$

We use polar coord.

$x = r\cos\theta \qquad y = r\sin\theta$

$x^2 + y^2 = r^2$

$\left.\begin{array}{l} \frac{dx}{d\theta} = -r\sin\theta \\ \frac{dy}{d\theta} = r\cos\theta \end{array}\right.$

$$L = \int_0^{2\pi} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(\sin^2\theta + \cos^2\theta)}\, d\theta = \int_0^{2\pi} r\, d\theta$$

$$= (r\theta)\Big|_0^{2\pi} = \boxed{2\pi r}$$

3.48 p: 322
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Calculus 28.11.2018

* Exercises

1) Find length of curve $y = x^{\frac{2}{3}}$, over $0 \le x \le 8$

! Since y is given in terms of x and interval of x is given; It seems as if, we will use the formula

(*) $\int_0^8 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dx$ to determine length of $y = x^{\frac{2}{3}}$ over $0 \le x \le 8$

!! Do not forget to check that $\frac{dy}{dx}$ is defined (or not) for every points over the interval we work $\rightarrow 0 \le x \le 8$

We differentiate both sides with respect to $x \Rightarrow y = x^{\frac{2}{3}} \Rightarrow \frac{dy}{dx} = \frac{2}{3\sqrt[3]{x}}$

$\Rightarrow$ For $x = 0$ $\frac{dy}{dx}$ is undefined

$\Rightarrow$ We can not apply (*) to find length of $y = x^{\frac{2}{3}}$ on $0 \le x \le 8$

Then we have to follow another way
We rearrange given equation and write x in terms of y and also we determine interval of y.
Then we have

$y^{\frac{3}{2}} = x$ For $x=0$ $y=0$
$\Downarrow$ For $x=8$ $y=4$
$0 \le y \le 4$

$$L = \int_c^d \sqrt{1 + \frac{dx}{dy}}\, dy \quad c \le y \le d$$

$$\frac{3}{2} y^{\frac{1}{2}} = \frac{dx}{dy}$$

It is continuous for every points on $0 \le y \le 4$

$$\Rightarrow L = \int_0^4 \sqrt{1 + \frac{9}{4}y}\, dy$$ By using substitution

$$1 + \frac{9}{4}y = u \Rightarrow \frac{9}{4}dy = du \Rightarrow dy = \frac{4}{9}du$$

for $\begin{cases} y=0 & u=1 \\ y=4 & u=10 \end{cases}$

$$\Rightarrow L = \frac{4}{9}\int_{u=1}^{u=10} \sqrt{u}\, du = \frac{4}{9}\left(\frac{2}{3}u^{\frac{3}{2}}\right)\Bigg|_1^{10}$$

$$= \frac{8}{27}\left[10^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$$

There is one more step!

By using min and max values of x we found min and max values of y respectively.

$y = x^{\frac{2}{3}} \Rightarrow y' = \frac{2}{3} x^{-\frac{1}{3}} > 0$ over $(0, 8]$

$\Rightarrow$ y is an increasing function over $(0, 8]$

2) Solve $\int x^3 \sqrt{1-x^2}\, dx$

! For the given integral we may follow different steps

1) Substitution 2) Trigonometric substitution
3) Integration by parts.

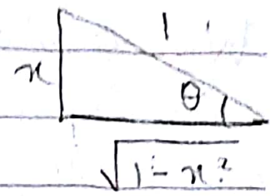
$\int x\, x^2 \sqrt{1-x^2}\, dx$; We assume that $1-x^2 = u$
$\Rightarrow -2x\,dx = du \Rightarrow x\,dx = -\frac{1}{2} du$

$\Rightarrow -\frac{1}{2}\int (1-u)\sqrt{u}\, du = -\frac{1}{2}\int u^{\frac{1}{2}} - u^{\frac{3}{2}}\, du$

$= -\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right] + C$

$= -\frac{1}{2}\left[\frac{2}{3}(1-x^2)^{\frac{3}{2}} - \frac{2}{5}(1-x^2)^{\frac{5}{2}}\right] + C$

By using (2) then integrand contains the term $\sqrt{1^2 - x^2}$ $\Rightarrow$ $x = 1.\sin\theta$



$dx = \cos\theta \, d\theta$

Then we have

$$\int \sin^3\theta \sqrt{1-\sin^2\theta} \, \cos\theta \, d\theta$$

$$= \int \sin^3\theta \sqrt{\cos^2\theta} \, \cos\theta \, d\theta$$

θ is an acute angle $\Rightarrow \cos\theta > 0$

$$\Rightarrow \int \sin^3\theta \cos^2\theta \, d\theta$$

Homework: Search solution ways of $\int \cos^n\theta \sin^m\theta \, d\theta$

Ali Nesin

3) Solve $\int \sqrt{\frac{x}{1-x^3}} \, dx$

$= \int \frac{\sqrt{x}}{\sqrt{1-x^3}} \, dx$ We assume that $u = x^{\frac{3}{2}}$

$\Rightarrow du = \frac{3}{2} x^{\frac{1}{2}} dx$

$= \frac{2}{3} \int \frac{du}{\sqrt{1-u^2}} du$ Final integral can be solved by using basic formula of integration

Reminde $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$

$$= \frac{2}{3} \sin^{-1} u + c$$

$$= \frac{2}{3} \sin^{-1}(x^{\frac{3}{2}}) + c$$

4) Solve $\int \sqrt{\frac{4-x}{x}}\, dx$

We assume that $x = u^2 \Rightarrow dx = 2u\,du$

$$\Rightarrow \int \sqrt{\frac{4-u^2}{u^2}}\; 2u\,du = 2\int \sqrt{4-u^2}\, du$$

We use $u = 2\sin\theta \Rightarrow du = 2\cos\theta\, d\theta$

Then we have:

$$2\int \sqrt{4-4\sin^2\theta}\; 2\cos\theta\, d\theta$$

$$= 4\int 2\sqrt{\cos^2\theta}\;\cos\theta\, d\theta = 8\int \cos^2\theta\, d\theta$$

$$= 8\int \frac{1+\cos 2\theta}{2}\, d\theta$$

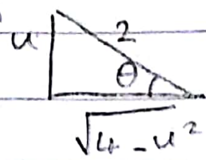
Reminder

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \frac{8}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

→ By half angle formula

$$= \frac{8}{2}\left(\theta - \sin\theta\cos\theta\right) + C$$



$$= \frac{8}{2}\left(\sin^{-1}\left(\frac{u}{2}\right) - \frac{u}{2}\,\frac{\sqrt{4-u^2}}{2}\right) + C$$

$$= \frac{8}{2}\left(\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) - \frac{\sqrt{x}}{2}\,\frac{\sqrt{4-x}}{2}\right) + C$$

5) Solve $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

We assume that $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\, d\theta$

$\Rightarrow \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{3\sqrt{1-\sin^2\theta}}$

$= \frac{9}{2}\int 1 - \cos 2\theta\, d\theta = \frac{9}{2}\left(\theta - \sin 2\theta\right)$

$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}\right) + C$

Defenite Integrals

H.W search Riemann Sums

AREA

Calculus 5.12.2018

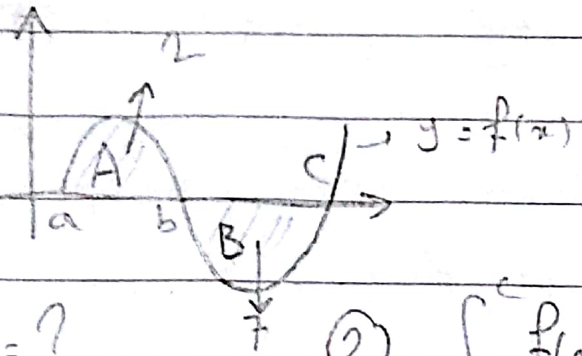
* Definite Integrals

$\int_a^b f(x)\,dx$ is called a definite integral if and only if both a and b are real numbers and $f(x)$ is continuous on $[a,b]$

① Result of defenite integral is a Real number

② If $f(x) \geq 0$ on $[a,b]$ then $\int_a^b f(x)\,dx \geq 0$

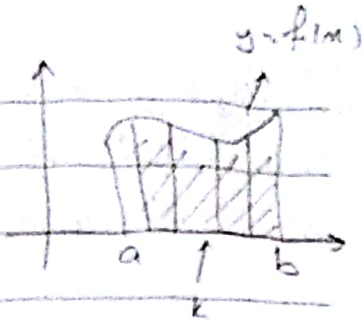
Example:



$f(x)$ is continuous on $[a,c]$

① $\int_a^b f(x)\,dx = ?$ ② $\int_b^c f(x)\,dx = ?$

③ $\int_a^c f(x)\,dx = ?$ ④ Total area $= ?$



To find shaded area:
By Remann sums we use vertical rectangles as given above.

k^{th} rectangle.
Area of k^{th} rectangle $(f(x_k - 0)\Delta x)$. Let's assume that we have n rectangles. Then total area of those rectangles:

$$\sum_{k=1}^{n} f(x_k).\Delta x$$

If $n \to \infty$ by using definite integral we can find shaded area: $\int_a^b f(x)\,dx$ = Shaded area

<u>Caution</u>: As given example (sometimes) definite integral will not give us Area (We are going to investigate in details).

$\Rightarrow$ ① $\int_a^b f(x)\,dx = 2$ ② $\int_b^c f(x)\,dx = -7$

③ $\int_a^c f(x)\,dx = 2 + (-7) = -5$

④ Total area $= 2 + 7 = 9$

<u>Caution</u>: We have different types of refference rectangles $\Rightarrow$ Result of ③ does not give us total area

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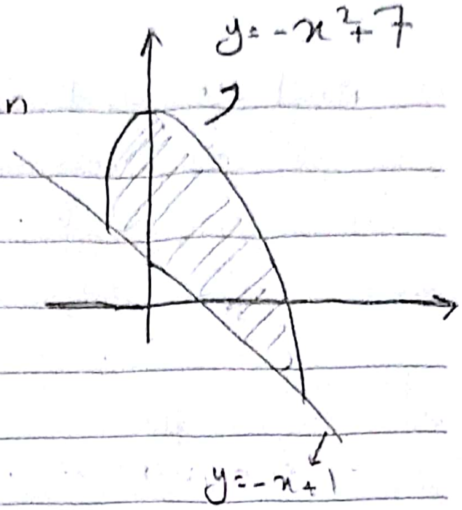
* Example 2: Find area of given region bounded by the parabola and given line.

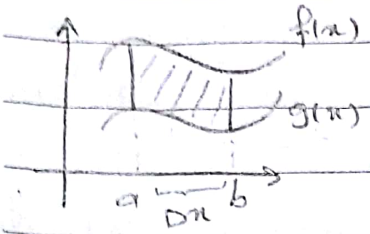


1st step: $x^2+7=-x+1$

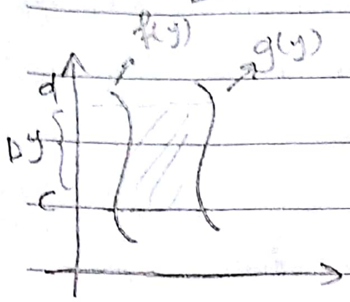
$\Rightarrow x=3,\ x=-2$

2nd step: Shaded area $=\int_{-2}^{3}\left[(-x^2+7)-(-x+1)\right]dx$

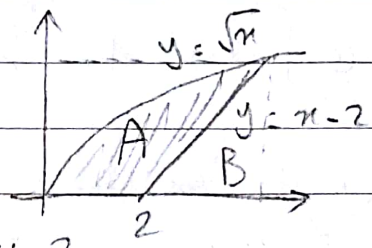
As a conclusion:



shaded area $=\int_a^b [f(x)-g(x)]\,dx$



Shaded area $=\int_{y=c}^{d} (g(y)-f(y))\,dy$



Example: Find shaded area.

1st way: $\sqrt{x}=x-2 \Rightarrow x=4 \Rightarrow y=2$

$\Rightarrow$ Shaded area $=\int_0^2 (y+2)-y^2\,dy = \left.\frac{y^2}{2}+2y-\frac{y^3}{3}\right|_0^2$

$=\boxed{\frac{10}{3}}$

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2nd way: We first evaluate A+B

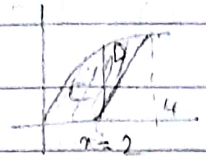
$$A+B = \int_0^4 \sqrt{x}\, dx = \left(\frac{2}{3} x^{\frac{3}{2}}\right)\Big|_0^4$$

$$= \frac{16}{3} - 0 = \frac{16}{3}$$

B is the area of right tringle $B = \frac{(4-2)\cdot 2}{2} = 2$

$\Rightarrow$ Shaded area $= \frac{10}{3}$.

3rd way: for the given example, if we choose integration with respect to x we have to separate A into two parts because we have two types of ref tringles.



$$\left.\begin{aligned} &\Rightarrow C = \int_0^2 \sqrt{x}\, dx = \left(\frac{2}{3} x^{\frac{3}{2}}\right)\Big|_0^2 \\ &D = \int_2^4 \sqrt{x} - (x-2)\, dx \end{aligned}\right\} \frac{10}{3}$$

If we use substitution for a definite integral we may follow two different ways which both meet at the same point as the result of given definite integral.

Example: $\int_2^5 2x e^{x^2}\, dx = ?$ we assume that
$x^2 = u \Rightarrow 2x\,dx = du$

Then $\int_4^{25} e^u\, du = e^{25} - e^4$

$\int e^u\, du = e^u = e^{x^2}\Big|_2^5 = e^{25} - e^4$

If $f(x) = f(-x) \Rightarrow f(x)$ is an even function ①

$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ ② a: real number

③ symmetrical interval $[-a, a]$

④ $f(x)$ is continuous o $[-a, a]$

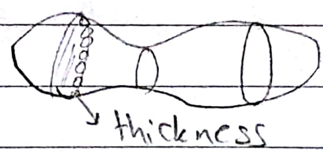
! If $f(x) = x^3 + 4x + 3$ $f(2) = 3$ $f(-2) = 3$

$\Rightarrow f(2) = f(-2)$

$\Rightarrow f(x)$ is an even function FALSE

② ③ ④ and $f(-x) = -f(x) \Rightarrow f(x)$ is an odd function
then $\int_{-a}^{a} f(x)\,dx = 0$

* Volume: For the given Solid, we think as it is the sum of all its slices (slabs)
For k^{th} slab volume of k^{th} slab is
(surface area) X (thickness)

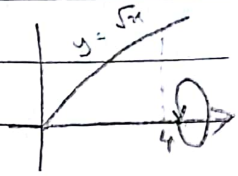


Then volume of solid is $V_{solid} = \int_{x=a \text{ or } y=c}^{x=b \text{ or } y=d} \text{surface area } dx \text{ or } dy$

* How can we find volume of a solid which is generated by rotation about a line (vertical or horizontal)

1st Method: Disk Method: We think as if the sum of infinitely many disks (there is no hole in the solid) Details are given in the ex.

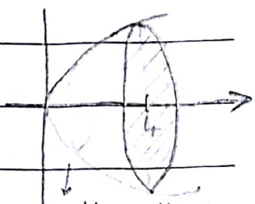
* Example: Find volume of generated solid.



$\Rightarrow V_{solid} = \int_0^4 \underbrace{\text{surface area}}_{\pi r^2} dx$

$= \pi \int_0^4 x \, dx = \pi \left. \frac{x^2}{2} \right|_0^4$

$= \boxed{8\pi}$



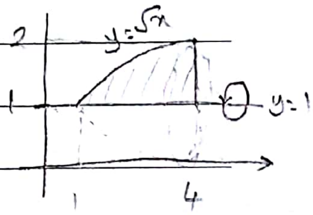
Δx is the thickness of indicated slab.

Conclusion: Disk method says that; if shaded area is rotated about horizontal line then we use integration with respect to x

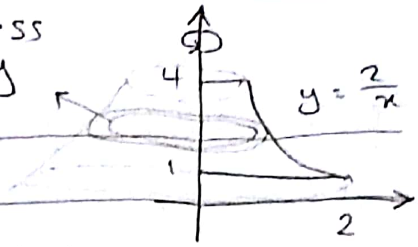
Example: No hole $\rightarrow$ Disk method

$V_{solid} = \pi \int_a^b r^2(x) dx \quad a \le x \le b$

$= \pi \int_1^4 (\sqrt{x} - 1)^2 dx$ ......



thickness is Dy



Example: $V_{solid} = ?$

By disk method; if given region is rotated about vertical line

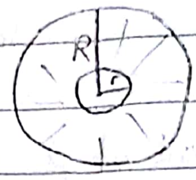
$$V_{solid} = \pi \int_c^d r^2(y)\, dy \qquad c \le y \le d$$

$$r(y) = \frac{2}{y} - 0 \Rightarrow V_{solid} = \pi \int_1^4 \left(\frac{2}{y}\right)^2 dy \; ----$$

Calculus 12.12.2018

* Washer Method

If there is a hole in the generated solid we may apply Washer Method.



(*) We integrate shaded area to find volume of generated solid

R: outer radius
r: inner radius

Shaded area $= \pi R^2 - \pi r^2$
$= \pi (R^2 - r^2)$

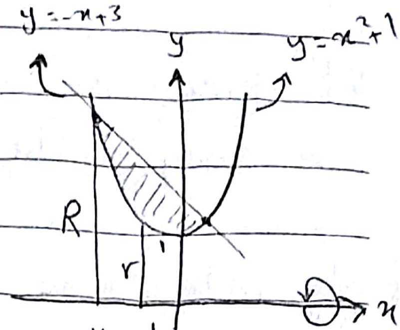
Similar with disk method (1) if we rotate given region around a horizontal line then we use integration with respect to x.

(2) If we rotate given region around a vertical line then we use integration with respect to y

By (*), (1) and (2) (1) $V_{solid} = \pi \int_a^b \left[R^2(x) - r^2(x) \right] dx \quad a \leq x \leq b$

(2) $V_{solid} = \pi \int_c^d \left[R^2(y) - r^2(y) \right] dy \quad c \leq y \leq d$

Example: The region is bounded by the parabola $y=x^2+1$ and the $y=-x+3$. The region is revolved about x axis. Find volume of generated solid.



1st step: We draw the graph of given region

2nd step: By the graph we see that there is a hole in the generated solid
$\Rightarrow$ We use Washer Method

Since rotation axis is horizontal we use ① st formula.

$$V_{solid}=\pi\int_a^b R^2(x)-r^2(x)\,dx$$

We determine $a, b, R(x), r(x)$

3rd step: $x^2+1=-x+3 \Rightarrow x^2+x-2=0$

$\Rightarrow x_1=-2 \quad x_2=1$
$\quad\quad\downarrow a \quad\quad \downarrow b$

$R(x)=-x+3-0 \qquad r(x)=x^2+1-0$

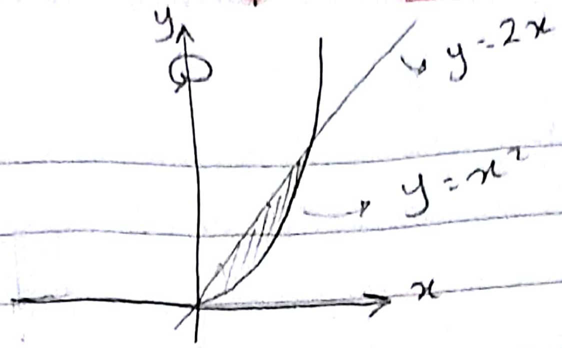
Final step: $V_{solid}=\pi\int_{-2}^{1}(-x+3)^2-(x^2+1)^2\,dx$

$=\pi\int_{-2}^{1} x^2-6x+9-x^4-2x^2+1\,dx$

$=\pi\int_{-2}^{1} -x^4-x^2-6x+10\,dx = \pi\left[-\frac{x^5}{5}-\frac{x^3}{3}-3x^2+10x\right]\Bigg|_{-2}^{1}$

$=\pi\left[-\frac{1}{5}-\frac{1}{3}-3+10-\left(\frac{32}{5}+\frac{8}{3}-12-20\right)\right]=$?

* Example:



By washer method

$$V_{solid} = \pi \int_{c=0}^{d=4} R^2(y) - r^2(y)\, dy$$

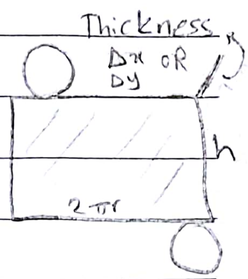
$$R(y) = \sqrt{y} - 0 \qquad r(y) = \frac{y}{2} - 0$$

$$V_{solid} = \pi \int_0^4 y - \frac{y^2}{4}\, dy = \pi \left(\frac{y^2}{2} - \frac{y^3}{12} \right)\Big|_0^4$$

$$= \frac{8}{3}\pi$$

* Shell Method: (kabuk yöntemi)

The solid is assumed as the sum of infinitely many cylindrical shells.
Similar with previous methods when we integrate surface area it gives us volume of generated solid.

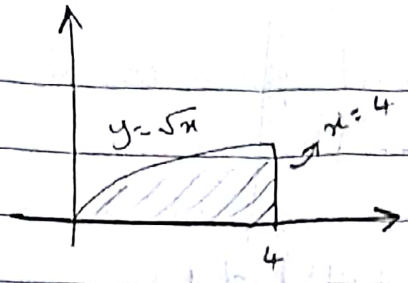


Surface area = $2\pi r h$

$$V_{solid} = \int 2\pi r h \, dx \text{ or } dy$$

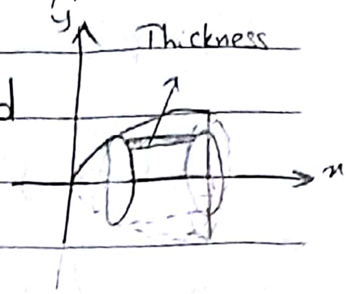
Details will be given in the examples.

Example:



Find V_{solid} by using shell method.

After rotation



By using shell method; generated solid is the sum of infinitely many horizontal cylindirical shells.

Shell method says that if given region is rotated about horizontal line, we the given formula:

$$V_{solid} = 2\pi \int_c^d r(y)\, h(y)\, dy$$

$c \le y \le d$

Min and Max value of y over the given region.

How do we determine radius and height of the shell?

We indicate any point (P) we choose on the boundry of given region.

Radius of the shell is the distance between P and rotation axis.

For the given example distance between P and and rotation axis is the ordinate of P

$r(y) = y$

To determine height of the shell from P, we draw a line segment (passing through given region) which is parallel to rotation axis.

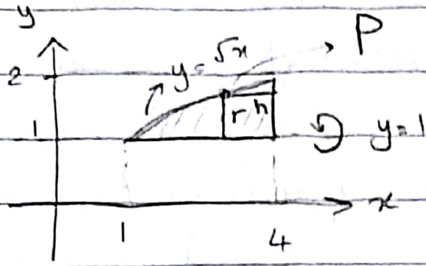
Length of line segment gives us height of the shell $h(y) = 4 - y^2$

$$V_{solid} = 2\pi \int_0^2 y \cdot (4 - y^2)\, dy$$

$$= 2\pi \left(2y^2 - \frac{y^4}{4}\right)\Big|_0^2$$

$$\Rightarrow V_{solid} = 8\pi$$

Example:



Shaded area is rotated about the line $y=1$. Find V_{solid} by using Shell Method.

$$V_{solid} = 2\pi \int_{c=1}^{d=2} r(y)\, h(y)\, dy \qquad c \le y \le d$$

$h(y) = 4 - y^2 \qquad r(y) = y - 1$

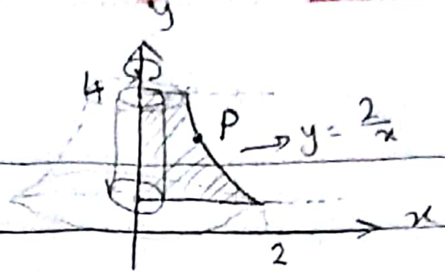
$$V_{solid} = 2\pi \int_1^2 (4 - y^2)(y - 1)\, dy$$

$$= 2\pi \int_1^2 4y - 4 - y^3 + y^2\, dy$$

$$= 2\pi \left(2y^2 - 4y - \frac{y^4}{4} + \frac{y^3}{3}\right)\Big|_1^2$$

$$= 2\pi \left(8 - 8 - 4 + \frac{8}{3} - 2 + 4 + \frac{1}{4} - \frac{1}{3}\right) = ?$$

Example:



By shell method:

$$V_{solid} = 2\pi \int_a^b r(x)h(x)\,dx$$

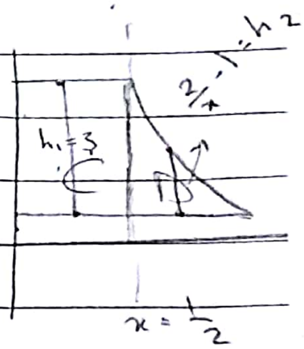
$r(x) = x \qquad h(x) = \frac{2}{x} - 1$

$$\Rightarrow V_{solid} = 2\pi \int_0^2 2 - x\,dx = 2\pi\left(2x - \frac{x^2}{2}\right)\Big|_0^2$$

$$= 4\pi$$

By Disk Method we have found that $V_{solid} = 3\pi$ ‼ The results must be equal. There is a mistake in the solution!

Since we have two types of height we have to separate given region into two parts. Then we have:



$V_{solid} = V_C + V_D$

$$V_C = \pi r^2 h = \pi \cdot \frac{1}{4} \cdot 3 = \frac{3\pi}{4}$$

By shell method $V_D = 2\pi \int_a^b r(x)h(x)\,dx$

$r(x) = x \qquad h(x) = \frac{2}{x} - 1$

$$V_D = 2\pi \int_{\frac{1}{2}}^{2} 2 - x\,dx = 2\pi\left(2x - \frac{x^2}{2}\right)\Big|_{\frac{1}{2}}^{2} = \frac{9\pi}{4}$$

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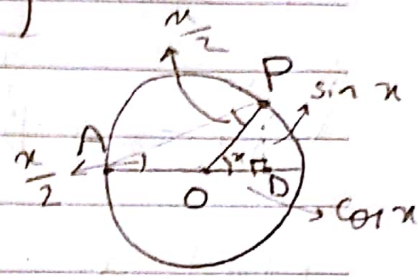
* The substitution $z = \tan\left(\frac{x}{2}\right)$



$\hat{AOP}$ is an isoscales tringle
$\hat{ODP}$ is a right tringle

By $\hat{APD}$ $\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1+\cos x}$
↓
right tringle

We use the given substitution for some types of integrals which contain $\sin x$ or $\cos x$. For those types of integrals using the given substitution may be useful.

We express all the terms $\sin x$, $\cos x$ and dx in terms of z.
By trigonometric identities:

$$\cos x = 2\cos^2\frac{x}{2} - 1 = \frac{2}{\sec^2\left(\frac{x}{2}\right)} - 1$$

$$\cos x = \frac{2}{z^2+1} - 1 = \frac{1-z^2}{1+z^2} \quad \text{---- } (1)$$

Reminder
$\tan^2\alpha + 1 = \sec^2\alpha$

By half angle formula

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cdot\cos\left(\frac{x}{2}\right)\cdot\frac{\cos\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}$$

$$= 2\tan\left(\frac{x}{2}\right)\cdot\cos^2\left(\frac{x}{2}\right)$$

$$= 2\,\frac{\tan\left(\frac{x}{2}\right)}{\sec^2\left(\frac{x}{2}\right)}$$

$$\Rightarrow \sin x = \frac{2z}{z^2+1} \quad \ldots\ldots (2)$$

Now we express dx in terms of z

$$z = \tan\left(\frac{x}{2}\right) \Rightarrow \tan^{-1} z = \frac{x}{2}$$

$$\Rightarrow 2\tan^{-1} z = x$$

We differentiate both sides with respect to z

Then we have:

$$\frac{2}{1+z^2}\,dz = dx \quad \ldots\ldots (3)$$

Example: Solve $\int \frac{dx}{2+\sin x}$

Sol: By using $z = \tan\left(\frac{x}{2}\right)$ we express the integrand and also dx in terms of z.

Then we have:

$$\int \frac{\frac{2}{1+z^2}}{2+\frac{2z}{1+z^2}}\,dz = \int \frac{\frac{\cancel{2}}{\cancel{1+z^2}}}{\cancel{2}\left(\frac{1+z^2+z}{\cancel{1+z^2}}\right)}\,dz = \int \frac{dz}{1+z^2+z}$$

$$\int \frac{dz}{1+z^2+z} = \int \frac{dz}{(z+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

Reminder

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

* **Exercises:**

1) Solve $\int \frac{dx}{1+e^x}$

$$= \int \frac{1+e^x-e^x}{1+e^x} = \int \frac{1+e^x}{1+e^x}\,dx - \int \frac{e^x}{1+e^x}\,dx$$

$$= x - \ln|1+e^x| + C$$

! Alternative solution ways may be available

2) $\int_{-1}^{1} \frac{x}{e^x+e^{-x}}\,dx$

$f(x) = \frac{x}{e^x+e^{-x}}$ $f(-x) = \frac{-x}{e^{-x}+e^x}$

$\Rightarrow f(x) = -f(-x) \Rightarrow f(x)$ odd func.

* $f(x)$ is continuous on $[-1,1]$
* Bounds of given integral are real numbers
* $[-1,1]$ Symmetric

$$\Rightarrow \int_{-1}^{1} f(x)\,dx = \boxed{0}$$

3) Solve $\int_0^1 \sqrt[3]{1-x^7} - \sqrt[7]{1-x^3}\, dx$ $\quad f \circ f^{-1}(x) = x$

Reminder: To determine $f^{-1}(x)$ we follow the given steps: $\left(f(x) = \sqrt[3]{1-x^7}\right)$

1. $\sqrt[3]{1-x^7} = y$
2. then we write x in terms of y.

$1 - x^7 = y^3 \Rightarrow x = \sqrt[7]{1-y^3}$

$\Rightarrow f^{-1}(x) = \sqrt[7]{1-x^3}$

------> ödev $\qquad$ Result = 0

4) Verify the given equation:

$$\underbrace{\int_0^a f(x)\,dx}_{①} = \int_0^a f(a-x)\,dx \qquad a \text{ is a real num.}$$

We use substitution $a - x = u \Rightarrow -dx = du$

$\Rightarrow dx = -du$

Caution: Since we use substitution we have to determine new bounds.

$a - x = u \Rightarrow \begin{cases} \text{for } x = 0 \Rightarrow u = a \\ \text{for } x = a \Rightarrow u = 0 \end{cases}$ Then we have:

$$\int_0^a f(a-x)\,dx = -\int_a^0 f(u)\,du = \int_0^a f(u)\,du \Big\}②$$

Caution: If we compare ① and ②; there bounds are the same (from 0 to a) & the integrands are denoted by f. Only integration parameters are different: x and u respectively $\Rightarrow$ Results of ① and ② are equal

By using results of 4^{th} question solve 5^{th} question.

5) Solve: $\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$ n: Real num

Let's assume that result is (A) (real num)

$= \int_0^{\frac{\pi}{2}} \frac{\sin^n\left(\frac{\pi}{2}-x\right)}{\sin^n\left(\frac{\pi}{2}-x\right)+\cos^n\left(\frac{\pi}{2}-x\right)}\, dx$

Recall that:

$\sin\left(\frac{\pi}{2}-x\right) = \cos x$

$\cos\left(\frac{\pi}{2}-x\right) = \sin x$

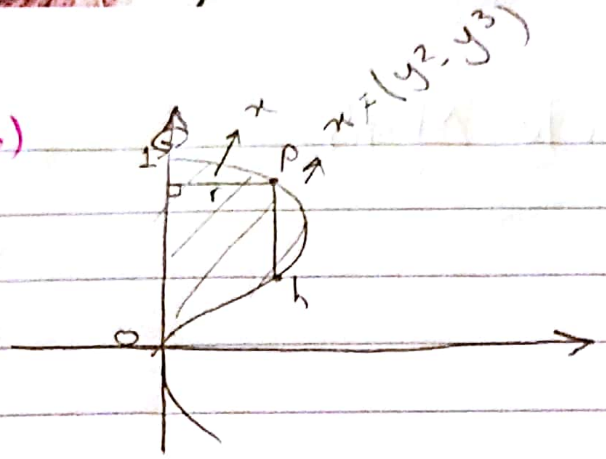
$= \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\cos^n x + \sin^n x}$ } result is A

$A + A = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\, dx$

$2A = \int_0^{\frac{\pi}{2}} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x}\, dx = \int_0^{\frac{\pi}{2}} dx$

$= (x)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}$

$\Rightarrow \boxed{A = \frac{\pi}{4}}$

6) 

Shaded area is rotated about y-axis. Find volume of generated solid

Sol: By shell method:

Given region is rotated about vertical line

=> We use integration with respect to x

$V_{solid} = 2\pi \int_a^b r(x)\,h(x)\,dx \qquad a \le x \le b$

Here we couldn't determine height of the shell.

By disk method:

$V_{solid} = \pi \int_{c=0}^{d=1} r(y)\,dy \qquad (r(y) = y^2 - y^3 - 0)$

$c \le y \le d$

→ egri ...

→ hacim

→ belirli int.

→ Liebniz

→ kismi int