3.2.1● The random variable N has PMF

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[N \leq 1]$?

a) $\sum_n P_N(n) = 1$

$$c \cdot \left(\frac{1}{2}\right)^0 + c \cdot \left(\frac{1}{2}\right)^1 + c \cdot \left(\frac{1}{2}\right)^2 = 1$$

$c \rightarrow$

b) $P[N \leq 1] = P[N=0] + P[N=1] = c \cdot \left(\frac{1}{2}\right)^0 + c \cdot \left(\frac{1}{2}\right)^1$

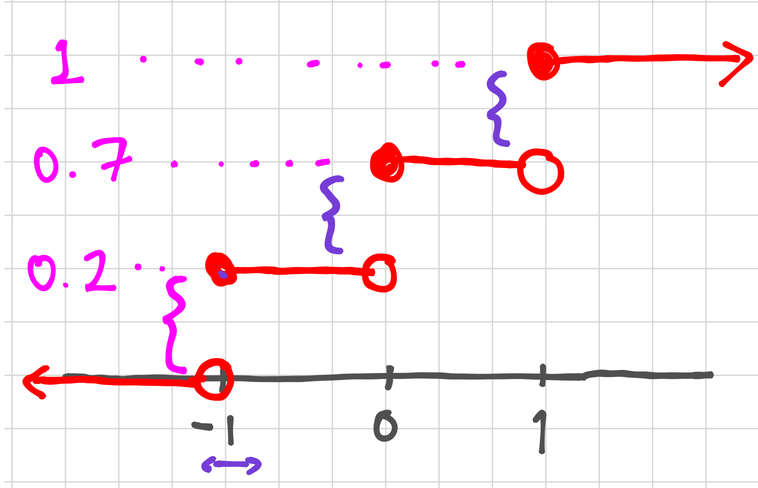
3.4.2● The random variable X has CDF

$$F_X(x) = \begin{cases} 0 & x < -1, \\ 0.2 & -1 \leq x < 0, \\ 0.7 & 0 \leq x < 1, \\ 1 & x \geq 1. \end{cases}$$

(a) Draw a graph of the CDF.

(b) Write $P_X(x)$, the PMF of X. Be sure to write the value of $P_X(x)$ for all x from $-\infty$ to ∞.

a)



b) PMF of X

for $x < -1 \Rightarrow P_X(x) = 0$

at $x = -1$ $\quad P_X(-1) = F_X(-1) - F_X(-1-\varepsilon)$

$= 0.2 - 0 = 0.2$

at $x = 0$ $\quad P_X(0) = F_X(0) - F_X(0-\varepsilon)$

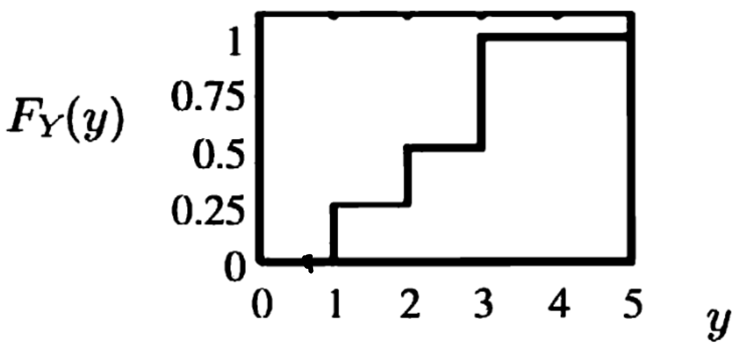
$= 0.7 - 0.2 = 0.5$

$P_X(1) = F_X(1) - F_X(1-\varepsilon)$

$= 1 - 0.7 = 0.3$

$$P_X(x) = \begin{cases} 0.2, & x = -1 \\ 0.5, & x = 0 \\ 0.3, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

3.4.1● Discrete random variable Y has the CDF $F_Y(y)$ as shown:



Use the CDF to find the following probabilities:

(a) $P[Y < 1]$ and $P[Y \leq 1]$

(b) $P[Y > 2]$ and $P[Y \geq 2]$

(c) $P[Y = 3]$ and $P[Y > 3]$

(d) $P_Y(y)$

d) $P_Y(y)$ PMF

for $y<1$ $\quad P_Y(y)=0$

@ $y=1$ $\quad P_Y(y) = 0.25-0 = 0.25$

$y=2$ $\quad P_Y(y) = 0.5-0.25 = 0.25$

$y=3$ $\quad P_Y(y) = \cancel{0.75}\ 1 - 0.5 = 0.5$

$$P_Y(y) = \begin{cases} 0.25, & y=1,2 \\ 0.5, & y=3 \\ 0, & \text{otherwise} \end{cases}$$

a) $P[Y<1] = 0$

$P[Y\leq 1] = F_Y[1] = 0.25$

b) $P[Y>2] = 1 - P[Y\leq 2] = 1 - F_Y(2) = 1 - 0.5 = 0.5$

$P[Y\geq 2] = 1 - P[Y<2] = 1 - \{P_Y[1]\} = 1 - 0.25 = 0.75$

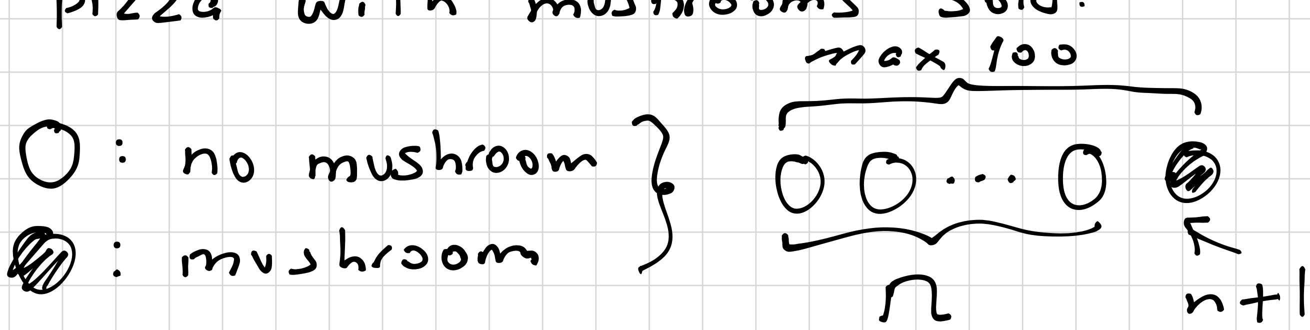
c) $P[Y=3] = P_Y[3] = 0.5$

$P[Y>3] = 1 - P[Y\leq 3] = 1 - F_Y[3] = 1 - 1 = 0$

3.4.5 At the One Top Pizza Shop, a pizza sold has mushrooms with probability $p = 2/3$. On a day in which 100 pizzas are sold, let N equal the number of pizzas sold before the first pizza with mushrooms is sold. What is the PMF of N? What is the CDF of N?

$P\{\text{Pizza has mushrooms}\} = \frac{2}{3}$

N: number of pizzas sold before the first pizza with mushrooms sold.



$S_N = \{0, 1, \ldots, 100\}$

$0 \leq n < 100 \qquad P_N(n) = \left(\frac{1}{3}\right)^n \cdot \left(\frac{2}{3}\right)$

$n = 100 \qquad P_N(100) = \left(\frac{1}{3}\right)^{100}$

PMF of N,

$$P_N(n) = \begin{cases} (1/3)^n \cdot (2/3), & 0 \leq n \leq 99,\ n \in \mathbb{Z} \\ (1/3)^{100}, & n = 100 \\ 0, & \text{otherwise} \end{cases}$$

CDF of N $\qquad F_N(n) = P[N \leq n]$

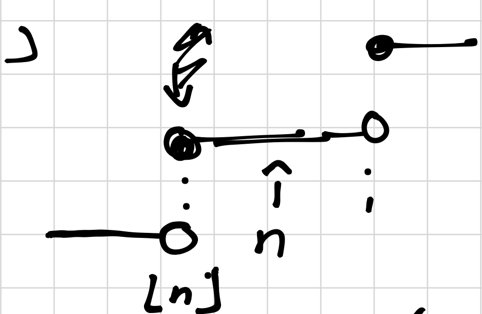
For $n < 100$ the CDF must obey

$$F_n(n) = \sum_{i=0}^{n} \left(\frac{1}{3}\right)^i \cdot \left(\frac{2}{3}\right) = 1 - \left(\frac{1}{3}\right)^{n+1}$$

~~Howe~~ For $n = 100 \rightarrow F_n(100) = 1$

However since $F_N(n)$ is defined on all values of n, including non-integer values

$$F_n(n) = \begin{cases} 0, & n < 0 \\ 1 - (1/3)^{\lfloor n \rfloor + 1}, & 0 \leq n < 100 \\ 1, & n \geq 100 \end{cases}$$



$F_N(n) = F_N(\lfloor n \rfloor)$

flo

$\lfloor n \rfloor$: floor.

3.5.2● It costs 20 cents to receive a photo and 30 cents to send a photo from a cellphone. C is the cost of one photo (either sent or received). The probability of receiving a photo is 0.6. The probability sending a photo is 0.4.

(a) Find $P_C(c)$, the PMF of C.

(b) What is $\mathrm{E}[C]$, the expected value of C?

C: Cost of one photo $\quad S_C = \{20, 30\}$

a) $P(C=20) = P\{\text{receive photo}\} = 0.6$

$P(C=30) = P\{\text{send photo}\} = 0.4$

$$P_C(c) = \begin{cases} 0.6, & c=20 \\ 0.4, & c=30 \\ 0, & \text{otherwise} \end{cases}$$

b) $E[C] = \sum_c c \cdot P_C(c) = 20 \times 0.6 + 30 \times 0.4$
$= 24$ cents.

3.5.3●

(a) The number of trains J that arrive at the station in time t minutes is a Poisson random variable with $E[J] = t$. Find t such that $P[J > 0] = 0.9$.

(b) The number of buses K that arrive at the station in one hour is a Poisson random variable with $E[K] = 10$. Find $P[K = 10]$.

(c) In a 1 ms interval, the number of hits L on a Web server is a Poisson random variable with expected value $E[L] = 2$ hits. What is $P[L \leq 1]$?

J: poisson

$E(J) = \alpha = t$

$$P_J(j) = \frac{t^j \cdot e^{-t}}{j!}, \quad j \geq 0, \ j \in \mathbb{Z}$$

$$P[J>0] = \sum_{j=1}^{\infty} P_J(j) = 1 - \underbrace{P[J \leq 0]} = 1 - P[J=0]$$

$$P[J=0] = e^{-t} \qquad P[J>0] = 1 - e^{-t} = 0.9$$

$$0.1 = e^{-t} \Rightarrow 10 = e^{t} = t = \ln(10)$$

$$= 2.3.$$

3.5.4● You simultaneously flip a pair of fair coins. Your friend gives you one dollar if both coins come up heads. You repeat this ten times and your friend gives you X dollars. Find E[X], the expected number of dollars you receive. What is the probability that you do "worse than average"?

A : both coins come up heads : SUCCESS

$P(A) = \frac{1}{4}$

X : number of successes in 10 trials

Binomial R.V $p = \frac{1}{4}$ $n = 10$

$P_X(x) = \binom{10}{x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{10-x}$, $x = 0, 1, \ldots, 10$

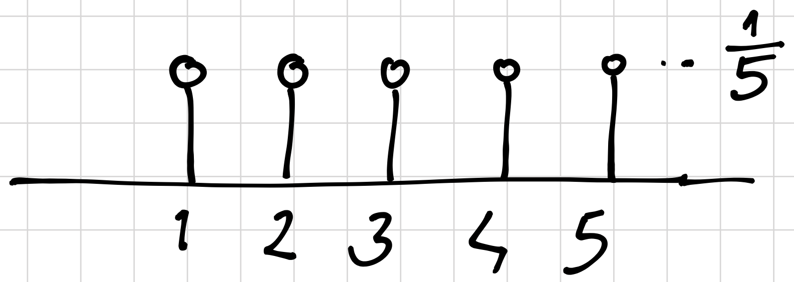
a) $E[X] = n \cdot p = 10 \times 0.25 = 2.5$

b) $P(X < E[X]) = P(X < 2.5) = P_X(0) + P_X(1) + P_X(2)$

3.5.10● X is the discrete uniform $(1, 5)$ random variable.

$k \quad \ell$

(a) What is $P[X = E[X]]$?



$P_X(x)$

$$E[X] = \frac{k+\ell}{2} = \frac{1+5}{2} = 3$$

$$P[X=3] = \frac{1}{5}$$

$$P(X > E[X]) = P(X>3) = P(X=4) + P(X=5)$$
$$= \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

Ödev

3.3.14 , 3.7.18 , . .

3.5.12● At a casino, people line up to pay \$20 each to be a contestant in the following game: The contestant flips a fair coin repeatedly. If she flips heads 20 times in a row, she walks away with $R = 20$ million dollars; otherwise she walks away with $R = 0$ dollars.

(a) Find the PMF of R, the reward earned by the contestant.

(b) The casino counts "losing contestants" who fail to win the 20 million dollar prize. Let L equal the number of losing contestants before the first winning contestant. What is the PMF of L?

(c) Why does the casino offer this game?

A: the event that has 20 heads in a row : success

$P(A) = \left(\frac{1}{2}\right)^{20} = 2^{-20}$

a) $S_R = \{0, 20\times10^6\}$ $\quad P_R(r) = \begin{cases} 1-2^{-20}, & r=0 \\ 2^{-20}, & r=20\times10^6 \\ 0, & \text{otherwise} \end{cases}$

b) $S_L = \{0, 1, 2, \ldots\}$ $\quad p = P(A) = 2^{-20}$

$P[L=\ell] = (1-p)^{\ell}\, p \qquad \ell = 0, 1, \ldots$

c) $E[R] = \sum_r r \cdot P_R(r) = 0\times(1-2^{-20}) + 20\times10^6 \times 2^{-20}$
$= 19.07$

3.7.4● Suppose an NBA basketball player shooting an uncontested 2-point shot will make the basket with probability 0.6. However, if you foul the shooter, the shot will be missed, but two free throws will be awarded. Each free throw is an independent Bernoulli trial with success probability p. Based on the expected number of points the shooter will score, for what values of p may it be desirable to foul the shooter?

Uncontested shot

X: number of points scored.

$$P_X(x) = \begin{cases} 0.4, & x = 0 \\ 0.6, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$S_X = \{0, 2\}$$

$$E[X] = 2 \times 0.6 + 0 \times 0.4 = \underline{1.2}$$

Free throws

- Each shot is a 1 point shot
- Y: the number of shots scored in 2 shots : Binomial, $n = 2$, $p = p$

$$E[Y] = np = \underline{\underline{2 \cdot p}}$$

$2p > 1.2 \Rightarrow p > 0.6 \Rightarrow$ we should not foul the shooter.

3.7.5● It can take up to four days after you call for service to get your computer repaired. The computer company charges for repairs according to how long you have to wait. The number of days D until the service technician arrives and the service charge C, in dollars, are described by

d	1	2	3	4
$P_D(d)$	0.2	0.4	0.3	0.1

and

$$C = \begin{cases} 90 & \text{for } \underline{\underline{1}}\text{-day service,} \\ 70 & \text{for 2-day service,} \\ 40 & \text{for 3-day service,} \\ 40 & \text{for 4-day service.} \end{cases}$$

$= g(D)$

$P_C(90) = 0.2$
$P_C(70) = 0.4$
$P_C(40) = P_D(3) + P_D(4)$
$= 0.3 + 0.1 = 0.4$

(a) What is the expected waiting time $\mu_D = E[D]$?

(b) What is the expected deviation $E[D - \mu_D]$?

(c) Express C as a function of D.

(d) What is the expected value $E[C]$?

a) $\mu_D = E[D] = \sum_d d \cdot P_D(d) = 1 \times 0.2 + 2 \times 0.4 + 3 \times 0.3 + 4 \times 0.1$
$= 2.3$

b) $E[D - \mu_D] = E[D] - E[\mu_D] = \mu_D - \mu_D = 0$

c) $E[C] = \sum g(d) \cdot P_D(D)$
$= \sum_c c \cdot P_C(c) = 90 \times 0.2 + 70 \times 0.4$
$+ 40 \times 0.4 = \sim$

3.7.8 A new cellular phone billing plan costs \$15 per month plus \$1 for each minute of use. If the number of minutes you use the phone in a month is a geometric random variable with expected value $1/p$, what is the expected monthly cost E[C] of the phone? For what values of p is this billing plan preferable to the billing plan of Problem 3.6.6 and Problem 3.7.7?

X: number of minutes used, geometric

$E[X] = 1/p$, $\underline{\underline{p}}$

C: monthly cost $\quad C = X \cdot 1 + 15$

$E[C] = E[X+15] = E[X] + 15 = \frac{1}{p} + \underline{\underline{15}}$

dollars.

3.7.9♦ A particular circuit works if all 10 of its component devices work. Each circuit is tested before leaving the factory. Each working circuit can be sold for k dollars, but each nonworking circuit is worthless and must be thrown away. Each circuit can be built with either ordinary devices or ultra-reliable devices. An ordinary device has a failure probability of $q = 0.1$ and costs \$1. An ultrareliable device has a failure probability of $q/2$ but costs \$3. Assuming device failures are independent, should you build your circuit with ordinary devices or ultra-reliable devices in order to maximize your expected profit E[R]? Keep in mind that your answer will depend on k.

- 10 devices must work.
- Each circuit can be sold for \$k if works.

- Ordinary devices fail with prob. $q = 0.1$ and costs \$1 per device.
- Ultra-reliable devices fail with prob. $\frac{q}{2} = 0.05$ and costs \$3 per device.

Ordinary devices

p: Probability that the circuit works.

$p = (1-q)^{10}$

W_s is R.V where

$W_s = 0$ if the circuit works

$W_s = 1$ " " " fails

$$P_{W_s}(w) = \begin{cases} 1-p, & w=0 \\ p, & w=1 \\ 0, & \text{otherwise} \end{cases}$$

R_s : total profit $R_s = g_s(W_s)$ is a function of W_s

$W_S = 0 \quad \Rightarrow \quad R_S = -10 \quad = \quad g_S(0)$

$W_S = 1 \quad \Rightarrow \quad R_S = k - 10 \quad = g_S(1)$

$$E[g_S[W_S]] = \sum g_S(W_S) \cdot P_{W_S}(w)$$

$$= -10 \times (1-p) + (k-10) \times p$$

$$= -10 + \cancel{10p} + pk - \cancel{10p}$$

$$= p \cdot k - 10$$

$$E[g_S[W_S]] = k \cdot (1-q)^{10} - 10 = 0.9^{10} \cdot k - 10$$

Ultra-reliable devices

W_U is R.V where

$W_U = 0$ if the circuit works

$W_U = 1$ " " " fails

$$P_{W_U}(w) = \begin{cases} 1-(1-q/2)^{10}, & w=0 \\ (1-q/2)^{10}, & w=1 \\ 0, & \text{otherwise} \end{cases}$$

R_U : total profit $R_U = g_U[W_U]$

$W_U = 0 \quad \Rightarrow \quad R_U = -30$

$W_U = 1 \quad \Rightarrow \quad R_U = k - 30$

$$P_{R_U}(r) = \begin{cases} 1-(1-q/2)^{10}, & r=-30 \\ (1-q/2)^{10}, & r=k-30 \\ 0, & \text{otherwise} \end{cases}$$

$$E(R_U) = E[g_U(W_U)] = -30 \times [1-(\overset{0.95}{1-q/2})^{10}] + (k-30) \cdot (1-q/2)^{10}$$

$$= \dots = 0.95^{10} k - 30$$

Use ultra-reliable devices if $E[R_U] > E(R_S)$

$$\Rightarrow \quad 0.95^{10} k - 30 > 0.90^{10} k - 10$$

$$\Rightarrow \quad k > \frac{20}{0.95^{10} - 0.90^{10}} = 80.21 \text{ dollars}$$

$\boxed{\text{Continuous Random Variables}}$

Continuous Sample Space

A random variable X is continuous if the range S_X consists of one ore more intervals

For each $x \in S_X$, $P[X=x]=0$

~ T: The arrival time of a particle

$$S_T = \{t \in \mathbb{R} \mid 0 \leq t < \infty\}$$

~ V: voltage across a resistor

$$S_V : \{v \in \mathbb{R} \mid -\infty < v < \infty\}$$

~ T: Arrival time in minutes of a professor at a class (8:55 − 9:05), let's say t is relative to 9 o'clock

$$S_T = \{t \in \mathbb{R} \mid -5 \leq t \leq 5\}$$