MATH2081: Math for Computing
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Lecture Notes 5: Sequences, Strings and Relations

(2024A, Week 5, Monday, April 1, 16:30-18:00)[5]

A **sequence** s is a function whose domain D is a subset of integers. The notation s_n is typically used instead of the more general function notation $s(n)$. The term n is the **index** of the sequence. If D is a finite set, we call s a **finite** sequence; otherwise, s is an **infinite** sequence.

(a) A sequence s is **increasing** (respectively, **decreasing**) if for all i and j in the domain of s, if $i < j$, then $s_i < s_j$ (respectively, $s_i > s_j$).

(b) A sequence s is **nondecreasing** (respectively, **nonincreasing**) if for all i and j in the domain of s, if $i < j$, then $s_i \leq s_j$ (respectively, $s_i \geq s_j$).

(c) A **subsequence** of a sequence s is a sequence obtained from s by choosing certain terms of s in the same order in which they appear in s.

(d) If $\{a_i\}_{i=m}^n$ is a sequence, we define

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n, \quad \prod_{i=m}^{n} a_i = a_m a_{m+1} \cdots a_n. \tag{2}$$

The formalism $\sum_{i=m}^n$ is the **sum** (or **sigma**) notation and $\prod_{i=m}^n$ is the **product** notation. In (2), i is the **index**, m is the **lower limit** and n is the **upper limit**.

A **string** over X, where X is a finite set, is a finite sequence of (repeatable) elements from X. Since a string is a sequence, order is taken into account.

(a) Repetitions in a string can be specified by **superscripts**. For example, the string '110001' over the binary set $\{0, 1\}$ can be written as '$1^2 0^3 1$'.

(b) The string with no element is the **null string** and is denoted by λ.

(c) X^* denotes the *set of all strings* over X including the null string, and X^+ denotes the set of all *nonnull* strings over X.

(d) The **length** of a string α, denoted by $|\alpha|$, is the number of elements in α.

(e) If α and β are two strings, the string consisting of α followed by β, written $\alpha\beta$ (or $\alpha + \beta$), is the **concatenation** of α and β.

(f) A string β is a **substring** of a string α if there are strings γ and δ such that $\alpha = \gamma\beta\delta$.

[5] Most of the content of this document is taken from the book [1].

A (binary) **relation** R from a set X to a set Y is a subset of the Cartesian product $X \times Y$. If $(x, y) \in R$, we write xRy and say that x is related to y. If $X = Y$, R is a relation on X.

An informative way to picture a relation on a set is to draw its **digraph**. To draw the digraph of a relation on a set X, we first draw dots or vertices to represent the elements of X. Next, if the element (x, y) is in the relation, we draw an arrow (a **directed edge**) from x to y. An element of the form (x, x) in a relation corresponds to a directed edge from x to x. Such an edge is a **loop**.

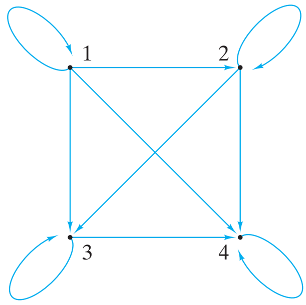

Figure 7: Digraph of the relation xRy if $x \leq y$ on $\{1, 2, 3, 4\}$.

Let R be a relation on a set X.

(a) R is **reflexive** if $(x, x) \in R$ for every $x \in X$.

(b) R is **symmetric** if for all $x, y \in X$, if $(x, y) \in R$, then $(y, x) \in R$.

(c) R is **antisymmetric** if for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$.

(d) R is **transitive** if for all $x, y, z \in X$, if (x, y) and $(y, z) \in R$, then $(x, z) \in R$.

(e) R is a **partial order** if R is *reflexive*, *antisymmetric* and *transitive*. If R is a partial order on a set X, the notation $x \preceq y$ is used to indicate that $(x, y) \in R$. This notation suggests that we are interpreting the relation as an ordering of the elements in X. Let R be a partial order on X. If $x, y \in X$ and either $x \preceq y$ or $y \preceq x$, we say that x and y are **comparable**. If $x, y \in X$ and $x \not\preceq y$ and $y \not\preceq x$, we say that x and y are **incomparable**. If every pair of elements in X is comparable, R is a **total order**.

(f) R is an **equivalence** relation if R is *reflexive*, *symmetric*, and *transitive*. Let R be an equivalence relation on a set X. For each $a \in X$, let $[a] = \{x \in X \mid xRa\}$. Then $S = \{[a] \mid a \in X\}$ is a partition of X. The set $[a]$ is the **equivalence class** of a in X given by the relation R.

Let R be a relation from X to Y. The **inverse** of R, denoted by R^{-1}, is the relation from Y to X defined by

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}.$$

Let R_1 be a relation from X to Y and R_2 be a relation from Y to Z. The **composition** of R_1 and R_2, denoted by $R_2 \circ R_1$, is the relation from X to Z defined by

$$R_2 \circ R_1 = \{(x, z) \mid (x, y) \in R_1 \text{ and } (y, z) \in R_2 \text{ for some } y \in Y\}.$$

A matrix is a convenient way to represent a relation R from X to Y. We label the rows with the elements of X (in some order) and the columns with the elements of Y (again, in some order). We then set the entry in row x and column y to 1 if xRy and to 0 otherwise. This matrix is the **matrix of the relation** R relative to the chosen orderings of X and Y.

References

1. Johnsonbaugh, R.: Discrete Mathematics - Eighth Edition. *Pearson Education*, New York (2018).