Some inequalities for the largest eigenvalue of a graph

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Abstract

Let $\lambda(G)$ be the largest eigenvalue of the adjacency matrix of a graph G. We show that if G is K_{p+1}-free then

$$\lambda(G) \leq \sqrt{2\frac{p-1}{p}e(G)}.$$

This inequality was first conjectured by Edwards and Elphick in 1983 and supersedes a series of previous results on upper bounds of $\lambda(G)$.

Let T_i denote the count of all i-cliques of G, $\lambda = \lambda(G)$ and $p = cl(G)$. We show

$$\lambda^p \leq T_2\lambda^{p-2} + ... + (i-1)T_i\lambda^{p-i} + ... + (p-1)T_p.$$

Let δ be the minimal degree of G. We show

$$\lambda(G) \leq \frac{\delta-1}{2} + \sqrt{2e(G) - \delta n + \frac{(\delta+1)^2}{4}}.$$

This inequality supersedes inequalities of Stanley and Hong. It is sharp for regular graphs and for a class of graphs which are in some sense maximally irregular.

1 Notation and conventions

We consider only simple finite, undirected graphs and use the standard graphic terminology of [2]. All graphs are assumed to be defined on the vertex set $\{1, 2, ...n\}$. $G(n)$ denotes a graph on n vertices and $G(n, e)$ denotes a graph on n vertices with e edges. For any two adjacent vertices i and j such that $i < j$ we write $i \sim j$. Given a vertex i, d_i denotes its degree and N_i denotes the set of its neighbours. A clique means a complete subgraph. The clique number of G is the size of a maximal clique in G and is denoted by $cl(G)$. By K_p we denote the complete graph on p vertices.

We deal only with eigenvalues of the adjacency matrix of a graph. The largest eigenvalue of a graph G is denoted by $\lambda(G)$.

2 An inequality involving the the edge count and the clique number

Let G be a graph on n vertices with $cl(G) \leq p$. The well-known Motzkin-Straus inequality [14] is stated as:

For any n-vector $(x_1, x_2, ..., x_n)$ with $x_i \geq 0$ $(1 \leq i \leq n)$ and $x_1 + x_2 + ... + x_n = 1$

$$\sum_{i \sim j} x_i x_j \leq \frac{p-1}{2p}. \tag{1}$$

Now, let λ be the largest eigenvalue of G and $(y_1, y_2, ..., y_n)$ be an eigenvector of unit length corresponding to λ. We have, by the Cauchy inequality,

$$\lambda^2 = \left(2 \sum_{i \sim j} y_i y_j\right)^2 \leq 2e \left(2 \sum_{i \sim j} y_i^2 y_j^2\right).$$

On the other hand, $y_1^2 + y_2^2 + ... + y_n^2 = 1$ and therefore, by (1),

$$2 \sum_{i \sim j} y_i^2 y_j^2 \leq \frac{p-1}{p}.$$

Hence,

$$\lambda^2 \leq 2e \frac{p-1}{p}$$

and we obtain the following assertion which apparently was first conjectured by Edwards and Elphick in 1983 [6].

Theorem 1 *For any graph $G = G(n, e)$,*

$$\lambda(G) \leq \sqrt{2\frac{cl(G) - 1}{cl(G)}e}. \tag{2}$$

The inequality (2) is sharp for any G consisting of a complete p-chromatic graph with equally sized chromatic classes and any number of isolated vertices (see [3], p. 73-74). It is interesting that it is sharp also for some essentially nonregular graphs such as for example the complete bipartite graphs with 0 or more isolated vertices.

It should be noted that the simplicity of the above proof is due exclusively to the power of Motzkin-Straus' approach and the inequality (2) is by no means a trivial result. Not only did Edwards and Elphick admit in [6] that they were unable to prove it but actually it supersedes several results on the largest graph eigenvalue, some of which we shall discuss in the sequel. In 1986 H. Wilf [20], using the same result of Motzkin and Straus came very close to the proof of (2) but for some reason overlooked it. He however, proved an important corollary of it (Corollary 2). Apparently Wilf was the first to notice the relation between the Motzkin-Straus result and graph spectra, which is actually a relation between some classical extremal graph problems and graph spectra.

Since, trivially $cl(G(n)) \leq n$, we obtain

Corollary 1 (Wilf[19]) *For any $G = G(n, e)$,*

$$\lambda(G) \leq \sqrt{2\frac{n-1}{n}e}.$$

By Turán's theorem, for any K_{p+1}-free graph $G = G(n, e)$

$$e \leq \frac{p-1}{2p}n^2.$$

Hence, we obtain

Corollary 2 (Wilf [20]) *For any G on n vertices,*

$$\lambda(G) \leq \frac{cl\,(G)-1}{cl\,(G)}n. \tag{3}$$

Now let $\chi(G)$ be the chromatic number of a graph G. By the trivial inequality $cl\,(G) \leq \chi(G)$, from (2) and (3), we obtain the following corollary.

Corollary 3 *For any $G = G(n,e)$,*

$$\lambda(G) \leq \sqrt{2\frac{\chi(G)-1}{\chi(G)}e} \tag{4}$$

and

$$\lambda(G) \leq \frac{\chi(G)-1}{\chi(G)}n. \tag{5}$$

The inequality (4) was previously proved in another way by Edwards and Elphick in [6] and by Hong in [10]. The inequality (5) was proved by Cvetković in [4] (see also [3], p. 92 Theorem 3.19).

Recently in [11] Hong and Shu, based on (4), proved the following theorem.

Theorem 2 (Hong, Shu) *Let G be a graph on n vertices and let $\overline{G}$ be its complementary graph. Then*

$$\lambda(G) + \lambda\left(\overline{G}\right) \leq \sqrt{\left(2 - \frac{1}{\chi(G)} - \frac{1}{\chi(\overline{G})}\right)n(n-1)}.$$

Hong and Shu note that their result in fact supersedes the results of Li [13], Zhou [21] and Hong [10]. Actually, following closely their arguments but using (2) instead of (4), we readily obtain an essentially stronger result.

Theorem 3 *Let G be a graph on n vertices and $\overline{G}$ be its complementary graph. Then*

$$\lambda(G) + \lambda\left(\overline{G}\right) \leq \sqrt{\left(2 - \frac{1}{cl(G)} - \frac{1}{cl(\overline{G})}\right)n(n-1)}.$$

Proof Indeed, applying Theorem 1 to G and $\overline{G}$, we obtain

$$\lambda(G)+\lambda\left(\overline{G}\right)\leq\sqrt{2\left(1-\frac{1}{cl(G)}\right)e(G)}+\sqrt{2\left(1-\frac{1}{cl(\overline{G})}\right)e\left(\overline{G}\right)}$$

and by the Cauchy-Schwarz inequality we are done.□

Let us now extend an interesting result of Nosal.

Theorem 4 *Let $G=G(n,e)$ be a graph with $cl(G)\leq p$ and let $\lambda_1\geq...\geq\lambda_n$ be its spectrum. Then*

$$\lambda_1^2\leq(p-1)\left(\lambda_2^2+...+\lambda_n^2\right). \tag{6}$$

Proof Indeed, it is known (see for instance [3], p. 85) that

$$\lambda_1^2+\lambda_2^2+...+\lambda_n^2=2e,$$

thus, by Theorem 1, we have

$$\lambda_1^2\leq\frac{p-1}{p}2e=\frac{p-1}{p}\left(\lambda_1^2+\lambda_2^2+...+\lambda_n^2\right)$$

and this is equivalent exactly to (6).□

The simplest case of Theorem 4 for $p=2$ was proved by Nosal in [16] (see also [3], p. 86. Theorem 3.9).

Close to this result is the following estimate.

Theorem 5 *For any graph G on n vertices with eigenvalues $\lambda_1\geq...\geq\lambda_n$ and $cl(G)\leq p$ we have*

$$\sum_{i=1}^{n}\lambda_i^3\leq\frac{p-2}{p-1}n\lambda_1^2. \tag{7}$$

Proof Let $T_3(G)$ denote the count of all triangles in G and $t_3(i)$ denote the count of all triangles containing the vertex i. It is well-known (see [3], p. 85) that

$$\sum_{i=1}^{n}\lambda_i^3=6T_3(G).$$

Since G contains no K_{p+1}, for any i, the graph induced by the neighbours of i contains no K_p. Thus, by virtue of Turán's theorem, for any vertex i we have

$$t_3(i) \leq \frac{p-2}{2(p-1)} d_i^2.$$

Hence, summing over all vertices, one obtains

$$\sum_{i=1}^{n} \lambda_i^3 = 6T_3 \leq \frac{p-2}{p-1} \sum_{i=1}^{n} d_i^2$$

and by the well-known inequality of Hofmeister [8]

$$\frac{1}{n} \sum_{i=1}^{n} d_i^2 \leq \lambda_1^2,$$

we obtain (7).□

In [5] Collatz and Sinogowitz raised the following problem (see also [3], p. 266, problem 3)

Problem 1 *Find the best upper bound of the function*

$$\varphi(n) = \max \left(\lambda(G) - \frac{2e(G)}{n} \right)$$

where the maximum is taken over all graphs G on n vertices.

In [1] F. Bell, using a deep result of Rowlinson [17], solves this problem in the following way

$$\varphi(n) = \begin{cases} \frac{1}{4}n - \frac{1}{2} & (n \text{ even}) \\ \frac{1}{4}n - \frac{1}{2} + \frac{1}{4n} & (n \text{ odd}). \end{cases}$$

In fact, using only (2) and some elementary calculus, it is not hard to obtain the following somewhat weaker but very close estimate

$$\varphi(n) = \begin{cases} \frac{1}{4}n - \frac{1}{2} & (n \text{ even}) \\ \frac{1}{4}n - \frac{1}{2} + O\left(\frac{1}{n}\right) & (n \text{ odd}). \end{cases}$$

We omit the tedious proof.

3 An inequality involving clique counts

Let $T_i(G)$ denote the count of all i-cliques of a graph G. In this section we prove the following theorem

Theorem 6 *Let G be a graph with $cl(G) = p \geq 2$ and let $\lambda = \lambda(G)$. Then*

$$\lambda^p \leq T_2(G)\lambda^{p-2} + ... + (i-1)T_i(G)\lambda^{p-i} + ... + (p-1)T_p(G). \quad (8)$$

Before proceeding to the proof of Theorem 6 let us note that the characteristic polynomial of any complete p-partite graph G of order n is exactly (see [3], p. 74)

$$\lambda^n - T_2(G)\lambda^{n-2} + ... + (i-1)T_i(G)\lambda^{n-i} + ... + (p-1)T_p(G)\lambda^{n-p}.$$

Hence, the inequality (8) is sharp for any G consisting of a complete p-partite graph and 0 or more isolated vertices.

Also note that for $cl(G) = 2$ the assertion of Theorem 6 appears as

$$\lambda^2 \leq T_2(G) = e(G).$$

Since this follows by Theorem 1 (or equivalently by the above mentioned result of Nosal), we have only to consider the case $cl(G) > 2$.

Let G be a graph and k be a positive integer. A k-walk in G is a sequence of vertices $i_1, i_2, ..., i_k$ such that i_j and i_{j+1} are adjacent for each $j = 1, 2, ..., k-1$. Denote by $W_k(G)$ the count of all k-walks of G. For any vertex i let $w_k(i)$ be the count of all k-walks which start with i. Similarly, let $t_p(i)$ be the count of all p-cliques containing i.

We note without proof the following easy lemma.

Lemma 1 *Let G be a graph on n vertices. For any $k \geq 1$*

$$\sum_{i=1}^{n} d_i w_k(i) = W_{k+1}(G).$$

The following lemma is the key element of the proof of Theorem 6.

Lemma 2 *Let G be a graph on n vertices with $cl(G) \geq p \geq 2$. For any $k \geq 1$*

$$\sum_{i=1}^{n} \left(t_p(i)\, w_{k+1}(i) - t_{p+1}(i)\, w_k(i)\right) \leq (p-1)\, T_p(G)\, W_k(G). \qquad (9)$$

Proof Note first the following obvious assertion.

Let $M_1, M_2, ..., M_p$ be subsets of the finite set M. Then

$$\sum_{i=1}^{p} |M_i| \leq (p-1)\,|M| + \left|\bigcap_{i=1}^{p} M_i\right|. \qquad (10)$$

Fix a p-clique, say $\{1, 2, ..., p\}$, in G and consider all $(k+1)$-walks of the type

$$v, i_1, i_2, ..., i_k$$

where $1 \leq v \leq p$. For any v $(1 \leq v \leq p)$ let M_v be the set of all k-walks beginning with a neighbour of v. Therefore, $w_{k+1}(v) = |M_v|$ and by (10), we have

$$\sum_{v=1}^{p} w_{k+1}(v) = \sum_{v=1}^{p} |M_v| \leq (p-1)\, W_k(G) + \left|\bigcap_{v=1}^{p} M_v\right|. \qquad (11)$$

Now, summing (11) over all vertices in all p-cliques of G, we obtain

$$\sum_{i=1}^{n} t_p(i)\, w_{k+1}(i) \leq (p-1)\, T_p(G)\, W_k(G) + \sum_{i=1}^{p} t_{p+1}(i)\, w_k(i)$$

and the proof is completed.□

Proof of Theorem 6 For the sake of simplicity we shall write W_m instead of $W_m(G)$ and T_k instead of $T_k(G)$. Fix a positive integer m and let $p = cl(G)$. According to Lemma 2, for any $q = 2, 3, ..., p$ the following inequality holds

$$\sum_{i=1}^{n} \left(t_q(i)\, w_{m+p-q+1}(i) - t_{q+1}(i)\, w_{m+p-q}(i)\right) \leq (q-1)\, T_q W_{m+p-q}.$$

Now, if we sum all those inequalities for $q = 2, 3, ..., p$, we obtain

$$\sum_{i=1}^{n} t_2(i)\, w_{m+p-1}(i) \leq T_2 W_{m+p-2} + 2T_3 W_{m+p-3} + ... + (p-1)\, T_p W_m.$$

Note that $t_2(i)$ is simply the degree d_i of i. Therefore, by Lemma 1, we have

$$W_{m+p} = \sum_{i=1}^{n} t_2(i)\, w_{m+p-1}(i)$$

and thus

$$W_{m+p} \leq T_2 W_{m+p-2} + 2T_3 W_{m+p-3} + \ldots + (p-1)\, T_p W_m. \tag{12}$$

Now let $\lambda_1 \geq \ldots \geq \lambda_n$ be the spectrum of G. It is known (see [3], p. 44, Theorem 1.10) that W_m can be expressed as

$$W_m = \sum_{i=1}^{n} c_i \lambda_i^m$$

where c_i are non-negative constants and $c_1 > 0$. If the modulus of any negative eigenvalue of G is less than λ_1, we easily obtain for any positive integer s

$$\lim_{m \to \infty} \frac{W_{m+s}}{W_m} = \lambda_1^s$$

and applying this to (12) for $s = 1, 2, \ldots, p$, we obtain the desired inequality. Consider the remaining case when G has a negative eigenvalue of modulus λ_1. Obviously, this implies that the modulus of the smallest eigenvalue λ_n is exactly λ_1. Then G is bipartite and thus $cl(G) = 2$. As we have noted this case is covered by Theorem 1, and so the proof is completed.□

In 1977 Khadziivanov and the author conjectured that for any graph G on n vertices with $cl(G) = p$ the following inequalities hold

$$\left(\frac{T_p}{\binom{p}{p}}\right)^{\frac{1}{p}} \leq \left(\frac{T_{p-1}}{\binom{p}{p-1}}\right)^{\frac{1}{p-1}} \leq \ldots \leq \frac{T_1}{\binom{p}{1}} = \frac{n}{p} \tag{13}$$

Later Khadziivanov in [12] (see also [15]) gave an elegant proof of (13) but his result remained unnoticed. In 1992 the same inequalities were rediscovered by D. Fisher and J. Ryan who settled the problem in [7] using a different method. Actually, it is possible to obtain Theorem 1 from Theorem 6 and (13). In this sense, Theorem 6 should be regarded as an extension of Theorem 1.

4 A generalization of Stanley's and Hong's inequalities for $\lambda(G)$

Denote by δ the minimal degree of $G = G(n,e)$. Stanley in [18] proved that

$$\lambda(G) \leq -\frac{1}{2} + \sqrt{2e + \frac{1}{4}} \tag{14}$$

and Hong in [9] proved that if G has no isolated vertices then

$$\lambda(G) \leq \sqrt{2e - n + 1}. \tag{15}$$

We shall show that actually both (14) and (15) are particular cases of the following more general inequality.

Theorem 7 *For any graph $G = G(n,e)$ with minimal degree δ the inequality*

$$\lambda(G) \leq \frac{\delta - 1}{2} + \sqrt{2e - n\delta + \frac{(1+\delta)^2}{4}} \tag{16}$$

holds.

Before proceeding to the proof of this theorem let us state without a proof the following claim.

Claim 1 *For $2e \leq n(n-1)$ the function*

$$f(x) = \frac{x-1}{2} + \sqrt{2e - nx + \frac{(1+x)^2}{4}}$$

is decreasing with respect to x.

Hence, by (16), we obtain,

$$\lambda(G) \leq f(\delta) \leq f(0) = -\frac{1}{2} + \sqrt{2e + \frac{1}{4}}$$

and this is precisely Stanley's inequality. Similarly for $\delta \geq 1$ we have

$$\lambda(G) \leq f(\delta) \leq f(1) = \sqrt{2e - n + 1}$$

and this is Hong's inequality.

Note that unlike the inequalities of Stanley and Hong, (16) is sharp for all regular graphs. Moreover, it is sharp also for a class of graphs which are in some sense maximally irregular. Indeed, let $n \geq k$ and $\mathcal{H}_{n,k}$ be the n-vertex graph which is the complement of K_{n-k}. It is not hard to compute that

$$\lambda\left(\mathcal{H}_{n,k}\right) = \frac{k-1}{2} + \sqrt{\frac{(k-1)^2}{4} + k\left(n-k\right)}.$$

On the other hand, the minimal degree of $\mathcal{H}_{n,k}$ is k and $2e\left(\mathcal{H}_{n,k}\right) = 2kn - k^2 - k$. Hence, for $G = \mathcal{H}_{n,k}$ we have equality in (16). As F. Bell has shown in [1] $\mathcal{H}_{n,k}$ maximizes the degree variance

$$\frac{1}{n}\sum_{i=1}^{n}\left(d_i - \frac{2e\left(G\right)}{n}\right)^2$$

over the class of all n-vertex graphs with the same edge count as $\mathcal{H}_{n,k}$ and thus, in this special respect, it is maximally irregular.

Moreover, (16) is sharp for any graph consisting of two vertex disjoint cliques as well. Indeed, let G be the union of K_p and K_q. Assume $p \geq q$. We have

$$p-1 = \lambda\left(G\right) \leq \frac{q-1}{2} + \sqrt{p\left(p-1\right) + q\left(q-1\right) - \left(p+q\right)\left(q-1\right) + \frac{q^2}{4}} = p-1.$$

Proof of Theorem 7 Let us exclude from the outset the trivial case of $\lambda\left(G\right) = 0$ which is possible only for edgeless G and is easily verified. It is also manifest that (16) follows from the inequality

$$\lambda^2 - \left(\delta - 1\right)\lambda - \left(2e - \left(n-1\right)\delta\right) \leq 0, \tag{17}$$

and so our goal hereafter will be the proof of (17). Note first that for $k \geq 4$ we have

$$W_k\left(G\right) = \sum_{i \sim j}\left(d_i w_{k-2}\left(j\right) + d_j w_{k-2}\left(i\right)\right).$$

Hence, by rearranging the summands, we obtain

$$W_k\left(G\right) = \sum_{i=1}^{n} w_{k-2}\left(i\right)\left(\sum_{j \in N_i} d_j\right). \tag{18}$$

Let V be the vertex set of G. For any vertex i, we have

$$\sum_{j\in N_i} d_j + \sum_{j\in V\setminus N_i} d_j = \sum_{i=1}^{n} d_i = 2e.$$

Hence, in view of

$$\sum_{j\in V\setminus N_i} d_j \geq (n-1-d_i)\,\delta + d_i,$$

we obtain

$$\sum_{j\in N_i} d_j \leq 2e - (n-1-d_i)\,\delta - d_i$$

and by (18),

$$\begin{aligned} W_k(G) &\leq \sum_{i=1}^{n} w_{k-2}(i)\,(2e-(n-1-d_i)\,\delta - d_i) \\ &= (2e-(n-1)\,\delta)\,W_{k-2}(G) + (\delta-1)\sum_{i=1}^{n} w_{k-2}(i)\,d_i \\ &= (2e-(n-1)\,\delta)\,W_{k-2}(G) + (\delta-1)\,W_{k-1}(G)\,. \end{aligned} \tag{19}$$

Let $\lambda_1 \geq ... \geq \lambda_n$ be the spectrum of G. As before we have

$$W_m = \sum_{i=1}^{n} c_i \lambda_i^m \tag{20}$$

where c_i are non-negative constants and $c_1 > 0$. If the modulus of any negative eigenvalue of G is less than λ_1, we easily obtain for any positive integer s

$$\lim_{m\to\infty} \frac{W_{m+s}}{W_m} = \lambda_1^s$$

and applying this to (19) for $s = 1$ and 2, we obtain (17).

Consider now the more subtle case when G has a negative eigenvalue of modulus λ_1. Then we have for some p and q $(1 \leq p < q \leq n)$

$$\lambda_1 = \lambda_2 = ... = \lambda_p > \lambda_{p+1} \geq ... \geq \lambda_{q-1} > \lambda_q = ... = \lambda_{n-1} = \lambda_n = -\lambda_1.$$

Thus, by (20),

$$W_m = \lambda_1^m \sum_{i=1}^{p} c_i + (-1)^m \lambda_1^m \sum_{i=q}^{n} c_i + \sum_{i=p+1}^{q-1} c_j \lambda_i^m.$$

Put

$$\alpha = \sum_{i=1}^{p} c_i \text{ and } \beta = \sum_{i=q}^{n} c_i.$$

Simple considerations show that $\alpha > \beta$, for otherwise W_{2m+1} would be negative for m sufficiently large. Hence, one easily verifies that

$$\lim_{m\to\infty} \frac{W_{m+2}}{W_m} = \lambda_1^2.$$

We also have

$$\lim_{m\to\infty} \frac{W_{2m}}{W_{2m-1}} = \frac{\alpha+\beta}{\alpha-\beta}\lambda_1 \geq \lambda_1.$$

Therefore, for $\delta \leq 1$, from (19) we obtain

$$\begin{aligned}\lim_{m\to\infty} \frac{W_{2m+1}}{W_{2m-1}} &\leq (\delta-1) \lim_{m\to\infty} \frac{W_{2m}}{W_{2m-1}} + (2e - (n-1)\,\delta) \\ &\leq (\delta-1)\,\lambda_1 + (2e - (n-1)\,\delta)\end{aligned}$$

and hence (17). Similarly, for $\delta > 1$, by virtue of

$$\lim_{m\to\infty} \frac{W_{2m+1}}{W_{2m}} = \frac{\alpha-\beta}{\alpha+\beta}\lambda_1 \leq \lambda_1,$$

from (19) we obtain

$$\begin{aligned}\lim_{m\to\infty} \frac{W_{2m+2}}{W_{2m}} &\leq (\delta-1) \lim_{m\to\infty} \frac{W_{2m+1}}{W_{2m}} + (2e - (n-1)\,\delta) \\ &\leq (\delta-1)\,\lambda_1 + (2e - (n-1)\,\delta)\end{aligned}$$

and hence (17). The proof is completed.□

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