spatiotemporal cats
or, try herding 7 cats

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Contents

> This cat has been skinned in more ways than any other cat in the history of cats.
>
> — Professore Gatto Nero

This is a project of many movable parts, so here is a guide where to blog specific topics (or where to find them)

- chapter **??** Sidney's blog

Blog fearlessly: this is your own lab-book, a chronology of your learning and research that you might find invaluable years hence.

Chapter 1

Sidney's blog

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The latest entry at the bottom for this blog

1.1 2020 blog

2020-05-20 Predrag to Sidney:

You can write up your narrative in this file. Can clip & paste anything from above sections you want to discuss, that saves you LaTeXing time.

2021-09-09 Predrag The 3rd line of *siminos/spatiotemp/blogCats.tex* says

```
\input{inputs/inclOnlyCats}  %process only the files you are editing
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you uncomment a single line in that file to "process only the files you are editing".

2021-07-04 Predrag to Sidney Pro tip: compile *blogCats.tex* often, as you write, and fix errors as you write. I had to go all they way back to May to find one of your unbalanced "{" and make the entire blog compile without errors...

2020-08-22 Predrag First task:

Start reading kittens/CL18.tex **[CL18]** sect. *Bernoulli map*. Everything up to CL18 sect. s:1D1dLatt *Temporal Bernoulli* you know from the Chaos-Book course.

New stuff starts here. See how much you understand. Write your study notes up here, ask questions - this is your personal blog.

You refer to an equation like this: CL18 eq. tempBern;

to figure like this: CL18 figure fig:BernCyc2Jacob;

to table like this: table **??**;

to a reference like this: Gutkin and Osipov **[GutOsi15]** (*GutOsi15* refers to an article listed in *../bibtex/siminos.bib*).

and to external link like this: "For great wallpapers, see overheads in Engel's course **[Engel11]**."

2020-08-22 Predrag An example of referring to the main text: Why do you write *orbit Jacobian matrix* CL18 eq. jacobianOrb as a partial derivative, when you already know $\mathcal{J}$, see CL18 eq. tempFixPoint?

2020-08-24 Sidney Started reading from the beginning as that only adds an additional 4 pages, and it would be beneficial to review.

General Notes: Showing what modern chaos calculations look like. The spatiotemporal cat is the arbitrary dimension generalization of the 1-D Bernoulli map.

(mod 1) subtracts the integer part of $s\phi_t$, this keeps ϕ_{t+1} within the unit interval (group theoretic analogue?). Also partitions the state-space into s sub-intervals.

2020-08-24 Predrag The group theory here compatifies translations on the (infinite) line $\phi \in (-\infty, \infty)$ to translations on the (compact) circle $\phi \in [0, 2\pi)$.

2020-08-24 Sidney Reminder to self: review the symbolic dynamics, and binary operations from chapter 14 of Chaosbook The unit interval is partitioned into s^n subintervals, each with one unstable period-n point, except the rightmost fixed point is the same as the fixed point at the origin. So there are $s^n - 1$ total period-n periodic points. r in (**??**) is a cyclic permutation that translates forward in time the lattice state by one site. Inverse r because the second term is always one step behind the first term and an inverse r moves the state back one.

Questions 1. I've pretty much never done modular arithmetic before, I understand CL18 eq. BerStretch in the idea that the circle map wraps in on itself and contributes the value of its slope after one go around, but I am unsure on how to use the modular arithmetic to do that, should I look into that?

2020-08-24 Predrag As I do not know what "modular arithmetic" is, don't worry about :)

2020-08-25 Sidney General Notes

CL18 eq. pathBern appears to be a vector of a periodic (or relative periodic orbit) through the Bernoulli map. Review Multishooting. Total number of of periodic points of period n is $N_n = s^n - 1$ but it also equals the magnitude of the determinant of the orbit Jacobian matrix. (got to page 7)

Q1 Is CL18 eq. tempBernFix the evolution function $f^t(y)$ that was referenced throughout ChaosBook?

Q2 What exactly is meant by a "lattice"?

2020-08-24 Predrag .

A1 The whole point of the paper us that ChaosBook is obsolete - in the new formulation, there is no 'time' evolution, no time trajectory $f^t(y)$, there are only sets of fields the live on lattice points that satisfy recurrence relations. CL18 eq. tempBernFix is *orbit Jacobian matrix*, the stability of a lattice state, to be related to stability forward in time in CL18 sect. s:Hill. This is a revolution: there is no more time, there is only spacetime.

A2 Temporal lattice $\mathbb{Z}$ is defined in CL18 eq. pathBern. Spacetime integer lattice $\mathbb{Z}^2$, (or more generally $\mathbb{Z}^d$) in CL18 eq. KanekoCML, CL18 eq. CatMap2d. When you get to it, a 2-dimensional *Bravais lattice* Λ is defined in CL18 eq. 2DBravaisLattice.

If this is unclear, read up on integer lattices, give your own precise definition.

2020-08-26 Sidney Point Lattice (integer lattice is a special case of point lattice) notes from Wolfram: "A point lattice is a regularly spaced array of points." The integer lattice is where all of these points are integers. I will look at the Barvinok lecture tomorrow, I have to finish moving to a different house today. (Stayed on page 7)

Q3 Please correct me if I am wrong, but a lattice seems to be a collection of points where are all regularly spaced, so does "regularly" mean that it is controlled by a deterministic law? If this is the case, the ϕ_n states in a periodic orbit can be grouped as a lattice and ordered by location along the periodic orbit, then the associated "winding" number m_t can be grouped in its own lattice, which in this case is an integer lattice. What is the "regular" spacing for the winding numbers? Have missed the point?

A3 Wolfram is right. When you have a discrete time map, time takes integer values $t = \cdots, -1, 0, 1, 2, \cdots$. That is called 1-dimensional integer lattice $\mathbb{Z}$. Once you are in $d = 2$ or higher, the name makes

sense, as you can visualize $\mathbb{Z}^2$ as a 'lattice'. It is regular, because all spacings between neighboring points are 1. There is nothing 'deterministic' about this, it just says that time takes its values on integers, rather than on a continuum.

There is only one lattice, but on each lattice site there is a real-valued field ϕ_t and the integer valued 'source' m_t.

2020-08-27 Sidney Thank you for A1, that makes complete sense now. Calculated the orbit Jacobian matrix using equation CL18 eq. tempBernFix, matched with the paper, yay. Orbit Jacobian matrix maps the basis vectors of the unit hyper-cube into a fundamental parallelepiped basis vectors, each of which is given by a column in the orbit Jacobian matrix. $|\mathrm{Det}\,(s/r| = s^n$ because r and its inverse are both unitary matrices, and if you multiply every row of an $[n \times n]$ matrix, the determinant is multiplied by the constant raised to the power n. Periodicity $r^n = 1$ accounts for $\overline{0}$ and $\overline{s-1}$ fixed points being a single periodic point. (got to page 9)

Q4 I was trying to calculate the orbit Jacobian matrix using the r matrix, but the delta function equation CL18 eq. hopMatrix for r doesn't seem to work for the Bernoulli map, I know that $r_{2,1} = 1$ and $r_{1,2} = 1$ which works with the delta function definition. However, $r_{2,1} = \delta_{3,1}$ from CL18 eq. hopMatrix, which should equal zero. Other than just the idea of being cyclic, I don't know why it yields one instead of zero, what am I missing?

A4 Work it out r matrices for $n = 1, 2, 3, \cdots$. It will start making sense.

Q5 So, does "lattice state" mean the set of all points (field of all points?) which running through the Bernoulli map requires the specific winding number at that lattice site?

A5 Interesting, grad students too seem to confuse coordinates (for example, $(x, t) = (3.74, -0.02)$ in continuum, $(n, t) = (7, -6)$ on a discrtized space) and the fields $\phi(n, t)$. Physical "state" refers to value of field ϕ over every (n, t) - is the grass high or low? rather than the coordinates of spacetime.

How would you state this precisely if you were trying to explain this paper to another student?

2020-08-30 Sidney

A5.1 Sidney: " $\mathsf{X_M}$ is the set of all values the field ϕ_z takes over the set of coordinates M. "

A5.2 Predrag: Please reread 2nd paragraph of CL18 sect. s:1D1dLatt and explain what is wrong with your answer A5.1

Notes: For an period-n lattice state $\mathsf{X_M}$ the Jacobian matrix is now a function of a [d x d] matrix J, so the formula for the number of periodic points

of period n (number of lattice states of period n) is now $|\det(1 - J_M)|$ where $J_M = \prod_{t=1}^{n} J_t$ where J_t is the one-step Jacobian matrix which is assumed to vary in time.

Note to self: look back over the topological zeta function, specifically try to understand derivation of:

$$\frac{1}{\zeta_{top}(z)} = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} N_n\right)$$

(got to CL18 page s:bernODE)

Predrag: ChaosBook (click here)

Q6 Is "there are s fundamental lattice states, and every other lattice state is built from their concatenations and repeats" is simply a restatement of the fact that the Bernoulli map is a full shift?

A6 For Bernoulli, yes. But search for word 'fundamental' in ChaosBook *Counting*. For example, 'We refer to the set of all non-self-intersecting loops $\{t_{p_1}, t_{p_2}, \cdots t_{p_f}\}$ as the *fundamental cycles*'. Write up here a more nuanced statement of 'fundamental' cycles might be (I do not have firm grip on this either...).

Q7 Is CL18 eq. bernN_n-s=2 a result of expanding in a Taylor the result of the derivative (and product of $1/\zeta_{top}$ and z)? Because the topological zeta function of the Bernoulli map is a closed form function, not an infinite sum.

2020-08-31 Sidney Via a finite difference method, CL18 eq. 1stepDiffEq can be viewed as a first order ODE dynamical system. Back-substituted with (**??**) to show that with $\Delta t = 1$ the velocity field does satisfy the diffeq (**??**). The Bernoulli system can be recast into a discretized ODE whose global linear stability is described by the orbit Jacobian matrix. (Stayed on CL18 page s:bernODE))

2020-09-01 Sidney Started reading CL18 sect. s:kickRot *A kicked rotor*.

(**??**) and (**??**) describe the motion of a rotor being subjected to periodic momentum pulses. The mod is present for the q equation to make sure that the angle varies from 0 to 2π. As in the Bernoulli map case, here mod is also added to the momentum equation to keep it bounded to a unit square. Cat maps with the stretching parameter s are the same up to a similarity transformation. An automorphism is an isomorphism of a system of objects onto itself. An isomorphism is a map that preserves sets and relations among elements.

Q8 Do the kicked rotor equations with Hooke's law force, and bounded momentum (mod 1 added to CL18 eq. PerViv2.1a) only take the form of CL18 eq. catMap if K is an integer?

A8 The text states: "The (mod 1) added to CL18 eq. PerViv2.1a makes the map a discontinuous 'sawtooth,' unless K is an integer." How would you make that clearer?

Q9 How does CL18 eq. catMap have a state space which is a 2-torus? I am having a hard time visualizing how this came about.

A9 Do you understand how (mod 1) operation turns unbounded stretch CL18 eq. BerStretch into a circle map CL18 eq. n-tuplingMap? Circle map is 1-torus. If both $(q_t, p_t) \in (0, 1] \times (0, 1]$ are wrapped into unit circles, the phase space (q_t, p_t) is not an infinite 2-dimensional plane, but a compact, doubly periodic unit square with opposite edges glued together, i.e., 2-torus.

2020-09-03 Sidney I was typing my description into "summary" textbox above the commit to master button. Obviously I was incorrect, I'll try to type in the "description" for this commit.

2020-09-02 Predrag "Tripping Through Fields" showed up :)

2020-09-03 Sidney

A5.3 Sidney: I'm not actually quite sure what's wrong with my given definition. From your answer A5 it seems that M is a set of coordinates (the location of the blade of grass) and X_M is the value at that coordinate (the height of the grass at that point). Perhaps I forgot that these lattice states are for periodic orbits, so I forgot the second coordinate (period of length n).

A5.4 Predrag: The textbook inhomogeneous *Helmoltz equation* is an elliptical equation of form

$$(\Box + k^2)\,\phi(z) = -m(z)\,, \qquad z \in \mathbb{R}^d\,, \tag{1.1}$$

where the *field* $\phi(z)$ is a C^2 functions of *coordinates* z, and $m(z)$ are *sources*. For example, charge density is a *source* of electrostatic *field*.

Suppose you are so poor, your computer lacks infinite memory, you only have miserly only 10 Tb, so you cannot store the infinitely many values that *coordinates* $z \in \mathbb{R}^d$ take. So what do you do?

Perhaps a peak at ChaosBook ChaosBook A24.1 Lattice derivatives can serve as an inspiration. And once you have done what a person must do, your Helmoltz equation (hopefully) has the form of CL18 eq. OneCat. What is a *field*, a *source*, a *coordinate* then?

2020-09-03 Sidney

A8.1 The sawtooth statement made sense, what made it unclear for me was the second sentence which started with "in this case" it was (again for me, I might not have been paying enough attention) ambiguous, I didn't know if it was talking about the integer case or the sawtooth case.

A8.2 Predrag: thanks, I rephrased that sentence.

A9.1 I understand, your explanation makes sense, thank you :).

Notes: The discrete time Hamiltonian system induces forward in time evolution on the 2-torus phase space. The orbit Jacobian matrix can take many different forms depending on the map. Despite this the Hill determinant can still count the number of lattice states. (got to page CL18 page s:tempCatCountTEMP)

2020-09-05 Sidney

A5.5 If I was so unlucky to only have 10Tb of memory, I would take a finite interval of points z that I was interested in, and discretize them (evenly, or unevenly) and then evaluate the field (that was probably the wrong wording) at a finite set of points, either of particular interest within the interval, or closely spaced enough so that the values were representative of the values the field took over a continuum. I think that a coordinate is a point in state space specified by specific values of state variables (position, time, momentum etc.). To try to answer source, and field, I'll be thinking of an electric charge, a source is what generates the medium by which other sources are effected, and the field is the medium which acts upon other sources.

A5.6 I did look at ChaosBook A24.1 Lattice derivatives, but it didn't seem to address quite the fundamental confusion I seem to be facing. I'm relatively confident in my coordinate definition, but not at all in my source, and field definition.

2020-09-05 Predrag .

A5.7 Expression ' state space' $\mathcal{M}$ refers to 'states': cats, dancers, i.e., 'fields' X and their names M. 'Coordinates' refer to markings on the floor that they stand on.

Does reading CL18 sect. s:lattState *Lattice states* now helps in distinguish a skater from the skating ring's ice? I rewrote it for your pleasure :)

A5.7 Re. the fundamental confusion of reading ChaosBook A24.1 Lattice derivatives: If you mark every inch on the floor, this is 'discretization'. But the floor is still a floor, no?

2020-09-05 Sidney

A5.8 I read the pink bits of CL18 sect. s:lattState *Lattice states* (as I assume that was the parts that you rewrote specially). From it I (think) I understand. We're looking at two coordinates for most of the Bernoulli and cat map stuff: a spatial one, and a temporal one, the maps only effect the temporal placement, but effect it differently depending on

where the point was in space when the map acted on it, because the field takes a different value at every point in space (and time). So the coordinates are the field point placement in time and space. The field is the value that is assigned to every lattice point. M keeps getting referred to as an alphabet, so that makes me think that it is similar (perhaps the multidimensional generalization) to the "alphabet" which was used to partition state space in the 1D maps of Chaosbook, such as 0 for the left half of the interval and 1 for the right, and then further partitioning the more the map is applied. Is that close at least?

2020-09-05 Predrag .

A5.9 Getting hotter. Look at CL18 eq. circ-m and CL18 eq. catMapNewt; ϕ_t and m_t are the same kind of a beast, m_t is just the integer part of the "stretched" field in CL18 eq. BerStretch. In this particular, linear map setting, this integer does double duty, as a letter of an "alphabet". It cannot possibly be a "coordinate", it like saying that a dancer's head is "floor."

A5.10 In temporal lattice formulation no "map is applied." That is the brilliance of the global spatiotemporal reformulation: there is no stepping forward in time, so there is no map - the only thing that exists is the global fixed point condition that has to be satisfied by field values everywhere on the lattice, simultaneously.

Time is dead.

2020-09-08 Sidney

Q11 So the temporal cat / spatiotemporal cat equations are moving around points in the lattice instead of through time?

Q12 Is something of the form of CL18 eq. tempFixPoint an example of the "global fixed point condition"?

2020-09-14 Predrag .

A11 An equation does not have to be "moving around" anything: think of a quadratic equation $x^2 + bx + c = 0$. Does it "move" anything? No. It's a condition that a single "field" x has to satisfy, and the solution is a root of that equation. The temporal cat / spatiotemporal cat equations are "equations" in the same sense, [bunch of terms involving ϕ_z]=0.

A12 Yes.

2020-09-09 Sidney Notes: Equations such as CL18 eq. catMapNewt can be solved using similar methods to linear odes: guessing a solution of the form Λ^t and finding the characteristic equation. Then assuming all terms are site independent because the difference of any two solutions of CL18

eq. catMapNewt solve its homogeneous counterpart CL18 eq. diffEqs:CatCharEq. Got to CL18 page s:tempCatZeta.

Notes: Topological zeta functions count orbits, i.e. time invariant sets of equivalent lattice states related by cyclic permutations. The "search for zeros" CL18 eq. tempCatFixPoint is the "fixed point condition." Which is a global statement which enforces CL18 eq. catMapNewt at every point in the lattice. Got to CL18 page s:catlatt

2020-09-13 Sidney The temporal cat is a special case of the spatiotemporal cat, defined on a one-dimensional lattice $\mathbb{Z}^1$. In this case the associated topological zeta function is known in a closed, analytic form.

Coupled map lattices: Starts with a review of finite difference methods for PDEs. The d dimensions in the lattice are d-1 spatial lattice points and 1 temporal one. The PDE is reduced to dynamics of a coupled map lattice, with a set of continuous fields on each site.

A5.11 I have experience with finite difference methods for solving a discretized form of a PDE, but I'm having a hard time visualizing the idea of having a discrete coordinate system in d different directions, but with a continuous field on each site. This may be valuable as it is a specific statement of where I'm getting stuck.

Q13 My current understanding is that at each point in the d-dimensional integer lattice ("point" as in a lattice node with d specified coordinates), but at each point (site) there is a continuous field. What is this field continuous over? It's at one point in a discrete coordinate system. And why is there a continuum at each point? And finally, I assume that these continuous fields are the values of the function being solved for at that point, however, shouldn't that just be a single value? Not a field? I'm sorry if this is a rather silly question, but I'll keep thinking about it and I'll make a note if my understanding (or lack thereof) changes.

A13 Predrag: In CL18 figure **??** field x_t or ϕ_t and $f(\phi_t)$ on the discrete site t run over continuous values. For example, at temporal lattice site $t = 7$ the field value is $\phi_7 = 0.374569263952942\cdots$. OK now?

Q13.1 Slight update, it seems that the field is the state of the system and at each discretized point there is a map acting on the state, although that conflicts with the notion that time is dead, so I'm probably still misunderstanding.

A13.1 Predrag: Yes.

Thinking of this as a spring mattress. Often starts out with chaotic on-site dynamics weakly coupled to neighboring sites. In this paper one sets the lattice spacing constant equal to one. Diffusive coupled map lattices introduced by Kaneko:

$$\phi_{n,t+1} = g(\phi_{n,t}) + \epsilon\left[g(\phi_{n-1,t}) - 2g(\phi_{n,t}) + g(\phi_{n+1,t})\right] ,$$

where each individual spatial site's dynamical system $g(x)$ is a 1D map, coupled to the nearest neighbors by the discretized second order *spatial* derivative. The form of time-step map $g(\phi_{n,t})$ is the same for all time i.e. invariant under the group of discrete time translations. Spatial stability analysis can be combined with temporal stability analysis, with orbit weights depending exponentially both on the space and the time variables: $t_p \propto e^{-LT\lambda_p}$. r_i translates the field by one lattice spacing in the i^{th} direction.

Q14 What is a lattice period?

A14 Predrag: Does the paragraph above CL18 eq. catlattFix answer you question? I would like to refer to the *set* of numbers $\{\ell_1, \ell_2, \cdots, \ell_d\}$ as the *period* of lattice Λ. Would that be confusing?

Q15 Is z in the definition of a lattice state both a temporal and a spatial index? So equivalent to both n and t?

A15 Predrag: after CL18 eq. CatMap2d I write "a 1-dimensional spatial lattice, with field ϕ_{nt} (the angle of a kicked rotor (**??**) at instant t) at spatiotemporal site $z = (n, t) \in \mathbb{Z}^2$." Should this "$z = (n, t) \in \mathbb{Z}^2$" be repeated elsewhere. If so, where?

Q16 Often a member of the alphabet can be a negative number, which I assume means that the state is taken out of unity in the negative direction.

A16 Do you understand CL18 figure fig:BernCyc2Jacob and CL18 figure fig:catCycJacob?

The spatiotemporal cat has the point-group symmetries of the square lattice. A lattice state is a set of all field values $\mathsf{X} = \{\phi_z\}$ over the d-dimensional lattice that satisfies the spatiotemporal cat equation, with all field values constrained between zero and one. A lattice state X_Λ is a *invariant 2-torus* if it satisfies $\mathsf{X}_\Lambda(z+R) = \mathsf{X}_\Lambda(z)$ for any discrete translation $R = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 \in \Lambda$. Got to CL18 page s:catLatt1x1.

2020-09-14 Sidney

A13.1 I think I'm OK now. I think what I was trying to visualize was a stack of an infinite number of values at each lattice point, which was confusing, but this makes sense.

A14.1 Unfortunately I don't think I quite understand. I understand the idea of the different directions, I understand treating $\mathsf{X}_\mathsf{M}(\phi_z)$ as a singular fixed point, but I do not understand ℓ_i.

A15.1 I think that I lost that definition of z around CL18 eq. dDCatsT, but I think that may have been a factor of how long it takes me personally to digest this material.

A16.1 After reading the descriptions and staring at it for awhile, I think that I do.

Q17 I tried a couple days back (Thursday or Friday I think, they all blend together) to log in to your bluejeans office. But it must have been one of the times that it had logged you off due to inactivity. There was also another person their I didn't recognize, and I didn't want to step on their toes if they were waiting for you to get back, so I logged off. So, when in general would good times to try hopping into your office?

2020-09-16 Sidney A Bravais lattice can be denoted $\Lambda = [L \times T]_S$ where L is the spatial lattice period, T is the temporal lattice period, S imposes the tilt to the cell. Basis vectors for the Bravais cell can be written as:

$$\mathbf{a}_1 = \begin{pmatrix} L \\ 0 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} S \\ T \end{pmatrix}$$

Q18 If something is written as $850[3 \times 2]_0$ what is the numerical value? More importantly, how is it found? I know it has to do with the cyclic permutations of the prime blocks, but I'm not sure how to get a numerical value.

Got to page CL18 page s:catLattCount

2020-09-17 Sidney For the Bernoulli map its stretching uniformity allows the use of combinatorial methods for lattice points. For temporal (not spatiotemporal) the number of lattice states is the same as the volume of the fundamental parallelepiped, so the magnitude of the determinant of the orbit Jacobian matrix. The block M can be used as a 2D symbolic representation of the lattice system state. For a given admissible source block M, the periodic field can be computed by:

$$\phi_{i_1 j_1} = \sum_{i_2=0}^{2} \sum_{j_2=0}^{1} \mathbf{g}_{i_1 j_1, i_2 j_2} \mathsf{M}_{i_2 j_2}$$

2020-09-19 Predrag Sorry, I've been a bit overwhelmed with lecture preparations, so I will not answer any of the questions quite yet. But I have rewritten the abstract, and the introduction to the paper, up to the start of CL18 sect. s:Bernoulli *Bernoulli map*. Can you have a critical look at the new text, report here if something does not make sense to you?

2020-09-19 Sidney

Update I read through, and aside from some very minor grammar issues (forgetting a "have" after "we") it all makes sense.

2020-09-20 Predrag .

A15.2 I now added the z definition to CL18 eq. dDCatsT, is that clearer?

2020-09-20 Sidney

A15.3 Yes, that makes it clearer.

$$-\sum_{r=1}^{\infty}\frac{1}{r}\mathrm{tr}\,\hat{\mathbf{J}}_p^r = \mathrm{tr}\,\left(-\sum_{r=1}^{\infty}\frac{1}{r}\hat{\mathbf{J}}_p^r\right) = \mathrm{tr}\quad \ln\left(\hat{1}_1-\hat{\mathbf{J}}_p^r\right) = \ln\quad \det\,\left(\hat{1}_1-\hat{\mathbf{J}}_p^r\right)$$

I liked the text cut from the introduction on page 44, it made the idea of time's death more easily digestible. Finished main paper, will look at the appendices for math.

2020-09-22 Sidney

Math Review Part 1

Updated 9/29/20

Bravais Lattice From Wikipedia: A Bravais lattice is an infinite array of discrete points generated by a set of discrete translation operations described in two dimensional space by:

$$\mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2$$

where n_i is any integer and $\mathbf{a}_i$ is a primitive vector, each $\mathbf{a}_i$ lie in different directions, but are not necessarily mutually perpendicular, but they do span the lattice. A fundamental aspect of a Bravais Lattice is that no matter the direction of the primitive vectors, the lattice will look exactly the same from each of the discrete lattice points when looking in that direction. A Lattice is is a periodic array of points where each point is indistinguishable from any other point and has identical surroundings. A unit cell expands the idea of the infinite array of discrete points to include the space inbetween the points, if we are looking at a physical system this includes the atoms in this space. There are two main types of unit cells: primitive unit cells and non-primitive unit cells. A unit cell is the smallest group of atoms of a substance that has the overall symmetry of a crystal of that substance, and from which the entire lattice can be built up by the repetition in three dimensions. A primitive cell must contain only one lattice point, generally, lattice points that are shared by n cells are counted as $\frac{1}{n}$ of the lattice points contained in each of those cells. So traditional primitive cells only contain points at their corners. The most obvious way to form a primitive cell is to use the basis vectors which the lattice is constructed from:

$$C(\mathbf{a}_1,\mathbf{a}_2) = \mathbf{r} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

$$0 \le x_i \le 1$$

The scaling factors are to ensure that lattice points are placed on the corners of the cell. In the current paper the primitive unit cell of a d-dimensional Bravais lattice tiles the spacetime. $C(\mathbf{a}_1,\mathbf{a}_2)$ is the Bravais cell of a Bravais Lattice spanned by basis vectors $(\mathbf{a}_1,\mathbf{a}_2)$. A

given Bravais Lattice Λ can be defined by an infinity of Bravais cells. Hermite normal form: the analogue of reduced echelon form for matrices over $\mathbb{Z}^n$. Each family of Bravais cells contains a unique cell of the Hermite normal form, this can be written in terms of L,T, and S, where L, and T are respectively the spatial, and temporal lattice periods, S is the "tilt" of the cell. Hence the lattice can be defined as $[L \times T]_S$.

Prime Bravais Lattices It may be possible to tile a given Bravais lattice Λ by a finer lattice Λ_p. A Bravais lattice is prime if there is no finer Bravais cell, other than the unit volume $[1 \times 1]_0$ that can tile it. If $det\Lambda$ is a prime number, then Λ is a *prime matrix*. If Λ is neither prime nor unimodular (a square integer matrix having determinant of ± 1), it is composit can can be decomposed into a product of two non-unimodular matrices $\Lambda = PQ$. In order to determine all prime lattices Λ_p that tiles a given Bravais lattice Λ:

$$\mathbf{a}_1 = k\mathbf{a}_1^p + l\mathbf{a}_2^p$$

$$\mathbf{a}_2 = m\mathbf{a}_1^p + n\mathbf{a}_2^p$$

observe that a prime tile $(\mathbf{a}_1^p, \mathbf{a}_2^p)$ tiles the large tile only if the larger tile's width L is a multiple of L_p, and the height T is a multiple of T_p, and the two tile "tilts" satisfy:

$$\mathbf{a}_2 = m\mathbf{a}_1^p + \frac{T}{T_p}\mathbf{a}_2^p \rightarrow S = mL_p + \frac{T}{T_p}S_p$$

A prime lattice only tiles the given lattice if the area spanned by the two tilted basis vectors:

$$\mathbf{a}_2 \times \mathbf{a}_2^p = ST_p - TS_p$$

is a multiple of the prime tile area L_pT_p. A lattice state is a set of all field values $\mathsf{X} = \{\phi_z\}$ over the d-dimensional lattice $z \in \mathbb{Z}$ that satisfies the spatiotemporal cat equation. Lattice state X is a periodic orbit if $\mathsf{X}(z + R) = \mathsf{X}(z)$ for any discrete translation $R = n_1\mathbf{a}_1 + n_2\mathbf{a}_2$. If a given periodic orbit over lattice Λ is not periodic under translations $R \in \Lambda_p$ for any sublattice Λ_p (except for Λ itself) we shall refer to it as an orbit: a lattice state of smallest periodicity in all spacetime directions.

Shift Operator Shift operator is a matrix: $r_{ij} = \delta_{i+1,j}$, this along with a periodic boundary condition assuming $[n \times n]$ matrix $r^n = I$ yields

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

A lattice state is a vector with all the values that the field takes on at each point on the lattice. Shift operator is cyclic permutation of a lattice state, changes only the coordinates of the lattice state.

$$r\mathsf{X} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_0 \end{bmatrix}$$

$r^T = r^{-1}$ cyclic permutation in the opposite direction, does not destroy anything, only changes the coordinates.

Lattice Derivatives Hypercube in d-dimensions with unit sides. Each side is described by a unit vector in direction μ $\hat{n}_\mu \in \{\hat{n}_1, \hat{n}_2, \hat{n}_3, \cdots, \hat{n}_d\}$ unit lattice cell, points along μ'th direction.

Forward Lattice Derivative (a is lattice spacing):

$$(\partial_\mu \phi)_l = \frac{\phi(x + a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{l+\hat{n}_\mu} - \phi_l}{a}$$

Backward Lattice Derivative (transpose of forward lattice derivative):

$$(\partial_\mu \phi)^T = \frac{\phi(x - a\hat{n}_\mu) - \phi(x)}{a} = \frac{\phi_{l-\hat{n}_\mu} - \phi_l}{a}$$

Lattice Discretization, Lattice State Divide interval of separation a creating a discrete coordinate system. At each point read off the value of the continuous counterpart. Field has a constant value over the interval. Lattice is a coordinate, set of points, the values of the field at each lattice point is a lattice state.

field $\phi = \phi(x) \quad x = al \quad l \in \mathbb{Z}$

Lattice State $\phi = \{\phi_0, \phi_1, \phi_2, \cdots, \phi_{n-1}\}$ "configuration".

N-Site Periodic Lattice After N steps, back

$$r^N = I$$

eigenvalues=$\omega = e^{\frac{i2\pi}{N}}$

$$r^N - I = \prod_{k=0}^{N-1} (r - \omega^k I)$$

N distinct eigenvectors, N-dim space (N irrep)

N projection operators

$$P_k = \prod_{j=k} \frac{r - \omega^j I}{\omega^k - \omega^j}$$

Discrete Fourier Transforms Have a lattice state $\phi = \{\phi_0, \phi_1, \cdots, \phi_{N-1}\}$ Kth Fourier Coeff=projection of ϕ onto eigen vector φ

$$\tilde{\phi}_k = \varpi_k^\dagger \cdot \phi = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{-\frac{i2\pi}{N} kl} \phi_l$$

Q19 I think I may have gotten to the point where I can go beyond exclusively reading the paper, what should I do beyond? As well, what times would be good for me to drop in on your Bluejeans office during the week?

Q20 I believe I've asked this before, or a form of it, but it seems that the periodic boundary condition is in direct conflict with the definition of the shift operator. Am I missing something?

2020-10-15 Sidney A reread.

The Bernoulli Shift map is a circle map due to the mod 1 operation for $[1/s, 1)$ where s is the "stretching parameter" of the general Bernoulli map: $\phi_{t+1} = s\phi_t \pmod 1$. $(\mod 1)$ subtracts the integer part of $s\phi_t$ yielding the "winding number" m_{t+1}. This keeps ϕ_{t+1} in the unit interval, and divides this interval into s subintervals. The winding number is also the alphabet of the system, denoting at time t, it visits interval m. Brief note from Chaosbook: we can represent a state as a base s decimal of the resulting visitation sequence: $\phi_0 = .m1m2m3\cdots$. The Bernoulli map operates on a state by shifting this itinerary over by one:$\phi_0 = .m_1m_2m_3\cdots \to \phi_1 = .m_2m_3\cdots$. The preimages of critical points (the point which when input into the map yield a maximum value on in the map) partition the map into s^n subintervals, where n is the orbit length. There is no pruning in the Bernoulli map, as its critical points are all unity, however, as it is a circle map the first and last fixed point (rightmost fixed point, and the fixed point at the origin) are the same, so they are counted as one fixed point, and thus the number of periodic orbits is $N_n = s^n - 1$. There can only be one periodic orbit per subinterval because each subinterval is treated as a single point where a certain orbit is possible, thus, there can only be one orbit. For the temporal Bernoulli, 'Temporal' here refers to the state (field) ϕ_t and the winding number m_t (source) taking their values on the lattice sites of a 1-dimensional temporal lattice $t \in \mathbb{Z}$. Over a finite lattice segment they can be written as a state, and a symbol block. The Bernoulli equation can be written as a first order difference equation $\phi_t - s\phi_{t-1} = -m_t$ where phi is contained within the unit interval. This is the condition which each point on the lattice must fulfill. This can then be written in terms of the orbit Jacobian matrix, which is a sum of the identity and cyclic permutation matrix which has the condition $r^n = I$. This permutation permutates forward in time the lattice state by one site. The temporal Bernoulli condition can be viewed as a search for zeros of the function involving the orbit Jacobian matrix operating on the lattice state

summed with the symbol block $\mathbf{M}$. This allows the entire lattice state which solves for zero X_M to be treated as a single fixed point. The orbit Jacobian matrix stretches the unit hyper cube such that every periodic point is mapped onto an integer lattice $\mathbb{Z}^n$ site, which is then translated by the winding numbers into the origin to satisfy the fixed point condition. Therefore N_n the number of solutions to the fixed point condition is the number of lattice sites within the fundamental parallelepiped (fp), which is equivalent to the volume of the fp because each unit cell in the lattice only contains one lattic4 e point. So N_n is the magnitude of the determinant of the orbit Jacobian matrix, this is called Hill's Determinant, or the Fundamental Fact. The orbit Jacobian matrix maps the unit hyper cube into the basis vectors of the fundamental parallelepiped which are given by columns of the orbit Jacobian matrix.

2020-10-18 Sidney My notes on Barvinok **[Barvinok04]** *Lecture notes*

The theory discussed in these lectures are inspired by a few series formulas, the first being:

$$\sum_{m=1}^{n} x^m = \frac{1-x^{n+1}}{1-x}$$

We take the interval $[0, n)$ and for every integer point in the interval we write the monomial x^m and then take the sum over each integer point on the interval. It gives a polynomial with n+1 terms, but can be written in the form given, later we will cover doing the same over a 2D plane (evaluating at each integer point on the plane and summing over every integer point $\mathbf{m} = (m_1, m_2)$ with bivariate monomials $\mathbf{x^m} = x^{m_1}x^{m_2}$. The second formula is the infinite geometric series:

$$\sum_{m}^{\infty} x^m = \frac{1}{1-x}$$

This makes sense if $|x| < 1$ similarly

$$\sum_{-\infty}^{0} x^m = \frac{1}{1-x^{-1}} = \frac{-x}{1-x}$$

This converges if $|x| > 1$

$$\sum_{m=-\infty}^{\infty} x^m$$

This converges for no values, so we will say that it equals zero, this can be reasoned through as every positive integer added to every negative integer is zero, we then subtract zero, as it was double counted:

$$\sum_{m=-\infty}^{\infty} = \sum_{m=0}^{\infty} x^m + \sum_{m=-\infty}^{0} x^m - x^0 = 0$$

This sugestively agrees with

$$\frac{-x}{1-x} + \frac{1}{1-x} - 1 = 0$$

Geometrically, the real line $\mathbb{R}^1$ is divided into two unbounded rays intersecting in a point. For every region (the two rays, the line and the point), we construct a rational so that the sum of x^m over the lattice points in the region converges to that rational function, if it converges at all.

2020-10-19 Sidney

Inclusion-exclusion principle

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{1.2}$$

where $|A \cap B|$ is the number of elements which are in both A and B. This avoids double counting.

If we think of a plane of points, we can draw lines which subdivide the plane, each line makes the plane two half planes, and every two lines form four angles, this forms several regions. Among these regions there are regions $\mathcal{R}$ where the sum:

$$\sum_{m \in \mathcal{R} \cap \mathbb{Z}^2} \mathbf{x^m}$$

converges for some $\mathbf{x}$, and some regions where the sum will never converge.

We shall show that it is possible to assign a rational function to every region simultaneously so that each series converges to the corresponding rational function, if it converges at all, it will also satisfy the inclusion-exclusion principle.

Definition 1 The scalar product in $\mathbb{R}^d$ is

$$\sum_{i=1}^{d} x_i y_i$$

for $x = (x_1, \cdots, x_d)$ and $y = (y_1, \cdots, y_d)$, and the same for $\mathbb{Z}^d \subset \mathbb{R}^d$.

Definition Polyhedron P is the set of solutions to finitely many linear inequalities:

$$P = \left\{ \phi \in \mathbb{R}^d : \quad \sum_{i=1}^{d} a_{ij} x_j \leq b_i \right\}$$

If all a_{ij} and b_j are integers the polyhedron is rational.

Barvinok notes concern themselves with the set $P \cap \mathbb{Z}^d$ of integer points in a rational polyhedron P. He introduces the algebra of polyhedra to account for all relations among polyhedra.

2020-11-29 Predrag We only need to understand parallelepipeds, not polyhedra in general. Should be easier.

2020-11-30 Sidney I understand your comments. Thank you, I am pretty sure that the general polyhedra stuff can be put in terms of parallelpipeds, so at least it wasn't wasted knowledge, but I'm glad that I don't need to know all of it, it's on the edge of my proof abilities.

2020-11-29 Predrag

Definition 2 For a set $\mathcal{B} \in \mathbb{R}^d$, the function

$$[\mathcal{B}](\phi) = \begin{cases} 1 & \text{if } \phi \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

is called the *indicator* of $\mathcal{B}$.

2020-10-24 Sidney

The intersection of finitely many (rational) polyhedra is a (rational) polyhedron. The union doesn't have to be, but may be a polyhedron.

Union: The union of a collection of sets is the set of all elements in the collection.

Intersection: $A \cap B$, is the intersection of two sets A and B, i.e., the set containing all elements of A that also belong to B.

The algebra of rational polyhedra is the vector space $\mathcal{P}\left(\mathbb{Q}^d\right)$ spanned by the indicators $[P]$ for all rational polyhedra $P \subset \mathbb{R}^d$

2020-11-29 Predrag $\mathbb{Q}$ is the field of *rationals*.

2020-10-24 Sidney

Valuations

Let V be a vector space. A linear transformation $\mathcal{P}\left(\mathbb{Q}^d\right) \to V$ is called a valuation. This course is on the particular valuation $\mathcal{P}\left(\mathbb{Q}^d\right) \to \mathbb{C}(x_1, \cdots, x_d)$, where $\mathbb{C}(x_1, \cdots, x_d)$ is the space of d-variate rational functions.

Theorem 1 There exists a unique valuation $\chi : \mathcal{P}\left(\mathbb{R}^d\right) \to \mathbb{R}$ called the Euler characteristic, such that $\chi([P]) = 1$ for any non-empty polyhedron $P \subset \mathbb{R}^d$

2020-11-30 Predrag Klain and Rota **[KlaRot97]** *Introduction to Geometric Probability*:

A *valuation* on a lattice L of sets is a function μ defined on L that takes real values, and that satisfies the following conditions:

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)\,, \tag{1.4}$$

$$\mu(\emptyset) = 0\,, \tag{1.5}$$

where $\emptyset$ is the empty set. By iterating the identity (1.4) we obtain the inclusion-exclusion principle for a valuation μ on a lattice L, namely

$$\begin{aligned}\mu(A_1 \cup A_2 \cup \ldots \cup A_n) &= \sum_i \mu(A_i) - \sum_{i<j} \mu(A_i \cap A_j) \\ &\quad + \sum_{i<j<k} \mu(A_i \cap A_j \cap A_k) + \ldots \end{aligned} \tag{1.6}$$

for each positive integer n.

Barvinok **[Barvinok04]** Lecture 1, Problem 1 statement of (1.6) is less intelligible: Take sets $A_1, A_2, \cdots, A_n \in \mathbb{R}^d$. The inclusion-exclusion formula is

$$\cup A_i = \sum_I (-1)^{|I|-1} \left[\cap_{i\in I} A_i\right]\,, \tag{1.7}$$

where the sum is taken over all non-empty subsets $I \subset \{1, \ldots, n\}$ and $|I|$ is the cardinality of I.

2020-10-25 Sidney

Identities in the Algebra of Polyhedra

The image of a polyhedron under a linear transformation is a polyhedron.

Theorem 1 Let $P \subset \mathbb{R}^d$ be a polyhedron and let T:$\mathbb{R}^d \to \mathbb{R}^k$ be a linear transformation. Then $T(P) \subset \mathbb{R}^k$ is a polyhedron. Furthermore, if P is a rational polyhedron and T is a rational linear transformation (that is, the matrix of T is rational), then $T(P)$ is a rational polyhedron.

Linear transformations preserve linear relations among indicators of polyhedra.

Theorem 2 Let T:$\mathbb{R}^d \to \mathbb{R}^k$ be a linear transformation. Then there exists a linear transformation $T : P(\mathbb{R}^d) \to P(\mathbb{R}^k)$ such that $T(P) = [T(P)]$ for every polyhedron $P \subset \mathbb{R}^d$.

Most sensible polyhedra have vertices, but some don't.

Definition 1 Let $P \subset \mathbb{R}^d$ be a polyhedron. A point $v \in P$ is called a vertex of P if whenever $v = (x + y)/2$ for some $x, y \in P$, we must have $x = y = v$.

If v is a point in P, we define the tangent cone of P at v as:

$$co(P,v) = \left\{x \in \mathbb{R}^d : \quad \epsilon x + (1-\epsilon)v \in P \quad \text{for all sufficiently small} \quad \epsilon > 0\right\}$$

Not all polyhedra have vertices. In fact, a non-empty polyhedron has a vertex if and only if it does not contain a line.

Definition 2 We say that a polyhedron P contains a line if there are points x and y such that $y \neq 0$ and $x+ty \in P$ for all $t \in \mathcal{R}$. $P_0(\mathbb{R}^d) \subset P(\mathbb{R}^d)$ is the subspace spanned by the indicators of rational polyhedra that contain lines.

Theorem 3 Let $P \subset \mathbb{R}^d$ be a polyhedron. Then there is a $g \in P_0(\mathbb{R}^d)$ such that

$$[P] = g + \sum_v [co(P,v)]\,,$$

where the sum is taken over all vertices v of P. If P is a rational polytope then we can choose $g \in P_0(\mathbb{Q}^d)$

Definition 3-0 A polytope is a high dimensional generalization of a polyhedron.

Definition 3 Let $A \subset \mathbb{R}^d$ be a non-empty set. the set

$$A^o = \left\{y \in \mathbb{R}^d :< x, y >\leq 1 \quad \text{for all} \quad x \in A\right\}$$

is called the polar of A, where $< x, y >$ is the inner product.

2020-10-27 Sidney

A set S in a vector space over $\mathbb{R}^d$ is convex, if the line segment connecting any two points in S lies entirely within S. If P is a rational polyhedron then P^o is also a rational polyhedron.

Theorem 4 There exists a linear transformation $D : P(\mathbb{Q}^d) \to P(\mathbb{Q}^d)$ such that $D[P] = [P^o]$ for every non-empty polyhedron P.

It follows from Theorem 4 that whenever we have a linear identity $\sum_{i=1}^m \alpha_p [P_i] = 0$ among the indicator functions of polyhedra, we have the same identity $\sum_{i=1}^m \alpha_p [P_i^0] = 0$ for the indicator functions of their polars.

Barvinok **[Barvinok04]** Lecture 3.
For an integer point $\mathbf{m} = (m_1, \cdots, m_d)$ we introduce the monomial $\mathbf{x^m} = x_1^{m_1} \cdots x_d^{m_d}$. Given a set $S \subset \mathbb{R}^d$, we consider the sum

$$f(S, \mathbf{x}) = \sum_{m \in S \cap \mathbb{Z}^d} \mathbf{x^m} \tag{1.8}$$

Our goal is to find a reasonably short expression for this sum as a rational function in $\mathbf{x}$.

Example 1 Let $\mathcal{R}_+^d$ be the non-negative orthant, that is the set of all points with all coordinates non-negative. We have

$$\sum_{m\in\mathbb{R}_+^d\cap\mathbb{Z}^d} \mathbf{x}^{\mathbf{m}} = \left(\sum_{m_1=0}^{\infty} x_1^{m_1}\right)\cdots\left(\sum_{m_1=0}^{\infty} x_1^{m_1}\right) = \prod_{i=1}^{d}\frac{1}{1-x_i}$$

provided that $|x_i| < 1$

Definition Let $u_1, \cdots, u_d \in \mathbb{Z}^d$ be linearly independent integer vectors. The simple rational cone generated by $u_1, \cdots, u_d$ is the set:

$$K = \left\{\sum_{i=1}^{d}\alpha u_i : \qquad \alpha_i \geq 0 \quad for \quad i = 1, \cdots, d\right\}$$

The fundamental parallelepiped of $u_1, \cdots, u_d$ is the set

$$\Pi = \left\{\sum_{i=1}^{d}\alpha u_i : \qquad 1 > \alpha_i \geq 0 \quad for \quad i = 1, \cdots, d\right\}$$

Theorem For a simple rational cone $K = K(u_1, \cdots, u_d)$ we have

$$f(K, \mathbf{x}) = \left(\sum_{m\in\Pi\cap\mathbb{Z}^d} \mathbf{x}^{\mathbf{m}}\right)\prod_{i=1}^{d}\frac{1}{1-\mathbf{x}^{u_i}}$$

Theorem The number of integer points in the fundamental parallelepiped is equal to its volume.

Sketch of Proof Let Λ be the set of all integer combinations of $u_1, \cdots, u_d$:

$$\Lambda = \left\{\sum_{i=1}^{d}\alpha_i u_i : \qquad \alpha_i \in \mathbb{Z} \quad for \quad i = 1, \cdots, d\right\}$$

Let us consider all translates $\Pi + u$ with $u \in \Lambda$. We claim that $\Pi + u$ can cover all $\mathbb{R}^d$ without overlapping, this can be extracted from the proof of theorem 1. Let us take a sufficiently large region $X \subset \mathbb{R}^d$ and let us count the number of integer point in X, the set is roughly covered by $volX/Vol\Pi$ translations of the parallelepiped, and each translation carries the same number of points hence we must have $|\Pi \cap \mathbb{Z}^d| = vol\Pi$.

Barvinok [Barvinok04] Lect. 3, Definition 2. Let $u_1, \cdots, u_d \in \mathbb{Z}^d$ be linearly independent vectors and let K be the simple cone generated by $u_1, \cdots, u_d$. We say that K is *unimodular* if the volume of the fundamental parallelepiped Π is 1. Equivalently, K is unimodular if the origin is the unique integer point in Π. Equivalently, (1.8) is of form

$$f(K, \mathbf{x}) = \prod_{i=1}^{d} \frac{1}{1 - x^{u_i}} \,. \tag{1.9}$$

2021-01-01 I have been looking at the flow conservation sum rule for the Hénon map, as of today I had my suspicions confirmed that the Hénon map is not flow conserving so, the sum rule will not go to 1. However, I am still investigating the relation between the orbit Jacobian matrix (the Hill matrix) and the local Jacobian J_M. To do this, I have been looking over the proof that was done with the cat map to show that $|\mathrm{Det}\, \mathcal{J}| = |\det(I - J_M)|$. I am hoping that I can find something similar to this identity for the Hénon map. I have been looking at section five of the cat paper to see if I can adapt anything. I am also going to look at the relaxation method for finding periodic orbits so I can start working within Orbithunter and finding periodic orbits.

Q21 Is XXX?

Q22 What exactly is meant by XXX?

2021-01-02 Sidney Here are my notes from section 5:

Kronecker product $A \otimes B$ A is $[n \times n]$ and B is $[m \times m]$

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}$$

for A, A′ $[n \times n]$ matrices and B,B′ $[m \times m]$

$$(A \otimes B)(A' \otimes B') = AA' \otimes BB'$$

$$tr(A \otimes B) = tr(A) tr(B)$$

$$\det(A \otimes B) = \det(A^m) \det(B^n)$$

the two $[mn \times mn]$ block matrices $A \otimes B$ and $A \otimes B$ are equivalent by a similarity transformation

$$B \otimes A = P^T (A \otimes B) P$$

where P is a permutation matrix, as $\det(P) = 1$

$$\det(A \otimes B) = \det(B \otimes A)$$

Consider a rectangular d=2 lattice $[L \times T]_0$ Bravais cell, for this cell, the spatiotemporal orbit Jacobian matrix is

$$\mathcal{J} = r_1 + r_2 - 2sI + r_2^{-1} + r_1^{-1}$$

The index 1 is the spacial direction, and the index 2 is the temporal direction. The $[LT \times LT]$ orbit Jacobian matrix can be rewritten using Kronecker products:

$$\mathcal{J} = I_1 \otimes (r_2 + r_2^{-1}) - sI_1 \otimes I_2 + (r_1 + r_1^{-1}) \otimes I_2$$

$$I_1 = [L \times L] \text{ Identity}$$

$$I_2 = [T \times T] \text{ Identity}$$

Hill determinant: fundamental parallelpiped example Consider the Bravais lattice with basis vectors $\vec{a}_1 =< 3, 0 >$ and $\vec{a}_2 =< 0, 2 >$ a periodic orbit over this Bravais cell has 6 field values, one for each lattice site $z = (n, t)$ on a $[3x2]_0$ rectangle:

$$\begin{bmatrix} \phi_{01} & \phi_{11} & \phi_{21} \\ \phi_{00} & \phi_{10} & \phi_{20} \end{bmatrix}$$

We can stack up the columns of this lattice state and the corresponding sources into 6-dimensional vectors

$$\begin{pmatrix} \phi_{01} \\ \hline \phi_{00} \\ \hline \phi_{11} \\ \phi_{10} \\ \hline \phi_{21} \\ \phi_{20} \end{pmatrix}, \quad \begin{pmatrix} m_{01} \\ \hline m_{00} \\ \hline m_{11} \\ m_{10} \\ \hline m_{21} \\ m_{20} \end{pmatrix}$$

The corresponding orbit Jacobian block-matrix:

$$\mathcal{J}_{[3x2]_0} = \left[\begin{array}{cc|cc|cc} -2s & 2 & 1 & 0 & 1 & 0 \\ 2 & -2s & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & -2s & 2 & 1 & 0 \\ 0 & 1 & 2 & -2s & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & -2s & 2 \\ 0 & 1 & 0 & 1 & 2 & -2s \end{array}\right]$$

The fundamental parallelpiped generated by the action of the orbit Jacobian matrix is spanned by $LT = 6$ basis vectors: the columns of the orbit Jacobian matrix. The fundamental fact now yields the Hill determinant as the number of lattice states

$$N_{[3x2]_0} = |\text{Det}\,(\mathcal{J}_{[3x2]_0})| = 4(s-2)s(2s-1)^2(2s+3)^2$$

In practice, one often computes the Hill determinant using a Hamiltonian or "transfer matrix" formulation. An example is the temporal cat 3-term recurrence

$$\phi_t = \phi_t$$
$$\phi_{t+1} = -\phi_{t-1} + s\phi_t - m_t$$

In the Percival-Vivaldi "two-configuration" cat map representation

$$\hat{\phi}_{t+1} = \hat{J}_1 \hat{\phi}_t - \hat{m}_t$$
$$\hat{J}_1 = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix}, \quad \hat{\phi}_t = \begin{bmatrix} \phi_{t-1} \\ \phi_t \end{bmatrix}, \quad m_t = \begin{bmatrix} 0 \\ m_t \end{bmatrix}$$

Similarly for the d=2 spatiotemporal cat lattice at hand, one can recast into a 5-term recurrence relation:

$$\phi_{nt} = \phi_{nt}$$
$$\phi_{n,t+1} = \phi_{n,t-1} - \phi_{n-1,t} + 2s\phi_{nt} - \phi_{n+1,t} - m_{nt}$$
$$\hat{J}_1 = \left[\begin{array}{c|c} 0 & I_1 \\ \hline -I_1 & -\mathcal{J} \end{array}\right]$$

This $[2L \times 2L]$ block matrix generates a "time" orbit by acting on a 2L-dimensional "phase space" lattice strip $\hat{\phi}_t$ along the "spatial" direction

2021-01-04 Sidney I am going to take a break from the sum rule proof. However, I do know that the Hénon map is not a closed system so the sum rule does not converge to 1. For this blog entry, I shall take notes on the relaxation method for finding cycles, I will also try to write some code today, if I do, I will post my initial python code here too.

Notes All methods for finding unstable cycles are based on the idea of constructing a new dynamical system such that (i) the position of the cycle is the same for the original system and the transformed on, and (ii) the unstable cycle in the original system is a stable cycle of the transformed system. For example, the Newton-Raphson method replaces iteration of f(x) by iteration of the Newton-Raphson map:

$$x_i' = g_i(x) = x_i - \left(\frac{1}{M(x) - I}\right)_{ij} (f(x) - x)_j$$

A fixed point x_* for a map $f(x)$ is also a fixed point of g(x), indeed a superstable fixed point since $\frac{\partial g_i(x_*)}{\partial x_j} = 0$. The relaxation methods start with a guess of not a few points along an orbit, but a guess of the entire orbit. The relaxation algorithm for finding cycles is based on the observation that a trajectory of a map such as the Hénon map (see the discussion leading up to (**??**)):

$$x_{i+1} = 1 - ax_i^2 + by_i$$

$$y_{i+1} = x_i$$

Is a stationary solution of the relaxation dynamics defined by the flow

$$\frac{dx_i}{d\tau}$$

for any vector field v_i which vanishes on the trajectory. Here τ is a "fictitious time" variable, unrelated to the dynamical time (in this example, the discrete time of map iteration). As the simplest example, take v_i to be the deviation of an approximate trajectory from the exact 2-step recurrence form of the Hénon map:

$$v_i = x_{i+1} - 1 + ax_i^2 - bx_{i-1}$$

For fixed x_{i-1} and x_{i+1} there are two values of x_i satisfying $v_i = 0$. These solutions are the two extremal points of a local "potential" function:

$$v_i = \frac{\partial}{\partial x_i} V_i(x)$$

$$V_i = x_i(x_{i+1} - bx_{i-1} - 1) + \frac{a}{3}x_i^3$$

Assuming that the two extremal points are real, one is a local minimum of $V_i(x)$ and the other is a local maximum. We can modify or vector field differential equation with

$$\frac{dx_i}{d\tau} = r_i v_i$$
$$r_i = \pm 1$$

The modified flow will be in the direction of the extremal point given by the local maximum of V_i if $r_i = 1$ is chosen, or in the direction of the one corresponding to the local minimum if we take $r_i = -1$. I think that this is because a negative slope seeks to minimize a a values, whereas a positive slope seeks to maximize it, therefore, if we can somehow keep the flow from going off into positive or negative infinity, it will go to either a local maximum or local minimum. The goal of the relaxation method is that instead of searching for an unstable periodic orbit of a map, one searches for a stable attractor of a vector field. More generally, consider a d-dimensional map $x' = f(x)$ with a hyperbolic fixed point x_*. Any fixed point x_* is by construction an equilibrium point of the fictitious time flow

$$\frac{dx}{d\tau} = f(x) - x$$

If all eigenvalues of the Jacobian matrix $J(x_*) = Df(x_*)$ have real parts smaller than unity, then x_* is a stable equilibrium point of the flow. If some of the eigenvalues have real parts larger than unity, then one needs to modify the vector field so that the corresponding directions of the flow

are turned into stable directions in a neighborhood of the fixed point. To do this, we can modify the flow by

$$\frac{dx}{d\tau} = \mathbf{C}(f(x) - x)\,,$$

where $\mathbf{C}$ is a $[dxd]$ invertible matrix. The aim is to turn x_* into a stable equilibrium point of the flow by an appropriate choice of $\mathbf{C}$. It can be shown that a set of permutation/reflection matrices with one and only one non-vanishing entry ± 1 per row or column (for d-dimensional systems, there are $d!2^d$ such matrices) suffices to stabilize any fixed point. In practice, one chooses a particular matrix $\mathbf{C}$, and the flow is integrated. For each choice of $\mathbf{C}$, one or more hyperbolic fixed points of the map may turn into stable equilibria of the flow. We can change the algorithm to a discrete method which solves the issue of lengthy integrations of the fictitious time method. The idea is to construct a very simple map g, a linear transformation of the original f, for which the fixed point is stable. We take the Newton-Raphson map and replace the Jacobian prefactor in it with a constant matrix prefactor:

$$x' = g(x) = x + \Delta\tau\mathbf{C}(f(x) - x)\,,$$

where $\Delta\tau$ is a positive real number, and $\mathbf{C}$ is a $[dxd]$ permutation and reflection matrix with one and only one non-vanishing entry ± 1 per row or column. A fixed point of f is also a fixed point of g. Since $\mathbf{C}$ is invertible, the inverse is also true. This construction is motivated by the observation that for small $\Delta\tau \to d\tau$ the map is the Euler method for integrating the modified flow with integration step $\Delta\tau$. The argument why a suitable choice of matrix $\mathbf{C}$ can lead to the stabilization of an unstable periodic orbit is similar to the one used to motivate the construction of the modified vector field. In fact, for very small $\Delta\tau$ this construction just becomes the flow. For a given fixed point of $f(x)$ we again chose a $\mathbf{C}$ such that the flow in the expanding directions of $M(x_*)$ is turned into a contracting flow. The aim is to stabilize x_* by a suitable choice of $\mathbf{C}$. In the case where the map has multiple fixed points, the set of fixed points is obtained by changing the matrix $\mathbf{C}$ (in general different for each unstable fixed point) and varying initial conditions for the map g. For example, for 2-dimensional dissipative maps it can be shown that the 3 matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

suffice to stabilize all kinds of possible hyperbolic fixed points. If $\Delta\tau$ is chosen sufficiently small, the magnitude of the eigenvalues of the fixed point x_* in the transformed system are smaller than one, one has a stable fixed point. However, $\Delta\tau$ should not be chosen too small: since the convergence is geometrical with a ratio $1 - \alpha\Delta\tau$ (where the value of the constant α depends on the stability of the fixed point in the original system), small $\delta\tau$ can slow down the speed of convergence. The critical value of $\Delta\tau$, which just suffices to make the fixed point stable can be read off from the quadratic equations relating the stability coefficients of the original system and those of the transformed system. In practice, one can find the optimal $\Delta\tau$ by iterating the dynamical system stabilized with a given **C** and $\Delta\tau$. In general, all starting points converge on the attractor provided $\Delta\tau$ is small enough. If this is not the case, the trajectory either diverges (if $\Delta\tau$ is far too large) or it oscillates in a small section of the state space (if $\Delta\tau$ is close to its stabilizing value). A fixed point can now be found by choosing a starting point in the global neighborhood of the fixed point, and iterating the map g which now converges to the fixed point due to its stability. The basin of attraction is very large. The step size $|g(x) - x|$ decreases exponentially when the trajectory approaches the fixed point. To get the coordinates of the fixed points with a high precision, one therefore needs a large number of iterations for the trajectory which is already in the linear neighborhood of the fixed point. To speed up the convergence of the final part of the approach to a fixed point, it is recommended to do a combination of this approach and the Newton-Raphson method. The fixed points of the nth iterate f^n are periodic points of a cycle of period n. If we consider the map

$$x' = g(x) = x + \Delta\tau\mathbf{C}(f^n - x)$$

the iterates of g converge to a fixed point provided that $\Delta\tau$ is sufficiently small and **C** is a $[dxd]$ constant matrix chosen such that it stabilizes the flow. As n grows, $\Delta\tau$ has to be chosen smaller and smaller.

2021-01-05 Sidney .

Q23 How does one choose what r or **C** to use? I know that for sigma, I chose 1 if I want to drive it to converge to a local maximum in the potential, and -1 if I want a local minimum, but how do I know if the fixed point I am dealing with is a maximum or minimum?

2021-01-05 Predrag .

A23

2021-01-05 Sidney I have been hard at work trying to understand relaxation for cyclists. And I have gotten somewhere, not to a solution just yet, but somewhere. I decided that it would be best if I tried to understand the numerical methods employed by OrbitHunter before I started using it, to

this end, I have constructed a crude python code that can find the fixed points of the Hénon map, and the two cycle. My next step is to make the two cycle program more efficient and cleaner, and then to generalize it to n-length orbits. After this has been done I can start work on the exercise that I was set. The code is in

`siminos/williams/python/relax1.py`

2021-01-14 Sidney I have contacted Matt about OrbitHunter and he says that he's working on fixing some things so it's easier for people that are not him to work with. He recommended that I try to work on my own code, so I have been doing that. I was able to get the Two_Cycle working with the modification that it can now determine the sigma itself, but the four_cycle code is still not working, I think I need to change the differential equation solver to a Runge-Kutta algorithm instead of the Euler one I've been using, I'll paste the code under here, please please help if you can.

2021-02-01 Predrag .

Do this Save and svn commit this code as `siminos/williams/python/XXX.py`, then remove the inset from here

```
\#def four_cycle(guesstrajectory,dt):
\#  henon = Henon(1.4, 0.3)
\#  x0=guesstrajectory
\#  vi=np.zeros(4)
\#  vi[0]=x0[0]-henon.oneIter(np.roll(x0,2))
\#  vi[1]=x0[1]-henon.oneIter(np.roll(x0,1))
\#  vi[2]=x0[2]-henon.oneIter(np.roll(x0,0))
\#  vi[3]=x0[3]-henon.oneIter(np.roll(x0,-1))
\#  ep=10**(-7)
\#  x=np.zeros(4)
\#  sigma=np.zeros(4)
\#  sigma[:]=1
\#  iglob=0
\#  while np.all(abs(vi))>ep:
\#      iglob+=1
\#      for i in range(0,4):
\#          x[i]=x0[i]-dt*vi[i]
\#          vi[i]=x[i]-henon.oneIter(np.roll(x,(2-i)))
\#          x0[i]=x[i]
\#          print(x[i])
\#          #print(i)
\#          #time.sleep(1)
\#          if abs(x[i])>5 or iglob>100000:
\#              print(x)
```

```
\#                return "Diverged"
\#                sigma[i]=-1
\#                x=np.zeros(4)
\#                x0=np.zeros(4)
\#                vi[0]=x0[0]-henon.oneIter(np.roll(x0,2))
\#                vi[1]=x0[1]-henon.oneIter(np.roll(x0,3))
\#                vi[2]=x0[2]-henon.oneIter(np.roll(x0,4))
\#                vi[3]=x0[3]-henon.oneIter(np.roll(x0,5))
\#                iglob=0
\#  return x
```

For the guess trajectory I put in "np.zeros(4)" and for dt I put in 0.1 to get: [1.11534978 -0.83649258 0.74365269 -0.33467244] close but no cigar. Also, this is not intended to be a "best practices" code, I shall work on that once I have made it work.

2021-01-15 Matt to Sidney Please comment your code so it is easier to read. Comments are lines which start with '#' Pain is a part of the learning process for programming, at least in my experience. Especially for interpreted languages like Python; compiled languages like C, C++, C#, F# etc. are must more explicit and "logical", at the expense of flexibility. You'll be able to look back at old code and be able to write it in a much clearer and nicer way in the future, that I can guarantee.

For now, I'm going to refactor your code; I do not know if it will give the desired results, but hopefully it will get you back on track. You can use this refactored code or simply use it as a guide, but you have to understand its incredibly hard to interpret someone else's code.

The way you have your cycle functions set up is not going to scale, as you'll have to write each component and index separately. Imagine doing this for a thirty two cycle.

Here are some signs that you need to vectorize or refactor your code. I'll explain what this means in the code itself.

1. you start labeling your variables with indices (point1, point2, etc.)
2. You have multiple functions that could be converted into a single function + parameter (two cycle and four cycle can be combined into a "cycle" function)
3. You have functions which are special cases of other functions. A single iteration oneIter should be produced by multiIter plus parameter that says iteration = 1.

One way of testing if you need a higher order integration scheme is to test large vs. small step sizes. I.e. the step size in Euler can control error, it simply requires a much smaller step size to do so. Also, you might look towards implicit integration schemes which are always stable, for example, backwards Euler would be the simplest.

2021-01-15 Sidney to Matt Thank you very much for the help, I agree that pain seems to be a necessary ingredient in coding, and it is definitely extremely difficult to interpret another person's code. I can try to re-write my code to make it more readable, along with references of where I am getting this method if that would be helpful, I shall wait for your response on that as I do not want to take more of your time. Also, I should definitely change the cycle functions, thank you, I will update that in the next iterate of the code. I have made my code all comments (I started each line with #).

2021-01-15 Matt:

Sidney the refactored code is available on an old branch of orbithunter github:
github.com/mgudorf/orbithunter/blob/henon/notebooks/sidney_refactoring.ipynb

The issue was setting sigma to be a constant and not dependent on i. In other words, to need a sigma for each dimension of the cycle.

2021-01-15 Sidney I have been looking at the refactored code, and I see how it works, in fact, I understand it enough to work with it, and have generated a good number of orbits. Unfortunately, the generated periodic points are not the same as they are in ChaosBook Table 34.2 the last 1 to 2 digits are different. I am really unsure as to why this discrepancy exists. The values of the error function v_i get to values below the cutoff of 10^{-7}, and my original Two_Cycle code gave me values equal to those of table 34.2, but with different values of sigma. I am tired, and I have a headache, so I will just take notes on the "Cartesian Product" that I had to use to construct a function to find all possible sigma matrices (**C** in Chaosbook), and paste in my current working code (the modified one that I took from Matt). My goal over the next few days is to better understand the code, and figure out why there is that small difference between my calculated values and table 34.2.

The Cartesian Product of two sets A and B, denoted by $A\times B$, is the set of all ordered pairs (a, b) where a is an element of set A and b is an element of set B or:

$$A\times B = \{(a, b)| \quad a \in A \quad and \quad b \in B\}$$

For two sets the Cartesian Product can be computed by constructing a table where one set is the row index and the other is the column index. or $A\times B_{i,j} = (a_i, b_j)$, due to the tabular nature of this product the number of ordered pairs is $L_A L_B$ where L_A is the number of elements in set A and L_B is the number of elements in set B (from Wikipedia).

2021-02-01 Predrag .

Do this Save and svn commit this code as `siminos/williams/python/XXX.py`, then remove the inset from here.

Update 2021-05-17: I removed the code here, and put the actual python file in williams/relax

1-17-2021 Sidney I was able to get the code to match the table in Chaosbook, I did this by changing the while loop condition to # np.any(abs(cycle.deviation())) > ep

Will discuss whether this is better or not at the next meeting

1-28-2021 Sidney Lots of coding issues later I finally have a working algorithm for calculating the periodic points and expanding eigenvalues for the Hamiltonian Hénon map ($b = -1, a = 6$). It is messy, and I have not written the loop to calculate, and then write to an external document all cycles up to length n, but that should not be difficult, I also need to write this loop for my (still working) non-Hamiltonian code. Matt has been helping me clean up my code, I have been finding that my skills in matrix manipulation within Python are sorely lacking, and I hope that I can fix this. Once I have cleaned up the code,added the last few loops, and added comments and other such things to make it more readable, I shall upload it to the repository. After that point I hope to read at least some of Han's blog and take notes on it so that I can better understand what the rest of the group is doing.

1.2 Frenkel-Kontorova model

2021-02-01 Predrag Sidney, I am putting this here mostly for Han and me not to remember, but equilibria and relative equilibria of Frenkel-Kontorova models might be closely related to your temporal Hénon lattice, and widely studied in literature.

It is difficult stuff, and you can safely ignore it, for now:)

Note the text below (**??**) and (1.10).

Search for **2019-12-12** *Meisinger and Ogilvie* notes in this blog.

2021-02-01 Anna Vainchtein (U. Pittsburgh)

Traveling waves in a driven Frenkel-Kontorova lattice: Variants of Frenkel-Kontorova model, originally proposed to describe dislocations in crystal lattices, have been widely used to study a variety of physical phenomena, including dynamics of twin boundaries and domain walls, crystal growth, charge-density waves, Josephson junctions and DNA denaturation. I discuss properties and stability of traveling waves in chains of Frenkel-Kontorova type driven by a constant external force. After reviewing some earlier studies for piecewise-smooth variants of the model, where exact and semi-analytical solutions can be constructed, I will describe numerical results for a fully nonlinear damped driven chain from a recent work with J. Cuevas-Maraver (U. of Sevilla), P. Kevrekidis (U. of

Mass.) and H. Xu (Huazhong U.). In this setting, traveling wave solutions are computed as fixed points of a nonlinear map. [...] Exploring the spectral stability of the obtained waveforms, we identify, at the level of numerical accuracy of our computations, a precise criterion for instability of the traveling wave solutions: monotonically decreasing portions of the kinetic curve always bear an unstable eigendirection.

The recorded talk will be available here. It is a difficult subject, but our case - equilibria and relative equilibria is perhaps a trivial case described in this literature.

2019-06-26 Predrag Mramor and Rink **[MraRin12]** *Ghost circles in lattice Aubry-Mather theory*, arXiv:1111.5963:

"Monotone lattice recurrence relations such as the Frenkel-Kontorova lattice, arise in Hamiltonian lattice mechanics as models for ferromagnetism and as discretization of elliptic PDEs. They are a multidimensional counterpart of monotone twist maps."

The paper has an appendix of twist maps, refers to Mather and Forni **[MatFor94]** *Action minimizing orbits in Hamiltomian systems*. Example of exact symplectic twist maps are the Chirikov standard map and convex billiards. I think the focus in all this work is on *integrable*, not chaotic: "Under generic conditions, the Poincaré return map of a 2 degree of freedom Hamiltonian system near an elliptic equilibrium point is an exact symplectic twist map. In this case, the corresponding twist map is close to *integrable*, so that it allows for the application of various kinds of perturbation theory **[MatFor94]**." Lectures on *Equidistribution of periodic orbits: An overview of classical VS quantum results* by Degli Esposti, Graffi, and Isola in *Transition to Chaos in Classical and Quantum Mechanics* **[Graffi94]** are also of interest.

Twist maps often admit a variational structure, so that the solutions $x : \mathbb{Z}^d \to \mathbb{R}$ are the stationary points of a formal action function $W(x)$. Given any rotation vector $\omega \in \mathbb{R}^d$, Aubry-Mather theory establishes the existence of a large collection of solutions of $\nabla W(x) = 0$ of rotation vector ω. For irrational ω, this is the *Aubry-Mather set*. It consists of global minimizers and it may have gaps.

The part relevant to our spatiotemporal cat is the idea of studying globally stationary solutions by means of a formal gradient. We do not really know how to find all invariant 2-tori in 2 or more dimensions, even though we know how to count them, right? They study the parabolic gradient flow $\frac{dx}{dt} = -\nabla W(x)$ and prove that every Aubry-Mather set can be interpolated by a continuous gradient-flow invariant family, the so-called 'ghost circle'. The existence of these ghost circles is known in dimension $d = 1$, for rational rotation vectors and Morse action functions.

d-dimensional Frenkel-Kontorova lattice: Here, the goal is to find a d-

dimensional "lattice configuration" $x : \mathbb{Z}^d \to \mathbb{R}$ that satisfies

$$V'(x_i) - (\Delta x)_i = 0 \text{ for all } i \in \mathbb{Z}^d . \tag{1.10}$$

The smooth function $V : \mathbb{R} \to \mathbb{R}$ satisfies $V(\xi + 1) = V(\xi)$ for all $\xi \in \mathbb{R}$. It has the interpretation of a periodic onsite potential.

I like their definition of the discrete Laplace operator $\Delta : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$, defined as

$$(\Delta x)_i := \frac{1}{2d} \sum_{||j-i||=1} (x_j - x_i) \text{ for all } i \in \mathbb{Z}^d . \tag{1.11}$$

where $||i|| := \sum_{k=1}^{d} |i_k|$. Thus, $(\Delta x)_i$ is the average of the quantity $x_j - x_i$ computed over the lattice points that are nearest to that with index i, i.e., the graph Laplacian [**Pollicott01**, **Cimasoni12**] (**??**) for the case of hypercubic lattice, or the "central difference operator" [**PerViv**].

One can think of (1.10) as a naive discretization of the nonlinear elliptic partial differential equation $V'(u) - \Delta u = 0$ for a function $u : \mathbb{R}^d \to \mathbb{R}$ and $x_i = u(i)$.

Eq. (1.10) is relevant for statistical mechanics, because it is related to the Frenkel-Kontorova Hamiltonian lattice differential equation

$$\frac{d^2 x_i}{dt^2} + V'(x_i) - (\Delta x)_i = 0 \text{ for all } i \in \mathbb{Z}^d. \tag{1.12}$$

This differential equation describes the motion of particles under the competing influence of an onsite periodic potential field and nearest neighbor attraction. Eq. (1.10) describes its stationary solutions.

In dimension $d = 1$, the solutions of equation (1.10) correspond to orbits of the Chirikov standard map $T_V : \mathbb{A} \to \mathbb{A}$ of the annulus.

The Frenkel-Kontorova problem (1.10) is an example from a quite general class of lattice recurrence relations to which the results of this paper apply. These are recurrence relations for which there exists, for every $j \in \mathbb{Z}^d$, a real-valued "local potential" function $S_j : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ so that the relation can be written in the form

$$\sum_{j \in \mathbb{Z}^d} \partial_i S_j(x) = 0 \text{ for all } i \in \mathbb{Z}^d. \tag{1.13}$$

It turns out that for the Frenkel-Kontorova problem (1.10), such local potentials exist and it is easy to check that they are given by

$$S_j(x) := V(x_j) + \frac{1}{8d} \sum_{||k-j||=1} (x_k - x_j)^2. \tag{1.14}$$

For the general problem (1.13), the functions $S_j(x)$ will be required to satisfy some rather restrictive hypotheses. Physically, the most important of these hypotheses is the *monotonicity* condition. It is a discrete analogue of ellipticity for a PDE. Among the more technical hypotheses is one that guarantees that the sums in expression (1.13) are finite. For the purpose of this introduction, it probably suffices to say that the potentials (1.14) of Frenkel-Kontorova are prototypical for the $S_j(x)$ that we have in mind.

It is important to observe that the solutions of (1.13) are precisely the stationary points of the formal sum

$$W(x) := \sum_{j \in \mathbb{Z}^d} S_j(x). \tag{1.15}$$

This follows because differentiation of (1.15) with respect to x_i produces exactly equation (1.13) and it explains why solutions to (1.13) are sometimes called *stationary* configurations.

In the case that the periodic onsite potential $V(\xi)$ vanishes, the Frenkel-Kontorova equation (1.10) reduces to the discrete Laplace equation $\Delta x = 0$, for which it is easy to point out solutions. For instance, when $\xi \in \mathbb{R}$ is an arbitrary number and $\omega \in \mathbb{R}^d$ is an arbitrary vector, then the linear functions $x^{\omega,\xi} : \mathbb{Z}^d \to \mathbb{R}$ defined by

$$x_i^{\omega,\xi} := \xi + \langle \omega, i \rangle$$

obviously satisfy $\Delta x = 0$. It moreover turns out that the $x^{\omega,\xi}$ are *action-minimizers*, in the sense that for every finite subset $B \subset \mathbb{Z}^d$ and every $y : \mathbb{Z}^d \to \mathbb{R}$ with support in B, it holds that

$$\sum_{j \in \mathbb{Z}^d} \big(S_j(x^{\omega,\xi} + y) - S_j(x^{\omega,\xi}) \big) \geq 0 \,.$$

Note that this sum is actually finite and can be interpreted as $W(x^{\omega,\xi} + y) - W(x^{\omega,\xi})$.

Definition 1.1. *Let $x : \mathbb{Z}^d \to \mathbb{R}$ be a d-dimensional configuration. We say that $\omega \in \mathbb{R}^d$ is the rotation vector of x if for all $i \in \mathbb{Z}^d$, the limit*

$$\lim_{n \to \infty} \frac{x_{ni}}{n} \text{ exists and is equal to } \langle \omega, i \rangle \,.$$

Clearly, the rotation vector of $x^{\omega,\xi}$ is equal to ω. On the other hand, in dimension $d \neq 1$, a solution to (1.10) does not necessarily have a rotation vector. An example is the hyperbolic configuration x^h defined by $x_i^h = i_1 i_2 \cdots i_{d-1} i_d$ which solves $\Delta x = 0$.

1.3 2021 blog

2021-02-01 Predrag to Sidney - sorry about the Frenkel-Kontorova interruption, not worth your time right now, but Han and I might profit from being the first to read it. We'll keep it here until then.

2021-02-07 Sidney No issues with the interruption. I have been working towards understanding the next step, which is applying a Fourier transform on my states (cycles) to get them into Fourier space. From Han and other resources (Strang Linear Algebra), I know that all I have to do is apply a discrete Fourier transform on the periodic cycle vector, so it boils down to matrix vector multiplication where the elements are $F_j k = w^{jk}$ where w is a complex root of unity. I think I could easily code this myself, however, I feel like I could use a prebuilt package to go much faster, so I will use the numpy fft package, and from there figure out how to create plots like Han did. I will hopefully soon have a full functional nice code to put into the repository.

Q24 Sidney I know that the Fourier modes make it a lot easier to see the symmetries, but is there a reason for that? Or is it just coincidence? Or, is it too far beyond my level of pure math to appreciate?

2021-02-09 Sidney I have modified my code and made it so I can generate the symbol sequences for all orbits (001101 etc.) up to a certain length n and store them in a list. I can then take this list and find the actual points of the cycle using my inverse iteration code, but for some reason the code diverges if I put the symbol sequence in the "wrong" order. For example, it gets the correct points if I put in 10 but not if I put in 01 which is weird. I still need to fix it. I have also used the fast Fourier transform from numpy to get the cycles into Fourier space. Now all I need to do is figure out why the code doesn't like some orders, and then figure out how to plot the points in Fourier space.

2021-02-24 Sidney I have officially completed my code, I will ask about how to upload my data and images here, I will also work on cleaning my code up a bit, and perhaps adding in the Fourier bit to the non-Hamiltonian Hénon map. Until that point though, it has become extremely obvious that I need to look at the theory about WHY I am doing Fourier transforms, so I am going to turn Michael Engel's course [**Engel11**], based on Sands [**Sands69**] *Introduction to crystallography* (1969) (click here). My notes:

Point Symmetry **Definition 1**: An Euclidean move $\mathcal{T} = \{A, b\}$ is a linear transformation that leaves space invariant:

$$x \mapsto \mathcal{T}(x) = Ax + b$$

Here, x is a vector, A a $[3 \times 3]$ orthogonal matrix and b a 3-vector.

Definition 2: The product of two transformations $\mathcal{T}_2 = \{A_1, b_1\}$ and $\mathcal{T}_2 = \{A_2, b_2\}$ is: $\mathcal{T}_2 \circ \mathcal{T}_1 = \{A_2A_1, A_2b_1 + b_2\}$($\mathcal{T}_1$ is applied first)
Definition 3: The order of a transformation $\mathcal{T}$ is the smallest integer n such that $\mathcal{T}^n(x) = \mathcal{T} \circ \mathcal{T} \circ \mathcal{T} \circ \cdots \circ \mathcal{T} = x$ one can also say this transformation is n-fold.
Observations:

1. The inverse is: $\mathcal{T}^{-1} = \{A^{-1}, -A^{-1}b\}$
2. Every transformation of finite order n (ie $\mathcal{T}^n = 1$)leaves at least one point invariant.

2021-02-27 Sidney Notes from Engel's course [**Engel11**].

A group G, together with an operation, that combines any two elements a and b to form another element $a \cdot b$. To qualify as a group, the set and the operation must satisfy the group axioms:

Closure	For all a and b in G, the result of the operation $a \cdot b$ is also in G
Associativity	For all a, b, and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
Identity Element	There must exist an identity element in G
Inverse Element	For each a in G there must exist an inverse which yields the identity element when the group operation is applied between the two elements.

A symmetry of an object in space is an Euclidean move which leaves the object indistinguishable. The order of the group is equal to the number of elements in that group.

2021-02-27 Sidney Figure 1.1 shows all Hamiltonian Hénon (**??**), $a = 6$ lattice states of period $n = 6$, in the C_6 reciprocal lattice.

2021-02-27 Sidney Notes from Engel's course [**Engel11**].

If G is a group and X is a set, then a left group action of G on X is a binary function:

$$GX \to X$$

denoted

$$(g, x) \mapsto g \cdot x$$

Which satisfies the following two axioms:

1.$(gh)x = g(hx)$

2.$ex = x$

The set X is called a left G-set. The group G is said to act on X on the left. When a group G acts on a set X the orbit of a point x in X is the set of elements of X to which x can be moved by the elements of G. The orbit of x is denoted as Gx:

$$Gx = \{g \cdot x | g \in G\}$$

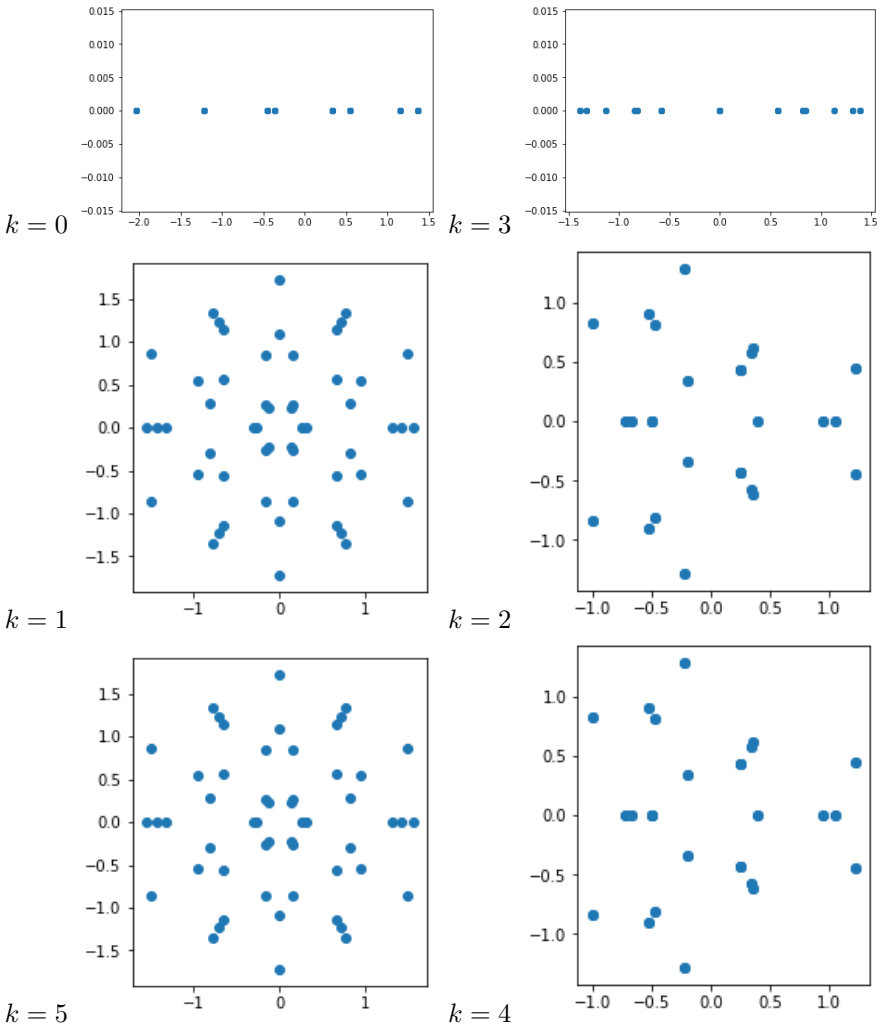

Figure 1.1: Hamiltonian Hénon (**??**), $a = 6$ lattice states of period $n = 6$, in the C_6 reciprocal lattice, $k = 0, 1, 2, 3, 4, 5$. $k = 0, 3$ are purely real. $k = 5, 4$ are the same as $k = 1, 2$, respectively, up to time reversal. Compare with Han's figure **??**, for example.

The properties of a group guarantee that the set of orbits of X under the action of G form a partition of X. The associated equivalence relation is x y if and only if there exists a g in G with $gx = y$. The orbits are then the equivalence classes under this relation, two elements x and y are equivalent if and only if their orbits are the same ie $Gx = Gy$. For every x in X, we define the stabilizer subgroup of x (also called the isotropy group or little group) as the set of all elements in G that fix x:

$$G_x = \{g \in G | g \cdot x = x\}$$

In short: the orbit consists of all points that are equivalent under symmetry. And the stabilizer consists of all symmetries that leave a point invariant.

Definition 6: A point symmetry is a symmetry which leaves a point x_0 invariant: $\mathcal{T}(x_0) = x_0$

So, we can see that translations cannot be point symmetries, symmetries with finite order are point symmetries, symmetries with infinite order cannot be point symmetries.

Definition 7: A point group is a group of point symmetries which leave a common point x_0 invariant.

So, we can see that a point group is a finite subgroup of $O(3)$, this space of three dimensional orthogonal matrices.

$$\text{Note: } O(3) = \{A \in \mathcal{R}^{3\times 3} : A^T A = 1\}$$

$$SO(3) = \{A \in \mathcal{R}^{3\times 3} : A^T A = 1, det(A) = 1$$

Definition 8: Two subgroups H_1 and H_2 of a group G are conjugated if there exists a $g \in G$, such that:

$$H_2 = g^{-1} H_1 g$$

Example: $G = O(3)$. Two point groups are conjugated, if there is a change of basis that maps them into each other.

Cyclic groups: $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, ...$ where C_n consists of all rotations about a fixed point by multiples of $360/n$

Dihedral groups: $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \mathrm{D}_4, ...$ where D_n (of order $2n$) consists of the rotations in C_n together with reflections in n axes that pass through the fixed point.

2021-03-07 Predrag For the current project, we need to understand D_n symmetry: see Ding thesis example 2.9.
pdflatex *siminos/lyapunov/blog.tex*, read sect. 7.11.2 *Factorization of C_n and D_n*.

See also sect. **??** *Pow wow 2021-01-08*.

2021-03-07 Sidney I shall now attempt some of the exercises in Engel's [**Engel11**] Point Groups lecture.

exercise 2

exercise 3

Exercise 1

Exercise 3

Determination of the point group of an object in space

1. Object linear: $C_{\infty v}$ or $D_{\infty h}$
2. High symmetry, non-axial: T, T_h, T_d, O, O_h, I, I_h
3. No rotation axis: C_1, C_i, C_s.

Carolyn's packings of small spheres on a big sphere

The point groups are for the sphere on the left: C_{2v} because it has 3 axes that it can rotate 180^{deg} around, and for each one it can then perform a flip and remain unchanged, I am unsure about the second sphere.

2021-03-07 Predrag For the current project, it will suffices to focus on 1-dimensional and 2-dimensional crystals (wallpaper groups). Engel is a chemist, so 3-dimensional symmetries are the most important thing, but we are good enough keeping to 2 dimensions.

2021-03-13 Sidney Here are my notes on Xiong Ding's example 2.9 and 2.8 in his thesis which is referenced above.

Example 2.8 Character table of dihedral group $D_n = C_{nv}$ n odd

$$D_n = \{e, C_n, C_n^2, \cdots, C_n^{n-1}, r, C_n r, \cdots, C_n^{n-1} r\}$$

D_n has n rotation elements and n reflections. Group elements satisfy $C_n^i C_n^j r = C_n^j r C_n^{n-i}$, so C_n^i and C_n^{n-i} form a class. Also, $C_n^{n-i} C_n^{2i+j} r = C_n^j r C_n^{n-i}$ implies that $C_n^j r$ and $C_n^{2i+j} r$ are in the same class. Therefore, there are only three different types of classes:
$\{e\}, \{C_n^k, C_n^{n-k}\}, \{r, C_n r, \cdots, C_n^{n-1} r\}$. The total number of classes is $(n+3)/2$. In this case, there are 2 one dimensional irreducible representations (symmetric A_1 and antisymmetric A_2) and $(n-1)/2$ two-dimensional irreducible representations. In the j^{th} two-dimensional irreducible representation, class $\{e\}$ has form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, class $\{C_n^k, C_n^{n-k}\}$ has form $\begin{pmatrix} \exp(\frac{i2\pi ki}{n}) & 0 \\ 0 & \exp(-\frac{i2\pi ki}{n}) \end{pmatrix}$, and class $\{r, C_n r, \cdots, C_n^{n-1} r\}$ has form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Definition of Irreducible Representation (and Representation): Will be using the Dresselhaus lecture notes from 2002 downloaded from the Chaosbook website:

Representation: A representation of an abstract group is a substitution group (matrix group with square matrices) such that the substitution group is homomorphic (or isomorphic) to the abstract group. We assign a matrix $D(A)$ to each element A of the abstract group such that $D(AB) = D(A)D(B)$.

Homomorphic/Isomorphic: Two groups are isomorphic or homomorphic if there exists a correspondence between their elements such that each

$$A \to \hat{A}$$
$$B \to \hat{B}$$
$$AB \to \hat{AB}$$

If the two groups have the same order, then they are isomorphic.

Irreducible Representation: If by one and the same equivalence transformation, all the matrices in the representation of a group can be made to acquire the same block form, then the representation is said to be reducible, otherwise it is irreducible. Thus, an irreducible representation cannot be expressed in terms of representations of lower dimensionality.

Aside: how to find a character table of a group. See here for the source, symbol A denotes symmetric with C_n so yields a 1 in the character table for all C. The subscripts 1, and 2 are symmetric (1) and antisymmetric (-1) respectively with respect to flips r. I am unsure about how the E_j column got its entries, it looks like it was a trace of the matrix representations, but I don't know the rule for that.

Q Sidney Conventionally, the irreps are the rows, and the symmetry operations are the columns, why are they transposed here?

2021-03-18 Sidney A review of the Group theory from Chaosbook, especially the irreps, and character tables. The goal of this is so that I can actually understand the character tables presented in the examples of Xiong Ding's thesis.

Q2 Sidney I am a little unsure about the statement in example 2.8 of Ding's thesis "in the jth two-dimensional irreducible representation class $\{r, rC_n, \cdots, rC_n^{n-1}\}$ has the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Shouldn't there a contribution from the rotation group C_n? This looks just like the inversion (reflection?) group r.

From Dresselhaus again.

The Unitary of Representations: This theorem which shows that in most physical cases, the elements of a group can be represented by unitary matrices.

Theorem: Every representation with matrices having non-vanishing determinants can be brought into unitary form by an equivalence

transformation. Skipping writing out the nitty gritty of the proof, we can first form a hermitian matrix using the representations of the group. We can than take advantage of the fact that any Hermitian matrix can be diagonalized by a suitable unitary transformation. We can construct this by doing a similarity transformation with some U from there we can redefine $\hat{\hat{A}}_x = d^{-1/2}U^{-1}A_xUd^{1/2}$ where d is the diagonal matrix. We can then show that $\hat{\hat{A}}_x$ is unitary by simple calculation, thus we have proved the theorem.

We can use this theorem to prove Schur's Lemma which is: A matrix which commutes with all matrices of an irreducible representation is a constant matrix (a constant times the unit matrix). Therefore, if a non-constant commuting matrix exits, the representation is reducible.

Proof: Let M be a matrix which commutes with all matrices of the representation $A_1, A_2, \cdots, A_h$:

$$MA_x = A_xM$$

Take the adjoint of both sides:

$$A_x^\dagger M^\dagger = M^\dagger A_x^\dagger$$

From the Unitary Representations theorem, with no loss of generality we can assume that A_x is unitary, so we can multiply on the left and right to obtain:

$$M^\dagger A_x = A_xM^\dagger$$

So, if M commutes with A_x then so does $M^\dagger$.

2021-03-22 Sidney I read through the rest of the proof, and I will continue Dresselhaus notes later, but now, I will review some Chaosbook material. I also looked at the decomposition of the cat map done by Predrag, but I do not understand it enough to see if it works with the Hénon map, so I will discuss in tomorrow's meeting. Anyway, here are notes for Week 13.

Video 1: Hard Work Builds Character Elementary examples of discrete groups: the character table characterizes the finite group.

2-element group: $\{e, g\}$ $g^2 = e$. $S_2, D_1, C_2, ...$ used to tile our space. Imagine butterfly, symmetry axis across the middle. Can separate the space into two spaces $\tilde{\mathcal{M}} \in \{\tilde{x}\}$ and $\tilde{\mathcal{M}}_2 = \{-\tilde{x}\}$ (top view of butterfly space). On the side, it is just a line, half of the line can be the fundamental domain, everything else is a copy of the fundamental domain. (Order of the group is the number of elements in it). Think of scalar functions defined on this domain $\rho(x)$. Can divide it into two functions $\rho_1(\tilde{x})$ (defined on fundamental domain) and $\rho_2 = \rho(-\tilde{x})$. Can decompose this function into a vector of functions evaluated on the

fundamental domain, the vector will have a length equal to the order of the group. Can write the original function on the fundamental domain as $\rho(\tilde{x}) = \frac{1}{2}(\rho(\tilde{x}) + \rho(-\tilde{x})) + \frac{1}{2}(\rho(\tilde{x}) - \rho(-\tilde{x}))$ (decompose into symmetric, and antisymmetric components). $\rho(\tilde{x}) = \rho_+(\tilde{x}) + \rho_-(\tilde{x})$. $\rho_\alpha = \frac{1}{|G|}\sum_g \chi_\alpha(g)\rho$. Where χ is the characters from the character table.

2: The Symmetry Group of a Propeller 3 elements. $G = \{e, g_2, g_3\}$ $g^3 = e$, rotation by $2\pi/3$ C_3 cyclic group with three elements. Can look at this as an abstract group (no physical, or geometrical realization): $g_2 = \omega$ $g_3 = \omega^3$ $\omega^3 = e$. Can write a multiplication table from these relationships with the inverses, there is the identity along the diagonal. Matrix representation: in 2D plane the identity is just the unit matrix, and the other elements are just the rotation matrices.

I shall end my notes here for today (it is getting late).

2021-04-19 Sidney I have been working through the chapter 10 ChaosBook problems and I'm pretty confident with all but problem 10.2, could I possibly have a hint on where to start?

2021-03-07 Predrag I put all solutions that I have here. They are pretty incomplete. If you have solutions that I do not have, or better / comparable solutions to those that I do have, let me know.

2021-04-24 Sidney Here I will attempt to derive the orbit Jacobian matrix, so that I can later find the eigenvalues and eigenvectors. I will start with analytically finding lattice states, and eigen-things. We will see how well that goes. First, the temporal Hénon is $x_{n+1} + ax_n^2 + x_{n-1} - 1 = 0$ this was drawn from the beginning of chapter **??** *Temporal Hénon*. Following the formalism of the CL18 paper, I can rewrite the map as $r + 2a\mathbb{X} + r^{-1} - I = 0$ where the 2 has come from the fact that finding the orbit Jacobian matrix is in effect taking a derivative, now the orbit Jacobian matrix is
[**2021-05-01 Sidney**: *this has wrong entries on the diagonal, (1.16), (**??**) is the correct formula.*]

$$\mathcal{J} = r + 2a\mathbb{X} + r^{-1} = \begin{bmatrix} 2ax_n & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 2ax_n & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2ax_n & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 2ax_n & 1 \\ 1 & 0 & \dots & \dots & \dots & 1 & 2ax_n \end{bmatrix},$$

where the ones on the upper right and lower left corners, are a result of the periodic boundary conditions. The goal is to adapt this so that we can apply the time reversal boundary conditions, and thus construct time-symmetric lattice states.

2021-05-01 Sidney Here is the correct orbit Jacobian matrix (in the above I made a mistake):

$$\begin{aligned}\mathcal{J} &= \sigma + 2a\mathbb{X} + \sigma^{-1} \\ &= \begin{bmatrix} 2ax_0 & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 2ax_1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2ax_2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 2ax_{n-2} & 1 \\ 1 & 0 & \dots & \dots & \dots & 1 & 2ax_{n-1} \end{bmatrix} \end{aligned} \tag{1.16}$$

I'll look at (and try to understand) the proof of the Hill's formula.

2021-05-05 Predrag Please modify your code so you are computing the (rescaled) temporal Hénon (**??**), orbit Jacobian matrix (**??**).

2021-05-05 Sidney I shall modify my code to do that, first, I will look at the rescaled temporal Hénon (**??**). Just for my own satisfaction, I shall rederive (**??**) and (**??**). The Hamiltonian Hénon map is given by

$$x_{n+1} = 1 - ax_n^2 - x_{n-1}$$

We note that an n-step recurrence relation is the discrete analogue to an order-n differential equation. We can now introduce the change of variables $\phi_n \equiv ax_n$ this turns our map into:

$$\frac{1}{a}\phi_{n+1} = 1 - \frac{1}{a}\phi_n^2 - \frac{1}{a}\phi_{n-1}$$

Rearranging, we get

$$a = \phi_{n+1} + \phi_n^2 + \phi_{n-1}\,. \tag{1.17}$$

The derivative of this map yields its orbit Jacobian matrix:

$$\mathcal{J}_p = r + 2\mathbb{X} + r^{-1}$$

Thus, we have rederived (**??**) and (**??**). I can now put it into my code. I am currently trying to understand the derivation of:

$$\hat{\mathcal{J}}_p = \begin{bmatrix} \mathbf{1} & & & & -J(\phi_0) \\ -J(\phi_1) & \mathbf{1} & & & \\ & \ddots & \ddots & & \\ & & -J(\phi_{n-2}) & \mathbf{1} & \\ & & & -J(\phi_{n-1}) & \mathbf{1} \end{bmatrix},$$

Any hints would be much appreciated.

2021-05-10 Predrag It's all in the blog, and the *siminos* repo. But why don't you go back where you started, and reread CL18 **[CL18]**:

sect. 1.5 *Stability of an orbit vs. its time-evolution stability*

appendix C *Spatiotemporal stability*

2021-05-10 Predrag One thing you can maybe help Han and me with; we find the block matrix formulation of sect. **??** *Spatiotemporal cat Hill's formula* quite reader-unfriendly. I like the latest version, sect. **??** *Han's Hénon map Hill's formula* better. Any suggestions how to make this easier to read are welcome!

2021-05-06 Matt to Sidney It's not clear to me what you need help with, but the origin of the derivation of the aforementioned matrix is the Jacobian of a multipoint shooting method (see ChaosBook sec. 16.2 *Multipoint shooting method* and ChaosBook example 16.2) vector $x_n - f(x_{n-1}), \quad n \in 0, ..., N$ (indices may be off depending on how matrix is ordered), which finds cycles of length n by solving a system of single steps. If the vector equals 0 then this means that we have found a cycle/orbit as we have found a set of points which is closed under evolution (we can take N steps and return to our original position, starting from any of the N points defining the cycle). The actual matrix, as with all Jacobians, tells us how the tangent space evolves. In terms of the multipoint shooting, this is the combination of N single step Jacobians. If you are on the cycle (i.e. no deviations, or components in the tangent space) which evolve other than the velocity, which is mapped into the velocity at the new point. This is a quick and dirty explanation which is probably over simplifying, but the near-tautological explanation is: if you're on the cycle, then you don't get pushed away from the cycle due to your deviation from the cycle.

tl;dr It's the Jacobian of the multipoint shooting equation $x_n - f^t(x_{n-1})$.

2021-05-10 Predrag Talking about multishooting, the material around eq. (**??**) might be worth revisiting.

2021-05-09 Sidney Thank you Matt, it makes sense now, given the definition of the single step Jacobian it just sort of pops out. So, at this point I know how to get both the traditional orbit Jacobian matrix (which I am currently working on coding in) and the orbit Jacobian matrix in terms of the single step Jacobian (which I also know how to derive).

2021-05-09 Sidney At this point, I am trying to understand the block matrix form that can be generated by applying a permutation matrix defined by the circular Kronecker delta. What I do not understand, is:

What is the difference between the circular Kronecker delta, and the regular one?

2021-05-10 Predrag Regular one is defined on $\mathbb{Z}$ (no restrictions), the circular one on C_n (periodic chain, so $\mod n$). (Re)read ChaosBook appendix X.3 *Discrete Fourier transforms*.

2021-05-09 Sidney And I am also having difficulty deriving the appropriate permutation matrix. I want to try turning the two-cycle orbit Jacobian matrix (in terms of single-step Jacobians) into the block matrix form. The following is my (incorrect) derivation of the 4x4 permutation matrix. I have assumed that when the index for the circular Kronecker delta reads something like $4, j$ it is equivalent to $0, j$, as that would be equivalent to going to the end of the columns (index j) and then starting back at the beginning. By doing this I get: P is a $[2n \times 2n]$ permutation matrix. Defined through the circular Kronecker delta: $P_{k,j} = \delta_{2k-1,j}$, for a periodic orbit of length $n = 2$ P is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(This is wrong)

What did I do wrong?

2021-05-10 Predrag By 'permutation matrix' you mean the one-step cyclic permutation r, or the shift matrix (**??**)? The circular Kronecker delta is just the $[n \times n]$ identity matrix. I leave it to Han and you to figure out what went wrong :)

2021-05-13 Han My Kronecker delta representation was wrong... (**??**) should be correct. I don't have a more compact way to write this permutation matrix. The permutation matrix for $n = 2$ is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

2021-05-10 Predrag To everybody - always number sites of length-n periodic chains (necklaces) as

$$\{0, 1, 2, \cdots, n-1\} ,$$

otherwise discrete Fourier transforms will go awkward on you.

2021-05-10 Predrag Regarding sect. **??** *"Center of mass" puzzle*:

Relevant Gallas papers are all in ChaosBook.org/library, for example (click here), with their names (always!) given by their BibTeX ID.

Predrag's unpublished 2004 drafts and calculations can be accessed by clicking on the link given there:

ChaosBook.org/projects/revHenon

There is no need to look at old drafts of papers, as the most of the relevant stuff is already included in that section, but the link includes some data files, orbit plots and programs that might be useful to Sidney as cross-checks on his calculations.

2021-05-12 Sidney Here are some notes from the last meeting: Any matrix can be written as a sum of projections into the different eigendirections. An anti-diagonal matrix, as well as the anti-diagonal matrix multiplied by sigma and its powers give the reflection matrix, and the reflection matrix across all axes.

Q What are these axes? A reflection matrix has two eigenvalues ± 1 (corresponding to either the symmetric, or antisymmetric subspace), and a projection operator can be constructed by subtracting out the opposite of the subspace you want to project onto. This projection operator can be applied to the left of the orbit Jacobian matrix to project it onto either the symmetric, or antisymmetric space. In this way, the orbit Jacobian matrix can be written as $\mathcal{J} = \mathcal{J}_- + \mathcal{J}_+$, it is worth noting that $\det(A+B) \neq \det(A) + \det(B)$. In the case of a linear map (where the fundamental fact still applies) the determinant of $\mathcal{J}_\pm$ counts the number of time symmetric/antisymmetric orbits. Note that we need to apply the time symmetric boundary conditions in order to enforce time symmetric orbits.

Q2 Is this last statement correct? I have now updated my code to work with the scaled Hénon map. Actually, I just did the lazy thing of dividing all values in all orbits by a. I also added a function to generate the (scaled) orbit Jacobian matrix, and I tried calculating its eigenvalues and vectors, it all seems to work, but I'm really not sure what I should be seeing for eigenvalues and vectors, I will paste my code below (I will past all of it, but it's mostly the same as before, just with a little added on the end): There is probably some extra code in there that just isn't being used, but hey, it's a work in progress. Update 2021-05-17: I have added this code to the williams/relax folder in the subversion, so I have now deleted the code here.

2021-05-13 Sidney With Han's help I was able to see the permutation matrix, and I was able to reproduce the Hill's formula "proof" for the specific case of n=2 (I use quotation marks because I only proved it for one case, and not all cases) I will probably keep thinking about how to generalize it, but I now feel better about applying Hill's formula to my orbit Jacobian matrix. I have been trying to work through the algebra with the reflection, and the corresponding projection matrices described in Han's most recent blog post. I understand qualitatively what's happening, effectively the projection operator is just the reflection operator with one of the eigenspaces subtracted out, with an additional weight that forces the

identity $\not\Vdash = \sum_i P_{A_i}$, this is a formula for a generic matrix. I cannot find anywhere in lectures, or textbooks where this is formalized, so I am not sure how the weights are constructed, but I'll keep looking at it.

2021-05-15 Matt I'll put a couple of responses here to hopefully help Sidney out. Response to questions in 2021-05-12 Sidney. It seems like the questions are with respect to the proceeding sentences but they are in different paragraphs so it is unclear to me if that is what you meant.

Q What are these axes? Is the question with respect to a specific system or just in general? If we're talking about reflection in D dimensional euclidean space then we can reflect over any $D-1$ dimensional hyperplane, so I'm assuming it's talking about reflection where the eigenvectors are the normal vectors to these planes, so the hyperplane which is normal to each eigenvector can define a reflection so maybe this is what you're referring to?

Q2 Is this last statement correct? What is "the last statement"? The last statement before the question, the last statement of the previous paragraph...? Is it the following?

> Note that we need to apply the time symmetric boundary conditions in order to enforce time symmetric orbits

To ensure that you get a time symmetric orbit then yes you have to impose boundary conditions which constrain you to that subspace. The unconstrained method (if variational in formulation) could possibly find these orbits, but there would be no guarantee.

Response to 2021-05-13 Sidney.

> I understand qualitatively what's happening, effectively the projection operator is just the reflection operator with one of the eigenspaces subtracted out, with an additional weight that forces the identity $\mathbb{I} = \sum_i P_{A_i}$, this is a formula for a generic matrix. I cannot find anywhere in lectures, or textbooks where this is formalized, so I am not sure how the weights are constructed, but I'll keep looking at it.

Where "this" is formalized: The "weights" are simply normalization coefficients. The sum of the projection operators equaling the identity essentially says that the "full" space can be decomposed into projection subspaces, each of which has its own projection operator. I.e. each subspace is a component of the full space. Predrag formalizes the projection operators *a lot*; especially in his group theory stuff which this is directly related to.

For a reflection operator we can decompose

$$\sum P_{\pm} = \frac{1}{2}(\mathbb{I} \pm R) = 1/2(\mathbb{I} + \mathbb{I} + R - R) = 1/2(2\mathbb{I}) = \mathbb{I}\,.$$

For a symmetry subspace that can be broken into 4 subspaces, the normalization would be $1/4$, etc.

2021-05-23 Sidney Thank you to Matt for the explanation, I also have now attended the group theory lecture on projection operators (I've actually been thinking about this a good bit, but taking the time to condense my thoughts onto my blog is not one of my strong suits). Anyway, the projection operator formalism for matrices can be derived from the Hamilton-Cayley theorem:

$$\prod_i (\mathbb{M} - \lambda_i \mathbb{I}) = 0$$

In words, this is just the statement that a matrix satisfies its own characteristic equation. However, if we take one element out of the product, the RHS will no longer be zero:

$$\prod_{i \neq j} (M - \lambda_i \mathbb{I}) = \prod_{i \neq j} (\lambda_j - \lambda_i)\, \mathbf{e}_i$$

We can rearrange to get:

$$\mathbb{P}_j = \prod_{i \neq j} \frac{\mathbb{M} - \lambda_i \mathbb{I}}{\lambda_j - \lambda_i}$$

Which is the definition of the projection operators that Han used, I was also able to reproduce his results. I also found out an issue with my code for finding time-reversal symmetric orbits. I was not taking into account permutations, but that is an easy fix, that I hope to fix quickly.

When I have, I can create a bank of orbits appropriate to use with the orbit Jacobian matrix projected into the symmetric subspace using projection operators, I know how to construct this both by hand, and by code, I just need to implement it. I also need to go through chapter 15 in Chaosbook, as it gives good visualization of what the Hamiltonian Hénon map does. I will add my notes here when I have a chance, hopefully tonight, or tomorrow. I also think that I should explore the volume of the parallelpiped represented by the orbit Jacobian matrix. I could probably use a package for that, but what I'll most likely do is just use the fact that the determinant of a matrix is just the product of its eigenvalues:

$$\det A = \prod_i \lambda_i$$

Then take the absolute value. I can then check the volumes generated by different orbits.

2021-05-26 Sidney I reviewed the spatiotemporal cat derivation to see if I could extend the idea to the Hénon map, I think I have.

If we enforce the following restrictions, we can qualitatively derive the spatiotemporal Hénon map:

- Each site couples to its nearest neighbors
- Spatial and temporal coupling is of the same strength
- Invariant under spatial translations
- Invariant under spatial reflections
- Invariant under space time exchange

With these conditions our temporal Hénon (**??**), (1.17)

$$a = \phi_{t+1} + \phi_t^2 + \phi_t$$

generalizes to my proposal for the spatiotemporal Hénon

$$\phi_{n,t+1} + \phi_{n,t-1} + 2\phi_{n,t}^2 + \phi_{n+1,t} + \phi_{n-1,t} = a\,. \tag{1.18}$$

The factor of two comes from adding together two Hénon maps (one spatial and one temporal), following the derivation of the spatiotemporal cat. I need to try to read Gutkin and Osipov **[GutOsi15]** arXiv:1503.02676.pdf to get a better idea of the derivation.

2021-05-28 Predrag I doubt you will find it in Gutkin and Osipov **[GutOsi15]** - if you do, cite their words in detail here. The above argument is mine, from Gutkin *et al.* **[GHJSC16]** and CL18 (kittens/ folder in this repository).

2021-05-28 Predrag Your spatiotemporal Hénon (1.18) looks OK to me, except maybe adding the maps for each direction of a d-dimensional lattice results in da on the RHS? In the spirit of eq. (80) in `CL18.pdf`?

How does your spatiotemporal Hénon compare to Politi and Torcini **[PolTor92]**, sect. **??** Periodic orbits in coupled Hénon maps?

2021-05-27 Sidney Here is my attempt at looking at the eigenvalues, eigenvectors, and the determinants of the orbit Jacobian matrix. My orbit Jacobian matrix constructor only works with orbits of length 3 and above, this is because is row has three values in it that permute, and I'm not sure how to scale it down, will discuss this at the next meeting. I cannot tell anything about the eigen stuff so far, it seems almost random, however, the determinant values are each approaching an integer value, so maybe that is something.

Note: I removed the table here because it was wrong

I will hopefully rearrange some of these tables to group them by orbit in a later post, but for now, here is the "raw" data, let me know if anything appears.

2021-05-31 Sidney I'm not sure why, but the large table with all of the eigenvalues does not appear in the pdf version of the blog. I am not sure how to fix that, if anyone could let me know, that would be great. I still need

to put it into a more readable format anyway. I also looked at eq. (80) in `CL18.pdf`, and it is the reason that I did not have $d * a$ on the RHS, the form of the spatiotemporal cat eq. (79), has only **M** not 2**M**, which it would have if both sides were added (right?), I need to read up on the coupled maps, that will be something I do next (along with the better tables).

That would also say that '1100' is symmetric under time inversion.

2021-06-03 Sidney I want to try to take advantage of Hill's formula:

$$\mathrm{Det}\,(\mathcal{J}_p) = \det\,(\mathbb{1} - \mathbb{J}_p) = (1-\Lambda_p)\left(1 - \frac{1}{\Lambda_p}\right) = 2 - \Lambda_p - 1/\Lambda_p\,. \quad (1.19)$$

So, I need to calculate the eigenvalues for the scaled time-step Jacobian matrix, but first I need to find the form of the scaled time-step Hénon map (1.20)

$$\begin{aligned} x_{n+1} &= 1 - a x_n^2 + b y_n \\ y_{n+1} &= x_n \end{aligned} \quad (1.20)$$

We scale $x \to \frac{1}{a}\phi$, $y \to \frac{1}{a}\varphi$, and the Hénon map becomes:

$$\begin{aligned} \phi_{n+1} &= a - \phi_n^2 + b\varphi_n \\ \varphi_{n+1} &= \phi_n \end{aligned} \quad (1.21)$$

Temporal stability of the nth iterate of the Hamiltonian Hénon map is [1]

$$M^n(\phi_0) = \prod_{m=n}^{1} \begin{bmatrix} -2\phi_m & -1 \\ 1 & 0 \end{bmatrix}, \qquad \phi_m = f_1^m(\phi_0, \varphi_0)\,. \quad (1.22)$$

It is very important to understand the Floquet multipliers are invariant under all smooth coordinate changes, see ChaosBook sect. 5.4, so they are not affected by this rescaling.. When we find the eigenvalues of this matrix they give contracting and expanding, we are interested in the expanding stability multiplier. There are three period 4 orbits: 1110, 1100, 1000, they have the following expanding Floquet multipliers:

2021-06-04 Predrag I do not seem to have handy list of the Hamiltonian $a = 6$ Hénon map Floquet multipliers. Floquet exponents for many $a = 1.4, b = 0.3$ orbits are listed in ChaosBook table 34.2. I believe that sect. **??** *Han's Hénon map Hill's formula* also applies to the dissipative case, might be worth by repeating the derivation with $b \neq -1$, or at least checking it for a few short periodic orbits.

[1] Predrag 2021-06-04: Was

$$\begin{bmatrix} -2\phi_n & \frac{b}{a} \\ 1 & 0 \end{bmatrix}$$

The correct form is (**??**). Hence, with Sidney's "scaled Jacobian," the four cycle Floquet multiplies were all wrong.

2021-06-11 Sidney I have read some of Wen's 2014 project ChaosBook.org/projects/Wen14.pdf, I should take some notes and put them here.

2021-06-11 Sidney The Hill determinant should equal $2 - \Lambda_p - 1/\Lambda_p$ by Hill's formula (1.19), where Λ_p is the expanding eigenvalue of the time-step Jacobian for that orbit, see table 1.1.

2021-06-11, 2021-06-23 Sidney Table 1.2 lists the Hill determinants computed from the orbit Jacobian matrices. Comparing with table 1.1 period-4 Hill determinants, we see that the Hill's formula (1.19) is satisfied to high precision.

2021-06-12 Predrag It looks like you now have the correct Hill determinants. Their numbers agree with ChaosBook Table 18.1; there are 6 period-5 orbits, and 9 period-6 orbits. The period-6 itineraries are not yet correctly assigned, fix that. The ones with even #'s of '1's are presumably the positive ones.

2021-06-12 Sidney I shall fix the itineraries for 6 cycles, and I'll add the itineraries of the other, shorter cycles later. Some were correct, and the negative determinants were in fact the odd number of 1s.

2021-06-13 Predrag Do include fixed points and the period 2 in table 1.2; might be helpful for understanding magnitudes of longer period Hill determinants.

2021-06-13 Predrag to Han and Sidney If you think of the Hénon map as a fattened parabola, then 0 fixed point has large positive slope (4 for the Ulam map), and 1 fixed point has small negative slope (-2 for the Ulam map). This explains the magnitudes, and should also determine the signs of Hill determinants in table 1.2.

However, either there is a - sign specific to Sidney's definition of the Hénon orbit Jacobian matrix, or Han and Predrag have to redefine both temporal cat and temporal Hénon orbit Jacobian matrix, so we do not pick up an extraneous '-' sign for odd period lattice states.

See also (**??**).

Fixing this is not essential, as we use only the absolute values of determinants in periodic orbit formulas.

2021-06-13 Sidney In the 2014 project paper by Haoran Wen, it is stated that for a range of a and b, the Hénon map is "structurally stable" and that means that the transport properties of the system have a smooth dependence on the parameters. And then it states that the lack of structural stability will result in the creation and destruction of infinitely many periodic orbits for any parameter change. Does that mean that a structurally stable system has relatively few bifurcations?

2021-06-13 Predrag It's a long story, starting with a wrong conjecture by Smale, but the answer is NO bifurcations for open intervals of system parameter values. Tends to be possible only for repellers, that is why you are working with Hénon $a > 6$ which has all possible binary symbolic dynamics cycles, and no bifurcations as you increase the parameter a.

Cat map has a finite grammar (is a "generating partition") for precisely s=*integer* >2, but change s to an open interval of real values around -let's say- s=3, and infinity of cycle get created and destroyed for any finite change in s.

2021-06-13 Sidney So, at a=6 we achieve all binary symbolic dynamics, and anything more than that we still have all binary symbols? No creation or destruction? Is there a proof for this in the blog or somewhere else easily linkable to, that wouldn't take me days to digest?

2021-06-13 Predrag No, not $a = 6$ - that's just the closest integer value. ChaosBook sect. 15.2 *Horseshoes* explains it, and -for example- Endler and Gallas **[EG05]** say "This classification is independent of the control parametera. Orbits are specially interesting for $a > 5.69931\ldots$, since beyond this value there is a complete Smale horseshoe **[HamSteDuMei99]** and all orbits are real."

2021-06-14 Sidney blogCats is being weird for me, it is only showing the most recently added parts, plus the table of contents, so right now when I build it, it only gives the dihedral groups chapter, even though all the include commands are not commented out.

2021-06-14 Predrag In `inclOnlyCats.tex` the uncommented line was

```
\includeonly{chapter/groups}
```

so `blogCats.tex` was doing what it should.

2021-06-14 Sidney to Predrag and Han I am having trouble defining the orbit Jacobian matrix for orbit length 1 and 2, this is because my code is designed to have a minimum width of 3 for the matrix, because there needs to be the terms for $\phi_{n+1}, \phi_{n-1}, \phi_n$, and I am not sure how to do that for 1 and 2 orbit lengths, should I just have the orbit repeat?

2021-06-14 Predrag The fixed points and period-2 lattices you can evaluate by hand, I believe. CL18 sect. 1.4 *Fundamental fact* and sect. 2.4 *Fundamental fact* evaluate the period-2 lattice orbit Jacobian matrices. If you understand how (**??**), (**??**), and CL18 (103) were derived, you will understand all such special cases.

2021-06-14 Sidney to Predrag and Han Is 6 digits of accuracy good enough? Is there a different method I should use? I am using the one from ChaosBook chapter 7 *Fixed points* (the link is right here, I'll use that style of referencing in the future).

2021-06-18 Sidney to Predrag and Han I am trying to reproduce the projection operator analysis for D_6 symmetric lattice states and I am not really sure how to do that, which of Han's posts talks about the lattice states? As well, doesn't the projection analysis apply for D_n in general, not just specific lattice states?

2021-06-19 Sidney to Predrag and Han I tried to read Endler and Gallas [**EG05a**] 2006 paper (not the Endler and Gallas paper [**EG05**] that Predrag mentioned) because Predrag said that it explained two period 3, one period 4, and two period 6 values in my table 1.2, but I am not quite sure how the authors are getting their P polynomials, or their S polynomials, specifically, I am not sure where the a value comes in, because when I just try to expand their eq. (3) and regroup it into eq. (6), I could not, so I am very confused. What obvious thing am I missing?

2021-06-19 Predrag To "expand their eq. (3) and regroup it into eq. (6)" you need to know analytically all period-4 periodic points, do you know them? They claim that "the solution of this problem is trivial because ref. [**EndGal02**] contains the solution for arbitrary a and b."

However, I was referring to the Endler and Gallas [**EG05**] Table 1. The invariant quantity that they associate with a periodic orbit is the orbital sum (**??**), the sum over the periodic points, while the Hill determinant involves various sums over values of fields raised to various powers. You will immediately note that the cases that have integer orbit sums correspond to your integer-valued Hill determinants. They had no reason to think of Hill determinants, so that would be a major reworking of their paper(s); I do not think you want to do that.

2021-06-20 Sidney to Predrag and Han Oops

I agree that a major reworking is probably not in the cards, I will go back to trying to understand orbit Jacobian matrices for the fixed point, and period 2 orbits. Endler and Gallas [**EG05a**, **EndGal02**] look exceptionally cool, I'll give them a whirl.

2021-06-23 Sidney I worked through both Endler and Gallas papers [**EG05a**, **EndGal02**], as much as I could. Here is what I came up with: in ref. [**EG05a**] the two important equations are $P_k(x)$ and $S_k(\sigma)$, where $S_k(\sigma)$ is found through manipulating the (scaled) Hénon map and σ is the sum of all points in the cycle, $P_k(x)$ is defined as

$$P_k = \prod_{\ell=1}^{k} (x - x_\ell)\,,$$

where x_l is the ℓth orbit point. The algorithm is simple: construct $P_k(x)$ by expanding the product and putting each coefficient in terms of σ, then use $S_k(\sigma) = 0$ to determine how many unique orbits of length k and 2.

what values σ can take, then combine with $P_k(x)$ to solve for each orbit point. I will work two examples, first an orbit of length 1:

$$\sigma = x_1$$

$$P_1 = x - \sigma$$

$$x_{t+1} = a - x_t^2 - x_{t-1} = x_1 = a - x_1^2 - x_1$$

Or

$$2\sigma + \sigma^2 - a = 0 = S_1(\sigma)$$

As this is a quadratic equation, we know that there are two different fixed points, and we can solve for them directly via $S_1(\sigma) = 0$, now for an orbit of length 2:

It is useful to first state that $a = 2x_2 + x_1^2 = 2x_1 + x_2^2$ by the Hénon map, and $\sigma = x_1 + x_2$

$$P_2(x) = (x - x_1)(x - x_2) = x^2 - \sigma x + x_1 x_2 = x^2 - \sigma x + \frac{1}{2}a - \frac{1}{2}\sigma^2 - \sigma$$

If we subtract the two a equations we get

$$0 = 2x_2 - 2x_1 + x_1^2 - x_2^2 = 2(x_2 - x_1) + (x_1 + x_2)(x_1 - x_2)$$

so

$$S_2(\sigma) = \sigma - 2 = 0$$

From this we know that there is one 2 cycle and this can be solved by solving for σ and solving for the roots of $P_2(x)$. There's a lot of algebra here, but I am sure there is some number theory trick that can be used that I am unaware of. I am also a little confused by the factorization (**??**)

$$S_k(\sigma) = C_k^2(\sigma) D_k(\sigma) N_k(\sigma)\,,$$

I think that it is a statement that the polynomial $S_k(\sigma)$ can be decomposed into polynomials that each give roots for the Chiral, Diagonal, and Non-diagonal orbits respectively. So it should be a way to count the number of each type of orbit. However, I am not sure how to tell the difference between any of them, I sort of understand that C_k has to be squared, and that could distinguish it, but I'm really not sure how to tell them apart.

Q16.1 Any suggestions for this?

A16.1 **Predrag 2021-07-04** Work through papers, they are clear and pedagogical

I have also figured out how to get the determinants of period 1 and period 2 cycles, I updated table 1.2 to include them. You'll notice that the determinant for $\overline{0}$ and $\overline{1}$ were negatives of each other up to six decimal places, neat. 10 was the integer -12 up to six decimal places, again, neat.

Q16.2 How do I calculate eigenvalues for periods 1 and 2?

A16.2 Here is how:

Period 1: For the fixed points, I get

$$\phi_{0,1} = -1 \pm \sqrt{1+a} \to -1 \pm \sqrt{7} \tag{1.23}$$

$\mathcal{J}_{0,1}$ are $[1 \times 1]$ matries

$$\phi_{t+1} + \phi_t^2 + \phi_{t-1} = a$$
$$F[\phi] = 2\phi + \phi^2 - a = 0\,,$$

Evaluating the Hill determinants (**??**) for both fixed points:

$$\mathcal{J}_{0,1} = 2(1+\phi)\,, \qquad \text{Det}\,\mathcal{J}_{0,1} = \pm 2\sqrt{1+a} \to \pm 5.2915026\,, \tag{1.24}$$

in agreement with the numerical estimates of table 1.2. The Hill determinants of the two fixed points are negatives of each other.

Period 2: The periodic points in the 10 orbit are (I did it on paper, do not want to reproduce it here):

$$\phi_{1,2} = 1 \pm \sqrt{a-3} \to 1 \pm \sqrt{3} \tag{1.25}$$

$\mathcal{J}_{10}$ follows

$$F[\phi]_1 = 2\phi_2 + \phi_1^2 - a = 0$$
$$F[\phi]_2 = 2\phi_1 + \phi_2^2 - a = 0\,,$$
$$\mathcal{J}_{10} = \begin{bmatrix} 2\phi_{01} & 2 \\ 2 & 2\phi_{10} \end{bmatrix}. \tag{1.26}$$

Hill determinants are *symmetric polynomials* in lattice fields $\{\phi_1, \phi_2, \cdots, \phi_n\}$, which are, by construction, all *prime* cycle p *invariants*. The orbital sum (**??**) is one example. Another one is (1.27).

In case at hand,

$$\text{Det}\,\mathcal{J} = 4(\phi_{01}\phi_{10} - 1) = 4(3-a) \to -12\,. \tag{1.27}$$

The Hill determinant is *exactly* 12, up to the annoying overall sign that cries out for a *redefinition* of orbit Jacobian matrixs.

2021-06-25 Sidney With the above analytic calculations, I feel very confident in stating that my code is accurate up to 6 decimal places.

Q16.3 Is there a way to rigorously prove that the code is accurate up to 6 decimal places?

Q16.4 Is this worth doing if it exists?

A16.4 **Predrag 2021-07-04** No.

Q16.5 What do these values mean? An integer Hill determinant should mean something right?

Table 1.1: Hill determinants for the Hamiltonian $a = 6$ Hénon map, period-4 lattice states, computed from time-evolution side of the Hill's formula (1.19). The pesky overall 'sign' presumably means we have to change the overall sign in the definition of orbit Jacobian matrix $\mathcal{J}$ everywhere.

Orbit	Hill determinant
1110	105.697960425014
1100	-576.000010077746
1000	1046.301985671792

Table 1.2: Hill determinants for the Hamiltonian $a = 6$ Hénon map, with correct symbolic dynamics. Indicated in red are values presumably explained by Endler and Gallas [**EG05**].

Period 1	
0	5.291502844
1	-5.291502494
Period 2	
10	-12.000000720
Period 3	
110	-53.914854639
100	133.914853323
Period 4	
1110	-105.697960425
1100	576.000010077
1000	-1046.301985671
Period 5	
11110	-388.996791481
11100	591.500599893
11010	712.689732105
00101	-768.203977660
00011	-4443.524089969
00001	7608.534459743

Period 6	
111110	-1045.3849327
111100	3899.9387739
111010	1092.9103354
111000	-4786.6149478
101000	5135.6190985
110100	-6396.0000670
001011	-6395.9999673
110000	32220.0609406
100000	-54576.5295457

A16.4 **Predrag 2021-07-04** It is well explained in papers you have been reading. Integer Hill determinant is a historical accident, due to our (arbitrary) choice $a = 6$. But it gave Gallas and collaborators a clue that something is going on. As does (1.32). Be Gallas.

And finally, I was thinking about the multidimensional Hénon map. Since this is not a physical problem, there is no "physical" definition about what makes a map "Hénon map" like, so it would be good to stick to the mathematical requirement of the folding being linearly related to b, so I was thinking that for the multidimensional map, the Hénon-ness could be satisfied if the appropriate time Jacobian matrix determinant would be $-b^d$, unfortunately, I don't know how to define the correct time Jacobian.

2021-07-03 Sidney I have not had much time outside of my internship lately, so I haven't done much. However, I did make an attempt at showing that the eigenvalues of the orbit Jacobian matrix are coordinate-choice independent. It did not go well. First, I must remember that

$$\mathcal{J}_{ij} = \frac{\delta F[\phi]_i}{\delta \phi_j} \,,$$

evaluated at a lattice state ϕ_M

$$F[\phi_M] = 0 \,.$$

This gives a problem when I try to repeat the proof done for time-step Jacobians, because I get

$$\mathcal{J}'(\phi'_M)_{ij} = \Gamma(0)_{ik} \mathcal{J}_{kl} \Gamma^{-1}(\phi_M)_{lj} \,, \tag{1.28}$$

which means that I cannot cancel the Γs in the determinant. So, I have failed to prove anything.

2021-06-24 Predrag Wow, I did not expect $\mathrm{Det}\, \mathcal{J}_0 = -\mathrm{Det}\, \mathcal{J}_1$! But Sidney's (1.23) nails it.

2021-06-13 Predrag Note the '.91485' decimal digits for period 3. Those are presumably explained by Endler and Gallas **[EG05]** analytic expressions, see their Table 1. For us they mean that Sidney's code is accurate to ca. 6 significant digits.

The 1100 Hill determinant is integer $576 = (6 \times 4)(-6 \times -4)$, (see (1.32)). Explain this factorization.

(1.26) explains 01 Hill determinant=12, but do you have an argument that this is the symmetry reduced Hill determinant=$2\sqrt{3}$ squared?

Show that 001011 and 001101 Hill determinants are (integer)^2 (are they?).

This is also a helpful check on the time-inversion factorization formulas Han and I are trying to establish.

2021-07-06 Predrag Regarding Sidney's attempt (1.28) to prove that the eigenvalues of the orbit Jacobian matrix $\mathcal{J}$ are coordinate-choice independent:

The time-evolution Jacobian matrix in general has a different left $\Gamma(\phi_t)_{ik}$ and right $\Gamma^{-1}(\phi_0)_{lj}$ $[d\times d]$ matrices, they line up only for the period value $t = n$, so the periodic boundary condition will have to be a part of your proof.

How the time-periodicity is built into orbit Jacobian matrices is explained by (**??**). From that you can perhaps see how the periodicity is imposed on the coordinate-change Jacobian $[nd \times nd]$ matrices $\Gamma(\mathsf{X})_{lj}$...

See whether you can prove it first for the Hill determinant $\det \mathcal{J}$?

To get it for individual eigenvalues, you'll have to write the eigenvalue, eigenvector equation for the orbit Jacobian matrix $\mathcal{J}$, then apply coordinate transformation $\Gamma(\mathsf{X})_{lj}$.

2021-07-06 Sidney I have considered showing that the Hill determinant is coordinate invariant, but I think that's just a matter of mentioning that Jacobian matrix has coordinate invariant eigenvalues. I'll formalize that in a later post (most likely the next one).

I have also come up with a proof that the $\mathcal{J}$ has the same set of eigenvalues evaluated at every point in the orbit, I suspect that there is a proof for the eigenvectors, but I don't know how to do it, again, will formalize on my next post.

2021-07-07 Predrag The orbit Jacobian matrix $\mathcal{J}$ is global, a property of the entire lattice state, so I do not understand "$\mathcal{J}$ has the same set of eigenvalues evaluated at every point in the orbit."

2021-07-06 Sidney I am not sure what was meant by "do you have an argument that this is the symmetry reduced Hill determinant=$2\sqrt{3}$ squared?". I assume this is something either from the group theory course, or blog which I have not yet gone over, but does this mean that I should look for some symmetry reduction of a matrix whose Hill determinant has the value $2\sqrt{3}$?

2021-07-07 Predrag Basically, yes. I'm referring to C_n^2 term in (**??**), $\sqrt{\zeta_{top}(t^2)}$ in (**??**), etc., throughout the time-reversal discussions in the blog.

Endler and Gallas **[EG05a]** eq. (9) and Table 1 has

$$D_{0011} = \sigma\,, \quad P_{0011} = (x^2 - a)^2\,, \tag{1.29}$$

There are two period 6 diagonal orbits

$$\mathrm{D}_6 = \sigma^2 + 4\sigma - 4a\,, \quad \text{orbits } 000111 \text{ and } ??\,, \tag{1.30}$$

but 000111 of figure **??** (c) belongs to N_6 messy polynomial eq. (14).

My reasoning is that due to the $\{1, s\}$ time-reversal symmetry, any $t < 0$ temporal lattice site can be mapped into $t > 0$ by time reversal s, so the D_∞ 'configuration' fundamental domain is the $t \geq 0$ temporal half-lattice. Full lattice periodic states Hill determinants for orbits such as 01, 0011 in table 1.2 are then (that's not quite right) squares (twice the relative periodic orbit period) of the fundamental domain orbit Hill determinants, as in example **??** *D_1 factorization*.

But there no reason why you should know that, Han and I are still working it out, will have more concrete suggestions for the Hénon case once we understand it better.

2021-07-07 Predrag Endler and Gallas [**EG05a**] eq. (13) presumably explains the integer valued pair of period-6 orbits in table 1.2:

$$C_6 = \sigma - 2\,, \qquad \text{orbits } 110100, 001011\,, \tag{1.31}$$

2021-06-25 Sidney My numerical values for period 4 eigenvalues:

$$\begin{array}{ll} 1000 & 6.77624515, -7.4374406, -4.23778399, -\sigma, \\ 1110 & -2.39080489,\ 1.58070478,\ 5.70907942,\ \ \sigma, \\ 1100 & 6,\ 4, -6, -4 \end{array} \tag{1.32}$$

2021-07-04 Predrag to Sidney and Han What's up with two distinct orbits $\overline{1000}$ and $\overline{1110}$ in (1.32), with different Hill determinants in table 1.2, sharing the eigenvalue $\sigma = 2\sqrt{6} = 4.89897947$ (see (1.33))? Well... when two numbers agree to 9 significant digits, it's usually a mere numerical coincidence. Happens $1/1\,000\,000\,000$ of time :)

2021-07-25 Predrag Endler and Gallas [**EndGal02**] have all period-4 periodic points:

$$S_4(\sigma) = \sigma(\sigma^2 - 4a) \tag{1.33}$$

The the 2-points on diagonal $\overline{0011}$ of figure **??** (b) has $\sigma = 0$. For $a = 6$ the $\overline{1000}$ has $\sigma = -2\sqrt{6} = -4.898979485566356$ and $\overline{1110}$ has $\sigma = 2\sqrt{6}$, which happens to be their common eigenvalue in (1.32). Endler and Gallas [**EG05a**] also define

$$\begin{aligned} \alpha &= \sqrt{6 + 2\sqrt{6}} = 3.3013602478 \\ \beta &= \sqrt{6 - 2\sqrt{6}} = 1.04929524655\,. \end{aligned} \tag{1.34}$$

and the corresponding orbital equations

$$\begin{aligned} P_{1000}(x) &= (x^2 - \alpha^2)(x + \sqrt{6})^2 \\ P_{0111}(x) &= (x^2 - \beta^2)(x - \sqrt{6})^2 \\ P_{0011}(x) &= (x^2 - 6)^2\,. \end{aligned} \tag{1.35}$$

2021-07-25 Predrag For a 4-cycle p, the Hill determinant of the orbit Jacobian matrix (**??**) is the polynomial

$$\mathrm{Det}\,(\mathcal{J}_p) = 2^2\left[2^2 x_0x_1x_2x_3 - x_0x_3 - x_1x_2 - x_2x_3 - x_1x_0\right]\ , \qquad (1.36)$$

not involving the orbit sum σ, not in the form currently written. Looks like one should rescale ϕ_i (again?).

The quadratic part is a sum of sequential pairs. There are two kinds of ways in which it can be time-reversal invariant:

- 2 on diagonal x_0x_2 fixed, $x_1 = -x_3$

$$\mathrm{Det}\,(\mathcal{J}_p) \;=\; 2^2\left[-2^2 x_0{x_1}^2x_2\right]\ , \qquad (1.37)$$

- none on diagonal, $x_0 = -x_1$, $x_2 = -x_3$

$$\begin{aligned}\mathrm{Det}\,(\mathcal{J}_p) &= 2^2\left[2^2{x_0}^2{x_2}^2 + 2x_0x_2 + {x_2}^2 + {x_0}^2\right]\\ &= 2^2\left[(2x_0x_2)^2 + (x_0+x_2)^2\right]\ , \qquad (1.38)\end{aligned}$$

At the first glance, Hill determinants do not seem to factorize, but the time reversal symmetry assumptions (and signs) have to be checked.

2021-07-25 Predrag Some while-falling-asleep reflections on orbits $\overline{1000}$ and $\overline{1110}$ in (1.32), sharing the eigenvalue $\sigma = 2\sqrt{6}$:

If an orbit has a symmetry H, all of its lattice states (periodic points) presumably live in an invariant subspace $\mathcal{M}_H$. An example is Kuramoto-Sivashinsky, where orbits that start in the antisymmetric subspace stay in this lower dimensional subspace.

Example **??** possibly explains this for binary symbolic dynamics (note: there our '$0,1$' are denoted '$-,+$'). In table **??**
the pair $\{-+++,+---\} \to$ fundamental domain $\overline{0011}$,
i.e., we are back to our perennial problem of mistaking internal dynamical symmetries for time reversal, not sure this symmetry reduction is the one we need.

My hunch is that *all* short orbits live in invariant subspace(s), with $\overline{110100}$ and $\overline{001011}$ being the first exceptions.

Orbit $\overline{1000}$ is like figure **??** (c), placed in the upper right corner, with no points on the diagonal. Orbit $\overline{1110}$ is in the lower left, also symmetric across the diagonal. Endler and Gallas **[EG05a]** plot them in their fig. 2. According to J. Montaldi table **??**, the D_4 permutation representation irreps are is $A_0 + B_1 + E$.

In the 2-dimensional invariant subspace E (?) orbits $\overline{1000}$ and $\overline{1110}$ are period-2 orbits (I'm guessing, have not checked) and their eigenvalues might be related, like, for example, $\overline{0}$ and $\overline{1}$ in table 1.1. Or there is shared eigenvalue in one of the 1-dimensional subspaces. Remember the z-axis

dynamics for the Lorenz flow? Read ChaosBook Desymmetrization of Lorenz flow (here example **??**) to understand that not every linearly independent space is invariant under time dynamics.

However, the orbit Jacobian matrix $\mathcal{J}$ is always a perturbation in the full $\mathcal{M}$, thus 3 other eigenvalues, all different. I would be happier if there were only 2 distinct eigenvalues per each cycle, but you cannot have everything. At least, not if you don't try.

 example **??** p. **??**

 example **??** p. **??**

Basically, also for nonlinear systems orbit Jacobian matrix is linear, so irreps of the symmetry group do block-diagonalize it. A stronger claim; symmetry can restrict entire orbits to flow-invariant subspaces of the state space $\mathcal{M}$, even for nonlinear flows. Then some of the orbit Jacobian matrix eigenvectors point into that subspace.

2021-07-25 Predrag Combination $r + r^{-1}$ in (1.16) commutes with σ, and σ conjugacy reverses $\mathbb{X}$

$$\begin{aligned}\sigma\mathcal{J}\sigma &= r + 2\sigma\mathbb{X}\sigma + r^{-1} \\ &= \begin{bmatrix} 2x_{n-1} & 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 2x_{n-2} & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2x_2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & \dots & 2x_2 & 1 \\ 1 & 0 & \dots & \dots & \dots & 1 & 2x_1 \end{bmatrix}\end{aligned} \tag{1.39}$$

Next, evaluate Hill determinant with a projection operator inserted. Should factorize?

$$\begin{aligned}\mathrm{Det}\,\mathcal{J} &= \mathrm{Det}\,(r + 2\sigma\mathbb{X}\sigma + r^{-1}) \\ &= \mathrm{Det}\,(r + 2\mathbb{X} + r^{-1})(P_+ + P_-) \\ &= \mathrm{Det}\,(P_+\mathcal{J})\mathrm{Det}\,(P_-\mathcal{J})\end{aligned} \tag{1.40}$$

2021-08-04 Han Yes! Read the text starting about (**??**) to see how that works.

2021-08-08 Sidney I have been out of the loop for awhile, so what I'm going to try to do is read up on what I missed, and take notes on that, and write it up in my blog, and then continue on with the work here, hopefully that can all happen in a timely fashion.

2021-08-20 Sidney I have been reading LC21 and other parts of Han's blog and taking notes as appropriate, as this already exists in this blog, I'll only talk about it when I can add something. Anyway, I've been thinking about the time reversal pairs in the temporal Hénon. In the case of 110100 and 001011, the determinants of the orbit Jacobian matrices were equal

and they are a time reversal pair. I think that maybe we can try to make a global statement about the relative weights of time reversal pairs. So, here is the first step in trying to see that.

First, I will remind everyone of the definition of the orbit Jacobian matrix:

$$\mathcal{J}_{ij} = \frac{\partial F[\mathsf{X}]_i}{\partial \phi_j}$$

This is defined for any lattice state: $\mathsf{X} = [\phi_1, \phi_2, \cdots, \phi_n]$, i defines the lattice point which is considered the "current" location, ie. for the temporal Hénon, $i = 1$ says that the first lattice state is what we should use for n^{th} in time, instead of $n + 1$ or $n - 1$. j defines the lattice point which the defining equation will be differentiated with respect to. By definition, any periodic orbit can be cyclically permuted and still be the same periodic orbit. When this happens the indices in the lattice state get shifted, say $1 \to 3$. In this case, the indices in the orbit Jacobian matrix must be redefined in accordance to the shift, if we want to know how the "original" orbit and the "permuted" Jacobians relate. As both indicies in the definitions of the orbit Jacobian matrix depend on the same index definition for the lattice state, when the lattice state is permuted by some number of steps p the indices in the definition for orbit Jacobian matrix change as follows

$$i \to \alpha \equiv i + r\,, \quad j \to \gamma \equiv j + r\,. \tag{1.41}$$

In fact, if we ignore the rules for "sameness" of orbit and say that the order of the lattice state can be arranged arbitrarily, the indices in the orbit Jacobian matrix definition are mapped as follows: $i \to \alpha \equiv f(i)$ and $j \to \gamma \equiv f(j)$, where $f(x)$ is some one-to-one map appropriate for the rearrangement applied to the lattice state. If we define a "diagonal entry" of the orbit Jacobian matrix as when the difference between the indices is zero, we can construct two indicator functions:

$$\Delta_{ij} = i - j$$

$$\Delta'_{\alpha\gamma} = \alpha - \gamma = f(i) - f(j)$$

As can be seen, when $i = j$ both indicator functions are zero, indicating that a diagonal entry in one index space, is a diagonal entry in the other index space. In fact, as $f(x)$ is one-to-one by definition there are an equal number of diagonal entries in each index space. And finally, as the definitions of the orbit Jacobian matrix in either index space are isomorphic:

$$\frac{\partial F[\mathsf{X}]_i}{\partial \phi_j} \simeq \frac{\partial F[\mathsf{X}]_\alpha}{\partial \phi_\gamma}$$

and as the individual lattice values are unchanged by the rearranging, not only are there an equal number of diagonal entries in each index space, but the same values exist in each. Thus, the (unordered) set of diagonal

orbit Jacobian matrix entries is invariant under one-to-one index mappings, ie. arbitrary rearrangements of lattice state order.

This shows that under time reversal, the diagonal entries of an orbit Jacobian matrix are preserved, even if the orbit is not time reversal symmetric. I may have made a mistake here, either in the math, or just basic notation, please let me know!

2021-08-23 Predrag I am not sure about this proof, discuss it with Matt and Han first.

Here how I think about (as always, I might be wrong): The beauty of our spatiotemporal, global approach is that every lattice state (ie, *a* solution of the defining equations of a particular problem) is a fixed point in its high-dimensional state space. So, if you can show that the eigenvalues of a fixed point problem in $1, 2, 3, \cdots, 63\,873, \cdots$, dimensions do not change under a smooth nonlinear change of fields, you have proven what we need to prove.

If you understand it 1 or 2 dimensions, you probably understand it any number of dimensions.

Reflection symmetry (sometime known as time reversal) comes in (see ChaosBook sect. 8.3) as an additional set of relations between the stability eigenvalues.

2021-08-23 Sidney At this point, I have pretty much settled on wanting to work on the mathematical physics end of plasma physics. Specifically with turbulence, and nonlinear aspects of fusion and astrophysics. But I quite frequently worry that I will miss out a great deal by not working with quantum, especially path integrals and field theories. I know that you transitioned from high energy to nonlinear dynamics and turbulence. How did you find that? And how analogous is the math? I know that for awhile there was quite a bit of overlap between turbulence and QFT methods, but that seems to have fallen by the wayside.

2021-08-23 Predrag Mhm. My impression is that much is going on, for example here, and you just happen to be on the most fearless and inventive team in the field.

Don't be Fritz Haake (who accepted the invitation on 30 May 2011). He, who hesitates, is lost.

2021-08-25 Predrag I keep saying that the proof of the invariance of orbit Jacobian matrix $\mathcal{J}$ eigenvalues for a nonlinear but nonsingular redefinition of fields ϕ_i is a variant of ChaosBook sect. 5.4 *Floquet multipliers are metric invariants*, but I'm not getting traction on that from anyone.

In today's group meeting, I interpreted Sidney's proof of the invariance of orbit Jacobian matrix $\mathcal{J}$ eigenvalues (1.41) as a permutation matrix on site labels, made a claim that any other permutation than D_n cyclic

ones or their reversals will change the value of temporal Hénon Hill determinant, and challenged Sidney to compute Hill determinant for other permutations, see that the resulting determinant is different.

But for temporal Hénon I am probably wrong, as Endler and Gallas [**EG05a**] prove that all their polynomials depend only on the orbital sum (**??**).

I believe that will not be true for the ϕ^4 theory (**??**) on d-dimensional lattice (**??**), with the Hill determinant of the same form (**??**), see sect. **??** *Classical ϕ^4 lattice field theory*, because in that case bilinear terms in lattice fields arising from (**??**) cannot be eliminated.

Sidney could easily write the program to find all of its 3^n lattice states for small n; it is mostly a replacement of the quadratic function in his temporal Hénon Biham-Wentzel program by a cubic one.

2021-08-26 Sidney I mostly tried to write the proof to see if I could show that the diagonal values were an invariant set for rearrangements of the orbits, because if I could, I could say something about the equal determinants of the length 6 time reversal pairs I calculated for the temporal Hénon. I need to look more at other cases.

As well, I tried again to look at varying the proof from ChaosBook sect. 5.4 *Floquet multipliers are metric invariants*. But I run into the issue that the fixed point condition for the map which is having its derivative taken for the orbit Jacobian matrix, is different that the fixed point condition for the map having its derivative taken for the time-step Jacobian. Namely, from CL18 eqn 13, it is stated that $F[\Phi] = 0$ is the fixed point condition, NOT $F[\Phi] = \Phi$ which would be required for the proof from Chaosbook to be carried out in the same way. I also tried to take Predrag's suggestion of looking directly at permutation matrices, and then taking the determinant to show that everything is invariant around an orbit. I tried the shift, i.e., the permutation matrix (**??**)

$$r^s_{ij} = \delta_{i,j+s}\,,$$

shifting each entry backwards by s steps. Shifting the n-dimensional *vector* $F[\mathsf{X}]$

$$f[\varphi] = F[r^{-s}\mathsf{X}]$$

a function of the n-dimensional lattice state vector X, such that the orbit Jacobian matrix

$$\mathcal{J} = \frac{\partial f[\varphi]}{\partial \varphi} = \frac{\partial r^s F[r^{-s}\mathsf{X}]}{\partial r^{-s}\mathsf{X}}\,.$$

And after this point I am stuck, mostly because I am not sure if this is right, and it's weird dividing by a matrix, although, I probably need to take the derivative of the change of coordinates that I defined.

2021-08-26 Sidney For the one-dimensional case (eqn 5.15 in Chaosbook) the final conclusion relies on the fixed point condition being $f(x) = x$. However, $F[\mathsf{X}]$ is effectively defined as $f(x)-x$. This causes a problem whether we're looking at scalars or vectors.

I thought a little more about the permutation proof, r^s is not position dependent, so there is no weird Jacobian shenanigans, using the chain rule we should just get

$$\mathcal{J} = \frac{\partial f[\varphi]}{\partial \varphi} = r^s \frac{\partial F[\mathsf{X}]}{\partial \mathsf{X}} r^{-s} .$$

I think that this is right, then we can just move around the determinant by the permutation property. I should think more how to relate this to eigenvalues.

2021-08-27 Sidney What is this "square root" thereof thou speaketh?

2021-08-05, 2021-08-28 Predrag It's been fuzzy all along, but roughly speaking it is this: Stability (at least, temporal evolution stability) is multiplicative along an orbit, so if you go twice as many time steps (lattice sites in our perspective), the stability gets squared.

Conversely, in going from period $n = 2m$ (**??**) D_8 symmetric orbit $\overline{\phi_1\phi_2\phi_3\phi_4|\phi_4\phi_3\phi_2\phi_1|}$ to the orbit Jacobian matrices evaluated on the m-dimensional $\phi_1\phi_2\phi_3\cdots\phi_m$ subspaces (**??**), the stability gets square-rooted. There are many little things that I do not understand about how this works in detail that you could easily work out

1. What type from the list (**??**)-(**??**) is each of the short temporal Hénon orbits that you have? Han tells me my guesses (1.42) to (1.44) are wrong.
2. What is its Hill determinant? Which block of orbit Jacobian matrices (**??**) goes with which orbit? What are its eigenvalues, the symmetries of eigenvectors?
3. What is the relation between our (**??**)-(**??**) and Endler and Gallas **[EG05a]** symmetry classifications?

Inspecting Endler and Gallas **[EG05a]** fig. 2: the Hénon period-4 orbits, $n = 2m$, are of form

$$\overline{1000}: \quad \overline{\boxed{\phi_0}\,\underline{\phi_1}\,\boxed{\phi_0}\phi_1}\,, \tag{1.42}$$

and

$$\overline{1110}: \quad \overline{\boxed{\phi_0}\,\underline{\phi_1}\,\boxed{\phi_2}\phi_1}\,, \tag{1.43}$$

with symmetric-antisymmetric subspace dimensions $d_+ = 3$, $d_- = 1$, and

$$\overline{1100}: \quad \overline{\underline{\phi_2}\,\underline{\phi_1}\,|\phi_1\phi_2|}\,, \tag{1.44}$$

with symmetric-antisymmetric subspace dimensions $d_+ = 2$, $d_- = 2$.

I leave it to the gentlepersons of this blog to compute the $[1\times 1]$ Hill determinants $\mathrm{Det}\,(\mathcal{J}_-)$ for $\overline{1000}$ and $\overline{1110}$ in (1.32), show their sole eigenvalue is $\pm\sigma = 2\sqrt{6}$.

8/29/2021 Sidney I am very confused. First off, I found out that I don't know how to block-diagonalize matrices using symmetry operators, I tried to recreate the CO_2 example in the group theory notes, but to no avail. So I turned to trying to show Hill determinants $\mathrm{Det}\,(\mathcal{J}_-)$ for $\overline{1000}$ and $\overline{1110}$ in (1.32) is, $\pm\sigma = 2\sqrt{6}$. But I am missing a factor of two, and I don't know why. I did the following:

The symmetry is an "even" reflection as defined in (**??**), so the symmetry matrix is

$$\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

From here, I can construct projection operators for the symmetric and anti-symmetric subspaces of this operator:

$$P_+ = \frac{1}{2}(I-\sigma) \quad P_- = \frac{1}{2}(I-\sigma)$$

Taking the trace of each of these operators gives me the dimension of each of these subspaces: $d_+ = 3, d_- = 1$. Now, let's look at $\overline{1000}$ $\phi_0, \phi_1, -\phi_0, \phi_1$. In this case the (incorrect! - see (1.46)) orbit Jacobian matrix is

$$\mathcal{J} = \begin{bmatrix} \phi_0 & 1 & 0 & 1 \\ 1 & \phi_1 & 1 & 0 \\ 0 & 1 & -\phi_0 & 1 \\ 1 & 0 & 1 & \phi_1 \end{bmatrix} \tag{1.45}$$

$$\sigma\mathcal{J} = \begin{bmatrix} \phi_0 & 1 & 0 & 1 \\ 1 & 0 & 1 & \phi_1 \\ 0 & 1 & -\phi_0 & 1 \\ 1 & \phi_1 & 1 & 0 \end{bmatrix}$$

The trace of $P_-\mathcal{J} = \frac{1}{2}(\mathcal{J} - \sigma\mathcal{J})$ should give me the eigenvalue for the asymmetric subspace, this gives ϕ_1 which equals $\sqrt{6}$ from Endler and Gallas [**EG05a**], which is missing a factor of -2, I have no idea what's wrong (since corrected in(1.46)).

2021-08-29 Predrag Getting it up to factor of 2 is a triumph! The rest is work :) Have you tried cross-checking formulas? One cannot trust anyone, one always makes sure that the formulas are as you yourself have derived them. I would not be surprised if there should be $2\phi_j$ along the diagonal...

In this context: (**??**) and the footnote next to it will amuse you.

A small aside - when you refer to an equation, like (**??**), refer to it, rather than having the reader try to figure out where it came from. It's much faster for everyone if you just do it.

A much less important thing at this stage: The macros such as σ [backslash Refl] are here for a reason - as the research progresses we often find that a better notation exists in literature. That can be fixed by editing a few characters in *siminos/inputs/defsSpatiotemp.tex*.

2021-08-29 Predrag to Sidney Here is a request that requires minimal work - you have it in your code or data sets: Plot the values of lattice state fields for the 6 lattice states of table **??** and figure **??** in the same format as figure **??** (b).

Note - temporal Hénon fields can be positive or negative. I'm particularly interested to see if any of your lattice states are antisymmetric under reflection across ϕ_0.

2021-09-01 Sidney In my last post I used the incorrect definition (1.45) of $\mathcal{J}$, the correct definition is

$$\mathcal{J} = \begin{bmatrix} 2\phi_0 & 1 & 0 & 1 \\ 1 & 2\phi_1 & 1 & 0 \\ 0 & 1 & -2\phi_0 & 1 \\ 1 & 0 & 1 & 2\phi_1 \end{bmatrix} \tag{1.46}$$

This, along with remembering that ϕ_1 is negative for the 1000 orbit, fixes the factor of negative 2 I was missing. I have also figured out my block diagonalization issue from before. Now the question is, why would $\overline{1000}$ and $\overline{0111}$ have equal but opposite eigenvalues for the asymmetric subspace of the reflection operator (is this the correct vocab?). I also think I know what is wanted for the plots, I will do that.

Back to an earlier project: eigenvalues of $\mathcal{J}$ in different coordinates. $\mathcal{J}$ is not a derivative on space (like the one time step Jacobian is), it is instead a derivative on periodic lattice points (again, could be incorrect vocab, will work on this). So, maybe instead of using a regular coordinate transform, we should a transform specifically on the periodic orbit? Maybe that would help with the issue of the fixed point condition being $F[\mathsf{X}] = 0$ instead of $F[\mathsf{X}] = \mathsf{X}$.

2021-09-01 Sidney to Predrag What colors should I use for the bars when I do your plotting suggestion for temporal Hénon, the colors mattered in the other bar graphs.

2021-09-01 Predrag Quality of a plot does not matter much at the exploratory stage, I know by plotting by hand the shapes of the 6 lattice states - they follow from their symbolic dynamics. However, if you have accurate numbers for fields and eigenvalues, you could discover the symmetries that I am missing in hand-sketches. Or you can put intelligible data files

in you computing folder *siminos/williams/* and make Predrag do your work, as in (1.32) :)

Plot the values of lattice state fields for the 6 lattice states of table **??** and figure **??** in the same format as figure **??** (b), using the same color scheme. When I plot them, I place the yellow bar at 0, then two red bars to the left and two blue to the right. You can also superimpose symbol dynamics code upon it - you will immediately understand '0's are negative and '1's are positive.

2021-09-03 Predrag To determine C_5 period-5 states of table **??** you only need to determine the D_5 length-3 block lattice state defined by boundary conditions of (**??**) - you might want to check whether the lattice states so obtained agree with the ones you already have.

Their Hill determinants are the determinants of the 3-dimensional orbit Jacobian matrix (**??**).

To determine C_6 period-6 states of table **??** you only need to determine the corresponding D_6 length-4 or -3 block lattice state defined by boundary conditions analogous to (**??**). Their Hill determinants are the determinants of 4- or 3–dimensional orbit Jacobian matrix (**??**) or (**??**).

2021-09-03 Predrag 2 Andrew afugett3 = fuggedaboutit

2021-09-06 Sidney 2 Everyone Sorry for the long silence, just settling back into everything. I will be working on the plots today, and will update as I go along, as well, I will show Andrew how to get the repository on his laptop.

Period $n = 5$ lattice states of table **??** plotted as in figure **??**. They are all reflection symmetric, with fixed lattice field $\boxed{\phi_0}$ colored gold. Note that the symbolic dynamics is given by the signs of lattice site fields. This plot is now superseded by figure **??**. (right)

2021-09-06 Sidney I completed the plots for both lattice field values, and the associated eigenvalues. The formatting is not finished yet, but I will change that later. Currently, I have the lattice state that remains fixed colored gold, and the ones that flip colored maroon.

2021-09-07 Predrag .

1. Figure 1.2 (a) agrees with my own sketch, but only examination of the actual ϕ_j values can reveal further symmetries and factorizations.
2. Figure 1.2 (b) currently does not make much sense to me. 3 of the eigenvalues should belong to the symmetric subspace (1.47), (**??**), 2 to the antisymmetric subspace (**??**).
3. 2021-09-27: Figure 1.2 (a) is now superseded by figure **??**.

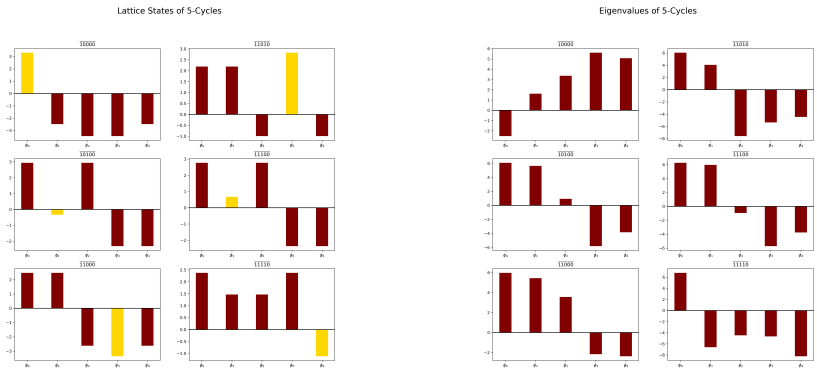

Figure 1.2: Temporal Hénon (**??**), $a = 6$: (left) The associated orbit Jacobian matrix eigenvalues (?). Currently we have no interpretation. Should't the corresponding eigenvectors be (anti)symmetric?

2021-09-06 Sidney For a D_5 lattice state, the following boundary conditions are respected: $\phi_i = \phi_{i+5}$ and $\phi_{-i} = \phi_i$, if we follow (**??**), we find the temporal Hénon orbit Jacobian matrix in the symmetric subspace is

$$\mathcal{J}_+ = \begin{pmatrix} 2\phi_0 & 2 & 0 \\ 1 & 2\phi_1 & 1 \\ 0 & 1 & 2\phi_2 + 1 \end{pmatrix} \tag{1.47}$$

As well, from (**??**), I can find the 3 equations that define this sort of orbit for temporal Hénon

$$\begin{aligned} \phi_0^2 + 2\phi_1 &= a \\ \phi_0 + \phi_1^2 + \phi_2 &= a \\ \phi_1 + \phi_2^2 + \phi_2 &= a \end{aligned} \tag{1.48}$$

Is there a good way of solving this system analytically? Otherwise, should I just check it by numerically finding solutions?

2021-09-07 Predrag No, for us getting into the Endler-Gallas annalytic solutions is getting too deep into the weeds. Use the same Biham-Wentzel program you have already written, with the boundary conditions added. The wonderful thing is that you are looking for the roots of an order 2^3 polynomial rather than 2^5.

To compute the Hill determinant $\mathrm{Det}\,\mathcal{J}_+$, Han would recommend doing the discrete Fourier transform first.

2021-09-07 Predrag Currently I prefer the (**??**) form of the temporal Hénon to (**??**), but that is not very important at this stage.

2021-09-07 Sidney The full orbit Jacobian matrix for the D_5 cycles of the temporal Hénon commutes with reflection about the center lattice state (**??**). Therefore, we can block diagonalize to find the orbit Jacobian matrices of the symmetric and antisymmetric subspaces:

$$\mathcal{J}_{D_5} = \begin{pmatrix} 2\phi_2 - 1 & 1 & 0 & 0 & 0 \\ 1 & 2\phi_1 & 0 & 0 & 0 \\ 0 & 0 & 2\phi_0 & 2 & 0 \\ 0 & 0 & 1 & 2\phi_1 & 1 \\ 0 & 0 & 0 & 1 & 2\phi_2 + 1 \end{pmatrix} \tag{1.49}$$

Which gives the same matrix for the symmetric subspace as (1.47), therefore either boundary conditions or projection operators are effective for these sort of calculations.

I took a look at my code, and it seems that I wrote it with no ability to scale a, I will fix that, and add in the functionality of using the traditional field theory formulation and the rescaled "Gallas" notation that I have been using for awhile.

2021-09-09 Predrag You sure about (1.49)? It might be correct, but how did you derive it?

2021-09-13 Sidney I am quite sure of (1.49). I noticed that the orbit Jacobian matrix for a time reversal invariant orbit commutes with the $[5 \times 5]$ reflection matrix

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.50}$$

So, I found all the eigenvectors associated with (1.50) by an online calculator and and arranged them so that the ones associated with -1 (antisymmetric) and 1 (symmetric) were grouped together in a matrix S, I then used an online calculator to perform

$$S^{-1}\mathcal{J}S = \mathcal{J}_{BD}$$

And after shuffling around the columns of S (which is allowed for diagonalization), I got (1.49).

2021-09-07 Sidney This brings me to what Predrag has asked me to do:

Predrag check list

1. I will need to reformulate everything back into the unscaled field values so that a multiplies the quadratic term.

2. Find eigenvalues of D_5 lattice states in the symmetric subspace to explain figure 1.2 (right).
3. Prove that the eigenvalues of the orbit Jacobian matrix are metric invariants. I am stuck here.
4. Implement Han's boundary conditions into my code so that I can directly find orbits with specific symmetries.

2021-09-12 Sidney I will start working on adding boundary conditions to find different symmetries (a task I have added to the list) after I clean up my code, and make it easy to switch between Gallas form, and Field Theory form.

2021-09-13 Sidney I have changed my code so that it can be easily switched between the Hénon **[henon]** (**??**), and Endler and Gallas **[EG05]** rescaled (**??**), I have added this updated code *Relaxation Method Henon with Orbit Jacobian.py* to *siminos/williams/python/relax*.

2021-09-12 to 2021-09-14 Sidney, Predrag As it turned to be a wild goose chase, not to distract Sidney further we moved all discussion of anti-integrable "perturbation theory" to sect. **??** *Temporal Hénon*.

2021-09-14 Predrag 2 Sidney Added exercise 4 *The matrix square root*.

2021-09-14 Sidney 2 Predrag and Han I looked at exercise 4 *The matrix square root*. I feel confident in being able to do that. However, I am not sure why I am looking at the square root here. From what I gathered during today's meeting, taking the square root of the time-step Jacobian at the boundary point did not give equality to the hill determinant of the symmetry reduced orbit Jacobian matrix. My impression was that I would instead need to look for a "square root" of the temporal Hénon is that incorrect?

2021-09-17 Sidney 2 Anyone Just for clarification, I should be looking at (**??**) for time symmetric orbits of the temporal Hénon. I was thinking that since this is an identity for the forward in time 2×2 Jacobian, for orbits symmetric with respect to time reversal, I would look at the Hill determinant $\mathrm{Det}\,\mathcal{J} = \det(\mathbf{1} - J)$ and check to see that the equality holds for when I take the half orbit of time-step J, compare to the symmetric part of (1.49). Is that a good strategy?

2021-09-17 Sidney I have completed exercise 4, the solution I got on pen and paper matches the solution that Predrag provided. I will take Andrew through it tomorrow.

2021-09-17 Sidney With respect to adding boundary conditions to my code: I do not know how to do it as I am not using the Biham-Wentzel method for the temporal Hénon, instead (a detail nowhere described in the blog) I invert it and feed the code a symbol sequence, see Vattay's ChaosBook exercise 7.2, copied to here as exercise **??**.

2021-09-17 Sidney I do not know how to change the boundary conditions.

2021-09-30 Sidney The above statement is still true, although, around studying for tests, and other homework, I been working on testing the formula (**??**) numerically. I have also been review some of the group theory lectures from over the summer. What I did, was say that the "factored" time-step Jacobian for a time symmetric period-5 is

$$\tilde{J} = J_2 J_1 \sqrt{J_0} \tag{1.51}$$

Where $\sqrt{J_0}$ can be calculated through the methods worked through in exercise 4. I then did the following calculation for every $\sqrt{J_0}$

$$|\mathrm{Det}\, \mathcal{J}_+| - |\det\,(I - \tilde{J})|(should) = 0\,, \tag{1.52}$$

where $\mathcal{J}_+$ is the $[3 \times 3]$ block in (1.49). This did not work. The closest I got to getting zero was 3.4, which isn't even an integer. So, I turned to Mathematica, and found that $\mathrm{Det}\, \mathcal{J}_+$ has a fundamentally different form from $\det\,(I - \tilde{J})$ for the temporal Hénon with a time symmetric five cycle, so, I went back to the drawing board. In the meeting at the beginning of the week, it was mentioned that the time-step Jacobians had to satisfy the time symmetry boundary conditions, and my thought was to try to force this by finding the Jacobian for each equation along a time symmetric period-5 which, with boundary conditions, yields a 3 equations:

$$2\phi_1 + \phi_0^2 = a \tag{1.53}$$

$$\phi_2 + \phi_1^2 + \phi_0 = a \tag{1.54}$$

$$\phi_2 + \phi_2^2 + \phi_1 = a \tag{1.55}$$

The time step Jacobian from (1.53) is $J_0 = -\phi_0$, and the time step Jacobian from (1.54) is just the normal one for the temporal Hénon. I am not sure how to get a Jacobian out of (1.55), perhaps the quadratic equation? If anyone has suggestions, that would be lovely.

2021-10-05 Sidney In this blog entry I describe Vattay's method of determining the periodic orbits of the Hamiltonian (phase space volume preserving) temporal Hénon (**??**), exercise **??** *Inverse iteration method for a Hénon repeller*, solution on page **??**.

First, I generate a list of all possible binary itineraries of a given length. This is a coding exercise, so I will not discuss that here. To determine the periodic orbits, Vattay inverts the temporal Hénon (**??**) as in (**??**):

$$\phi_i^{(m+1)} = \sigma_i \sqrt{\frac{1 - \phi_{i+1}^{(m)} - \phi_{i-1}^{(m)}}{a}}\,, \tag{1.56}$$

where S_i is a sign generated by the symbol sequence. I start the iteration by setting the initial guess orbit to

$$\phi_i^{(0)} = \sigma_i a^{-1/2}$$

(remember you $a \to \infty$ lattice states estimates?) and then evaluating (1.56). I then subtract the RHS from the LHS of the temporal Hénon for each point on the cycle and if it is smaller than a certain predetermined tolerance, the loop terminates, and the cycle is found. The meat of the method is contained in these two loops:

```
for i in range(0,len(symbols)):
cycle[i]=signs[i]*np.sqrt(abs(1-np.roll(cycle,1)[i]-np.roll(cycle,-1)[i])/a)
for i in range(0,len(symbols)):
deviation[i]=np.roll(cycle,-1)[i]-(1-a*(cycle[i])**2-np.roll(cycle,1)[i])
```

I could probably set time symmetric boundary conditions in the second loop, perhaps through an if statement.

I tried stating that (1.55) could be written as $\phi_3 = a - \phi_2^2 - \phi_1$, which would give just the normal temporal Hénon Jacobian, but it did not match with the determinant of the $[3 \times 3]$ block in (1.49). So my hunch was wrong. Not quite sure where to go from here in that area.

2021-10-12 Sidney I am currently trying to address "Find eigenvalues of D_5 lattice states in the symmetric subspace to explain figure 1.2 (right)." from 1.3. I found a equation for the symmetric part of the orbit Jacobian matrix $\mathcal{J}_+$ (1.49), it is of the form $a\lambda^3 + b\lambda^2 + c\lambda + d$ where the coefficients are all inelegant sums of the lattice field values of a given orbit, it is not particularly helpful.

2021-10-15 Predrag You have to check that the "inelegant sums" are invariant under D_n symmetries. Han knows and explains how to compute the eigenvalues and eigenvectors (irreps of D_n) on the reciprocal lattice.

I think you will eventually end up with evrything being expressible in terms of traces of powers of $\mathrm{Tr}\, \mathcal{J}_+^k$.

I am curious how many orbit Jacobian matrix $\mathcal{J}$ eigen-directions are expanding, what do they look like, stuff like that.

2021-10-12 Sidney I think that part of the confusion of the right hand side of figure 1.2 is that I tried to assign eigenvalues to individual lattice sites, which is just incorrect, right?

2021-10-15 Predrag Eigenvalues are properties of the whole matrix, not a single site. Only if the matrix is diagonalized are they associates with eigenstates ('lattice sites' of the reciprocal lattice).

2021-10-12 Sidney I am going to look at (1.34) again to see if I can see some pattern. But as of right now, I think the main conclusion is that the eigenvalues do not necessarily have the same symmetries as the orbit they belong to.

2021-10-15 Predrag Eigen*vectors* have symmetries, not the eigenvalues.

2021-10-25 Sidney I have generated some good data for the eigenstuff, see figure **??**. I need to do further analysis to see which eigenstate(s) is most important. As well, I think I have some insight into why (**??**) does not work. The orbit Jacobian matrix can be block diagonalized into symmetric and antisymmetric blocks. As this is the case, Hill's formula can be written as

$$\mathrm{Det}\,(\mathcal{J}_-)\mathrm{Det}\,(\mathcal{J}_+) = \det|I - J|$$

If we assume we can write J as $(J')^2$ (which is what (**??**) assumed), we can then write Hill's formula as

$$\mathrm{Det}\,(\mathcal{J}_-)\mathrm{Det}\,(\mathcal{J}_+) = \det|I - J'|\det|I + J'|$$

Which does not imply that $\mathrm{Det}\,(\mathcal{J}_+) = \det|I - J'|$ which I numerically showed to be incorrect a few weeks ago. This does not necessarily help find a correct factorization, but it at least shows us what is wrong.

2021-11-11 Sidney I added the plots of the eigenstates for every 5-cycle, as well as the decomposition for each period lattice state. Unfortunately, there seems to be no correlation between the important eigenstates and the size of the eigenvalues. I suppose that almost makes sense because treating the whole cycle as a fixed point removes iteration from our calculations, perhaps they will be useful for global stability analysis?

1.4 Sidney exercises

1. **Hénon temporal lattice.**

   1-dimensional temporal Hénon lattice (see ChaosBook Example 3.5) is given by a 3-term recurrence

   $$\phi_{n+1} + a\phi_n^2 - b\,\phi_{n-1} = 1\,.$$

   The parameter a quantifies the "stretching" and b quantifies the "contraction".

   The single Hénon map is nice because the system is a nonlinear generalization of temporal cat 3-term recurrence CL18 eq. catMapNewt, with no restriction to the unit hypercube XXX, but has binary dynamics.

   There is still a tri-diagonal orbit Jacobian matrix $\mathcal{J}$ CL18 eq. tempCatFixPoint, but CL18 eq. (Hessian) is now lattice state dependent. Also, I beleive Han told me that CL18 sect. s:Hill *Hill determinant: stability of an orbit vs. its time-evolution stability* block matrices derivation of Hill's formula does not work any more. Neither does the 'fundamental fact', as each lattice state's orbit Jacobian matrix is different, and presumably does not count periodic states, as there is no integer lattice within the Hill determinant volume.

   Does the ChaosBook flow conservation sum rule ChaosBook ed. (27.15) (or CL18.tex eq. Det(jMorb)eights) still work?

   The assignment: Implement the variational searches for periodic states in Matt's OrbitHunter, find all lattice states up to $n = 6$.

   (a) $a = 1.4$ $b = 0.3$, compare with ChaosBook Table 34.2.

   (b) For $b = -1$ the system is time-reversible or 'Hamiltonian', see ChaosBook Example 8.5. For definitiveness, in numerical calculations in examples to follow we fix (arbitrarily) the stretching parameter value to $a = 6$, a value large enough to guarantee that all roots of the periodic point condition $0 = f^n(x) - x$ are real.

   Note also ChaosBook sect A10.3 *Hénon map symmetries* and ChaosBook Exer. 7.2 *Inverse iteration method*.

   The deviation of an approximate trajectory from the 3-term recurrence is

   $$v_n = \phi_{n+1} - (1 - a\phi_n^2 + b\,\phi_{n-1})$$

   In classical mechanics force is the gradient of a potential, which Biham-Wenzel **[afind]** construct as a cubic potential

   $$V_n = \phi_{n+1}\phi_n - b\,\phi_n\phi_{n-1} + (a\phi_n^3 - \phi_n)\,. \tag{1.57}$$

   With the cubic potential at lattice site n we can start to look for orbits variationally. Note that the potential is time-reversal invariant for $b = 1$.

   Compare with XXX

2. **Engel Point Groups 1.** (Engel's [**Engel11**] Point Groups Exercise 1): The molecule on the left has C_{1s} which signifies that it has reflection symmetry over one axis. The molecule on the right has C_3 symmetry, signifying that it is symmetric by rotations of $\frac{2\pi}{3}$ or one third of a full circle.

   M. Engel

3. **Engel Point Groups 3.** (Engel's [**Engel11**] Point Groups Exercise 3): Three point groups for C_2H_6: a. C_{3v} because rotating it by 1/3 of a circle leaves it invariant, and one can cut the molecules into three identical pieces. b. C_s because the top and bottom have the same orientation, it is like looking in a mirror, so can apply reflection symmetry. c. Unsure, perhaps inversion symmetry C_i.

   M. Engel

4. **The matrix square root.** Consider matrix

$$A = \begin{bmatrix} 4 & 10 \\ 0 & 9 \end{bmatrix} .$$

   Generalize the square root function $f(x) = x^{1/2}$ to a square root $f(A) = A^{1/2}$ of a matrix A.
   a) Which one(s) of these is/are the square root of A

$$\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & 10 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} -2 & -2 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 2 & -10 \\ 0 & -3 \end{bmatrix} ?$$

   b) Assume that the eigenvalues of a $[d \times d]$ matrix are all distinct. How many square root matrices does such matrix have?
   c) Given a [2×2] matrix A with a distinct pair of eigenvalues $\{\lambda_1, \lambda_2\}$, write down a formula that generates all square root matrices $A^{1/2}$. Hint: one can do this using the 2 projection operators associated with the matrix A. 2 points

Solution 1 - Hénon temporal lattice.

(a) Here's my initial attempt, I'm trying to see if the flow conservation law CL18 eq. Det(jMorb)eights still works for the Hénon map:

$$\phi_{n+1} + a\phi_n^2 - b\phi_{n-1} = 1$$

The first step seems to be to construct the orbit Jacobian $\mathcal{J}$:

$$F[\mathbf{\Phi}] = \mathcal{J}\mathbf{\Phi} - I \tag{1.58}$$

Where I is the identity matrix, and F is the function where we want to find the zeros for (the orbits). We can rewrite this as:

$$\left(\sigma + aI\mathbf{\Phi} - b\sigma^{-1}\right)\mathbf{\Phi} = I \tag{1.59}$$

Therefore, $\mathcal{J}$ is

$$\mathcal{J} = \sigma + aI\mathbf{\Phi} - b\sigma^{-1} = \begin{bmatrix} a\phi_1 & 1 & 0 & \cdots & -b \\ -b & a\phi_2 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & -b & a\phi_{n-1} & 1 \\ 1 & 0 & \cdots & -b & a\phi_n \end{bmatrix} \tag{1.60}$$

Before I take a crack at seeing if this flow conservation still holds, I do have some questions:

Q1 Sidney *It appears that the derivation from chapter 23 (eqn 23.17, I don't know how to cite that specifically) the denominator of the sum rule is a product of the eigenvalues Λ_{pi}, which (if I remember correctly) are just the eigenvalues of the orbit Jacobian of the flow or map, which from basic linear algebra I know to be just the determinant of the orbit Jacobian. It cannot be that straightforward, where is the flaw in my logic?*

Q2 Sidney *How do I go from the periodic orbit formulation of the sum rule from Ch 23 to the lattice formulation? My initial thought is that since lattice states are akin to a periodic orbit (right?) that the sum can just be immediately changed from a sum over all periodic orbits, to a sum over all lattice states. Is this reasoning correct?*

Comment Sidney *I now realize that the flow sum rule involving the orbit Jacobian (NOT the Hill matrix) is a fundamental property that applies to all systems (at least all closed systems), what I now know is that I need to work out if I can convert between the determinant of the orbit Jacobian and the determinant of the Hill matrix.*

Plan Sidney *I am going to try to see what I can do with the block matrix proof, and I will get back to everyone on Friday*

Update Sidney *I tried working out the proof with the block matrices for just the regular Bernoulli map, I understand everything except the sentence "For a period-n lattice state $\mathbf{\Phi}_M$, the orbit Jacobian matrix (15) is now a $[ndXnd]$ matrix function of the $[dXd]$ block matrix J." It sort of seemed like it was much like "poof! And then a miracle happens!" I will keep exploring.*

I shall now correct my mistake with the derivation of the orbit Jacobian/Hill matrix $\mathcal{J}$ I shall use the differential definition:

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_j}{\delta \phi_i}$$

Which gives us that $\mathcal{J} = \sigma + 2aI\Phi_n - b\sigma^{-1}$. Now I will use the differential definition of the local Jacobian, where f is a functions such that $f(\phi_n) = \phi_{n+1}$

$$J_{ij} = \frac{\partial f(\phi_n)_i}{\partial \phi_{nj}}$$

Which gives us that $J(\phi_n) = -2a\phi_n$. So we can rewrite $\mathcal{J} = \sigma - J(phi_n)I - b\sigma^{-1}$, with the understanding that J changes along the diagonal. I am not quite sure how to bring this to the sum rule, but I will soon (hopefully), how do I math things like ϕ and Φ bold?

Update Sidney *I need to do a proper mathematical look at the flow conservation, but the Hénon map is not flow conserving (some trajectories are inadmissible) so the sum rule does not equal* 1*, I will try later to look at what it does equal analytically, but until then I will tackle the computation. I have made great progress with that, with help from Matt I was able to create a working code that gave me the correct orbits up to length 10 (I could not check past that). The code is in my blog. Once Matt has completed the current round of OrbitHunter updates I shall try to use that to reproduce my results.*

Solution Sidney *(a) The flow conservation sum rule does not sum to* 1 *so it does not work as before, I still need to try to relate the global Hill matrix to the local Jacobian matrix, I think I may be close to reworking the block matrix proof. Anyway, here are the periodic points I found (please note that the code cannot be used to find fixed points (n=1) so I just did it analytically, I will try to add that to the code later):*

$$n = 1 \quad -1.13135447$$

$$n = 1 \quad 0.63135447$$

$$n = 2 \quad [0.97580005, -0.47580005]$$

$$n = 4 \quad [1.12506994, -0.70676678, 0.63819399, 0.21776177]$$

$$n = 6 \quad [1.03805954, -0.41515894, 1.07011813, -0.72776163, 0.57954366, 0.31145232]$$

$$n = 6 \quad [1.1579582, -0.8042199, 0.44190995, 0.48533586, 0.80280173, 0.2433139]$$

When I tried to find $n = 3$ *and* $n = 5$ *the code returned nothing, this matches with what is tabulated in table 34.2. I will try using some of the analytical pruning techniques to prove that* $n = 3$ *and* $n = 5$ *are not allowed.*

(Sidney Williams 2021-01-20)

Solution 2 - Engel Point Groups 1. *The molecule on the left has* C_i *symmetry which is inversion symmetry NOT reflection symmetry because the top and bottom arrangements are not like they would be if placed in front of a mirror. The molecule on the right has* C_{3v} *symmetry, which is pyramidal symmetry which corresponds to the fact that one could take three slices of the molecule and they would each be identical, not just the configuration would be preserved by a rotation.*

(Sidney Williams 2021-03-07)

Solution 3 - Engel Point Groups 3. *Apparently, it depends based on whether we are dealing with staggered Ethane or not. If it is staggered, then it has inversion symmetry, if it is not, it has reflection symmetry, it should have* C_3 *symmetry instead of* C_{3v} *which confuses me a great deal. If I remember correctly it should correspond to a reflection vertical plane, which it should have, so I do not understand. The third one is* C_2 *which I do not understand how it is different from the inversion symmetry. Although, looking at simulations from here, it looks like the* C_2 *group can be used to rotate about axes that are different in orientation from just right through the middle. (Sidney Williams) 2021-03-07*

Solution 4 - The matrix square root. . now in CB

a) It is easy to check that

$$A = \begin{bmatrix} 4 & 10 \\ 0 & 9 \end{bmatrix} = \left(A_{ij}^{1/2}\right)^2$$

for the matrices

$$
\begin{aligned}
A^{1/2}_{++} &= \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}, \qquad A^{1/2}_{+-} = \begin{bmatrix} -2 & 10 \\ 0 & 3 \end{bmatrix} \\
A^{1/2}_{--} &= \begin{bmatrix} -2 & -2 \\ 0 & -3 \end{bmatrix}, \quad A^{1/2}_{-+} = \begin{bmatrix} 2 & -10 \\ 0 & -3 \end{bmatrix}
\end{aligned} \tag{1.61}
$$

Being upper-triangular, the eigenvalues of the four matrices can be read off their diagonals: there are four square root $\pm$ eigenvalue combinations {3,2}, {-3,2}, {3,-2}, and {-3,-2}.

Associated with each set $\lambda_i \in \{\lambda_1, \lambda_2\}$ is the projection operator

$$
P^{(1)}_{ij} = \frac{1}{\lambda_1 - \lambda_2}(A^{1/2}_{ij} - \lambda_2 \mathbf{1}) = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \tag{1.62}
$$

$$
P^{(2)}_{ij} = \frac{1}{\lambda_2 - \lambda_1}(A^{1/2}_{ij} - \lambda_1 \mathbf{1}) = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}. \tag{1.63}
$$

Note that all 'square root' matrices have the same projection operators / eigenvectors as the matrix A itself, so one can drop the ij subscripts on $P^{(1)}, P^{(2)}$.
b) If the eigenvalues of a [$d \times d$] matrix are all distinct, the matrix is diagonalizable, so the number of square root $\pm$ combinations is 2^d. However, for general matrices things can get crazy - there can be no, or some, or ∞ of 'square root' matrices.
c) We know $\{\lambda_1, \lambda_2\}$ and $P^{(\alpha)}$ for A, and the four 'square root' eigenvalues are clearly $\{\pm\lambda_1^{1/2}, \pm\lambda_2^{1/2}\}$. That suggest finding the 'square root' matrices (1.61) by reverse-engineering (1.62), (1.63):

$$
A^{1/2}_{ij} = (\lambda_1 - \lambda_2)P^{(1)} + \lambda_2 \mathbf{1}\,,
$$

which is, of course, how the problem was cooked up. For example,

$$
A^{1/2}_{+-} = (+3 - (-2)) \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} + (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$