Boris Pusher Theory Booklet

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1 Introduction

This booklet covers all the theory that I needed during my 2021 SULI when I was writing my Boris pusher code. This is by no means a replacement for any course, nor a statement of a completed project. Instead, I hope that this booklet allows anyone to "pick up and play" with the code that I have written this summer. I do say "I have written", I actually took the base code from this video, it was nothing close to what I needed for my project, so I changed its character completely, but the basic Boris mechanism that was originally in place is still there, so I need to cite where that came from. In the nature of modification though, I hope that this will be a living, breathing animal. I will be uploading the code and this theory booklet to Git with the intention of other people modifying both this booklet and the code to their purposes, all I ask is that unless some functionality is outright incorrect, that it is not taken out. As of now, I feel that what should be added is an extension to relativity, and particle-particle interactions, I am no means an expert on everything that this algorithm can do, but that felt like the logical next step as I was working. Happy coding!

-Sidney Williams August 2021

2 Electricity and Magnetism

[?]

2.1 Magnetic Field of a Coil

The off-axis magnetic field produced by a coil with radius a and current I is

$$B_r(r,z) = \frac{\mu_0 I k z}{4\pi(ar^3)^{1/2}}\left[-K(k) + \frac{1-\frac{1}{2}k^2}{1-k^2}E(k)\right] \tag{1}$$

$$B_z(r,z) = \frac{\mu_0 I k}{4\pi(ar)^{1/2}}\left[K(k) + \frac{(a+r)k^2 - 2r}{2r(1-k^2)}E(k)\right] \tag{2}$$

Where k is a parameter, and K and E are elliptic integrals of the first and second kind, respectively [**?**]

$$k \equiv \left[\frac{4ra}{z^2+(r+a)^2}\right]^{1/2} \tag{3}$$

$$E(k) = \int_0^{\pi/2}(1-ksin^2\theta)^{1/2}d\theta \tag{4}$$

$$K(k) = \int_0^{\pi/2}(1-ksin^2\theta)^{-1/2}d\theta \tag{5}$$

The process of deriving this is long, and drawn out, but it boils down to finding the vector potential

$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint\frac{d\vec{l}}{s}$$

The full procedure can be found in this video. By taking the limit $r \to 0$ of (1) and (2) we get

$$B_r(0,z) = 0 \quad B_z(0,z) = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}} \tag{6}$$

It is important to note that (6) possesses a non-zero divergence, but it is only the value of the magnetic field in a limiting case, and the full magnetic field (1), (2) has a zero divergence everywhere in space, thus there is no violation of Maxwell's equations.

In the case of a magnetic mirror, there are two coils, equidistant from a central point. The distance from this center point is called the "mirror half length" and is denoted by L. With this in mind, the full magnetic field in a classical magnetic mirror is

$$\begin{aligned}\vec{B} = &\left(B_r(a_1, I_1, z-L, r) + B_r(a_2, I_2, z+L, r)\right)\hat{r} \\ &+ \left(B_z(a_1, I_1, z-L, r) + B_z(a_2, I_2, z+L, r)\right)\hat{z}\end{aligned} \tag{7}$$

This form accounts for both off-axis effects, and possible asymmetries in the coils. In magnetic mirrors, we are concerned with the "mirror ratio", R, which is the ratio of the maximum magnitude of the magnetic field to its minimum magnitude:

$$R \equiv \frac{B_{max}}{B_{min}} \tag{8}$$

As the fields from coils are strongly dependent on radial position, and the radial position is determined by the Lorentz equation which is usually not solvable by hand, R can often be quite hard to determine. If we start the particle at the origin, we can numerically produce plots of mirror ratios, such as the following, which use coils of radius $0.3m$ and a mirror half length of $L = 0.1m, 03m, \text{and} 1m$ respectively

These figures were produced using the section of the code which was written by Tal Rubin. As can be seen, the mirror ratio is pretty uniform throughout the middle areas of the mirror, and only changes a lot at the extreme areas, such as at the edge of the coil, or nearing $z = L$. In general, we want the largest possible R, and so for most cases, we would choose $L = 0.3$.

The coil fields are very difficult to work with, with pen and paper, so it is common to make approximations. An extremely useful one, takes advantage of the assumption of a symmetric mirror machine. In this case, there is a minimum at the center, and at that point the first derivative is zero, so the Taylor expansion is simply quadratic:

$$B_z = B_0 + B''(0)z^2$$

We can approximate the second derivative as B_0/L^2, and so taking into account the requirement for a zero divergence:

$$\vec{B} = B_0\left(1 + \frac{z^2}{L^2}\right)\hat{z} - B_0\frac{rz}{L^2}\hat{r} \tag{9}$$

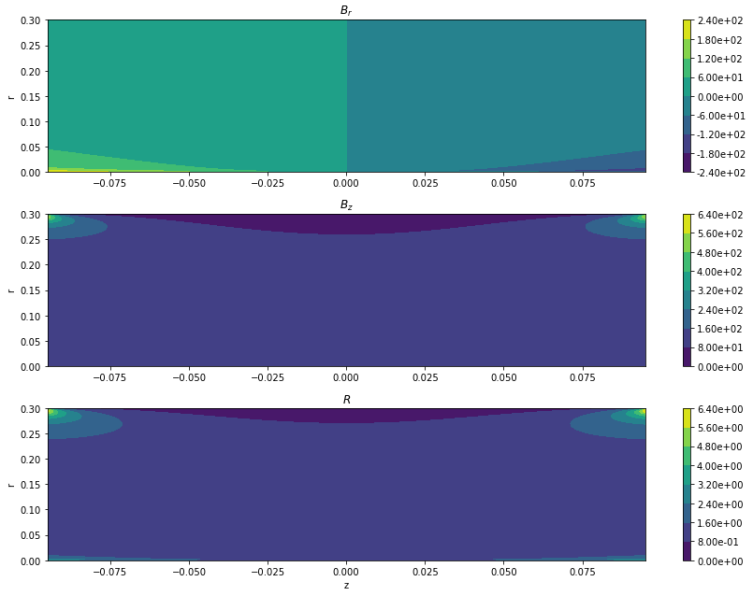


Figure 1: Mirror Ratio, a=0.3m, L=0.1m

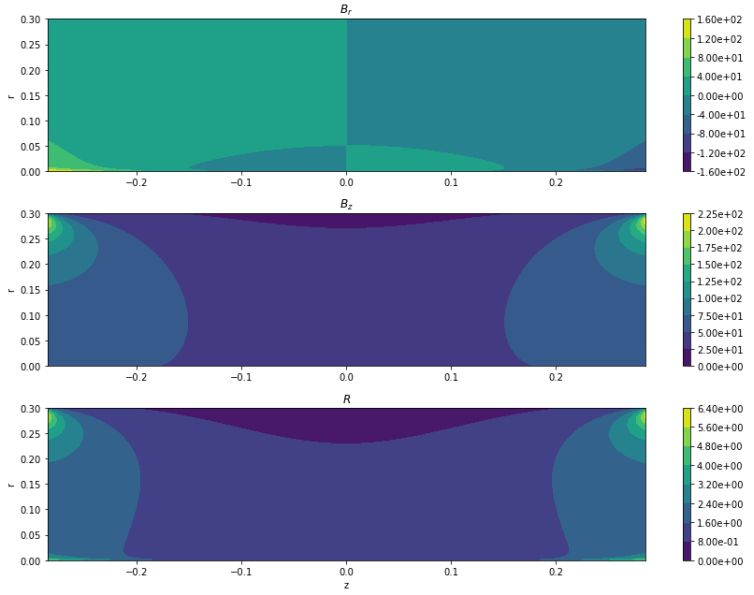


Figure 2: Mirror Ratio, a=0.3m, L=0.3m

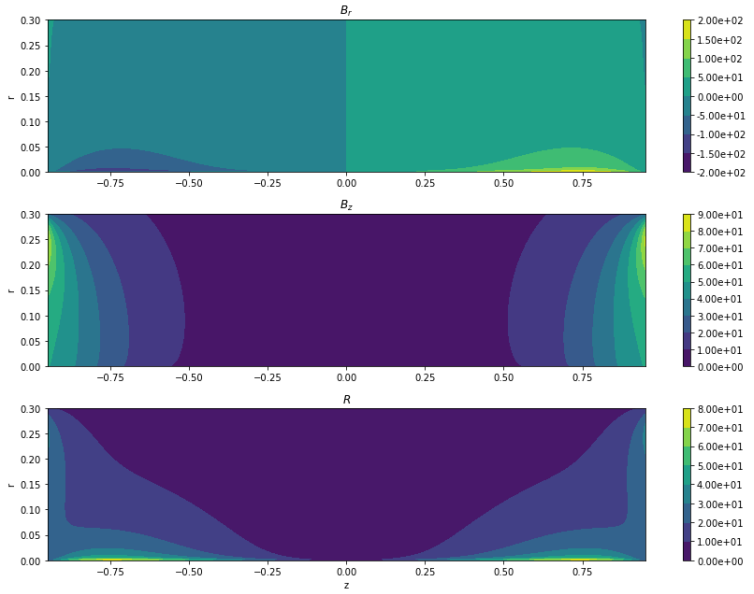


Figure 3: Mirror Ratio, a=0.3m, L=1m

Through a similar procedure, we can also find an approximate field for an asymmetric mirror machine

$$\vec{B} = B_0 \left(1 + \frac{z}{L} + \frac{z^2}{L^2}\right) \hat{z} - B_0 \left(\frac{r}{2L} + \frac{rz}{L^2}\right) \hat{r} \tag{10}$$

These fields provide quite good approximations to real-world behavior, and they have the nice property of being workable by hand.

2.2 Relativistic Effects

In this code's current iteration, it cannot deal with relativistic particles. Therefore, we need to know when a particle becomes relativistic, for the most part I will be using this website as the reference for this section. In general, a good rule of thumb for the limit of non-relativistic theory, is when the error between classical, and relativistic calculations reaches 1%. Relativistic kinetic energy is $(1 - \gamma)m_0c^2$ where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{11}$$

So, this means that the range of energy that is allowed is governed by

$$\frac{(1-\gamma)m_0c^2 - \frac{1}{2}m_0v^2}{(1-\gamma)m_0c^2} \leq 0.01 \tag{12}$$

Which can be solved numerically to find that the allowed γ is about $\gamma = 1.00673$.

3 Plasma Physics

3.1 Gyro-Orbit

At any point in space and time, we can evaluate the strength of the magnetic field experienced by a particle as a constant, and construct a basis such that the velocity is broken up into a direction perpendicular to the magnetic field line, and parallel to the magnetic field line. In other words, we can construct a basis for a particle such that $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ and $\vec{B} = B\hat{z}$. Plugging these vectors into the Lorentz force equation we get

$$\vec{F} = q\left(v_y B\hat{x} - v_x B\hat{y}\right) \tag{13}$$

Which can be written in the form of three coupled ordinary differential equations

$$\begin{aligned} \frac{dv_x}{dt} = \frac{qB}{m}v_y \rightarrow \frac{d^2v_x}{dt^2} &= -\frac{q^2B^2}{m^2}v_x \\ \frac{dv_y}{dt} = -\frac{qB}{m}v_x \rightarrow \frac{d^2v_y}{dt^2} &= -\frac{q^2B^2}{m^2}v_y \\ \frac{dv_z}{dt} &= 0 \end{aligned} \tag{14}$$

Recognizing these as simple harmonic oscillator equations, we can define the gyrofrequency [**?**]

$$\omega_B = \frac{qB}{m} \tag{15}$$

We can find the following arbitrary initial conditions

$$\begin{aligned} v_x(0) &= v_{x0} & v_y(0) &= v_{y0} \\ \frac{dv_x}{dt}(0) &= \omega_B v_{y0} & \frac{dv_y}{dt}(0) &= -\omega_B v_{x0} \end{aligned} \tag{16}$$

Solving (14) we get

$$v_x(t) = v_{x0}\cos(\omega_B) + v_{y0}\sin(\omega_B t)$$

$$v_y(t) = v_{y0}\cos(\omega_B) - v_{x0}\sin(\omega_B t)$$

We can put this in amplitude form to get

$$\begin{aligned} v_x &= A^* \cos(\omega_B t + \phi) \\ v_y &= -A^* \sin(\omega_B t + \phi) \\ A^* = \sqrt{v_{x0}^2 + v_{y0}^2} = v_\perp \quad \phi &\equiv \arctan\left(\frac{v_{y0}}{v_{x0}}\right) \end{aligned} \tag{17}$$

Integrating (17) we get the parametric equations for a circle of radius r_g centered at the point (x_0, y_0)

$$\begin{aligned} x(t) &= r_g \sin(\omega_B t + \phi) + x_0 \\ y(t) &= r_g \cos(\omega_B t + \phi) + y_0 \end{aligned} \tag{18}$$

Where we have defined the gyroradius as [**?**]

$$r_g = \frac{A^*}{\omega_B} = \frac{m v_\perp}{qB} \tag{19}$$

In practice, we initialize particles in a magnetic mirror with (18) with different phases ϕ so that they start distributed about the same point. As well, since drifts in magnetic mirrors act only in the $\hat{\theta}$ direction as discussed in 3.2, the guiding center (the point which is at the center of the gyro-orbit) does not change over time. As this is the case, we can use (19) to determine the current which should be run through the coils at either end of the mirror. This can be done because at the coil we do not want the particle's radial position to be larger than the radius of the coil, so (19) can be solved for field strength, and current can then be solved for from (1) and (2).

3.2 Drifts

Taken primarily from Weston Stacey's Introduction to Fusion book

3.2.1 $\vec{E} \wedge \vec{B}$ Drift

If we add an electric field to the system which is perpendicular to the magnetic field it will accelerate particles traveling on their gyro orbits. This prevents the orbits from closing because at somepoint in the orbit the particle will be fighting the E-field, and at another point, it will be being pushed by it. This yields a drift perpendicular to the magnetic field, as orbiting

through the electric field accelerates and decelerates an equal amount, if we average over the gyro-orbit the overall acceleration is zero, an so we can write

$$\vec{0} = \vec{E} + \vec{v}_{E\wedge B} \wedge \vec{B}$$

This is the same as looking at the force balance on the gyrocenter. If we take the lefthanded cross product of $\vec{B}$ on either side of the equation we obtain

$$\vec{0} = \vec{B} \wedge \vec{E} + \vec{B} \wedge \vec{v}_{E\wedge B} \wedge \vec{B}$$

We can then use the vector triple product, and the fact that $\vec{v}_{E\wedge B}$ is perpendicular to $\vec{B}$ to obtain

$$\vec{v}_{E\wedge B} = \frac{\vec{E} \wedge \vec{B}}{B^2} \tag{20}$$

For our purposes, we want $\vec{E} \perp \vec{B}$, so we can find the magnitude of the drift velocity to be

$$v_{E\wedge B} = \frac{E}{B} \tag{21}$$

Of course, normally, we look at drifts in Cartesian coordinates, so the drift just goes off in a straight line, but in cylindrical coordinates, it is possible to isolate the $\hat{\theta}$ basis vector, and thus produce a rotational plasma. But how do we do this? First, we see that the direction of the drift is dictated by $\vec{E}\wedge\vec{B}$, so we must force this cross product to be only in the $\hat{\theta}$ direction, or else particles will leave the system. This can be accomplished by ensuring that the E-field does not have a $\hat{\theta}$ component, as the B-field for a magnetic mirror only has $\hat{r}$ and $\hat{z}$ components, this yields a cross product of $(B_r E_z - E_r B_z)\hat{\theta}$, so that answers the directional question. However, we want the electric field to be perpendicular to the magnetic field so that we can get predictable results. This can be accomplished systematically by taking a dot product of the two fields and setting it to zero, which yields:

$$E_r = -\frac{B_z}{B_r} E_z \tag{22}$$

As an electric field is already built into the Boris algorithm, all we need to do to implement this is to define the electric field according to (22).

3.2.2 ∇B Drift

A charged particle traveling in a gyro-orbit produces a loop of current which has a magnetic moment,μ, which is the first adiabatic invariant, and thus assumed to be conserved in most cases. The force on the magnetic moment (not the particle, but the magnetic moment itself) is given as

$$\vec{F} = -\mu \nabla B$$

This form assumes that μ is conserved. If we look at the force balance on the magnetic moment (the gyrocenter) we get:

$$0 = -\mu \nabla B + e(\vec{v}_{\nabla B} \wedge \vec{B})$$

We can then take advantage of the vector triple product again to obtain:

$$\vec{v}_{\nabla B} = \frac{m v_{\perp}^2}{2e} \frac{\vec{B} \wedge \nabla B}{B^3} \tag{23}$$

The force that produces this drift is only acting on the gyrocenter, so a full orbit calculation accounts for this effect without needing to add an additional force to the particle's Lorentz force equation.

3.2.3 Curvature Drift

From classical mechanics we know that a particle traveling along a curved arc with radius of curvature R experiences the fictitious "centrifugal" force:

$$\vec{F} = -\frac{m v_{\parallel}^2}{R} \vec{n}_c$$

Acting towards the center of curvature, $\vec{n}_c$ is a unit vector in the direction of the radius of curvature. If we look at the force balance on the gyrocenter:

$$0 = -\frac{m v_{\parallel}^2}{R} \vec{n}_c + e(\vec{v}_c \wedge \vec{B})$$

Doing the vector triple product:

$$\vec{v}_c = -\frac{m v_{\parallel}^2}{eR} \frac{\vec{B} \wedge \vec{n}_c}{B^2} \tag{24}$$

Note that in a magnetic mirror, all of these drifts are in the $\hat{\theta}$ direction, so containment is not breached.

3.2.4 Charge Separation

The curvature drift and the ∇B drift are charge dependent, so they produce a current density:

$$\vec{j} = n_i e_i \vec{v}_i + n_e e_e \vec{v}_e \tag{25}$$

Where n is the density, subscript i indicates ions, and subscript e indicates electrons. This creates charge separation which then creates an electric field which is perpendicular to $\vec{B}$ and thus creates an $E \wedge B$ drift.

3.3 Adiabatic Invariants

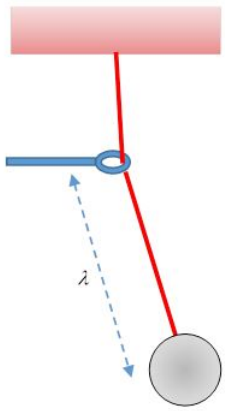


Figure 4: Variable Length Pendulum

The following argument is valid for any potential in which a trapped particle executes periodic motion of period T. If we also enforce that the potential depends on some variable parameter λ, which changes slowly ($T\frac{d\lambda}{dt} << \lambda$). For a concrete example, we consider a pendulum of variable length: Fig. 4.

If λ were fixed, both the period T and the energy E would be constant, but in general, the variable length exchanges energy with the system and thus varies both quantities. As explained in this Stack Exchange post, when the path of satisfies Hamilton's equations (which would be the case in this

pendulum example):

$$\begin{aligned}\dot{q}(t) &= \frac{\partial H}{\partial p}(q(t), p(t), t) \\ \dot{p}(t) &= -\frac{\partial H}{\partial q}(q(t), p(t), t)\end{aligned} \tag{26}$$

the full time derivative of the Hamiltonian is given by the partial time derivative of the Hamiltonian:

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{27}$$

For the case of the variable length pendulum, $H(q, p; \lambda)$, where ; is denoting what are parameters, and what are variables, see mathisfun for a painless explanation of this notation. We can use the chain rule on (27) to obtain

$$\frac{dE}{dt} = \frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt} \tag{28}$$

where the time derivatives of $p(t)$ and $q(t)$ cancel as explained in the above Stack Exchange link. Note that it was assumed that the system is such that the Hamiltonian is equivalent to the total energy of the system. Since λ varies slowly, if we average 28 over one period $\frac{d\lambda}{dt}$ is unaffected, and the equation becomes:

$$\overline{\frac{dE}{dt}} = \overline{\frac{\partial H}{\partial \lambda}}\frac{d\lambda}{dt} \tag{29}$$

Where

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda} dt \tag{30}$$

Due to Hamilton's equations (26) $dt = \frac{dq}{(\partial H)/\partial p)_\lambda}$ and the period becomes

$$T = \int_0^T dt = \oint \frac{dq}{(\partial H/\partial p)} \tag{31}$$

plugging (31) into (29) we obtain

$$\overline{\frac{dE}{dt}} = \frac{d\lambda}{dt}\frac{\oint \frac{(\partial H/\partial \lambda)dq}{\partial H/\partial p}}{\oint \frac{dq}{(\partial H/\partial p)_\lambda}} \tag{32}$$

Since E and λ both are varying slowly, over a single cycle they can be considered constant and then at a point q, $p = p(q; E, \lambda)$, thus regarding E and λ as constant, independent parameters.

We can now partially differentiate $H(q, p, \lambda) = E$ with respect to λ while keeping E constant. This yields:

$$\frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial \lambda} = 0$$

or

$$\left(\frac{\partial p}{\partial \lambda}\right)_E = -\frac{\partial h/\partial \lambda}{\partial H/\partial p} \tag{33}$$

plugging (33) into (32) we get

$$\overline{\frac{dE}{dt}} = -\frac{d\lambda}{dt}\frac{\oint(\partial p/\partial \lambda)_E dq}{\oint(\partial p/\partial E)_\lambda dq} \tag{34}$$

Where $1/(\partial H/\partial p)_\lambda = (\partial p/\partial E)_\lambda$. We can rearrange (34) to get:

$$\oint \left[\left(\frac{\partial p}{\partial E}\right)_\lambda \overline{\frac{dE}{dt}} + \left(\frac{\partial p}{\partial \lambda}\right)_E \frac{d\lambda}{dt}\right] dq = 0 \tag{35}$$

(35) can be written as $\frac{\overline{dI}}{dt} = 0$ where

$$I = \oint p(q, \lambda, E) dq \tag{36}$$

Where I is the adiabatic invariant and the general form of the action integral. The slower the cycle we are concerned with is in relation to the change in the parameter, the less accurate conservation is. This is actually the lowest order approximation to a Poincare invariant.

3.3.1 First Adiabatic Invariant (μ Conservation)

Magnetic moment conservation is useful because it establishes a relationship between the perpendicular kinetic energy and the magnetic field. In the case of a magnetic field varying slowly in comparison with the gyrofrequency of the trapped particle, we can replace p and q in (36) with $mv_\perp r_L$ and ϕ respectively. In this case, ϕ is the gyrophase of the particle, it varies from 0 to 2π over the course of the particle's gyromotion. With that in mind, (36) becomes:

$$I = \oint mv_\perp r_L d\phi \tag{37}$$

taking advantage of $\omega r = v$ and centripetal acceleration with the Lorentz equation, we integrate and get

$$I = \frac{2\pi}{q}\frac{m^2 v_\perp^2}{B} = \frac{4\pi}{q}\frac{W_\perp}{B} \propto \mu \tag{38}$$

So, with sufficiently slow B variation, μ is a constant.

3.3.2 Second Adiabatic Invariant

Using (36), with generalized momentum $Mv_\parallel$ and generalized position as the distance along the mirror axis, we get the second adiabatic invariant:

$$J = \oint Mv_\parallel dz \tag{39}$$

The following is a slightly reworded excerpt from from Fitzpatrick: this invariant is associated with the periodic bouncing motion of a particle trapped between two mirror points on a magnetic field line. The value is only conserved if the bounce time is much smaller than the timescale at which the magnetic field varies. The invariance of J is very important for charged particle dynamics in Earth's inner magnetosphere. This is because Earth's magnetic field is distorted from pure axisymmetry due to the action of the solar wind. Thus, there is no immediate reason to believe that the particle will return to its original trajectory after making the full rotation around Earth. In other words, a particle may well end up on a different field line after returning to its original azimuthal angle. However, at a given azimuthal angle, each field line has a different length between mirror points, and a different variation of the field strength for a particle with given energy and magnetic moment. Thus, each field line has a different J value, and if J is conserved, it means the particle must stay on the same field line throughout its motion. In other words, the conservation of J prevents charged particles from spiraling radially in or out of the Van Allen belts as they rotate around Earth. This helps to explain the persistence of these belts. Note: A Van Allen belt is a zone of energetic charged particles, most of which originate from solar wind, that are captured by and held around a planet by that planet's magnetosphere.

Example:

What happens when a ball elastically bounces between two parallel walls which are slowly closing in. Assume no gravity

As the walls are closing in slowly, we can assume that the second adiabatic invariant is conserved, so

$$J = \oint mv_{\parallel} dz = const$$

If we look at orders of magnitude we get

$$J\ mv_{\parallel} L = const$$

Where L is the distance between the walls. m is constant and L is decreasing, so to keep J constant $v_{\parallel}$ must increase as the walls close in. So, the kinetic energy is increasing.

3.4 Magnetic Mirror

3.4.1 Rotating Magnetic Mirror

The following is a transcription of Tal Rubin's handwritten notes, thank you Tal.

We can write the kinetic energy of a particle as

$$W_{kin} = W_{\parallel} + W_{\perp} + W_D \tag{40}$$

Where $W_{\parallel} = \frac{1}{2}m(\vec{v} \cdot \hat{B})^2$ is the kinetic energy related to the motion along the field, $W_D = \frac{1}{2}mv_D^2$ is the energy associated with the gyrocenter drift velocity:

$$\vec{v}_D = \frac{\vec{E} \wedge \vec{B}}{B^2} - \frac{\mu}{q}\frac{\nabla B \wedge \vec{B}}{B^2} + \frac{2W_{\parallel}}{q}\frac{\vec{B} \wedge (\hat{B} \cdot \vec{v})\hat{B}}{B^2} \approx \frac{\vec{E} \wedge \vec{B}}{B^2} \tag{41}$$

We can make the approximation that there is only $E \wedge B$ drift because the other two drifts have a factor of mass to charge which is very small. And $W_{\perp} = \frac{1}{2}mv^2 - W_{\parallel} - W_D$ is the remaining perpendicular energy, primarily contributed by gyromotion. We want to know something about W_D. To do this, we must define the concept of a Magnetic Surface, which is simply: "A surface defined by its normal at any point $\hat{n}(\vec{r})$ is a "Magnetic Surface" if this normal is paralle to $\vec{B}(\vec{r})$, eg. $\hat{n} \cdot \vec{B} = 0$. It is useful to note that $rA_{\theta} = const.$ is a magnetic surface, where A_{θ} is the angular component of the vector potential. We also have to be aware of the "Isorotation Theorem" which states that

Theorem 1 *If magnetic surfaces are also equipotential surfaces, then particles under the effect off $E \wedge B$ drift rotate around the axis with the same frequency.*

We also need to state the "cooling corollary"

Corollary 1 *Particles moving along equipotential surfaces gain potential energy equal to exactly twice the energy lost from the azimuthal drift.*

Proof 1 *Canonical angular momentum is conserved: $L_\theta = r(mv_\theta + qA_\theta) = const$. Rotation frequency around mirror is $\Omega = \frac{E}{Br}$. If we imagine looking at two different radii in the plasma, we can state $\Omega_a = \Omega_b = \frac{E_a}{B_a r_a} = \frac{E_b}{B_b r_b}$. Taking the difference of the canonical angular momentum at these to locations we get $\Delta[qrA_\theta] = -\Delta[rmv_\theta] = -m(r_a v_{\theta a} - r_b v_{\theta b}) = -m\left(r_a \frac{E_a}{B_a} - r_b \frac{E_b}{B_b}\right)$, this last step is taken assuming that the only angular velocity of the guiding center is from drifts. We can rewrite this in terms of rotation frequency: $\Delta[qrA_\theta] = m\Omega[r_b^2 - r_a^2]$. From the definition of the vector potential: $\iint \vec{B} \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{l}$ which can be directly integrated: $2\pi[r_b A_\theta(r_b) - r_a A_\theta(r_a)] \approx \pi B[r_b^2 - r_a^2] \approx 2\pi r B\delta$, where $\delta = r_b - r_a$. Using the definition of the difference in potential $-\delta E = \Delta\Phi$, so $\Delta[qrA_\theta] \approx \frac{qrB\Delta\Phi}{E} = \frac{-q\Delta\Phi}{\Omega} = m\Omega(r_b^2 - r_a^2) = 2(W_{Db} - W_{Da})\frac{1}{\Omega}$, or $-q\Delta\Phi = 2\left(\frac{1}{2}m(\Omega r_b)^2 - \frac{1}{2}m(\Omega r_a^2)\right)$*

We can now write out the energy balance

$$E = W_{kin} + q\Phi = W_\parallel + W_\perp + W_D + q\Phi \tag{42}$$

This leads us directly to our new trapping condition by taking the difference of energy at two points in space, E cancels and we get

$$\begin{aligned} W_\parallel = W_{\parallel 0} - W_\perp + W_{\perp 0} - W_D + W_{D0} - q\Phi + q\Phi_0 = \\ W_{\parallel 0} + (1-R)W_{\perp 0} + \left(1 - \frac{r^2}{r_0^2}\right) W_{D0} \end{aligned} \tag{43}$$

This then leads to the trapping condition

$$W_{\parallel 0} \leq (R-1)W_{\perp 0} + \left(\frac{r^2}{r_0^2} - 1\right) W_{D0} \tag{44}$$

This leads to a larger value on the right hand side. Therefore, adding rotation to a magnetic mirror makes trapping easier.

3.4.2 Worked Asymmetric Magnetic Mirror Problem

a. Derive the trapping condition in midplane (z=0) energy coordinates for ions in a magnetic mirror machine with mirror ratio $R \equiv B_{max}/B_{min}$ and mirror axis in the $\hat{z}$ direction.

I shall assume that "midplane" refers to the center of the magnetic "potential" (minimum B)

$$R \equiv \frac{B_{max}}{B_{min}}$$

Assuming no external work, torque, and a negligible potential energy, we have that both angular momentum and kinetic energy is conserved:

$$KE = \frac{1}{2} m\vec{v} \cdot \vec{v} = const$$

$$\vec{L} = m\vec{v} \wedge \vec{r}$$

Therefore, we can take advantage of the explanation in Sect.**??** to state that the velocity along the magnetic field lines can be written as:

$$v_{\parallel}^2 = \frac{2}{m}(KE - \mu B)$$

The turning points are defined as the location where the parallel velocity vanishes, or the locations $s_{\pm}^*$ where:

$$KE = \mu B(s_{\pm}^*)$$

In terms of minimum B and R, the maximum kinetic energy that a particle can posses while still being trapped is:

$$KE = \mu B_{max} = \mu R B_{min}$$

We can derive a trapping condition in terms of $W_{\parallel 0}$ and $W_{\perp 0}$ via a energy balance:

$$W_{\perp 0} + W_{\parallel 0} = W_{\parallel 0} + \mu B_0 = \mu B_1$$

where $\frac{B_1}{B_0} = R$, we can rearrange to get

$$W_{\parallel 0} = \frac{W_{\perp 0}}{B_0}(B_1 - B_0) = W_{\perp 0}(R - 1)$$

And so

$$\boxed{W_{\parallel 0} \leq W_{\perp 0}(R-1)}$$

Is the trapping condition

b. Sketch the trapping condition in the $W_{\perp 0} - W_{\parallel 0}$ plane, where $W_{\perp 0}$ and $W_{\parallel 0}$ are the perpendicular and parallel energies as the midplane is crossed.

From the previous problem, we get a straight line with a slope of $(R-1)$ (which is definitionally positive), and any particle with energy below the line, will be trapped.

c. Suppose that the magnetic field near the axis can be approximated as

$$\vec{B} = \begin{cases} B_0(1 + z^2/L^2)\hat{z} & z^2 < c^2L^2 \\ B_0(1+c^2)\hat{z} & z^2 > c^2L^2 \end{cases}$$

Show that the turning points for trapped ions obey: $z_T^2/L^2 = W_{\parallel 0}/W_{\perp 0}$

From part a. we know $W_{\parallel 0} + W_{\perp 0} = \mu B$, as well, a trapped particle must be contained in a finite region, therefore

$$W_{\parallel 0} + W_{\perp 0} = \mu B_0 + \frac{\mu B_0 z^2}{L^2}$$

From part a. we know that the perpendicular component of the energy at the midplane is $W_{\perp 0} = \mu B_0$ so the trapping condition becomes:

$$W_{\perp 0}\left(\frac{W_{\parallel 0}}{W_{\perp 0}} + 1\right) = W_{\perp 0} + \frac{W_{\perp 0} z^2}{L^2}$$

$$\frac{W_{\parallel 0}}{W_{\perp 0}} + 1 = 1 + \frac{z^2}{L^2}$$

Therefore

$$\boxed{\frac{W_{\parallel 0}}{W_{\perp 0}} = \frac{z^2}{L^2}}$$

We can use a similar method to derive bounce frequency. We start with an energy balance with the left hand side NOT at the midplane, and the right hand side at one of the turning points:

$$W_{\parallel} + W_{\perp} = \mu B(z_T)$$

If we expand the magnetic field term we get

$$\frac{1}{2}mv_{\parallel}^2 + \frac{\mu B_0}{L^2}z^2 = \frac{\mu B_0}{L^2}z_T^2$$

We can solve for $\frac{dz}{dt}$ to find an ordinary differential equation

$$\frac{dz}{dt} = \sqrt{C - \frac{2W_{\perp 0}}{mL^2}z^2}$$

If we take a time derivative of both sides we get

$$\frac{d^2z}{dt^2} = -\frac{v_{\perp 0}^2}{L^2}z$$

Therefore, the bounce frequency of a trapped particle is

$$\omega_T = \frac{v_{\perp 0}}{L}$$

By using the same method, we find that introducing a gravitational field does not change the bounce frequency.

Suppose now that the mirroring particles are in an axial gravitational field: $-g\hat{z}$

d. Write the new trapping condition in the midplane (z=0)

With the influence of gravity, the total energy gains a term accounting for the gravitational potential: mgh, so the total energy is now

$$E = KE + mgh = \frac{1}{2}m(v_{\perp}^2 + v_{\parallel}^2) + mgh$$

From (38), we know that $\mu = \frac{1}{2}\frac{mv_{\perp}^2}{B}$ is conserved, and therefore $\frac{1}{2}mv_{\perp}^2 = \mu B$ and

$$E = \mu B + \frac{1}{2}mv_{\parallel}^2 + mgh$$

and

$$\boxed{v_{\parallel}^2 = \frac{2}{m}[E - \mu B - mgz]}$$

However, it is much more useful to repeat part a:

$$W_{\parallel 0} + W_{\perp 0} = W_{\parallel 0} + \mu B_0 = \mu B_1 - mgL$$

Where L is the mirror machine's half length, and it was evaluated at the negative point from the mid-plane because that is where the gravitational potential is at its minimum, so particles will gather there. As well, if the particle does not turn before reaching the end of the mirror machine, it will definitely not turn, therefore, we can use $-cL$ as the minimum turning point. If we again define $R \equiv \frac{B_1}{B_0}$, the trapping condition becomes:

$$\boxed{W_{\parallel 0} \leq W_{\perp 0}(R-1) - cmgL}$$

We can redefine the coordinate system as $L' \equiv cL$, so the trapped B-field condition becomes:

$$B = B_0 \left(1 + c^2 \frac{z^2}{L'^2}\right)$$

So, $R = 1 + c^2$ and $c^2 = R - 1$, so

$$B = B_0 \left(1 + (R-1)\frac{z^2}{L'^2}\right)$$

Note that if $R = 2$ the form of B is the same in either coordinate system. In this coordinate system:

$$\boxed{W_{\parallel 0} \leq W_{\perp 0}(R-1) - mgL'}$$

Note that

$$L = \frac{L'}{\sqrt{R-1}}$$

Will be an important identity throughout the rest of the problem.

e. Derive the new turning points, defining the high turning point as z_H and the low turning point as z_L. Express z_H and $z - L$ in terms of z_T and z_g, where $z_g \equiv mgL^2/W_{\perp 0}$. Show that trapped particles always cross the midplane.

The turning points occur when $v_\parallel = 0$ so $0 = E - \mu B - mgz_{L/H}$, the energy is conserved, so I'll evaluate it at $z = 0$, therefore:

$$\mu B + mgz = W_{\perp 0} + W_{\parallel 0}$$

I'll also assume that the magnetic field has the same form as in part c. such that

$$\mu B_0(1 + z^2/L^2) + mgz = W_{\perp 0} + W_{\parallel 0}$$

Evaluating at $z = 0$ and taking advantage of the first adiabatic invariant we get $\mu B_0 = W_{\perp 0}$ so,

$$(1 + (R-1)z^2/L'^2) + \frac{mgz}{W_{\perp 0}} = 1 + \frac{W_{\parallel 0}}{W_{\perp 0}}$$

rearranging, we get a quadratic equation:

$$(R-1)\frac{z^2}{L'^2} + \frac{mg}{W_{\perp 0}}z - (R-1)\frac{z_T^2}{L'^2} = 0$$

We can multiply through by $\frac{L'^2}{(R-1)}$ to get

$$z^2 + z_g z - z_T^2 = 0$$

Which can be solved via the quadratic equation

$$\boxed{z_{L,H} = \frac{-z_g \pm \sqrt{z_g^2 + 4z_T^2}}{2}}$$

For the + case of the quadratic equation, the root is positive because $\sqrt{z_g^2 + 4z_T^2} > z_g$ and thus is a positive value, and so the particle must cross the midplane.

Suppose now that the axial gravitational field is not constant, but instead increases very slowly in time from zero at time t = 0 to a finite value g at time $t = t_f$. (Hint: You may find it convenient to write the parallel energy as a function of $W_{\perp 0}$, z_H, z_L and z)

f. What is the change in the perpendicular midplane energy $W_{\perp 0}$?

We know that $\mu = \frac{W_{\perp 0}}{B}$ is the first adiabatic invariant from (38) so it does not vary with g, so

$$\boxed{\Delta W_{\perp 0} = 0}$$

g. What is the change in the parallel midplane energy $W_{\parallel 0}$?

From (39) we can rearrange to obtain

$$J = \sqrt{2m} \oint W_{\parallel}^{1/2} ds = const$$

Since the potential added by the variable g is conservative, $E(t)$, the total energy is conserved at any fixed time t. So we can write $E(t) = W_{\parallel}(t) + W_{\perp} + mgz = W_{\parallel}(t) + \mu B(z) + mg(t)z = W_{\perp 0} + W_{\parallel 0}(t) = \mu B_0 + W_{\parallel 0}(t)$ so,

$$W_{\parallel 0} + \mu B_0 = W_{\parallel}(t) + \mu B_0(1 + (R-1)z^2/L'^2) + mg(t)z$$

and

$$W_{\parallel}(t) = W_{\parallel 0}(t) - (R-1)W_{\perp 0}\frac{z^2}{L'^2} - mg(t)z$$

Taking into account the total loop J becomes

$$J = 2\sqrt{2m}\int_{z_L}^{z_H}\sqrt{W_{\parallel 0}(t) - (R-1)W_{\perp 0}\frac{z^2}{L'^2} - mg(t)z}dz =$$

$$\frac{2}{L'}\sqrt{2mW_{\perp 0}(R-1)}\int_{z_L}^{z_H}\sqrt{\frac{W_{\parallel 0}L'^2}{W_{\perp 0}(R-1)} - z^2 - \frac{mgL'^2}{W_{\perp 0}(R-1)}}dz$$

Or

$$J = \frac{2}{L}\sqrt{2mW_{\perp 0}(R-1)}\int_{z_L}^{z_H}\sqrt{z_T^2 - z^2 - z_g z}dz$$

If we look at the polynomial under the square root, and within the integral, and set it to zero $z_T^2 - z^2 - z_g z = 0$, then we can multiply both sides by -1 to get $z^2 + z_g z - z_T^2 = 0$, which is the equation that was solved in part h, so we know the roots, and can factor the equation

$$z^2 + z_g z - z_T^2 = (z - z_L)(z - z_H)$$

But we want the negative of this equation

$$z_T^2 - z^2 - z_g z = (z - z_L)(z_H - z)$$

So $J = \frac{2}{L'}\sqrt{2mW_{\perp 0}(R-1)}\int_{z_L}^{z_H}\sqrt{(z - z_L)(z_H - z)}dz$, which can be evaluated with the given integral:

$$J = \frac{2\pi}{8L'}\sqrt{2mW_{\perp 0}(R-1)}(z_H - z_L)^2$$

At $g = 0$

$$z_H = z_T = L'\sqrt{\frac{W_{\parallel 0}}{W_{\perp 0}(R-1)}}$$

$$z_L = -z_T$$

$$(z_H - z_L)^2 = 4L'^2 \frac{W_{\|0}}{W_{\perp 0}(R-1)}$$

and

$$J = \frac{\pi L'}{\sqrt{W_{\perp 0}(R-1)}} W_{\|0}(0)\sqrt{2m}$$

At $g \neq 0$

$$z_H = \frac{-\frac{mgL'^2}{W_{\perp 0}(R-1)} + \sqrt{\frac{m^2 g^2 L'^4}{W_{\perp 0}^2 (R-1)^2} + \frac{4W_{\|0}L'^2}{W_{\perp 0}(R-1)}}}{2}$$

$$z_L = \frac{-\frac{mgL'^2}{W_{\perp 0}(R-1)} - \sqrt{\frac{m^2 g^2 L'^4}{W_{\perp 0}^2 (R-1)^2} + \frac{4W_{\|0}L'^2}{W_{\perp 0}(R-1)}}}{2}$$

$$(z_H - z_L)^2 = \frac{m^2 g^2 L'^4}{W_{\perp 0}^2 (R-1)^2} + 4L'^2 \frac{W_{\|0}}{W_{\perp 0}(R-1)} = z_g^2 + 4L'^2 \frac{W_{\|0}}{W_{\perp 0}(R-1)}$$

So

$$J = \frac{\pi}{4} \frac{\sqrt{2mW_{\perp 0}(R-1)}}{L'} (z_g^2 + 4L'^2 \frac{W_{\|0}(t_f)}{W_{\perp 0}(R-1)})$$

Let's do $J(t_f) - J(0) = 0 = \frac{\pi z_g^2}{4L'}\sqrt{2mW_{\perp 0}(R-1)} + \sqrt{\frac{2m}{W_{\perp 0}(R-1)}}\pi L' \Delta W_{\|0}$ or

$$\boxed{\Delta W_{\|0} = -\frac{z_g^2 W_{\perp 0}(R-1)}{4L'^2}}$$

(Note: Tal Rubin gave me the solution with only math, I added explanation and worked through all the algebra explicitly, thank you Tal)

h. In terms of the initial (t = 0) $W_{\perp 0}$ and $W_{\|0}$, write down the condition for particles that are initially trapped but then detrapped. If there are any such particles, what is the minimum $W_{\perp 0}$ and what is the minimum $W_{\|0}$ (not necessarily corresponding to the minimum $W_{\perp 0}$) that they can have?

From the previous problem

$$W_{\|0}(t_f) = W_{\|0}(0) - \frac{z_g^2 W_{\perp}(R-1)}{4L'^2}$$

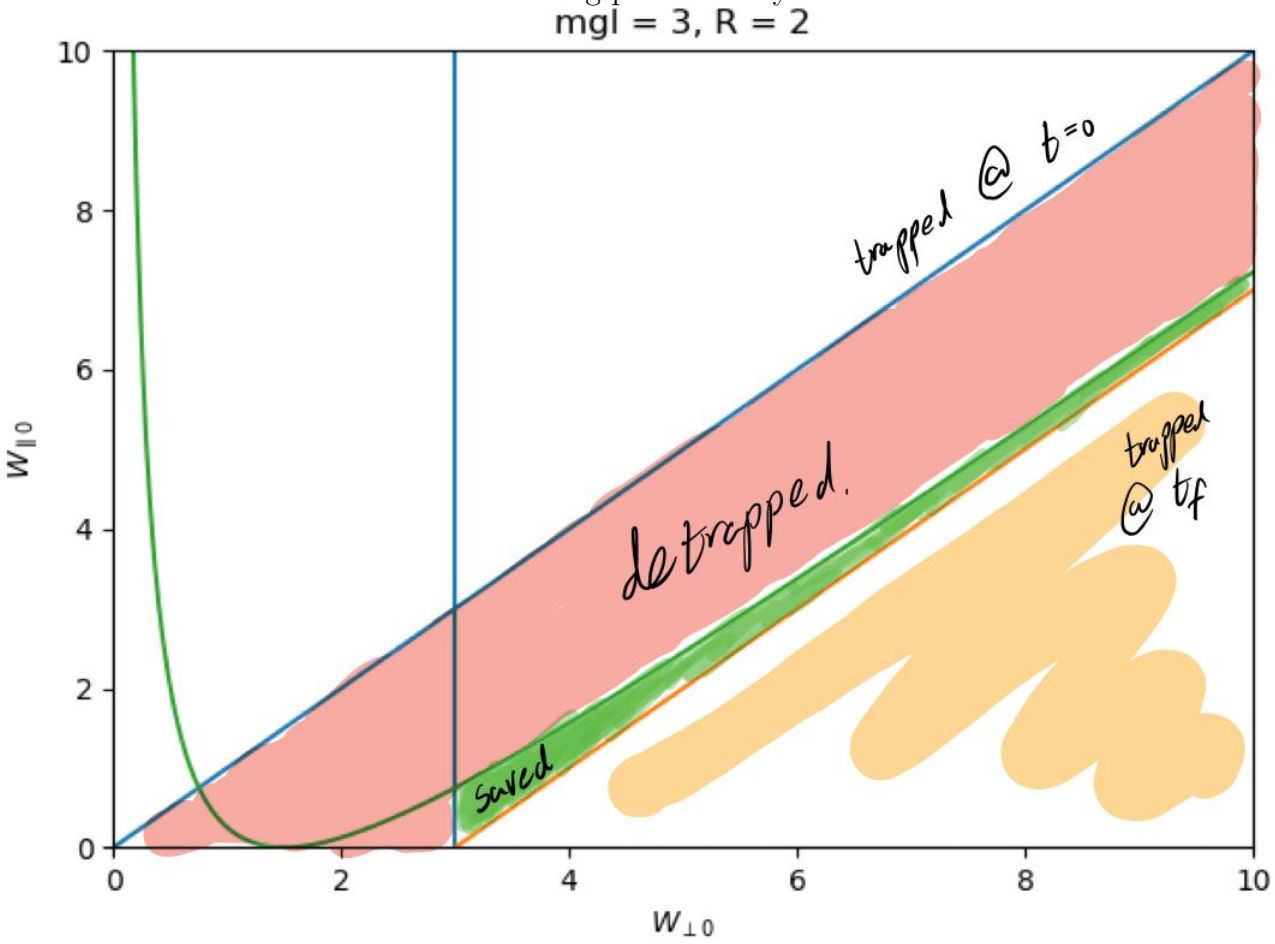
So, pulling from part d.

$$W_{\|0}(0) \leq W_{\perp 0}(R-1) + \frac{m^2 g^2 L'^2}{4W_{\perp 0}(R-1)} - mgL'$$

If we want a condition for particles to be initially trapped at $t = 0$ and then detrapped by $t = t_f$ we have the double inequality:

$$W_{\perp 0}(R-1) \geq W_{\parallel 0}(0) \geq W_{\perp 0}(R-1) + \frac{m^2 g^2 L'^2}{4W_{\perp 0}(R-1)} - mgL'$$

This can be visualized in the following plot courtesy of Tal Rubin:



The minimum $W_{\perp 0}$ which remains trapped is given by the zero of the orange line (the trapping condition in the $W_{\parallel 0} - W_{\perp 0}$ plane), it is

$$W_{\perp 0}^{min} = \frac{mgL'}{R-1}$$

Then the maximum $W_{\parallel 0}(0)$ at that value is:

$$W_{\parallel 0}^{max}(0) = \frac{mgL'}{4}$$

We can write this in terms of c if we like:

$$W_{\perp 0}^{min} = \frac{mgL}{c}$$

$$W_{\parallel 0}^{max}(0) = \frac{cmgL}{4}$$

This gives a range of allowed $W_{\parallel 0}$ values at the minimum $W_{\perp 0}$ location which remain trapped as gravity is turned on. For detrapping, we know that the trapping condition at $t = t_f$ has shifted right by $\frac{mgL'}{R-1}$ in comparison with the condition at $t = 0$ so anything to the left of $W_{\perp 0} = \frac{mgL'}{R-1}$ will be detrapped, there is no such condition for $W_{\parallel 0}(0)$ so we are free to set it to zero, so:

$$\boxed{W_{\perp 0}^{min} = 0}$$

$$\boxed{W_{\parallel 0}^{min}(0) = 0}$$

In other words: The question asks for the minimum values of $W_{\parallel 0}$ and $W_{\perp 0}$ which are initially trapped and then detrapped. The minimum values that can be trapped initially are 0,0 since it satisfies the trapping condition without gravity. When gravity is turned on the trapping condition moves the the right by $\frac{mgL'}{R-1}$ and hence detraps anything to the left of that, this includes the point 0,0. Since 0,0 is the minimum value initially trapped and it is subsequently detrapped, it must be the minimum value initially trapped and then detrapped.

i. Suppose instead the axial gravitational field starts off at a finite value g at time t = 0, and then decreases very slowly in time to zero at time $t = t_f$. In terms of the initial (t = 0) $W_{\perp 0}$ and $W_{\parallel 0}$, write down the condition for particles that are initially trapped but then detrapped. If there are any such particles, what is the minimum $W_{\perp 0}$ and what is the minimum $W_{\parallel}$ (not necessarily corresponding to the minimum $W_{\perp 0}$) that they can have?

Since there is no additional additional potential in this problem, it is just the reverse of what was done above. meaning that instead of having the trapping condition go from a form of $W_{\parallel 0} = x$ to one of the form $W_{\parallel 0} = x - mgL$ we are going from $W_{\parallel 0} = x - mgL$ to $W_{\parallel 0} = x$, which is a shift to the left, and encloses more space, so there cannot be any trapped particles that are detrapped. (Equivalent of shifting from the orange line to the blue line above, no particles are lost)

Hint: You may wish to use the integral:

$$\int_a^b [(s-a)(b-s)]^{1/2} ds = \frac{\pi}{8}(b-a)^2$$

4 Code and Numerical Theory

4.1 Boris Algorithm

This code is a modification of the code presented in this video. The derivation of the algorithm will be taken from Birdsall's book, unless otherwise mentioned everything is sourced from there[**?**]. The particle equations of motion to be integrated are (assuming no force outside of electric and magnetic):

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right)$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}$$

We want a centered difference scheme, and we do this by evaluating the velocity vector at half-steps:

$$\frac{\mathbf{v}_{t+\Delta t/2} - \mathbf{v}_{t-\Delta t/2}}{\Delta t} = \frac{q}{m}\left(\mathbf{E} + \frac{\mathbf{v}_{t+\Delta t/2} + \mathbf{v}_{t-\Delta t/2}}{2} \wedge \mathbf{B}\right) \tag{45}$$

The Boris algorithm introduces new velocity variables to separate the solution steps of the magnetic and electric fields:

$$\begin{aligned} \mathbf{v}^- &= \mathbf{v}_{t-\Delta t/2} + \frac{q\mathbf{E}}{m}\frac{\Delta t}{2} \\ \mathbf{v}^+ &= \mathbf{v}_{t+\Delta t/2} - \frac{q\mathbf{E}}{m}\frac{\Delta t}{2} \end{aligned} \tag{46}$$

Plugging this into (45) yields

$$\frac{\mathbf{v}^+ - \mathbf{v}^-}{\Delta t} = \frac{q}{2m}(\mathbf{v}^+ + \mathbf{v}^-) \wedge \mathbf{B} \tag{47}$$

This is a pure rotation of $\mathbf{v}$. To show this, let's take the dot product of (47) with $(\mathbf{v}^+ + \mathbf{v}^-)$, doing this cancels out the RHS because the dot product of perpendicular vectors is zeros, so we get

$$\mathbf{v}^+ \cdot (\mathbf{v}^+ + \mathbf{v}^-) = \mathbf{v}^- \cdot (\mathbf{v}^+ + \mathbf{v}^-)$$

Taking advantage of the commutative nature of the dot product we get

$$|v^+|^2 = |v^-|^2$$

Since $\mathbf{v}^+$ is the time evolution of $\mathbf{v}^-$ the only thing that (47) can change is orientation, and thus it is only a rotation. Therefore, if we know what angle is traversed we can use rotation matrices in the solution process. In fact the solution process is the following:

- Add half the electric impulse to $\mathbf{v}_{t-\Delta t/2}$ via (46) to obtain $\mathbf{v}^-$
- Rotate through the angle dictated by (47) to get $\mathbf{v}^+$
- Add the remaining half of the electric impulse to $\mathbf{v}^+$ to obtain $\mathbf{v}_{t+\Delta t/2}$

Now we need to derive the angle by which the velocity is rotated, we can do it by drawing the following figure:

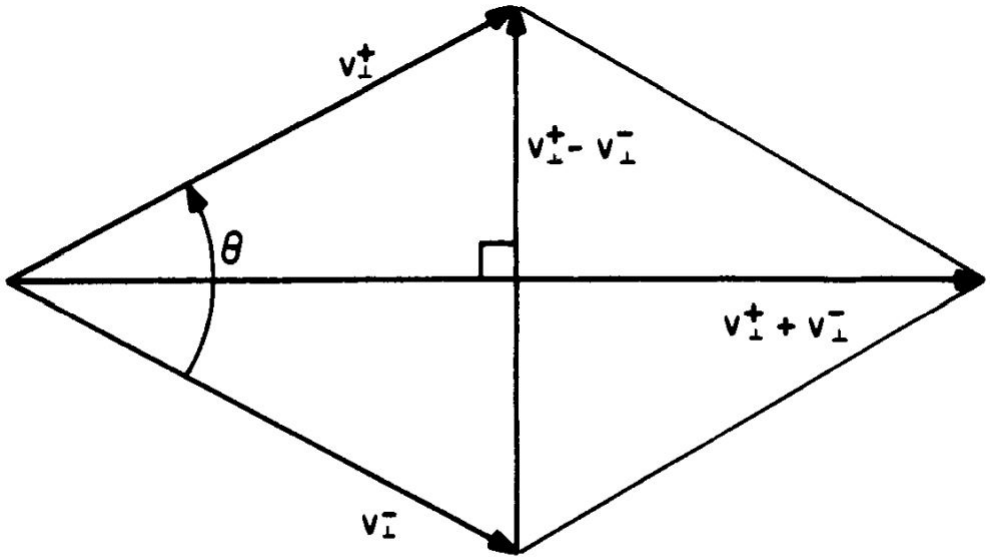


Figure 4-3a Knowing that (9) represents a rotation, we construct this diagram, from which tan (θ / 2) is readily obtained.

By construction, if we take half of $\mathbf{v}_\perp^+ - \mathbf{v}_\perp^-$ and $\mathbf{v}_\perp^+ + \mathbf{v}_\perp^-$ we get the legs of the right triangle in the upper left, and therefore:

$$tan\left(\frac{\theta}{2}\right) = \frac{|\mathbf{v}_\perp^+ - \mathbf{v}_\perp^-|}{|\mathbf{v}_\perp^+ + \mathbf{v}_\perp^-|}$$

We can then use (47) and take the magnitude of the cross product with the components of the velocity perpendicular to $\mathbf{B}$ and we get

$$\frac{|\mathbf{v}_\perp^+ - \mathbf{v}_\perp^-|}{|\mathbf{v}_\perp^+ + \mathbf{v}_\perp^-|} = \frac{qB\Delta t}{2m}$$

Therefore:

$$\left| tan\left(\frac{\theta}{2}\right) \right| = \frac{qB\Delta t}{2m} = \frac{\omega_c \Delta t}{2}$$

Now, by using the Lorenz force and the right hand rule, we realize that a particle gyrates with the opposite sign of the charge (if one defines counter-clockwise as positive, as is customary), so we get a relative minus sign for θ and as tan is an odd function we can define t as

$$t \equiv tan\left(\frac{\theta}{2}\right) = -\frac{qB\Delta t}{2m} \tag{48}$$

if we use the half angle formulas for sin and cos we get:

$$\begin{aligned} s \equiv -sin\theta &= \frac{2t}{1+t^2} \\ c \equiv cos\theta &= \frac{1-t^2}{1+t^2} \end{aligned} \tag{49}$$

Therefore, the rotation becomes

$$v_x^+ = cv_x^- + sv_y^-$$

$$v_y^+ = -sv_x^- + cv_y^-$$

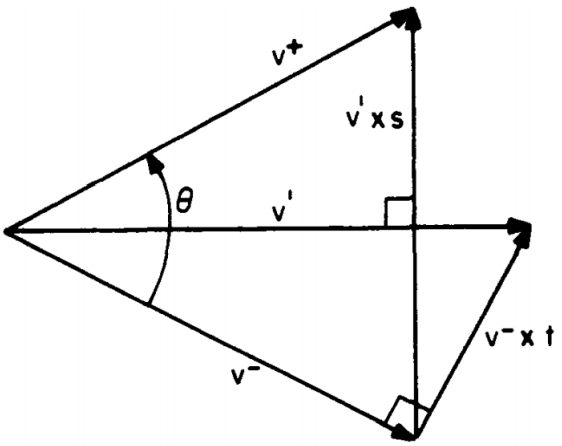
We can reduce the number of multiplications involved in this system by defining a new variable:

$$\begin{aligned} v_x' &= v_x^- + v_y^- t \\ v_y^+ &= v_y^- - v_x' s \\ v_x^+ &= v_x' + v_y^+ t \end{aligned} \tag{50}$$

Boris (cite 1970 paper) generalized this to a case in which $\mathbf{v}$ and $\mathbf{B}$ have arbitrary directions via a cross product:

$$\mathbf{v}' = \mathbf{v}^- + \mathbf{v}^- \wedge \mathbf{t} \tag{51}$$

By construction $\mathbf{v}'$ is perpendicular to both $\mathbf{B}$ and $\mathbf{v}^+ - \mathbf{v}^+$, which means that the angle between $\mathbf{v}^-$ and $\mathbf{v}'$ is $\theta/2$, thus, we can draw the following figure:



which shows us that

$$\mathbf{t} \equiv -\hat{\mathbf{b}} tan\left(\frac{\theta}{2}\right) = \frac{q\mathbf{B}}{m}\frac{\Delta t}{2} \tag{52}$$

As $\mathbf{v}'$ is by construction perpendicular to both $\mathbf{B}$ and $\mathbf{v}^+ - \mathbf{v}^-$ and we are in a three-dimensional space, $\mathbf{v}' \wedge \mathbf{B}$ is parallel to $\mathbf{v}^+ - \mathbf{v}^-$ and so if we define a vector $\mathbf{s}$ which is parallel to $\mathbf{B}$ we can make the statement:

$$\mathbf{v}^+ = \mathbf{v}^- + \mathbf{v}' \wedge \mathbf{s} \tag{53}$$

If we take the dot product of each side we get

$$|v^+|^2 = \mathbf{v}^+ \cdot \mathbf{v}^- + \mathbf{v}^+ \cdot (\mathbf{v}^- \wedge \mathbf{s}) + \mathbf{v}^+ \cdot (\mathbf{v}^- \wedge \mathbf{t}) \wedge \mathbf{s}$$

Let's look at just the triple product term:

$$(\mathbf{v}^- \wedge \mathbf{t}) \wedge \mathbf{s} = -\mathbf{v}^-(ts)$$

Where $\mathbf{v}^- \cdot \mathbf{s}$ cancel because they are perpendicular. Which, when plugged into the above expression yields:

$$|v^+|^2 = (1-ts)|v^-|^2 cos\theta + \mathbf{v}^+ \cdot (\mathbf{v}^- \wedge \mathbf{s}) = (1-ts)|v^-|^2\frac{1-t^2}{1+t^2} + \mathbf{s} \cdot (\mathbf{v}^+ \wedge \mathbf{v}^-)$$

Where I have used the identity derived earlier: $|v^+|^2 = |v^-|^2$ as well as (49) and properties of the scalar product. As $\mathbf{v}^+$ and $\mathbf{v}^-$ form a plane, their cross product gives a vector antiparallel to $\mathbf{s}$ by the right hand rule, therefore we get (again using (49))

$$|v^+|^2 = (1-ts)|v^-|^2\frac{1-t^2}{1+t^2} - s|v^-|^2 sin\theta = (1-ts)|v^-|^2\frac{1-t^2}{1+t^2} + s|v^-|^2\frac{2t}{1+t^2}$$

Pulling out a common factor:

$$|v^+|^2 = |v^-|^2 \frac{(1-ts)(1-t^2)+2st}{1+t^2}$$

By the fact that we want $|v^+|^2 = |v^-|^2$

$$1 + t^2 = 1 + ts - t^2 + st^3$$

$$2t = (1+t^2)s$$

And since $\mathbf{s}$ is definitionally parallel to $\mathbf{B}$ we get

$$\mathbf{s} = \frac{2\mathbf{t}}{1+t^2} \tag{54}$$

Finally, we can combine (46), (51), (52), (53), and (54) to get the full Boris algorithm for the Lorenz force with no additional forces:

$$\begin{aligned}
\mathbf{t} \equiv -\hat{\mathbf{b}} tan\left(\frac{\theta}{2}\right) &= \frac{q\mathbf{B}}{m}\frac{\Delta t}{2} \\
\mathbf{s} &= \frac{2\mathbf{t}}{1+t^2} \\
\mathbf{v}^- &= \mathbf{v}_{t-\Delta t/2} + \frac{q\mathbf{E}}{m}\frac{\Delta t}{2} \\
\mathbf{v}' &= \mathbf{v}^- + \mathbf{v}^- \wedge \mathbf{t} \\
\mathbf{v}^+ &= \mathbf{v}^- + \mathbf{v}' \wedge \mathbf{s} \\
\mathbf{v}_{t+\Delta t/2} &= \mathbf{v}^+ + \frac{q\mathbf{E}}{m}\frac{\Delta t}{2}
\end{aligned} \tag{55}$$

Of course, the goal of this algorithm is to find the evolution of the position of the particles over time, and there is some subtlety with that due to the half step notation of $\mathbf{v}$. The main loop is run with $\mathbf{x}$ leading $\mathbf{v}$ by $\Delta t/2$. Hence, at the start, $\mathbf{v}(0)$ is moved backwards to $\mathbf{v}(-\Delta t/2)$ (due to $\mathbf{v}_{t-\Delta t/2} = \mathbf{v}(t-\Delta t/2)$ so the initial point used for velocity is $\mathbf{v}_{0-\Delta t/2} = \mathbf{v}(-\Delta t/2)$)by first applying a rotation through the angle $\Delta\theta = \omega_c \Delta t/2$ then applying a half acceleration using $-\Delta t/2$ based on $\mathbf{E}(0)$ obtained from $\mathbf{x}(0)$. Alternatively, we can start the main loop by progressing $\mathbf{v}$ forward by a half step in a traditional euler form, an example of this in python is

```
for i in range(0, self.N):

   QM = self.Q[i]/self.M[i]

   self.Vx[i] += QM* (Ex[i] + Bz[i]* self.Vy[i] - By[i]* self.Vz[i])* self.dt /2
   self.Vy[i] += QM* (Ey[i] + Bx[i]* self.Vz[i] - Bz[i]* self.Vx[i])* self.dt /2
   self.Vz[i] += QM* (Ez[i] + By[i]* self.Vx[i] - Bx[i]* self.Vy[i])* self.dt /2
```

These can be treated as $\mathbf{v}_{\Delta t/2}$ in

$$\mathbf{x}_1 = \mathbf{x}_0 + \Delta t \mathbf{v}_{\Delta t/2}$$

The above equation can be derived from centering the first derivative of $\mathbf{x}$ at the half step:

$$\frac{\mathbf{x}_{t+\Delta t} - \mathbf{x}_t}{\Delta t} = \mathbf{v}_{t+\Delta t/2} \tag{56}$$

This is an example of a leapfrog method, with its characteristic half-step evaluations. Although the Boris algorithm as a whole is not a leapfrog method because of the fact that a leapfrog scheme cannot have the acceleration force depend on velocity (cite PPPL Boris letter), it is worth learning more about the leapfrog scheme as it describes the half-step evaluations for velocity while position is at a full step.

4.1.1 Generic Velocity-Independent Force

In Section 3.4.2 a gravitational field was introduced into the mirror. This exerts a special force on the particle, which is velocity independent. As this is the case, we can simply redefine (46) to account for additional velocity independent forces:

$$\begin{aligned} \mathbf{v}^- &= \mathbf{v}_{t-\Delta t/2} + \frac{q\mathbf{E}}{m}\frac{\Delta t}{2} + \mathbf{F}\frac{\Delta t}{2m} \\ \mathbf{v}^+ &= \mathbf{v}_{t+\Delta t/2} - \frac{q\mathbf{E}}{m}\frac{\Delta t}{2} - \mathbf{F}\frac{\Delta t}{2m} \end{aligned} \tag{57}$$

This maintains the form of the Lorenz equation for the Boris algorithm (47), and so has all of its properties, and thus we just need to appropriately modify (55), to simulate systems under the influence of these types of forces.

4.2 Leapfrog Scheme

For this section, I will begin by taking notes from this Wikipedia page.

A leapfrog method is one which is used to integrate systems of the following form:

$$\frac{dv}{dt} = A(x)$$

$$\frac{dx}{dt} = v$$

These differential equations are discretized in a centered scheme such that the velocity is evaluated at half steps and the position is evaluated at whole steps. The integrated progresses by interweaving position and velocity in a "leapfrog" pattern. Note that the derivative of velocity cannot depend on velocity to be considered a "leapfrog" method and come with all of the method's inherent advantages. Which includes the fact that it is symplectic in nature, which means that it conserves the (slightly modified) energy of dynamical systems.

Further notes on this subject will be taken from the 2013 lectures on the leapfrog method written by UCSC's Peter Young. (Included as a PDF in the "Individual" folder)

When looking at the slope of a chord between two points on a function, we find that it is a much better approximation of the derivative at the midpoint than at either end. For differential equations of a single degree of freedom (ie. where $\frac{dx}{dt}$ does not involve x, and $\frac{dv}{dt}$ does not involve v):

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = F(x) = -\frac{dU(x)}{dx}$$

The Euler method would integrate the first DE via $x_1 = x_0 + hv_0$, but a better approximation (second order) would be evaluating at the midpoint of v: $x_1 = x_0 + hv_{1/2}$, and so doing something similar for the velocity equation: $v_{3/2} = v_{1/2} + hF(x_1)$. From here, we can repeat the cycle, letting x and v leapfrog over each other after starting at the point x_0, $v_{1/2}$ for the increased accuracy. How accurate is this approach? Well, $x_1 - x_0$ is of order h, for the midpoint approximation the leading error of h^2 vanishes so the error for one interval is h^3. To integrate for a finite time T the number of intervals is T/h

and so the overall error is proportional to h^2 and the leapfrog method is a second order method.

To make the leapfrog useful, however, two questions have to addressed: first, how do we start at $v_{1/2}$, since we only know x_0 and v_0? The simplest approximation is just to do a single half step of Euler: $v_{1/2} = v_0 + \frac{1}{2}hF(x_0)$. Although this is not a midpoint method and has an order of h, we only do this once, so it does not lower the order of the method, which remains second order. The second question is how get the velocity at the same time as the position, which is needed to produce "phase space" plots and to compute the energy and angular momentum. The simplest approach is to consider $v_{n+3/2} = v_{n+1/2} + hF(x_{n+1})$ to be made up of two equal half steps, which successively relates v_{n+1} to $v_{n+1/2}$ and $v_{n+3/2}$ to v_{n+1}. The leapfrog algorithm with a means of starting the algorithm and determining x and v at the same times is called velocity Verlet. A single time step can be written as

$$v_{n+1/2} = v_n + \frac{1}{2}hF(x_n)$$

$$x_{n+1} = x_n + hv_{n+1/2}$$

$$v_{n+1} = v_{n+1/2} + \frac{1}{2}hF(x_{n+1})$$

This is the same as the leapfrog method because we have just separated $v_{n+3/2} = v_{n+1/2} + hF(x_{n+1})$ into two steps. We could also do it so that we start with a half step of position followed by a full step of velocity followed by a half step of velocity.

There are several advantages to this algorithm, one of them is that the method is time reversible. It conserves angular momentum exactly. The leapfrog algorithm is "symplectic" ie area preserving. To better understand this, consider a small rectangular region of phase space of area dA. Let the four corners of the square, $(x, p), (x + dx, p), (x, p + dp), (x + dx, p + dp)$ represent four possible coordinates of a particle at time t. These are labeled 1,2,3,4. Then , at a later time t' each of thse four points will have changed, to form the corners of a parallelogram. Let the area of the parallelogram be dA'. An important theorem (Liouville's theorem) states that the areas are equal: $dA = dA'$. (x, p) transforms to (x', p') where x' and p' are some functions of x, and p:

$$x' = X(x, p)$$

$$p' = P(x, p)$$

A set of equations like this, in which values of one set of variable is transformed to new values, is called a map. Thus, the result of integration of Newton's laws by a finite amount of time can be represented as an area preserving map. Since the area preserving property is an exact feature of the equations, it is desirable that a numerical approximation preserve it. Such approximations are called *symplectic*. What is the condition for a map to be symplectic? To see this we need to compute the area dA', and set it equal to $dA = dxdp$. The area dA' is given by

$$dA' = |d\vec{e}_1' \wedge d\vec{e}_2'|$$

Where $d\vec{e}_1'$ and $d\vec{e}_2'$ are the vectors describing the two sides of the parallelogram. Now, the components of $d\vec{e}_1'$ are just the changes in x' and p' when x is changed by dx put p is fixed, ie:

$$d\vec{e}_1' = \left(\frac{\partial x'}{\partial x}\hat{x} + \frac{\partial p'}{\partial x}\hat{p}\right) dx$$

and similarly

$$d\vec{e}_2' = \left(\frac{\partial x'}{\partial p}\hat{x} + \frac{\partial p'}{\partial p}\hat{p}\right) dp$$

The vector product representing dA' can be represented by a determinant where we get

$$dA' = detJdA$$

where

$$J = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial p} \\ \frac{\partial p'}{\partial x} & \frac{\partial p'}{\partial p} \end{bmatrix}$$

$detJ$ is the Jacobian of the transformation from (x', p') to (x, p) which occurs when you change variables in an integral:

$$\iint \ldots dx'dp' = \iint \ldots detJdxdp$$

Hence, a symplectic algorithm has $detJ = 1$ to show that the leapfrog method is symplectic, we should consider each step of the Verlet algorithm separately. If we compare the first step at two nearby points: (x_0, v_0) and $(x_0 + \delta x_0, v_0 + \delta v_0)$:

$$v_{1/2} + \delta v_{1/2} = v_0 + \delta v_0 + \frac{h}{2}F(x_0 + \delta x_0), \quad v_{1/2} = v_0 + \frac{h}{2}F(x_0)$$

Subtracting and letting δx_0 and δv_0 tend to zero, we get a matrix equation:

$$\begin{pmatrix} \delta x_0 \\ \delta v_{1/2} \end{pmatrix} = A \begin{pmatrix} \delta x_0 \\ \delta v_0 \end{pmatrix}$$

Where

$$A = \begin{bmatrix} 1 & 0 \\ \frac{h}{2}F'(x_0) & 1 \end{bmatrix}$$

Using the other steps in the Verlet algorithm

$$\begin{pmatrix} \delta x_1 \\ \delta v_{1/2} \end{pmatrix} = B \begin{pmatrix} \delta x_0 \\ \delta v_{1/2} \end{pmatrix} \quad \begin{pmatrix} \delta x_1 \\ \delta v_1 \end{pmatrix} = C \begin{pmatrix} \delta x_1 \\ \delta v_{1/2} \end{pmatrix}$$

Where

$$B = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ \frac{h}{2}F'(x_1) & 1 \end{bmatrix}$$

Since the Jacobian is the evolution matrix of perturbations (a stretching matrix), we can write

$$\begin{pmatrix} \delta x_1 \\ \delta v_1 \end{pmatrix} = J \begin{pmatrix} \delta x_0 \\ \delta v_0 \end{pmatrix}$$

Where we know J from the previous analysis: $J = CBA$, as the determinant of a product of matrices is the product of determinants of the individual matrices, and the determinants of A, B, and C are all 1 by inspection

$$detJ = 1$$

And the leapfrog is symplectic. The advantage of symplectic algorithms is that possess global stability. Since the area bounded by adjacent trajectories is preserved, we can never have the situation that the coordinates increase without bound, because this would expand the area. As well, the numerically calculated energy oscillates with small amplitude around the correct value, instead of diverging like the Euler method would. Note that the leapfrog algorithm is formulated for position dependent (conservative) forces only, and thus, it is not automatic that the Boris algorithm would have this symplectic property. The assumption that the force is velocity independent is because if this assumption is broken, the leapfrog method becomes implicit, and no long functional as originally formulated. A method for generic velocity dependent forces is the Taijama inversion method. This method is expensive,

and so treated only as a last resort. The Boris algorithm posseses the long-time stability characteristic of a leapfrog method, despite having a velocity dependent force. This includes exact energy conservation when the electric field is zero, and bounded energy error when the electric field is non-zero[**?**]. As well, the Boris algorithm has been found to solve for a particle's trajectory accurately for an arbitrarily large number of steps[**?**]. It, as well, is phase-space volume preserving.

References

[1] A.B. Langdon C.K. Birdsall. *Plasma Physics via Computer Simulation*. CRC Press, 1st edition, 2004.

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[3] Weston M. Stacey. *Fusion: An Introduction to the Physics and Technology of Magnetic Confinement Fusion*. Wiley, 2nd edition, 2010.