1. Explain why non-dimensionalization is valuable within Engineering and Science, then explain it could be applied in our current project.

2. Non-Dimensionalize the Navier Stokes equations, any left-over parameter should be labeled $\frac{1}{Re}$ (Hints: Use $\hat{p} = \frac{p}{\rho U_\infty}$ and $\hat{t} = \frac{tU_\infty}{L}$, the length scale L, and the velocity scale U_∞ are both physical, so you may not choose them so that they cancel with other values)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Note that these are only two of the four equations, but the final two are very similar to the given second equation, so the non-dimensionalization is the same for all.

3. Non-Dimensionalize the damped oscillator equation, and analyze what the process tells you about the system. If there are any remaining constant(s) explain what they do/represent. (Hint: k is the spring "strength" and q is the resistance "strength").

$$m\frac{d^2x}{dt^2} + q\frac{dx}{dt} + kx = 0$$

4. Non-Dimensionalize both the undamped and the damped wave equation. Compare the two, and comment on any remaining parameters.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - \lambda\frac{\partial u}{\partial t}$$

Note that c has units of velocity.

5. Work through the second Non-Dimensionalizing example from the lecture but annotate it.

6. Given the ODE model for spreading Diseases:

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \nu I$$

$$\frac{dR}{dt} = \nu I$$

Where S is the number of people susceptible to the disease, I is the number infected, and R are the ones who have recovered. Go through every term of each equation and describe what they represent and do, then determine the correct scaling factors by which to non-dimensionalize. From there, explain what the remaining represents (Hint: answer the question "under what conditions does the infection spread")

7. Explain Big O notation.

8. Re-do the final scale analysis example, but annotate the steps, and solve the resulting differential equation given

$$u(x, y, 0) = 5x^3 + 2e^{-}2y$$

$$u(x, y, D = 0.001) = 4x + y$$

(Hint: the "constants" of integration are functions of x and y)

9.Given the conditions of a boundary layer: Velocity (w) and Velocity fluctuations in the z-direction are small. Velocity in the x-direction is much larger than Velocity in the y-direction (u≫v). And characteristic distance in the y-direction is much smaller than characteristic distance in the x-direction ($\delta \ll$ L). Derive the Boundary Layer Equations from the Navier Stokes Equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

(Hint: Use the non-dimensionalization of t and p given in problem 2).

10. Using the same method as in lecture, find the terminal velocity of an object that obeys the equation:

$$m\frac{dv}{dt} = mg - kv$$

Then, do the same but with damping being proportional to v^2.

11. Given the Boundary Layer assumptions (question 9, but U and V instead of u and v) as well as the following assumptions: velocities U and V are small (but U remains much larger than V, note that velocities being small does NOT mean that their derivatives are small), and there is no pressure gradient. Apply scale analysis to the following equation:

$$\begin{aligned}(S'''\{y^+\})^+ = \left(\frac{\partial U^+}{\partial y^+}\right)^2 + \frac{\partial^2}{\partial (y^+)^2}\left(k^+ + \overline{(v^+)^2}\right) \\ -U^+\frac{1}{2}\frac{\partial\left(\overline{q^2}\right)^+}{\partial y^+} - V^+\frac{1}{2}\frac{\partial\left(\overline{q^2}\right)^+}{\partial y^+} \\ -\left(\left(\overline{u^2}\right)^+ - \left(\overline{v^2}\right)^+\right)\frac{\partial U^+}{\partial x^+} - (\overline{uv})^+\frac{\partial U^+}{\partial y^+} \\ -\frac{\partial}{\partial y}\overline{v\left[k^+ + p/\rho\right]}\end{aligned} \tag{1}$$

given that

$$k = \frac{1}{2}(u^2 + v^2 + w^2)$$

$$-\left(\overline{uv}\right)^+ = (v_\tau)^+\frac{\partial U^+}{\partial y^+}$$

$$(v_\tau)^+ = \frac{k^+}{\omega^+}$$

$$\omega^+ = \frac{\epsilon^+}{k^+}$$

$$\begin{aligned}\epsilon^+ = 2\overline{\left(\frac{\partial u^+}{\partial x^+}\right)^2} + 2\overline{\left(\frac{\partial v^+}{\partial y^+}\right)^2} + 2\overline{\left(\frac{\partial w^+}{\partial z^+}\right)^2} + \\ \overline{\left(\left(\frac{\partial u^+}{\partial y^+}\right) + \left(\frac{\partial v^+}{\partial x^+}\right)\right)^2} + \overline{\left(\left(\frac{\partial u^+}{\partial z^+}\right) + \left(\frac{\partial w^+}{\partial x^+}\right)\right)^2} \\ + \overline{\left(\left(\frac{\partial v^+}{\partial z^+}\right) + \left(\frac{\partial w^+}{\partial y^+}\right)\right)^2}\end{aligned} \tag{2}$$

Ignore the plus signs and overbars, they have no bearing on this problem. u,v,w are turbulent velocity fluctuations, they are all O(U). It is useful to know that $O(\delta^2 U^2) = O(1)$ and $O(\frac{U^2}{\delta^2}) = O(U^4)$

12. Prove that

$$\int_0^L sin\left(\frac{n\pi x}{L}\right) sin\left(\frac{m\pi x}{L}\right) dx$$

is zero if m does not equal n and $L/2$ if m does equal n. Do the same for an interval of -L to L, note that if m does equal n it is L. Do the same analysis for an integral involving cosines instead of sines. Then show that

$$\int_0^L sin\left(\frac{n\pi x}{L}\right) cos\left(\frac{m\pi x}{L}\right) dx = 0$$

no matter what n and m are.

13. Find the Fourier coefficients for the first Fourier series example. Use MATLAB to plot the solution. Compute using four different numbers of terms in the series (four separate graphs) and compare between them. What do you notice?

14. Find the Fourier series for x^2 on the interval -L to L. Plot it using MATLAB, compare the graph of the series with the graph of the function. What does the graph look like with a small number of terms? A large number of terms? About how many terms does adding more terms start to make negligible difference to the accuracy?

15. Repeat the last problem, but with e^x, the integrals here are quite difficult, you need to use integration by parts until you get an algebraic relationship with which you can solve for the initial integral. Try it by hand first, but if you cannot solve it after giving it the good ol' college try you can use an online calculator.