1. Solve the Laplacian:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Subject to the boundary conditions:

$$u(0, y, z) = y + z$$

$$u(1, y, z) = y^2$$

$$u(x, 0, z) = z$$

$$u(x, 1, z) = x$$

$$u(x, y, 0) = 5x^3 + 2e^{-2y}$$

$$u(x, y, 0.001) = 4x + y$$

Then nondimensionalize and apply scale analysis, solve the resulting equation and compare resulting numerical values. (When you solve this problem pay attention to the eigenvalues λ and explain like in the lecture why they must be a certain sign)

2. Solve the Wave Equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(0, t) = u(L, t) = 0$$

$$u(x, 0) = e^x$$

$$\frac{\partial u}{\partial t}(x, 0) = 0$$

Plot snapshots of the x vs. u graph for c=1 and L=1 as time progresses and compare them.

3. Solve the cylindrically symmetric wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$$

$$u(a, t) = 0$$

$$|u(0,t)| < \infty$$
$$u(r,0) = \alpha(r)$$
$$\frac{\partial u}{\partial t}(r,0) = \beta(r)$$

Note that the second boundary condition is a statement that the equation cannot diverge at zero. Explain what you interpret "cylindrically symmetric" as.

4. Use scale analysis to solve the system of equations given that u and ρ are of the same order but x is much larger than y (Hint: $L * \delta = O(1)$, use $\tau = \frac{tc}{\delta}$, if by scale analysis you reduce the order of the equation, you can ignore some conditions, only use the ones you need):

$$\frac{\partial \rho}{\partial t} + \frac{\partial^4 u}{\partial x^3 \partial y} + \left(\frac{\partial \rho}{\partial x} + y^2 \frac{\partial u}{\partial y}\right)^2 + \frac{\partial \rho}{\partial x} = c^2 \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial y \partial x}$$

$$\frac{\partial^4 \rho}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} + \left(\frac{\partial u}{\partial x} + \frac{\partial^2 \rho}{\partial y^2}\right)^2 = \frac{\partial^2 \rho}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2}$$
$$u(0,y,t) = u(L,y,t) = u(x,0,t) = u(x,L,t) = 9$$
$$\frac{\partial^2 u}{\partial x^2}(x,0,t) = \frac{\partial^2 u}{\partial x^2}(x,L,t) = 0$$
$$\rho(0,y,t) = \rho(L,y,t) = \rho(x,0,t) = \rho(x,L,t) = 9$$
$$\frac{\partial^2 \rho}{\partial x^2}(x,0,t) = \frac{\partial^2 \rho}{\partial x^2}(x,L,t) = 0$$
$$u(x,y,0) = 3x^2$$
$$\rho(x,y,0) = 3x^2$$
$$\frac{\partial u}{\partial t}(x,y,0) = 0$$

5. According to the Kutta-Joukowski Lift Theorem, in a steady flow past a two-dimensional body, the cross section of which is a simple closed curve, if the flow is uniform with a speed U in the x-direction at infinity then the force on the body is:

$$F_x = 0$$

$$F_y = -\rho U \Gamma$$

where Γ is the circulation which is defined to be:

$$\int_0^{2\pi} u_\theta r d\theta$$

We can find u_{theta} by using conservation of mass in the following way:

$$\nabla \cdot \vec{u} = 0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

We can now define a "stream function" by taking advantage of the fact that for mixed derivatives the order does not matter, so a y partial derivative and then an x partial derivative is the same the other way around:

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

In irrotational fluids the stream function ψ satisfies the Laplace equation:

$$\nabla^2 \psi = 0$$

For a simple application of the Kutta-Joukowski theorem let's look at the lift of a simple 2-D circle with radius a, so all of our equations will be in polar coordinates:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$u_\theta = -\frac{\partial \psi}{\partial r}$$

$$\nabla^2 \psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0$$

For fluid-flow, the no-slip principal mandates that the fluid speed must be equal to the surface speed that the fluid is flowing over, as well, the stream functions must be parallel to the boundary so:

$$\psi(a, \theta) = 0$$

This system is also symmetric, so we must employ periodic boundary conditions:

$$\psi(r, -\pi) = \psi(r, \pi)$$

$$\frac{\partial \psi}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

As well, the system must be bounded at the origin:

$$|\psi(0, \theta)| < \infty$$

Given that at infinity we have a velocity of U solely in the x-direction and because of that the stream function at infinity is written as:

$$\psi = Ursin\theta$$

(Why is this equation valid? Hint: use this equation to simplify the Fourier series). Solve the Laplacian for the stream function and given that a=1 find a velocity which allows for positive lift according to the Kutta-Joukowski theorem. You will have an arbitrary constant, set it to positive 1. (Hint: in the final stream equation, the logarithm $ln\frac{r}{a}$ goes to zero in the limit to infinity, in the process of solving via separation of variables, you will get an ODE that you have not seen before and may appear quite difficult, it is the equipotential equation, google it.)

6. The following is the beam equation, which models the deflection w of a load bearing beam subjected to a constant load q. This equation became widely used in civil engineering in the 1800s being used in the Eiffel Tower. In this case the beam is anchored at one end and free at the other, so the boundary conditions are as follows:

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} + q = 0$$

$$w(0, t) = 0$$

$$\frac{\partial w}{\partial x}(0, t) = 0$$

$$\frac{\partial^2 w}{\partial x^2}(L, t) = 0$$

$$\frac{\partial^3 w}{\partial x^3}(L, t) = 0$$

Draw a picture of the system and solve the PDE.

7. In Fluid Dynamics, a common homework problem is solving for the velocity profile in a so-called fully-developed Couette flow. A Couette flow is where a fluid is inbetween two plates, the one on the bottom is stationary, the one on the top is being pulled along at a constant velocity hence driving flow. The fully developed problem is a simple ODE, but we will be dealing with the "start-up" problem where it it starting from rest and developing into normal flow. Under the "start-up" conditions, velocity (in all directions) is zero, the driving force is exclusively in the x-direction, so there is no velocity in any other direction except the x-direction, and no change in velocity in anywhere but the y-direction (please try to explain this). The bottom plate is stationary (velocity zero) and set at y=0 and the top plate is moving with velocity U and at height h. Given the information here, derive the governing equation, initial condition(s), and boundary conditions from the starting point of the time-dependent Navier Stokes equations (written out for you in HW 1). Comment on the solution, what is the steady state? Non-dimensionalize the problem using h and U for the length and velocity scales and graph the solution to this system in MATLAB with a number of different Reynolds numbers. You will have to solve the equation again after non-dimensionalizing.

8. Let's say you are tracking the movement of a small object within the flow described in the previous problem, formulate the ODE which describes its change in position (x-direction) if its height is half of h. Derive the analytical solution to the quasi-static equation governing the change in position. Saying it starts at x=0, using MATLAB, compare the quasi-static form to the full form at different Reynolds numbers, comment. (Hint: the velocity of the movement will be equal to the velocity of the fluid at the height of the object)

9. Consider an infinitely tall cylindrical cup of coffee. Let's make the assumption that you just finished stirring it with an infinitely long stirring stick at the edge of the cup (r=a). From the system we can assume that there is only azimuthal velocity, and at zero it is zero so at t=0:

$$u_\theta = \Omega r$$

However, we know that because of the no-slip condition, at the walls of our cup the velocity is zero, as well, we don't want our solution to explode, so our BCs are:

$$u_\theta(a,t) = 0$$

$$|u_\theta(0,t)| < \infty$$

In this system the Navier Stokes equations become:

$$\frac{\partial u_\theta}{\partial t} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)$$

Solve this equation and then find how long it takes for the speed to reach $\frac{1}{e}$ of its initial speed. (Hint: you only need to deal with the exponential term, and there are tables of the appropriate zeros) Explain why your infinite cylinder cup of coffee takes so much longer to slow down than a normal cup of coffee.

10. Nondimensionalize the following equations, boundary conditions and initial conditions:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{q}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$k_{liq}\frac{\partial T_{liq}\,(x,t)}{\partial x}\bigg|_{x=s} + \rho_{sol}\Delta h_f\frac{ds(t)}{dt} = k_{sol}\frac{\partial T_{sol}\,(x,t)}{\partial x}\bigg|_{x=s}$$

Liquid BCs

$$\begin{aligned} \frac{\partial T_{liq}}{\partial r}(0,t) &= 0 \\ T_{liq}(s(t),t) &= T_m \\ T_{liq}(r,0) &= \phi_{liq} \end{aligned} \tag{1}$$

Solid BCs

$$\begin{aligned} T_{sol}(r_0,t) &= T_o \\ T_{sol}(s(t),t) &= T_m \\ T_{sol}(r,0) &= \phi_{sol} \end{aligned} \tag{2}$$

Label any remaining constants on the first equation Q and any on the second equation $\frac{1}{St}$ (Hint: Use $\tau = \frac{\alpha t}{r_0^2}$ and $\theta = \frac{T(r,t)-T_0}{T_m-T_0}$). This is the spherical Stefan problem, which models the movement of an interface (s(t)) due to internal heating. You solve the non-homogeneous heat equation twice, once applying liquid boundary conditions, and then again but applying solid boundary conditions. You then plug the resulting equations into the given interface ODE. What I want you to do beyond solving the PDE is find the quasi-static equation, and then plot it for St=0.01 St=0.1 St=1 St=10 using the initial interface position as 0.99 for solidification, and 0.01 for melting, comment on what changing St does. As well, find the analytic form for the zeros in both the liquid and the solid equation.(Hint: Check your ODE notes, and use google for trig identities).