On the Stefan Problem with Internal Heat Generation and Prescribed Heat Flux Conditions at the Boundary

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Applications

- Nuclear accident/meltdown scenario analysis
- Metal casting/materials processing industry
- Cryosurgical applications
- Geophysical modeling

Objective

Find an analytic solution to the Stefan problem (solid-liquid phase change) with internal heat generation, and fixed heat flux in both cylindrical and plane wall geometries

Problem Geometries

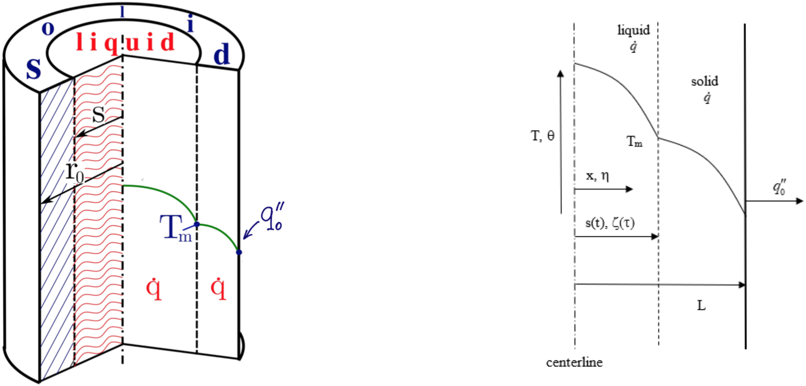


Figure: Cylindrical, and Plane Wall Schematic

Problem Assumptions

- The internal heat generation and the thermophysical properties of the materials are equal and constant in both the solid and liquid phases.
- The melting temperature T_m of the material is fixed and known.
- The phase change occurs at a single temperature T_m and there is no “mushy zone” at the interface in between phases.
- The heat transfer is by conduction. Convection in the liquid phase is neglected.

Governing Equations

Conduction Equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Interface Equation

$$-k_{liq}\frac{\partial T_{liq}(r,t)}{\partial r}\bigg|_{r=s}+\rho_{sol}\Delta h_f\frac{ds(t)}{dt}=-k_{sol}\frac{\partial T_{sol}(r,t)}{\partial r}\bigg|_{r=s}$$

$$T=T(r,t)$$

$$0\leq s\leq r_0$$

$$t\geq 0$$

Boundary and Initial Conditions

BCs and IC in the liquid phase

$$\left.\frac{\partial T_{liq}(r,t)}{\partial r}\right|_{r=0} = 0,$$

$$T_{liq}(s(t),t) = T_m, \quad T_{liq}(r,0) = \phi_{liq}(r)$$

BCs and IC in the solid phase

$$T_{sol}(s(t),t) = T_m,$$

$$-k\left.\frac{\partial T_{sol}(r,t)}{\partial r}\right|_{r=r_0} = q_0'', \quad T_{sol}(r,0) = \phi_{sol}(r)$$

Nondimensional Variables

$$\eta = \frac{r}{r_0}, \quad \zeta = \frac{s(t)}{r_0}, \quad \tau = \frac{\alpha t}{r_0^2}$$

$$\theta(\eta, \tau) = \frac{T(r,t) - T_m}{T^*}$$

$$T^* = \frac{\alpha \rho_{sol} \Delta h_f}{k_{sol}}, \quad \dot{Q} = \frac{\dot{q} {r_o}^2}{\alpha \rho_{sol} \Delta h_f}, \quad Q'' = \frac{q'' r_0}{k_{sol} T^*}$$

Dimensionless Governing Equations

Conduction Equation

$$\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \theta(\eta,\tau)}{\partial \eta}\right) + \dot{Q} = \frac{\partial \theta(\eta,\tau)}{\partial \tau},$$

Interface Equation

$$\left.\frac{\partial \theta_{liq}(\eta,\tau)}{\partial \eta}\right|_{\eta=\zeta} + \frac{d\zeta(\tau)}{d\tau} = \left.\frac{\partial \theta_{sol}(\eta,\tau)}{\partial \eta}\right|_{\eta=\zeta}$$

$$0 \leq \zeta \leq 1$$

$$\tau \geq 0$$

Dimensionless Boundary and Initial Conditions

BCs and IC in the liquid phase:

$$\frac{\partial \theta_{liq}(0,\tau)}{\partial \eta} = 0$$

$$\theta_{liq}(\zeta(\tau),\tau) = 0, \theta_{liq}(\eta,0) = \Phi_{liq}(\eta)$$

BCs and IC in the solid phase:

$$\theta_{sol}(\zeta(\tau),\tau) = 0$$

$$\frac{\partial \theta_{sol}(1,\tau)}{\partial \eta} = -Q'', \quad \theta_{sol}(\eta,0) = \Phi_{sol}(\eta)$$

Solution in liquid phase

$$\theta_{liq}(\eta,\tau) = \sum_{n=1}^{\infty} A_n \, \mathrm{e}^{-\lambda_n^2 \tau} \, J_0(\lambda_n \eta) + \frac{\zeta^2 - \eta^2}{4} \dot{Q}$$

where

$$A_n = \frac{\int_0^{\zeta} [\Phi_{liq}(\eta) - \theta_{liq,ss}(\eta)] J_0(\lambda_n \eta) \eta \, d\eta}{\int_0^{\zeta} J_0^2(\lambda_n \eta) \eta \, d\eta}$$

$$\lambda_n = \frac{z_{0n}}{\zeta(\tau)}, \quad n = 1, 2, \ldots$$

Solution in Solid Phase

The full solution in solid phase is:

$$\theta_{sol}(\eta,\tau) = \theta_{sol,tr}(\eta,\tau) + \theta_{sol,ss}(\eta), \quad \zeta(\tau) < \eta < 1$$

$$\theta_{sol}(\eta,\tau) = \sum_{n=1}^{\infty} B_n \tilde{f}_n(\tilde{\lambda}_n \eta)\, \mathrm{e}^{-\tilde{\lambda}_n^2 \tau} + \frac{\dot{Q}}{4}\left[\zeta^2 - \eta^2\right] + \left(\frac{\dot{Q}}{2} - Q''\right) \ln \frac{\eta}{\zeta}$$

where

$$B_n = \frac{\int_{\zeta(\tau)}^{1} (\Phi_{sol}(\eta) - \theta_{sol,ss}(\eta)) \tilde{f}_n(\tilde{\lambda}_n \eta) \eta\, d\eta}{\int_{\zeta(\tau)}^{1} [\tilde{f}_n(\tilde{\lambda}_n \eta)]^2 \eta\, d\eta}$$

The eigenvalues λ_n are the roots of

$$Y_1(\lambda) J_0(\zeta\lambda) - Y_0(\zeta\lambda) J_1(\lambda) = 0$$

$$\tilde{f}_n(\tilde{\lambda}_n \eta) = J_0(\tilde{\lambda_n}\eta) - \frac{J_1(\tilde{\lambda_n})}{Y_1(\tilde{\lambda_n})} Y_0(\tilde{\lambda_n}\eta) : \quad \text{associated e'functions}$$

Interface Equation

The interface equation is:

$$\left.\frac{\partial\theta_{liq}(\eta,\tau)}{\partial\eta}\right|_{\eta=\zeta} + \frac{d\zeta(\tau)}{d\tau} = \left.\frac{\partial\theta_{sol}(\eta,\tau)}{\partial\eta}\right|_{\eta=\zeta}$$

Hence we have:

$$\frac{d\zeta(\tau)}{d\tau} = \sum_{n=1}^{\infty} A_n\lambda_n\,\mathrm{e}^{-\lambda^2\tau}\,J_1(\lambda_n\zeta) + \sum_{n=1}^{\infty} B_n\tilde{\lambda_n}\bar{f_n}(\zeta\tilde{\lambda_n})\,\mathrm{e}^{-\tilde{\lambda_n^2}\tau} + \left(\frac{\dot{Q}}{2} - Q''\right)\frac{1}{\zeta}$$

where

$$\bar{f_n}(\tilde{\lambda_n}\eta) = -J_1(\tilde{\lambda_n}\eta) + \frac{J_1(\tilde{\lambda_n})}{Y_1(\tilde{\lambda_n})}Y_1(\tilde{\lambda_n}\eta)$$

Quasi-Static Interface Equation

$$\frac{d\zeta(\tau)}{d\tau} = \left(\frac{\dot{Q}}{2} - Q''\right)\frac{1}{\zeta}$$

Melting: $Q'' < \frac{\dot{Q}}{2}$.
Solidification: $Q'' > \frac{\dot{Q}}{2}$.

J. Crepeau, A. Siahpush, Approximate solutions to the Stefan problem with internal heat generation, *Heat and Mass Transfer* **44** (7) (2008).

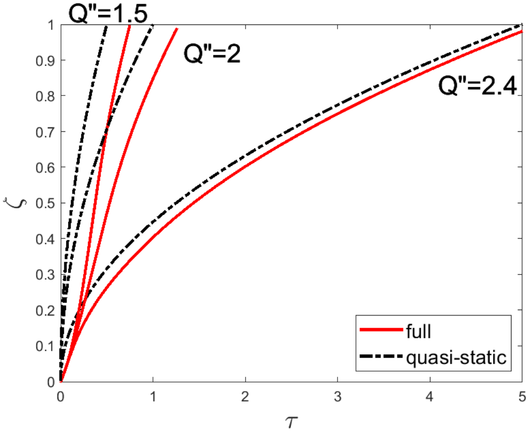
Interface Evolution for Melting

Initial conditions for melting:

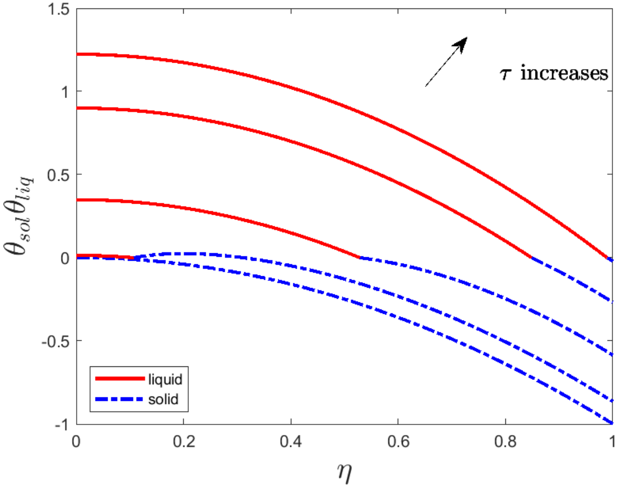
$$\text{At} \quad \tau = 0, \quad \Phi_{liq}(\eta) = 1, \quad \Phi_{sol}(\eta) = 1 - \eta^2, \quad \zeta(\tau) = 0$$

$\dot{Q} = 5$

Q'' varies



Temperature Evolution During Melting



$\dot{Q} = 5 \qquad Q'' = 2$

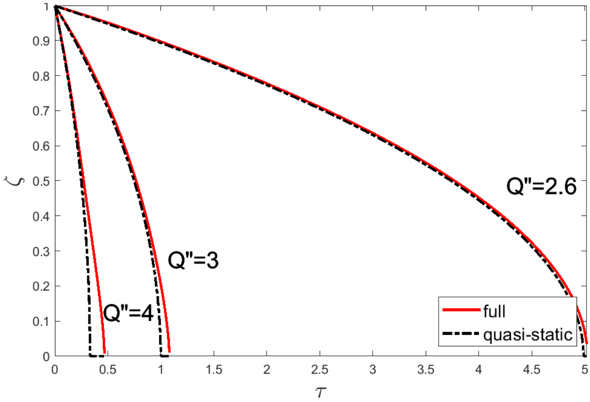
Interface Evolution for Solidification

Initial conditions for solidification:

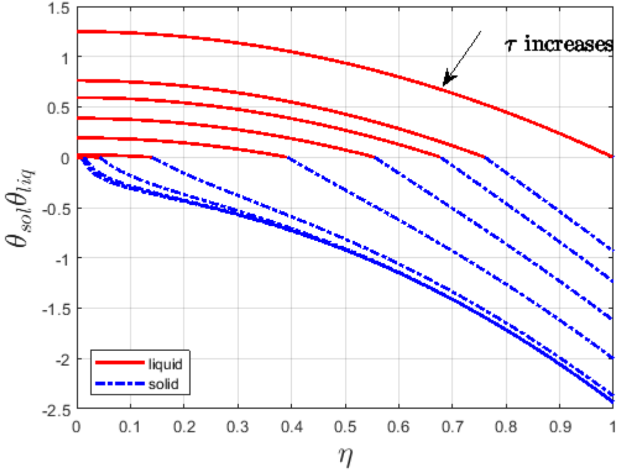
$$\text{At} \quad \tau = 0, \quad \Phi_{liq}(\eta) = \frac{\dot{Q}}{4}(1-\eta^2)+1, \quad \Phi_{sol}(\eta) = 1, \quad \zeta(\tau) = 1$$

$\dot{Q} = 5$

Q'' varies



Temperature Evolution for Solidification



$$\dot{Q} = 5 \qquad Q'' = 4$$

Conclusions

Defined the Governing Equations

Solution Through Classical Techniques

Interface Equation with Quasi-Static Approximation

Numerical Results