We start with our non-homogeneous heat equation and our interface equation along with boundary conditions:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{q}}{k}=\alpha\frac{\partial T}{\partial t} \tag{1}$$

$$k_{liq}\frac{\partial T_{liq}}{\partial r}\bigg|_{r=s}+\rho_{sol}\Delta h_f\frac{ds}{dt}=k_{sol}\frac{\partial T_{sol}}{\partial r}\bigg|_{r=s} \tag{2}$$

$$\frac{\partial T_{liq}(0,t)}{\partial r}=0 \quad T_{liq}(s(t),t)=T_m \quad T_{liq}(x,0)=\phi_{liq}(x) \tag{3}$$

$$\frac{\partial T_{sol}(R,t)}{\partial r}=-\frac{q_0''}{k_{sol}} \quad T_{sol}(s(t),t)=T_m \quad T_{sol}(x,0)=\phi_{sol}(x) \tag{4}$$

We shall assume $k=k_{sol}=k_{liq}$. Then we will apply the following non-dimensional variables:

$$\begin{gathered}\eta=\frac{r}{R} \quad \zeta=\frac{s}{L} \quad \tau=\frac{\alpha t}{L^2} \quad \theta(\eta,\tau)=\frac{T(x,t)-T_m}{T^*}\\ \dot{Q}=\frac{\dot{q}L^2}{\alpha\rho_{sol}\Delta h_f} \quad Q''=\frac{q_0''L}{\alpha\rho_{sol}\Delta h_f}\\ \Phi_{liq}(\eta)=\frac{\phi_{liq}(R\eta)-T_m}{T^*} \quad \Phi_{sol}(\eta)=\frac{\phi_{sol}(R\eta)-T_m}{T^*}\\ T^*=\frac{\alpha\rho_{sol}\Delta h_f}{k_{sol}}\end{gathered} \tag{5}$$

Inputting these into the governing equations and boundary conditions yields:

$$\begin{gathered}\frac{1}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial\theta}{\partial\eta}\right)+\dot{Q}=\frac{\partial\theta}{\partial\tau}\\ \frac{\partial\theta}{\partial\eta}\bigg|_{\eta=\zeta}+\frac{d\zeta}{d\tau}=\frac{\partial\theta_sol}{\partial\eta}\bigg|_{\eta=\zeta}\\ \frac{\partial\theta_{liq}}{\partial\eta}(0,\tau)=0 \quad \theta_{liq}(\zeta(\tau),\tau)=0 \quad \theta(\eta,0)=\Phi_{liq}(\eta)\\ \frac{\partial\theta_{sol}}{\partial\eta}(1,\tau)=-Q'' \quad \theta_{sol}(\zeta,\tau)=0 \quad \theta_{sol}(\eta,0)=\Phi_{sol}(\eta)\end{gathered} \tag{6}$$

Now we can go on to looking at the steady state differential equations.

$$\frac{1}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2\frac{d\theta_{ss}}{d\eta}\right)=-\dot{Q} \tag{7}$$

Liquid BCs

$$\begin{aligned} \frac{d\theta_{liq,ss}}{d\eta}(0) &= 0 \\ \theta_{liq,ss}(\zeta) &= 0 \end{aligned} \tag{8}$$

Yields

$$\theta_{liq,ss} = -\frac{1}{6}\eta^2\dot{Q} + \frac{1}{6}\zeta^2\dot{Q} \tag{9}$$

Solid BCs

$$\begin{aligned} \frac{d\theta_{liq,ss}}{d\eta}(1) &= -Q'' \\ \theta_{liq,ss}(\zeta) &= 0 \end{aligned} \tag{10}$$

Yields

$$\begin{aligned} \theta_{sol,ss} &= -\frac{\dot{Q}}{6}\eta^2 - \frac{C_1}{\eta} + C_2 \\ C_2 &= \frac{\dot{Q}}{6}\zeta^2 + \frac{C_1}{\zeta} \\ C_1 &= \frac{\dot{Q}}{3} - Q'' \end{aligned} \tag{11}$$

From here, we can plug these steady state equations into the interface equation, and apply a quasi-static approximation in order to neglect the transient portions we get:

$$\frac{d\zeta}{d\tau} = \left(\frac{\dot{Q}}{3} - Q''\right)\zeta^{-2} \tag{12}$$

As ζ cannot be negative, we know that melting occurs when $Q'' < \frac{\dot{Q}}{3}$, and solidification when $Q'' > \frac{\dot{Q}}{3}$. Now, we can apply separation of variables to the transient partial differential equation:

$$\begin{gathered} \frac{1}{\eta^2}\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial\theta}{\partial\eta}\right) = \frac{\partial\theta}{\partial\tau} \\ \frac{\partial\theta_{liq}}{\partial\eta}(0,\tau) = 0 \quad \theta_{liq}(\zeta(\tau),\tau) = 0 \quad \theta(\eta,0) = \Phi_{liq}(\eta) \\ \frac{\partial\theta_{sol}}{\partial\eta}(1,\tau) = 0 \quad \theta_{sol}(\zeta,\tau) = 0 \quad \theta_{sol}(\eta,0) = \Phi_{sol}(\eta) \end{gathered} \tag{13}$$

The general solution is:

$$
\begin{aligned}
f(\eta) &= Aj_0(\lambda\eta) + By_0(\lambda\eta) \\
g(\tau) &= Ce^{-\lambda^2\tau}
\end{aligned} \tag{14}
$$

Where j_0 and y_0 are spherical Bessel functions of the first and second kind respectively. Applying the liquid boundary conditions and using the fact that $j_0(\lambda\eta) = \frac{sin(\lambda\eta)}{\lambda\eta}$:

$$
\begin{aligned}
f_n(\eta) &= A'_n j_0(\lambda_n\eta) \\
\lambda_n &= \frac{n\pi}{\zeta} \qquad n = 1, 2, \ldots
\end{aligned} \tag{15}
$$

Putting all of this into a series, and using the steady state solution for the liquid phase we get:

$$
\begin{aligned}
\theta_{liq}(\eta,\tau) &= \frac{1}{6}\dot{Q}\zeta^2 - \frac{1}{6}\dot{Q}\eta^2 + \sum_{n=1}^{\infty} A_n j_0(\lambda_n^{liq}\eta)e^{-\left(\lambda_n^{liq}\right)^2\tau} \\
A_n &= \frac{\int_0^\zeta \left[\Phi_{liq} - \theta_{liq,ss}\right] j_0\left(\lambda_n^{liq}\eta\right)\eta^2 d\eta}{\int_0^\zeta j_0^2\left(\lambda_n^{liq}\eta\right)\eta^2 d\eta} \\
\lambda_n^{liq} &= \frac{n\pi}{\zeta} \qquad n = 1, 2, \ldots
\end{aligned} \tag{16}
$$

Applying the solid boundary condition at $\eta = R$ we find:

$$
\begin{aligned}
f(\eta) &= A'\tilde{f}\left(\lambda^{sol},\eta\right) \\
\tilde{f}\left(\lambda^{sol},\eta\right) &= j_0\left(\lambda^{sol}\eta\right) - \frac{j_1\left(\lambda^{sol}\right)}{y_1\left(\lambda^{sol}\right)} y_0\left(\lambda^{sol}\eta\right)
\end{aligned} \tag{17}
$$

To find λ_n^{sol} we must apply the boundary condition at the interface which yields:

$$
0 = y_1\left(\lambda^{sol}\right) j_0\left(\lambda^{sol}\zeta\right) - j_1\left(\lambda^{sol}\right) y_0\left(\lambda^{sol}\zeta\right) \tag{18}
$$

We will define $\lambda^{sol} \equiv \lambda$ for the time being and then use the following properties of spherical Bessel functions:

$$\begin{aligned} j_0(z) &= \frac{sin(z)}{z} \\ j_1(z) &= \frac{sin(z)}{z^2} - \frac{cos(z)}{z} \\ n_0(z) &= -\frac{cos(z)}{z} \\ n_1(z) &= -\frac{cos(z)}{z^2} - \frac{sin(z)}{z} \end{aligned} \tag{19}$$

Which, when plugging into (18) and applying some trig identities yields:

$$\begin{aligned} 0 &= sin\,(\lambda - \lambda\zeta) - \lambda cos\,(\lambda - \lambda\zeta) \\ 0 &= tan\,((1-\zeta)\lambda) - \lambda \\ 0 &= tan(\gamma) - \frac{\gamma}{1-\zeta} \\ \gamma &\equiv (1-\zeta)\lambda \end{aligned} \tag{20}$$

(20) is in the form of the transcendental equation $tan(x) = Cx$, whose roots need to be solved for numerically. However, we can get an arbitrarily accurate estimate of the n^{th} root by utilizing the Lagrange Inversion Theorem in a similar fashion to N.G. De Bruijn's method for solving $cot(x) = x$ presented in his 1970 book "Asymptotic Methods in Analysis". In the book the Lagrange Inversion Theorem is stated as follows:

Let the function $f(z)$ be analytic in some neighborhood of the point z=0 of the complex plane. Assuming that $f(0) \neq 0$ we consider the equation

$$w = \frac{z}{f(z)}$$

Where z is the unknown. Then there exist positive numbers a and b such that for $|w| < a$, the equation has just one solution in the domain $|z| < b$, and this solution is an analytic function of w:

$$z = \sum_{k=1}^{\infty} C_k w^k$$

Where

$$C_k = \frac{1}{k!}\left\{\left(\frac{d}{dz}\right)^{k-1}(f(z))^k\right\}\Big|_{z=0}$$

In order to use this method, we need to give a baseline by analyzing the geometry of the tangent function. First, we notice, that as n becomes large, the associated root necessarily approaches infinity, so, to get $tan(x_n)$ to approach infinity, we find that at large n $x_n \approx \left(n + \frac{1}{2}\right)\pi$, where x_n is the n^{th} root. We want z to be bounded, so we will define it such that it approaches zero. So we will set $z \equiv \pi n + \frac{\pi}{2} - x_n$, we also want w to be small when z is, so we will set $w \equiv (\pi n + \frac{\pi}{2})^{-1}$. So $x = w^{-1} - z$, and we can now proceed to the root derivation:

$$tan(w^{-1} - z) = C(w^{-1} - z)$$

Using trig identities

$$cot(z) = C(w^{-1} - z)$$

$$\frac{\frac{1}{C}cos(z) + zsin(z)}{sin(z)} = w^{-1}$$

$$w = \frac{Csin(z)}{cos(z) + Czsin(z)} = \frac{z}{f(z)}$$

$$f(z) = \frac{z(cos(z) + Czsin(z))}{Csin(z)}$$

Note that $f(0) = \frac{1}{C}$. Now we can plug all of this into the series from the definition of the Lagrange Inversion Theorem, as we will be calculating the roots numerically, and only need an initial guess of a root, we will only keep the first term of the series:

$$z \approx \frac{1}{C}\left(\frac{1}{\left(n + \frac{1}{2}\right)\pi}\right)$$

Which we can plug in for x as:

$$x_n \approx \left(n + \frac{1}{2}\right)\pi - \frac{1}{C}\left(\frac{1}{\left(n + \frac{1}{2}\right)\pi}\right)$$

Now, this equation is normalized for the first non-zero root being larger than $\frac{\pi}{2}$, however, that is only true if the slope of Cx is smaller than or equal to the slope of $tan(x)$ near zero, otherwise Cx intersects with the first positive asymptote of $tan(x)$. Near zero $tan(x) \approx x$ so, if C is larger than 1 there is a root smaller than $\frac{\pi}{2}$. In our case $C = \frac{1}{1-\zeta}$ which is always larger than 1, so we will renormalize our root equation to take that into account so:

$$\begin{aligned} \gamma_n \approx \left((n-1)+\frac{1}{2}\right)\pi - (1-\zeta)\left(\frac{1}{((n-1)+\frac{1}{2})\pi}\right) \\ \lambda_n = \frac{\gamma_n}{1-\zeta} \end{aligned} \tag{21}$$

Now we can write our solid solution

$$\begin{aligned} \theta_{sol}(\eta,\tau) = -\frac{\dot{Q}}{6}\eta^2 - \frac{C_1}{\eta} + C_2 + \sum_{n=1}^{\infty} B_n \tilde{f}_n e^{-\left(\lambda_n^{sol}\right)^2\tau} \\ B_n = \frac{\int_\zeta^1 \left[\Phi_{liq} - \theta_{liq,ss}\right]\tilde{f}_n \eta^2 d\eta}{\int_\zeta^1 \tilde{f}_n^2 \eta^2 d\eta} \\ \tilde{f}_n\left(\lambda_n^{sol},\eta\right) = j_0\left(\lambda_n^{sol}\eta\right) - \frac{j_1\left(\lambda_n^{sol}\right)}{y_1\left(\lambda_n^{sol}\right)} y_0\left(\lambda_n^{sol}\eta\right) \\ C_2 = \frac{\dot{Q}}{6}\zeta^2 + \frac{C_1}{3}\zeta^{-3} \\ C_1 = \frac{\dot{Q}}{3} - Q'' \\ \lambda_n^{sol} = \frac{\gamma_n}{1-\zeta} \\ \gamma_n \approx \left((n-1)+\frac{1}{2}\right)\pi - (1-\zeta)\left(\frac{1}{((n-1)+\frac{1}{2})\pi}\right) \\ n = 1,2,\ldots \end{aligned} \tag{22}$$

Interface Equation

$$\begin{aligned} -\sum_{n=1}^{\infty} A_n \lambda_n^{liq} j_1\left(\lambda_n^{liq}\zeta\right) e^{-\left(\lambda_n^{liq}\right)^2\tau} + \frac{d\zeta}{d\tau} = C_1\zeta^{-2} \\ +\sum_{n=1}^{\infty} B_n \lambda_n^{sol}\left(\frac{j_1\left(\lambda_n^{sol}\right)}{y_1\left(\lambda_n^{sol}\right)} y_1\left(\lambda_n^{sol}\zeta\right) - j_1\left(\lambda_n^{sol}\zeta\right)\right) e^{-\left(\lambda_n^{sol}\right)^2\tau} \end{aligned} \tag{23}$$

Quasi Static

$$\frac{d\zeta}{d\tau} = \left(\frac{\dot{Q}}{3} - Q''\right)\zeta^{-2} \tag{24}$$

$$\zeta(\tau) = \sqrt[3]{\left(\dot{Q} - 3Q''\right)\tau + \zeta_0^3} \tag{25}$$

From here we can determine an upper bound for the time it will take for the interface to reach a certain position:

$$\tau_f = \frac{\zeta_f^3 - \zeta_0^3}{\dot{Q} - 3Q''} \tag{26}$$

This will be used as a cutoff for our numerical simulations.

For the melting case

$$\begin{aligned} \zeta(0) = \zeta_0 &= 0.01 \\ \Phi_{sol} &= -\frac{\dot{Q}}{8}\eta^2 \\ \Phi_{liq} &= 0 \end{aligned} \tag{27}$$

For the solidification case

$$\begin{aligned} \zeta(0) = \zeta_0 &= 0.99 \\ \Phi_{sol} &= 0 \\ \Phi_{liq} &= -\frac{1}{6}\dot{Q}\eta^2 + \frac{1}{6}\dot{Q}\zeta_0^2 \end{aligned} \tag{28}$$