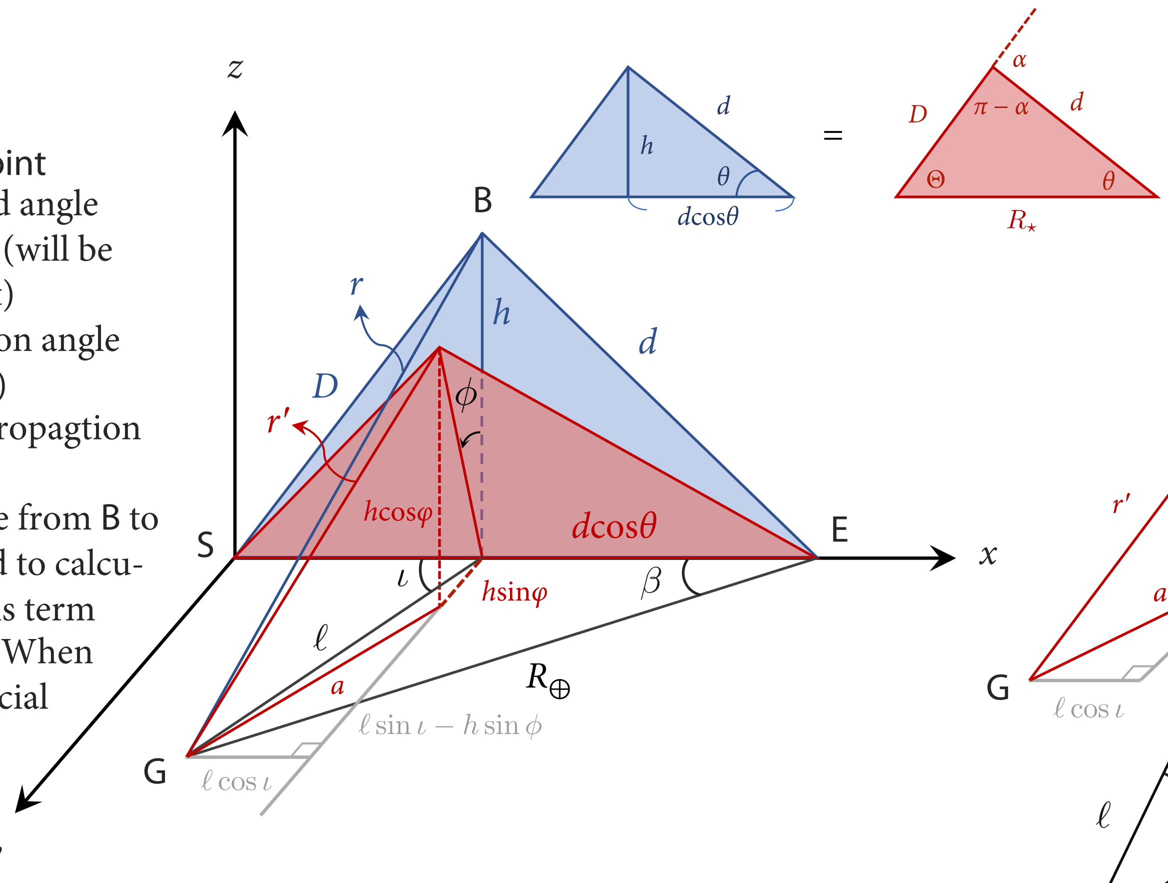
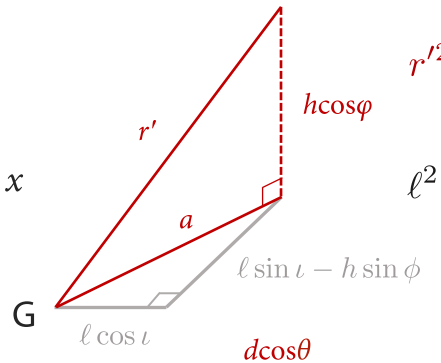
G: GC
E: Earth
S: SN
B: Boosted point
β: off-centered angle
θ: open-angle (will be integrated out)
α: the deflection angle (θ-dependent)
$D = ct$ (SNν propagtion length)
r': the distance from B to G, will be used to calculate $n_\chi(r')$. This term varies with φ. When $\varphi = 0$, it is special with $r' = r$.



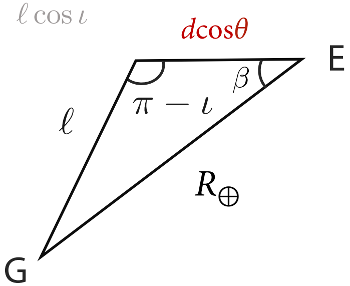
$$\frac{R_\star}{\sin\alpha} = \frac{D}{\sin\theta} = \frac{d}{\sin\Theta} \;\rightarrow\; \Theta = \sin^{-1}\frac{d}{D}\sin\theta \quad \text{and} \quad -1 \le \frac{d}{D}\sin\theta \le 1$$

$$\begin{aligned} d^2 &= R_\star^2 + D^2 - 2R_\star D\cos\Theta \\ &= R_\star^2 + D^2 - 2R_\star D\sqrt{1 - \frac{d^2}{D^2}\sin^2\theta} \\ d^2 &= R_\star^2 + D^2 - 2R_\star^2\sin^2\theta \pm \sqrt{2[R_\star^2\cos^2\theta(2D^2 - R_\star^2 + R_\star^2\cos 2\theta)]} \end{aligned}$$



$$\begin{aligned} r'^2 &= a^2 + h^2\cos^2\phi \\ &= \ell^2\cos^2\iota + (\ell\sin\iota - h\sin\phi)^2 + h^2\cos^2\phi \end{aligned}$$

$$\ell^2 = R_\oplus^2 + d^2\cos^2\theta - 2R_\oplus d\cos\theta\cos\beta$$



$$\cos\iota = \frac{R_\oplus^2 - \ell^2 - d^2\cos^2\theta}{2\ell d\cos\theta}$$

In the limit of SN at GC (S = G), we have $\beta = 0$ and $R_\star = R_\oplus$. Thus we recover

$$\begin{aligned} \ell &= R_\oplus - d\cos\theta \\ r^2 &= h^2 + \ell^2 = d^2\sin^2\theta + (R_\oplus - d\cos\theta)^2 \\ &= R_\oplus^2 + d^2 - R_\oplus d\cos\theta = D^2 \\ \therefore r &= D \quad \text{as desired!} \end{aligned}$$

$$\begin{aligned} \cos\iota &= \frac{R_\oplus^2 - (R_\oplus - d\cos\theta)^2 - d^2\cos^2\theta}{2\ell d\cos\theta} \\ &= \frac{2R_\oplus d\cos\theta - 2d^2\cos^2\theta}{2(R_\oplus - d\cos\theta)d\cos\theta} = 1 \text{ which implies } \sin\iota = 0 \end{aligned}$$

$$\begin{aligned} r'^2 &= \ell^2\cos^2\iota + (\ell\sin\iota - h\sin\phi)^2 + h^2\cos^2\phi \\ &= \ell^2 + h^2\sin^2\phi + h^2\cos^2\phi \\ &= \ell^2 + h^2 = r^2 \quad \text{as desired!} \end{aligned}$$