The Surface Stochastic Volatility Inspired model and its calibration for arbitrage-free volatility surfaces

Executive summary

1. Goal

Modelling an implied volatility surface using SSVI model (Gatheral and Jacquire 2013)

- Fits to the given quoted data as close as possible
- Satisfies no static arbitrage conditions

2. Applications

Implied volatility surface: for derivatives pricing and risk management

- Pricing and fair comparison of derivatives across strikes and expiration not quoted
- Avoid arbitrage opportunities

3. Why SSVI model?

SSVI: benchmark model in Equity market:

- Satisfy Roger Lee's moment formula, for extrapolation purposes
- Conditions for no static arbitrage

Outline

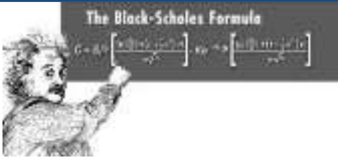
Theory

Black-Scholes Model and Implied Volatility

1. Black- Scholes Model

- Give an intuition on option price:
  European call option $C(t, S_t) = S_t N(d_1) - Ke^{-r\Delta t} N(d_2)$

- Flawed assumptions: S_t follows a Geometric Brownian Motion:
  $dS_t = \mu S_t \, dt + \sigma \, S_t \, dW_t$ with ***constant drift and volatility***



Notation:
S_t is underlying price at t
K is strike price, T is maturity.
N(·) is the cdf of standard normal distribution

$$d_1 = \frac{ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\Delta t}{\sigma\sqrt{\Delta t}}$$

$$d_2 = d_1 - \sigma\sqrt{\Delta t}$$

2. Implied Volatility: $\sigma = \sigma_{BS}$

- Provide a forward-looking
- Not constant in reality

Black-Scholes model predicts a flat implied volatility curve, but it is a smile or skew in practice

→ Black-Scholes model: no use for option pricing.

3. Volatility smile

- A smile: near-term equity options and forex options
- A skew: index or commodities options, and long-term options.

(Source: Investopedia)

Stochastic Volatility Model

1. Extending the Black-Scholes Model

- Constant volatility σ: in Black and Scholes model
- Time dependent volatility σ (t)
- The time-and state dependent σ (t, S_t): local volatility
- Stochastic volatility σ (t, ω)

Notation:

μ_t is drift

η is volatility of volatility

(w_1, w_2) a 2-dimensional Brownian motion

ρ is the correlation between stock price returns and changes in v_t.

-1< ρ < 1

2. Stochastic Volatility Model

- It models volatility σ = $\sqrt{v}$ as:

$$dS_t = \mu_t S_t dt + \sqrt{v_t}\, S_t dw_1$$

$$dv_t = \alpha(S_t, v_t, t)dt + \eta\beta(S_t, v_t, t)\sqrt{v_t}\, dw_2$$

and $(dw_1\, dw_2) = \rho dt$

- ρ assumption:
  - In practice: $\rho < 0$, leverage effect
  - SSVI model: ρ is constant

Static arbitrage

1. Calendar spread arbitrage

- The price of an option with a longer time to expiration is priced cheaper than an option with a shorter time to expiry. (same characteristics (underlying, moneyness, ...)

2. Butterfly arbitrage

- **Long put butterfly strategy:** options with same expiry

a) long one put at strike $K+a$
b) short two puts at strike K
c) long one put at strike $K-a$

if the price of the strategy is zero or less and the final price within $K-a$ and $K+a$ one has made a profit at no cost and no risk of a loss:

Payoff	$S_T < K-a$	$K-a < S_T < K$	$K < S_T < K+a$	$K+a < S_T$
a)	$-S_T+K+a$	$-S_T+K+a$	$-S_T+K+a$	0
b)	$2(S_T-K)$	$2(S_T-K)$	0	0
c)	$-S_T+K-a$	0	0	0
Σ	0	$S_T-K+a>0$	$K+a-S_T>0$	0

A costless trading strategy which at some future time provides a positive profit but has no possibility of a loss

it is an assumption in most methods for asset pricing

If at a discrete point in time, arbitrage is possible, this is called static arbitrage.

Total Variance Surface

- For any $k \in R$ and $T > 0$, $C_{BS}(k, \sigma^2_T)$ is Black-Scholes price of a European Call option of the underlying stock S_T with strike $K = F_T e^k$, maturity T and volatility $\sigma > 0$

- Denoting the Black-Scholes implied volatility by $\sigma_{BS}(k,T)$, and the ***total implied variance*** by

$$\mathbf{w(k,T) = \sigma^2_{BS}(k,T).T}$$

- The two-dimensional ***map (k,T) → w(k,T): the surface of the total implied variance.*** We call it volatility surface

- The function ***k → w(k,T): a slice*** (for a fixed maturity T)

- The notation $w(k; \chi)$: a given maturity slice, where χ is a set of parameters, and drop the T-dependence

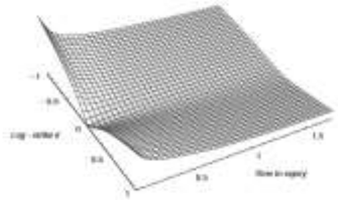


Figure 1: SPX-implied volatility surface *(Gatheral)*

The surface: a three-dimensional plot Oxyz, where:

x-axis: time to maturity

z-axis: log strike price $k = ln(\frac{K}{F_T})$

y-axis: implied volatility.

Total Variance Surface – Static arbitrage

Conditions for the volatility surface to ensure the absence of arbitrage:

- **Calendar spread arbitrage**

In the absence of dividends the volatility surface w is free of calendar spread arbitrage

$$\frac{\partial w(k,T)}{\partial T} \geq 0, \text{ for all } k \in R \text{ and } T > 0.$$

This condition is violated when two given slices $w_1(k)$ and $w_2(k)$ cross

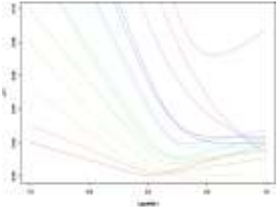


Figure 2: Total variance plot gives rise to calendar spread arbitrage (crossed lines)

(Gatheral)

- **Butterfly arbitrage**

A slice is free of butterfly arbitrage if and only if the corresponding density g(k) of that slice is non-negative

$$g(k) = \left(1 - \frac{k\,w'(k)}{2w(k)}\right)^2 - \frac{w'(k)^2}{4}\left(\frac{1}{w(k)} + \frac{1}{4}\right) + \frac{w''(k)}{2}$$

$\forall\, k \,\epsilon\, R$, and $\lim_{k \to +\infty} d_+(k) = -\infty$, in which $w' = \frac{\partial w}{\partial K}$

$d_+(k)$ is indeed d_1 in Black-Scholes model

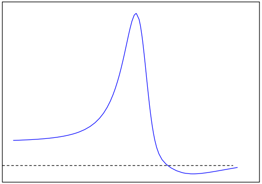


Figure 3: Plots of the function g(k) - butterfly arbitrage occurs

(Gatheral)

Raw SVI slice parametrization

A slice parameterization based on parameter set $\chi_{raw} = \{a, b, \rho, m, \sigma\}$, Gatheral[1]

$$w(k, \chi_R) = a + b\left\{\rho(k-m) + \sqrt{(k-m)^2 + \sigma^2}\right\}$$

where $a \in R$, $b \geq 0$, $|\rho| < 1$, $m \in R$, $\sigma > 0$

Effects on the smile as parameter changes:

- Increasing a leads to a vertical translation of the smile in the positive direction
- Increasing b increases the slopes of the wings (tightening the smile)
- Increasing ρ results in a counter-clockwise rotation of the smile.
- Increasing m translates the slice to the right
- Increasing σ reduces the at-the-money curvature of the smile.

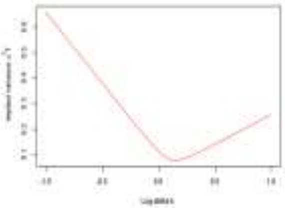


Figure 4: With $a = 0.04$, $b = 0.4$, $\sigma = 0.1$, $\rho = -0.4$, $m = 0$

(Gatheral)

[1] Gatheral "A parsimonious arbitrage-free implied volatility parameterization with application to the valuation of volatility derivatives", Presentation at Global Derivatives (2004)

Raw SVI slice parametrization – Static arbitrage

The raw slice parameterization is prone to arbitrage:

- **Calendar spread arbitrage**
  Given two slices with parameters χ_1 and χ_2, setting them equal: $w(k, T_1; \chi_1) = w(k, T_2, \chi_2)$ and by rearranging and squaring twice one arrives at a quartic polynomial in k:

$$\alpha_4 k^4 + \alpha_3 k^3 + \alpha_2 k^2 + \alpha_1 k + \alpha_0 = 0$$

  whose roots are known. If there are no real roots, the slices do not cross.
  If they do cross, we define the crossed-ness of two slices as the maximum of the differences in w at certain distances from the crossing points.

- **Butterfly arbitrage**
  There are no known conditions on w to eliminate butterfly arbitrage

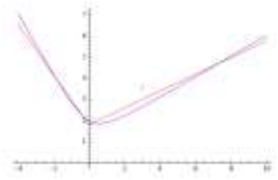


Figure 5: SVI slices may cross at no more than four points *(Gatheral)*

Surface SVI (SSVI) parameterization

- **Why SSVI instead of SVI?**

  SVI has two shortcomings:

  - Not model the whole surface
  - No explicit condition on w for non arbitrage

- **SSVI model**

  Gatheral and Jacquier (2013) develop Surface SVI model:

$$w(k,\theta_T) = \frac{\theta_T}{2}\left\{1 + \rho\,\phi(\theta_T)k + \sqrt{(\phi(\theta_T)k\ +\ \rho)^2 + (1-\ \rho)^2}\right\}$$

A model with 3 parameters for each slice.

Where + ρ: constant correlation (the surface has the same ρ for all slices)

+ $\theta_T = \sigma^2_{BS}(0,T)\,T$: ATM total implied variance

+ $\phi\ (\theta_T)$: a smooth curvature function

SSVI parameterization - Static arbitrage

1. Conditions for no static arbitrage

- **Calendar spread arbitrage**

1. $\partial_t \theta_t \geq 0$, for all t > 0

2. $0 \leq \partial_\theta \left(\theta\, \phi(\theta)\right) \leq \frac{1}{\rho^2}\left(1 + \sqrt{1-\rho^2}\,\right)\phi(\theta)$, for all θ > 0

- **Butterfly arbitrage**

1. $\theta\, \phi(\theta)(1 + |\rho|) < 4$, for all θ > 0

2. $\theta\, \phi(\theta)^2(1 + |\rho|) \leq 4$, for all θ > 0

Condition 1 is necessary and Condition 2 is tight.

2. Choice of function ϕ (θ_T)

We use complex-power law form $\phi(\theta) = \frac{\eta}{\sqrt{\theta}\sqrt{1+\theta}}$

Under the condition $\eta(1 + |\rho|) \leq 2$ this SSVI surface is free of static arbitrage

Empirical part

General recipe

1. General recipe

- Data preparation
- Fit SSVI for the whole surface, then convert parameters (ρ, η) → (a, b, ρ, m, σ). : ***initial guess***
- Fit SVI slice by slice (using initial guess as the starting) with penalty for arbitrage: ***optimal calibration***
- Interpolation between calibrated slices and extrapolation beyond given maturities in data set

2. Black-Scholes formula expressed in discounted factor (DF) and forward price (F)

$$C(0, S_0) = S_0\, N(d_1) - K e^{-rT} N(d_2) \quad \textit{(Original B\&S formula)}$$

$$C(0, S_0) = DF\, (F\, N(d_1) - K N(d_2)) \quad \textit{(B\&S formula expressed in DF \& F)}$$

- The 2nd formula is used in both cases: the asset pays dividends and does not pay dividends. (Hull)
- The 2nd formula is used by practitioners: no need for information of dividend and interest rate

Data preparation

- **Data**
  - SPX options data in May 2019 (CBOE website)

- **Data handle**
  - Select ***liquid options*** which are: bid price ≥ 0, ask price ≥ 0, volume ≥ 0
  - Select options which have the ***time to maturity is 5 days*** $\leq$ ***T*** $\leq$ ***2 years***
    - T is so small: intrinsic value
    - T is large: not reliable
  - ***Calculate mid-price*** of option: average of ask and bid price

- **Raw data**

quotedate	Expirationdate	ttm	strike	optiontype	lastsale	net	bid	ask	vol	openint	pricemid
2019-05-13	2019-05-24	0.03	2200	C	658.32	-9.98	647.10	651.40	1	3	649.250
2019-05-13	2019-05-24	0.03	2650	C	186.15	-37.8	201.30	205.00	1	88	203.150
2019-05-13	2019-05-24	0.03	2675	C	186	-14.6	177.80	181.40	4	0	179.600
2019-05-13	2019-05-24	0.03	2750	C	101.2	-32.75	110.90	113.80	8	167	112.350
2019-05-13	2019-05-24	0.03	2755	C	98	-33.85	106.90	109.60	2	15	108.250
2019-05-13	2019-05-24	0.03	2760	C	100.8	-34.7	102.60	105.40	1	14	104.000
2019-05-13	2019-05-24	0.03	2765	C	88.7	-32.6	98.50	101.20	3	13	99.850
2019-05-13	2019-05-24	0.03	2770	C	93	-24.25	94.40	97.10	5	56	95.750
2019-05-13	2019-05-24	0.03	2775	C	82.25	-30.85	90.40	93.00	8	19	91.700
2019-05-13	2019-05-24	0.03	2780	C	80.9	-28.2	86.40	89.00	4	35	87.700
2019-05-13	2019-05-24	0.03	2790	C	86	-14.95	78.70	81.10	3	27	79.900
2019-05-13	2019-05-24	0.03	2795	C	87.1	-9.95	75.00	77.20	17	59	76.100
2019-05-13	2019-05-24	0.03	2800	C	83.8	-9.35	71.30	73.40	28	61	72.350

Figure 6: SPX options raw data: the first 13 options in the data set

Inputs calculation

- **Discount factor and Forward price of the index**

  Use Put-call parity: $C_t + Ke^{-r\,(T-t)} = P_t + S_t$

  $\rightarrow C_t - P_t = -DF.\,K + DF\,.\,F$

  Remind that $DF = e^{-r\,(T-t)}$

  Run linear regression $y = ax + b$

  So, y is $C_t - P_t$

  x is K

  $\rightarrow$ Get (DF, F) for each maturity.

	ttm	DF	forward	w0
1	0.03	0.9983622	2849.657	0.001151431
2	[illegible]	[illegible]	[illegible]	[illegible]
3	0.11	[illegible]	2852.718	[illegible]
4	[illegible]	[illegible]	[illegible]	[illegible]
5	0.36	[illegible]	2855.745	[illegible]
6	0.61	[illegible]	[illegible]	[illegible]
7	[illegible]	[illegible]	[illegible]	[illegible]
8	[illegible]	0.9616304	[illegible]	[illegible]

Figure 7: Inputs calculation

- **Implied volatility (IV):** use Black-Scholes formula
- **Theta:** use interpolation.

  $\theta_T = \sigma_{BS}(0,T).T$

  From the data, we only have values of σ_{BS} for our given maturities, use interpolation to find σ_{BS} at $K = F$ ($k = 0$) which is ATM implied volatility $\sigma_{BS}(0,T)$

	ttm	strike	DF	forward	w0	pricemid	iv
1	0.03	2650	0.9983622	2849.657	0.001151431	203.150	0.33880
2	0.03	2675	0.9983622	2849.657	0.001151431	179.600	0.31960
3	0.03	2750	0.9983622	2849.657	0.001151431	112.350	0.26820
4	0.03	2755	0.9983622	2849.657	0.001151431	108.250	0.26580
5	0.03	2760	0.9983622	2849.657	0.001151431	104.000	0.26340
6	0.03	2765	0.9983622	2849.657	0.001151431	99.850	0.25900
7	0.03	2770	0.9983622	2849.657	0.001151431	95.750	0.25590
8	0.03	2775	0.9983622	2849.657	0.001151431	91.700	0.25280
9	0.03	2780	0.9983622	2849.657	0.001151431	87.700	0.24960
10	0.03	2790	0.9983622	2849.657	0.001151431	79.900	0.24370
11	0.03	2795	0.9983622	2849.657	0.001151431	76.100	0.24080
12	0.03	2800	0.9983622	2849.657	0.001151431	72.350	0.23790
13	0.03	2805	0.9983622	2849.657	0.001151431	68.600	0.23460

Figure 8: SPX options cleaned data: the first 13 options in the data set

Implied volatility

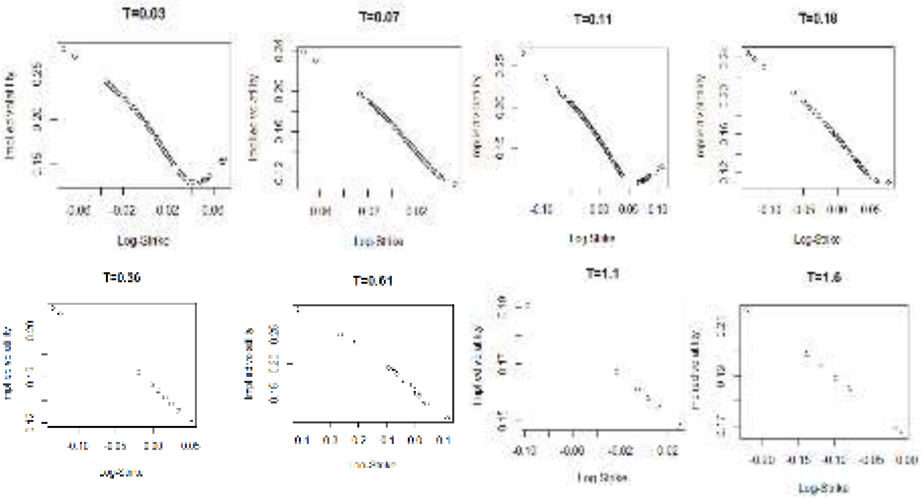


Figure 9: Implied volatility

SSVI fit: Initial guess

Remind: Condition for no static arbitrage in our SSVI surface: $\eta(1+|\rho|) \leq 2$

- **Objective function: minimization function of (η, ρ)**

$$\sum_{T \in \mathrm{T}} \sum_{k \in \mathrm{K}} \frac{(SSVI\ price - market\ price)^2}{market\ price} + \max(0, \eta(1+|\rho|) - 2)$$

- **Algorithm:** Generalized Simulated Annealing Function, both for initial guess and optimal calibration.

- **Parameters results:**

  $(\eta, \rho) = (0.9774298, -0.7456917)$

  → condition $\eta(1+|\rho|) \leq 2$ is satisfied

  Convert to parameters (a, b, ρ, m, σ)

  for 8 slices

	a	b	rho	m	sig
1	0.0002555655	0.01857390	-0.7456917	0.02590256	0.02314448
2	0.0004348362	0.02160948	-0.7456917	0.03379967	0.03020077
3	0.0006211836	0.02581723	-0.7456917	0.04041490	0.03611156
4	0.0009771332	0.03235412	-0.7456917	0.05072389	0.04532733
5	0.0018471203	0.04439710	-0.7456917	0.06980304	0.06244196
6	0.0031851191	0.05612569	-0.7456917	0.09204099	0.08224055
7	0.0051633199	0.08032796	-0.7456917	0.12887813	0.11515531
8	0.0098803725	0.10068730	-0.7456917	0.16450081	0.14698492

Figure 10: Initial guess - Parameter results

SSVI fit: Initial guess

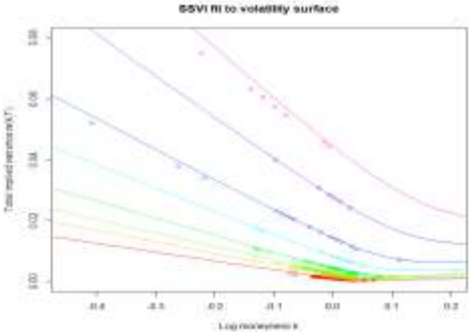


Figure 11: Initial guess – Fitting result

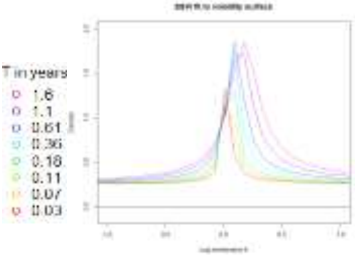


Figure 12: Initial guess – Density plot

- **Dots**: market data points & **line**: SSVI fitting line. Plot also shows no crossing lines
- **No arbitrage fitting**
    - The crossed-ness of the fit is 0, no crossing lines
    - Density of the slices: positive
- **SSVI price error:** 4.506323. Model fits well at the near time to maturity options, worse for the long- term options (last slices)

SVI fit: Optimal calibration

- **Objective function: minimization function of (a, b, ρ , m, σ)**

$$\sum_{T\,€\,\mathrm{T}}\sum_{k\,€\,\mathrm{K}}\frac{(SVI\;price - market\;price)^2}{market\;price} + \sum_{T\,€\,\mathrm{T}} c_i + \sum_{T\,€\,\mathrm{T}} |\min(0, g(k)|$$

C_i :Penalty for calendar spread arbitrage (maximum crossed-ness of 2 consecutive slices)

|min (0, g(k)|: Penalty for butterfly arbitrage

Remind:
Arbitrage conditions
- Calendar spread : no crossing line s
- Butterfly: g(k) is non negative

- **Parameters results:**

Parameters (a, b, ρ , m, σ) for 8 slices

1	0.0004814773	0.01100839	-0.7453005	0.03377439	0.005400929
2	0.0006276907	0.01796750	-0.7455901	0.04018193	0.017253739
3	0.0011653770	0.02128091	-0.7441270	0.04312464	0.010951375
4	0.0017973754	0.02818632	-0.7478784	0.05088391	0.020020707
5	0.0021799765	0.04192062	-0.7459039	0.07617485	0.045828709
7	0.0045674571	0.07789552	-0.7466929	0.15337900	0.099551176
6	0.0066287535	0.05284784	-0.7439628	0.08307917	0.014934992
8	0.0101013679	0.08377889	-0.7481887	0.21698056	0.093407477

Figure 13: Optimal calibration - Parameter results

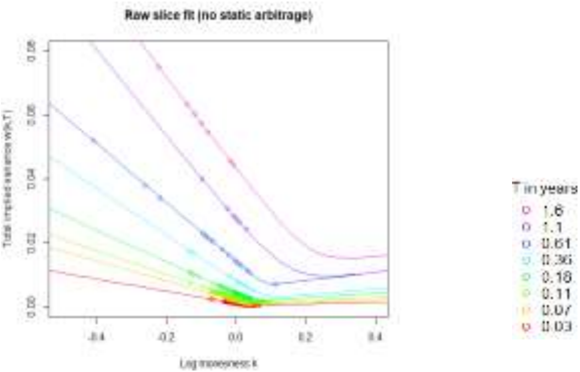


Figure 14: Optimal calibration – Fitting result

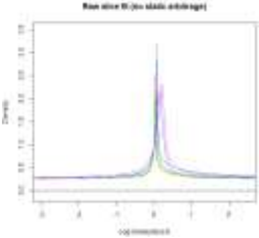


Figure 15: Optimal calibration – Density plot

- **No arbitrage fitting**
  - The crossed-ness of the fit is near zero (2.909224e-08), no crossing lines
  - Density of the slices: positive

- **SVI price error:** 0.5193063. SVI fits better than SSVI (price error: 4.50632). SVI fits well even at long-term maturity options

Interpolation

Given set of arbitrage-free calibrated slices, we can determine prices for options with any time to maturity T:

- For $T_1 < T < T_2$ compute the (undiscounted) option prices $C_i(F_i, K_i, T_i)$ ($i = 1, 2$) for two consecutive slices with expiry T_i
- Use monotonic extrapolation of θ_T ATM implied total variance (e.g. linear in T)
- For C_1, C_2 and C_T with identical log moneyness k, define

$$\alpha_T = \frac{\sqrt{\theta_{T2}} - \sqrt{\theta_T}}{\sqrt{\theta_{T2}} - \sqrt{\theta_{T1}}} € [0,1]$$

$$\frac{C_T}{K_T} = \alpha_T \frac{C_1}{K_1} + (1 - \alpha_T)\frac{C_2}{K_2}$$

- where $C_T = C(F_T, K_T, T)$ is the desired price of an option for any strike and expiry
- The resulting prices are again free of static-arbitrage (Gatheral)

Extrapolation

One example of algorithms for extrapolation of volatility surface:

When going beyond or beneath the time range spanned by our calibrated slices:

- For expiries between $T = 0$ and the first slice, use the intrinsic value of the option to interpolate (interpolation method in previous section)

- For extrapolation beyond the final slice Tn, expand the volatility surface as:
  - Re-calibrate the final slice using SSVI form
  - Then, fix a monotonic increasing extrapolation of θ_T and extrapolate slimes as:

$$w(k, \theta_T) = w(k, Tn) + \theta_T - \theta_{Tn}$$

- The slice $w(k, \theta_t)$ is free of static arbitrage if the last slice is free of static arbitrage

Conclusion

1. Conclusion

We have presented:

- Surface SVI, a benchmark model for implied volatility surface in Equity market
- SSVI calibration method to recent SPX options data whilst ensuring no arbitrage. The fitting result has been shown with high quality

2. Insights on SSVI model

- **Advantages**: simple parameterization and sufficient conditions for no static arbitrage
- **Shortcomings:**
    - Correlation parameter ρ is constant across expirations → eSSVI model
    - Entirely based on the market price rather than fundamentals.

However, SSVI often fits well in practice

3. Future research

- Calibration using bid-ask range instead of exact mid-price of options


Thank you!
Questions?